# Bell's theorem without the hypothesis of locality 

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Dedicated to the memory of Yurii Yakovlevich Yushin

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#### Abstract

One of Bell's assumptions in the original derivation of his inequalities was the hypothesis of locality, i.e. of the absence of the influence of two remote measuring instruments on one another. That is why violations of these inequalities observed in experiments are often interpreted as a manifestation of the nonlocal nature of quantum mechanics, or a refutation of local realism. In this paper, Bell's inequality is derived in its traditional form, without resorting to the hypothesis of locality, the only assumption being that the probability distributions are nonnegative. These probability distributions are calculated, for a specific optical experiment, in the framework of quantum theory and it is shown that they can take on negative values. This can therefore be regarded as a rigorous proof that the hypothesis of locality is not relevant to violations of Bell's inequalities. The physical meaning of the obtained results is examined.


## 1. Introduction

Despite the fact that the questions associated with the Einstein - Podolsky - Rosen (EPR) paradox [1] and Bell's theorem [2] appear to have been largely elucidated, the stream of publications on this topic has recently appreciably increased (see, for example, Belinskii and Klyshko [3] and the literature quoted therein). The failure of Bell's inequalities predicted by quantum theory and frequently tested experimentally is treated by the vast majority of investigators as the manifestation of the nonlocality of quantum theory. The point is that Bell [2] derived the original inequalities on the basis of the theory of hidden variables [1], one of the assumptions of which is the

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hypothesis of locality, i.e. of the absence of the influence of two remote measuring instruments on one another. The logical inconsistency of the need to resort to the concept of nonlocality in order to account for the failure of Bell's inequalities has been demonstrated in a recent review [3] (see also Refs [4-7]). My aim in the present study is to formulate a rigorous proof of Bell's theorem without resorting to the hypothesis of locality.

## 2. Bell's inequality

Using an algorithm analogous to that described by De Muynck [6], I shall derive Bell's inequality in its traditional form

$$
\begin{equation*}
\left|\langle A B\rangle+\left\langle A^{\prime} B\right\rangle+\left\langle A B^{\prime}\right\rangle-\left\langle A^{\prime} B^{\prime}\right\rangle\right| \leq 2 \tag{1}
\end{equation*}
$$

without the assumption of locality. Another form of the inequality has been obtained by De Muynck [6]. Here $A, A^{\prime}$, $B$, and $B^{\prime}$ are dichotomous variables assuming unit values:

$$
\begin{equation*}
A, A^{\prime}, B, B^{\prime}= \pm 1 \tag{2}
\end{equation*}
$$

The averaging is carried out with respect to events in the experiment described below. In order to demonstrate the validity of inequality (1), it is only necessary that the normalised probability distribution functions are nonnegative:

$$
\begin{equation*}
W\left(A, A^{\prime}, B, B^{\prime}\right) \geq 0, \quad W\left(A, B, B^{\prime}\right) \geq 0, \text { etc. } \tag{3}
\end{equation*}
$$

$$
\sum_{A, A^{\prime}, B, B^{\prime}} W\left(A, A^{\prime}, B, B^{\prime}\right)=1
$$

$$
\begin{equation*}
\sum_{A, B, B^{\prime}} W\left(A, B, B^{\prime}\right)=1, \text { etc. } \tag{4}
\end{equation*}
$$

Naturally,

$$
\begin{align*}
& W\left(A, A^{\prime}, B, B^{\prime}\right)+W\left(-A, A^{\prime}, B, B^{\prime}\right) \\
& =W\left(A^{\prime}, B, B^{\prime}\right) \geq W\left(A, A^{\prime}, B, B^{\prime}\right) \tag{5}
\end{align*}
$$

similarly to other variables and distributions of lower dimensions.

According to the property defined by Eqn (5), one can write

$$
\begin{align*}
W\left(A, B, B^{\prime}\right) & =W\left(A, A^{\prime}, B, B^{\prime}\right)+W\left(A,-A^{\prime}, B, B^{\prime}\right) \\
& \leq W\left(A^{\prime}, B^{\prime}\right)+W\left(-A^{\prime}, B\right) \\
& =W\left(A^{\prime}, B^{\prime}\right)+W(B)-W\left(A^{\prime}, B\right) . \tag{6}
\end{align*}
$$

Similarly

$$
\begin{align*}
0 & \leq W\left(A,-B,-B^{\prime}\right) \\
& =W(A,-B)-W\left(A,-B, B^{\prime}\right) \\
& =W(A)-W(A, B)-W\left(A, B^{\prime}\right)+W\left(A, B, B^{\prime}\right) \tag{7}
\end{align*}
$$

Let us transfer the last term of inequality (7) to the lefthand side (to zero) and add the resulting relation to inequality (6). We obtain as a result

$$
\begin{align*}
0 \leq W(A) & +W(B)-W(A, B) \\
& -W\left(A^{\prime}, B\right)-W\left(A, B^{\prime}\right)+W\left(A^{\prime}, B^{\prime}\right) \tag{8}
\end{align*}
$$

or

$$
\begin{align*}
\Lambda\left(A, A^{\prime}, B, B^{\prime}\right) & \equiv W(A, B)+W\left(A^{\prime}, B\right)+W\left(A, B^{\prime}\right) \\
& -W\left(A^{\prime}, B^{\prime}\right)-W(A)-W(B) \leq 0 \tag{9}
\end{align*}
$$

Further

$$
\begin{equation*}
W\left(B, B^{\prime}\right)=W(B)-W\left(B,-B^{\prime}\right) \tag{10}
\end{equation*}
$$

Similarly

$$
\begin{align*}
W\left(-B,-B^{\prime}\right) & =W\left(-B^{\prime}\right)-W\left(B,-B^{\prime}\right) \\
& =1-W\left(B^{\prime}\right)-W\left(B,-B^{\prime}\right) \tag{11}
\end{align*}
$$

If we subtract Eqn (10) from Eqn (11) we obtain as a result

$$
\begin{equation*}
W\left(-B,-B^{\prime}\right)=1-W(B)-W\left(B^{\prime}\right)+W\left(B, B^{\prime}\right) \tag{12}
\end{equation*}
$$

Let us substitute this relation, together with Eqn (7), in the following inequality:

$$
\begin{equation*}
0 \leq W\left(-A,-B,-B^{\prime}\right)=W\left(-B,-B^{\prime}\right)-W\left(A,-B,-B^{\prime}\right) \tag{13}
\end{equation*}
$$

Then

$$
\begin{align*}
0 \leq & 1-W(A)-W(B)-W\left(B^{\prime}\right)+W(A, B)+W\left(A, B^{\prime}\right) \\
& +W\left(B, B^{\prime}\right)-W\left(A, B, B^{\prime}\right)=1-W(A)-W(B) \\
& -W\left(B^{\prime}\right)+W(A, B)+W\left(A, B^{\prime}\right)+W\left(-A, B, B^{\prime}\right) . \tag{14}
\end{align*}
$$

The last term is in this instance subject to the inequality

$$
\begin{align*}
W\left(-A, B, B^{\prime}\right) & =W\left(-A, A^{\prime}, B, B^{\prime}\right) \\
+W(-A & \left.-A^{\prime}, B, B^{\prime}\right) \leq W\left(A^{\prime}, B\right)+W\left(-A^{\prime}, B^{\prime}\right) \\
& =W\left(A^{\prime}, B\right)+W\left(B^{\prime}\right)-W\left(A^{\prime}, B^{\prime}\right), \tag{15}
\end{align*}
$$

whence

$$
\begin{gather*}
0 \leq 1-W(A)-W(B)+W(A, B)+W\left(A^{\prime}, B\right) \\
+W\left(A, B^{\prime}\right)-W\left(A^{\prime}, B^{\prime}\right), \tag{16}
\end{gather*}
$$

or, taking into account inequality (9),

$$
\begin{equation*}
-1 \leq \Lambda\left(A, A^{\prime}, B, B^{\prime}\right) \leq 0 \tag{17}
\end{equation*}
$$

We shall now express the averages in inequality (1) in terms of combined probabilities, for example:

$$
\begin{equation*}
\langle A B\rangle=W_{A B}(++)+W_{A B}(--)-W_{A B}(+-)-W_{A B}(-+), \tag{18}
\end{equation*}
$$

where

$$
W_{A B}(++) \equiv W(A=+1, B=+1), \text { etc. }
$$

As a result of direct substitution, one can show that

$$
\begin{align*}
\langle A B\rangle+ & \left\langle A^{\prime} B\right\rangle+\left\langle A B^{\prime}\right\rangle-\left\langle A^{\prime} B^{\prime}\right\rangle=\Lambda(++++) \\
& +\Lambda(----)-\Lambda(+-+-)-\Lambda(-+-+) . \tag{19}
\end{align*}
$$

According to inequality (17), we have

$$
\begin{align*}
& -2 \leq \Lambda(++++)+\Lambda(----) \leq 0  \tag{20}\\
& 0 \leq-\Lambda(+-+-)-\Lambda(-+-+) \leq 2 \tag{21}
\end{align*}
$$

After adding together inequalities (20) and (21) and taking into account Eqn (19) we obtain the final result (1). We emphasise that the hypothesis of locality was not used in this derivation.

## 3. An example of the failure of Bell's inequality (1) and its cause

The question arises why inequality (1), which is based on extremely general postulates, is violated in practice. The lack of an answer to this question in De Muynck's communication [6] apparently led to this work being undeservedly ignored.

We shall consider the scheme for the simplest experiment designed to test inequality (1) $[3,8,9]$. Two observers (Fig. 1) A and B each record simultaneously one photon on ' + ' or ' - ' detectors assigning to these events the values $A$, $B=+1$ or -1 . By changing the phase delays $\alpha$ by $\alpha^{\prime}$ and $/$ or $\beta$ by $\beta^{\prime}$, a transition from the variables $A$ and $B$ to $A^{\prime}$ and/or $B^{\prime}$ is achieved. Numerous repetitions of the measurements make it possible to calculate the averages in inequality (1).

The quantum state of the photons reaching the observers is described by the wave vector [3]


Figure 1. Schematic illustration of the intensity interferometer with parametric sources of radiation for two observers A and B. The correlated photons are created simultaneously in the nonlinear elements $l$ or 2 under the influence of the pumping P and are directed to A and B via two modes, one of which undergoes a phase delay (circles). The modes are mixed in $50 \%$ light dividers (dashed lines) and are detected.

$$
\begin{align*}
|\psi\rangle & =(1 / \sqrt{2})\left(a_{1}^{+} b_{1}^{+}+a_{2}^{+} b_{2}^{+}\right)|0\rangle \\
& \equiv(1 / \sqrt{2})\left(|10\rangle_{a}|10\rangle_{b}+|01\rangle_{a}|01\rangle_{b}\right) \tag{22}
\end{align*}
$$

where $a_{j}^{+}$and $b_{j}^{+}$are the photon generation operators in two signal (received by the observer A) and idler (received by the observer B) modes; $j=1,2$ corresponds to the number of the crystal emitting the given mode (Fig. 1); and $|0\rangle$ denotes the vacuum state.

The photon number operators recorded by the detectors ' + ' and ' - ' in channel A assume the form

$$
\begin{equation*}
n_{ \pm}^{a} \equiv a_{ \pm}^{+} a_{ \pm}=(1 / 2)\left[n_{1}^{a}+n_{2}^{a} \pm\left(\sigma_{-}^{a} \mathrm{e}^{\mathrm{i} \alpha}+\sigma_{+}^{a} \mathrm{e}^{-\mathrm{i} \alpha}\right)\right] \tag{23}
\end{equation*}
$$

where $n_{j}^{a} \equiv a_{j}^{+} a_{j}, \sigma_{-}^{a} \equiv a_{1} a_{2}^{+}$, and $\sigma_{+}^{a}=\left(\sigma_{-}^{a}\right)^{+}, \quad j=1,2$. Similar relations define $n_{ \pm}^{b}$ in channel B.

We now find the distribution function $W\left(A, A^{\prime}, B, B^{\prime}\right)$ having calculated the combined probabilities as quantum moments:

$$
\begin{align*}
& \begin{array}{l}
W_{A A^{\prime} B B^{\prime}}(++++) \\
\\
\quad \equiv W\left(A=+1, A^{\prime}=+1, B=+1, B^{\prime}=+1\right) \\
\\
\quad=\langle\psi| n_{+}^{a} n_{+}^{a^{\prime}} n_{+}^{b} n_{+}^{b^{\prime}}|\psi\rangle
\end{array} \\
& W_{A A^{\prime} B B^{\prime}}(+++-)=\langle\psi| n_{+}^{a} n_{+}^{a^{\prime}} n_{+}^{b} n_{-}^{b^{\prime}}|\psi\rangle \text { etc. }
\end{align*}
$$

The primes denote here the replacement of $\alpha$ by $\alpha^{\prime}$ in Eqn (23) and/or $\beta$ by $\beta^{\prime}$ in channel B.

We establish the following phases in the channels:

$$
\begin{equation*}
\alpha=0, \quad \alpha^{\prime}=\pi / 2, \quad \beta=-\pi / 4, \quad \beta^{\prime}=\pi / 4, \tag{25}
\end{equation*}
$$

which corresponds to failures of inequality (1). As a result, we obtain the matrix elements (the lower indexes are omitted)

$$
\begin{align*}
& \begin{array}{l}
W(++++)=W(+--+) \\
\quad=W(-++-)=W(----)=\sqrt{2} / 16 \\
\\
\quad W(++--)=W(+-+-)=W(-+-+) \\
\quad=W(--++)=-\sqrt{2} / 16 \\
W(++-+)=W(+---)=W(-+++) \\
\quad=W(--+-)=(2-\sqrt{2}) / 16 \\
W(+++-)=W(+-++)=W(-+--) \\
\quad=W(---+)=(2+\sqrt{2}) / 16
\end{array}
\end{align*}
$$

Some three-dimensional probabilities are also negative, for example

$$
\begin{align*}
& W_{A^{\prime} B B^{\prime}}(+++)=W(-++) \\
& \quad=W(+--)=W(---)=1 / 8 \\
& W_{A^{\prime} B B^{\prime}}(--+)=W(++-)=(1+\sqrt{2}) / 8 \\
& W_{A^{\prime} B B^{\prime}}(+-+)=W(-+-)=(1-\sqrt{2}) / 8 \tag{27}
\end{align*}
$$

Thus the only cause of the failure of inequality (1) is the negative sign of the probability distribution functions, i.e. failure of inequality (3) and hence of inequality (5). According to Eqns (26) and (27), the normalisation conditions (4) and equations of type (5) hold in this situation.

## 4. Conclusion

In connection with the EPR paradox and Bell's theorem, negative probability distributions have been encountered in the literature [ $10-15$ ]. However, the definition of the distribution function in the form $W\left(A, A^{\prime}, B, B^{\prime}\right)$ makes it possible to reach an unambiguous conclusion concerning the role of locality or, more precisely, the absence of such in the failure of inequality (1) by comparing directly the results (26) and (27) with the initial postulates (3)-(5). There is also no need to resort to 'hidden variables'.

The probability distribution function $W\left(A, A^{\prime}, B, B^{\prime}\right)$ is analogous to the Wigner distribution. Not all the observables in it are described by commutating operators, for example $A$ and $A^{\prime}$. They cannot be measured in a single event (the observer $A$ can in no way record a single photon at different phase delays $\alpha$ and $\alpha^{\prime}$ ). Consequently, direct measurements of $W\left(A, A^{\prime}, B, B^{\prime}\right)$ are impossible. However, indirect methods for the measurement of distribution functions of this type are nevertheless permissible. Thus a novel method, admittedly designed for two-dimensional continuous Wigner distributions, including negative ones, has been proposed [16] and applied experimentally. Perhaps one should become recon-ciled to a negative probability, regarding it, after Dirac [17], as a well-defined mathematical analogue of a negative sum of money (see also Muckenheim [12]).

Indeed, by following the proof of inequality (1) in the opposite direction, from the experimentally recorded averages, we obtain joint probabilities of types (26) and (27); although they do not have concrete values, nevertheless some of them are bound to be negative.

We have yet another analogy. Negative temperatures do not exist on the Kelvin scale, but a formal description of the inversion of population with the aid of a negative temperature is widely used in quantum electronics. A negative temperature cannot be measured with a thermometer, but it can be calculated and the state of the active levels can be elucidated. Thus the formal recognition of the existence of the distribution function $W\left(A, A^{\prime}, B, B^{\prime}\right)$ does not leave room for the requirement that it should be nonnegative.

The present study does not claim to embrace all possible quantum effects and to elucidate the problem of nonlocality in quantum theory in general. For example, the behaviour of a single photon in a Mach-Zender interferometer can be treated as nonlocal in the sense that it belongs simultaneously to two modes (arms) of the interferometer separated in space (see also Belinskii and Klyshko [3]). The present study nevertheless makes it possible to claim that the violation of Bell's inequalities does not provide grounds for the invocation of the inexplicable nonlocality as a characteristic of quantum mechanics and for seeking help from mysticism in this connection.

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