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Localised nontopological structures: construction of solutions and stability problems

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Contents

Abstract. Possible methods are discussed for describing structures localised in finite region (solitons, vortices, defects and so on) within the framework of both integrable and nonintegrable field models. For integrable models a universal algorithm for the construction of soliton-like solutions is described and discussed in detail. This algorithm can be generalised to many-dimensional cases and its efficacy for several examples exceeds that of the standard inverse scattering transform method. For non-integrable models we focus mainly on methods of studying the stability of soliton-like solutions, since stability problems become essential when one turns to a description of many-dimensional solitons. Special attention is paid to those stable localised structures that are not endowed with topological invariants, since for topologically nontrivial structures there exist effective methods of stability analysis, based on energy estimates. Here the principal topic is that of Lyapunov's direct method as applied to distributed systems are discussed. Effective stability criteria for stationary solitons, endowed with one or more charges, (the Q-theorem) are derived. Several examples are presented that illustrate the applicability of the method of functional estimates, and the stability of plasma solitons of the electron phase hole type is discussed.

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1. Introduction

Localised structures, or *soliton-like excitations* arise in dynamical systems either under the influence of sufficiently strong external forces, or as a result of nonlinear self-interaction effects. Indeed, under weak influence (or, equivalently, when it is possible to ignore self-interaction effects), the evolution of the system is well described by linear relationships. But linear equations yield only spreading wave packets as solutions for the *Cauchy problem* with regular boundary conditions, localised in a small space region. The basis for this spreading is the superposition principle, which is characteristic of linear systems. However, under sufficiently strong perturbation, or under non-negligible self-interaction effects, this principle breaks down, since further evolution of the system is governed by substantially nonlinear relations. As a result, structures with properties unfamiliar to linear physics are derived. In particular, these objects might be incredibly stable. Studies of such localised structures became the subject of *soliton physics*.

In connection with an intensive development of solitonic themes, starting in the late 1960s, the problem of the description of localised structures (vortices, defects, textures and so on) at a new qualitative level has again arisen, as well as the description of particles as extended objects in condensed matter physics, in astrophysics and cosmology, and in particle and nuclear physics. In time this coincided with the appearance of experimental proof of the existence of an internal structure for strongly interacting particles: in the experiments of R Hofstadter (1956) on elastic scattering of electrons on protons, the electric charge distribution within the proton was found; in the experiments of E Blume et al. (1969) on the deep inelastic scattering of electrons on nucleons, *the scaling phenomenon* was discovered, that is the scale invariance of the scattering cross-section. The latter observation served as a basis for *the parton model*, suggested by R Feynman, the model which provided an explanation why, in elastic scattering experiments nucleons manifest themselves as extended objects, whereas for deep inelastic processes this picture is no longer valid. Indeed the results for an inelastic process are analogous to scattering on a pointlike (structureless) object [1]. In such modern theories as quantum chromodynamics, electro-weak and standard models, the role of partons is played by quarks. This means that the particle structure is described within the framework of the so-called *composite models*, when extended particles are constructed from pointlike ones. On the one hand, it is clearly hard to imagine a logical completion for this process, and, on the other hand the presence of structureless particles in a model leads to the appearance of divergences. The elimination of these divergences requires ever more ingenious schemes at each successive level of the theory.

For this reason, alternative approaches to the description of particles as extended objects, i.e. beyond the framework of composite models, deserve special attention. It is appropriate here to note that a search for such an alternative description is in a sense traditional in the evolution of ideas in physics. Very similar considerations led Lord Kelvin (W Thomson) at the end of the last century to suggest the existence of 'vortex atoms' of finite extension instead of pointlike atoms. Similar ideas were proposed by O Heaviside, J J Thomson and G Mie. In a more concrete form these ideas have been formulated by A Einstein, who suggested describing particles in terms of regular solutions of field equations, as, in effect bunched fields that occupy ''...a bounded region in space, where the field strength and the energy density are particularly high...'' ([2], p. 725). The notion of a particle as a regular physical field, localised in a small region of space and endowed with finite energy and all other dynamical attributes appeared in the literature under several names: particle-like solutions in articles by N Rosen, R Finkelstein, Ya P Terletskii et al.; le *champ a bosse* in L de Broglie's papers; *kinks* as named by D Finkelstein, and *lumps* by S Coleman. The concept of a many-dimensional soliton endowed with nontrivial topological structure arose in the late 1950s in T H R Skyrme's papers (see Ref. [3] and original papers cited therein), and one can consider this as an appropriate generalisation of all these earlier notions. It might be well to point out a beautiful (nontopological) concept, introduced by T D Lee [4], which in a particular form combines the approaches listed above into a description of particle structure. As the basis for this concept a nonlinear mechanism of quark confinement was chosen, whereby bosons strongly interacting with quarks form a confining potential of *a solitonic bag* type. Recent trends are toward the development of this concept in the framework of the so-called *hybrid bag models*, where an external soliton (a topological one, as a rule, and providing the correct spectral data) is used for the confinement of quarks inside the bag. In the present review the current status of the problem of the description of localised coherent structures is discussed, and, in particular, a description of particles as extended objects on the basis of integrable as well as nonintegrable field models is given. In doing this we are restricting ourselves to problems of constructing explicit solutions for integrable models, and to studies of stability problems for nonintegrable models which possess many-dimensional localised structures. We intend to discuss in a separate publication the problems of the existence and stability of topologically nontrivial localised structures.

PART I. INTEGRABLE MODELS

A dozen or so substantial monographs and reviews [5 – 15] are devoted to integrable dynamical systems. Therefore we feel free here to limit ourselves to mentioning only some aspects of this theory (those most important for the following presentation). Since in the existing literature there are some disagreements on the very definition of integrable systems (as well as that of solitons), in what follows we shall use the definitions:

1. *Integrable systems* — systems which possess a Lax representation (or, in a more recent sense, a zero curvature representation), which have a countable number of integrals of motion, and for which in the investigation of their dynamics one can apply the inverse spectral transform method, the Riemann problem, the $\bar{\partial}$-problem, and the finite-zone integration method. This is called *S-integrability*. To a similar category belong dynamical systems which one can integrate by a change of variables or by means of an ansatz (C-integrability).

2. *Completely integrable systems* — Hamiltonian integrable systems, for which one can find action-angle variables and rewrite the Hamiltonian of the system in terms of these variables.

Let us list without exhaustive details, which one can find in monographs [9, 14], some properties of integrable systems using as an example *the nonlinear Schrödinger equation*† (NSE) pursuing the twofold aim of defining the terminology to be used later and of reminding the reader of some facts on NSE properties, which might be useful in what follows. A dynamical system associated with the NSE

$$\mathrm{i}\partial_t\psi + \partial_x^2\psi + 2g|\psi|^2\psi = 0 , \tag{2.1}$$

where $\psi(x,t)$ is a complex-valued function, is classified as a Hamiltonian system, since it is provided with a set of canonically conjugate variables $\psi, \bar{\psi}$ (here the bar means complex conjugation). Consequently, for the system (2.1) the Poisson bracket (in its generalised form for the continuous case) is defined as

$$\{\psi, \bar{\psi}\} = \mathrm{i}\delta(x-y) , \tag{2.2}$$

and Hamiltonian

$$H = \int_{-\infty}^{\infty} \mathrm{d}x \left[(\partial_x\bar{\psi} \cdot \partial_x\psi) - g(\bar{\psi}\psi)^2 \right] , \tag{2.3}$$

for which the Hamiltonian equations hold

†Note, that this name given to Eqn (2.1), has in fact nothing to do with (fundamentally linear!) quantum mechanics, but was given to it only because in its linear limit $g = 0$ it coincides with the Schrödinger equation for the wave function of a free particle.

$$\partial_t\psi = \{H, \psi\} = -\mathrm{i}\frac{\delta H}{\delta\bar{\psi}},$$

$$\partial_t\bar{\psi} = \{H, \bar{\psi}\} = \mathrm{i}\frac{\delta H}{\delta\psi}. \tag{2.4}$$

Taken together the facts (2.2)–(2.4) prove that the system (2.1) is Hamiltonian. Note that to define the dynamical system one must also specify boundary conditions. In the following discussion we will mainly consider rapidly decreasing cases, i.e. where

$$\psi(x,t) \to 0 \quad \text{for} \quad |x| \to \infty. \tag{2.5}$$

In addition we will assume that $\psi(x,t)$ is an infinitely smooth function, decreasing at spatial infinity, together with its derivatives, faster than any power of $|x|^{-1}$.

One can also represent equation (2.1) as *the consistency condition* for the following overdetermined system of linear matrix equations

$$\partial_t y = A_0(x, t; \lambda)y,$$
$$\partial_x y = A_1(x, t; \lambda)y \tag{2.6}$$

on the vector-function

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which one can derive from Eqn (2.6) by equating the cross derivatives $\partial_t\partial_x y = \partial_x\partial_t y$. As a result one obtains the *zero curvature condition*†

$$\partial_t A_1 - \partial_x A_0 + [A_1, A_0] = 0, \tag{2.7}$$

which is one of the principal relations in the inverse spectral transform method. Note that the 2×2-matrices A_0, A_1 in Eqns (2.6) are dependent on an arbitrary complex-valued parameter λ, known as the *spectral parameter of the problem*, and condition (2.7) has to be valid for all λs. The explicit form for matrices A_0, A_1 is given as:

$$A_1 = -\mathrm{i}\lambda\sigma_3 + P, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \mathrm{i}\begin{pmatrix} 0 & \bar{\psi} \\ \psi & 0 \end{pmatrix},$$

$$A_0 = B - 2\lambda P + 2\mathrm{i}\lambda^2\sigma_3, \quad B = \begin{pmatrix} -\mathrm{i}|\psi|^2 & \partial_x\bar{\psi} \\ -\partial_x\psi & \mathrm{i}|\psi|^2 \end{pmatrix}. \tag{2.8}$$

It is known, that a system possessing the *zero curvature representation* is endowed with an infinite (but countable!) set of additive integrals of motion (or conservation laws), or, in the presence of internal (isotopic) symmetries, with a series of such sets. Formally these laws can be written as the continuity equation

$$\dot{\rho}_n + \partial_x j_n = 0, \quad n = 1, 2, \dots, \tag{2.9}$$

where the functionals ρ_n and j_n are polynomials in the field function and its spatial derivatives, associated with the 'densities' and 'currents' of the system, respectively.‡ On integrating Eqn (2.9) over x we obtain the integrals of motion

$$I_n = \int_{-\infty}^{+\infty} \rho_n(x,t)\,dx, \quad j_n \to 0 \quad \text{when} \quad |x| \to \infty. \tag{2.10}$$

If the integrals I_n are in involution with respect to the Poisson bracket (2.2) and one is able to introduce angle variables canonically conjugate with them, then the corresponding system would be *completely integrable*, and the integrals (2.10) would play the role of the action variables.

In some cases, one can solve the inverse scattering problem for the operator $L = \mathrm{i}\sigma_3(\partial_x + \mathrm{i}\lambda\sigma_3 - A_1)$, i.e. via scattering data to find an explicit form of the potential, the required function $\psi(x,t)$ playing its role. This means that for the situation discussed here one can solve the Cauchy problem, so that the behaviour of this integrable system will be strictly determined. The localised regular solutions to integrable systems (if they exist), which correspond to the discrete part of the spectrum of the operator L are usually called *solitons*. For integrable systems the quantum inverse scattering method has also been developed, which enables one to find the ground state and excitation spectra [17, 18].

Here we have sketched a rough outline of the current possibilities of describing localised structures as solitons in integrable models. It should be noted at once that in spite of intensive efforts undertaken in the development of this approach, the scheme presented may be successfully realised only in the case of $(1+1)$-dimensional completely integrable models, such as the Korteveg–de Vries (KdV) equation, NSE, sine-Gordon, and so on. This restricts the range of possible applications, but at the same time it prompts an active search for other methods of studying integrable models, which would enable one to describe many-dimensional solitons. Among them should be mentioned the Riemann problem [5, 6] and the finite-zone integration methods [5, 8, 19], the Darboux transformation method [20], and various methods of group-theoretical and algebraic–geometric analysis [8, 15, 21]. In what follows we will consider only one of the possibilities listed above of extending the methods for studying integrable models to many-dimensional cases, choosing as the basic model the nonlinear Schrödinger equation.

2. Multisoliton solutions to Schrödinger-type nonlinear equations

The nonlinear Schrödinger equation (NSE) is one of the fundamental equations of nonlinear mathematical physics, describing the evolution of a weakly nonlinear and strongly dispersed quasimonochromatic wave. In particular, NSE describes the evolution of hydrodynamic waves in deep water, that of optical waves in nonlinear crystals and light guides, that of Langmuir waves in plasma and heat waves in solids, the evolution of spin waves in magnets, and so on. For a number of reasons, partly outlined in monographs [9, 11, 14], NSE might be considered as exceptional among integrable models. Strictly speaking, the fact that $\psi(x,t)$ in Eqn (2.1) is a complex, instead of a real-valued function as in the KdV or sine-Gordon equations distinguishes the NSE among integrable equations. On the other hand, this exceptionality of NSE is related to the quadratic form of its dispersion, which in the vacuum state coincides with that of a free nonrelativistic particle. At the same time, for strongly nonlinear states, for example for a condensate, NSE provides the correct expression for linear excitations (first obtained by N N Bogolubov). One of the

†This name is related to the geometrical interpretation of the system (2.6) together with condition (2.7) in terms of fibre-bundle spaces (see [9]).

‡These densities are referred to as local ones. In a number of models along with local conservation laws there are also nonlocal conservation laws, with 'densities' as integrals over x.

most attractive features of the NSE is that it enables us to describe the evolution of wave packet envelopes for carrier waves in media with quadratic dispersion. Thus, NSE allows one to rehabilitate the L de Broglie idea of representating of particles as wave packets, which has not met with success in linear theories, where the wave packets spread out owing to dispersion.

2.1 Nonlinear Schrödinger equations and envelope solitons

The NSE arises, as a rule, when one describes nonlinear phenomena in various circumstances where solutions in the form of harmonic wave packets are found to be acceptable

$$\psi(x,t) = A\exp\{\mathrm{i}[kx - \omega(k)t]\}\ , \tag{2.1.1}$$

with a sufficiently small amplitude A. Nonlinearity of the medium manifests itself in a back action on the amplitude in Eqn (2.1.1). As a result, the wave envelope slowly (compared with the carrier wave) varies in space, as well as in time, i.e. it modulates the fast (high-frequency) carrier wave. The key point of the NSE approach consists in finding weakly nonlinear expansions of the dispersion relations†, which, in contrast to the pure linear state, can take into account the dependence on the amplitude. Two different specifications of the problem are frequently given:

$$\omega = \omega(k;|A|^2),\quad k \in R,\quad \text{initial-value problem}\ , \tag{2.1.2a}$$

or

$$k = k(\omega;|A|^2),\quad \omega \in R,\quad \text{boundary-value problem}\ . \tag{2.1.2b}$$

A Taylor expansion of Eqn (2.1.2) in the vicinity of some $(\omega_0,\ k_0)$ gives

$$\omega = \omega_0 + \frac{\partial\omega}{\partial k}\Big|_0 (k-k_0) + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\Big|_0 (k-k_0)^2 + \frac{\partial\omega}{\partial|A|^2}\Big|_0 |A|^2 + \ldots\ ,$$

or

$$k = k_0 + \frac{\partial k}{\partial\omega}\Big|_0 (\omega-\omega_0) + \frac{1}{2}\frac{\partial^2 k}{\partial\omega^2}\Big|_0 (\omega-\omega_0)^2 + \frac{\partial k}{\partial|A|^2}\Big|_0 |A|^2 + \ldots\ .$$

In Fourier-transform space for the waves (2.1.1), this expansion might be represented in operator form by the relations

$$(\omega-\omega_0) \to \mathrm{i}\frac{\partial}{\partial t}\ ,$$

$$(k-k_0) \to -\mathrm{i}\frac{\partial}{\partial x}\ ,$$

$$(k-k_0)^2 \to -\frac{\partial^2}{\partial x^2}\ ,$$

i.e. in the form of a NSE operator, acting on amplitude A:

$$\left[\mathrm{i}\left(\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k}\Big|_0\frac{\partial}{\partial x}\right) + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\Big|_0\frac{\partial^2}{\partial x^2} - \frac{\partial\omega}{\partial|A|^2}\Big|_0|A|^2\right]A = 0 \tag{2.1.3a}$$

or as

$$\left[\mathrm{i}\left(\frac{\partial}{\partial x} + \frac{\partial k}{\partial\omega}\Big|_0\frac{\partial}{\partial t}\right) - \frac{1}{2}\frac{\partial^2 k}{\partial\omega^2}\Big|_0\frac{\partial^2}{\partial t^2} + \frac{\partial k}{\partial|A|^2}\Big|_0|A|^2\right]A = 0\ . \tag{2.1.3b}$$

Equation (2.1.3a) describes the time evolution of an envelope for a narrow packet of carrier waves with real-valued k. The crude way of deriving of the latter equation displayed here has been generalised to the method of *multiscale* (*or two time variables*) *decomposition*, where together with the 'fast' variables x, t for the carrier wave, a set of 'slow' variables $X_n = \varepsilon^n x, T_n = \varepsilon^n t$, $(\varepsilon \ll 1)$ is introduced for a description of the envelope motion (see detailed description of the multiscale expansion methods with a number of references in Ref. [11], Ch. 8). Equation (2.1.3b) describes the propagation in space (for example, in a waveguide) of a narrow wave packet with a given carrier wave frequency $\omega = \omega_0 \in R$.

Equations of the type (2.1), (2.1.3), also called *scalar nonlinear Schrödinger equations*, themselves represent the simplest mathematical models for a description of weakly nonlinear wave packets of high-frequency, and, in particular, models for self-interacting spin waves (magnons) in ferromagnets, for excitations in molecular crystals, for the Langmuir waves in plasma, for two-body interactions of boson gas particles at zero temperature and so on. Detailed derivations of the NSE together with a description of the models listed above can be found in Refs [11] and [14].

As a natural generalisation for Eqn (2.1) one can consider the system which describes the interaction of high-frequency wave packets $\psi(x,t)$ with low-frequency waves $U(x,t)$. For this type of situation a complex function $\psi(x,t)$ is subject to the same scalar NSE

$$\mathrm{i}\partial_t\psi + \partial_x^2\psi + U\psi + g|\psi|^2\psi = 0\ , \tag{2.1.4}$$

with a low-frequency wave $U(x,t)$ as potential, the latter in turn beingdescribed by one of the following self-consistency equations:

$$\Box U = -\partial_x^2(|\psi|^2)\quad \text{(Zakharov, 1972)}\ , \tag{2.1.4a}$$

$$\partial_t U + \partial_x(U - |\psi|^2) = 0\quad \text{(Yajima-Oikawa, 1976)}, \tag{2.1.4b}$$

$$(\partial_t + \alpha\partial_x^3)U + \partial_x(\beta U^2 - |\psi|^2 + U) = 0 \tag{2.1.4c}$$

(Nishikawa, et al., 1974) ,

$$(\Box + \alpha\partial_x^4)U + \partial_x^2(\beta U^2 + |\psi|^2) = 0 \tag{2.1.4d}$$

(Makhankov, 1974) .

Of the systems listed above, in the first two cases (2.1.4a,b) the low-frequency excitations are described by linear equations, but only the second is integrable (for $g = 0$). The remaining two equations are nonlinear. The integrability of Eqn (2.1.4d), with an appropriate choice of parameters α and β, has been established by I M Krichever [22], and in turn the non-integrability of Eqn (2.1.4c) has been proved by E S Benilov and S P Burtsev [23]. Systems like (2.1.4a – d) with $g = 0$ occur in plasma physics, where they serve as a basis for the description of coupled Langmuir and ion-acoustic waves. In general, for $g \neq 0$

†In physics this corresponds, for example, to the observation that the refractive index in nonlinear optics or the dielectric constant of a plasma can be represented as polynomial functions of electric field strength.

they appear in the description of spin waves and phonon interactions in ferromagnets [24], and in a description of excitons and phonons in molecular crystals [25], and so on.

Another natural generalisation of NSE (2.1) is related to its transformation into a vector version by the rule: $\psi \to (\psi_1, \psi_2, ..., \psi_n)^{\rm tr}$ along with a simultaneous replacement of $|\psi|^2$ in (2.1) by the inner product

$$(\psi, \psi) \stackrel{\rm def}{=} \sum_{i,j=1}^{n} g_{ij} \overline{\psi}_i \psi_j \ ,$$

where g_{ij} is the metric tensor in some internal symmetry space of the model under investigation [26] and [37]. Applying the hermiticity condition to the Hamiltonian of the system, we arrive at non-compact groups of internal symmetry $U(p,q)$. Physical models of this type are used to describe the dynamics of a boson gas with internal quasispin (or 'coloured') degrees of freedom or that of boson gas mixtures (related to the superconductivity phenomenon at $T \neq 0$), as well as to describe the propagation in plasmas of high-frequency plane waves with circular polarisation and that of spin waves in multilayered ferromagnets. (For details and references one may consult Ref. [14], Ch. 2). The integrability of some systems described by the vector NSE has been established in [26] and [35]. Finally, if we combine the generalisations listed above, we obtain the vector NSE

$$\mathrm{i}\partial_t \psi + \partial_{xx}^2 \psi + U\psi + g(\psi \cdot \psi)\psi = 0 \ , \qquad (2.1.5)$$

with a self-consistent potential (the low-frequency mode), which in turn is governed by one of equations (2.1.4 a – d).

A whole range of applications is associated with the so-called derivative NSE (i.e. NSE with a potential that contains a derivative):

$$[-\mathrm{i}\partial_t + \partial_x^2 + \mathrm{i}U(x,t)\partial_x]\psi(x,t,k) = 0 \ , \qquad (2.1.6)$$

which has attracted considerable interest in relation to the study of (2+1)-dimensional systems, such as the modified Kadomtsev – Petviashvili and Ishimori equations [27].

It is a remarkable fact that for all the variants of NSE enumerated above a general-purpose algebraic method of constructing exact solitonic solutions exists and was first propounded in Ref. [12] (see also Ref. [14], Ch. 8).

2.2 Algebraic method of constructing exact solitonic solutions for Schrödinger-type nonlinear equations ($D = 1$)

Conceptually the method outlined below is a specific case of the general algebraic – geometric scheme of *finite-zone integration*, as described in Ref. [5]. Contrary to the standard inverse scattering problem method, where for each equation considered there is its inherent auxiliary linear spectral problem, in the proposed construction a universal auxiliary role is played by a linear Schrödinger equation with a time-dependent potential $U(x,t)$:

$$\mathrm{i}\partial_t + \partial_x^2 + U(x,t)\psi(x,t,k) = 0 \ . \qquad (2.2.1)$$

It should also be noted, that the method presented proves effective in those cases where the inverse scattering method fails to be helpful. In particular this is the case for the boson-gas models with non-compact internal symmetry groups, for the isotropic Landau – Lifshitz model with the $SU(1,1)$ group, for nonlinear σ-models, and others (see Ref. [14], Part III). In further discussion, the problem will be considered at two levels: the linear and the nonlinear one.

At the *linear level* for given spectral data (SD) we find a special class of localised reflectionless (Bargman) potentials $U(x,t)$ along with the corresponding wave functions $\psi(x,t,k)$. The spectral data consist of a set of complex numbers κ_i, $i = \overline{1,N}$, and a complex-valued-constant $N \times N$-matrix c_{ij}, i.e. these SD in fact provide a solution for a specific inverse problem. Furthermore, we derive the conditions which κ_i and c_{ij} have to satisfy in order to provide the real value and regularity of the obtained potentials $U(x,t)$ together with the corresponding wave functions $\psi(x,t,k)$. We also discuss the degeneracy of solutions with respect to the SD and note two possible representations for wave functions: the polynomial and the rational (pole-type) one. Investigations of the asymptotic behaviour of the solutions allows us to find explicit expressions for structural units ('bricks'), of which the potentials and wave functions are composed.

At the *nonlinear level*, *self-consistency* conditions are found that relate the potentials to the wave functions and their residues. Here a choice of boundary conditions for nonlinear fields plays a crucial role and the fields are expressed in the form of direct sums of the aforementioned structural units.

2.2.1 Linear level The potential $U(x,t)$ in the nonstationary Schrödinger equation will be called the *integrable* potential (associated with a rational algebraic curve) if equation (2.2.1) admits solutions in the form of the *plane-wave ansatz*:

$$\begin{aligned}\psi(x,t,k) &= P_N(x,t,k)\exp\left[\mathrm{i}k(x+kt)\right] \\ &\equiv \left(k^N + a_{N-1}(x,t)k^{N-1} + ... \right. \\ &\qquad \left. ... + a_0(x,t)\right)\exp\left[\mathrm{i}k(x+kt)\right] \ . \end{aligned} \qquad (2.2.2)$$

Let us introduce, as the free parameters of our construction, the complex numbers $\kappa_1, ..., \kappa_M$ with nontrivial imaginary parts and a matrix of coefficients α_{ij}, $i = \overline{1,N}$, $j = \overline{1,M}$. For any set of these parameters we can uniquely determine the function $\psi(x,t,k)$ having the form (2.2.2) with the help of the following system of linear conditions

$$\sum_{j=1}^{M} \alpha_{ij}\psi(x,t,k)\Big|_{k=\kappa_j} = 0, \quad i = \overline{1,N} \ . \qquad (2.2.3)$$

The conditions (2.2.3), which themselves represent a system of N linear inhomogeneous algebraic equations, are solvable, if the corresponding matrix of coefficients $A(x,t) = [\alpha_{ij}]$ is nonsingular, or, in other words, if $\operatorname{rank}[\alpha_{ij}] = N$. The search for the potential $U(x,t)$ is based on the theorem, proved in Ref. [12]:

Theorem 2.1. *If the matrix $A(x,t)$ of the system* (2.2.3) *is not identically singular* (*in x,t*), *then the function $\psi(x,t,k)$ of the form* (2.2.2) *under conditions* (2.2.3) *satisfies Eqn* (2.2.1) *with the potential*

$$U(x,t) = 2\mathrm{i}\partial_x a_{N-1}(x,t) = 2\partial_x^2 \ln\det A(x,t) \ . \qquad (2.2.4)$$

In general, potentials $U(x,t)$, corresponding to an arbitrary set of parameters κ_i and $[\alpha_{ij}]$, will be complex meromorphic functions of (x,t). In order to obtain real-valued and regular potentials as functions of the real variables x,t, one needs to put some restrictions on the choice of parameters. Let us assume that $M = 2N$ and that it is possible to subdivide the parameter set $\kappa_1, ..., \kappa_{2N}$ into complex conjugate pairs of the type $\kappa_{N+i} = \bar{\kappa}_i$, $i = \overline{1, N}$. We can further assume, and without loss of generality, that the minor of the matrix $[b_{ij}] \equiv [\alpha^c_{ij}]$, consisting of the columns with indices $j = N+1, ..., 2N$, is nonsingular (in general this minor can be reduced to the unit matrix). In this case the conditions (2.2.3) take the form

$$\psi(\bar{\kappa}_i) = -\sum_{j=1}^{N} b_{ij}\psi(\kappa_j), \quad i = \overline{1, N} \ , \tag{2.2.5}$$

where b_{ij} is a constant $N \times N$ matrix . The ansatz (2.2.2) gives a polynomial representation for the wave function of the NSE (2.2.1), and when coupled with the condition (2.2.5), it can be regarded as a generalisation of the Beiker–Akhiezer function [5].

The *rational* or *pole-type representation* for the wave function of Eqn (2.2.1) may be given in the form

$$\Psi(x, t, k) = \frac{\psi(x, t, k)}{\prod_{i=1}^{N}(k - \kappa_i)} \equiv \left(1 + \sum_{j=1}^{N}\frac{r_j(x, t)}{k - \kappa_j}\right) \exp\left[\mathrm{i}k(x + kt)\right] . \tag{2.2.6}$$

In this case, the condition (2.2.5) may be rewritten as

$$\Psi(x, t, \bar{\kappa}_i) = -\sum_{j=1}^{N} c_{ij}\Psi_j(x, t) \ , \tag{2.2.7}$$

where

$$\Psi_j(x, t) = \mathrm{res}_{k=\kappa_j}\Psi(x, t, k) = \lim_{k\to\kappa_j}\left[(k - \kappa_j)\Psi(x, t, k)\right] \tag{2.2.8}$$

with the introduction of the matrix

$$c_{ij} = b_{ij}\frac{R'(\kappa_j)}{R(\bar{\kappa}_j)}, \quad R' = \frac{\mathrm{d}}{\mathrm{d}k}R(k) \ , \tag{2.2.9}$$

by means of the function

$$R(k) = \prod_{j=1}^{N}(k - \kappa_j) \ .$$

T h e o r e m 2.2. *In order for the potential $U(x, t)$ of the NSE (2.2.1) to be a real and non-singular function of the real variables x, t, the following conditions are sufficient*: (1) *The matrix c_{ij} in* (2.2.9) *should be skew-Hermitian*: $[c_{ij}] = -[c_{ij}]^\dagger$; (2) *On the assumption that for parameters κ_i the following conditions hold: $\mathrm{Im}\,\kappa_i > 0$ for $i = \overline{1, p}$ and $\mathrm{Im}\,\kappa_j < 0$ for $j = \overline{p+1, N}$, the Hermitian matrix $i^{-1}[c_{ij}]$ for $i, j = \overline{1, p}$ should be positive definite whereas $i^{-1}[c_{ij}]$ for $i, j = \overline{p+1, N}$ it should be a negative definite matrix*.

This is proved in Ref. [12]. In practice the conditions of Theorem 2.2 contain the first substantial limitations on the location of poles κ_i of the sought-for function, related to the form of the matrix c_{ij} from (2.2.9).

Let us enumerate some properties of the solutions obtained.

1. T h e d e g e n e r a t e c a s e. In the polynomial representation (2.2.2) both the wave function $\psi(x, t, k)$, and the potential $U(x, t)$ are 2^N-fold degenerate with respect to SD changes. At the same time in the pole-type representation (2.2.6) only the potential $U(x, t)$ remains 2^N-fold degenerate, i.e. for 2^N different sets of the SD we obtain one and the same potential $U(x, t)$.

Here we point out the explicit form of transformations from one set of SD to another. Let the matrix b_{ij} be given in the block form:

$$[b_{ij}] = \begin{pmatrix} \alpha_+ & \beta \\ \gamma & \alpha_- \end{pmatrix} , \tag{2.2.10}$$

where the square matrices α_+ and α_- are $p \times p$ and $(N-p) \times (N-p)$ matrices, respectively (recall, that $\mathrm{Im}\,\kappa_i > 0$ for $i = \overline{1, p}$, and $\mathrm{Im}\,\kappa_j < 0$ for $j = \overline{p+1, N}$) and $\det\alpha_- \neq 0$. Then the transformations from one set of SD to another $\{\kappa_i, b_{ij}\} \Longrightarrow \{\kappa'_i, b'_{ij}\}$ are written in the form

$$\begin{aligned} \kappa'_i &= \kappa_i \quad \text{for} \quad i = \overline{1, p}, \\ &= \bar{\kappa}_i \quad \text{for} \quad i = \overline{p+1, N} \ , \end{aligned} \tag{2.2.11}$$

and

$$[b'_{ij}] = \begin{pmatrix} \alpha_+ - \beta\alpha_-^{-1}\gamma & -\beta\alpha_-^{-1} \\ \alpha_-^{-1}\gamma & \alpha_-^{-1} \end{pmatrix} . \tag{2.2.12}$$

2. A s y m p t o t i c b e h a v i o u r. Let us consider the asymptotic behaviour of the solution in x and t for various N.

(a) In the simplest case for $N = 1$, $\kappa = \alpha + \mathrm{i}\beta$, the potential $U(x, t)$ assumes a soliton-like form (for details consult Ref. [14], Ch. 8):

$$U(x, t) = -2\beta^2\cosh^{-2}[\beta(x - x_0 + 2\alpha t)] \ , \tag{2.2.13}$$

while the wave function is

$$\Psi = \left[1 + \frac{\mathrm{i}\beta}{k - \kappa}\left\{1 + \tanh\beta(x - x_0 + 2\alpha t)\right\}\right] \times \exp\left[\mathrm{i}k(x + kt)\right] \ , \tag{2.2.14}$$

where it is assumed that

$$2\mathrm{i}\beta c = -\mathrm{e}^{2\beta x_0} \ .$$

For $N > 1$ and for all κ_i with $\mathrm{Im}\,\kappa_i > 0$ and $\mathrm{Re}\,\kappa_i \neq \mathrm{Re}\,\kappa_j$ for $i \neq j$, the potential asymptotically decays to a direct sum of the potentials (2.2.13). Hence, such $N = 1$ potentials can be regarded as simple building bricks for complexes with $N > 1$. Below we call them *solibricks*.

(b) Another fundamental type of building brick of which NSE potentials for more complicated systems are constructed, are *breathers*, or string-like solutions, arising for $N > 1$, $\mathrm{Re}\,\kappa_i = \mathrm{Re}\,\kappa_j$, periodic or quasiperiodic in time [29].

(c) Now we proceed to a new type of building bricks for NSE potentials, called *bions*. This type of solution is defined by the off-diagonal matrices $[c_{ij}]$; for example, for $N = 2$ we have

$$[c_{ij}] = \begin{pmatrix} 0 & c \\ -\bar{c} & 0 \end{pmatrix} . \tag{2.2.15}$$

The wave function for this case is given in Refs [33, 36]:

$$\Psi = \left[1 + \frac{c_3\cos(q\xi + \Omega t + \theta_3) + c_4\mathrm{e}^{\beta\xi}}{c_2\cosh(\beta\xi + \theta_2) + c_1\cos(q\xi + \Omega t + \theta_1)}\right] \times \exp\left[\mathrm{i}k(\xi + k't)\right] \ , \tag{2.2.16}$$

where the coefficients are

$$c_1 = -\left|\frac{c}{\tilde{\kappa}_{12}}\right|, \quad e^{\theta_2} = \left|\frac{\tilde{\kappa}_{12}}{c\kappa_{12}}\right|\left(\frac{1}{\tilde{\kappa}_{11}\tilde{\kappa}_{22}}\right)^{1/2},$$

$$c_2 = \left|\frac{c\kappa_{12}}{\tilde{\kappa}_{12}}\right|\left(\frac{1}{\tilde{\kappa}_{11}\tilde{\kappa}_{22}}\right)^{1/2}, \quad e^{2i\theta_1} = \frac{c\tilde{\kappa}_{12}}{\bar{c}\bar{\tilde{\kappa}}_{12}},$$

$$c_3 = -|c|\left(-\frac{1}{(k-\kappa_1)(k-\kappa_2)}\right)^{1/2}, \quad e^{2i\theta_3} = -\frac{c(k-\kappa_2)}{\bar{c}(k-\kappa_1)},$$

$$c_4 = -\frac{1}{2}\left[\frac{\bar{\kappa}_{21}}{(k-\kappa_1)\tilde{\kappa}_{12}\tilde{\kappa}_{22}} + \frac{\bar{\kappa}_{12}}{(k-\kappa_2)\tilde{\kappa}_{21}\tilde{\kappa}_{11}}\right].$$

The solution (2.2.16) is defined by the following parameters:

$$\kappa_j = \alpha_j + i\beta_j, \quad q = \alpha_2 - \alpha_1, \quad \Omega = \omega - qv,$$
$$k' = k - v, \quad \beta = \beta_1 + \beta_2, \quad \xi = x + vt,$$
$$\omega = \alpha_2^2 - \alpha_1^2 + \beta_1^2 - \beta_2^2, \quad v = 2\frac{\alpha_1\beta_1 + \alpha_2\beta_2}{\beta_1 + \beta_2},$$
$$\tilde{\kappa}_{ij} = \bar{\kappa}_i - \kappa_j, \quad \kappa_{ij} = \kappa_i - \kappa_j .$$

Here α_i may be interpreted as velocities of the constituents (bions), and β_i as their 'masses'.

Note that the bionic solutions (2.2.16), as well as breathers, are formed from two 'solibricks' and both types of solution are periodic (or quasiperiodic) functions with respect to time. Nevertheless, the nature of breathers and bions is substantially different in accordance with their interpretation in physics. This will be clearly shown at the nonlinear level. Here we simply note that breathers can be easily broken down into their constituents, whereas for bions such a process is strongly forbidden. In this sense, the constituents of the bion behave like quarks in a meson. Therefore one can say that there are three types of building bricks: solibricks, breathers, and bions†, and the NSE potentials asymptotically assume the following symbolic form:

$$U(x,t)_{t\to\infty} = \sum \text{solibricks} + \sum \text{breathers} + \sum \text{bions} + \ldots .$$

One can use the scheme given above to construct soliton-like solutions for (2 + 1)-dimensional Kadomtsev–Petviashvili (KP) and Davey–Stewartson-I (DS-I) equations. As is known, attempts to find such solutions in Refs [31] and [32] have led to the discovery of *dromions*. The scheme given above has been applied to the DC-I equation in Ref. [33]. In this sense one can consider the above technique as a constructive way for producing solitonic solutions for (2+1)-dimensional KP and DS-I models. It can also be easily generalised to relativistic models of the Dirac type, in particular, to the Wax–Larkin and Thirring models [41]. Let us now consider a modification of the present scheme in order to study the derivative NSE (DNSE) (2.1.6).

(a) *Solutions of the derivative NSE*. Below we primarily follow the arguments in papers [33] and [39]. Let us consider equation (2.1.6) with a potential of arbitrary sign i.e.

$$[(-i\partial_t + \partial_x^2 \pm iU(x,t)\partial_x]\psi(x,t,k) = 0 . \tag{2.2.17}$$

†In doing so, we do not deny the possible existence of other building bricks.

Accordingly, we choose the plane-wave ansatz in a form slightly different from (2.2.2), namely:

$$\psi(x,t,k) = Q_N(x,t,k)\exp[ik(x+kt)] , \tag{2.2.18}$$

with

$$Q_N = a_N(x,t)k^N + a_{N-1}(x,t)k^{N-1} + \ldots + a_1(x,t)k + 1 . \tag{2.2.19}$$

If we specify again the location of N poles κ_i on the complex plane, as well as the complex $N \times N$ matrix b_{ij}, it is not difficult to check the validity of Theorem 2.1, with the only difference that now

$$U(x,t) = 2i\partial_x \ln a_N(x,t) . \tag{2.2.20}$$

We write down the pole-type representation for the wave function in the form

$$\hat{\Psi}(x,t,k) = \left(a_N + \sum_{j=1}^{N}\frac{\hat{r}_j(x,t)}{k-\kappa_j}\right)\exp[ik(x+kt)] \equiv \frac{\psi(x,t;k)}{\prod_{j=1}^{N}(k-\kappa_j)}, \tag{2.2.21}$$

so that we have $N+1$ unknown functions $a_0, a_1, \ldots, a_N$ (or a_N and N functions $\hat{r}_j$) and N additional equations (2.2.5) [or (2.2.7)].

T h e o r e m 2.3. *The potential* $U(x,t)$ *of the equation* (2.2.17) *is a real-valued nonsingular function of arbitrary real variables* x *and* t *under conditions* (2.2.26).

P r o o f. We define an additional condition for $k = 0$ as

$$\frac{Q_N(x,t;k=0)}{\prod_{j=1}^{N}\kappa_j} \equiv a_N - \sum_{j=1}^{N}\frac{\hat{r}_j}{\kappa_j} = 1 . \tag{2.2.22}$$

We now have a complete system of equations for $a_N, \hat{r}_j$:

$$a_N = 1 + \sum_{j=1}^{N}\frac{\hat{r}_j}{\kappa_j}, \tag{2.2.23}$$

$$\Psi(x,t,\bar{\kappa}_i) = -\sum_{j=1}^{N} c_{ij}\Psi_j(x,t) . \tag{2.2.24}$$

Consider a meromorphic function

$$\Omega = \overline{\Psi(\bar{x},t,\bar{k})}\Psi(x,t,k)/k . \tag{2.2.25}$$

Applying the *residue theorem*, we obtain

$$1 - |a_N|^2 - \sum_{i,j}\left(\frac{\bar{c}_{ij}}{\kappa_j} + \frac{c_{ij}}{\bar{\kappa}_j}\right) = 0 ,$$

whence subject to the condition

$$c_{ij} + \frac{\bar{\kappa}_j}{\kappa_j}\bar{c}_{ij} = 0 , \tag{2.2.26}$$

we have

$$|a_N|^2 = 1 , \tag{2.2.27}$$

to maintain the real-valuedness and nonsingularity of the potential $U(x,t)$ for arbitrary real x and t.

When $N = 1$, from formula (2.2.5) one finds

$$a = \left(-\frac{1}{\kappa}\right)\frac{1 + b\exp[i(\theta - \bar{\theta})]}{1 + b(\kappa/\bar{\kappa})\exp[i(\theta - \bar{\theta})]} . \tag{2.2.28}$$

The potential U will be a real function on condition that $|a| = \text{const}$ or, bearing in mind that $\kappa = \alpha + i\beta$, $b = b_1 + ib_2$, one can rewrite this condition in an equivalent form

$$b\kappa = \overline{b\kappa}, \quad \text{or} \quad b_1\beta + b_2\alpha = 0 . \tag{2.2.29}$$

By differentiating (2.2.28) and taking into account condition (2.2.29), we find

$$U = \frac{8\beta^2 \operatorname{sgn} b_2}{2\beta\cosh 2\eta - \alpha[2 - (\alpha/\beta)e^{-2\eta}]}, \quad |b_2| = e^{-2\beta x_0} , \tag{2.2.30}$$

or for $b_1, \alpha \neq 0$:

$$U = \frac{8|\alpha|\beta^2 \operatorname{sgn}\alpha}{4\alpha^2\cosh^2\eta + \beta^2 e^{-2\eta}}, \quad b_1 = e^{-2\beta x_0} > 0 , \tag{2.2.31a}$$

$$U = -\frac{8|\alpha|\beta^2 \operatorname{sgn}\beta}{4\alpha^2\sinh^2\eta + \beta^2 e^{-2\eta}}, \quad b_1 = -e^{-2\beta x_0} < 0 , \tag{2.2.31b}$$

whereas for $\alpha = b_1 = 0$

$$U = \frac{4\beta \operatorname{sgn} b_2}{\cosh 2\beta(x + x_0)}, \quad |b_2| = e^{-2\beta x_0} , \tag{2.2.31c}$$

where

$$\eta = \beta(x + 2\alpha t + x_0) . \tag{2.2.32}$$

In turn, the wave function

$$\Psi = \left[1 - \frac{k\{1 + b\exp[-2(\eta - \beta x_0)]\}}{\bar{\kappa}\{1 + \bar{b}\exp[-2(\eta - \beta x_0)]\}}\right]\exp[ik(x + kt)] \tag{2.2.33}$$

for the case $k = \bar{\kappa}$ takes the form

$$\Psi = 2i\frac{b_1^{1/2}\beta e^{i\theta}}{2\alpha\cosh\eta + i\beta e^{-\eta}}, \quad \theta = i\alpha x + i(\alpha^2 - \beta^2)t . \tag{2.2.34}$$

A similar expression (up to a constant factor) is obtained for the case $k = \kappa$. Note that expression (2.2.33), which is dependent on five real parameters $(\alpha, \beta, b_1, k_1, k_2)$, itself represents a general formula for the wave function of the derivative NSE for an arbitrary complex number k.

(b) *Solutions of the Ishimori*-II *equations*. The solution obtained for the DNSE allows us in particular to find solitons in the (2 + 1)-dimensional Ishimori-II model, which is described in [27], with the following equations

$$\partial_t \mathbf{S}(x,y,t) + \mathbf{S} \wedge (\partial_x^2 \mathbf{S} + \partial_y^2 \mathbf{S}) + \partial_x\phi\partial_y \mathbf{S} + \partial_y\phi\partial_x \mathbf{S} = 0, \tag{2.2.35a}$$

$$\partial_x^2\phi - \partial_y^2\phi + 2\mathbf{S}(\partial_x \mathbf{S} \wedge \partial_y^2 \mathbf{S}) = 0 , \tag{2.2.35b}$$

where $\mathbf{S} = (S_x, S_y, S_z)$ is a vector of unit length $\mathbf{S}^2 = 1$ and $\phi(x,t)$ is a real function. On passing to the cone variables $\xi = \frac{1}{2}(x + y)$, $\eta = \frac{1}{2}(x - y)$, solving the problem (2.2.35) may be reduced, following Ref. [34], to a solution of the linear system of equations

$$i\partial_t X(\xi,t) + \tfrac{1}{2}\partial_\xi^2 X + iU_2(\xi,t)\partial_\xi X = 0 , \tag{2.2.36a}$$

$$i\partial_t Y(\eta,t) + \tfrac{1}{2}\partial_\eta^2 Y - iU_1(\eta,t)\partial_\eta Y = 0 , \tag{2.2.36b}$$

with real-valued potentials $U_i = \bar{U}_i$. Solutions of the type (2.2.33), related to degenerate spectral data (factorised), in accord with the results of Ref. [34] are written as:

$$S_x + iS_y \equiv S_+ = \frac{2XY}{|1 - AB|^2}(1 + \bar{A}B) , \tag{2.2.37a}$$

$$S_- = \bar{S}_+ ,$$

$$S_3 = -\left[1 + 2\frac{(A + \bar{A})(B + \bar{B})}{|1 - AB|^2}\right] , \tag{2.2.37b}$$

$$\phi(\xi,\eta,t) = 2\left(i\ln\det\Delta + \partial_\xi^{-1}U_2(\xi,t) + \partial_\eta^{-1}U_1(\eta,t)\right] , \tag{2.2.37}$$

where

$$A = \int_{-\infty}^{\eta} dy\,\bar{Y}(y,t)\partial_y Y(y,t) , \tag{2.2.38a}$$

$$B = -\int_{-\infty}^{\xi} dx\,X(x,t)\partial_x \bar{X}(x,t) , \tag{2.2.38b}$$

$$\Delta = \frac{1 - \bar{A}B}{1 + AB} . \tag{2.2.38c}$$

Taking into account (2.2.36), we find

$$Y(y,t) = \bar{X}(y,-t) .$$

When $b_1 > 0$, $k = \kappa$, we obtain the solutions

$$X(x,t) = \frac{\exp\{i[\alpha_1 x + (\beta_1^2 - \alpha_1^2)]\}}{2\alpha_1\cosh z_1 + i\beta_1 e^{-z_1}} , \tag{2.2.39a}$$

$$z_1 = \beta_1(x - 2\alpha_1 t + x_0) ,$$

$$Y(y,t) = \bar{X}(y,-t) = \frac{\exp\{-i[\alpha_2 y - (\beta_2^2 - \alpha_2^2)]\}}{2\alpha_2\cosh z_2 - i\beta_2 e^{-z_2}} , \tag{2.2.39b}$$

$$z_2 = \beta_2(y + 2\alpha_2 t + y_0),$$

and, accordingly, the expressions

$$A = \frac{1}{2}\frac{1 - i(\alpha_2/\beta_2)(1 + e^{2z_2})}{4\alpha_2^2\cosh^2 z_2 + \beta_2^2 e^{-2z_2}} , \tag{2.2.40a}$$

$$B = -\frac{1}{2}\frac{1 - i(\alpha_1/\beta_1)(1 + e^{2z_1})}{4\alpha_1^2\cosh^2 z_1 + \beta_1^2 e^{-2z_1}} . \tag{2.2.40b}$$

Solutions (2.2.37)–(2.2.40) describe a soliton which moves with velocity $v = 2(\alpha_1 - \alpha_2)$. To obtain solutions corresponding to solitons at rest, it is sufficient to proceed to a moving frame. Note that obtaining two-soliton solutions on the basis of Eqns (2.2.26) and (2.2.27) presents no special problems. In particular, an expression for Ψ will be just a slightly modified expression (2.2.16).

2.2.2 Nonlinear level Let us now consider the results obtained in the light of paper [12], where the main problem of the suggested algebraic method for constructing solitonic solutions has been emphasised. This problem resides in pinpointing the connection between the potential $U(x,t)$ and the wave function $\psi(x,t,k)$ (or its residues $\Psi_i(x,t)$). Since in calculating the solution we deal with meromorphic functions, it seems natural to solve this problem by applying the residue theorem. This suggests the form of a rational function $E(k)$ allowing us to find the self-consistency conditions by calculating the residues of some auxiliary function

$$\Omega = E(k)\overline{\psi(x,t,\bar{k})}\psi(x,t,k) . \tag{2.2.41}$$

In particular, if we specify $E(k)$ as the polynomials

(i) $E_1 = k$,

(ii) $E_2 = k^2 + ak$,

(iii) $E_3 = k^3 + 2bk^2 + 2dk$, (2.2.42)

we arrive at the following relations between the potential $U(x,t)$ and the wave functions $\psi(x,t,k)$:

(i) $U = -2F(x,t)$,

(ii) $\partial_t U + a\partial_x U = 2\partial_x F(x,t)$,

(iii) $(\partial_t^2 - \frac{1}{3}\partial_x^4)U + 2\partial_x^2 U^2 + \frac{8}{3}(b\partial_{xt}^2 + d\partial_x^2)U = -\frac{8}{3}\partial_x^2 F(x,t)$, (2.2.43)

where $F(x,t)$ denotes the quadratic form

$$F(x,t) = \sum_{i,j=1}^{N} \bar{\Psi}_i E_{ij} \Psi_j \qquad (2.2.44)$$

with Hermitian matrix

$$E_{ij} = \left[E(\bar{\kappa}_j) - E(\kappa_j)\right] c_{ij} . \qquad (2.2.45)$$

The matrix (2.2.45) along with the set of poles κ_i completely defines the solutions of the nonlinear equation

$$\left[\mathrm{i}\partial_t + \partial_x^2 + U(x,t)\right]\Psi_i(x,t) = 0 \qquad (2.2.46)$$

with a corresponding self-consistency condition from (2.2.43). The role of nonlinear field variables in Eqn (2.2.46) is played by the residues Ψ_i of wave functions $\psi(x,t,k)$, that possess the 'correct' asymptotic behaviour at spatial infinity $x \to \pm\infty$ for certain sets of the SD (see Theorem 2.2). In these cases

$$\Psi_i(x \to \pm\infty) \to 0 \qquad (2.2.47)$$

and we have nonlinear problems with *trivial boundary conditions* (TBC). For other sets of the SD, the residues Ψ_i grow infinitely and are not usually of interest in problems of physics (at least for homogeneous systems).

In order to use this approach for problems with nontrivial or, as they are frequently called, *condensate boundary conditions* (CBC)

$$|\Phi_i(t, x \to \pm\infty)| \to \text{const} , \qquad (2.2.48)$$

instead of functions (2.2.42) one has to consider functions $E = \tilde{E}$ of the type

$$\tilde{E} = \sum_{i=1}^{n} \frac{\varepsilon_i b_i^2}{k - k_i} + E_j, \quad j = 1,2,3 , \qquad (2.2.49)$$

where $\varepsilon_i = \pm 1, b_i$ and k_i are arbitrary real constants. Calculating the residues of the function Ω, defined in (2.2.41), we again find conditions (2.2.43), where now, instead of $F(x,t)$, we have

$$\tilde{F}(x,t) = \sum_{i,j=1}^{N} \bar{\Psi}_i E_{ij} \Psi_j + \sum_{m=1}^{n} \varepsilon_m (|\Phi_m|^2 - b_m^2) , \qquad (2.2.50)$$

with nonlinear fields

$$\Phi_i(x,t) = b_i \psi(x,t,k = k_i) , \qquad (2.2.51)$$

themselves representing wave functions at fixed points $k = k_i$. Studies of asymptotic behaviour at $x \to \pm\infty$ show that $\psi(t, k_i x \to \pm\infty) = 1$, and as a result we have CBC (2.2.48) for the nonlinear fields $\Phi_i(x,t)$.

In general one can consider a $(n+m)$-component vector field

$$\varphi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \\ \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix} , \qquad (2.2.52)$$

satisfying the equation

$$\left[\mathrm{i}\partial_t + \partial_x^2 + U(x,t)\right]\varphi(x,t) = 0 \qquad (2.2.53)$$

with self-consistency conditions of the form (2.2.43) and (2.2.50). It is clear that in the case of pure condensate fields the quadratic form $F(x,t) = \sum_{i,j}^{N} \bar{\Psi}_i E_{ij} \Psi_j$ must be equal to zero for every nontrivial 'solibrick' Ψ_i. This puts some extra (nonlinear) restrictions on the choice of SD, namely, on the location of the poles. Let us illustrate this taking as an example the scalar NSE [40]

$$\left[(\mathrm{i}\partial_t + \partial_x^2 + \varepsilon(|\Phi|^2 - b^2)\right]\Phi(x,t) = 0 , \qquad (2.2.54)$$

for $N = 1$ and subject to condensate boundary conditions (2.2.48). From formula (2.2.45) we find in this case

$$\tilde{E}(\bar{\kappa}_1) - \tilde{E}(\kappa_1) = 0 , \qquad (2.2.55)$$

or in a more particular form

$$(\bar{\kappa}_1 - \kappa_1)\left(\varepsilon \frac{b^2}{|\kappa_1 - k_1|^2} - 1\right) = 0 . \qquad (2.2.56)$$

One can easily see from the above that the equation

$$\varepsilon \frac{b^2}{|\kappa_1 - k_1|^2} = 1 \qquad (2.2.57)$$

has a solution when $\varepsilon = 1$, i.e. when the NSE has a repulsive potential, while the allowed poles are located on the circle $|\kappa_1 - k_1|^2 = b^2$. As for the NSE with an attractive potential (when $\varepsilon = -1$), one-pole condensate solutions are absent.

As another example, we consider a two-pole solution of the bion type (2.2.16). In this case, instead of (2.2.56) we have

$$\tilde{E}(\bar{\kappa}_1) - \tilde{E}(\kappa_2) = 0 ,$$

or

$$(\bar{\kappa}_1 - \kappa_2)\left[\varepsilon \frac{b^2}{(\bar{\kappa}_1 - k_1)(\kappa_2 - k_1)} - 1\right] = 0 . \qquad (2.2.58)$$

The first solution $\bar{\kappa}_1 = \kappa_2$ coincides with the well-known Zakharov–Shabat breather (or string-type) solution [29], so we pass at once to the second solution

$$\kappa_2 = k_1 + \varepsilon \frac{b^2}{|\kappa_1 - k_1|^2}(\kappa_1 - k_1) . \qquad (2.2.59)$$

From formula (2.2.59) it follows that the condition $\operatorname{Im}\kappa_2 \times \operatorname{Im}\kappa_1 < 0$, which is required for the nonsingularity of the solution in accordance with Theorem 2.2, is valid for $\varepsilon < 0$, i.e. it is permissible only within the framework of the NSE with an attractive potential.

Summing up we come to the following conclusion:

P r o p o s i t i o n 2.1. In systems with condensate boundary conditions and with a plane-wave ansatz, one-pole solutions (kinks) for the repulsive type of interactions exist which are otherwise absent. Meanwhile, two-pole solutions (bions) appear within the framework of the NSE with an attractive potential as elementary nonlinear excitations, composed of *invisible* (quark like) constituents called 'solibricks'.

Let us point out, that the method considered is applicable as well to other versions of the nonlinear vector Schrödinger equation [12, 26], including those with off-diagonal potentials [37]. Consider as an example the following simple system

$$\mathrm{i}\partial_t\phi_i + \partial_x^2\phi_i + U(x,t)\phi_i = 0, \quad i = 1,2\,, \tag{2.2.60}$$

$$U = \bar{\phi}_1\phi_2 + \phi_1\bar{\phi}_2\,, \tag{2.2.61}$$

which has the solution [40]

$$\phi_i = \frac{A_i\mathrm{e}^{\mathrm{i}\vartheta_1}\cosh\Theta_1 + B_i\mathrm{e}^{\mathrm{i}\vartheta_2}\cosh\Theta_2}{C_1\cosh\eta^+ + C_2\cosh\eta^- + C_3\cos(\vartheta+\omega_0)}\,, \tag{2.2.62}$$

where the following notations is used

$$\Theta_i = \beta_i(x+v_it) + b_i \qquad \vartheta_i = q_ix + \omega_it$$

$$\eta^\pm = \beta^\pm(x + v^\pm t) + h_i, \qquad i = 1,2\,,$$

and TBC are imposed in the form

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{x\to\pm\infty} = \begin{pmatrix}0\\0\end{pmatrix}, \tag{2.2.63}$$

with the coefficients in (2.2.62) analogous to those given in (2.2.16). Equations of the type (2.2.60) with the potential (2.2.61)

$$U_\pm = |\phi_1|^2 \pm |\phi_2|^2$$

occur in nonlinear optics (laser beams in light-guides), as well as (for $U_-(x,t)$) in the phenomenological description of superconductivity when $T \neq 0$. In the latter case we have a system of two coupled components: a normal and a superfluid one, such that their density ratio is defined by the temperature [38]. The two-pole solutions found correspond to a new type of localised excitations in this system, known as [*Sov. Phys. Solid State* **30**(12) 2119 (1989)]double vortices (endowed with the topological charge $Q = 2$ or $Q = 0$).

2.3 Many-dimensional systems and how they relate to the nonstationary SchroSdinger equation

Another feature of the nonstationary SchroSdinger equation, is that it can be generalised to include the many-dimensional variants, which allow for localised solitonic solutions [42]. The point of the suggested constructive algorithm is to consider linear Schrödinger equation of the type (2.2.1) in $N+1$ dimensions:

$$\left[\mathrm{i}\partial_t + \triangle + U(\boldsymbol{x},t)\right]\psi(\boldsymbol{x},t) = 0; \quad \boldsymbol{x} = (x_1,\dots,x_N) \in R^N\,, \tag{2.3.1}$$

where $\triangle$ is the N-dimensional Laplace operator. Solutions are found in the standard form

$$\psi(\boldsymbol{x},t) = A(\boldsymbol{x},t)\exp\left[\mathrm{i}\varphi(\boldsymbol{x},t)\right]\,, \tag{2.3.2}$$

which leads to the system of equations

$$\partial_tA + 2\vec{\nabla}\varphi\cdot\vec{\nabla}A + A\triangle\varphi = 0, \tag{2.3.3a}$$

$$\triangle A + \left(U - \partial_t\varphi - \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi\right)A = 0 \tag{2.3.3b}$$

with the assumption that the potential $U(\boldsymbol{x},t)$ is a real-valued function. Noting that the potential $U(\boldsymbol{x},t)$ does not enter the equation (2.3.3a), one can easily finds that in the new variables

$$\Phi = \varphi, \qquad R = A/F(w) \tag{2.3.4}$$

the arbitrary positive definite function $F(w) > 0$ satisfies the equation

$$\partial_tR + 2\vec{\nabla}\varphi\cdot\vec{\nabla}R + R\triangle\varphi = 0\,, \tag{2.3.5a}$$

analogous to (2.3.3a), if the auxiliary function $w = w(\boldsymbol{x},t)$ is a solution of the first-order homogeneous equation

$$\partial_tw + 2\vec{\nabla}\varphi\cdot\vec{\nabla}w = 0\,. \tag{2.3.6}$$

For any given solution $w = w(\boldsymbol{x},t)$ of equation (2.3.6) the function $R(\boldsymbol{x},t)$ is obtained from (2.3.4) and the equation

$$\triangle R + \left(V - \partial_t\Phi - \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi\right)R = 0\,, \tag{2.3.5b}$$

where the new potential $V(\boldsymbol{x},t)$ is defined by the relation

$$V = U + \triangle\ln F(w) + \vec{\nabla}\ln F(w)\cdot\vec{\nabla}\ln\left[A^2/F(w)\right]\,. \tag{2.3.7}$$

Equation (2.3.5) implies that the new complex wave function

$$\Psi(\boldsymbol{x},t) = R(\boldsymbol{x},t)\exp\left[\mathrm{i}\Phi(\boldsymbol{x},t)\right]$$

satisfies the equation

$$\mathrm{i}\partial_t\Psi(\boldsymbol{x},t) + \triangle\Psi(\boldsymbol{x},t) + V(\boldsymbol{x},t)\Psi(\boldsymbol{x},t) = 0 \tag{2.3.8}$$

with potential (2.3.7).

In the next step of the algorithm one arrives at a solution of the equation (2.3.6) for the auxiliary function $w = w(\boldsymbol{x},t)$. To this end *separability conditions* are introduced for the solutions $\psi(\boldsymbol{x},t)$ of equation (2.3.1) in the form

$$U(\boldsymbol{x},t) = U_1(x_1,t) + \dots + U_N(x_N,t)\,, \tag{2.3.9a}$$

$$\psi(\boldsymbol{x},t) = \psi_1(x_1,t) + \dots + \psi_N(x_N,t)\,, \tag{2.3.9b}$$

whence from Eqn (2.3.2) the separation of the phase and amplitude of the wave function follows

$$\varphi(\boldsymbol{x},t) = \varphi_1(x_1,t) + \dots + \varphi_N(x_N,t)\,, \tag{2.3.10a}$$

$$A(\boldsymbol{x},t) = A_1(x_1,t)\dots A_N(x_N,t)\,, \tag{2.3.10b}$$

where the components are related by

$$\psi_n(x_n,t) = A_n(x_n,t)\exp\left[\mathrm{i}\varphi_n(x_n,t)\right], \qquad n = \overline{1,N}\,. \tag{2.3.11}$$

In the case considered, a separable solution of equation (2.3.6) in the form

$$w(\boldsymbol{x},t) = w_1(x_1,t)\dots w_N(x_N,t)\,, \tag{2.3.12}$$

exists, if each of its factors satisfies the $(1+1)$-dimensional equation

$$\partial_tw_n + 2\partial_{x_n}\varphi_n\partial_{x_n}w_n = 0, \qquad n = \overline{1,N}\,. \tag{2.3.13}$$

By using the expressions for the conservation laws

$$\partial_tA_n^2 + 2\partial_{x_n}(\partial_{x_n}\varphi_nA_n^2) = 0; \qquad n = \overline{1,N}\,, \tag{2.3.14}$$

one can easily write down an explicit form of solutions for equations (2.3.13)

$$w_n(x_n,t) = c_n + \int_{\bar{x}_n}^{x_n}\mathrm{d}\xi A_n^2(\xi,t) - 2\int_{t_0}^{t}\mathrm{d}\tau\,\partial_{x_n}\varphi_n^2(\bar{x}_n,\tau)A_n^2(\bar{x}_n,\tau)\,, \tag{2.3.15}$$

where c_n, $\bar{x}_n$, t_0 are real-valued arbitrary constants. A further observation is that $\partial_{x_n} w_n = A_n^2$; therefore equations (2.3.12), (2.3.10b) and (2.3.4) imply the equation

$$\partial_{x_1 \ldots x_n} w = F^2(w)|\Psi|^2 . \tag{2.3.16}$$

Next, following the line of arguments in paper [43], one can introduce the notion of *AC-integrability*†.

D e f i n i t i o n 2.1. A nonlinear evolution equation (or a dynamical system) is called *AC-integrable* if it possesses an infinite class of multisoliton-like solutions, which can be obtained by a change of variables from the solutions of an integrable equation. In particular this includes the well-known Davey–Stewartson equation (DS-I) [28], which describes quasimonochromatic wave packets on the surface of a liquid with small depth:

$$\mathrm{i}\partial_t \Psi(\boldsymbol{x},t) + \triangle\Psi(\boldsymbol{x},t) + V(\boldsymbol{x},t)\Psi(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in R^2 , \tag{2.3.17a}$$

$$\partial_{x_1 x_2} V(\boldsymbol{x},t) = 2\varepsilon \triangle |\Psi|^2 , \tag{2.3.17b}$$

which in the given scheme corresponds to the case $N = 2$. Indeed, a simple choice $F(w) = 1 + \varepsilon w$ in equation (2.3.16), and taking into account the separability conditions (2.3.12) and the form of solutions (2.3.15) of the corresponding equation from (2.3.16), leads to the expression for the potential

$$V = U_1(x_1,t) + U_2(x_2,t) + 2 \triangle \ln(1+\varepsilon w) , \tag{2.3.18}$$

which coincides with the result of integrating Eqn (2.3.17b). The known result, obtained in Refs [30] and [31], is that soliton solutions of the DS-I equation arise if one chooses for $U_1(x_1,t)$ and $U_2(x_2,t)$ in Eqn (2.3.18) appropriate potentials of the linear nonstationary Schrödinger equation

$$\mathrm{i}\partial_t\psi + \triangle\psi + \left[U_1(x_1,t) + U_2(x_2,t)\right]\psi = 0 ,$$
$$\psi = \psi(\boldsymbol{x},t),\ \boldsymbol{x} \in R^2 , \tag{2.3.19}$$

which appears to be the linear limit case of the DS-I equation for $\varepsilon \to 0$. From this it follows; in particular, that, contrary to the (1 + 1)-dimensional case, the solitons of the DS-I equation remain well localised objects even in the linear limit as $\varepsilon \to 0$. However, in this limit they behave like free particles, and the nonlinearity for $\varepsilon \neq 0$ leads to the establishment of a nontrivial interaction among them. A detailed description of the DS-I soliton interaction, based on numerical experiments can be found in Ref. [44].

In order to complete the presentation of the Degasperis scheme for the $(N + 1)$-dimensional case, let us write down the general expression for the new potential

$$V = U_1 + \ldots + U_N + \triangle \ln F(w) + \vec{\nabla} \ln F(w) \cdot \vec{\nabla} \ln\left[F(w)|\Psi|^2\right] , \tag{2.3.20}$$

which results in the system of equations (2.3.8) and (2.3.16) with respect to the functions $\Psi(\boldsymbol{x},t)$ and w being AC-integrable, since on changing the variables in the form

$$\Psi(\boldsymbol{x},t) = \frac{\psi(\boldsymbol{x},t)}{F(w)} \tag{2.3.21}$$

its solutions can be obtained from the solutions (2.3.9b) of the linear equation (2.3.1) with the potential (2.3.9a).

It is clear, that the scheme presented here substantially widens the possibilities for constructing integrable many-dimensional models, which could be of use for the description of localised objects existing in reality. Nevertheless, in practice, we more often come across models which arise from various concepts in physics and which, as a rule, belong to the class of nonintegrable models.

†Almost C-integrability.

PART II. STABLE STRUCTURES IN NONINTEGRABLE MODELS

The widespread use of nonlinear equations in contemporary physics has revealed an important characteristic property of nonlinear wave processes: when a nonlinear dynamical system is strongly excited, it produces *stable localised structures*, known as *solitons*. It is precisely these structures which survive as the evolution of the system progresses and which define the principal features of the dynamics of the system (see, for example, Refs [13, 14, 45]).

3. The Lyapunov direct method in the theory of soliton stability

One of the most important problems in the theory of solitons is the study of their stability. The customary approach to this problem is to consider small initial perturbations of solitons that permit us to linearize the equations of motion. However, this method does not always lead to the correct answer, as has been shown by A M Lyapunov [46], who developed a rigorous method of treating stability, the so-called *direct method*. The main point of this method consists in choosing some special functions whose properties allow us to draw conclusions on the character of the evolution of the system. An extension of this method to distributed systems (in particular, to the field models) is presesnted in Refs [47]–[49]. Several modifications of the Lyapunov method as applied to the theory of solitons are known: *the method of functional estimates of Zakharov and Kuznetsov* [50], *the Arnold energy method* [51, 52], *the method of Shatah and Strauss* [53], *the Benjamin method* [54], etc.

It is our aim to apply the Lyapunov method to the treatment of soliton stability in some physical field models. To this end, we begin with the analysis of the stability concept in the general theory of dynamical systems, and then shall concentrate on the special aspects of its application to soliton physics.

3.1 Definition of stability and the principal theorems of the direct method

Stability is one of the important concepts in practice which arises when studying real dynamical systems. As a qualitative notion, stability can be associated with the continuity of the motion with respect to perturbations of unknown origin. Depending on the type of these perturbations we can distinguish a few kinds of stability. We shall consider mainly the stability of many-dimensional solitons, that is of regular solutions to nonlinear field equations, localised in space with dimension $D \geqslant 2$. Let $\phi(t,\boldsymbol{x})$ be a many-component field function with values in R^n, which is considered as an element of some Banach space B and which satisfies an evolution equation

$$\partial_t \phi = \hat{F}(\phi) , \tag{3.1.1}$$

with $\hat{F}$ a nonlinear operator. Suppose that for an initial condition of the type $\phi_{t=0} = \phi_0(\boldsymbol{x})$ equation (3.1.1) admits the unique soliton-type solution

$$\phi(t,\boldsymbol{x}) = \hat{S}_t[\phi_0] , \tag{3.1.2}$$

where $\hat{S}_t$ denotes an evolution operator with semigroup properties, i.e.

$$\hat{S}_{t_1}\big[\hat{S}_{t_2}[\phi_0]\big] = \hat{S}_{t_1+t_2}[\phi_0], \quad t_i \geqslant 0 . \tag{3.1.3}$$

There is a very close relationship between the concept of stability for the given unperturbed motion $\phi \equiv u = u(t,\boldsymbol{x})$ and that of *the correctly posed Cauchy problem in the sense of Hadamard*. To define the latter notion, let us introduce two functional metrics which describe field perturbations

$$\xi(t,\boldsymbol{x}) = \phi(t,\boldsymbol{x}) - u(t,\boldsymbol{x}) . \tag{3.1.4}$$

Namely, let the metric $\rho_0(\xi_0)$ determine the distance in the space of initial perturbations ξ_0, while the metric $\rho(\xi)$ measures that in the space of current perturbations† ξ.Under the standard assumptions

$$\rho_0(\xi_0) \geqslant \rho(\xi) , \tag{3.1.5}$$

that is, the metric $\rho_0(\xi_0)$ is stronger than the metric $\rho(\xi)$.

Definition 3.1. The Cauchy problem for equation (3.1.1) is said to be *correctly posed in the sense of Hadamard*, if $\forall t \in [0,T]$, $T < \infty$, $\rho_0(\xi_0) \to 0$ implies $\rho(\xi) \to 0$.

To illustrate this definition, consider the well-known example of Hadamard:

Example 3.1. *The ill-posed Cauchy problem* for the equation

$$\partial_t^2\phi(t,x) + \partial_x^2\phi(t,x) = 0, \quad t \geqslant 0, \quad x \in [-\pi/2, \pi/2] . \tag{3.1.6}$$

Consider the following initial and boundary conditions:

$$\phi(t, x = \pm\pi/2) = \phi(t = 0, x) = 0 ,$$
$$\partial_t\phi\big|_{t=0} = \mathrm{e}^{-\sqrt{n}}\cos nx, \quad n = 2k+1 .$$

The corresponding solution to problem (3.1.6) reads

$$\phi(t,x) = \frac{1}{n}\mathrm{e}^{-\sqrt{n}}\cos nx \sinh nt . \tag{3.1.7}$$

If one chooses two coincident metrics $\rho = \rho_0 = \sup_x(|\xi| + |\partial_t\xi|)$ then putting $n \to \infty$ one finds that the metric for initial perturbations behaves as follows:

$$\rho_0(\phi_0) = \sup_x\left(\mathrm{e}^{-\sqrt{n}}|\cos nx|\right) = \mathrm{e}^{-\sqrt{n}} \to 0 ,$$

while for the current metric $\forall t > 0$, $n \to \infty$ (3.1.7) implies

$$\rho(\phi) = \sup_x\left[\frac{1}{n}\mathrm{e}^{-\sqrt{n}}|\cos nx|(\sinh nt + n\cosh nt)\right] \to \infty .$$

Definition 3.2. A soliton solution $u(t,\boldsymbol{x})$ is called *stable in the Lyapunov sense* with respect to the metrics ρ_0, ρ, if $\forall\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho_0(\xi_0) < \delta$ implies the inequality $\rho(\xi) < \varepsilon$, $\forall t > 0$.

Therefore, *the correctly posed Cauchy problem in the sense of Hadamard* is equivalent to stability over the finite *time interval T*.

Finally, the following typical problem is encountered when the perturbation appears on the right hand side of equation (3.1.1), i.e. when one assumes

$$\partial_t\phi - \hat{F}(\phi) = \hat{f}(\phi) . \tag{3.1.8}$$

If one introduces the special metric ρ_f to measure the perturbation $\hat{f}(\phi)$, i.e. $\rho_f = \rho_f\big[\hat{f}(\phi)\big]$, then the following definition seems to be reasonable.

Definition 3.3. The solution $u(t,\boldsymbol{x})$ to equation (3.1.8) is called stable with respect to the metrics ρ_0, ρ, ρ_f under the action of permanent perturbations $\hat{f}(\phi)$, if $\forall\varepsilon > 0$ there exist $\delta_1(\varepsilon) > 0, \delta_2(\varepsilon) > 0$ such that $\rho_0(\xi_0) < \delta_1$ and $\rho_f\big[\hat{f}(\phi)\big] < \delta_2$ imply the inequality $\rho(\xi) < \varepsilon$, $\forall t > 0$.

A coarser stability definition given by Lagrange should also be mentioned.

Definition 3.4. The solution $u(t,\boldsymbol{x})$ to equation (3.1.1) is *stable in the sense of Lagrange*, if there exists $\delta > 0$, such that $\rho(\xi) < \infty, \forall t > 0$ under the condition $\rho_0(\xi_0) < \delta$.

Hence the boundedness of the perturbations at any instant is sufficient for stability in the sense of Lagrange. Note that a finer concept of asymptotic stability is often used in practice.

Definition 3.5. The solution $u(t,\boldsymbol{x})$ is called *asymptotically stable in the sense of Lyapunov*, if it is stable according to Definition 3.2 and also if $\rho(\xi) \to 0$ as $t \to \infty$.

However, in soliton physics one frequently deals not with a single soliton solution $u(t,\boldsymbol{x})$ but with a set $U = \{u\}$ of such solutions, usually labelled by some group parameters α, i.e.

$$U = \left\{\hat{T}_g(\alpha)u \big| g \in G\right\} , \tag{3.1.9}$$

with G being the symmetry group of the dynamical system. In this case, the stability is called *orbital*, with the current metric being $\inf_{u\in U}\rho(\phi - u)$, that is the distance between the field ϕ and the set U which is the orbit of group G. One should also distinguish between *stable sets* and *attractors* (attracting sets), for which $\rho(\xi) \to 0$ as $t \to \infty$. It is obvious that an asymptotically stable set is both attracting and stable set. Note that a set might be attracting and unstable since $\rho(\xi)$ can increase in a finite time interval, though $\rho(\xi) \to 0$ as $t \to \infty$.

In view of the complexity of a rigorous treatment of the stability problem, in practice one is restricted to the linearised equations

$$\partial_t\xi = \hat{A}(\xi) \equiv \hat{F}'(u)\xi . \tag{3.1.10}$$

The stability for the linear problem (3.1.10) is called *linearised stability*, or stability to the first approximation (or to the first order), and that for the original equation (3.1.1) is called *nonlinear stability*. It is clear that nonlinear stability implies first order stability, though generally with respect to a weaker metric. The converse is valid if only $\mathrm{Re}\,\lambda < 0\ \forall\lambda \in \sigma(\hat{A})$, where $\sigma(\hat{A})$ stands for the spectrum of the operator $\hat{A}$ (which is called *dissipative* in this case). More precisely, the spectrum $\mathrm{Re}\,\lambda \leqslant 0$ corresponds to *spectral stability*, and $\mathrm{Re}\,\lambda = 0$ in its turn to *neutral stability*. (A typical example is provided by stable Hamiltonian systems.) Note that spectral stability follows from linearised stability owing to the fact that $\mathrm{Re}\,\lambda > 0$ implies the existence of increasing modes. The converse is not valid as one can see from the following problem in mechanics.

Example 3.2. Consider a mechanical system with the Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{4}q^4$$

and the relevant equation of motion

†As will be shown later (cf. Example 3.3), the introduction of two metrics seems to be necessary to pose correctly the stability problem for distributed systems.

$$\ddot{q} = -q^3 \, .$$

The linearised equation reads

$$\ddot{\xi} = 0$$

and corresponds to the spectrum $\lambda = 0$ which indicates neutral stability. However, the solution $\xi = at + b$ to the linearised equation is linearly increasing. Thus we infer linearised instability, though the original system is stable in the nonlinear sense. Note that the linearised system turns out to be stable with respect to the velocities only, that is in the weaker metric.

It is a well-known fact [55] that for a relatively wide class of dynamical systems spectral instability implies nonlinear instability. In fact, let us rewrite the equation (3.1.1) as follows

$$\partial_t \xi = \hat{A}\xi + \delta\hat{F}(\xi), \quad \delta\hat{F} \equiv \hat{F} - \hat{A} \, , \tag{3.1.11}$$

assuming that $\| \delta\hat{F}(\xi) \| \leqslant C \| \xi \|^2$, where $\| \cdot \|$ denotes the norm in the space B. Let the operator $\hat{A}$ have an eigenvector y with $\|y\| = 1$, corresponding to an eigenvalue λ having a maximal real part $\mathrm{Re}\,\lambda = 1$. Let also $\xi_0 = y\delta$, $\| \xi_0 \| = \delta$ be an initial perturbation. To prove the instability we argue by a reductio ad absurdum assuming that the motion in question is stable, that is $\| \xi(t) \| < \varepsilon \; \forall t > 0$. Rewriting equation (3.1.11) in integral form

$$\xi(t) = \mathrm{e}^{\hat{A}t}\xi_0 + \int_0^t \mathrm{e}^{\hat{A}(t-s)} \delta\hat{F}[\xi(s)] \, \mathrm{d}s \, , \tag{3.1.12}$$

we infer the validity of the following estimate for the perturbation norm

$$\begin{aligned} \| \xi(t) \| \; &\leqslant \; \| \mathrm{e}^{\hat{A}t}\xi_0 \| + \left\| \int_0^t \mathrm{e}^{\hat{A}(t-s)} \delta\hat{F}[\xi(s)] \, \mathrm{d}s \right\| \\ &\leqslant \delta \mathrm{e}^t + \int_0^t \left| \mathrm{e}^{\lambda(t-s)} \right| \, \| \delta\hat{F}[\xi(s)] \| \, \mathrm{d}s \\ &\leqslant \delta \mathrm{e}^t + C \int_0^t \mathrm{e}^{t-s} \, \| \xi(s) \|^2 \, \mathrm{d}s \, . \end{aligned}$$

It can be seen from the above expression, that there exists $T_1 > 0$, such that $\forall t \in [0, T_1]$ the inequality $\| \xi(t) \| \leqslant 2\delta \mathrm{e}^t$ holds . In fact, the latter can be deduced if one supposes, that

$$\delta \mathrm{e}^t + 4C\mathrm{e}^t \delta^2 \int_0^t e^s \mathrm{d}s \leqslant 2\delta \mathrm{e}^t \, ,$$

whence

$$\mathrm{e}^{T_1} - 1 = \frac{1}{4C\delta} \, . \tag{3.1.13}$$

However, equation (3.1.12) implies the validity of yet another inequality:

$$\begin{aligned} \| \xi(t) \| \; &\geqslant \; \| \mathrm{e}^{\hat{A}t}\xi_0 \| - \left\| \int_0^t \mathrm{e}^{\hat{A}(t-s)} \delta\hat{F}[\xi(s)] \, \mathrm{d}s \right\| \\ &\geqslant \delta \mathrm{e}^t \left[1 - 4C\delta(\mathrm{e}^t - 1) \right] \, , \end{aligned}$$

that permits us to choose T_2 from the equality

$$\mathrm{e}^{T_2} - 1 = \frac{1}{8C\delta} \, , \tag{3.1.14}$$

and to conclude, by comparing expression (3.1.13) with expression (3.1.14) that $T_2 < T_1$, and $\| \xi(t) \| \geqslant \mathrm{e}^t \delta/2 \; \forall t \leqslant T_2$, or equivalently

$$\| \xi(T_2) \| \; \geqslant \tfrac{1}{2} \mathrm{e}^{T_2} \delta = \tfrac{1}{2}\delta + \frac{1}{16C} \, .$$

Thus, choosing $\varepsilon \leqslant 1/16C, \| \xi_0 \| = \delta$, we infer that $\| \xi(T_2) \| > 1/16C \geqslant \varepsilon \; \forall \delta > 0$, which implies instability.

Let us formulate the main theorem of the direct method.

Theorem 3.1 (The Lyapunov–Movchan stability theorem). *A solution $u \in U$ is stable with respect to the metrics ρ_0, ρ, if and only if there exists, in some vicinity $\rho_0 < \alpha$, the Lyapunov functional $V[\phi]$ with the following properties*:

(i) *V is positive-definite with respect to $\rho(\xi)$*,

(ii) *V is continuous in ρ_0*,

(iii) *V is nonincreasing in time.*

The conditions of the theorem mean that there exist two continuous monotonic functions $m(\rho) > 0$ and $M(\rho_0) > 0$, $m(0) = M(0) = 0$, called the lower and upper *comparison functions* respectively, such that

$$m(\rho) \leqslant V[\phi] - V[u] \leqslant M(\rho_0) \, . \tag{3.1.15}$$

Let $\rho_0 < \delta$; then inequality (3.1.15) implies that $M(\delta) > M(\rho_0) \geqslant m(\rho)$, whence $\rho < \varepsilon$, i.e. the motion is stable.

The choice of the metrics ρ and ρ_0 is distorted by the structure of the Lyapunov functional. For example, let V be an additive functional of the form

$$V[\phi] = \int \mathrm{d}x \, F(\phi, \dot{\phi}, \nabla\phi) \, . \tag{3.1.16}$$

One can use the Taylor expansion with integral remainder:

$$f(x + \xi) = f(x) + f'(x)\xi + \int_0^1 \mathrm{d}s f''(x + s\xi)(1 - s) \, .$$

In the case considered $\delta V[u] = 0$ which implies

$$\begin{aligned} V[u + \xi] = V[u] + \int \mathrm{d}x \int_0^1 \mathrm{d}s(1 - s) \Big[& F_{\phi\phi}\xi^2 + F_{\dot{\phi}\dot{\phi}}\dot{\xi}^2 \\ & + F_{(\nabla\phi)^2}(\nabla\xi)^2 + 2F_{\phi\dot{\phi}}\xi\dot{\xi} \\ & + 2F_{\phi\nabla\phi}\xi\nabla\xi + 2F_{\dot{\phi}\nabla\phi}\dot{\xi}\nabla\xi \Big] \\ = V[u] + \int_0^1 \mathrm{d}s(1 - s)\delta^2 V[u + s\xi] \, . \end{aligned}$$

If $V[\phi]$ is a globally convex functional, then $\delta^2 V[u + s\xi] > 0$, which allows us to choose the current metrics as follows

$$\rho^2(\xi) = \int_0^1 \mathrm{d}s(1 - s)\delta^2 V[u + s\xi] \, .$$

This particular choice of metric forms the foundation of the method of V I Arnold [36, 40, 41], who assumed that $V[\phi] = H + C$, with H being the Hamiltonian, and C an integral of motion (the Casimir invariant) specified by the condition $\delta V[u] = 0$.

The notion of *formal* or *energetic* stability is also often used when the conservation law

$$E = \int \mathrm{d}x \, F(\phi, \dot{\phi}, \nabla\phi) = \mathrm{const} \, ,$$

or the evolution law $\dot{E} \leqslant 0$ holds such that $\delta E = 0$, $\delta^2 E > 0$ in the vicinity of the solution in question. It is clear that from energetic stability one can derive linearised stability, because the linear equations of motion imply the inequality $\delta^2 \dot{E} \leqslant 0$, and stability follows if one takes $\rho^2 = \rho_0^2 = \delta^2 E$. The converse does not hold, which can be ascertained by a simple counter-example from classical mechanics, with Hamiltonian of the form

$$H = \tfrac{1}{2}\left(p_1^2 + q_1^2 - p_2^2 - q_2^2 - q_1^2 q_2^2\right) \equiv E = \text{const} .$$

Since in linearisation we are dealing here with two independent harmonic oscillators, linearised stability is in this case obvious. On the other hand, the quadratic form

$$\delta^2 E = \delta p_1^2 + \delta q_1^2 - \delta p_2^2 - \delta q_2^2$$

is sign-indefinite, that is, the system is energetically unstable.

R e m a r k 3.1. It should be emphasised that in a finite-dimensional theory (mechanical systems with a finite number of degrees of freedom) both energetic stability and the analyticity of the Hamiltonian imply Lyapunov stability in the small, as can be deduced from the Lyapunov stability theorem. Under these conditions, the inequality $E > E_0$ (here E_0 is the unperturbed value of energy) is valid in some vicinity of the unperturbed solution which stems from from $\delta^2 E > 0$. However, in an infinite-dimensional theory (for distributed systems), this is not the case since then $\delta^2 E > 0$ does not necessarily imply that $E > E_0$ in some vicinity of the solution. A typical counter example is given by:

$$E = \int \mathrm{d}x \left[|\nabla\phi|^2 - |\nabla\phi|^4 + \phi^2 \right] .$$

Finally, the notion of *global stability* is used, when the system turns out to be stable for values of ρ however large. *This is the strongest possible stability one can define*.

To illustrate the peculiarities of stability analysis for distributed systems, let us consider the following simple example.

E x a m p l e 3.3. *The stability of a homogeneous unloaded string with fixed end points*. Let us solve the wave equation

$$\partial_t^2 \phi(t,x) - \partial_x^2 \phi(t,x) = 0 , \qquad (3.1.17)$$

with the boundary conditions

$$\phi(t,0) = \phi(t,1) = 0, \quad t \geqslant 0, \quad x \in [0,1] ,$$

and initial conditions

$$\partial_t \phi(0,x) = v(x), \quad \phi(0,x) = u(x) .$$

The solution to this familiar problem is given by the d'Alembert formula:

$$2\phi(x,t) = u(x-t) + u(x+t) + \int_{x-t}^{x+t} v(s)\,\mathrm{d}s , \qquad (3.1.18)$$

with functions u, v being skew-symmetrically extended over the whole real axis. To treat the stability of the steady state of the string $\phi = 0$, the following metrics may be introduced:

$$\rho_1 = \int_0^1 \phi^2 \mathrm{d}x, \qquad \rho_2 = \int_0^1 \left[\phi^2 + (\partial_t\phi)^2\right] \mathrm{d}x ,$$

$$\rho_3 = \int_0^1 \left[(\partial_x\phi)^2 + (\partial_t\phi)^2\right] \mathrm{d}x, \qquad \rho_4 = \rho_1 + \rho_3 .$$

As follows from the solution (3.1.18), the equilibrium $\phi = 0$ is stable with respect to the metrics ρ_2, ρ_3, ρ_4 separately (here ρ_3 is the integral of motion), and to the pairs of metrics (ρ_2, ρ_1), (ρ_3, ρ_2), (ρ_4, ρ_3). However, it is unstable with respect to the metric ρ_1 (the ill-posed problem in the Hadamard sense), because the fixing of ρ_1 imposes no constraints on the velocity $\partial_t \phi$.

In the light of the aforesaid, it is clear that stability analysis of many-dimensional distributed systems requires a very careful selection of metrics, given the fact that formulae of the kind of (3.1.18) do not work in this case and the smoothness of the initial data proves to be of critical importance. That is why from the outset a stronger metric should be taken.

Now let us formulate *the main instability criterion* which is given by the following theorem:

T h e o r e m 3.2. (The Chetaev–Movchan instability theorem). *A solution $u \in U$ is unstable with respect to the metrics ρ_0, ρ, if and only if there exists a Chetaev functional $W[\phi]$ with the following properties*: (i) $W[\phi]$ *is continuous with respect to* ρ_0; (ii) $W[\phi]$ *is bounded with respect to* ρ; (iii) $W[\phi]$ *increases in time in the domain* $W > 0$.

3.2 Energetic instability of many-dimensional stationary solitons

Once Lyapunov's functional $V[\phi]$ has been chosen, it is necessary to verify its global convexity, i.e. the validity of the inequality $\delta^2 V[u + \xi] \geqslant m(\rho)$. However, in practice only the local inequality $\delta^2 V[u] > 0$ is verifiable and even this not always. Thus, in all cases the structure of the second variation of the Lyapunov functional needs to be studied. To this end, some virial theorems, similar to those established by Hobart and Derrick [56, 57], [though limited to the case of static soliton configurations $u(\boldsymbol{x})$] might be useful (for the exposition and development of this approach see Refs [3], Ch. 3; and [14], Ch. 9). Let the functional $V[\phi]$ possess the critical point $\phi = u(\boldsymbol{x})$, that is $\delta V[u] = 0$. Consider the simplest perturbation of the soliton, generated by the scale transformation $\phi_\lambda = u(\lambda \boldsymbol{x})$. Then we get

$$\delta\phi = \xi = \delta\lambda \left(\frac{\partial u}{\partial \lambda}\right)\bigg|_{\lambda=1} .$$

Let us assume that the functional $V[\phi]$ can be represented as the sum

$$V[\phi] = \sum_{\nu=-n_1}^{n_2} V^{(\nu)}(\lambda) , \qquad (3.2.1)$$

where $V^{(\nu)}(\lambda)$ is a homogeneous function of the scale parameter λ of order ν. In view of Eqn (3.2.1) we find

$$\frac{\delta V[u]}{\delta\lambda} = \sum_{\nu=-n_1}^{n_2} \frac{\partial V^{(\nu)}}{\partial\lambda}\bigg|_{\lambda=1} = \sum_{\nu=-n_1}^{n_2} \nu V^{(\nu)}\bigg|_{\lambda=1} = 0 . \qquad (3.2.2)$$

The identity (3.2.2) is known as *the first virial theorem of Hobart and Derrick*. Calculating the second variation $\delta^2 V$, taking into account the identity (3.2.2), we obtain:

$$\frac{\delta^2 V[u]}{\delta\lambda^2} = \sum_{\nu=-n_1}^{n_2} \frac{\partial^2 V^{(\nu)}}{\partial\lambda^2}\bigg|_{\lambda=1} = \sum_{\nu=-n_1}^{n_2} \nu(\nu-1) V^{(\nu)}\bigg|_{\lambda=1}$$

$$= \sum_{\nu=-n_1}^{n_2} \nu^2 V^{(\nu)}\Big|_{\lambda=1} . \qquad (3.2.3)$$

The identity (3.2.3) is known as *the second virial theorem of Hobart and Derrick*.

E x a m p l e 3.4. Consider the following functional defined in a space of dimension D:

$$V[\phi] = \int \mathrm{d}^D x |\nabla\phi|^2 + \int \mathrm{d}^D x F(\phi) . \qquad (3.2.4)$$

Then, in accordance with (3.2.1), we have

$$V[\phi] = \int \mathrm{d}^D x |\nabla\phi|^2 + \int \mathrm{d}^D x F(\phi) . \qquad (3.2.4)$$

The virial identities (3.2.2) and (3.2.3) imply the relations:

$$\frac{\delta V}{\delta\lambda} = (2-D)V^{(2-D)} - DV^{(-D)} = 0 ,$$

$$\frac{\delta^2 V}{\delta\lambda^2} = (2-D)^2 V^{(2-D)} + D^2 V^{(-D)} = 2(2-D)V^{(2-D)} .$$

Thus we infer from the first equation that for $D \geqslant 3$, $V^{(-D)} < 0$; and from the second one, that $\delta^2 V < 0$ for scale deformations (dilatations). This means that for the models described by functionals of type (3.2.4) the static solitons are unstable for $D \geqslant 3$.

R e m a r k 3.2. As already noted by Hobart [56], this situation can be improved by inserting into the functional (3.2.4) terms containing higher degrees of field derivatives. This remedy was employed in the models of Skyrme and Faddeev (cf. Ref. [3], Ch. 3), which are usually associated with the notion of *topological stability*. Moreover, solitons with neutral scaling behaviour can be realised in 'exotic' models with functionals of the form

$$\Phi[\phi] = \int \mathrm{d}^3 x (\nabla\phi\cdot\nabla\phi)^{3/2} + \dots ,$$

where the power $3/2$ was chosen in order to satisfy the Hobart–Derrick criterion [58].

D e f i n i t i o n 3.6. The soliton solution $u(t,\boldsymbol{x})$ is called *stationary* if it satisfies the equations

$$\frac{\delta V}{\delta\dot\phi} = 0, \quad \frac{\delta V}{\delta\phi} = 0 , \qquad (3.2.5)$$

where $V[\phi]$ is an additive functional of the form (3.1.16), (3.2.4) [to meet more general requirements, we consider here equations of the second order in time derivatives, in contrast with (3.1.1), while first-order equations emerge if one chooses $F = F(\phi, \nabla\phi)$] in (3.1.16). In what follows all the soliton solutions $u(t,\boldsymbol{x})$ are supposed to decrease at spatial infinity according to the law

$$|\nabla u| = \mathrm{O}\left(r^{-[(D/2)+\alpha]}\right), \quad \alpha > 0 .$$

Then the following theorem is valid

T h e o r e m 3.3. *The second variation of an additive Lyapunov functional is sign-indefinite in the neighbourhood of stationary soliton solution for a spacedimension $D \geqslant 2$.*

P r o o f. For the sake of convenience we choose $D = 3$. Let us write down equation (3.2.5) for the field u with components u^s, $s = \overline{1,n}$,

$$F_{\dot s} = 0, \quad F_s - \partial_i F_s^i = 0, \quad i = 1,2,3 , \qquad (3.2.6)$$

where the derivatives are as follows

$$F_{\dot s} = \frac{\partial F}{\partial \dot u^s}, \quad F_s^i = \frac{\partial F}{\partial(\partial_i u^s)}, \quad F_s = \frac{\partial F}{\partial u^s} .$$

Let us now calculate the second variation of the functional V at the point u:

$$\delta^2 V = \int \mathrm{d}^3 x \Big(F_{\dot s\dot r}\dot\xi^s\dot\xi^r + 2F_{\dot s r}\dot\xi^s\xi^r + F_{sr}\xi^s\xi^r$$

$$+ 2F_{\dot s r}^{\;i}\dot\xi^s\partial_i\xi^r + F_{rs}^{ik}\partial_i\xi^r\partial_k\xi^s + 2F_{rs}^i\partial_i\xi^r\xi^s\Big) .$$

Now insert the special perturbations $\dot\xi^s = f^i(\boldsymbol{x})\partial_i\dot u^s$, $\xi^s = f^i(\boldsymbol{x})\partial_i u^s$ in $\delta^2 V$ transforming it, with account of Eqn (3.2.6), to the form

$$\delta^2 V[f] = \int \mathrm{d}^3 x \left[\partial_i f^l A_{lj}^{ik}\partial_k f^j + \left(\partial_i f^l\cdot f^j - \partial_i f^j\cdot f^l\right)B_{jl}^i\right] , \qquad (3.2.7)$$

where the following notation is used:

$$A_{lj}^{ik} = \partial_l u^r F_{rs}^{ik}\partial_j u^s, \quad 2B_{jl}^i = -2B_{lj}^i = \partial_{[j}F_r^i\partial_{l]}u^r . \qquad (3.2.8)$$

Note that the second term in Eqn (3.2.7) is obviously sign-indefinite, and, owing to Eqn (3.2.6), the following equality holds

$$\partial_i B_{jl}^i = 0 , \qquad (3.2.9)$$

whence we obtain the representation

$$2B_{jl}^i = \varepsilon^{ikm}\partial_k a_{mjl} , \qquad (3.2.10)$$

with the notation

$$a_{mjl} = \frac{1}{2\pi}\varepsilon_{mki}\partial^k \int \mathrm{d}^3 x' \frac{B_{jl}^i(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} . \qquad (3.2.11)$$

Taking into account Eqn (3.2.10) and integrating by parts in Eqn (3.2.7), we get

$$\delta^2 V[f] = \int \mathrm{d}^3 x \left[\partial_i f^l\left(A_{lj}^{ik} + \varepsilon^{ikm}a_{mjl}\right)\partial_k f^j\right] . \qquad (3.2.12)$$

Now verify the sign-definiteness of the integrand in (3.2.12). Consider, in particular, the asymptotic region $r\to\infty$, $r = |\boldsymbol{x}|$, where, in accordance with expressions (3.2.8),

$$B_{jl}^i = \mathrm{O}\left(r^{-(3+2\alpha)}\right) , \qquad (3.2.13)$$

showing that $a_{mjl} = \mathrm{O}(r^{-3})$. To prove this asymptotic behaviour we first deduce from expressions (3.2.9) and (3.2.13), through integration by parts, the identity

$$\int \mathrm{d}^3 x\, B_{jl}^i = 0 .$$

Taking the latter into account, one can rewrite Eqn (3.2.11) as follows

$$a_{mjl} = \frac{1}{2\pi}\varepsilon_{mki}\partial^k \int \mathrm{d}^3 x' B_{jl}^i(\boldsymbol{x}')\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} - \frac{1}{r}\right) . \qquad (3.2.14)$$

By using the mean value theorem one now easily finds from expressions (3.2.13) and (3.2.14) that

$$|a_{mjl}| < C_1\left|\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} - \frac{\boldsymbol{x}}{r^3}\right| = \mathrm{O}(r^{-3}) . \qquad (3.2.15)$$

Finally, comparing Eqn (3.2.15) with the estimate

$$A_{jl}^{ik} = \mathrm{O}\left[r^{-(3+2\alpha)}\right] ,$$

emerging from expressions(3.2.8), we infer that the quadratic form in (3.2.12) is sign-indefinite.

The proof given above can easily be extended to the case of solitons with the spatial dimension $D = N + 2 \geqslant 2$. The only modification concerns the expressions (3.2.10) and (3.2.11) which should be replaced by the following ones:

$$2B^{\,i}_{\,jl} = \varepsilon^{ik\alpha_1\ldots\alpha_N}\partial_k a_{\alpha_1\ldots\alpha_N jl} ,$$

$$a_{\alpha_1\ldots\alpha_N jl} = \frac{2}{N!}\varepsilon_{ik\alpha_1\ldots\alpha_N}\partial^k \int \mathrm{d}^{N+2}x'\, G(\boldsymbol{x}-\boldsymbol{x}')B^{\,i}_{\,jl}(\boldsymbol{x}') ,$$

where $G(\boldsymbol{x}-\boldsymbol{x}')$ stands for the Green function of the D-dimensional Laplace operator. Thus, the Hobart–Derrick theorem can be extended to the case $D \geqslant 2$, so that only one-dimensional solitons are excluded. The proof of Theorem 3.3 was first given in Ref. [59], and then cited in Refs [3, 60, 61].

P r o p o s i t i o n 3.1. Nontopological many-dimensional ($D \geqslant 2$) stationary solitons are energetically unstable. Therefore, within the scope of additive Lyapunov functionals of the form (3.1.16), only *conditionally-stable* many-dimensional stationary solitons may exist i.e. those stable under certain subsidiary conditions on the initial perturbations ξ_0.

As is known, all possible conditions on the perturbations can be included in the definition of the metric, though this procedure makes the stability analysis more complicated. To treat the conditional stability of a set U of stationary solutions it proves convenient to pick out a single solution u (or any narrow subset of these solutions, labelled with param-eters ω), and to consider all the rest as generated by transformations which form the invariance group G of the equation (3.1.1)†. Let G_0 be the invariance group of the functional V in expressions (3.1.16) and (3.2.4), with group parameters α_0, so that G_0 is a subgroup of a group G parametrised by $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}_0, \boldsymbol{\beta}\}$, where $\boldsymbol{\beta}$ stands for the complementary parameters. In general, the stationary solution might depend both on group parameters $\boldsymbol{\alpha}$, and on some nongroup parameters $\boldsymbol{\omega}$ (frequencies), that is $u = u(t, \boldsymbol{x}|\boldsymbol{\alpha}, \boldsymbol{\omega}) \in U$. If one takes $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ one now obtains the stationary solutions to Eqn (3.2.5), which form a subset $U_0 \subset U$. Let us denote the set of stationary solutions with fixed parameters $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, by $U_\omega \subset U_0$. The soliton configuration will be called perturbed, if $\varphi \notin U$.

The treatment of orbital stability, or the stability of any set of solutions, requires a particular choice of the current metric ρ. If one fixes the Banach norm $d = \|\varphi - u\|_{\mathrm{B}}$, then it is possible to define the following metrics:

$$\rho = \inf_{U_\omega} d, \qquad \rho_1 = \inf_{U_0} d, \qquad \rho_2 = \inf_{U} d \ , \tag{3.2.16}$$

$$\rho_3 = \inf_{\boldsymbol{\alpha}_0} d, \qquad \rho_4 = \inf_{\boldsymbol{\alpha}} d, \qquad \rho_5 = \inf_{\boldsymbol{\alpha},\boldsymbol{\omega}} d \ . \tag{3.2.17}$$

It is worth emphasising the difference between the metrics (3.2.16) and (3.2.17). In version (3.2.16) the minimisation procedure is carried out within the class of stationary solutions to the equation of motion, which is equivalent to fixing the parameters $\boldsymbol{\alpha}, \boldsymbol{\omega}$. In contrast, in version (3.2.17) the parameters prove to be functions of time; consequently the comparison function u is not forced to belong to the manifold of solutions of the equations of motion. This fact distinguishes, in any case from the physical point of view, the choice of (3.2.16) as the preferred metric. Then the minimisation in the metrics ρ_1 and ρ_2 is carried out at some fixed moment of time $t = T$, thus permitting us, as will be shown later, to get rid of undesirable zero perturbative modes ($\delta^2 V = 0$) as well as of negative ($\delta^2 V < 0$) ones .

By virtue of Theorem 3.3, for many-dimensional solitons, there are no additive Lyapunov functionals which are positive-definite with respect to the metric $\rho = \inf_{U_\omega} d$, and according to Proposition 3.1 only conditionally stable solitons can be realised. In other words, stable localised structures in many-dimensional dynamical systems can exist only if some subsidiary physical conditions are imposed on the initial perturbations ξ_0. In many cases the conditions mentioned here can be formulated as *fixing certain integrals of motion* (generalised charges) Q_α. The corresponding conditional stability is known as *Q-stability* [14, 45, 61]. Noting that $Q_\alpha = Q_\alpha(\boldsymbol{\omega})$, one concludes from definitions (3.2.16) that the charges Q_α can be fixed if one uses the metrics ρ_1 or ρ_2. The latter choice is equivalent, in turn, to studying nonadditive Lyapunov functionals, such as those quadratic in the charges.

Thus we have extended the domain of validity for the Hobart–Derrick energetic-instability theorem [56, 57] to the case of stationary solitons, the model-independent character of the results having also been established. However, for static solutions the result proves to be even stronger when energetic instability is replaced by a linearised one. In particular, the following theorem [61] is valid.

T h e o r e m 3.4. *In any local model possessing a translationally invariant Lagrangian, which is positive-definite with respect to velocities $\dot\varphi$, static, topologically trivial‡ many-dimensional solitons are linearly unstable.*

P r o o f. Suppose that the equation of motion for the real n-component field φ admits a static soliton solution $u(\boldsymbol{x})$. Define the perturbation $\xi = \varphi - u$ and metrics ρ_0, ρ in the form

$$\rho_0(\xi_0) = \|\dot\xi_0\| + \|\xi_0\|', \quad \rho(\xi) = \inf_{u\in U_0}\left(\|\dot\xi\| + \|\xi\|\right) ,$$

where $\|\cdot\|$ denotes the norm in the Hilbert space $L_2(R^D)$, and $\|\cdot\|'$ is that in the Sobolev space $W_2^1(R^D)$, i.e. $\|\xi\|' = \|\nabla\xi\| + \|\xi\|$. The metric ρ_0 can also correspond to a narrower space, defined by the continuity requirements of the functional under consideration with respect to ρ_0.

Let us write down the second variation of the energy functional, denoting the scalar product in $L_2(R^D)$ by (,):

$$\delta^2 E = (\dot\xi, \hat A\dot\xi) + (\xi, \hat B\xi) , \tag{3.2.18}$$

where $\hat A$ is a positive symmetric matrix and $\hat B$ a Hermitian operator, both local in u. Let us introduce the Chetaev functional

$$W = -\tfrac{1}{2}(\pi, \xi)(E - E_0) , \tag{3.2.19}$$

where $E_0 = E[u]$ denotes the unperturbed energy and $\pi = 2\hat A\dot\xi$. Taking into account that, by virtue of the

†To define such a group it is enough to specify relations of the type (3.1.9).

‡In what follows the solitons will be called *topologically trivial* if their *topological charges* are equal to zero (see Ref. [3], Ch. 2).

linear-ised equations of motion $\dot{\pi} = -\delta E/\delta\xi$, in the vicinity of the solution u one gets

$$\dot{W} = (\xi, \hat{B}\xi)^2 - (\dot{\xi}, \hat{A}\dot{\xi})^2 \ . \tag{3.2.20}$$

According to the statement of Theorem 3.3, one deduces from (3.2.18) that , given a number $\delta > 0$, there must be an initial perturbation ξ_0 and a number $\delta_1(\delta) > 0$, such that at the initial moment the following inequalities hold:

$$\rho_0(\xi_0) < \delta, \quad (\pi_0, \xi_0) > 0, \quad (\delta^2 E)_0 < -\delta_1 < 0 \ . \tag{3.2.21}$$

Now the positivity of the matrix $\hat{A}$ implies that in the domain $W > 0$, determined by the conditions (3.2.21), the following inequality holds

$$|(\xi, \hat{B}\xi)| \geqslant -(\xi, \hat{B}\xi) > (\dot{\xi}, \hat{A}\dot{\xi}) + \delta_1 \ .$$

From relations (3.2.20), (3.2.21) and the last inequality, it follows that $\dot{W} > \delta_1^2 > 0$ in the domain $W > 0$. Therefore, the conditions of the Chetaev–Movchan theorem will be satisfied if one proves the boundedness of the functional (3.2.19) with respect to the metric ρ in some neighbourhood $\rho < \varepsilon$. To this end notice that for the initial conditions (3.2.21) the following estimate is valid

$$\sup_{\rho<\varepsilon} W \leqslant |(\delta^2 E)_0| \cdot \|\hat{A}\| \sup_{\rho<\varepsilon} \Big(\|\dot{\xi}\| \cdot \|\xi\| \Big) \ . \tag{3.2.22}$$

Furthermore, as $\dot{\xi} = \dot{\varphi}$ is independent of u, for $\rho < \varepsilon$ the following estimate applies:

$$\|\dot{\xi}\| = \inf_{u\in U_0} \|\dot{\xi}\| \leqslant \rho < \varepsilon \ . \tag{3.2.23}$$

Moreover, the triangle inequality gives us

$$\|\varphi\| - \|u\| \leqslant \|\varphi - u\| \leqslant \|\varphi\| + \|u\| \ ,$$

whence

$$\sup_{u\in U_0} \|\varphi - u\| - \inf_{u\in U_0} \|\varphi - u\| \leqslant 2\|u\| \ ,$$

and therefore

$$\|\xi\| \leqslant \sup_{u\in U_0} \|\varphi - u\| \leqslant 2\|u\| + \inf_{u\in U_0} \|\xi\| \leqslant 2\|u\| + \varepsilon \ . \tag{3.2.24}$$

Inserting the estimates (3.2.23) and (3.2.24) into (3.2.22), we get

$$\sup_{\rho<\varepsilon} |W| < |(\delta^2 E)_0| \cdot \|\hat{A}\| \varepsilon (2\|u\| + \varepsilon) \equiv \overline{W} \ .$$

Making the natural assumption of the boundedness of the norm $\|u\|$ together with that of $\|\hat{A}\|$, we infer that W is bounded, which proves the theorem.

Using the inequalities obtained while proving the theorem, one can easily estimate the time τ for the perturbation to reach the sphere $\rho = \varepsilon$:

$$\tau < \frac{1}{\delta_1^2} (\overline{W} - W) \ .$$

3.3 Stability of scalar charged solitons (Q-theorem)

We begin with the case that is simple to analyse, when the soliton is described by a complex scalar field ψ, defined in four-dimensional Minkowski space–time†. Let the nonperturbed soliton be

$$\psi_0 = u(\boldsymbol{x}) \exp(-\mathrm{i}\omega t), \quad u^* = u \ , \tag{3.3.1}$$

where the function $u(\boldsymbol{x})$ is assumed to decrease sufficiently fast at $r = |\boldsymbol{x}| \to \infty$. Consider the class of models given by the $U(1)$- and Lorentz-invariant Lagrangian density

$$L = -F(p, q, s) \ .$$

Here the following invariants are introduced

$$p = -\partial_\mu \psi^* \partial^\mu \psi, \quad q = j_\mu j^\mu, \quad s = \psi^* \psi \ ,$$

where j_μ stands for the ordinary 4-current expression $j_\mu = \frac{1}{2}\mathrm{i}\,[\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*]$. Now construct the invariant set U_0 of nonperturbed soliton solutions, that is the variety of orbits for the group $G = T(3) \otimes_s SO(3) \otimes U(1)$, where $\otimes_s$ is a semidirect product. In other words,

$$U_0 = \{u(\hat{O}\boldsymbol{x} + \boldsymbol{a}; \omega)\mathrm{e}^{\mathrm{i}\beta}\} \ , \tag{3.3.2}$$

where $\hat{O}$ denotes the matrix of 3-rotations, $\boldsymbol{a} \in R^3$, $\beta \in [0, 2\pi)$. It should be emphasised that the frequency ω in set (3.3.2) is not fixed. The perturbed soliton can be described by the field

$$\psi = \varphi(t, \boldsymbol{x}) \exp(-\mathrm{i}\omega t), \quad \varphi \notin U_0 \ ,$$

with the perturbation ξ being defined as follows

$$\xi = \varphi - u = \xi_1 + \mathrm{i}\xi_2, \quad \xi_i^* = \xi_i \ .$$

We now choose the metrics ρ_0, ρ as

$$\rho_0(\xi_0) = \sum_{i=1}^{2} \Big(\|\dot{\xi}_{0i}\| + \|\xi_{0i}\|' \Big)_C \ ,$$

$$\rho(\xi) = \inf_{u\in U_0} \sum_{i=1}^{2} \Big(\|\dot{\xi}_i\| + \|\xi_i\| \Big) \ , \tag{3.3.3}$$

where the subscript C denotes the common norm in $L_2 \cap C$.

As it turns out, for the soliton solutions (3.3.1), Theorem 3.3 also proves to be valid in the one-dimensional ($D = 1$) case [63]. To support this argument, first recall a useful Lemma in the calculus of variations. Let the functional

$$V[\phi] = \int_{R^D} \mathrm{d}x\, S(\partial_i \phi, \phi)$$

be determined in the class of piecewise smooth functions ϕ: $R^D \to R^n, \phi(\infty) = 0$, and admit the field of extremals $u(\boldsymbol{a}; \boldsymbol{x})$ given by the set of continuous parameters α_i, $i = \overline{1, l}$. Then the following Lemma is valid.

Lemma 3.1. *There exist constants c_i, not all equal to zero, and such that the function*

$$f = \sum_{i=1}^{l} c_i \frac{\partial u}{\partial \alpha_i}\Big|_{\boldsymbol{a}\ =\boldsymbol{0}} \tag{3.3.4}$$

vanishes on a surface Σ, which encloses in R^D a domain Ω with nonzero measure, and where $\nabla f|_\Sigma \neq 0$, then $\delta^2 V$ is sign-indefinite in the neighbourhood of the extremal $u(0; \boldsymbol{x})$.

Proof. Consider the particular perturbation

$$\xi(\boldsymbol{x} \in \Omega) = 0, \qquad \xi(\boldsymbol{x} \notin \Omega) = f \ . \tag{3.3.5}$$

Then, owing to the properties of extremal fields, $\delta^2 V[\xi] = 0$. However, the perturbation (3.3.5) is not an extremal of the functional $\delta^2 V$, as it violates the matching conditions on the boundary Σ. Hence, one can find perturbations, close to (3.3.5), such that $\delta^2 V$ takes values of either sign.

†One can find an alternative account of the Q-stability theory in monographs [3], §3.3.3 and [14], Ch. 10, 14. Our version is close to that in the review [61], §3.

Theorem 3.5. *The stationary soliton solutions* (3.3.1) *cannot be energetically stable.*

Proof [63]. Let the Lyapunov functional V be of the form (3.1.16). Then it admits the field of extremals

$$\psi_0 = u(\boldsymbol{x} + \boldsymbol{a}) \exp(-\mathrm{i}\omega t) .$$

Taking into account that $|u| < \infty$, $u(\infty) = 0$, we infer that the equation $f = \partial_1 u(\boldsymbol{x}) = 0$ is satisfied on some surface Σ. Hence, the conditions of Lemma 3.1 are fulfilled, and $\delta^2 V$ is sign-indefinite for any dimension D.

In view of the aforesaid statements, let us study the Q-stability of the soliton solutions (3.3.1), under *the assumption of fixed charge*, which was already presupposed in the definition (3.3.3) of the metric ρ:

$$Q = \int \mathrm{d}^3 x\, (F_p - F_q s) j_0 = Q[\psi_0] \equiv Q_0 \, . \qquad (3.3.6)$$

Let us write condition (3.3.6) in a linear approximation:

$$(hu, \dot{\xi}_2) = (g, \xi_1) \, , \qquad (3.3.7)$$

with the following notation

$$h = -2\omega^2 s (F_{pp} - 2F_{pq} s + F_{qq} s^2) + F_p - F_q s \, ,$$

$$g = \mathrm{div}(u\boldsymbol{a}) + uc,$$

$$\boldsymbol{a} = \omega (F_{pq} s - F_{pp}) \nabla s \, ,$$

$$c = 2\omega [F_p + s(F_{ps} - 2F_q - F_{qs} s) - \omega^2 s (F_{pp} - 3F_{pq} s + 2F_{qq} s^2)] \, .$$

It turns out that under condition (3.3.7) the non-nodal solitons ($u > 0$) can become stable (in fact Q-stable). As for the nodal solitons, the following theorem is valid.

Theorem 3.6. *Nodal solitons of the form* (3.3.1) *are energetically Q-unstable.*

Proof. In line with Kumar et al. [64], assume that the Lyapunov functional V admits a field of extremals of the form

$$\psi_0(\beta) = u \exp[\mathrm{i}(\beta - \omega t)] \, .$$

Since for the nodal solitons $u = 0$ on a certain closed nodal surface Σ, the function (3.3.4) takes the form

$$f = \psi'_0(\beta)|_{\beta=0} = \mathrm{i}\psi_0(0)$$

and satisfies the condition of Lemma 3.1. Therefore $\delta^2 V$ is sign-indefinite in a neighbourhood of perturbations

$$\dot{\xi}_i = \xi_1 = 0, \quad \xi_2 = u \, ,$$

that is for $\|\xi_1\| \ll \|\xi_2\|$. As can be easily verified, condition (3.3.7) can also be satisfied in this class of perturbations.

In view of the aforementioned results, all that is left is to obtain the conditions of Q-stability for the non-nodal solitons. To this end choose the following integral of motion as the Lyapunov functional

$$V = E - \omega Q \equiv \tilde{E}[\varphi] \, , \qquad (3.3.8)$$

where E is the field energy. The second variation of V has the form

$$\delta^2 V = \left(\dot{\xi}_1, F_p \dot{\xi}_1\right) + \left(\dot{\xi}_2, h\dot{\xi}_2\right) + \sum_{i=1}^{2} \left(\xi_i, \hat{L}_i \xi_i\right) , \qquad (3.3.9)$$

where the following self-conjugated operators are introduced:

$$\begin{aligned} \hat{L}_1 &= 2\omega^4 s (F_{pp} + 4F_{qs} s^2 - 4F_{pq} s) + F_s + 2F_{ss} s \\ &\quad + \omega^2 (-F_p + 6F_q s - 4F_{ps} s + 8F_{qs} s^2) \\ &\quad + \mathrm{div}\{-F_p \nabla - 2F_{pp} \nabla u \cdot (\nabla u \nabla) \\ &\quad + [\omega^2 (F_{pp} - 2F_{pq} s) - F_{ps}] \nabla s\} \, , \\ \hat{L}_2 &= F_s - \omega^2 F_p + F_q (\omega^2 s - p) \\ &\quad - \mathrm{div}[(F_p - F_q s) \nabla + F_q u \nabla u] \, . \end{aligned}$$

It follows from (3.3.9), that for $\delta^2 V$ to be positive-definite the inequalities $F_p > 0$, $h > 0$ must hold. Here and later on the following theorem of R Courant [65] will be of help.

Theorem 3.7. *The first eigenfunction of a self-conjugate elliptic differential operator of the second order has no zeroes and the corresponding eigenvalue is non-degenerate.*

Notice that the spectrum of the operator $\hat{L}_2$ is non-negative, because, owing to the field equation $\hat{L}_2 u = 0$, where $u > 0$, and therefore, according to the Courant theorem, the function u is the first eigenfunction of the operator $\hat{L}_2$. At the same time the zero mode $\xi_2 = u$ is excluded because of the choice of the metric ρ.

Using the Schwartz inequality and the condition (3.3.7), we get the constraint

$$\left(\dot{\xi}_2, h\dot{\xi}_2\right) \geqslant \left(g, \xi_1\right)^2 (u, hu)^{-1} \, ,$$

which permits us to obtain the estimate

$$\delta^2 V \geqslant \left(\dot{\xi}_1, F_p \dot{\xi}_1\right) + \left(\dot{\xi}_2, h\dot{\xi}_2\right) + \left(\xi_1, \hat{K}\xi_1\right) ,$$

where the self-conjugate operator $\hat{K}$ is given by:

$$\hat{K}\xi_1 = \hat{L}_1 \xi_1 + g(g, \xi_1)(u, hu)^{-1} \, . \qquad (3.3.10)$$

We shall now specify the conditions for the spectrum of the operator $\hat{K}$ to be positive or, in other words, for the quadratic form

$$(\xi, \hat{K}\xi) = (\xi, \hat{L}_1 \xi) + a^{-1} (g, \xi) \, , \qquad (3.3.11)$$

to be positive-definite. Here $a = (u, hu)$.

Consider first the case when $a > 0$. The case $a = 0$, relevant to nonrelativistic systems, will be studied separately. We now calculate the action of the operator (3.3.10) on the function

$$v = \frac{\partial u}{\partial \omega} \, . \qquad (3.3.12)$$

After cumbersome calculations, taking into account the field equations for u, differentiated with respect to ω, one finds

$$\hat{K} v = \frac{g}{a} \frac{\partial Q}{\partial \omega} \, . \qquad (3.3.13)$$

Via the inner product of Eqn (3.3.13) with v we obtain

$$(v, \hat{K} v) = (b - a) \frac{b}{a} \, , \qquad (3.3.14)$$

with $b = \partial Q / \partial \omega$.

Clearly, if $\lambda(\omega)$ is the minimal eigenvalue of the operator $\hat{K}$, the boundary of the stability domain is given by the equation

$$\lambda(\omega) = 0 \, . \qquad (3.3.15)$$

However, as one can see from Eqn (3.3.13), if the condition

$$b(\omega) = \frac{\partial Q}{\partial \omega} = 0 , \tag{3.3.16}$$

is fulfilled, then the operator $\hat{K}$ has a zero eigenvalue, with the eigenfunction given by the expression (3.3.12). Let us verify the equivalence of conditions (3.3.15) and (3.3.16). It should be noted that if the operator $\hat{L}_1$ has two or more negative eigenvalues, then the equality

$$(g, \xi) = 0 , \tag{3.3.17}$$

can always be satisfied by taking as a perturbation the linear superposition of the corresponding eigenfunctions. This results in the sign-indefinitness of the quadratic form (3.3.11). Therefore to ensure stability under the singlet subsidiary condition (3.3.6) it is necessary to impose the requirement that in the stability domain (i.e. for the corresponding values of the parameter ω), the operator $\hat{L}_1$ has a single negative eigenvalue. Let us denote the corresponding eigenfunction by ψ_-. If now

$$(g, \psi_-) \neq 0 , \tag{3.3.18}$$

the operator $\hat{K}$ will have a positive spectrum in some ω-domain. As follows from expressions (3.3.14) and (3.3.16), this domain is determined by the condition

$$\frac{\partial Q}{\partial \omega} < 0 , \tag{3.3.19}$$

because its boundary is determined by the equation (3.3.15) and at interior points $(v, \hat{K}v) > 0$ must be true. Thus, we arrive at the following sufficient criterion for the conditional stability of non-nodal stationary solitons, which is known as the *Q-theorem* [3, 14, 45, 63, 64, 66–72].

T h e o r e m 3.8. (The Q-theorem). *Non-nodal stationary soliton solutions* (3.3.1) *are Q-stable in the Lyapunov sense in the domain* (3.3.19), *if for all ω taking values within this domain the operator $\hat{L}_1$ has a single negative eigenvalue and the corresponding eigenfunction satisfies the condition* (3.3.18).

R e m a r k 3.3. The operator $\hat{K}$ has zero eigenvalue corresponding to translations of the solution (3.3.1) by the parameters of the invariance group G of the model (*the zero-modes*). Under such a perturbation the solitons do not leave the invariant set U_0 and therefore the corresponding perturbations must be excluded by the choice of the current metric ρ in the form (3.3.3).

R e m a r k 3.4. It is not difficult to demonstrate the limits of the applicability of the Q-theorem, if one takes into account the particular dependence of the Lagrangian density on the parameter ω, as this dependence was crucial to the proof. For instance, this ω-dependence changes its form when one incorporates the gauge vector fields via the covariant derivative. Furthermore the Q-theorem cannot be applied if the solutions (3.3.1) exist only for some discrete values of ω, since in this case the differentiation with respect to ω cannot be performed.

Now consider the very important particular case $a = 0$ corresponding to nonrelativistic models. It turns out that the Q-theorem proves to be valid in this case and the stability domain is still determined by the inequality (3.3.19). In fact, condition (3.3.7) reduces to (3.3.17) as $a \to 0$, and thus the problem comes down to finding the spectrum of the operator $\hat{L}_1$ under condition (3.3.17). Introducing the Lagrange multiplier χ, we get the following representation for the minimal eigenvalue λ of the operator $\hat{L}_1$:

$$\lambda = \min_{\|\psi\|=1} \left[(\psi, \hat{L}_1 \psi) + \chi(g, \psi) \right] . \tag{3.3.20}$$

Using the condition $(g, \psi) = 0$ and the eigenvalue equation

$$2\hat{L}_1 \psi + \chi g = 2\lambda \psi$$

we obtain the Lagrange multiplier

$$\chi = -2 \frac{(g, \hat{L}_1 \psi)}{\|g\|^2} .$$

Inserting it into expression (3.3.20), we obtain

$$\lambda = \min_{\|\psi\|=1} \left[(\psi, \hat{L}_1 \psi) - 2(g, \psi) \frac{(g, \hat{L}_1 \psi)}{\|g\|^2} \right] \equiv \min_{\|\psi\|=1} P[\psi] . \tag{3.3.21}$$

Thus, we have reduced the problem to the unconditional minimisation of the functional $P[\psi]/\|\psi\|^2$. Let us prove that the new and the old problem are equivalent. Minimising the functional (3.3.21), we obtain the equation

$$\hat{L}_1 \psi - \left[g(g, \hat{L}_1 \psi) + \hat{L}_1 g(g, \psi) \right] \|g\|^2 = \lambda \psi . \tag{3.3.22}$$

Note that the possible solution $\psi = g$ to Eqn (3.3.22) has to be excluded as it violates the condition (3.3.17). Therefore, putting $\psi = g$ into Eqn (3.3.22), we obtain

$$g \left[\frac{(g, \hat{L}_1 g)}{\|g\|^2} + \lambda \right] \neq 0 . \tag{3.3.23}$$

Finally, the scalar multiplication of both sides of Eqn (3.3.22) by g leads to

$$\left[\frac{(g, \hat{L}_1 g)}{\|g\|^2} + \lambda \right] (g, \psi) = 0 ,$$

whence in view of the inequality (3.3.23), we deduce that $(g, \psi) = 0$. This proves the equivalence of problems (3.3.20) and (3.3.21).

Now for the particular substitution $\psi = v$ in accordance with the definition (3.3.12) we find $\hat{L}_1 v = -g$, thus deriving from Eqn (3.3.21) the equality

$$P[v] = -(g, v) = -\frac{\partial Q}{\partial \omega} .$$

The minimisation in expression (3.3.21) then leads to the equation

$$-\hat{L}_1 g \frac{(g, v)}{\|g\|^2} = \lambda v ,$$

from which it follows that the eigenvalue $\lambda = 0$ still corresponds to the surface (3.3.16) in the ω-space, and the stability domain, where $P[\psi] > 0$, is determined by the inequality (3.3.19) in the following explicit form:

$$\frac{\partial Q}{\partial \omega} = \frac{\partial}{\partial \omega} \left[\omega \int \mathrm{d}^3 x (F_p - F_q s) s \right] < 0 . \tag{3.3.24}$$

As will be shown later, the conditions of Theorem 3.8 appear to be necessary for the Q-stability of non-nodal solitons [62, 73].

T h e o r e m 3.9. *The conditions of Theorem 3.8 are necessary and sufficient for the Q-stability of scalar non-nodal solitons.*

P r o o f. Let us show that a violation of the conditions of the Q-theorem implies the linearised instability of non-nodal solitons with respect to the metrics (3.3.3). To this end, consider the following Chetaev functional:

$$W = (\tilde{E}_0 - \tilde{E})\Big[(\xi_1, F_p\dot{\xi}_1) - (\xi_2, h\dot{\xi}_2) + (\xi_1, c\xi_2) - (\xi_2, \boldsymbol{a}\cdot\nabla\xi_1)\Big] \,. \tag{3.3.25}$$

Using the ellipticity of the operators $\hat{L}_i$, which is necessary for the positive definiteness of $\delta^2\tilde{E}$, one can establish that the functional (3.3.25) is bounded with respect to the metric ρ in the domain $W > 0$, where $\delta^2\tilde{E} < -\delta_1 < 0$. In fact, in such a domain $(\xi_1, \hat{L}_1\xi_1) < -\delta_1$, whence, in view of the general structure of the operator

$$\hat{L}_1 = -\partial_i[A_{ik}\partial_k] + B \,,$$

we obtain

$$(\partial_i\xi_1, A_{ik}\partial_k\xi_1) < -\delta_1 - (\xi_1, B\xi_1) < M\|\xi_1\|^2 \,,$$

where $\max(-B) = M$. The latter inequality, together with the ellipticity condition

$$(\partial_i\xi_1, A_{ik}\partial_k\xi_1) \geqslant m\|\nabla\xi_1\|^2, \quad m > 0 \,,$$

and with the inequality (3.2.24) yields finally the desired estimate. Therefore, to verify the applicability of the Chetaev–Movchan instability theorem (Theorem 3.2), it remains only to calculate the time derivative $\dot{W}$. From the linearised equations of motion

$$F_p\ddot{\xi}_1 + c\dot{\xi}_2 + \mathrm{div}(\boldsymbol{a}\dot{\xi}_2) + \hat{L}_1\xi_1 = 0 \,,$$
$$h\ddot{\xi}_2 - c\dot{\xi}_1 + (\boldsymbol{a}\cdot\nabla)\dot{\xi}_1 + \hat{L}_2\xi_2 = 0 \tag{3.3.26}$$

we get

$$W = (\tilde{E}_0 - \tilde{E})\Big[(\xi_1, F_p\dot{\xi}_1) - (\xi_2, h\dot{\xi}_2) + (\xi_1, c\xi_2) - (\xi_2, \boldsymbol{a}\cdot\nabla\xi_1)\Big] \,.$$

However, in the domain $\delta^2\tilde{E} < -\delta_1$, in accordance with Eqn (3.3.9) the following inequality obtains:

$$-(\dot{\xi}_2, h\dot{\xi}_2) - (\xi_1, \hat{L}_1\xi_1) > \delta_1 + (\dot{\xi}_1, F_p\dot{\xi}_1) + (\xi_2, \hat{L}_2\xi_2) \,.$$

Therefore

$$\dot{W} > \delta_1\big[\delta_1 + 2(\dot{\xi}_1, F_p\dot{\xi}_1) + 2(\xi_2, \hat{L}_2\xi_2)\big] \geqslant \delta_1^2 > 0 \,.$$

Now to complete the presentation we come back to studying the nodal solitons for which the following theorem is valid [61]:

T h e o r e m 3.10. *The nodal stationary solitons are linearly unstable.*

P r o o f. For the nodal solitons, it follows, on the basis of Lemma 3.1 and the Courant theorem (Theorem 3.7), from the zero mode equations

$$\hat{L}_2 u = 0, \quad \hat{L}_1\nabla u = 0$$

that the operators $\hat{L}_1$ and $\hat{L}_2$ have negative eigenvalues. Solving the equation (3.3.26) with respect to ξ_2, regarding ξ_1 as a given source, we get $\xi_2 = \eta + A(\xi_1)$, where η satisfies the homogeneous equation

$$h\ddot{\eta} + \hat{L}_2\eta = 0 \,, \tag{3.3.27}$$

admitting the sign-indefinite 'energy' integral

$$E = (\dot{\eta}, h\dot{\eta}) + (\eta, \hat{L}_2\eta) \,.$$

Constructing the Chetaev functional

$$W = -E(\eta, h\dot{\eta}) \,,$$

we infer that its time derivative $\dot{W}$ is positive in the domain $E < -\delta_1 < 0$:

$$\dot{W} = E\big[(\eta, \hat{L}_2\eta) - (\dot{\eta}, h\dot{\eta})\big] > \delta_1^2 \,.$$

Notice that Theorem 3.10 deals in fact with the stronger spectral instability.

It would also be of interest to consider the limiting (non-relativistic) case, when in the equations of motion (3.3.26) $F_p = h = \vec{a} = 0, \quad c = 1$. To prove instability in this case it is convenient to use the spectral decomposition

$$\hat{L}_2 = \hat{L}_2^{(+)} + \hat{L}_2^{(-)}, \quad \xi_1 = \xi_1^{(+)} + \xi_1^{(-)} \,,$$

with respect to the sign $(\geqslant, <)$ of the spectrum of the operator $\hat{L}_2$.

The equation thus arising is

$$\ddot{\xi}_1^{(+)} = -\hat{L}_2^{(+)}\hat{L}_1\left(\xi_1^{(+)} + \xi_1^{(-)}\right)$$

and admits a solution of the form

$$\xi_1^{(+)} = \left[\hat{L}_2^{(+)}\right]^{1/2}\xi + \hat{S}\left(\xi_1^{(-)}\right) \,,$$

where the function ξ satisfies an equation similar to (3.3.27):

$$\ddot{\xi} = -\left[\hat{L}_2^{(+)}\right]^{1/2}\hat{L}_1\left[\hat{L}_2^{(+)}\right]^{1/2}\xi \,,$$

for which the instability has been already established.

For the application of the Q-theorem it is necessary to make sure that the operator $\hat{L}_1$ possesses only a single negative eigenvalue (negative mode).

L e m m a 3.2. *For spherically symmetric solitons the nega-tive mode is unique if and only if the function $u(r)$, $r = |\boldsymbol{x}|$, is a monotonically decreasing one, and the solution $w(r)$ of the equation $\hat{L}_1(w/r) = 0$, with the boundary conditions $w(0) = 0$, $w'(0) = 1$ has a single internal zero (for $r > 0$).*

P r o o f [62]. For spherically symmetric solitons the operator $\hat{L}_1$ commutes with the generators $\hat{\boldsymbol{J}}$ of the rotation group and therefore can be expressed in terms of the Casimir operator $\hat{\boldsymbol{J}}^2$ with eigenvalues $l(l+1)$. Furthermore, the ellipticity of the operator $\hat{L}_1$ implies that its spectrum $\lambda(l)$ increases with l. At the same time from $\hat{L}_1\partial_i u = 0$ it follows that the zero mode $\partial_i u = u'(r)x_i/r$ corresponds to the angular momentum $l = 1$, or $\lambda(1) = 0$. Now, since $u'(r) < 0$, in accordance with the *Courant theorem* (Theorem 3.7), $\lambda = 0$ is the minimal eigenvalue for $l = 1$. Thus, the states with $\lambda < 0$ cannot be other than spherically-symmetrical, and their number will be equal, in accordance with the *Sturm comparison theorem* [65], to that of the internal zeros of the solution $w(r)$ to the equation $\hat{L}_1(w/r) = 0$ with the boundary conditions $w(0) = 0$, $w'(0) = 1$.

Consider some simple physical models for which the conditions of Lemma 3.2 are satisfied, that is, the negative mode is unique and the Q-theorem is applicable.

E x a m p l e 3.5. *The power model.* In this case the Lagrangian function can be written as $F = p + s - s^n/n$ and admits a soliton-like solution, determined as the field function $u(\boldsymbol{x})$ satisfying the equation

$$\big[\triangle - 1 + \omega^2 + u^{2(n-1)}\big]u = 0 \,. \tag{3.3.28}$$

Equation (3.3.28) admits, in particular, the non-nodal solution $u(r)$ if the conditions

$$|\omega| < 1, \quad 0 < 1 - 1/n \leqslant 2/D \qquad (3.3.29)$$

are fulfilled. After a change of variables in Eqn (3.3.28)

$$r = s(1-\omega^2)^{-1/2}, \quad u = v(1-\omega^2)^{1/[2(n-1)]},$$

the charge $Q(\omega)$ of the nonperturbed soliton can be found as

$$Q(\omega) = \omega\|u\|^2 = C\omega(1-\omega^2)^{[1/(n-1)-D/2]}. \qquad (3.3.30)$$

As follows from (3.3.30), condition (3.3.24) is satisfied for frequencies in the interval

$$1 > |\omega| > \left(\frac{n+1}{n-1} - D\right)^{-1/2}. \qquad (3.3.31)$$

Condition (3.3.18) is also fulfilled since $g = 2\omega u \neq 0$, and the eigenfunction $\psi_- \neq 0$ (recall that ψ_- is the first eigenfunction of the operator $\hat{L}_1$). Thus we infer that the inequality (3.3.31) determines the domain of Q-stability for the non-nodal solitons in the model.

Example 3.6. *The logarithmic model* [74]. This model is determined by the Lagrangian function $F = p + s(1 - \ln s)$, admitting soliton solutions (3.3.1) of the form

$$u(r) = \exp\left[\tfrac{1}{2}(D - \omega^2 - r^2)\right].$$

The expression for the charge

$$Q(\omega) = C\omega\exp(-\omega^2)$$

determines the domain of Q-stability for the non-nodal solitons of the model [64], [75] through the inequality

$$|\omega| > 1/\sqrt{2}. \qquad (3.3.32)$$

Example 3.7. *The nonlinear Schrödinger equation*. We shall consider it here in its most general form (see Part I)

$$\mathrm{i}\partial_t\psi = -\left[\triangle + |\psi|^{2(n-1)}\right]\psi, \quad n > 1. \qquad (3.3.33)$$

It admits the soliton solutions (3.3.1) with the amplitude u, subject (in turn) to an equivalent equation one can derive from expression (3.3.28) by substituting $\omega^2 - 1 \to \omega > 0$. Performing the change of variables

$$r = s|\omega|^{-1/2}; \quad u = v|\omega|^{1/2(n-1)}$$

we can reduce the aforementioned equation to the form (3.3.28) with $\omega = 0$, which gives an explicit expression for the charge

$$Q(\omega) = \|u\|^2 = C|\omega|^{\{[1/(n-1)]-(D/2)\}}. \qquad (3.3.34)$$

As follows from expression (3.3.34), the *stability domain* is determined by the conditions

$$1 < n < 1 + \frac{2}{D}, \qquad (3.3.35)$$

and the *instability domain* can be characterised by the inequalities

$$1 + \frac{2}{D} < n < \frac{D}{(D-2)}, \qquad (3.3.36)$$

stemming from conditions (3.3.29) and having a meaning when $D \geqslant 2$. The instability of the non-nodal solitons in the domain (3.3.36) can be established by means of the Chetaev functional

$$W = (\tilde{E}_0 - \tilde{E})(\xi_1, \xi_2),$$

which is the limiting case of the functional (3.3.25) (the non-relativistic limit). Instability of the nodal solitons with respect to the metrics

$$\rho_0 = \sum_{i=1}^{2}\|\xi_{i0}\|'_C, \quad \rho = \inf_{u\in U_0}\sum_{i=1}^{2}\|\xi_i\|$$

is a consequence of Theorem 3.8.

3.4. Stability of multiply charged solitons

Let us consider a natural generalisation of the Q-theorem for many-component fields $\psi^s(\boldsymbol{x}, t)$, $s = \overline{1, n}$, with the Lagrangian density

$$L = -F\left(\psi^s, \dot{\psi}^s, \partial_i\psi^s\right), \qquad (3.4.1)$$

admitting the internal symmetry group G of rank l. In other words, in the group G one has l diagonal generators $\hat{\Gamma}_\alpha$, $\alpha = \overline{1, l}$, corresponding to the conserved charges Q_α, and the stationary (in this case, *multiply charged*) solitons are described by the functions

$$\psi_s^{(0)}(\boldsymbol{x}, t) = [\exp(\hat{\omega}t)u(\boldsymbol{x})]_s, \quad \hat{\omega} = \omega_\alpha\hat{\Gamma}_\alpha. \qquad (3.4.2)$$

This allows us to use in the expression for density (3.4.1) more appropriate field variables $\varphi(\boldsymbol{x}, t)$, by means of the ansatz

$$\psi = \exp(\hat{\omega}t)\varphi,$$

and the Lagrangian density in terms of the new variables reads

$$L = -F(\varphi^s, \dot{\varphi}^s + (\hat{\omega}\varphi)^s, \partial_i\varphi^s).$$

For the Lyapunov functional we choose the integral

$$V = E - \omega_\alpha Q_\alpha,$$

where

$$E = \int \mathrm{d}x\{F - F_s^{\cdot}[\dot{\varphi}^s + (\hat{\omega}\varphi)^s]\}, \quad Q_\alpha = -\int \mathrm{d}x\, F_s^{\cdot}\left(\hat{\Gamma}_\alpha\varphi\right)^s.$$

The second variation $\delta^2 V$ and conditions $\delta Q_\alpha = 0$ are written in the form:

$$\delta^2 V = -\left(F_{sr}^{\cdot\cdot}\dot{\xi}^s, \dot{\xi}^r\right) + \left(\xi^r, \hat{L}_{rs}\xi^s\right), \qquad (3.4.3)$$

$$\delta Q_\alpha = -\left(\dot{\xi}^r, F_{sr}^{\cdot\cdot}(\hat{\Gamma}_\alpha u)^s\right) - (g_\alpha^r, \xi^r) = 0. \qquad (3.4.4)$$

Since the quadratic form with respect to the velocities in equation (3.4.3) should be positive definite, we introduce the positive-definite matrix

$$A_{\alpha\beta} = -\left(F_{sr}^{\cdot\cdot}(\hat{\Gamma}_\alpha u)^s, (\hat{\Gamma}_\beta u)^r\right)$$

and, taking into account equation (3.4.4) with the help of the Lagrange multiplier method, we eliminate the velocities $\dot{\xi}^s$ from Eqn (3.4.3):

$$\delta^2 V \geqslant (A^{-1})_{\alpha\beta}(g_\alpha^r, \xi^r)(g_\beta^s, \xi^s) + \left(\xi^r, \hat{L}_{rs}\xi^s\right) \equiv \left(\xi, \hat{K}\xi\right). \qquad (3.4.5)$$

From expression (3.4.5) it is obvious that the stability condition reduces to the requirement that the spectrum of the operator $\hat{K}$ should be positive, i.e. $\lambda_0 = \lambda_{\min} > 0$. In order to define the boundary of the stability region, let us differentiate the equation in u^s with respect to ω_α, leading to the relation

$$\hat{L}_{rs}u_\alpha^s = -g_\alpha^r,$$

where

$$u^s_\alpha \equiv \frac{\partial u^s}{\partial \omega_\alpha} . \qquad (3.4.6)$$

In view of the identity (3.4.6), the condition $\lambda_0 = 0$, or equivalently $\hat{K}\xi = 0$, takes the form

$$\hat{L}\xi + (A^{-1})_{\alpha\beta}(\hat{L}u_\alpha)(\xi, \hat{L}u_\beta) = 0 . \qquad (3.4.7)$$

In turn, from expression (3.4.7) it follows that $\xi = \xi_0 + a_\alpha u_\alpha$, where $\hat{L}\xi_0 = 0$. On the other hand, by virtue of the identity (3.4.6) we have

$$(g^r_\alpha, u^s_\alpha) = A_{\alpha\beta} + F_{\alpha\beta}, \quad F_{\alpha\beta} \equiv -\frac{\partial Q_\alpha}{\partial \omega_\beta} , \qquad (3.4.8)$$

and therefore equation (3.4.7) is equivalent to the algebraic system

$$(A^{-1})_{\alpha\beta} a_\gamma F_{\gamma\beta} = 0 ,$$

possessing the nontrivial solution $a_\gamma \neq 0$ only under the condition

$$\det[\![F_{\alpha\beta}]\!] = 0 . \qquad (3.4.9)$$

Hence, the condition $\lambda = 0$ holds along with the condition (3.4.9). At the same time, from the identity (3.4.6) we deduce that

$$K_{\alpha\beta} \equiv (u_\alpha, \hat{K}u_\beta) = F_{\alpha\beta} + (A^{-1})_{\mu\nu} F_{\alpha\mu} F_{\beta\nu} ,$$

and therefore the range of stability is located inside the region where the matrix $F_{\alpha\beta}$ is positive definite, or in brief $F > 0$.

We represent the Hilbert space $\mathcal{H}$ of the functions ξ as: $\mathcal{H} = N \oplus \mathrm{Ker}\hat{L} \oplus P$, where N and P correspond to the negative and positive eigenvalues of the operator $\hat{L}$, respectively. Eqn (3.4.5) leads to a further necessary condition for $\lambda_0 > 0$:

$$\mathrm{Lin}\{g^r_\alpha\} \equiv \mathcal{H}_g \supset N , \qquad (3.4.10)$$

and, in particular, $\dim N \equiv \nu \leqslant l$.

Let us demonstrate that the conditions $F > 0$ and (3.4.10) are *necessary and sufficient* for $\lambda_0 > 0$ to hold. Indeed, let us assume that $u_\alpha = x_\alpha + y_\alpha + z_\alpha$, $\xi = a + b + c$, where $x_\alpha, a \in N$; $y_\alpha, b \in \mathrm{Ker}\hat{L}$; $z_\alpha, c \in P$. On minimising Eqn (3.4.5) with respect to c, we find

$$\delta^2 V \geqslant (a, \hat{L}x_\alpha)(A + B)^{-1}_{\alpha\beta}(a, \hat{L}x_\beta) + (a, \hat{L}a) \equiv (a, \hat{M}a) , \qquad (3.4.11)$$

where $B_{\alpha\beta} = (z_\alpha, \hat{L}z_\beta)$. Furthermore, by virtue of condition (3.4.10) the following decomposition $a = s^\alpha x_\alpha$ holds, and therefore from expression (3.4.11) we derive the estimate

$$\delta^2 V \geqslant s^\alpha M_{\alpha\beta} s^\beta \equiv s^T \hat{M} s ,$$

where $M_{\alpha\beta} = (x_\alpha, \hat{M}x_\beta)$. Observing that

$$B_{\alpha\beta} = (u_\alpha, \hat{L}u_\beta) - (x_\alpha, \hat{L}x_\beta)$$

is a non-negative matrix, we find in accordance with expression (3.4.8),

$$A_{\alpha\beta} + B_{\alpha\beta} = -F_{\alpha\beta} - (x_\alpha, \hat{L}x_\beta) = -F_{\alpha\beta} + C_{\alpha\beta} , \qquad (3.4.12)$$

where $C_{\alpha\beta} = -(x_\alpha, \hat{L}x_\beta)$ is a positive matrix. Thus, from Eqns (3.4.11) and (3.4.12) we deduce that

$$M = -C - C(F - C)^{-1}C = F(I - C^{-1}F)^{-1} ,$$

or $M > 0$, if $F > 0$.

Consider now an important special case, when only one negative mode occurs, with $\nu = \dim N = 1$. Then, instead of expression (3.4.5), it is more appropriate to use the estimate

$$\delta^2 V \geqslant a^{-1}(g, \xi)^2 + (\xi, \hat{L}\xi) \equiv (\xi, \hat{K}'\xi) , \qquad (3.4.13)$$

where $g = \omega_\alpha g_\alpha$, $a = A_{\alpha\beta}\omega_\alpha\omega_\beta$. As a result, instead of relation (3.4.6), we arrive at the relation

$$\hat{L}(u_\alpha, \omega_\alpha) = -g ,$$

and the condition for the spectrum of operator $\hat{K}'$ to be positive-definite can be expressed by the inequality

$$F_{\alpha\beta}\omega_\alpha\omega_\beta = -\omega_\alpha\omega_\beta \frac{\partial Q_\beta}{\partial \omega_\alpha} > 0 . \qquad (3.4.14)$$

Thus, the following generalisation of the Q-theorem for multiply charged solitons holds [3, 59, 61, 70, 73, 76, 77]:

Theorem 3.11. *The stationary solitonic solutions* (3.4.2) *are Q-stable in the region* $F > 0$ [*see Eqns* (3.4.8) *and* (3.4.14)], *if the condition* (3.4.10) *is fulfilled, i.e. if the space spanned by the vectors* g_α *contains negative modes.*

3.5 The method of functional estimates for studying stability problems

The idea of conditional stability for many-dimensional solitons has been developed by V E Zakharov and E A Kuznetsov, who demonstrated that in various situations one can with relative ease be assured of the existence of a lower bound for the energy functional of the system subject to the condition that some additional integrals of motion are fixed [78], [79]. It is true, that in so doing some delicate questions on the attainability of the lower bound, on the convergence of the minimising sequence, and on the regularity and smoothness of the minimal energy configuration are left open. Nevertheless such an approach fulfills the requirements of the so-called 'physical level of rigour' and is found to work efficiently in various applications. Since there are excellent descriptions of this method in the literature (see, for example, Refs [50], [80]), we shall confine ourselves to just one demonstration of the method for some physical models.

Example 3.8. *The nonlinear Schrödinger equation.* Using the definitions of the dynamical variables of this model, already introduced in Example 3.7, we shall demonstrate that the energy E in R^3 is estimated from below by the charge Q as defined in (3.3.34). In fact,

$$E[\psi] = \int d^3x \left(|\nabla\psi|^2 - \frac{1}{n}|\psi|^{2n} \right) \equiv \|\nabla\psi\|^2 - \frac{1}{n}\|\psi^n\|^2 .$$

Introducing the notation $I_{2k} = \|\psi^k\|^2$, $k = 1, 2, \ldots$, and making use of the well-known inequalities:

$$\|\nabla\psi\|^2 \geqslant \alpha I_6^{1/3}, \quad \alpha = 3(\pi/2)^{4/3} ,$$
$$I_{2n} \leqslant I_2^{(3-n)/2} I_6^{(n-1)/2} ,$$

we arrive at the estimate

$$E[\psi] \geqslant \alpha I_6^{1/3} - \frac{1}{n} I_2^{(3-n)/2} I_6^{(n-1)/2} . \qquad (3.5.1)$$

If $5 > 3n$, then the right-hand side of expression (3.5.1) attains its minimum for

$$I_6 = \left[\frac{3(n-1)}{2\alpha n} \right]^{6/(5-3n)} I_2^{3(3-n)/(5-3n)} .$$

Therefore the energy functional $E[\psi]$ for the fixed value of charge (or that of a number of particles) $I_2 = Q$ attains its minimum as well, which is realised in some stable configuration.

Example 3.9. *The Korteveg–de Vries equation* in R^1 reads

$$\partial_t \varphi + \partial_x^3 \varphi + 6\varphi \partial_x \varphi = 0, \tag{3.5.2}$$

and describes waves in shallow water. As is known, this model assumes the energy conservation law

$$E[\varphi] = \int \mathrm{d}x \left[\frac{1}{2}(\partial_x \varphi)^2 - \varphi^3\right] \equiv \frac{1}{2}\|\partial_x \varphi\|^2 - I_3$$

and also the momentum conservation law

$$P = \int \mathrm{d}x\, \varphi^2 = I_2 \ .$$

Making use of the Gagliardo–Nirenberg–Ladyzhenskaya inequality

$$I_3 \leqslant C I_2^{5/4} \|\partial_x \varphi\|^{1/2}, \quad C = \mathrm{const}\ ,$$

we obtain an estimate of the energy functional from below:

$$E[\varphi] \geqslant \tfrac{1}{2}\|\partial_x \varphi\|^2 - C I_2^{5/4}\|\partial_x \varphi\|^2 \ . \tag{3.5.3}$$

Minimising the right hand side of expression (3.5.3) with respect to $\|\partial_x \varphi\|$, we get a new estimate

$$E[\varphi] \geqslant C_0 I_2^{5/3}, \quad C_0 = \mathrm{const}\ .$$

Thus, for a fixed value of momentum $P = I_2$ the energy appears to be bounded from below and therefore it has a minimum, which is realised in some stable configuration.

Example 3.10. *The Kadomtsev–Petviashvili equation* in the space R^2 has the form

$$\partial_x\left(\partial_t \varphi + \partial_x^3 \varphi + 6\varphi\partial_x \varphi\right) = 3\partial_y^2 \varphi$$

and is commonly considered as a two-dimensional generalisation of the Korteveg–de Vries equation. It also assumes conservation of energy

$$E[\varphi] = \int \mathrm{d}x \left[\frac{1}{2}(\partial_x \varphi)^2 + \frac{3}{2}(\partial_y w)^2 - \varphi^3\right],$$

$$\partial_x w = \varphi\ ,$$

and of momentum

$$P = \int \mathrm{d}x\, \varphi^2 = I_2 \ .$$

Here we use the HoSlder inequality

$$I_3 \leqslant (I_2 I_4)^{1/2}\ ,$$

together with the obvious inequalities:

$$I_4 \leqslant 4\int \mathrm{d}^2 x\, |\varphi\partial_x \varphi| \int \mathrm{d}^2 x\, |\varphi \partial_y \varphi|\ ,$$

$$\int \mathrm{d}^2 x\, |\varphi \partial_y \varphi| = \int \mathrm{d}^2 x\, |\varphi \partial_{xy}^2 w|$$

$$\leqslant \int \mathrm{d}^2 x\, |\partial_x \varphi| \cdot |\partial_y w| \leqslant \|\partial_x \varphi\| \cdot \|\partial_y w\| \ .$$

On combining them, we arrive at the relation

$$I_3 \leqslant 2 I_2^{3/4} \|\partial_x \varphi\| \cdot \|\partial_y w\|^{1/2}\ ,$$

which allows us to derive an estimate for the energy functional from below:

$$E[\varphi] \geqslant \tfrac{1}{2}\|\partial_x \varphi\|^2 + \tfrac{3}{2}\|\partial_y w\|^2 - 2I_2^{3/4}\|\partial_x \varphi\| \cdot \|\partial_y w\|^{1/2}\ . \tag{3.5.4}$$

On minimising the right hand side of expression (3.5.4) with respect to $\|\partial_x \varphi\|$ and $\|\partial_y w\|$, we obtain the following inequality

$$E[\varphi] \geqslant -\tfrac{2}{3} I_2^3\ ,$$

which means that for a fixed value of momentum $P = I_2$ the energy minimum is realised in a stable solitonic configuration.

3.6. Stability of plasma solitons (BGK-structures)

Here we use the direct Lyapunov method to investigate the stability of plasma solitons of the electron-phase-hole type, known also as Bernstein–Greene–Kruskal waves [81], [82]. To this end we give the Vlasov–Poisson equation with the electron distribution function $f(t,x,v)$ and the electric field strength $E(t,x)$ in plasma in the heavy-ion approximation:

$$\partial_t f + v\partial_x f - E\,\partial_v f = 0\ , \tag{3.6.1}$$

$$\partial_x E = 1 - \int \mathrm{d}v f\ . \tag{3.6.2}$$

Taking account of the boundary conditions

$$E(t, \pm\infty) = 0, \quad f(t, \pm\infty, v) = f_\infty(v),$$

$$\int \mathrm{d}v f_\infty(v) = 1\ ,$$

corresponding to an electrically neutral system, and choosing a frame of reference tied to the centre of the distribution f_∞, one can eliminate the electric field with the help of Eqn (3.6.2):

$$E(t,x) = -\int_{-\infty}^{x} \mathrm{d}x' \int \mathrm{d}v f(t,x',v') - f_\infty(v')\ ,$$

on rewriting Eqn (3.6.1) in the form

$$\partial_t f + v\partial_x f + \partial_v f \int_{-\infty}^{x} \mathrm{d}x' \int \mathrm{d}v' \left[f(t,x',v') - f_\infty(v')\right] = 0\ . \tag{3.6.3}$$

Let us assume that the Eqn (3.6.3) has the stationary solution

$$f_0 = f_0(w,\mu), \quad E_0(x) = -\phi'_0(x+a), \quad a = \mathrm{const}\ ,$$

where $w = \frac{1}{2}v^2 - \phi_0(x+a)$ is the electron energy, $\mu = \mathrm{sgn}v$. Since the distribution function is a positive-definite one, one can put $f = \chi^2$, $f_0 = \chi_0^2$, where it is assumed that the function $\chi_0(x+a, v)$ is a solution to the equation

$$\hat{D}_0 \chi_0 = 0\ , \tag{3.6.4}$$

where the stationary Liouville operator

$$\hat{D}_0 = -v\partial_x + E_0 \partial_v\ .$$

has been used. Let us now introduce the perturbation $\xi = \chi - \chi_0$, and take into account that it has to satisfy the linearised normalisation condition

$$\int \mathrm{d}x \int \mathrm{d}v\, \chi_0 \xi = 0\ ; \tag{3.6.5}$$

it proves convenient to present the perturbation ξ in the form

$$\xi = \hat{D}_0[S(2f_0)^{-1/2}\varphi], \quad S = |\partial_w f_0|^{1/2} .$$

It is then easy to ascertain that Eqn (3.6.5) is satisfied in view of Eqn (3.6.4). The new unknown function $\varphi(t,x,v)$ satisfies the linearised equation

$$\hat{L}\partial_t\varphi = \hat{H}\varphi , \qquad (3.6.6)$$

in which the following operators

$$\hat{L} = \varepsilon\hat{D}_0, \quad \varepsilon = \operatorname{sgn}\partial_w f_0 ,$$

$$\hat{H} = \varepsilon\hat{D}_0^2 + vS\int dv' v' S(x,v') ,$$

have been introduced. Observing that $\int dx\, dv\, \chi_0^2 = \infty$, and taking into account Eqn (3.6.4), we verify that $\hat{L}\chi_0 = 0$, and therefore that the zero eigenvalue of the operator $\hat{L}$ belongs to the continuous part of the spectrum. This allows us to invert of the operator $\hat{L}$ and to reduce Eqn (3.6.6) to its normal form:

$$\partial_t\varphi = \hat{D}_0\varphi - \varepsilon S\int_{-\infty}^{x} dx' \int dv' v' S'\varphi' . \qquad (3.6.7)$$

From Eqn (3.6.7) it follows that the integral of motion exists

$$V = \int dx\left[-\int dv\,\varepsilon\left(\hat{D}_0\varphi\right)^2 + \left(\int dv\, vS\varphi\right)^2\right] , \qquad (3.6.8)$$

which for monotonic distributions, i.e. for $\varepsilon = -1$, is positive definite. Therefore, it is reasonable to choose the metrics ρ_0, ρ as follows:

$$\rho_0^2 = \int dx\left[\int dv\left(\hat{D}_0\varphi\right)^2 + \left(\int dv\, vS\varphi\right)^2\right], \quad \rho = \inf_a \rho_0 .$$

In this case for $\varepsilon = -1$ the functional $V = \rho_0^2 \geqslant \inf_a \rho_0^2 = \rho^2$ can be regarded as a Lyapunov functional. Therefore the stability of the electron distributions in the Vlasov–Poisson plasma is established when they are monotonic with respect to the energy w. The result obtained is known as the *Newcomb–Gardner theorem* for homogeneous distributions (i.e. for $\partial_x f_0 = 0$).

Let us show that the monotonic distributions are not only locally but also globally stable. To this end, we choose as Lyapunov functional

$$V_1 = \int dx\left\{\frac{1}{2}E^2 + \int dv\left[\frac{1}{2}fv^2 + \lambda(f - f_\infty) + G(f)\right]\right\} ,$$

where $\lambda = G'(f_0) - w$ is a Lagrange multiplier, found from the condition $\delta V_1(f_0) = 0$. Since $\lambda = \text{const}$, on differentiating λ with respect to w, one finds

$$\frac{\partial\lambda}{\partial w} = S^2 G''(f_0) - 1 = 0 . \qquad (3.6.9)$$

Eqn (3.6.9) allows us to obtain an expression for the function $G(f)$ and to conclude about the convexity of the functional $V_1[f]$, and in so doing on the stability of the monotonic distributions.

However, if the distribution is not monotonic, i.e. when ε is sign-indefinite, then the functional (3.6.8) is also sign-indefinite and this indicates instability. Indeed, considering the Chetaev functional

$$W = V\int dx\; dv\,\varepsilon F(x,v)\left(\hat{D}_0\varphi\right)^2 , \qquad (3.6.10)$$

where $F(x,v)$ is a solution of the auxiliary equation, we have

$$\hat{D}_0 F = 1 + \varepsilon F^2\int dv\, v^2 S^2 . \qquad (3.6.11)$$

Making use of equations (3.6.7), (3.6.11) and expression (3.6.10), we obtain

$$\frac{dW}{dt} = -V\left\{-V + \int dx\left[\int dv\, vS(\varphi - F\hat{D}_0\varphi)\right]^2 + \int dx\left[\left(\int dv\, v^2S^2\right)\int dv\, F^2\left(\hat{D}_0\varphi\right)^2 - \left(\int dv\, vSF\hat{D}_0\varphi\right)^2\right]\right\} ,$$

whence it is evident that in the region where $V < 0$ the inequality

$$\frac{dW}{dt} \geqslant V^2$$

holds. The latter statement means that the conditions of the Chetaev–Movchan theorem (Theorem 3.2) on instability with respect to metrics ρ_0', ρ' are satisfied, where

$$\rho_0'^2 = \rho_0^2 + \int dx\, dv|F|\left(\hat{D}_0\varphi\right)^2, \quad \rho' = \inf_a \rho_0' .$$

It should be emphasised, that the known frequency criteria of instability [83] are not applicable here owing to the substantial inhomogeneity of the distributions.

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