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# Statistical description of the diffusion of a passive tracer in a random velocity field

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## Contents



**Abstract.** A single functional approach is used to treat the problem of the diffusion of a tracer in a random velocity field, posed in a general form. Approximate methods leading to various approximate solutions and the conditions of their validity are examined. Plane parallel average flow is considered in detail and some features of statistical solutions are discussed for the simplest case.

## 1. Introduction

The problem of propagation of a passive tracer in a random velocity field is of major importance in ecological problems of oceanology and atmospheric physics. It has been studied since the end of the fifties starting with the pioneering papers of Batchelor et al [1, 2]. Later, many researchers derived a variety of equations describing statistical characteristics of the tracer field both in *Eulerian* and in *Lagrangian* description (e.g. Refs [3 – 7]). The derivation of such equations is still progressing at full pace at the present time. Meanwhile many researchers are, in essence, often repeating each other's work, deriving equations similar to ones obtained long ago. The assumptions which underly these derivations also do not differ in principle. It is quite easy to write down equations describing the statistical characteristics of a tracer field in the so called *delta-correlated approach* for a random velocity field (see, e.g., Refs [8 – 10] in which in Lagrangian representation the particles behave like ordinary Brownian particles. The most difficult problem is to take into approximate account the finiteness of the *temporal correlation radius* of the velocity field. Here, a single *functional approach* will be used to treat the problem of the diffusion of a tracer in a random velocity field, posed in a general form. Approximate methods leading to various approximations and the conditions of their validity will be examined. The case of plane parallel average flow will be considered in detail and special features of statistical solutions will be discussed for the simplest problem used as an example.

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## 2. Formulation of the problem

The principal equations governing the diffusion of a tracer in a random velocity field are of the following two types:

$$\left(\frac{\partial}{\partial t}+\boldsymbol{U}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)q(\boldsymbol{r},t)=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}q(\boldsymbol{r},t),\qquad q(\boldsymbol{r},0)=q_0(\boldsymbol{r}), \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{U}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)p_i(\boldsymbol{r},t)=-\frac{\partial u_k(\boldsymbol{r},t)}{\partial r_i}p_k(\boldsymbol{r},t)+\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}p_i(\boldsymbol{r},t),\qquad p(\boldsymbol{r},0)=p_0(\boldsymbol{r}). \tag{1'}$$

Equation (1) describes a scalar field $q(\boldsymbol{r},t)$ of such quantities as temperature, salinity, etc. of interest in geophysics,
and equation (1′) describes its spatial gradient $p(\boldsymbol{r},t)=\partial q(\boldsymbol{r},t)/\partial \boldsymbol{r}$. Let us note also the additional equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{U}(\boldsymbol{r},t)\right)\rho(\boldsymbol{r},t)=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\rho(\boldsymbol{r},t),$$
$$\rho(\boldsymbol{r},0)=\rho_0(\boldsymbol{r}) \tag{2}$$

for the 'material' density of the passive tracer. The quantity $\kappa$ denotes here the *molecular diffusion coefficient*.

The difference between equations (1) and (2) is that the latter is conservative, while the former is not. The velocity field is assumed to be a random field with the mean value $\boldsymbol{V}(\boldsymbol{r},t)=\langle\boldsymbol{U}(\boldsymbol{r},t)\rangle$ $(\boldsymbol{U}(\boldsymbol{r},t)=\boldsymbol{V}(\boldsymbol{r},t)+\boldsymbol{u}(\boldsymbol{r},t))$, and a fluctuating component with an ensemble average $\langle\boldsymbol{u}(\boldsymbol{r},t)\rangle=0$.

The flowing fluid can be *compressible* or *incompressible* $(\nabla\boldsymbol{U}(\boldsymbol{r},t)=0)$; in the latter case equations (1) and (2) are identical and the quantity $Q=\int \mathrm{d}\boldsymbol{r}\,q(\boldsymbol{r},t)$ is conserved. In the one-dimensional case equations (1′) and (2) are also identical. Let us note that in this case the fluid is always compressible.

Although equations (1), (1′), (2) are linear, solutions $q,p$, and $\rho$ depend in a complicated, implicit, nonlinear functional way on the velocity field $\boldsymbol{U}(\boldsymbol{r},t)$, i.e. $q=q([\boldsymbol{U}],\boldsymbol{r},t)$. The main problem here is to find the statistics of solutions such as the mean, correlation functions, probability distributions, and so on:

$$\langle q(\boldsymbol{r},t)\rangle,\quad \langle q(\boldsymbol{r}_1,t)q(\boldsymbol{r}_2,t)\rangle,\quad \langle q(\boldsymbol{r}_1,t_1)q(\boldsymbol{r}_2,t_2)\rangle,$$

in terms of the statistics of $\boldsymbol{U}$.

Equations (1), (1′), and (2) yield the *Eulerian description* of the system. We cannot study directly the probability distribution of $q(\boldsymbol{r},t)$ because equation (1) contains a second-order (diffusion) term in $\boldsymbol{r}$. But we can write a *variational* (*Hopf*) *equation* for the characteristic functional, which corresponds to a statistical analysis of solutions of (1), (1′), (2) in the infinite-dimensional functional space [8–11].

Let us introduce an auxiliary field $\widetilde{q}(\boldsymbol{r},t)$ described by the stochastic equation

$$\left(\frac{\partial}{\partial t}+\boldsymbol{U}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\widetilde{q}(\boldsymbol{r},t)=-\boldsymbol{\alpha}(t)\frac{\partial}{\partial \boldsymbol{r}}\widetilde{q}(\boldsymbol{r},t),$$
$$\widetilde{q}(\boldsymbol{r},0)=q_0(\boldsymbol{r}), \tag{3}$$

where $\boldsymbol{\alpha}(t)$ is a delta-correlated Gaussian random vector process (independent of $\boldsymbol{U}$) with parameters

$$\langle\boldsymbol{\alpha}(t)\rangle=0,\quad \langle\alpha_i(t)\alpha_j(t')\rangle=2\kappa\delta_{ij}\delta(t-t'),\quad i,j=1,2,3.$$

In that case solution of equation (1) corresponds to ensemble averaging of equation (3) relative to the $\boldsymbol{\alpha}$-process, so that

$$q(\boldsymbol{r},t)=\langle\widetilde{q}(\boldsymbol{r},t)\rangle_{\boldsymbol{\alpha}}\,. \tag{4}$$

Formula (4) gives a *path-integral representation* of solution (1).

Let us note that the first-order stochastic partial differential equation (3) can be solved by the *method of characteristics*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t)=\boldsymbol{U}(\boldsymbol{r}(t),t)+\boldsymbol{\alpha}(t),\quad \boldsymbol{r}(0)=\boldsymbol{\xi},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{q}(t)=0,\quad \widetilde{q}(0)=q_0(\boldsymbol{\xi}). \tag{5}$$

Solution of equations (5) depends on the initial parameter $\boldsymbol{\xi}$:

$$\boldsymbol{r}(t)=\boldsymbol{r}(t|\boldsymbol{\xi}),\quad \widetilde{q}(t)=\widetilde{q}(t|\boldsymbol{\xi}).$$

This is the *Lagrangian description*. Eliminating the parameter $\boldsymbol{\xi}$ from the solution of the set of equations (5) we get the *Eulerian description* of the tracer concentration field:

$$\boldsymbol{\xi}=\boldsymbol{\xi}(t|\boldsymbol{r}),\quad \widetilde{q}(\boldsymbol{r},t)=\widetilde{q}(t|\boldsymbol{\xi}(t|\boldsymbol{r})).$$

To find the statistics of solution (5) we shall introduce the function

$$\widetilde{\Phi}(\boldsymbol{r},t)=\delta(\boldsymbol{r}(t)-\boldsymbol{r}),$$

which satisfies *Liouville's equation* with the initial condition [8–10]

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{U}(\boldsymbol{r},t)\right)\widetilde{\Phi}(\boldsymbol{r},t)=-\boldsymbol{\alpha}(t)\frac{\partial}{\partial \boldsymbol{r}}\widetilde{\Phi}(\boldsymbol{r},t),$$
$$\widetilde{\Phi}(\boldsymbol{r},0)=\delta(\boldsymbol{r}-\boldsymbol{\xi}).$$

Once again ensemble averaging relative to $\boldsymbol{\alpha}$ yields a stochastic equation for the mean field $\phi(\boldsymbol{r},t)=\langle\widetilde{\Phi}(\boldsymbol{r},t)\rangle_{\boldsymbol{\alpha}}$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{U}(\boldsymbol{r},t)\right)\phi(\boldsymbol{r},t)=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\phi(\boldsymbol{r},t),$$
$$\phi(\boldsymbol{r},0)=\delta(\boldsymbol{r}-\boldsymbol{\xi}). \tag{6}$$

Of course, expression (6) is still a stochastic equation relative to the random field $\boldsymbol{U}$. Dynamic equation (6) coincides with equation (2), but has a point-source initial condition. So the probability density (6) behaves like the particle density (2), localised initially at a point source.

The transition from the Lagrangian to the Eulerian statistical description, i.e. from equation (5) to equation (1) involves the *Jacobian*

$$j(t|\boldsymbol{\xi})=\mathrm{Det}\left\|\frac{\partial r_i}{\partial \xi_j}\right\|,\quad j(0|\boldsymbol{\xi})=1.$$

The reciprocal of the Jacobian

$$\widetilde{\rho}(\boldsymbol{r},t)=J^{-1}(t|\boldsymbol{\xi}(t|\boldsymbol{r}))$$

satisfies the standard continuity equation in Eulerian coordinates

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{U}(\boldsymbol{r},t)\right)\widetilde{\rho}(\boldsymbol{r},t)=-\boldsymbol{\alpha}(t)\frac{\partial}{\partial \boldsymbol{r}}\widetilde{\rho}(\boldsymbol{r},t),$$
$$\widetilde{\rho}(\boldsymbol{r},0)=1, \tag{7}$$

which after additional ensemble averaging yields an equation for the mean density (over $\boldsymbol{\alpha}$), $\rho(\boldsymbol{r},t)=\langle\widetilde{\rho}(\boldsymbol{r},t)\rangle_{\boldsymbol{\alpha}}$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{U}(\boldsymbol{r},t)\right)\rho(\boldsymbol{r},t)=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\rho(\boldsymbol{r},t). \tag{8}$$

As mentioned above, for incompressible fluids equation (8) coincides with equation (1).

The basic equations (1), (2) allow us to compute the mean concentration $\langle q(\boldsymbol{r},t)\rangle$ or a one-particle probability density $\phi(\boldsymbol{r},t)$. To compute correlation, higher moments, or two- particle distributions, we must use the product of

fields $\widetilde{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = q(\boldsymbol{r}_1,t)q(\boldsymbol{r}_2,t)$, described by the linear equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \boldsymbol{U}(\boldsymbol{r}_1,t)\frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{U}(\boldsymbol{r}_2,t)\frac{\partial}{\partial \boldsymbol{r}_2}\right)\widetilde{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\\ &\quad = \kappa\left(\frac{\partial^2}{\partial \boldsymbol{r}_1^2} + \frac{\partial^2}{\partial \boldsymbol{r}_2^2}\right)\widetilde{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2,t),\\ &\widetilde{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2,0) = q_0(\boldsymbol{r}_1)q_0(\boldsymbol{r}_2),\end{aligned} \tag{9}$$

with subsequent averaging. Let us note that averaging products $\widetilde{q}(\boldsymbol{r}_1,t)\widetilde{q}(\boldsymbol{r}_2,t)$ from equation (3) over the $\boldsymbol{a}$-ensemble does not yield solutions of equation (9) since

$$\langle \widetilde{q}(\boldsymbol{r}_1,t)\widetilde{q}(\boldsymbol{r}_2,t)\rangle_{\boldsymbol{a}} \neq \langle \widetilde{q}(\boldsymbol{r}_1,t)\rangle_{\boldsymbol{a}}\langle \widetilde{q}(\boldsymbol{r}_2,t)\rangle_{\boldsymbol{a}}\,.$$

So to interpret the solution of equation (9) as a probability density we shall model the continuous field $q$ by a system of *particles*. The $k$-th particle of the passive scalar will then be described by the stochastic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}^{(k)}(t) = \boldsymbol{U}(\boldsymbol{r}^{(k)}(t),t) + \boldsymbol{a}^{(k)}(t), \quad \boldsymbol{r}^{(k)}(0) = \boldsymbol{\xi}^{(k)}, \tag{10}$$

where the sources $\boldsymbol{a}^{(k)}(t)$ are assumed to be statistically independent random processes. Next, let us introduce a measure valued (random) function

$$\widetilde{\phi}(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = \delta(\boldsymbol{r}^{(1)}(t) - \boldsymbol{r}_1)\delta(\boldsymbol{r}^{(2)}(t) - \boldsymbol{r}_2)$$

that describes the joint probability density for two particles. Function $\widetilde{\phi}$ satisfies the Liouville-type equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}_1}\boldsymbol{U}(\boldsymbol{r}_1,t) + \frac{\partial}{\partial \boldsymbol{r}_2}\boldsymbol{U}(\boldsymbol{r}_2,t)\right)\widetilde{\phi}(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\\ &\quad = -\left(\boldsymbol{a}^{(1)}(t)\frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{a}^{(2)}(t)\frac{\partial}{\partial \boldsymbol{r}_2}\right)\widetilde{\phi}(\boldsymbol{r}_1,\boldsymbol{r}_2,t),\end{aligned}$$

which after averaging over ensembles $\boldsymbol{a}^{(k)}$ for the function

$$\phi(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = \langle \widetilde{\phi}(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\rangle_{\boldsymbol{a}^{(1)},\boldsymbol{a}^{(2)}}$$

assumes the form

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}_1}\boldsymbol{U}(\boldsymbol{r}_1,t) + \frac{\partial}{\partial \boldsymbol{r}_2}\boldsymbol{U}(\boldsymbol{r}_2,t)\right)\phi(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\\ &\quad = \kappa\left(\frac{\partial^2}{\partial \boldsymbol{r}_1^2} + \frac{\partial^2}{\partial \boldsymbol{r}_2^2}\right)\phi(\boldsymbol{r}_1,\boldsymbol{r}_2,t),\\ &\phi(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = \delta(\boldsymbol{\xi}^{(1)}(t) - \boldsymbol{r}_1)\delta(\boldsymbol{\xi}^{(2)}(t) - \boldsymbol{r}_2).\end{aligned} \tag{11}$$

For an incompressible fluid, the latter equation is identical to equation (9). For compressible fluids, equation (11) describes the product of two densities $\widetilde{R}(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = \rho(\boldsymbol{r}_1,t)\rho(\boldsymbol{r}_2,t)$ satisfying the continuity equation (2). Equation (11) determines the *two-point probability density* for Lagrangian coordinates (10).

Our goal now is to average the linear stochastic equations (1), (2), (9)–(11) over the random ensembles of velocity fields $\boldsymbol{U}(\boldsymbol{r},t)$, to get the effective evolution of mean fields $\langle q(\boldsymbol{r},t)\rangle_{\boldsymbol{u}}$, two-point correlations

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = \langle \widetilde{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\rangle = \langle q(\boldsymbol{r}_1,t)q(\boldsymbol{r}_2,t)\rangle_{\boldsymbol{u}}\,,$$

etc.

## 3. Exact solutions

Averaging equations (1), (2) over the ensemble $\{\boldsymbol{u}\}$ yields an evolution equation for the mean field, where the random velocity field $\boldsymbol{u}$ is coupled through the fluctuation term

$$\left\langle \boldsymbol{u}\frac{\partial}{\partial \boldsymbol{r}}q\right\rangle \tag{12}$$

to random solution $q$, itself a functional of $\boldsymbol{u}$, i.e. $q = q[\boldsymbol{u};\ldots]$. So to get effective mean field evolution it is necessary to decouple (split) the cross-correlation term (12). The decoupling methods depend on the nature of the random field $\boldsymbol{u}$. In the Gaussian case the decoupling is carried out with the use of the so called *Furutsu–Novikov formula* [12, 13] (see also Refs [8–10])

$$\langle u_i(\boldsymbol{r},t)R[\boldsymbol{u}]\rangle = \int \mathrm{d}\boldsymbol{r}' \int \mathrm{d}t'\langle u_i(\boldsymbol{r},t)u_j(\boldsymbol{r}',t')\rangle \times \left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',t')}R[\boldsymbol{u}]\right\rangle, \tag{13}$$

which holds for any functional $R(\boldsymbol{u})$ of a random Gaussian field $\boldsymbol{u}$. Formula (13) is essentially integration by parts in functional space [14]. The case of non-Gaussian fluctuations of fluid flow velocity has been discussed by Samokhin and Chechetkin [15].

Applying formula (13) to the cross-correlation term (12) of the mean-field equation (2) we get the mean tracer concentration

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle\\ &\quad + \int_0^t \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}' B_{ij}(\boldsymbol{r},t;\boldsymbol{r}',t')\frac{\partial}{\partial r_i}\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',t')}q(\boldsymbol{r},t)\right\rangle\\ &\quad = \kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle,\end{aligned} \tag{14}$$

where $B_{ij}(\boldsymbol{r},t;\boldsymbol{r}',t') = \langle u_i(\boldsymbol{r},t)u_j(\boldsymbol{r}',t')\rangle$ is the space–time correlation function of field $\boldsymbol{u}$. Equation (14) is exact for any zero mean Gaussian field $\boldsymbol{u}$, but is not closed since evolution of the mean field is coupled to the mean variational derivative of $\boldsymbol{u}$. The latter, $\delta q/\delta \boldsymbol{u}$, is described by a stochastic differential equation, obtained by varying equation (1) in $\boldsymbol{u}$

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{u}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\delta}{\delta u_j(\boldsymbol{r}',t')}q(\boldsymbol{r},t)\\ &\quad = \kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\frac{\delta}{\delta u_j(\boldsymbol{r}',t')}q(\boldsymbol{r},t),\\ &\frac{\delta}{\delta u_j(\boldsymbol{r}',t')}q(\boldsymbol{r},t)\bigg|_{t\to t'+0} = -\delta(\boldsymbol{r}-\boldsymbol{r}')\frac{\partial}{\partial r_j}q(\boldsymbol{r},t').\end{aligned} \tag{15}$$

Let us note that equation (15) is essentially equivalent to the Green function for the original problem (1). Applying the ensemble average to equation (15) along with the Furutsu–Novikov formula leads to the appearance of higher-order derivatives $\langle \delta^2 q/\delta u_i \delta u_j\rangle$ etc. The resulting system of equations requires a suitable closure, which can be implemented exactly in some special cases. Two of them will be discussed in detail: the delta-correlated field in the time- variable $\boldsymbol{u}(\boldsymbol{r},t)$ and the telegraph process.

### 3.1 Delta-correlated field

Let $\boldsymbol{u}(\boldsymbol{r},t)$ be a Gaussian delta-correlated field with parameters

$$\begin{aligned}\langle \boldsymbol{u}(\boldsymbol{r},t)\rangle &= 0,\\ B_{ij}(\boldsymbol{r},t;\boldsymbol{r}',t') &= \langle u_i(\boldsymbol{r},t)u_j(\boldsymbol{r}',t')\rangle\\ &= 2B_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{r}';t)\delta(t-t').\end{aligned} \tag{16}$$

Then for delta-correlated fields $\boldsymbol{u}(\boldsymbol{r},t)$ given by equation (16) we get

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle\\ &\quad+2\int_0^t \mathrm{d}t'\int \mathrm{d}\boldsymbol{r}' B_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{r}',t)\delta(t-t')\\ &\quad\times\frac{\partial}{\partial r_i}\left\langle\frac{\delta}{\delta u_j(\boldsymbol{r}',t')}q(\boldsymbol{r},t)\right\rangle=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle\end{aligned}$$

and the variational derivative is expressed through $\delta q(\boldsymbol{r},t)/\delta u_j(\boldsymbol{r}',t')$ at $t\to t'$, i.e. through the initial conditions of equation (15). This yields

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle\\ &\quad-2\int_0^t \mathrm{d}t'\int \mathrm{d}\boldsymbol{r}' B_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{r}',t)\delta(t-t')\\ &\quad\times\frac{\partial}{\partial r_j}\delta(\boldsymbol{r}-\boldsymbol{r}')\frac{\partial}{\partial r_i}\langle q(\boldsymbol{r},t)\rangle=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle.\end{aligned}$$

Integration in $t'$ and $\boldsymbol{r}'$ gives the closed-form equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle-A_j(\boldsymbol{r},t)\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r},t)\rangle\\ &\quad=\left(B_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{r},t)\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_i}+\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\right)\langle q(\boldsymbol{r},t)\rangle,\end{aligned}$$

with coefficients

$$A_j(\boldsymbol{r},t)=\frac{\partial}{\partial r_i}B_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{r}',t)\bigg|_{\boldsymbol{r}'=\boldsymbol{r}}.$$

If the random field $\boldsymbol{u}(\boldsymbol{r},t)$ is homogeneous and isotropic (or the original flow $\boldsymbol{U}(\boldsymbol{r},t)$ is that of an incompressible fluid), then coefficients $A_j(\boldsymbol{r},t)=0$. In this case $B_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{r}',t)=B_{ij}(\boldsymbol{r}-\boldsymbol{r}',t)$, and $B_{ij}^{\text{eff}}(0,t)=\delta_{ij}B(t)$, where $B(t)=N^{-1}B_{ii}^{\text{eff}}(0,t)$ and $N$ denotes the dimensionality of the space: $N=1;2;3$. Hence we get

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle=\big(B(t)+\kappa\big)\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle. \tag{17}$$

In a similar way we obtain an equation for the correlation function $\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t)=\langle q(\boldsymbol{r}_1,t)q(\boldsymbol{r}_2,t)\rangle_{\boldsymbol{u}}$:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r}_1,t)\frac{\partial}{\partial \boldsymbol{r}_1}+\boldsymbol{V}(\boldsymbol{r}_2,t)\frac{\partial}{\partial \boldsymbol{r}_2}\right)\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\\ &\quad+\int_0^t \mathrm{d}t'\int \mathrm{d}\boldsymbol{r}'\left(B_{ij}(\boldsymbol{r}_1,t;\boldsymbol{r}',t')\frac{\partial}{\partial r_{1i}}\right.\\ &\quad\left.+B_{ij}(\boldsymbol{r}_2,t;\boldsymbol{r}',t')\frac{\partial}{\partial r_{2i}}\right)\left\langle\frac{\delta}{\delta u_j(\boldsymbol{r}',t')}q(\boldsymbol{r}_1,t)q(\boldsymbol{r}_2,t)\right\rangle\\ &\quad=\kappa\left(\frac{\partial^2}{\partial \boldsymbol{r}_1^2}+\frac{\partial^2}{\partial \boldsymbol{r}_2^2}\right)\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t),\end{aligned}$$

which for a homogeneous isotropic and delta-correlated field $\boldsymbol{u}(\boldsymbol{r},t)$ becomes

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r}_1,t)\frac{\partial}{\partial \boldsymbol{r}_1}+\boldsymbol{V}(\boldsymbol{r}_2,t)\frac{\partial}{\partial \boldsymbol{r}_2}\right)\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t)\\ &\quad=\left[\big(B(t)+\kappa\big)\left(\frac{\partial^2}{\partial \boldsymbol{r}_1^2}+\frac{\partial^2}{\partial \boldsymbol{r}_2^2}\right)\right.\\ &\qquad\left.+2B_{ij}^{\text{eff}}(\boldsymbol{r}-\boldsymbol{r}',t)\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}\right]\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t).\end{aligned} \tag{18}$$

In the particular case when the mean flow velocity $\boldsymbol{V}(\boldsymbol{r},t)=0$ and $q(\boldsymbol{r},0)=q_0$ is constant, then the random field $q(\boldsymbol{r},t)$ will also be homogeneous and isotropic. Hence

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2,t)=\Gamma(\boldsymbol{r}_1-\boldsymbol{r}_2,t),$$

and equation (18) simplifies to

$$\frac{\partial}{\partial t}\Gamma(\boldsymbol{r},t)=2\big(B(t)+\kappa\big)\frac{\partial^2}{\partial \boldsymbol{r}^2}-2B_{ij}^{\text{eff}}(\boldsymbol{r},t)\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\Gamma(\boldsymbol{r},t), \tag{19}$$

where $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$.

Let us note that equations (17), (19) for incompressible fluid flow have the form of the standard *Fokker–Planck equation* for one-point and two-point distribution densities of Lagrangian coordinates; furthermore the Lagrangian description of problem (10) is a *Markov process*. For the case of turbulent fluid flow, equation (19) has been analysed by Lutovinov and Chechetkin [16].

### 3.2 The telegraph process

Let us now assume that the random velocity field has the form $\boldsymbol{u}(\boldsymbol{r},t)=\boldsymbol{g}(\boldsymbol{r},t)z(t)$ with a deterministic factor $\boldsymbol{g}(\boldsymbol{r},t)$, and the standard telegraph process $z(t)$ (see, for example, Refs [9, 10]) is given by

$$z(t)=a(-1)^{n(0,t)}.$$

The random variable $a$ takes on values $\mp a_0$ with probability $\frac{1}{2}$ and $n(t_1,t_2)$ is a standard random integer process of *Poisson point-flux* with mean value $\langle n(t_1,t_2)\rangle=\nu|t_2-t_1|$ and with the following properties:

1. $n(t_1;t_3)=n(t_1;t_2)+n(t_2;t_3)$ for any $t_1<t_2<t_3$;
2. $n(t_1;t_2)$ and $n(t_2;t_3)$ are statistically independent for $t_1<t_2<t_3$;
3. the probability of the existence of $m$ events in the interval $[t_1,t_2]$ is

$$P\big(n(t_1;t_2)=m\big)=\frac{\langle n(t_1;t_2)\rangle^m}{m!}\exp\big(-\langle n(t_1;t_2)\rangle\big).$$

Thus $z(t)$ is a stationary Markov process with the correlation function

$$\langle z(t)z(t')\rangle=a_0^2\exp(-2\nu|t_1-t_2|)$$

and correlation radius $l_0=1/2\nu$.

As before, let us average solutions of equations (1), (2) over the ensemble $z(t)$. This time we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle_z+\boldsymbol{g}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\langle z(t)q(\boldsymbol{r},t)\rangle_z\\ &\quad=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle_z\end{aligned}$$

or

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle_z+\boldsymbol{g}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{\Psi}(\boldsymbol{r},t)$$
$$=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle_z\,, \tag{20}$$

with a new unknown cross-correlation function

$$\boldsymbol{\Psi}(\boldsymbol{r},t)=\langle z(t)q(\boldsymbol{r},t)\rangle_z\,.$$

It can be shown that the latter is described by the equation

$$\left(\frac{\partial}{\partial t}+2\nu+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\boldsymbol{\Psi}(\boldsymbol{r},t)+a_0^2\boldsymbol{g}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\langle q(\boldsymbol{r},t)\rangle_z$$
$$=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\boldsymbol{\Psi}(\boldsymbol{r},t), \tag{21}$$

which plays a role analogous to that of the Furutsu–Novikov formula in the previous section (delta-correlated field). The derivation of expression (21) is based on the following *differentiation rule* for telegraph processes

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+2\nu\right)\langle z(t)R_t[z(\tau)]\rangle=\left\langle z(t)\frac{\mathrm{d}}{\mathrm{d}t}R_t[z(\tau)]\right\rangle,$$

valid for the cross-correlation of $z(t)$ with an arbitrary functional $R_t[z(\tau)]$, defined for $\tau\leqslant t$ (see, for example, Refs [9, 10, 17]). Thus we get a closed system of equations (20), (21) for the unknown mean field $\langle q(\boldsymbol{r},t)\rangle$ and correlation $\boldsymbol{\Psi}$.

Let us note that in the limiting case $\nu\to\infty$, $a_0^2\to\infty$, $a_0^2/2\nu\to$ const the telegraph process turns into the Gaussian delta-correlated one. Since in this limiting case

$$\boldsymbol{\Psi}(\boldsymbol{r},t)=-a_0^2\boldsymbol{g}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\langle q(\boldsymbol{r},t)\rangle_z\,,$$

we arrive at a single equation for the mean field $\langle q(\boldsymbol{r},t)\rangle_z$, which corresponds to the Fokker–Planck equation for an incompressible fluid.

The above procedure is appropriate for the telegraph process but the underlying method is in reality more general in nature. To illustrate this, let us now examine the derivation of these equations from a somewhat different viewpoint. Once again we apply ensemble averaging to equation (1) to get equation (20)

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle_z+g(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\langle z(t)q(\boldsymbol{r},t)\rangle_z$$
$$=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle_z\,.$$

To decouple the cross-correlation term $\langle z(t)q(\boldsymbol{r},t)\rangle$ we shall use the identity [9, 10]

$$\langle z(t)R_t[z(\tau)]\rangle$$
$$=a_0^2\int_0^t \mathrm{d}t'\exp[-2\nu(t-t')]\left\langle\frac{\delta}{\delta z(t')}\widetilde{R}_t[t';z(\tau)]\right\rangle, \tag{22}$$

where the functional

$$\widetilde{R}_t[t';z(\tau)]=R_t[z(\tau)\theta(t'-\tau)]$$

and

$$\theta(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$$

denotes the standard Heavyside step-function. The decoupling relation (22) closely resembles formula (13) for Gaussian fields with exponential correlation functions. The only difference is that the right-hand side of expression (22) contains a truncated functional $\widetilde{R}_t$ extending over the interval $[t',t]$ rather than $R_t[z(\tau)]$ extending over the entire range of $t$, namely

$$\widetilde{R}_t[t';z(\tau)]=\begin{cases}R_t[z(\tau)], & \tau<t',\\ R_t[0], & \tau>t'.\end{cases}$$

Thus we get from expression (22)

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle_z+a_0^2\boldsymbol{g}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\int_0^t \mathrm{d}t'$$
$$\times\exp[-2\nu(t-t')]\left\langle\frac{\delta}{\delta z(t')}\widetilde{q}[t';z(\tau)]\right\rangle=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle_z\,.$$

The functional $\widetilde{q}[t';z(\tau)]$ is described by equation (1) with the random component of velocity

$$\boldsymbol{u}(\boldsymbol{r},t)=\boldsymbol{g}(\boldsymbol{r},t)z(t)\theta(t'-t).$$

Hence for $t>t'$ we obtain the equation

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\widetilde{q}(\boldsymbol{r},t)=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\widetilde{q}(\boldsymbol{r},t) \tag{24}$$

with the initial condition

$$\widetilde{q}(\boldsymbol{r},t)\Big|_{t=t'}=q(\boldsymbol{r},t'). \tag{25}$$

Varying (24), (25) in $z(t')$ we get for $t>t'$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\delta}{\delta z(t')}\widetilde{q}(\boldsymbol{r},t)=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\frac{\delta}{\delta z(t')}\widetilde{q}(\boldsymbol{r},t) \tag{26}$$

with the initial condition

$$\frac{\delta}{\delta z(t')}\widetilde{q}(\boldsymbol{r},t)\Big|_{t=t'}=-\boldsymbol{g}(\boldsymbol{r},t')\frac{\partial}{\partial \boldsymbol{r}}q(\boldsymbol{r},t'),$$

i.e. an equation of the type of equation (15) but without the fluctuating component of velocity. If we now introduce the function

$$\boldsymbol{\Psi}(\boldsymbol{r},t)=a_0^2\int_0^t \mathrm{d}t'\exp[-2\nu(t-t')]\left\langle\frac{\delta}{\delta z(t')}\widetilde{q}[t';z(\tau)]\right\rangle,$$

then it is easily seen from expressions (24), (25) that $\boldsymbol{\Psi}$ is a solution of equation (21). Thus in the case of telegraph-type velocity fluctuations we are able to produce a closed-form system of equations (20), (21).

Note that if we consider velocity fluctuations of the above type $\boldsymbol{u}(\boldsymbol{r},t)=g(\boldsymbol{r},t)z(t)$ with the *Markov Gaussian process* $z(t)$, having the same parameters $a_0$ and $\nu$, then the stochastic equation for variational derivative $\delta q/\delta z$ becomes

$$\left(\frac{\partial}{\partial t}+\boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}+\boldsymbol{u}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\delta}{\delta z(t')}q(\boldsymbol{r},t)$$
$$=\kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\frac{\delta}{\delta z(t')}q(\boldsymbol{r},t).$$

This resembles expression (26) but has an additional random advection term $\boldsymbol{u}(\boldsymbol{r},t)\partial/\partial\boldsymbol{r}$ in the left-hand side. If we drop this extra term we get back to the stochastic equation corresponding to the telegraph process. One can

therefore regard the telegraph equation as an approximation of the Gaussian Markov process. Another way to see this relation is to represent a Gaussian Markov process $z(t)$ with the correlation function

$$\sigma^2 \exp[-2\nu(t-t')],$$

as a limit of the sum of independent telegraph processes $\{z_j(t)\}$

$$z(t) = \lim_{N\to\infty} \sum_1^N z_k(t)$$

with correlation functions [9, 10]

$$\langle z_i(t) z_j(t')\rangle = \delta_{ij} \frac{\sigma^2}{N} \exp\left(-2\nu|t-t'|\right).$$

## 4. Approximate methods

Let us now discuss approximate methods of analysis of propagation of a tracer in a random velocity field and their contribution to solutions of stochastic equations, as in reality fluctuations of the velocity field are not likely to behave like delta-correlated or telegraph processes. The latter can arise only as certain asymptotic limits. In other words, asymptotic expansions of exact solutions should yield certain simplified field equations statistically equivalent to the delta-correlated or telegraph models.

### 4.1 Method of successive iterations

Let $\boldsymbol{u}(\boldsymbol{r},t)$ be a Gaussian field with the correlation function $B_{ij}(\boldsymbol{r},t;\boldsymbol{r}',t')$. We have then the exact equation (14) which contains the variational derivative $\delta q/\delta \boldsymbol{u}$. The latter is in turn described by the stochastic equation (15). Averaging expression (15) over the ensemble we get another exact equation which contains the second variational derivative $\delta^2 q/\delta \boldsymbol{u}\,\delta \boldsymbol{u}$ etc. The method of successive iterations requires the resulting system to be closed at the $n$-th level, i.e. for $\delta^n q/\delta \boldsymbol{u}\ldots\delta \boldsymbol{u}$. Usually the closure is based on a suitable delta-correlation assumption (see, for example, Refs [8–10, 18]. The method consists of a step by step improvement of the functional dependence of $q$ on $\boldsymbol{u}$. In some cases one can argue on physical grounds that the $n$-th order corrections to $q[\boldsymbol{u}]$ give a negligible contribution to the general solution, and therefore can be dropped. Let us note that in the case when there is no average flow ($\boldsymbol{V}=0$) and the correlation functions $B_{ij}(\boldsymbol{r},t)$ decrease sufficiently rapidly in space–time variables, the delta-correlation approximation predicts the correct asymptotics of solutions as $t\to\infty$. This was also confirmed by numerical simulations [19]. If, however, correlation functions $B_{ij}(\boldsymbol{r},t)$ have a more complicated structure (for instance in turbulent velocity fields [11]), or when there is an average flow, the delta-correlated approximation is clearly insufficient.

For practical purposes it is often sufficient to close the averaged equation (15) at the second step, when using delta-correlation hypothesis. Although such closures are widely used in the physics of plasma and ionosphere, in particular in problems related to magnetic field generation by turbulent gas or fluid flows [20] there is no satisfactory mathematical justification for the closure and the validity of the resulting approximation of $q$ (even for large $n$).

### 4.2 Telegraph approximation

Let the correlation function $B_{ij}(\boldsymbol{r},t;\boldsymbol{r}',t')$, as a function of time difference $t-t'$, be characterised by the correlation radius $t_0$,

$$B = B\left(\frac{t-t'}{t_0}\right).$$

If account is taken of the fact that equation (14) contains an integral in $t'$, the principal range of integration of $\delta q/\delta \boldsymbol{u}$ is on the scale of $t_0$. If we make a physically justified assumption that on such scales the random velocity component $\boldsymbol{u}$ does not enter into $\delta q/\delta \boldsymbol{u}$ (i.e. the latter remains functionally independent of $\boldsymbol{u}$), then we may drop the fluctuating term in expression (15). Thus we arrive at a closed-form description that could be called the telegraph approximation, as we have shown it earlier to be the exact description of the telegraph process. Now we get a system of coupled equations for two means $\langle q(\boldsymbol{r}',t)\rangle$ and $\langle \delta q/\delta \boldsymbol{u}\rangle$,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r},t)\rangle + \int_0^t \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}' B_{ij}(\boldsymbol{r},t;\boldsymbol{r}',t') \times \frac{\partial}{\partial r_i}\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',t')} q(\boldsymbol{r},t)\right\rangle = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},t)\rangle,$$

and

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right)\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',t')} q(\boldsymbol{r},t)\right\rangle = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2}\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',t')} q(\boldsymbol{r},t)\right\rangle, \tag{27}$$

$$\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',t')} q(\boldsymbol{r},t)\right\rangle\Big|_{t\to t'} = -\delta(\boldsymbol{r}-\boldsymbol{r}')\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r},t')\rangle.$$

We have already mentioned that the equation for the variational derivative is essentially equivalent to Green's function of the original problem (1) in the absence of the velocity fluctuations. Let us note that system (27) is equivalent to the one obtained by Lipscombe et al [21], although their method is entirely different. Another well-known equation follows from expressions (27) in the absence of average flow and molecular diffusion [6]. The system of equations (27) is too complicated for direct analysis, so it needs to be further simplified.

### 4.3 Diffusion approximation

Let us make an additional assumption with respect to the fluctuating velocity component. Namely, let us assume that on the time scale of the correlation radius $t_0$ velocity $\boldsymbol{u}$ has a negligible effect on the dynamics of $q$ as well as on the functional dependence of $q$ on $\boldsymbol{u}$. On such a scale the dynamics of a passive scalar field can be approximately described by the equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}(\boldsymbol{r},t)\frac{\partial}{\partial \boldsymbol{r}}\right) q(\boldsymbol{r},t) = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2} q(\boldsymbol{r},t),$$

$$q(\boldsymbol{r},t)\Big|_{t\to t'} = q(\boldsymbol{r},t').$$

As a consequence we get an additional relation between quantities $q(\boldsymbol{r},t)$ and $q(\boldsymbol{r},t')$, which allows us to eliminate the second of the two equations (27). Hence we obtain a closed first-order equation in $t$ for the mean concentration $\langle q(\boldsymbol{r},t)\rangle$. For large time scales ($t \gg t_0$) we note that $q(\boldsymbol{r},t)$ behaves like a Markov random field in $t$, which justifies the

name of this approximation as the diffusion random field approximation (diffusion approximation) [22].

## 5. The case of plane parallel average flow

Let us apply the formalism developed above to the case of planar incompressible fluid flow with mean velocity

$$\boldsymbol{V}(\boldsymbol{r},\, t) = v(y)\mathbf{1}\,,$$

where $\boldsymbol{r} = (x, y), \mathbf{1} = (1, 0)$. In this case equation (1) can be written in the form

$$\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x} + \boldsymbol{u}(\boldsymbol{r},\, t)\frac{\partial}{\partial \boldsymbol{r}}\right) q(\boldsymbol{r},\, t) = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2} q(\boldsymbol{r},\, t) \tag{28}$$

and the corresponding Lagrangian equations for a 'particle' become

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x(t) &= v(y) + u_1(\boldsymbol{r},\, t) + \alpha_1(t), \\ \frac{\mathrm{d}}{\mathrm{d}t} y(t) &= +u_2(\boldsymbol{r},\, t) + \alpha_2(t), \end{aligned} \tag{29}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are statistically independent stochastic 'white noise' processes.

The random field $\boldsymbol{u}(\boldsymbol{r}, t)$ is assumed to be an incompressible Gaussian homogeneous and isotropic and stationary random field with a space–time correlation function

$$B_{ij}(\boldsymbol{r} - \boldsymbol{r}',\, t - t') = \langle u_i(\boldsymbol{r},\, t) u_j(\boldsymbol{r}',\, t)\rangle,$$

characterised by the following quantities: variance $\sigma_u^2 = B_{ii}(0, 0)$, and space–time correlation radii $l_0$ and $t_0$. Let us introduce instead of $B_{ij}$ its spectral space density $E_{ij}(\boldsymbol{k}, t)$ according to the formula

$$B_{ij}(\boldsymbol{r},\, t) = \int \mathrm{d}\boldsymbol{k}\, E_{ij}(\boldsymbol{k},\, t) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}).$$

For a homogeneous and isotropic 'turbulence'

$$E_{ij}(\boldsymbol{k},\, t) = E(k,\, t)(\delta_{ij} - k_i k_j k^{-2}) \tag{30}$$

and consequently

$$B_{ij}(\boldsymbol{r},\, t) = \int \mathrm{d}\boldsymbol{k}\, E(k,\, t)(\delta_{ij} - k_i k_j k^{-2}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}),$$

and the variance of the field $\boldsymbol{u}(\boldsymbol{r}, t)$ is found from the formulae

$$\sigma_u^2 = 2\pi \int_0^\infty \mathrm{d}k\, kE(k, 0);\; B_{ij}(0, t) = \pi \int_0^\infty \mathrm{d}k\, kE(k, t)\delta_{ij}. \tag{31}$$

We shall consider equation (28) with the initial condition corresponding to 'point distribution'

$$q(\boldsymbol{r},\, 0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{32}$$

In this case, the solution of problem (28) is a function of the parameter $\boldsymbol{r}_0$, i.e. $q(\boldsymbol{r}, t) = q(\boldsymbol{r}, t|\boldsymbol{r}_0)$, and in the case of an arbitrary random initial distribution of the tracer

$$q(\boldsymbol{r},\, 0) = q_0(\boldsymbol{r})$$

the solution is determined by the convolution

$$q(\boldsymbol{r},\, t) = \int \mathrm{d}\boldsymbol{r}_0 q(\boldsymbol{r},\, t|\boldsymbol{r}_0) q_0(\boldsymbol{r}_0).$$

As has been demonstrated earlier, under the initial condition (32) the solution of problem (28) for the mean value $\langle q(\boldsymbol{r}, t|\boldsymbol{r}_0)\rangle$ coincides with 'one-particle' probability density for the Lagrangian coordinate of the particle (29), and the quantity $\langle q(\boldsymbol{r}_1, t|\boldsymbol{r}_0^{(1)}) q(\boldsymbol{r}_2, t|\boldsymbol{r}_0^{(2)})\rangle$ coincides with 'two-particle' probability density.

Let us then consider problem (28), (32). We are interested in the mean concentration of the tracer $\langle q(\boldsymbol{r}, t)\rangle$. Averaging equation (28) over the ensemble of the random field $\boldsymbol{u}$ we get

$$\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)\langle q(\boldsymbol{r},\, t)\rangle + \int_0^t \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}' B_{ij}(\boldsymbol{r} - \boldsymbol{r}',\, t - t') \times \frac{\partial}{\partial r_i}\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t)\right\rangle = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r},\, t)\rangle. \tag{33}$$

For the variational derivative $\delta q(\boldsymbol{r}, t)/\delta u_j(\boldsymbol{r}', t')$ we have the following stochastic equation

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x} + \boldsymbol{u}(\boldsymbol{r},\, t)\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t) \\ &\quad + \delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t')\frac{\partial}{\partial r_j} q(\boldsymbol{r},\, t') \\ &\quad = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2}\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t)\,, \end{aligned} \tag{34}$$

with

$$\left.\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t)\right|_{t=0} = 0,$$

$$\left.\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t)\right|_{t\to t'+0} = -\delta(\boldsymbol{r} - \boldsymbol{r}')\frac{\partial}{\partial r_j} q(\boldsymbol{r},\, t')$$

or an equation with the initial condition

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x} + \boldsymbol{u}(\boldsymbol{r},\, t)\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t) \\ &\quad = \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2}\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t) \quad (t > t'), \\ &\left.\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t)\right|_{t=t'} = -\delta(\boldsymbol{r} - \boldsymbol{r}')\frac{\partial}{\partial r_j} q(\boldsymbol{r},\, t'). \end{aligned} \tag{35}$$

In geophysical problems, the quantity $\kappa$ — the coefficient of molecular diffusion — is usually rather small. Therefore the term containing $\kappa$ may be left out from equations (34), (35) (in any case we are interested in the limit $\kappa \to 0$), i.e. for the variational derivative we can write down the equation

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x} + \boldsymbol{u}(\boldsymbol{r}, t)\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\delta}{\delta u_j(\boldsymbol{r}', t')} q(\boldsymbol{r},\, t) = 0 \quad (t > t'), \\ &\left.\frac{\delta}{\delta u_j(\boldsymbol{r}',\, t')} q(\boldsymbol{r},\, t)\right|_{t=t'} = -\delta(\boldsymbol{r} - \boldsymbol{r}')\frac{\partial}{\partial r_j} q(\boldsymbol{r},\, t'). \end{aligned} \tag{36}$$

But let us keep the term containing $\kappa$ in equation (33), because in some cases it can play the role of a regularisation factor.

Let us now consider various approximations.

1. In the approximation of the delta-correlated random field $\boldsymbol{u}(\boldsymbol{r}, t)$, the quantity $\delta q/\delta u_j$, which enters into expres-

sion (33), is determined by the initial condition (36) for $t' = t$, i.e. by the expression

$$\frac{\delta}{\delta u_j(\boldsymbol{r}', t')} q(\boldsymbol{r}, t) = -\delta(\boldsymbol{r} - \boldsymbol{r}') \frac{\partial}{\partial r_j} q(\boldsymbol{r}, t) \tag{37}$$

and equation (33) for $t \gg t_0$, where $t_0$ is the temporal correlation radius of the field $\boldsymbol{u}$, is transformed into

$$\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)\langle q(\boldsymbol{r}, t)\rangle = \int_0^t \mathrm{d}t' B_{ij}(0, t') \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r}, t)\rangle + \kappa \frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r}, t)\rangle,$$

which can be written in the form

$$\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)\langle q(\boldsymbol{r}, t)\rangle = D\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r}, t)\rangle, \tag{38}$$

where, according to equation (31), the quantity

$$D = \kappa + D^{\mathrm{T}}, \tag{39}$$

with

$$D^{\mathrm{T}} = \pi \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}k\, kE(k, t)$$

is the coefficient of 'turbulent' diffusion.

Equation (39) now takes on the form of the Fokker–Planck equation for the probability density of a Lagrangian particle coordinate (29).

2. In the telegraph stochastic process approximation we get the following equation for the mean value of the variational derivative

$$\begin{gathered}\left(\frac{\partial}{\partial t} + v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}', t')} q(\boldsymbol{r}, t)\right\rangle = 0 \qquad (t > t'),\\ \left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}', t')} q(\boldsymbol{r}, t)\right\rangle\Big|_{t=t'} = -\delta(\boldsymbol{r} - \boldsymbol{r}')\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r}, t')\rangle.\end{gathered} \tag{40}$$

In this approximation we have a closed system of equations (33) and (40) which is of the second order with respect to time. From equation (40) we can obtain the relationship between the quantities $\langle \delta q/\delta u\rangle$ and $\langle q\rangle$ in the form

$$\left\langle \frac{\delta}{\delta u_j(\boldsymbol{r}', t')} q(\boldsymbol{r}, t)\right\rangle = -\exp\left[-(t - t')v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right] \times \delta(\boldsymbol{r} - \boldsymbol{r}')\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r}, t')\rangle. \tag{41}$$

Substituting expression (41) into equation (33) we obtain an integro-differential equation for $\langle q(\boldsymbol{r}, t)\rangle$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)&\langle q(\boldsymbol{r}, t)\rangle\\ &= \kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r}, t)\rangle + \int_0^t \mathrm{d}t' \int \mathrm{d}\boldsymbol{r}' B_{ij}(\boldsymbol{r} - \boldsymbol{r}', t - t')\\ &\quad\times \frac{\partial}{\partial r_i}\exp\left[-(t - t')v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right]\delta(\boldsymbol{r} - \boldsymbol{r}')\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r}, t')\rangle,\end{aligned} \tag{42}$$

which, on taking into account the incompressibility of the field $\boldsymbol{u}$, after integration with respect to $\boldsymbol{r}'$, can be written in a final form as

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)&\langle q(\boldsymbol{r}, t)\rangle = \kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r}, t)\rangle\\ &+ \frac{\partial}{\partial r_i}\int_0^t \mathrm{d}\tau\, B_{ij}(\tau v(y)\mathbf{1}, \tau)\\ &\times \exp\left(-\tau v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r}, t - \tau)\rangle.\end{aligned} \tag{43}$$

Notice that in equation (43)

$$\begin{aligned}&\exp\left(-\tau v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)\frac{\partial}{\partial r_j}\langle q(\boldsymbol{r}, t - \tau)\rangle\\ &\quad= \left(\frac{\partial}{\partial r_j} + \frac{\mathrm{d}v(y)}{\mathrm{d}y}\delta_{j2}\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r} - \tau v(y)\mathbf{1}, t - \tau)\rangle.\end{aligned}$$

3. In the diffusion approximation the quantity $\langle q(\boldsymbol{r}, t')\rangle$ on the right-hand side of equation (42) can be determined from the original dynamical system (28) in the absence of the fluctuation term and the term containing the parameter $\kappa$

$$\left(\frac{\partial}{\partial t} + v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)q(\boldsymbol{r}, t) = 0,$$

$$q(\boldsymbol{r}, t)\Big|_{t=t'} = q(\boldsymbol{r}, t')$$

and, consequently,

$$\langle q(\boldsymbol{r}, t)\rangle = \exp\left[-(t - t')v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right]\langle q(\boldsymbol{r}, t')\rangle$$

or

$$\langle q(\boldsymbol{r}, t')\rangle = \exp\left[(t - t')v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right]\langle q(\boldsymbol{r}, t)\rangle. \tag{44}$$

Substituting expression (44) into equation (43) we obtain a closed equation of the first order in time

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)&\langle q(\boldsymbol{r}, t)\rangle = \kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r}, t)\rangle\\ &+ \frac{\partial}{\partial r_i}\int_0^t \mathrm{d}\tau\, B_{ij}(\tau v(y)\mathbf{1}, \tau)\exp\left(-\tau v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)\\ &\times \frac{\partial}{\partial r_j}\exp\left(\tau v(y)\mathbf{1}\frac{\partial}{\partial \boldsymbol{r}}\right)\langle q(\boldsymbol{r}, t - \tau)\rangle,\end{aligned}$$

which can be rewritten in the form

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v(y)\frac{\partial}{\partial x}\right)&\langle q(\boldsymbol{r}, t)\rangle = \kappa\frac{\partial^2}{\partial \boldsymbol{r}^2}\langle q(\boldsymbol{r}, t)\rangle\\ &+ \frac{\partial}{\partial r_i}\int_0^t \mathrm{d}\tau\left(D_{ij}^{(1)}(\boldsymbol{r}, \tau)\frac{\partial}{\partial r_j} + D_{i2}^{(2)}(\boldsymbol{r}, \tau)\frac{\partial}{\partial x}\right)\langle q(\boldsymbol{r}, t - \tau)\rangle.\end{aligned} \tag{45}$$

Here

$$D_{ij}^{(1)}(\boldsymbol{r}, \tau) = B_{ij}(\tau v(y)\mathbf{1}, \tau),$$

$$D_{i2}^{(2)}(\boldsymbol{r}, \tau) = \tau B_{i2}(\tau v(y)\mathbf{1}, \tau)\frac{\mathrm{d}v(y)}{\mathrm{d}y}$$

are diffusion coefficients. Equation (45) describes correctly the dynamics of the quantity $\langle q(\boldsymbol{r}, t)\rangle$ also for time scales $t \leqslant t_0$, $t_0$ being the temporal correlation radius of the random fields $\boldsymbol{u}(\boldsymbol{r}, t)$. However, in this case the statistical solution of equation (29) for the particle does not satisfy

the Markov property. If the problem is simplified and one considers the behaviour of the system for time $t \gg t_0$, then one can replace the upper limits of the integrals in equation (45) by infinity, and in this case, the solution of equation (29) in such a time scale will be a Markov process.

Above, we have considered approximate methods of description of diffusion of a passive tracer in a plane parallel average fluid flow. The equations we have obtained are rel-atively complex. In the general case they cannot be solved analytically. However, in geophysical applications there are several simpler problems of immediate interest that admit a more complete analysis. Among these problems let us mention the following:

1. $v(y) = \beta y$ — *linear shear flow*;
2. $v(y) = v_0\theta(y - y_0) - v_0\theta(y_0 - y)$ — *tangential shock detachment*;
3. $v(y) = v_0 \sin(\beta y)$ — *Kolmogorov flow*;
4. $v(y) = \tilde{v}(y)\theta(y_0 - |y|)$ — *jet flow*.

We leave aside the question about their stability (see e.g. Refs [11, 23, 24]).

In some cases equation (38) can be easily solved for an initial point condition $\langle q(\boldsymbol{r}, 0)\rangle = \delta(\boldsymbol{r} - \boldsymbol{r}_0)$ and its solution corresponds to Gaussian probability distribution for the system of equations (29), which is statistically equivalent to the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = v(y) + \tilde{\alpha}_1(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \tilde{\alpha}_2(t), \tag{29'}$$

where $\tilde{\alpha}_i(t)$ are statistically independent stochastic 'white noise' processes with correlation functions

$$\langle\tilde{\alpha}_i(t)\tilde{\alpha}_j(t')\rangle = 2D^{1/2}\delta(t - t').$$

It is easy to write the solution of system (29′):

$$y(t) = y_0 + w_2(t),$$
$$x(t) = x_0 + w_1(t) + \int_0^t \mathrm{d}\tau\, v\big(y_0 + w_2(\tau)\big), \tag{46}$$

where

$$w_i(t) = \int_0^t \mathrm{d}\tau\,\tilde{\alpha}_i(\tau)$$

are independent *Wiener processes* with the characteristics

$$\langle w_i(t)\rangle = 0, \quad \langle w_i(t)w_j(t')\rangle = 2D\delta_{ij}\min\{t, t'\}.$$

It follows from equalities (46) that the $y(t)$ coordinate has a Gaussian probability density with the parameters

$$\langle y(t)\rangle = y_0, \quad \langle y^2(t)\rangle = y_0^2 + 2Dt,$$

which corresponds to usual *Brownian motion* with the turbulent diffusion coefficient $D$.

Now, from equalities (46) we can easily calculate any momentum functions $\langle x^n(t)\rangle$ and correlations $\langle x^n(t)y^m(t)\rangle$ for Lagrangian particles. From the point of view of an Eulerian description for average concentrations these values characterise the divergence of the tracer 'cloud' since the equations for mean concentration $\langle q(\boldsymbol{r}, t)\rangle$ and one-particle probability distributions are the same as has been repeatedly demonstrated above. Thus the value

$$\langle \boldsymbol{r}(t)\rangle = \frac{1}{Q}\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{r}\langle q(\boldsymbol{r}, t)\rangle$$

defines the position of the 'centre of gravity' of the tracer cloud in time, whereas higher momenta, such as

$$\langle r_i(t)r_j(t)\rangle = \frac{1}{Q}\int \mathrm{d}\boldsymbol{r}\, r_i r_j\langle q(\boldsymbol{r}, t)\rangle\ ,$$

characterise the deformation of this cloud.

Thus in the simplest example of shear flow the equalities (46) correspond to the joint Gaussian probability density with the parameters [25, 26]

$$\langle x(t)\rangle = \beta y_0 t + x_0, \quad \langle y(t)\rangle = y_0,$$
$$\sigma_{xx}^2 = 2Dt\left(1 + \beta t + \tfrac{1}{3}\beta^2 t^2\right),$$
$$\sigma_{yy}^2 = 2Dt, \quad \sigma_{xy}^2 = 2Dt(1 + \beta t), \tag{46'}$$

where

$$\sigma_{xx}^2 = \langle(x - \langle x\rangle)^2\rangle, \quad \sigma_{yy}^2 = \langle(y - \langle y\rangle)^2\rangle,$$
$$\sigma_{xy}^2 = \langle xy\rangle - \langle x\rangle\langle y\rangle.$$

Solution (46′) is also well known in the absence of shear ($\beta = 0$), and in this case corresponds to the usual joint Brownian motion in the $(x, y)$ plane with a turbulent diffusion coefficient.

In the case of Kolmogorov flow we have

$$\langle y(t)\rangle = y_0,$$
$$\langle x(t)\rangle = x_0 + \frac{v_0}{\beta^2 D}\sin(\beta y_0)\big[1 - \exp(-\beta^2 Dt)\big]. \tag{46''}$$

If now $t \gg 1/D\beta^2$, then

$$\langle x(t)\rangle_{t\to\infty} = x_0 + \frac{v_0}{\beta^2 D}\sin(\beta y_0),$$

that is the particle is located on average in a finite part of space. In this case the correlations $x(t)$ and $y(t)$ also do not depend on time:

$$\langle(x(t) - x_0)(y(t) - y_0)\rangle_{t\to\infty} = x_0 + \frac{4v_0}{\beta^3 D}\cos(\beta y_0).$$

But in this limit the quantity $x(t)$ behaves like a Brownian particle with a turbulent diffusion coefficient $D$, i.e. $\sigma_{xx}^2 \sim 2Dt$.

Let us note that after the loss of equilibrium of the Kolmogorov flow a quasiperiodical flow is established in plane $(x, y)$. Tracer diffusion in flows of this kind with $\boldsymbol{V} = (B\cos y, A\sin x)$ has been examined by Crisanti and Vulpiani [27].

## 6. Special features of statistical solutions

In order to identify special features of statistical solutions we shall confine ourselves to the simplest problem — we shall consider a one-dimensional problem with zero average flow and we shall neglect molecular diffusion. In this case we have the following equations

$$\left(\frac{\partial}{\partial t} + u(x, t)\frac{\partial}{\partial x}\right)q(x, t) = 0, \quad q(x, 0) = q_0(x), \tag{47}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}u(x, t)\right)\rho(x, t) = 0, \quad \rho(x, 0) = \rho_0(x) \tag{48}$$

instead of equations (1), (2).

Let us recall that a one-dimensional problem always describes compressible fluid flow and the quantity

$$p(x,t)=\frac{\partial}{\partial x}q(x,t),$$

for the spatial concentration gradient is also described by equation (48).

On solving equation (47) with the aid of the method of characteristics we obtain, instead of equation (5), corresponding equations providing a Lagrangian description

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t|\xi)=u(x(t|\xi),t),\quad x(0|\xi)=\xi,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}q(t|\xi)=0,\quad q(0|\xi)=q_0(\xi). \tag{49}$$

Hence $q(t|\xi)=q_0(\xi)$. As mentioned above, divergence, described by the quantity $j(t|\xi)=|\partial x(t|\xi)/\partial\xi|$, plays an important role in going over to Eulerian description. The above quantity satisfies the following equation derived from equation (49)

$$\frac{\mathrm{d}}{\mathrm{d}t}j(t|\xi)=\frac{\partial u(x,t)}{\partial x}j(t|\xi),\quad j(0|\xi)=1. \tag{50}$$

It has been often stated above that to describe statistical properties of $x(t|\xi)$ one should introduce the function

$$\Phi_t(x,j|\xi)=\delta(x(t|\xi)-x)\delta(j(t|\xi)-j),$$

satisfying Liouville's equation

$$\frac{\partial}{\partial t}\Phi_t(x,j|\xi)=-\left(\frac{\partial}{\partial x}u(x,t)+\frac{\partial}{\partial j}j\frac{\partial u(x,t)}{\partial x}\right)\Phi_t(x,j|\xi),$$
$$\Phi_0(x,j|\xi)=\delta(x-\xi)\delta(j-1). \tag{51}$$

For the sake of simplicity we shall consider the random field $u(x,t)$ to be a Gaussian homogeneous random field, isotropic in space and stationary in time, with the parameters

$$\langle u(x,t)\rangle=0,$$
$$B(x-x',t-t')=\langle u(x,t)u(x',t')\rangle.$$

In this case

$$\left\langle\frac{\partial u(x,t)}{\partial x}u(x,t)\right\rangle=\frac{\partial}{\partial x}B(x-x',t-t')\bigg|_{x'=x}=0. \tag{52}$$

For the sake of simplicity we shall use the approximation of the field $u(x,t)$ delta-correlated in time, in which the correlation function $B(x,t)$ may be approximated by the expression

$$B(x,t)=2B^{\mathrm{eff}}(x)\delta(t),\quad 2B^{\mathrm{eff}}(x)=\int_{-\infty}^{\infty}\mathrm{d}\tau\,B(x,\tau). \tag{53}$$

On averaging equation (51) over the ensemble of the field $u(x,t)$ using the Furutsu–Novikov formula, we obtain an equation for joint offset probability density of the 'particle' and its divergence $P_t(x,j|\xi)=\langle\Phi_t(x,j|\xi)\rangle$

$$\frac{\partial}{\partial t}P_t(x,j|\xi)=\int\mathrm{d}x'\left(\frac{\partial}{\partial x}B^{\mathrm{eff}}(x-x')+\frac{\partial}{\partial j}j\frac{\partial B^{\mathrm{eff}}(x-x')}{\partial x}\right)\times\left\langle\frac{\delta}{\delta u(x',t)}\Phi_t(x,j|\xi)\right\rangle.$$

Taking into account that

$$\frac{\delta}{\delta u(x',t)}\Phi_t(x,j|\xi)=-\left(\frac{\partial}{\partial x}\delta(x-x')+\frac{\partial}{\partial j}j\frac{\partial\delta(x-x')}{\partial x}\right)\Phi_t(x,j|\xi),$$

and using expression (52) we can rewrite the last equality as a Fokker–Planck equation

$$\frac{\partial}{\partial t}P_t(x,j|\xi)=D_1\frac{\partial^2}{\partial x^2}P_t(x,j|\xi)+D_2\left(\frac{\partial}{\partial j}j+\frac{\partial}{\partial j}j\frac{\partial}{\partial j}j\right)P_t(x,j|\xi),$$
$$P_0(x,j|\xi)=\delta(x-\xi)\delta(j-1),$$

or in the form

$$\frac{\partial}{\partial t}P_t(x,j|\xi)=D_1\frac{\partial^2}{\partial x^2}P_t(x,j|\xi)+D_2\frac{\partial}{\partial j}\frac{\partial}{\partial j}j^2P_t(x,j|\xi),$$
$$P_0(x,j|\xi)=\delta(x-\xi)\delta(j-1), \tag{54}$$

where the diffusion coefficients $D_i$ are determined by the equalities

$$D_1=B^{\mathrm{eff}}(0),\quad D_2=-\frac{\partial^2}{\partial x^2}B^{\mathrm{eff}}(x)\bigg|_{x=0}. \tag{55}$$

Let us note that the use of the diffusion approximation instead of the delta-correlated in time approximation yields the same equation (54) but the diffusion coefficients depend now on time

$$D_1(t)=\int_0^t\mathrm{d}\tau\,B(0,\tau),\quad D_2(t)=-\int_0^t\mathrm{d}\tau\frac{\partial^2}{\partial x^2}B(x,\tau)\bigg|_{x=0}.$$

If $t\gg t_0$, where $t_0$ is the temporal correlation radius, these equations yield equalities (55).

From equation (54) it is clear that the diffusion of a 'particle' does not depend upon divergence statistics and is described by a Gaussian probability distribution with the parameters

$$\langle x(t|\xi)\rangle=\xi,\quad \sigma_x^2(t)=\left\langle(x(t|\xi)-\langle x(t|\xi)\rangle)^2\right\rangle=2D_1t\,,$$

i.e. it corresponds to usual Brownian motion. As regards the probability distribution for the divergence, it is logarithmically normal and statistically equivalent to the representation of divergence without dependence on the parameter $\xi$ [28]

$$j(t)=j(t|\xi)=\exp(-D_2t+w(t)), \tag{56}$$

where $w(t)$ is a *Wiener process* with the parameters

$$\langle w(t)\rangle=0,\quad\langle w^2(t)\rangle=2D_2t\,.$$

From formula (56) as well as from equation (54) it follows that

$$\langle j(t)\rangle=1,\quad\langle j^n(t)\rangle=\exp[D_2n(n-1)t], \tag{57}$$

i.e. the mean value of divergence is constant and higher moments, starting from the second ones, grow exponentially in time. Let us note that for the quantity equal to the inverse of divergence, $\rho(t)=1/j(t)$, which has the meaning of particle density and satisfies in Lagrangian description the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\rho}(t) = -\frac{\partial u(x,\,t)}{\partial x}\widetilde{\rho}(t), \quad \widetilde{\rho}(0) = 1,$$

one can also obtain a logarithmically normal probability distribution, the momentum functions of which are determined by the equality

$$\langle \widetilde{\rho}^n(t)\rangle = \exp[D_2 n(n+1)t]. \tag{58}$$

Thus the mean density of the passive tracer grows exponentially in time and this applies also to its higher momenta.

The paradoxical behaviour of the statistical characteristics of divergence and particle density which consists in simultaneous growth of the statistical characteristics in time may be explained by the property of logarithmically normal probability distribution [29]. Thus a typical expression for random divergence $j(t)$ is an exponentially decaying curve

$$j(t) = \exp(-D_2 t),$$

whilst there are top majorant estimates for the expressions for the random process $j(t)$. In particular, with probability $p = \frac{1}{2}$

$$j(t) < 4\exp\left(-\tfrac{1}{2}D_2 t\right)$$

for any time period. Correspondingly, for density we have a typical expression and bottom majorant estimates as follows

$$\widetilde{\rho}(t) = \exp\left(D_2 t\right), \quad \widetilde{\rho}(t) > \tfrac{1}{4}\exp\left(\tfrac{1}{2}D_2 t\right).$$

The estimates presented above show that the statistics of the random values $j(t)$ and $\widetilde{\rho}(t)$ are formed by jumps of their realisations with respect to their typical realisations. At the same time the particles are being compressed, forming clusters located mainly in low-density zones.

Let us consider now the Eulerian description of our problem. Let us introduce the functions

$$\Phi_{t,x}(q) = \delta(q(x,\,t) - q), \quad \Phi_{t,x}(\rho) = \delta(\rho(x,\,t) - \rho),$$

satisfying Liouville's equations

$$\left(\frac{\partial}{\partial t} + u(x,\,t)\frac{\partial}{\partial x}\right)\Phi_{t,x}(q) = 0,$$

$$\Phi_{0,x}(q) = \delta(q_0(x) - q),$$

$$\left(\frac{\partial}{\partial t} + u(x,\,t)\frac{\partial}{\partial x}\right)\Phi_{t,x}(\rho) = \frac{\partial}{\partial \rho}\rho\frac{\partial u(x,\,t)}{\partial x}\Phi_{t,x}(\rho),$$

$$\Phi_{0,x}(\rho) = \delta(\rho_0(x) - \rho). \tag{59}$$

On averaging now expressions (59) over the ensemble of random field $u(x,t)$ we obtain the following equations for density probabilities $P_{t,x}(q) = \langle \Phi_{t,x}(q)\rangle$, $P_{t,x}(\rho) = \langle \Phi_{t,x}(\rho)\rangle$:

$$\frac{\partial}{\partial t}P_{t,x}(q) = D_1\frac{\partial^2}{\partial x^2}P_{t,x}(q),$$

$$P_{0,x}(q) = \delta(q_0(x) - q), \tag{60}$$

$$\frac{\partial}{\partial t}P_{t,x}(\rho) = D_1\frac{\partial^2}{\partial x^2}P_{t,x}(\rho) + D_2\frac{\partial}{\partial \rho}\frac{\partial}{\partial \rho}\rho^2 P_{t,x}(\rho),$$

$$P_{0,x}(\rho) = \delta(\rho_0(x) - \rho). \tag{61}$$

The solution of equation (60) corresponds to spatial diffusion of the initial distribution. In the simplest case of a homogeneous initial condition $q_0(x) = q_0 - \text{const}$, the distribution of probabilities does not depend on $x$ and $P_t(q) = \delta(q - q_0)$.

For homogeneous initial conditions for density (61) $\rho_0(x) = \rho_0 - \text{const}$, the distribution of probabilities also does not depend on $x$ and equation (61) can be simplified:

$$\frac{\partial}{\partial t}P_t(\rho) = D_2\frac{\partial}{\partial \rho}\frac{\partial}{\partial \rho}\rho^2 P_t(\rho),$$

$$P_0(\rho) = \delta(\rho_0 - \rho). \tag{62}$$

The solution of equation (62) corresponds to a logarithmically normal distribution and then

$$\langle \rho(x,\,t)\rangle = \rho_0, \quad \langle \rho^n(x,\,t)\rangle = \rho_0^n\exp[D_2 n(n-1)t]. \tag{63}$$

From expressions (62), (63) one can obtain a typical form of expression for the field $\rho(z,t)$ at any fixed point in space

$$\rho(x,\,t) = \rho_0\exp(-D_2 t)$$

and Eulerian statistics reflect density fluctuations relative to this curve, which confirms the cluster nature of the density fluctuations of the medium.

As has been mentioned earlier, the spatial concentration gradient of the tracer

$$p(x,\,t) = \frac{\partial}{\partial x}q(x,\,t)$$

is described by an equation which coincides with the equation for the density of the medium. In this case joint probability density for the quantities $q(x,t)$ and $p(x,t)$ — $P_{t,x}(q,p) = \langle \delta(q(x,t) - q)\delta(p(x,t) - p)\rangle$ is also described by equation (61), i.e. by the equation

$$\frac{\partial}{\partial t}P_{t,x}(q,\,p) = D_1\frac{\partial^2}{\partial x^2}P_{t,x}(q,\,p) + D_2\frac{\partial}{\partial p}\frac{\partial}{\partial p}p^2 P_{t,x}(q,\,p),$$

$$P_{0,x}(q,\,p) = \delta(q_0(x) - q)\delta\left(\frac{\partial}{\partial x}q_0(x) - p\right), \tag{64}$$

from which it follows that the joint momentum functions are

$$\langle q^n(x,\,t)p^m(x,\,t)\rangle \sim \exp[D_2 m(m-1)t]\ .$$

Hence, the statistics of the concentration gradients are formed by jumps with respect to a typical form of expression that exponentially decays in time at a fixed point in space.

It is clear from the above discussion that to describe tracer diffusion in detail it is not enough to know the behaviour of individual momentum functions of tracer concentration and its gradient or density in space and time. One must also examine the probability distribution for these quantities. It was demonstrated above that in the general case of a three- dimensional problem the term taking account of molecular diffusion makes this impossible, so approximate approaches must be used. Some of these approaches are now in the process of being developed [30–32].

As regards the one-dimensional problem under consideration here, if molecular diffusion is taken into account we obtain the following equation instead of equation (47)

$$\left(\frac{\partial}{\partial t} + u(x,\,t)\frac{\partial}{\partial x}\right)q(x,\,t) = \kappa\frac{\partial^2}{\partial x^2}q(x,\,t),$$

$$q(x,\,0) = q_0(x) \tag{65}$$

and a corresponding equation for the spatial tracer concentration gradient

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}u(x,t)\right)p(x,t)=\kappa\frac{\partial^2}{\partial x^2}p(x,t),$$
$$p(x,0)=\frac{\partial}{\partial x}q_0(x). \tag{66}$$

In accordance with what has been said above, in the absence of molecular diffusion the behaviour of a typical expression for the spatial gradient of passive tracer concentration is characterised by exponential decay in time at any point in space. Hence we can regard the diffusional term in the right-hand side of equation (66) to be immaterial. On leaving out this term we obtain the following system of first-order equations:

$$\left(\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x}\right)q(x,t)=\kappa\frac{\partial}{\partial x}p(x,t),$$
$$q(x,0)=q_0(x),$$
$$\left(\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x}\right)p(x,t)=-p(x,t)\frac{\partial}{\partial x}u(x,t), \tag{67}$$
$$p(x,0)=\frac{\partial}{\partial x}q_0(x),$$

i.e. a value described by a closed equation has been added as a source in the right-hand side of the equation for $q(x,t)$. However, it is clear that the quantities $q(x,t)$ and $p(x,t)$ are statistically related.

Let us consider the function

$$\Phi_{t,x}(q,p)=\delta(q(x,t)-q)\delta(p(x,t)-p).$$

On differentiating it with respect to $x$ we obtain

$$\frac{\partial}{\partial x}\Phi_{t,x}(q,p)=-p\frac{\partial}{\partial q}\Phi_{t,x}(q,p)-\frac{\partial}{\partial p}\frac{\partial p(x,t)}{\partial x}\Phi_{t,x}(q,p). \tag{68}$$

Differentiating now $\Phi_{t,x}(q,p)$ with respect to $t$ we obtain

$$\left(\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x}\right)\Phi_{t,x}(q,p)$$
$$=\frac{\partial u(x,t)}{\partial x}\frac{\partial}{\partial p}\Phi_{t,x}(q,p)-\kappa\frac{\partial}{\partial q}\frac{\partial p(x,t)}{\partial x}\Phi_{t,x}(q,p). \tag{69}$$

Averaging equations (68), (69) over the ensemble of random field $u(x,t)$ we obtain for joint probability density $P_{t,x}(q,p)=\langle\Phi_{t,x}(q,p)\rangle$ the equalities

$$\frac{\partial}{\partial x}P_{t,x}(q,p)=-p\frac{\partial}{\partial q}P_{t,x}(q,p)-\frac{\partial}{\partial p}\Psi_{t,x}(q,p),$$
$$\frac{\partial}{\partial t}P_{t,x}(q,p)=D_1\frac{\partial^2}{\partial x^2}P_{t,x}(q,p)+D_2\frac{\partial}{\partial p}\frac{\partial}{\partial p}p^2P_{t,x}(q,p)$$
$$-\kappa\frac{\partial}{\partial q}\Psi_{t,x}(q,p),$$

where

$$\Psi_{t,x}(q,p)=\left\langle\frac{\partial p(x,t)}{\partial x}\Phi_{t,x}(q,p)\right\rangle.$$

On excluding the unknown function $\Psi_{t,x}(q,p)$ we obtain a closed equation for probability density

$$\frac{\partial}{\partial t}\frac{\partial}{\partial p}P_{t,x}(q,p)$$
$$=D_1\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial p}P_{t,x}(q,p)+D_2\frac{\partial}{\partial p}\frac{\partial}{\partial p}\frac{\partial}{\partial p}p^2P_{t,x}(q,p)$$
$$+\kappa\frac{\partial}{\partial q}\frac{\partial}{\partial x}P_{t,x}(q,p)+\kappa p\frac{\partial^2}{\partial q^2}P_{t,x}(q,p). \tag{70}$$

In particular, if we multiply expression (70) by $p$ and integrate over $p$ we obtain an equation for the probability density of passive tracer concentration

$$\frac{\partial}{\partial t}P_{t,x}(q)=D_1\frac{\partial^2}{\partial x^2}P_{t,x}(q)-\kappa\frac{\partial}{\partial q}\frac{\partial}{\partial x}\langle p|q\rangle-\kappa\frac{\partial^2}{\partial q^2}\langle p^2|q\rangle,$$

where

$$\langle p^n(x,t)|q\rangle=\int_{-\infty}^{\infty}\mathrm{d}p\,p^nP_{t,x}(q,p).$$

Equation (70) has not yet been studied.

Let us note that equation (70) leads to a closed system of equations for momentum functions of the type $\langle p^n(x,t)q^m(x,t)\rangle$, which includes as a source the quantities $\langle p^l(x,t)\rangle$ described in a closed form by an independent equation of the kind of equation (62).

## 7. Conclusion

A functional approach has been used here for the detailed examination of various approximate methods of describing the statistical characteristics of a scalar tracer field in a random velocity field. Special features of statistical solutions have been illustrated by taking the simplest problem as an example. The approach outlined above is based essentially on the conditions of finiteness of the temporal correlation radius of the velocity field, while the conditions of applicability of this approach are governed by various limitations in relation to the correlation radius (the limitations are different for different approximations). The equations obtained are not valid for an unlimited temporal correlation radius (*random stationary velocity field*). In fact, this case has so far hardly been investigated.

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