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Bell's paradoxes without the introduction of hidden variables

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Contents

Abstract. A hidden-variable theory is the traditional, but not unique, basis for constructing various types of Bell's theorem. The starting point may also be a recognition of the existence of a positive-definite probability distribution function. This assumption alone is used to formulate and prove Bell's paradoxes of different types. A specific example is used to show that a formal quantum calculation can sometimes give negative values of the joint probabilities that are used in the proof. An attempt is made to identify the physical meaning of this result and an algorithm for determination of negative joint probabilities of this type is proposed.

1. Introduction

Bell's theorem or paradox is usually assumed to be some quantitative criterion of a *hidden-variable theory* (HVT), which can be checked in practice. However, this criterion is not simple: it must contradict the results of *quantum theory*. The resultant conflict is an interesting field for the activity of experimentalists and the results obtained by them usually support quantum theory, but they are continuously being disputed by adherents of the HVT that are continuously looking for new 'loopholes' to save this theory.

The history of the HVT begins with the famous paper of Einstein, Podolsky, and Rosen (EPR) [1], who reached the conclusion that quantum theory does not provide a complete description of the physical reality. Bohm [2] proposed an experiment illustrating the EPR reasoning. Bell [3] was able to 'formalise' the main propositions of the HVT in the form of an inequality that may break down within the quantum theory framework.

A plethora of different types of such inequalities has been proposed and some of them have been checked by experimentalists who have reported results mostly in conflict with the HVT. However, the conflict is not dying out: it is becoming overgrown with new details and has basically transformed into an independent topic in modern physics, but it seems to have lost its initial attraction. The complexity of the problem is enhanced also by the fact that the HVT is based on several postulates (including the postulate of *locality* i.e. the absence of the influence of two instruments on one another when they are far apart; this possibility is still in hot dispute). Therefore, the reasons for the disagreement between the local HVT and the experimental results may be sought in failure of any one of these postulates. The situation is thus ambiguous.

The purpose of the present paper is not to decide the right and wrong of the long-drawn-out controversy but to provide a somewhat different view on the problem which may possibly reconcile both sides. Without invoking hidden variables expli-citly, we shall begin simply from 'common sense' which tells us that the probabilities, including joint probabilities, cannot be negative. This sole assumption is sufficient to formulate a number of paradoxes initially developed specifically for the local HVT.

I would not wish to claim any priority because the situation is not very clear. In a sense a similar approach has been followed by d'Espagnat [4] (see also Ref. [5]), but it has been implemented most consistently by Fine [6] and de Muynck [7], who have demonstrated Bell's inequalities for two ($N = 2$) observers (see also Refs [8 – 10] and the literature cited there).

In Section 2 I shall propose a new simple algorithm for proving Bell's inequalities for the case when $N = 2$. In

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Section 3 an experimental scheme for checking this inequality will be considered and the results will be given of quantum calculations confirming the possibility of its breakdown. In the next section an analysis will be made of the meaning (if it exists) of the negative joint probabilities, which may then be encountered and an algorithm for indirect measurement of these probabilities will be given, i.e. a method for reconstructing these probabilities from experimental results will be described. In Section 5 there is a proof of Bell's inequalities for any number of observers N and Section 6 provides a formulation of the interesting Greenberger–Horne–Zeilinger paradox of the $+1 = -1$ type, again starting only from the assumption of the existence of a positive-definite probability distribution function. Finally, in Section 7 Bell's inequalities for any number N of observers are generalised to the case of nondichotomous variables. The concluding section summaries the main results obtained.

2. Bell's inequality for two observers

Let us consider a random process characterised by four dichotomous variables which assume unity values:

$$A = \pm 1, \quad A' = \pm 1, \quad B = \pm 1, \quad B' = \pm 1 . \tag{1.1}$$

Let us assume that there is a positive-definite normalised probability distribution function

$$P(A, A', B, B') \geqslant 0 , \tag{1.2}$$

$$\sum_A \sum_{A'} \sum_B \sum_{B'} P(A, A', B, B') = 1 , \tag{1.3}$$

which satisfies the correspondence conditions of the type

$$P(A, A', B, B') + P(-A, A', B, B') = P(A', B, B') , \tag{1.4}$$

which are similar for other variables and distributions of lower dimensionalities.

We shall prove Bell's inequality of the type

$$|\Pi| \equiv \tfrac{1}{2} \left| \langle AB \rangle + \langle A'B \rangle + \langle AB' \rangle - \langle A'B' \rangle \right| \leqslant 1 \tag{1.5}$$

on the basis of Eqns (1.1)–(1.4). This was first done by Fine [6]. The proposed variant of a proof of Bell's inequalities of Eqn (1.5) is much simpler than those already published [6, 10]

The discrete probability distribution function (1.2) consists of 2^4 joint probablities:

$$\begin{aligned} &P_{AA'BB'}(++++) \\ &\quad \equiv P(A = +1, A' = +1, B = +1, B' = +1) , \\ &P_{AA'BB'}(+++-) \\ &\quad \equiv P(A = +1, A' = +1, B = +1, B' = -1) \text{ etc.} \end{aligned} \tag{1.6}$$

We shall use them to express the averages occurring in Eqn (1.5), such as

$$\langle AB \rangle = P_{AB}(++) + P_{AB}(--) - P_{AB}(+-) - P_{AB}(-+) , \tag{1.7}$$

where, for example, it follows from Eqn (1.4) that

$$\begin{aligned} P_{AB}(++) = {} & P_{AA'BB'}(++++) + P_{AA'BB'}(+++-) \\ & + P_{AA'BB'}(+-++) + P_{AA'BB'}(+-+-) . \end{aligned}$$

We can substitute these expansions on the left-hand side of Eqn (1.5). The result is (the subscripts of P's are omitted)

$$\begin{aligned} \Pi \equiv {} & \tfrac{1}{2} \left(\langle AB \rangle + \langle A'B \rangle + \langle AB' \rangle - \langle A'B' \rangle \right) \\ = {} & P(++++) + P(+++-) - P(++-+) \\ & - P(++--) + P(+-++) - P(+-+-) \\ & + P(+--+) - P(+---) - P(-+++) \\ & + P(-++-) - P(-+-+) + P(-+--) \\ & - P(--++) - P(--+-) + P(---+) \\ & + P(----) . \end{aligned} \tag{1.8}$$

If all the terms on the right-hand side of Eqn (1.8) had appeared in the sum with the plus sign, the result would have been an expansion of the unity in Eqn (1.3). Since half the components have the minus sign, it follows from Eqns (1.2) and (1.3) that the sum of Eqn. (1.8) lies within the interval $[-1, +1]$. This proves Eqn (1.5).

3. Quantum theory results

Why the inequality of Eqn (1.5), based on very general assumptions described by Eqns (1.1)–(1.4), breaks down in practice?

Let us consider the simplest experimental setup needed to check Bell's inequality of Eqn (1.5) [11–13]. Two observers A and B (Fig. 1) detected simultaneously one photon each with the aid of '+' or '−' detectors. If the observer A after a phase lag α records a photocount at the detector '+', then this event is assigned the value $A = +1$. If this event occurs subject to phase lag α', then $A' = +1$. Photocounts recorded by the detector '−' are labelled similarly and this applies also to the channels of the observer B. The 'primed' variables correspond to the 'primed' phase lags. Many repetitions of such measurements make it possible to calculate the averages occurring in Eqn (1.5).

The quantum state of photons reaching the observers is described by the wave vector [11]

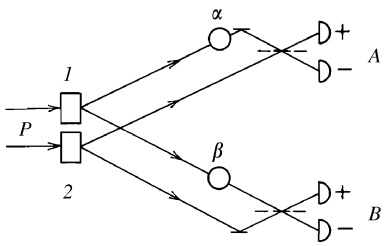

Figure 1. Schematic diagrams of an intensity interferometer with parametric radiation sources for two observers A and B. Correlated photons are created simultaneously in nonlinear components (*1* or *2*) by the action of a shared coherent pump P and directed to A and B in the form of two modes, one of which is subject to a phase lag (circles). The modes are mixed in 50% beam splitters (dashed segements) and are detected by identical detectors '+' or '−'.

$$|\psi\rangle = \sqrt{\tfrac{1}{2}}(a_1^+ b_1^+ + a_2^+ b_2^+)|0\rangle$$
$$\equiv \sqrt{\tfrac{1}{2}}\left(|10\rangle_a|10\rangle_b + |01\rangle_a|01\rangle_b\right), \tag{2.1}$$

where a_j^+ and b_j^+ are the operators describing creation of photons in two signal (travelling towards the observer A) and idler (travelling towards B) modes; $j = 1, 2$ is the number of the crystals emitting a given mode (see Fig. 1) and $|0\rangle$ denotes vacuum.

The operators of the numbers of photons recorded by the detectors '+' and '−' in the channel A are [11]

$$n_\pm^a \equiv a_\pm^+ a_\pm = \tfrac{1}{2}\left[n_1^a + n_2^a \pm (\sigma_-^a e^{i\alpha} + \sigma_+^a e^{-i\alpha})\right], \tag{2.2}$$

where $n_j^a \equiv a_j^+ a_j$, $\sigma^a \equiv a_1 a_2^+$, $\sigma_+^a \equiv (\sigma_-^a)^+$. Similar relationships define $n_\pm^b$ in the channel B.

We can now find the probability distribution function $P_\psi(A, A', B, B')$ predicted by quantum theory and we shall calculate all the components of its 16 joint probabilities. Since the variables in Eqn (1.1) are discrete, these joint probabilities are equal to the following moments:

$$P_{AA'BB'}(++++) = \langle\psi|n_+^a n_+^{a'} n_+^b n_+^{b'}|\psi\rangle,$$
$$P_{AA'BB'}(+++-) = \langle\psi|n_+^a n_+^{a'} n_+^b n_-^{b'}|\psi\rangle \text{ etc.} \tag{2.3}$$

The primes denote here the replacement of α with α' in Eqn (2.2) and/or β with β' in the channel B. The result is

$$P_\psi(A, A', B, B') = \tfrac{1}{16}\left[1 + AA'\cos(\alpha - \alpha')\right.$$
$$+BB'\cos(\beta - \beta') + AA'BB'\cos(\alpha + \beta - \alpha' - \beta')$$
$$+AB\cos(\alpha + \beta) + A'B\cos(\alpha' + \beta)$$
$$\left.+AB'\cos(\alpha + \beta') + A'B'\cos(\alpha' + \beta')\right]. \tag{2.4}$$

Let us now assume that the phases in the channels are as follows:

$$\alpha = 0, \quad \alpha' = \frac{\pi}{2}, \quad \beta = -\frac{\pi}{4}, \quad \beta' = \frac{\pi}{4}, \tag{2.5}$$

which violates Bell's inequality of Eqn (1.5). Then some of the joint probabilities prove to be negative (their subscripts are omitted):

$$P(++++) = P(+--+) = P(-++-)$$
$$= P(----) = \frac{\sqrt{2}}{16},$$
$$P(++--) = P(+-+-) = P(-+-+)$$
$$= P(--++) = \frac{\sqrt{2}}{16}, \tag{2.6}$$
$$P(++-+) = P(+---) = P(-+++)$$
$$= P(--+-) = \frac{2-\sqrt{2}}{16},$$
$$P(+++-) = P(+-++) = P(-+--)$$
$$= P(---+) = \frac{2+\sqrt{2}}{16},$$

and direct substitution of Eqn (2.6) into Eqn (1.8) gives $\sqrt{2}$, i.e. Bell's inequality (1.5) is not obeyed.

It therefore follows that determination of the probability distribution function in the form $P_\psi(A, A', B, B')$ makes it possible, by direct comparison of Eqns (2.4) and (2.6) with the initial assumptions of Eqns (1.2)–(1.4), to draw an unambiguous conclusion about the nature of breakdown of Eqn (1.5). Thus the *only* reason for the existence of the distribution function of the probabilities $P(A, A', B, B')$ would seem to be failure to observe the self-evident requirement of Eqn (1.2), because according to Eqns (2.4) and (2.6) the normalisation (1.3) and correspondence (1.4) conditions are satisfied. In other words, a formal recognition of the existence of the distribution function of the probabilities $P(A, A', B, B')$ does not leave any scope for the unconditional requirement that the function should be nonnegative.

4. What is the meaning of negative probabilities and how can they be measured?

Negative [10, 14–20] and even complex-variable [2, 21] probability distributions have been considered frequently in connection with the EPR paradox and Bell's theorem. In a wider sense similar questions have been raised a long time ago by Dirac [22] (see also Ref. [20]). The crux of the matter is that the distribution function of random quantities, described by noncommuting operators, is not always positive definite. A striking example is the Wigner distribution function for the coordinate and momentum of a quantum particle. In the orthodox Copenhagen understanding of quantum mechanics there are no such probability distribution functions since it is impossible to carry out experiments in which they would be measured directly. Nevertheless, indirect measurement methods are possible. This has been demonstrated [23] by reconstructing a continuous two-dimensional Wigner distribution function of quadrature components of light in a squeezed state. Although negative distributions have not been observed, there are no fundamental quantum-theoretical reasons why these distributions should not exist.

The probability distribution function $P_\psi(A, A', B, B')$ is similar to the Wigner distribution function in the sense that not all the observables occurring in it can be described by commuting operators, for example, A and A'. They cannot be measured in one realisation (a single photon can never be recorded by observer A when the phase lags α and α' are different). Consequently, direct measurements of $P_\psi(A, A', B, B')$ are impossible. However, does it mean that we should give up? I shall try to answer this question in the present section.

The results of the experiment mentioned above can be described formally by a discrete four-dimensional probability distribution function which in this section can be denoted more conveniently by $P_{\alpha\alpha'\beta\beta'}^{AA'BB'}$. Quantum theory predicts the possibility of obtaining not only a variable-sign $P_{\alpha\alpha'\beta\beta'}^{AA'BB'}$, but also variable-sign three-dimensional probability distribution functions of the type $P_{\alpha\alpha'\beta}^{AA'B}$, $P_{\alpha\alpha'\beta'}^{AA'B'}$, etc. This can be demonstrated quite easily with the aid of Eqns (1.4) and (2.6) [10]. Calculation of the latter from the experimental results will be tackled in the later sections. It should be stressed that direct measurements are possible only in the case of the two-dimensional distribution functions $P_{\alpha\beta}^{AB}$, $P_{\alpha'\beta}^{A'B}$, $P_{\alpha\beta'}^{AB'}$, $P_{\alpha'\beta'}^{A'B'}$, and the one-dimensional functions P_α^A and P_β^B.

Let us now identify some of the properties of $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$, which will allow us to solve the formulated inverse problem.

In view of the arbitrary nature of the selection of the signs of the variables, which depends on the observer's decision in the case of identical detectors, the following symmetry should be observed:

$$P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = P^{\bar{A}\bar{A}'\bar{B}\bar{B}'}_{\alpha\alpha'\beta\beta'}, \tag{3.1}$$

$$\bar{A} = -A, \quad \bar{A}' = -A', \quad \bar{B} = -B, \quad \bar{B}' = -B'.$$

Consequently

$$\begin{aligned} P^{A}_{\alpha} &= \sum_{A'}\sum_{B}\sum_{B'} P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = \sum_{A'}\sum_{B}\sum_{B'} P^{\bar{A}\bar{A}'\bar{B}\bar{B}'}_{\alpha\alpha'\beta\beta'} \\ &= P^{\bar{A}}_{\alpha} = P^{B}_{\beta} = P^{\bar{B}}_{\beta} = \tfrac{1}{2}, \end{aligned} \tag{3.2}$$

which agrees with the experimental results [13]. The correspondence (1.4) and normalisation (1.3) conditions are used here.

In view of the symmetry of the 'primed' and 'unprimed' indices, it is found that

$$P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = P^{AA'BB'}_{\alpha'\alpha\beta'\beta}. \tag{3.3}$$

The interference nature of the experiment means that the dependence of $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ on one of the phase lags should be a harmonic of the type give by

$$P^{AA'BB'}_{\alpha\alpha'\beta\beta'} = G(\alpha', \beta, \beta') + H(\alpha', \beta, \beta')\cos[\alpha + \varphi(\alpha', \beta, \beta')] \tag{3.4}$$

The above relationship applies also to the probability distribution functions of lower dimensionalities. It follows from the classical stochastic description of mixing of interfering waves in beam splitters. In fact, the double intensity of the radiation reaching the detector '+' belonging to the observer A is

$$\begin{aligned} 2n^{a}_{+} &= |a_1^{-\mathrm{i}\alpha} + a_2|^2 \\ &= ||a_1|\exp(-\mathrm{i}\alpha - \mathrm{i}\varphi_1) + |a_2|\exp(-\mathrm{i}\varphi_2)|^2 \\ &= |a_1|^2 + |a_2|^2 + 2|a_1a_2|\cos(\alpha + \varphi_1 - \varphi_2), \end{aligned} \tag{3.5}$$

where a_j are the complex amplitudes of the signals being mixed and φ_j are their phases. Similarly, in the case of the '−' detector, this intensity is

$$2n^{a}_{-} = |a_1|^2 + |a_2|^2 - 2|a_1a_2|\cos(\alpha + \varphi_1 - \varphi_2). \tag{3.6}$$

The above expressions match Eqn (3.4) if we assume that the joint probabilities are proportional to the intensities. Moreover, Eqn (3.4) does not contradict quantum theory, i.e. it does not contradict the result given by Eqn (2.4).

For the same reasons, the relationship

$$P^{\bar{A}A'BB'}_{\alpha\alpha'\beta\beta'} = P^{AA'BB'}_{(\alpha\pm\pi)\alpha'\beta\beta'} \tag{3.7}$$

follows from Eqns (3.5) and (3.6), because the addition of $\pm\pi$ to α is equivalent to transposition of n^{a}_{+} and n^{a}_{-}. Similar reasoning applies to other variables and to the corresponding phase lags.

Let us start with the reconstruction of the two-dimensional joint probability,

$$P^{++}_{\alpha\alpha'} \equiv P^{A=+1,A'=+1}_{\alpha\alpha'}. \tag{3.8}$$

The correspondence condition of Eqn (1.4) gives

$$P^{+}_{\alpha} = P^{++}_{\alpha\alpha'} + P^{+-}_{\alpha\alpha'}. \tag{3.9}$$

Let $\alpha = \alpha'$; then $P^{+-}_{\alpha\alpha'} = 0$ since the identity $A \equiv A'$ then applies. Consequently, subject to Eqn (3.2), the result is

$$P^{++}_{\alpha\alpha} = P^{+}_{\alpha} = \tfrac{1}{2}. \tag{3.10}$$

It follows from Eqn (3.4) that transposition of α and α' gives

$$P^{++}_{\alpha\alpha'} = G(\alpha) + H(\alpha)\cos[\alpha' + \varphi(\alpha)]. \tag{3.11}$$

According to Eqn (3.7), the result is

$$P^{+-}_{\alpha\alpha} = P^{++}_{\alpha\alpha+\pi} = 0. \tag{3.12}$$

It is evident from Eqns (3.10)–(3.12) that

$$G(\alpha) + H(\alpha)\cos[\alpha + \varphi(\alpha)] = \tfrac{1}{2}, \tag{3.13}$$

$$G(\alpha) - H(\alpha)\cos[\alpha + \varphi(\alpha)] = 0, \tag{3.14}$$

and hence,

$$G(\alpha) = \tfrac{1}{4}, \tag{3.15}$$

$$H(\alpha) = \{4\cos[\alpha + \varphi(\alpha)]\}^{-1}, \tag{3.16}$$

where in the last relationship it is assumed that

$$\cos[\alpha + \varphi(\alpha)] \neq 0. \tag{3.17}$$

Substitution of Eqns (3.15) and (3.16) into Eqn (3.11) yields

$$\begin{aligned} P^{++}_{\alpha\alpha'} &= \tfrac{1}{4}\left\{1 + \cos(\alpha' + \varphi(\alpha))\left[\cos(\alpha + \varphi(\alpha))\right]^{-1}\right\} \\ &\equiv P^{++}_{\alpha'\alpha}. \end{aligned} \tag{3.18}$$

The last identity is obtained on the basis of the property described by Eqn (3.3) and is derived similarly to Eqn (3.2)

It follows from Eqn. (3.18) that

$$\varphi(\alpha) \equiv -\alpha \pm m\pi, \quad m = 0, 1, 2, \ldots, \tag{3.19}$$

which is not in conflict with Eqn (3.17). Consequently, if we allow for Eqn (3.17), the result is

$$P^{AA'}_{\alpha\alpha'} = \tfrac{1}{4}\left[1 + AA'\cos(\alpha - \alpha')\right]. \tag{3.20}$$

According to Eqns (2.1) and (2,2) a formal quantum calculation of the moments $\langle\psi|n^{a}_{\pm}n^{a'}_{\pm}|\psi\rangle$, corresponding to the joint probabilities $P^{\pm\pm}_{\alpha\alpha'}$, gives the same result. It can also be obtained with the aid of Eqn (2.4) and by consistent application of Eqn (1.4).

The three-dimensional probability distribution functions are related to the two-dimensional distributions by simple

expressions of the type

$$P^{AA'B}_{\alpha\alpha'\beta} = \tfrac{1}{2}\left(P^{AB}_{\alpha\beta} + P^{A'B}_{\alpha'\beta} - P^{A\bar{A}'}_{\alpha\alpha'}\right) , \tag{3.21}$$

which can be readily checked by substitution of all the following quantities on the right-hand side of Eqn (3.21):

$$P^{AB}_{\alpha\beta} = P^{AA'B}_{\alpha\alpha'\beta} + P^{A\bar{A}'\bar{B}}_{\alpha\alpha'\beta} \text{ etc.}$$

The probability distribution functions $P^{AB}_{\alpha\beta}$ and $P^{A'B}_{\alpha'\beta}$ are found directly from experiments and are given by

$$\begin{aligned} P^{AB}_{\alpha\beta} &= \tfrac{1}{4}\left[1 + AB\cos(\alpha+\beta)\right] , \\ P^{A'B}_{\alpha'\beta} &= \tfrac{1}{4}\left[1 + A'B\cos(\alpha'+\beta)\right] . \end{aligned} \tag{3.22}$$

Substitution of the system (3.22) into Eqn (3.21) yields

$$\begin{aligned} P^{AA'B}_{\alpha\alpha'\beta} = \tfrac{1}{8}\big[&1 + AB\cos(\alpha+\beta) + A'B\cos(\alpha'+\beta) \\ &+ AA'\cos(\alpha-\alpha')\big] . \end{aligned} \tag{3.23}$$

A quantum calculation of the corresponding moments gives the same result.

It is obvious that the joint probabilities, of which the probability distribution function (3.23) is composed, can be negative. For example,

$$P^{+-+}_{\alpha\alpha'\beta} = \tfrac{1}{8}(1-\sqrt{2}) \tag{3.24}$$

for $\alpha = \pi/2$, $\alpha' = 0$, and $\beta = \pi/4$.

It thus follows from the experimental results [Eqn (3.22)] and from the properties of Eqns (3.1)–(3.4) and (3.7) that the discrete probability distribution functions of the $P^{AA'B}_{\alpha\alpha'\beta}$ type can be reconstructed. They link together the observable quantities, some of which are described by non-commuting operators (A and A' in the present case). Therefore, as pointed out above, it is not possible to determine them directly, which applies also to the Wigner distribution functions. In this sense such distributions do not have an *operational* meaning. However, as shown for the continuous Wigner distribution [23] and demonstrated in the present paper for the discrete distributions, indirect methods can be used to find these distributions. Then $P^{AA'B}_{\alpha\alpha'\beta}$ and functions of the same kind need not be positive definite.

What does the negative joint probability mean? Dirac [20, 23] regards this probability as a "fully defined mathematical analogue of a negative amount of money". It should be mentioned that in describing the results of the experiment in question the negative probability *reduces* the probability of events corresponding to it and *increases* the probability of opposite events. For example,

$$\langle AB\rangle = P^{++}_{\alpha\beta} + P^{--}_{\alpha\beta} - P^{+-}_{\alpha\beta} - P^{-+}_{\alpha\beta}, \tag{3.25}$$

$$P^{++}_{\alpha\beta} = P^{+++}_{\alpha\alpha'\beta} + P^{+-+}_{\alpha\alpha'\beta}. \tag{3.26}$$

The negative nature of $P^{+-+}_{\alpha\alpha'\beta}$ means here that the probability of the result $AB = +1$, one of whose components if $P^{+-+}_{\alpha\alpha'\beta}$, falls as the probability of the opposite result ($AB = -1$) rises. Like the Wigner distribution function, multidimensional distributions of the $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ type are convenient in the calculations and make it possible to present the results in a clear manner. In fact, suitable summation of Eqns (2.4) and (2.6) gives the correct values of the two-dimensional probabilities, for example those occurring in Eqn (3.22), which can be measured directly. One further advantage of the use of $P^{AA'BB'}_{\alpha\alpha'\beta\beta'}$ is the ability to resolve unambiguously Bell's paradox which consists in violation of Bell's inequality of Eqn (1.5).

5. Bell's inequalities for any number N of observers

The attraction of Bell's inequalities with $N > 2$ is primarily due to the quantitative increase in the discrepancies from quantum-theoretical predictions by a factor of $2^{(N-1)/2}$ [11, 24, 25]. However, we shall begin this section from one further variant of the proof of Bell's inequality of Eqn (1.5) for two observers, which can easily be generalised to any N.

Let us number consecutively the joint probabilities of Eqn (1.6), which are components of the probability distribution function of Eqn (1.2), for example, in the sequence adopted on the right-hand side of Eqn (1.8):

$$\begin{aligned} P_1 &\equiv P_{AA'BB'}(++++) \\ &\equiv P(A_1 = +1, A'_1 = +1, B_1 = +1, B'_1 = +1), \\ P_2 &\equiv P_{AA'BB'}(+++-) \\ &\equiv P(A_2 = +1, A'_2 = +1, B_2 = +1, B'_2 = -1) \text{ etc.} \end{aligned} \tag{4.1}$$

up to

$$P_{16} \equiv P_{AA'BB'}(----).$$

The initial notation of the joint probabilities, adopted in the first two sections, is used again above.

The normalisation condition (1.3) now looks as follows:

$$\sum_M P_M = 1, \quad M = 1, 2, \ldots, 2^{2N} . \tag{4.2}$$

Let us consider the average products of two variables, for example A and B, in the form of the sum

$$\langle AB\rangle = \sum_M A_M B_M P_M \tag{4.3}$$

and proceed similarly for other averages occurring in Eqn (1.5). We thus have

$$\Pi \equiv \tfrac{1}{2}\left(\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle\right) = \sum_M S^{(2)}_M P_M, \tag{4.4}$$

where Bell's observable for two observers

$$\begin{aligned} S^{(2)}_M &\equiv \tfrac{1}{2}(A_M B_M + A'_M B_M + A_M B'_M - A'_M B'_M) \\ &\equiv \tfrac{1}{2}\left[(B_M + B'_M)A_M + (B_M - B'_M)A'_M\right] \end{aligned} \tag{4.5}$$

can assume only the values ± 1 in view of Eqn (1.1). Consequently, Eqn (1.5) follows from Eqns (1.2), (4.2), and (4.4).

This proof represents in fact a modification of the derivation of Bell's inequalities [11, 25, 26] based on the concepts of the local HVT. A further natural generalisation involves going over from two to an arbitrary number of observers N.

Let us consider a random process described by $2N$ dichotomous variables, assuming unity values:

$$A^{(1)} = \pm 1, \quad A^{(1)'} = \pm 1, \quad A^{(2)} = \pm 1,$$
$$A^{(2)'} = \pm 1, \ldots, A^{(N)} = \pm 1, \quad A^{(N)'} = \pm 1. \tag{4.6}$$

Let us assume that there is a positive definite normalised probability distribution function

$$P[A^{(1)}, A^{(1)'}, A^{(2)}, A^{(2)'}, \ldots, A^{(N)}, A^{(N)'}] \geqslant 0, \tag{4.7}$$

$$\sum_{A^{(1)}} \sum_{A^{(1)'}} \cdots \sum_{A^{(N)'}} P[A^{(1)}, A^{(1)'}, A^{(2)}, A^{(2)'}, \ldots, A^{(N)}, A^{(N)'}] = 1, \tag{4.8}$$

which satisfies the correspondence conditions of the type

$$\sum_{A^{(1)}} P[A^{(1)}, A^{(1)'}, A^{(2)}, A^{(2)'}, \ldots, A^{(N)}, A^{(N)'}]$$
$$= P[A^{(1)'}, A^{(2)}, A^{(2)'}, \ldots, A^{(N)}, A^{(N)'}] \tag{4.9}$$

with similar generalisations to other variables and to distributions of lower dimensionalities.

We shall introduce also a Bell's observable of the type

$$S_M^{(N)} = \tfrac{1}{2}\big[(A_M^{(N)} \pm A_M^{(N)'}) S_M^{(N-1)}$$
$$\pm (A_M^{(N)} \mp A_M^{(N)'}) S_M^{(N-1)'}\big] = \pm 1, \tag{4.10}$$

which is similar to that used in Eqns [11, 25]. However, the variables of Eqn (4.6) labelled with the index M do not correspond to the same set of hidden variables $\{\lambda\}$, but are numbered in accordance with the corresponding joint probabilities, as has been done in Eqn (4.1).

The recurrence relationships (4.10) may go over from a Bell's observable for $N = 2$ to a Bell's observable for $N = 3$, etc. The prime in the last term of Eqn (4.10) means redesignation of the primed variables occurring in Bell's observables for the observer $(N-1)$ and vice versa, for example,

$$S_M^{(2)} = \tfrac{1}{2}\big[(A_M^{(2)} \pm A_M^{(2)'}) A_M^{(1)} \pm (A_M^{(2)} \mp A_M^{(2)'}) A_M^{(1)'}\big], \tag{4.11}$$

$$S_M^{(2)'} = \tfrac{1}{2}\big[(A_M^{(2)'} \pm A_M^{(2)}) A_M^{(1)'} \pm (A_M^{(2)'} \mp A_M^{(2)}) A_M^{(1)}\big], \tag{4.12}$$

where $S_M^{(1)} = A_M^{(1)}$.

The signs in Eqn (4.10) are arbitrary, but if the first parentheses contain a sum, then the second parentheses should contain the difference, and vice versa.

A comparison of Eqns (4.7) and (4.8) [or Eqn (4.2)], and Eqn (4.10) makes it possible to conclude that

$$\left| \sum_M S_M^{(N)} P_M \right| \leqslant 1. \tag{4.13}$$

This is the prototype of a nonlocal Bell's inequality for any number N of observers. Its specific form is obtained by calculation from Eqn (4.10). The following formal transformations have to be made: the subscripts M have to be removed from Eqn (4.10), both S_{N-1} and $S_{N-1'}$ have to be expressed in terms of the variable of Eqn (4.6), all the parentheses have to be opened, and each term should be enclosed inside the averaging symbols. The absolute value of the resultant expression should not exceed unity. For example, if $N = 3$, one of the variants of the combination of signs in Eqn (4.10) gives

$$\tfrac{1}{2}|\langle A'BC\rangle + \langle AB'C\rangle + \langle ABC'\rangle - \langle A'B'C'\rangle| \leqslant 1. \tag{4.14}$$

The following substitutions are made here for clarity: $A \equiv A^{(1)}$, $B \equiv A^{(2)}$, $C \equiv A^{(3)}$; similar substitutions are also made for the 'primed' variables.

The relationship (4.14) is identical with the corresponding Bell's inequality derived on the basis of the local HVT [11, 25, 26]. This applies also to the Bell's inequality for an arbitrary value of N because of the formal identity of the expression given by Eqn (4.10) for a Bell's observable and the corresponding expression derived from the concept of the local HVT [11, 25].

As pointed out above, an increase in N increases quantitatively the deviations from Bell's inequalities by the factor $2^{(N-1)/2}$ [11, 24, 25]. For example, the left-hand side of Eqn (4.14) assumes the value 2 when a quantum analysis is made under certain conditions [11]. Moreover, beginning with $N = 3$ we can formulate clearly the Greenberger–Horne–Zeilinger paradox, which is dealt with in the next section.

6. Greenberger–Horne–Zeilinger paradox

This interesting paradox demonstrates a contradiction of the $+1 = -1$ type [11, 25–27]. It implies a total correlation of the results of measurements, for example,

$$\langle A'BC\rangle = A'BC = \langle AB'C\rangle = AB'C = \langle ABC'\rangle$$
$$= ABC' = -\langle A'B'C'\rangle = -A'B'C' = 1, \tag{5.1}$$

which is permitted by quantum theory. It is precisely in this case that Bell's inequality of Eqn (4.14) breaks down, as mentioned at the end of the preceding section. Eqn (5.1) leads to the product

$$(A'BC)(AB'C)(ABC')(A'B'C') = -1. \tag{5.2}$$

It will be shown later that this is impossible if the requirements of Eqns (4.6)–(4.9) are satisfied.

Under complete correlation conditions, corresponding to Eqn (5.1), the components P_M which give $A'_M B_M C_M = -1$ should vanish. This applies also to the components which give $A_M B'_M C_M = A_M B_M C'_M = -1$. Then, out of 64 components P_M of the function $P(A, A', B, B', C, C')$, only 8 nonzero components remain:

$$P(+++++++),\ P(++---),\ P(+-+--+),$$
$$P(+--++-),\ P(-++-+-),\ P(-+-+-+),$$
$$P(--++--),\ P(----++).$$

but all of them give (!) $A'_M B'_M C'_M = +1$. Moreover, there is not a single component P_M which for any of the three out of the investigated four factors will give the same sign, and for the fourth one the opposite sign, because the product is

$$(A'_M B_M C_M)(A_M B'_M C_M)(A_M B_M C'_M)(A'_M B'_M C'_M)$$
$$= (A_M A'_M B_M B'_M C_M C'_M)^2 = +1. \tag{5.3}$$

Consequently, if the limitations (4.7)–(4.9) permit the complete correlation required under these conditions, this happens only in the case when the product (5.2) is $+1$. The above reasoning is readily generalised to arbitrary values $N \geqslant 3$. The Greenberger–Horne–Zeilinger paradox formulated here does not require the hypothesis of locality and rejection of this hypothesis thus does not solve this paradox.

7. Bell's inequalities for nondichotomous variables

The algorithm for the proof of Bell's inequalities described in Section 5 in the case of an arbitrary N permits also generalisation of the investigated inequalities to the case of nondichotomous variables contained within the interval $[-1, +1]$:

$$|A^{(1)}| \leqslant 1, \quad |A^{(1)'}| \leqslant 1, \quad |A^{(2)}| \leqslant 1, \quad |A^{(2)'}| \leqslant 1, \ldots, \quad |A^{(N)}| \leqslant 1, \quad |A^{(N)'}| \leqslant 1 . \tag{6.1}$$

Let us consider first Bell's inequalities for two observers under these conditions. Let us return to the Bell's observable of Eqn (4.11) and prove that if Eqn (6.1) is obeyed, then

$$|S_M^{(2)}| \leqslant 1 . \tag{6.2}$$

Subject to Eqn (6.1), we find that

$$\begin{aligned} \left|(A_M^{(2)} \pm A_M^{(2)'})A_M^{(1)}\right| &= \left|A_M^{(2)} \pm A_M^{(2)'}\right|\left|A_M^{(1)}\right| \\ &= \left|A_M^{(2)} \pm A_M^{(2)'}\right| - \left(1 - \left|A_M^{(1)}\right|\right)\left|A_M^{(2)} \pm A_M^{(2)'}\right| \\ &\leqslant \left|A_M^{(2)} \pm A_M^{(2)'}\right| . \end{aligned} \tag{6.3}$$

Similarly, we obtain

$$\left|(A_M^{(2)} \mp A_M^{(2)'})A_M^{(1)'}\right| \leqslant \left|A_M^{(2)} \mp A_M^{(2)'}\right| . \tag{6.4}$$

Let us add Eqn (6.3) to Eqn (6.4). If allowance is made for Eqn (4.11), the result is

$$\begin{aligned} 2\left|S_M^{(2)}\right| &= \left|(A_M^{(2)} \pm A_M^{(2)'})A_M^{(1)} \pm (A_M^{(2)} \mp A_M^{(2)'})A_M^{(1)'}\right| \\ &\leqslant \left|(A_M^{(2)} \pm A_M^{(2)'})A_M^{(1)}\right| + \left|(A_M^{(2)} \mp A_M^{(2)'})A_M^{(1)'}\right| \\ &\leqslant \left|A_M^{(2)} \pm A_M^{(2)'}\right| + \left|A_M^{(2)} \mp A_M^{(2)'}\right| \leqslant 2 , \end{aligned} \tag{6.5}$$

which leads to Eqn (6.2).

The iterational nature of the algorithm of the algorithm of formation of the Bell's observable of Eqn (4.10) makes it possible to draw the conclusion from Eqns (6.1) and (6.2) that $|S_M^{(3)}| \leqslant 1$ and so on:

$$\left|S_M^{(N)}\right| \leqslant 1 . \tag{6.6}$$

We shall now assume that the variables of Eqn (6.1) assume a finite number K of their values, which is always satisfied in practice if only because of the finite number of experimental realisations. The number of components P_M of the probability distribution function P then increases to 2^{2KN}, but the validity of Eqn (4.13) is retained because of Eqn (6.6). Therefore, if the conditions described by Eqns (4.7)–(4.9) are obeyed, Bell's inequalities described above should be satisfied also in the case of nondichotomous variables obeying Eqn (6.1).

8. Conclusions

We shall now summarise some of our results. In a wider sense the family of different variants of Bell's theorem (or of Bell's paradox) is obtained in two stages. The first stage involves a certain relationship based on classical representations such as the HVT or simply 'common sense'. This relationship is frequently in the form of an inequality, but sometimes it can be simply an algebraic formula as is true, for example, of Greenberger–Horne–Zeilinger or Kochen–Specker paradoxes (see, for example Ref. [11]). The object of the description can then be a specific concrete model, as well as an arbitrary random process in general.

The second stage of Bell's theorem refutes the first stage and is based on a quantum-mechanical description of the process. Since to refute a given view it is sufficient to find just one concrete example, the second stage usually involves a concrete physical model, but sometimes the refutation can be very general as is true, for example, of the original version of the Kochen–Specker paradox just mentioned (for details see, for example, Ref. [11]).

If we start from 'common sense' and recognise the existence of a nonnegative probability distrbution function, the result is a noncorrespondence of quantum-mechanical results of fundamentally possible and already carried out (for $N = 2$) experiments capable of demonstrating breakdown of Bell's inequalities or the Greenberger–Horne–Zeilinger paradox. The *only* argument which can resolve these contradictions is that in such cases there is no positive-definite probability distribution function.

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