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The stimulated Cherenkov effect

V M Arutyunyan, S G Oganesyan

Contents

Abstract. The interaction between free electrons and a laser field in an unbounded dielectric medium and above the surface of a dielectric waveguide is discussed in detail. Both classical and quantum approaches are applied. The feasibilities of modulation and polarisation of an electron beam by laser radiation are also discussed. Theories of the Cherenkov laser and Cherenkov klystron are developed.

1. Introduction

The advent of a powerful source of electromagnetic radiation — the laser — marked the beginning of active research of electromagnetic effects in high-intensity fields. The first works along this line were conducted as early as 1933 by Kapitsa and Dirac [*Proc. Cambridge Philos. Soc.* **29** 297 (1933)]. They considered the scattering of an electromagnetic wave by an electron in the presence of another wave (the stimulated Compton effect). Then the theories of stimulated braking, magnetobraking, Cherenkov, and other effects were developed in parallel with the cited problem. We have systematically studied the stimulated Cherenkov, transient, diffraction, and Compton effects. In this review we present the results on the stimulated Cherenkov and Cherenkov surface effects (in the first case electrons travel in an unbounded dielectric medium, in the second they travel over the surface of a dielectric). For brevity we refer to either effect as the stimulated Cherenkov effect (SCE).

Note that along with the scientific aspect of the issue — how powerful fields affect the course of electrodynamic effects — the problems in question are of extreme practical importance: design of new sources of electromagnetic radiation (free-electron lasers) and laser-driven charged-particle accelerators.

The idea that the stimulated Compton effect could be used for developing the Compton laser was suggested by Pantell and co-authors [*IEEE J. Quantum Electron.* **4** (11) 905 (1968)] and Madey (*J. Appl. Phys.* **42** (3) 1906 (1971)). They proposed an undulator laser. Since then this last scheme has been worked out in detail and at present it is implemented experimentally. The main bulk of publications on this laser is systematically surveyed in reviews [1 – 6], the collected volume [7], and monograph [8]. However, the undulator laser is not efficient in the optical and shorter

V M Arutyunyan, S G Oganesyan Research and Production Company ''Lazernaya Tekhnika'' 375090 Yerevan, ul. Shoprona 21
Tel. (7-8852) 65-0552, 61-5398. Fax (7-8852) 15-1057
E-mail: bogl@adonis.iasnet.com

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wavelength range. Therefore, there was a need to study other electromagnetic effects. We have been developing the theory of how to amplify an electromagnetic radiation on the basis of the Cherenkov, transient, diffraction, and Compton effects. In this review we present the analysis of various operational modes of the Cherenkov laser.

The energy of an electron is changed by its interaction with intensive electromagnetic radiation. At present there is a variety of suggestions of how to accelerate electrons on the basis of the Compton (including the undulator case), Cherenkov, diffraction, and other effects. The present state of the art of laser-driven accelerators is well reflected in the collected volume [9] and the review [10]. We have confined our study to the mechanisms of emission and absorption of a photon on the basis of the listed effects. These results are of key importance for an understanding of all effects in the field of laser radiation.

One of the ways of increasing the efficiency of interaction of an electron beam with radiation lies in a preliminary modulation of its density. The research of modulation of an electron beam at optical frequencies was initiated by Schwarz and Hora [*Appl. Phys. Lett.* **15** 349 (1969)]. Modulated electron beams are the basis for developing a free-electron laser of the klystron type. In this review we present an analysis of the characteristics of the Cherenkov klystron.

The polarisation states of an electron beam and laser radiation are changed by their interaction. The laser radiation will magnetise a nonpolarised electron beam or modulate its magnetisation if it is polarised before the interaction. As an elliptically polarised wave propagates in an electron beam, the plane of polarisation rotates and the ellipse is deformed.

The systematic analysis of the cited effects revealed the following underlying general rules:

1. The possibility for modulation of electron beams, amplification of electromagnetic waves, and magnetisation of a particle beam occurs only if photons are emitted and absorbed by electrons with different energies or if the photons have different projections of wave vectors. In what follows, for brevity, we refer to these processes as asymmetric. If there is no asymmetry, then it must be created.

2. Quantitatively, the listed effects depend on the increment in the energy of an electron after the interaction with radiation, in the linear approximation with respect to the field.

Note that the references fall into two parts: the first part consists of works which deal directly with the problems in question [1–73], and the second part consists of works on the stimulated Cherenkov [74–91] and Cherenkov surface [92–97] effects.

2. The stimulated Cherenkov effect

The spontaneous emission of a charged particle in a dielectric — the Cherenkov radiation (C) — has been thoroughly studied theoretically and experimentally [11–15]. It occurs only if the velocity of the charged particle is greater than the velocity of the electromagnetic wave in the medium ($v > c/n$).

If the same particle travels in a dielectric medium in the presence of an external electromagnetic wave, then the radiation becomes stimulated. In this case the dynamics of the process acquires an essential feature which is absent in the spontaneous effect: the particle may not only decelerate, radiating its energy to the wave (the stimulated Cherenkov radiation), but it may also accelerate, absorbing energy from the external field (the stimulated Cherenkov absorption). As a rule these processes cannot be separated completely for particle beams. Clearly, the competition between absorption and emission will significantly restrict the transfer of energy from the particle beam to the electromagnetic wave.

In analysis of the SCE we use two models: (a) electrons interact with radiation which can be described by a plane monochromatic wave; and (b) an electron beam interacts with a monochromatic spatially bounded wave. In Sections 2.1 and 2.2, the dynamics of electrons is studied for both cases. A simple analysis of the classical equations of motion shows that an electron can decelerate or accelerate depending on initial conditions. For an electron beam this effect results in modulation of its density and current at the frequency of the electromagnetic wave (see Section 2.3). In Sections 2.4 and 2.5 the quantum theory of the SCE is presented [16–19]. This approach makes it possible to consider absorption and emission separately.

Analysis shows that the asymmetry of these processes lies at the heart of the modulation effect (see Section 1). The asymmetry is responsible for magnetisation of the electron beam [18]. Note that these effects are studied with account being taken of angular and energy spreads of the electron beam and also of angular and frequency spreads of the photon beam (see Section 2.7). In Sections 2.8–2.12, 2.14 we develop the theory of how to amplify an electromagnetic wave on the basis of the SCE — the Cherenkov klystron [16, 20] and the Cherenkov laser [21–24]. Clearly, amplification is possible only if emission dominates over absorption. An interesting possibility for total suppression of absorption is considered in Section 2.12 [23]. In Sections 2.9, and 2.14 the SCE is considered in a constant magnetic field [20, 24]. In this scheme the negative effect of the angular spread of an electron beam can be neutralised and the range of operation of the laser can be extended significantly at the cyclotron resonance (see Section 2.15) [25].

In section 2.13 the optical polarisation effects are considered in a system of the Cherenkov laser type, related to anisotropy and polarisation of the electron beam [22]. At the end of Section 2 we discuss conditions under which the effects in question can be observed experimentally.

2.1 Motion of an electron in the presence of a plane wave in a dielectric medium

If an electron travels in an unbounded dielectric medium, spontaneous Cherenkov emission occurs when the velocity of the particle v, the wave vector $\boldsymbol{k}$, and the frequency ω of the emitted electromagnetic wave are related by the equation

$$\omega - \boldsymbol{k}\cdot v = 0 \,. \tag{1}$$

Let an electromagnetic wave with the vector potential

$$\boldsymbol{A} = \boldsymbol{A}_0 \cos(\omega t - \boldsymbol{k}\cdot r) \tag{2}$$

propagate in the same medium, i.e. it is the same wave as that induced by the SC effect. (In the analysis which follows we suppose that the magnetic permeability is $\mu = 1$.)

We shall determine the changes in momentum and energy of a particle using the classical equations of motion,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\boldsymbol{E} + \frac{e}{c} v \times \boldsymbol{H}, \quad \frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = ev\cdot\boldsymbol{E}, \tag{3}$$

in the linear approximation with respect to the field. Remembering that $\boldsymbol{E} = -c^{-1}\partial\boldsymbol{A}/\partial t$, $\boldsymbol{H} = \mathrm{curl}\boldsymbol{A}$ and substituting the unperturbed trajectory of the particle in the form

$$\boldsymbol{r} = \boldsymbol{r}_0 + v_0 t, \tag{4}$$

into the right-hand sides of Eqn (3), we obtain

$$\boldsymbol{p} = \boldsymbol{p}_0 + \Delta\boldsymbol{p}', \quad \mathscr{E} = \mathscr{E}_0 + \Delta\mathscr{E}'. \tag{5}$$

Here,

$$\Delta\boldsymbol{p}' = \left[-\frac{e}{c}\boldsymbol{A}_0 - \frac{e\boldsymbol{k}(v_0\cdot\boldsymbol{A}_0)}{c(\omega - \boldsymbol{k}\cdot v_0)}\right]\cos[(\omega - \boldsymbol{k}\cdot v_0)t - \boldsymbol{k}\cdot\boldsymbol{r}_0], \tag{6}$$

$$\Delta\mathscr{E}' = -\frac{e\omega(v_0\cdot\boldsymbol{A}_0)}{c(\omega - \boldsymbol{k}\cdot v_0)}\cos[(\omega - \boldsymbol{k}\cdot v_0)t - \boldsymbol{k}\cdot\boldsymbol{r}_0], \tag{7}$$

$v_0 = p_0 c^2/\mathscr{E}_0$ is the initial velocity of the particle, and n is the refractive index. The field is assumed to be switched on adiabatically slowly at $t = -\infty$. The amplitudes of the quantities $\Delta\mathscr{E}'$ and $\Delta\boldsymbol{p}'\cdot\boldsymbol{k}/|\boldsymbol{k}|$ contain the Cherenkov pole $\omega - \boldsymbol{k}\cdot v_0$ which coincides with the spontaneous emission condition (1).

At first we suppose that the quantity $\omega - \boldsymbol{k}\cdot v_0 > 0$ (or $v_0\cos\theta < c/n$) and the inner product $\boldsymbol{A}_0\cdot v_0 > 0$. The strength of the electric field of the wave [Eqn (2)] is $\boldsymbol{E} = \boldsymbol{E}_0\sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})$. Hence, the electrons lag behind the wave and are decelerated and accelerated in turn within the intervals of time

$$\frac{\pi(2N+1) + \boldsymbol{k}\cdot\boldsymbol{r}_0}{\omega - \boldsymbol{k}\cdot v_0} > t > \frac{2\pi N + \boldsymbol{k}\cdot\boldsymbol{r}_0}{\omega - \boldsymbol{k}\cdot v_0},$$

$$\frac{2\pi(N+1) + \boldsymbol{k}\cdot\boldsymbol{r}_0}{\omega - \boldsymbol{k}\cdot v_0} > t > \frac{\pi(2N+1) + \boldsymbol{k}\cdot\boldsymbol{r}_0}{\omega - \boldsymbol{k}\cdot v_0}$$

(here $\boldsymbol{E} = \omega\boldsymbol{A}_0/c$, $N = 0, \pm 1, \pm 2...$ are arbitrary integers). If the quantity $\omega - \boldsymbol{k}\cdot v_0 < 0$ (or $v_0\cos\theta > c/n$), then the electron overtakes the wave and is accelerated and decelerated in turn within the intervals of time

$$\frac{|\pi(2N+1) - \boldsymbol{k}\cdot\boldsymbol{r}_0|}{|\omega - \boldsymbol{k}\cdot v_0|} > t > \frac{|2\pi N - \boldsymbol{k}\cdot\boldsymbol{r}_0|}{|\omega - \boldsymbol{k}\cdot v_0|},$$

$$\frac{|2\pi(N+1) - \boldsymbol{k}\cdot\boldsymbol{r}_0|}{|\omega - \boldsymbol{k}\cdot v_0|} > t > \frac{|\pi(2N+1) - \boldsymbol{k}\cdot\boldsymbol{r}_0|}{|\omega - \boldsymbol{k}\cdot v_0|}.$$

According to the above inequalities the intervals of time within which these processes occur coincide and are equal to $T_{\mathrm{f}} = \pi/|\omega - \boldsymbol{k}\cdot v_0|$. For this time the electron travels the distance $\boldsymbol{L}_{\mathrm{f}} = \pi v_0/|\omega - \boldsymbol{k}\cdot v_0|$. In these terms the amplitude of the energy increment (7) is $\Delta\mathscr{E} = (1/\pi)|e|\boldsymbol{E}_0\cdot\boldsymbol{L}_{\mathrm{f}}$. Clearly, the maximal increment (or decrement) in the energy of particles is equal to the work done by the constant electric field with the strength E_0 along the path $\boldsymbol{L}_{\mathrm{f}}$.

Thus, the quantities T_{f} and $\boldsymbol{L}_{\mathrm{f}}$ give intervals of time and parts of the trajectory, along which processes of stimulated emission of energy from an electron to the wave or stimulated absorption of energy of the wave by the electron proceed. They coincide with the intervals of time and the forming zones, which were introduced to describe spontaneous emission of particles [26–29].

Let velocities of electrons be such that the quantity $\omega - \boldsymbol{k}\cdot v_0$ tends to zero. Then the parameters T_{f} and $\boldsymbol{L}_{\mathrm{f}}$ increase indefinitely. This result is concurrent with the analysis of the spontaneous SC effect: emission of uniformly moving particle occurs along its whole trajectory in an unbounded transparent medium. If the quantity $\omega - \boldsymbol{k}\cdot v_0 = 0$, then the energy and momentum of the particle are divergent. The divergence is caused by the infinitely long action of the electromagnetic field of the wave [Eqn (2)] on the electron. We shall consider the case of $\omega - \boldsymbol{k}\cdot v_0 = 0$ in more detail. We suppose that the field is switched on for a time $\Delta t \sim 2\tau$ in accordance with the law

$$\boldsymbol{A}_0(t) = \frac{1}{2}\boldsymbol{A}_0'\left(1 + \frac{\tanh t}{\tau}\right). \tag{8}$$

Substituting Eqns (2), (4), (8) into Eqn (3) and taking into account Eqn (1) we obtain

$$\Delta\mathscr{E}' = -\frac{1}{2}e\boldsymbol{E}_0\cdot v_0\left[t + \tau\ln\left(2\frac{\cosh t}{\tau}\right)\right]\sin(\boldsymbol{k}\cdot\boldsymbol{r}_0). \tag{9}$$

Here $\boldsymbol{E}_0 = \omega\boldsymbol{A}_0'/c$ is the amplitude of the strength of the electromagnetic field of the wave [Eqn (2)]. If the time of the electron – wave interaction $\Delta t \gg \tau$, then

$$\Delta\mathscr{E}' = -e\boldsymbol{E}_0\cdot v_0 t\sin(\boldsymbol{k}\cdot\boldsymbol{r}_0). \tag{10}$$

Clearly, the sign of the expression depends on the phase $\phi = \boldsymbol{k}\cdot\boldsymbol{r}_0$. If $\Delta\mathscr{E}' < 0$, the particle is decelerated and transfers its energy to the field [Eqn (2)]. If $\Delta\mathscr{E}' > 0$, the particle is accelerated at the expense of the energy of the electromagnetic wave.

Expressions (6), (7), and (10) are true provided that the synchronism condition $\omega - \boldsymbol{k}\cdot v_0 = \mathrm{const}$ is satisfied for the electron and the wave. Actually the equality is violated even by a small change in velocity. Therefore, the correct expression for the energy and momentum of a particle in the presence of a plane wave can be obtained only by solving Eqns (3) exactly [30]. However, formulae (6) and (7) are of practical importance. This is because of the fact that actual beams have certain spreads in energies and angles and the Cherenkov divergence can be eliminated by averaging over these spreads.

2.2 The stimulated Cherenkov effect in a finite laser beam

The stimulated Cherenkov effect in a laser beam of a finite diameter was studied theoretically and experimentally in Refs [31–38]. We consider the relatively simple case of an electromagnetic wave propagating along the z axis and which has a finite width in the x direction only:

$$A_{x,y} = \int A_{x,y}(\boldsymbol{q}')\, q_z'\,\delta\left(\frac{\omega^2}{c^2}n^2 - |\boldsymbol{q}'|^2\right) \times \exp(\mathrm{i}\boldsymbol{q}'\cdot\boldsymbol{r} - \mathrm{i}\omega t)\,\mathrm{d}\boldsymbol{q}' + \mathrm{c.c.}, \tag{11}$$

$$A_{x,y}(\boldsymbol{q}') = \frac{1}{2\sqrt{\pi}}a_{x,y}\,d\exp\left(-\frac{q_x'^2 d^2}{4}\right), \tag{12}$$

$$a_x = -\mathrm{i}A_{0x}, \quad a_y = A_{0y},$$

$$A_z(\boldsymbol{r}, t) \approx 0.$$

Here $\omega = 2\pi c/\lambda$ is the frequency of laser radiation, λ is its wavelength in vacuum, and $\boldsymbol{q}'$ is the wave vector of the Fourier component of the field.

The Fourier transform of the vector potential [Eqn (12)] is chosen so that, in the plane $z = 0$, the amplitude of the field is attenuated on increase in $|x|$ as the Gaussian distribution of a width $2d$. For simplicity the diffraction divergence of the beam is assumed to be small ($\lambda/d \ll 1$) so that z-projections of the field can be neglected.

We want to determine the change in the energy and momentum of an electron travelling in the xz plane and intersecting the field [Eqn (11)] at an angle θ, which is much greater than the angular divergence of the laser beam. It follows from Eqns (3), in the linear approximation with respect to field, that

$$\Delta\mathscr{E}' = \Delta\mathscr{E} \cos\phi \ . \tag{13}$$

Here the amplitude of the energy change is

$$\Delta\mathscr{E} = \xi_x \, \Delta E \ ,$$

$$\Delta E = 2\pi\sqrt{\pi}\, mc^2 \, \frac{d}{\lambda} \exp\left(-\frac{q_x^2 d^2}{4}\right) , \tag{14}$$

the phase is $\phi = \boldsymbol{q}\cdot\boldsymbol{r}_0$, and the dimensionless parameter is $\xi_x = eA_{0x}/mc^2$.

The wave vector is found from the system of equations

$$\omega - \boldsymbol{q}\cdot v_0 = 0 \ , \tag{15}$$

$$q_x^2 + q_z^2 = \left(\frac{\omega}{c}\, n\right)^2 , \tag{16}$$

and its projections are

$$q_x = \frac{\omega}{v_0^2}\left[v_{0x} - v_{0z}\sqrt{(n\beta_0)^2 - 1}\,\right] ,$$

$$q_z = \frac{\omega}{v_0^2}\left[v_{0z} + v_{0x}\sqrt{(n\beta_0)^2 - 1}\,\right] , \tag{17}$$

where $\beta_0 = v_0/c$.

The system of equations (15), (16) has another pair of solutions, different from Eqns (17) in the signs in front of the radical signs. Since the amplitude of the energy change $\Delta\mathscr{E}$ drops on increase in q_x, their contribution may be neglected. The quantity $\Delta\mathscr{E}$ is very important in the theory of interaction between free electrons and laser radiation. As will be shown later, the characteristics of all processes in which the electron is involved depend on $\Delta\mathscr{E}$.

If the velocity of a particle is such that

$$q_x = \frac{\omega}{v_0^2}\left[v_{0x} - v_{0z}\sqrt{(n\beta_0)^2 - 1}\,\right] = 0 \ , \tag{18}$$

then its interaction with the field occurs under the optimal conditions. In this case,

$$\Delta E = 2\pi\sqrt{\pi}\, mc^2 \, \frac{d}{\lambda} , \tag{19}$$

and the amplitude of the change in energy of the particle $\Delta\mathscr{E}$ peaks.

We want to express $\Delta\mathscr{E}$ in terms of the width of the angular distribution of the strength of the field [Eqn (11)]. Let the angle between the vector $\boldsymbol{q}'$ and the axis z be θ. If $\theta \ll 1$, then

$$q'_x = n\frac{\omega}{c}\sin\theta \approx n\frac{2\pi}{\lambda}\,\theta \ .$$

By substituting the last expression into Eqn (12), we obtain the intensity of the luminous flux represented by the Gaussian distribution in angles such that

$$\frac{\mathrm{d}I}{\mathrm{d}\theta} \sim \exp\left[-\frac{4(\ln 2)\theta^2}{\delta_\phi^2}\right] ;$$

the quantity $\delta_\phi = \lambda\sqrt{2\ln 2}/\pi n d$ characterises the angular width of the distribution. We substitute the expression $d/\lambda = 2\sqrt{\ln 2}/\pi n \delta_\phi$ into

$$\Delta E = \frac{4\sqrt{\pi \ln 2}}{n\delta_\phi}\, mc^2 \ , \tag{20}$$

and compare formulae (13), (14), (20), and (7). Clearly, if a photon beam has an angular spread, the Cherenkov pole $\omega/(\omega - \boldsymbol{k}\cdot v_0)$ is replaced by the angular width δ_ϕ according to the remark at the end of Section 2.1.

Formula (13) is true provided that

$$|\Delta\mathscr{E}| \ll \mathscr{E} \ . \tag{21}$$

The condition (21) imposes a constraint on the parameters of the laser beam. Note that the z-projection of the vector potential can be neglected in Eqns (3) when the inequality

$$\frac{v_{0x}}{v_{0z}} \gg \frac{q_x}{q_z} \tag{22}$$

holds. It holds automatically when $q_x = 0$ [see Eqn (18)].

If the electron beam crosses the field [Eqn (1)] then, depending on the phase ϕ, its energy spectrum contains either accelerated ($\Delta\mathscr{E}' > 0$) or decelerated ($\Delta\mathscr{E}' < 0$) particles [Eqn (13)]. This results in a wider energy spectrum of electrons, which was observed in experiments [27–30].

2.3 Modulation of an electron beam (the classical theory)

The stimulated Cherenkov effect causes the modulation of energy [Eqn (13)] and, consequently, of the velocity of the particle beam. In accordance with the theory of the klystron, electrons can overtake those electrons which have left the region of interaction at earlier instances of time [35]. As a result the electron beam becomes inhomogeneous and its density and current break into oscillation at the frequency of the external field and the harmonics of the frequency. We shall neglect quantum effects in considering the features of the modulated density and current of an electron beam, with the aid of the kinetic equation

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0 \ , \tag{23}$$

where $\boldsymbol{F} = e\boldsymbol{E} + e/c(v \times \boldsymbol{H})$ is the Lorentz force.

Let an electron beam cross the electromagnetic wave [Eqn (11)] propagating along the axis z at an angle θ. If the electric and magnetic fields are of low strength, then the electron distribution function can be determined by the perturbation theory, in the region $x \gg d$, as

$$f = f_0 + f_1 \ . \tag{24}$$

Here f_0 is the initial distribution function

$$f_1 = -\frac{\sqrt{\pi}}{2}\frac{e}{c} A_{0x} d \exp\left(-\frac{q_x^2 d^2}{4}\right)$$

$$\times\left(q_x\frac{\partial f_0}{\partial p_x} + q_z\frac{\partial f_0}{\partial p_z}\right)\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r} - \mathrm{i}\omega t) + \mathrm{c.c.} \ , \tag{25}$$

and the projections of the vector $\boldsymbol{q}$ are specified by the expressions (17) [in calculations we take into account inequality (22)].

By substituting the function (24) into the expressions

$$\rho = \rho_0 \int f(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}\,, \quad \boldsymbol{j} = e\rho_0 \int v f(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}$$

for the density and current of the electron beam, we have

$$\rho = \rho_0 + \rho_1\,, \quad \boldsymbol{j} = \boldsymbol{j}_0 + \boldsymbol{j}_1\,, \tag{26}$$

where ρ_0 and $\boldsymbol{j}_0 = e\rho_0 \int v f_0\,\mathrm{d}\boldsymbol{p}$ are the initial density and current of the electron beam,

$$\rho_1 = -\frac{\sqrt{\pi}}{2}\frac{e}{c}\,\rho_0 A_{0x} d \int \exp\left(-\frac{q_x^2 d^2}{4}\right)$$
$$\times \left(q_x \frac{\partial f_0}{\partial p_x} + q_z \frac{\partial f_0}{\partial p_z}\right) \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r} - i\omega t)\,\mathrm{d}\boldsymbol{p} + \text{c.c.}\,, \tag{27}$$

$$\boldsymbol{j}_1 = -\frac{\sqrt{\pi}}{2}\frac{e^2}{c}\,\rho_0 A_{0x} d \int v \exp\left(-\frac{q_x^2 d^2}{4}\right)$$
$$\times \left(q_x \frac{\partial f_0}{\partial p_x} + q_z \frac{\partial f_0}{\partial p_z}\right) \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r} - i\omega t)\,\mathrm{d}\boldsymbol{p} + \text{c.c.} \tag{28}$$

For simplicity we suppose that all particles of the beam have the same momentum, $f_0 = \delta(p_x - p_{0x})\,\delta(p_z - p_{0z})\,\delta(p_y)$, before the interaction. Integration of Eqns (27) and (28) by parts yields

$$\rho = \rho_0 \left[1 + x\,\frac{\Delta\mathscr{E}}{\mathscr{E}_0}\,\frac{\omega(n^2-1)}{v_{0x}} \sin(q_z z - \omega t) + \frac{\Delta\mathscr{E}}{\mathscr{E}_0} \cos(q_z z - \omega t)\right], \tag{29}$$

$$j_x = j_{0x}\left[1 + x\,\frac{\Delta\mathscr{E}}{\mathscr{E}_0}\,\frac{\omega(n^2-1)}{v_{0x}} \sin(q_z z - \omega t)\right],$$

$$j_y = 0\,,$$

$$j_z = j_{0z}\left[1 + x\,\frac{\Delta\mathscr{E}}{\mathscr{E}_0}\,\frac{\omega(n^2-1)}{v_{0x}} \sin(q_z z - \omega t) + \frac{\Delta\mathscr{E}}{\mathscr{E}_0}\,\frac{1}{\beta_{0z}^2} \cos(q_z z - \omega t)\right]. \tag{30}$$

Here we have taken into account condition (18); the wave vector $q_z = \omega/v_{0z}$ and the quantity $\Delta\mathscr{E}$ are specified by the expressions (14) and (19).

Clearly, the density and current of the electron beam break into oscillation at the laser radiation frequency ω, and also the depth of modulation increases in direct proportion to the drift distance x (the second terms in brackets). This result is well known from the theory of the klystron. As for the third terms, they are responsible for rearranging the density of particles when they move in the field [Eqn (11)]; the rearrangement remains in the drift region. However, the depth of modulation as a result of this mechanism is not large.

In Section (2.7) the role of the angular, frequency, and energy spreads of the light beam and particle beam will be examined. We shall also give the conditions under which formulae (29) and (30) are applicable. These conditions are found by the perturbation theory. Since $\rho_1 \ll \rho_0$ and $\boldsymbol{j}_1 \ll \boldsymbol{j}_0$, we have

$$x\,\frac{\Delta\mathscr{E}}{\mathscr{E}_0}\,\frac{\omega(n^2-1)}{v_{0x}} \ll 1\,. \tag{31}$$

2.4 Modulation of an electron beam (the quantum theory)

Let us now study the modulation effect for an electron beam on the basis of a more general, quantum mechanical approach. From the quantum standpoint a change in the energy of a particle is caused by emission or absorption ($\nu = |\Delta\mathscr{E}|/\hbar\omega$) of photons. Clearly, in the region $x \gg d$ the wave function of a particle beam is a superposition of states describing different multiphoton processes. The result is modulation of the density and current of the electron beam. Since the amplitudes of emission and absorption make opposite contributions, the depth of modulation of the particle beam depends on the difference between them.

We shall determine the depth of modulation of a relativistic electron beam as a result of the SCE in the simplest case — at the first harmonic of the field [Eqn (11)], neglecting the spin effects. Let the electron beam be described by the plane wave

$$\psi_0 = \sqrt{\frac{\rho_0}{2\mathscr{E}}} \exp\left(-\mathrm{i}\frac{\mathscr{E}}{\hbar}\,t + \mathrm{i}\frac{\boldsymbol{p}}{\hbar}\cdot\boldsymbol{r}\right) \tag{32}$$

before the interaction. Here ρ_0 is the density of electrons, $\mathscr{E}$ and $\boldsymbol{p}$ are their energy and momentum.

We shall determine the wave function of the electron beam after the interaction with the use of the Klein – Gordon equation

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left[c^2\left(\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right)^2 + (mc^2)^2\right]\psi\,. \tag{33}$$

Its solution can be presented in the form

$$\psi = \psi_0 + \psi_+ + \psi_-\,.$$

Here ψ_0 is the initial wave function of the electron beam [Eqn (32)], the terms ψ_+ and ψ_- describe emission and absorption of a photon. By substituting them into Eqn (33) we have

$$\left[-(mc^2)^2 - \hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} + \hbar^2 c^2 \frac{\partial^2}{\partial z^2}\right]\psi_\pm$$
$$= -2ecp_x \int A_\pm q'_z \delta\left(\frac{\omega^2}{c^2}\,n^2 - |\boldsymbol{q}'|^2\right) \times \exp[\pm\mathrm{i}(\boldsymbol{q}'\cdot\boldsymbol{r} - \omega t)]\,\mathrm{d}q'_x\,\mathrm{d}q'_z\,\psi_0\,, \tag{34}$$

where $A_+ = A_x(q)$, $A_- = A_x^*(q)$ [see Eqn (12)]. We suppose that electrons travel in the xz plane before the interaction.

The solution to Eqn (34) is sought in the form

$$\psi_\pm = \int \varphi_\pm(x, q'_z) \exp\left(-\mathrm{i}\frac{\mathscr{E} \pm \hbar\omega}{\hbar}\,t + \mathrm{i}\frac{p_z \pm \hbar q'_z}{\hbar}\,z\right) \mathrm{d}q'_z\,. \tag{35}$$

Then the partial differential equation (34) can be reduced to the second order ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\varphi_\pm + \gamma_\pm^2 \varphi_\pm = -2\frac{e}{c}\frac{p_x}{\hbar^2}\sqrt{\frac{\rho_0}{2\mathscr{E}}}$$
$$\times \int A_\pm q'_z \delta\left(\frac{\omega^2}{c^2}\,n^2 - |\boldsymbol{q}'|^2\right) \exp\left(\mathrm{i}\frac{p_x \pm \hbar q'_x}{\hbar}\,x\right) \mathrm{d}q'_x\,, \tag{36}$$

where,

$$\gamma_{\pm} = \frac{1}{\hbar c}[(\mathscr{E} \pm \hbar\omega)^2 - (p_z \pm \hbar q_z')^2 c^2 - (mc^2)^2] \,.$$

Its solution has the form

$$\varphi_{\pm} = -\frac{2e}{c}\frac{p_x}{\hbar^2}\sqrt{\frac{\rho_0}{2\mathscr{E}}}\frac{1}{\gamma_{\pm}}\int_{-\infty}^{x} \mathrm{d}\alpha \int_{-\infty}^{+\infty} \mathrm{d}q_x' \sin[\gamma_{\pm}(x-\alpha)]$$
$$\times A_{\pm} q_z' \delta\left(\frac{\omega^2}{c^2} n^2 - |\boldsymbol{q}'|^2\right) \exp\left(\mathrm{i}\,\frac{p_x \pm \hbar q_x'}{\hbar}\,\alpha\right) . \quad (37)$$

In the region $x \gg d$ the upper limits in the integral of Eqn (37) can be replaced by $+\infty$. Integrating with respect to α yields

$$\varphi_{\pm} = \mathrm{i}\pi \frac{2e}{c}\frac{p_x}{\hbar^2}\sqrt{\frac{\rho_0}{2\mathscr{E}}}\frac{1}{\gamma_{\pm}}$$
$$\times\left\{\int A_{\pm} q_z' \,\delta\left(\frac{\omega^2}{c^2} n^2 - |\boldsymbol{q}'|^2\right)\delta\left(\frac{p_x \pm \hbar q_x'}{\hbar} - \gamma_{\pm}\right)\exp(\mathrm{i}\gamma_{\pm}x)\,\mathrm{d}q_x'\right.$$
$$\left.-\int A_{\pm} q_z' \delta\left(\frac{\omega^2}{c^2} n^2 - |\boldsymbol{q}'|^2\right)\delta\left(\frac{p_x \pm \hbar q_x'}{\hbar} + \gamma_{\pm}\right)\exp(-\mathrm{i}\gamma_{\pm}x)\,\mathrm{d}q_x'\right\}. \quad (38)$$

The second terms in Eqn (38) are responsible for one-photon reflection of electrons from the laser beam. If the x-projection of the momentum of an electron is $p_x \gg \hbar/d$, then the probability of such processes is exponentially low and they can be neglected. The projections of the wave vectors of photons $\boldsymbol{q}_{\pm}$ involved in absorption and emission are found from the laws of conservation of energy and momentum, and from the dispersion equation

$$\mathscr{E} \pm \hbar\omega = \mathscr{E}^{\pm}, \quad \boldsymbol{p} \pm \hbar\boldsymbol{q}^{\pm} = \boldsymbol{p}^{\pm} ,$$
$$\frac{\omega^2}{c^2} n^2 - |\boldsymbol{q}^{\pm}|^2 = 0 \,. \quad (39)$$

Given the initial energy and momentum of the electron we have

$$q_x^{\pm} = \frac{\omega}{v^2}\left\{v_x\left[1 \pm \frac{\hbar\omega}{2\mathscr{E}}(1-n^2)\right]\right.$$
$$\left.-v_z\left(n^2\beta^2 - \left[1 \pm \frac{\hbar\omega}{2\mathscr{E}}(1-n^2)\right]^2\right)^{1/2}\right\},$$
$$q_z^{\pm} = \frac{\omega}{v^2}\left\{v_z\left[1 \pm \frac{\hbar\omega}{2\mathscr{E}}(1-n^2)\right]\right.$$
$$\left.+v_x\left(n^2\beta^2 - \left[1 \pm \frac{\hbar\omega}{2\mathscr{E}}(1-n^2)\right]^2\right)^{1/2}\right\}. \quad (40)$$

The second pair of solutions, which are different from Eqns (40) in that they have different signs in front of the radical signs, can be neglected as in the case of Eqn (17). If $n^2\beta^2 - 1 \gg \hbar\omega(n^2-1)/2\mathscr{E}$, then the roots in Eqns (40) can be expanded into the Taylor series,

$$\boldsymbol{q}^{\pm} = \boldsymbol{q} \pm \Delta\boldsymbol{q} \,. \quad (41)$$

The projections of the vector $\boldsymbol{q}$ are specified by expressions (17), and we can write

$$\Delta q_x = \frac{\hbar\omega}{2\mathscr{E}}\frac{n^2-1}{\sqrt{n^2\beta^2-1}}\,q_z\,, \quad \Delta q_z = -\frac{\hbar\omega}{2\mathscr{E}}\frac{n^2-1}{\sqrt{n^2\beta^2-1}}\,q_x\,. \quad (42)$$

By neglecting the term in the conservation law [Eqn (39)] we find the expression for the momenta of electrons which have emitted or absorbed a photon

$$\boldsymbol{p}^{+} = \boldsymbol{p} + \hbar\boldsymbol{q}\,, \quad \boldsymbol{p}^{-} = p - \hbar\boldsymbol{q} \,.$$

The quantum corrections [Eqn (42)] introduce an asymmetry in these formulae and play a leading part in the effect of quantum modulation of an electron beam. The momenta can then be written as

$$\boldsymbol{p}^{+} = (\boldsymbol{p} + \hbar\boldsymbol{q}^{+}) - \hbar\,\Delta\boldsymbol{q}\,, \quad \boldsymbol{p}^{-} = (\boldsymbol{p} - \hbar\boldsymbol{q}^{-}) - \hbar\,\Delta\boldsymbol{q} \,.$$

Taking into account Eqns (38) and (40) we arrive at the final expression for the wave function of the electron beam in the region $x \gg d$,

$$\psi = \sqrt{\frac{\rho_0}{2\mathscr{E}}}\left[1 + \frac{1}{2}\frac{\Delta\mathscr{E}_{+}}{\hbar\omega}\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{q}^{+}\cdot\boldsymbol{r})\right.$$
$$\left.-\frac{1}{2}\frac{\Delta\mathscr{E}_{-}}{\hbar\omega}\exp(\mathrm{i}\omega t - \mathrm{i}\boldsymbol{q}^{-}\cdot\boldsymbol{r})\right]\psi_0\,, \quad (43)$$
$$\Delta\mathscr{E}_{\pm} = 2\pi\sqrt{\pi}\,mc^2\xi_x\,\frac{d}{\lambda}\exp\left[-\frac{(q_x^{\pm})^2 d^2}{4}\right] .$$

The density of the electron beam is

$$\rho = \mathrm{i}\hbar\psi^{*}\frac{\partial}{\partial t}\psi + \mathrm{c.c.}$$
$$= \frac{1}{2}\rho_0\left[1 + \frac{1}{2}\frac{\Delta\mathscr{E}_{+}}{\hbar\omega}\frac{\mathscr{E}+\hbar\omega}{\mathscr{E}}\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{q}^{+}\cdot\boldsymbol{r})\right.$$
$$\left.-\frac{1}{2}\frac{\Delta\mathscr{E}_{-}}{\hbar\omega}\frac{\mathscr{E}-\hbar\omega}{\mathscr{E}}\exp(\mathrm{i}\omega t - \mathrm{i}\boldsymbol{q}^{-}\cdot\boldsymbol{r})\right] + \mathrm{c.\,c.}\,, \quad (44)$$

in the linear approximation with respect to field. If an electron interacts with the wave under the optimal conditions:

$$q_x = 0\,, \quad q_z = \frac{\omega}{v_z} = \frac{\omega}{c}\,n\,,$$
$$\Delta q_z = 0\,, \quad \Delta q_x = \frac{1}{2}\frac{\hbar\omega^2(n^2-1)}{\mathscr{E}v_x}\,, \quad (45)$$

then

$$\rho = \rho_0\left[1 + 2\frac{\Delta\mathscr{E}}{\hbar\omega}\sin(\Delta q_x x)\sin(q_z z - \omega t)\right.$$
$$\left.+\frac{\Delta\mathscr{E}}{\mathscr{E}}\cos(\Delta q_x x)\cos(q_z z - \omega t)\right] . \quad (46)$$

Here the quantity $\Delta\mathscr{E}$ is specified by the expressions (14) and (19): the terms of order of $\hbar\omega/\mathscr{E}$ are omitted.

We will now analyse the expression in brackets in Eqn (46). The second term is proportional to the difference of amplitudes of emission and absorption of a photon, and describes the quantum modulation of the electron beam. Interestingly enough, this difference depends both on the asymmetric part of the loss Δq_x and on the distance to the point of observation along the x axis. In the region $\Delta q_x x \ll 1$ the modulation is classical in nature and the expression for the density of electrons [Eqn (46)] coincides with Eqn (29). Since the second term is proportional to x in this limit, the associated modulation can be called the klystron modulation. In the region $\Delta q_x x \sim 1$ the difference between the amplitudes of emission and absorption peaks, and the classical modulation goes into a quantum one with depth $2\Delta\mathscr{E}/\hbar\omega$. Subsequently the factor $\sin(\Delta q_x x)$ causes the spatial modulation of the density of the electron beam with the period $L = 2\pi/\Delta q_x$ along the x axis, whereas the depth of modulation remains constant.

The amplitude of modulation of the density of electrons, owing to the third term, is classical in nature and is well below the amplitude in the previous case:

$$\frac{\Delta\mathscr{E}}{\mathscr{E}} \ll \frac{\Delta\mathscr{E}}{\hbar\omega} . \tag{47}$$

As is noted in Section 2.3, it is associated with the rearrangement of the density of electrons in the presence of the laser radiation, and the rearrangement remains in the drift region $x > d$. The quantum correlations cause an additional spatial modulation of the density of electrons with the period $L = 2\pi/\Delta q_x$ along the x axis. The modulation of the electron current occurs similarly, and is given by

$$\boldsymbol{j} = \boldsymbol{j}_0 + \boldsymbol{j}_1 , \tag{48}$$

where $\boldsymbol{j}_0 = e\rho_0 v$ is its initial value and

$$\boldsymbol{j}_1 = e\rho_0\left[2v\frac{\Delta\mathscr{E}}{\hbar\omega}\sin(\Delta q_x x)\sin(q_z z - \omega t) + \frac{\boldsymbol{q}c^2}{\omega}\frac{\Delta\mathscr{E}}{\mathscr{E}}\cos(\Delta q_x x)\cos(q_z z - \omega t)\right] . \tag{49}$$

The gap between the regions of the classical and quantum modulations can be determined from the condition $\Delta q_x x_1 = 1$:

$$x_1 = \frac{\lambda}{\pi}\frac{\mathscr{E}}{\hbar\omega}\frac{\beta_x}{n^2 - 1} . \tag{50}$$

The expressions (46) and (49) are true for

$$\frac{\Delta\mathscr{E}}{\hbar\omega} \ll 1 . \tag{51}$$

Note in conclusion that the experimental and theoretical research of the quantum modulation was initiated in Refs [40, 41]. A comprehensive review of the results is given in Ref. [42].

2.5 Modulation of a polarised electron beam

The expressions for the depths of modulation of the current and density of an electron beam [Eqns (46), (48), and (49)] are generally applicable.

Using these formulae, we will evaluate the contribution of the magnetic moment of an electron in the modulation of the electron beam. Since the quantity $\Delta\mathscr{E} \sim \mu H$ in the case of a spin interaction (here $\mu = e\hbar/2mc$ is the magnetic moment of the electron and H is the magnetic field strength), the amplitudes of terms responsible for the klystron modulation are classical in nature. Interestingly enough, the Planck constant $\hbar$ enters only into the asymmetric part of the loss and has no effect over distances $x \sim x_1$.

We shall find the expressions for the density and current of a polarised electron beam from the Dirac equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \left[c\boldsymbol{\alpha}\cdot\left(\hat{\boldsymbol{q}} - \frac{e}{c}\boldsymbol{A}\right) + mc^2\beta\right]\psi . \tag{52}$$

In order to extract the pure spin contribution we assume that the electromagnetic wave is polarised along the y axis such that

$$A_y = \int A_y(\boldsymbol{q}')\,q'_z\delta\left(\frac{\omega^2}{c^2}n^2 - |\boldsymbol{q}'|^2\right) \times \exp[\mathrm{i}(\boldsymbol{q}'\cdot\boldsymbol{r} - \omega t)]\,\mathrm{d}q' + \text{c.c.} , \tag{53}$$

$$A_y(\boldsymbol{q}') = \frac{1}{2\sqrt{\pi}}A_{0y}d\exp\left(-\frac{q_x'^2 d^2}{4}\right) ,$$

and the electron beam moves in the xz plane.

If we solve the Dirac equation in the linear approximation with respect to the field, the wave function of the electron beam takes the form,

$$\psi = \psi_0 + \psi_+ + \psi_- , \tag{54}$$

in the region $x \gg d$. Here

$$\psi_0 = \sqrt{\frac{\rho_0}{2\mathscr{E}}}\,u\exp\left[-\mathrm{i}\left(\frac{\mathscr{E}}{\hbar}t - \frac{\boldsymbol{p}}{\hbar}\cdot\boldsymbol{r}\right)\right] \tag{55}$$

is the wave function of the initial electron beam and the terms

$$\psi_\pm = -\mathrm{i}\frac{e\pi}{2\hbar c p_x}\exp[\mathrm{i}(\boldsymbol{q}^\pm\cdot\boldsymbol{r} \mp \omega t)](\hat{p}_\pm + mc)\hat{A}_\pm(\boldsymbol{q}^\pm)\psi_0 \tag{56}$$

describe absorption and emission of a photon. The quantities $\boldsymbol{q}^\pm$ are specified by the expressions (40) and (45); the operators

$$\hat{p}_\pm = (p_\mu \pm \hbar q_\mu^\pm)\gamma^\mu , \quad \hat{A}_\pm = A_\pm^\mu\gamma_\mu , \quad A_+^\mu = [0, 0, A_y(\boldsymbol{q}), 0] ,$$
$$A_-^\mu = (A_+^\mu)^* , \quad p^\mu = (\mathscr{E}/c, \boldsymbol{p}) , \quad (q^\pm)^\mu = (\omega/c, \boldsymbol{q}^\pm)$$

are the four momenta of electrons and photons involved in absorption (+) and emission (−).

We determine the density, $\rho = \psi^+\psi$, and the current, $\boldsymbol{j} = ec\psi^+\boldsymbol{\alpha}\psi$, of the electron beam under the assumption that the polarisation matrix has the form

$$\hat{\rho} = \tfrac{1}{2}(\hat{p} + mc)(1 - \gamma^5\hat{a}) \tag{57}$$

before the interaction [Eqn (43)]. The four-dimensional vector a^μ is related to the electron polarisation vector $\boldsymbol{\zeta}$ by the equations

$$a_0 = \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{mc} , \quad \boldsymbol{a} = \boldsymbol{\zeta} + \frac{(\boldsymbol{\zeta}\cdot\boldsymbol{p})\boldsymbol{p}}{(\mathscr{E} + mc^2)m} . \tag{58}$$

Taking Eqns (54)–(57) into account we have

$$\rho = \rho_0\left[1 + \frac{\Delta E}{\mathscr{E}}\xi_y\frac{mc^2}{\mathscr{E}}\frac{a_x}{\beta_x\beta_z}\sin(\Delta q_x x)\sin\phi\right] , \tag{59}$$

$$j_x = j_{0x}\left[1 + \frac{\Delta E}{\mathscr{E}}\xi_y\frac{mc^2}{\mathscr{E}}\frac{a_0 - \beta_z a_z}{\beta_x^2\beta_z}\sin(\Delta q_x x)\sin\phi\right] , \tag{60}$$

$$j_y = 0 ,$$

$$j_z = j_{0z}\left[1 + \frac{\Delta E}{\mathscr{E}}\xi_y\frac{mc^2}{\mathscr{E}}\frac{a_x}{\beta_x\beta_z}\sin(\Delta q_x x)\sin\phi\right] .$$

Here ρ_0 and $\boldsymbol{j}_0 = e\rho_0 v$ are the initial density and current of the electron beam; the quantity $\Delta\mathscr{E}$ is specified by Eqn (19); $\xi_y = eA_{0y}/mc^2$; $\beta_{x,z} = v_{x,z}/c$; and the phase $\phi = q_z z - \omega t$.

In calculations we do not consider the terms of order of $(\Delta E/\mathscr{E})\xi_y\hbar\omega/\mathscr{E}$ which are associated with the induced magnetisation of the electron beam by the laser radiation (see Section 2.6). The cited approximation is true when

$$|\boldsymbol{a}| \gg \frac{\hbar\omega}{\mathscr{E}} . \tag{61}$$

Note also that the constraints on the angular and energy spreads of the electron beam are specified by inequalities (82). It is clear from Eqns (59) and (60) that the modulation of a polarised electron beam is specified by the asymmetric part of the loss Δq_x.

Since an actual laser beam always has an angular spread, it acquires an addition to its state, with polarisation along the x axis, upon traversing the polariser. In order to assess the possibility of extracting the spin contribution we shall give the expressions for the density and current of an electron beam in the general case of an elliptically polarised electromagnetic wave [see Eqn (11)]. As done earlier, we will determine the wave function and then the density and current of the electron beam, given by

$$\rho = \rho_0\bigg[1 + 2\frac{\Delta E}{\hbar\omega}\xi_x \sin(\Delta q_x x)\sin\phi + \frac{\Delta E}{\mathscr{E}}\xi_x \cos(\Delta q_x x) \times\cos\phi + \frac{\Delta E}{\mathscr{E}}\frac{mc^2}{\mathscr{E}}\frac{1}{\beta_x\beta_z}\sin(\Delta q_x x) \times(a_x\xi_y\sin\phi + a_y\xi_x\cos\phi)\bigg] , \tag{62}$$

$$j_x = j_{0x}\bigg[1 + 2\frac{\Delta E}{\hbar\omega}\xi_x\sin(\Delta q_x x)\sin\phi + \frac{\Delta E}{\mathscr{E}}\xi_y\frac{mc^2}{\mathscr{E}}\frac{a_0 - \beta_z a_z}{\beta_x^2\beta_z}\sin(\Delta q_x x)\sin\phi\bigg] ,$$

$$j_y = e\rho_0 c\frac{\Delta E}{\mathscr{E}}\xi_x\frac{mc^2}{\mathscr{E}}\frac{a_0-\beta_z a_z}{\beta_x\beta_z}\sin(\Delta q_x x)\cos\phi , \tag{63}$$

$$j_z = j_{0z}\bigg[1 + 2\frac{\Delta E}{\hbar\omega}\xi_x\sin(\Delta q_x x)\sin\phi + \frac{\Delta E}{\mathscr{E}}\xi_x\frac{1}{\beta_z^2}\cos(\Delta q_x x)\cos\phi + \frac{\Delta E}{\mathscr{E}}\frac{mc^2}{\mathscr{E}}\frac{1}{\beta_x\beta_z} \times\sin(\Delta q_x x)(a_x\xi_y\sin\phi + a_y\xi_x\cos\phi)\bigg] .$$

In calculations we assume that inequality (61) holds. Note that the expressions (62) and (63) go into formulae (59) and (60) for $\xi_x = 0$ and that they coincide with Eqns (46) and (48) for $a^\mu = 0$.

For interpretation of the above results it is convenient to make use of nonrelativistic quantum mechanics. In this limit the Hamiltonians of the orbital ($\hat{H}_1 = -e\boldsymbol{A}\cdot\boldsymbol{p}/mc$) and spin ($\hat{H}_2 = -\hat{\boldsymbol{\mu}}\cdot\boldsymbol{H}$, where $\hat{\boldsymbol{\mu}} = e\hbar\boldsymbol{\sigma}/2mc$ and $\boldsymbol{\sigma}$ is the Pauli matrix) interactions enter independently. Thus each contribution can be examined separately. We solve the Pauli equation [44] in the linear approximation with respect to the field and substitute the wave function ψ into the definitions of the current and density of electrons,

$$\rho = \psi^+\psi + \text{c.c.} , \tag{64}$$

$$\boldsymbol{j} = \frac{e}{2m}\psi^+\hat{\boldsymbol{p}}\,\psi + \text{c.c.} + c\,\mathrm{curl}\boldsymbol{I} . \tag{65}$$

Here $\boldsymbol{I} = (e\hbar/2mc)\psi^+\boldsymbol{\sigma}\psi$ is the magnetisation of the particle beam. Since the polarisation matrix takes the form $\hat{\rho} = (1 + \boldsymbol{\zeta\sigma})/2$ in the nonrelativistic limit, we obtain the same expressions as (62), (63), for $v/c \ll 1$.

The calculations show that the second and third terms in Eqn (62) are associated with the orbital motion, and the fourth term with the spin motion. Analysis of the projections of the current [Eqn (63)] is more complicated. It follows from Eqn (64) that the current modulation is caused both by the modulation of the density of the particle beam (the terms of the type $\psi^+\boldsymbol{p}\psi$) and by the magnetisation modulation (the terms of the type of curl$\boldsymbol{I}$). The first effect is associated with the oscillating terms in the x- and z-projections of the current. It is shown in Section 2.6 that the magnetisation breaks into oscillation due to: (a) the induced magnetisation of the electron beam; (b) the magnetisation modulation associated with the modulation of the density of electrons; and (c) the oscillation of magnetic moments of electrons about the magnetic field of the laser radiation. The contribution of the first effect is negligible [see Eqn (61)], therefore it is not considered. The effects (b) and (c) are responsible for the emergence of the y-projection of the current. Note that both effects also make a contribution to the x-projection of the current. However, the terms which describe them are cancelled, with a part of the terms responsible, as is noted above, for the density modulation.

Clearly, the spin effects are large when the quantities $\xi_{x,y}\Delta E/\mathscr{E} \sim 1$. However, the perturbation theory is not applicable for the orbital motion in this case ($\xi_{x,y}\Delta E/\hbar\omega \gg 1$). To avoid complicating the problem with the analysis of multiphoton processes, we assume that the field is polarised along the y axis and that electrons travel in the xz plane [see Eqns (59), (60)]. If the conditions are optimal, i.e., the particle beam is fully polarised along the x axis, then it follows from Eqns (62), (63), that the ratio of the x- and y-projections of the field obeys the inequality

$$\frac{\xi_x}{\xi_y} \leqslant \frac{\hbar\omega}{2\mathscr{E}}\frac{mc^2}{\mathscr{E}}\frac{a_x}{\beta_x\beta_z} . \tag{66}$$

2.6 Magnetisation of an electron beam by laser radiation

The density, current, and state of polarisation of an electron beam are changed as a result of the interaction between the beam and the laser radiation in a dielectric medium. Clearly, the last effect is associated with the magnetic field of the electromagnetic wave. It is well known that a constant magnetic field causes the magnetic moment of a particle to precess. If, in addition, the electron experiences an inelastic collision with the surrounding medium, its magnetic moment is gradually oriented along the magnetic field so that the potential energy $U = -\boldsymbol{\mu}\cdot\boldsymbol{H}$ is minimum.

The pattern is complicated when the electron interacts with the magnetic field of the laser radiation. Since the direction and magnitude of the magnetic field oscillate with a frequency ω, the magnetic moment of a particle-at-rest oscillates (not precesses) about the magnetic field at the frequency ω.

If an electron beam crosses the wave under the synchronism condition (1), the magnetic moment of each particle turns through the angle $\Delta\varphi$ which depends on the phase of the field [Eqn (11)] and on the duration of interaction $T = d/v_x$. Clearly, the magnetisation of the electron beam oscillates at the frequency ω at a fixed point ($x \gg d$, y, z) after the interaction. If an electron is involved in nonelastic processes of emission and absorption of a photon, a magnetisation of the electron beam along the magnetic field of the laser radiation results.

The magnetisation of the electron beam is determined by use of the Dirac equation (52). In the linear approximation with respect to the field, the wave function of the electron beam takes the form of Eqns (64)–(65) where the vector $\boldsymbol{A}(\boldsymbol{q})$ is specified by Eqn (12). In the same approximation the magnetisation of the electron beam is

$$\boldsymbol{I} = \frac{e}{mc}\,\psi^{+}\boldsymbol{\Sigma}\psi = \boldsymbol{I}_0 + \boldsymbol{I}_1\ , \tag{67}$$

where

$$\boldsymbol{\Sigma} = \frac{\hbar}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \quad \text{and} \quad \boldsymbol{I}_0 = \frac{e}{mc}\,\psi_0^{+}\boldsymbol{\Sigma}\psi_0$$

is the spin operator and the initial magnetisation of the electron beam, $\boldsymbol{I}_0$ is the linear correlation with respect to the field.

If the polarisation matrix of the particle beam has the form of Eqn (57) before the interaction, then the projections of the vector $\boldsymbol{I}_1$ are

$$\begin{aligned}
I_{1x} = \rho_0\mu\biggl\{ & \frac{\Delta E}{v_z p_x}\,\xi_y(1-\beta_z^2)\sin(\Delta q_x x)\sin\phi \\
& - \frac{\Delta E}{\mathscr{E}}\,\frac{mc^2}{\mathscr{E}}\,\frac{a_z - \beta_z(\boldsymbol{\beta}\cdot\boldsymbol{a})}{\beta_x\beta_z}\cos(\Delta q_x x)\cos\phi \\
& + 2\frac{\Delta E}{\hbar\omega}\,\frac{mc^2}{\mathscr{E}}\,\xi_x a_x \sin(\Delta q_x x)\sin\phi \biggr\}\ , \\
I_{1y} = \rho_0\mu\biggl\{ & \frac{\Delta E}{v_z p_x}\,\xi_x(1-\beta_z^2)\sin(\Delta q_x x)\cos\phi \\
& + \frac{\Delta E}{\mathscr{E}}\,\frac{mc^2}{\mathscr{E}}\,\frac{a_z - \beta_z(\boldsymbol{\beta}\cdot\boldsymbol{a})}{\beta_x\beta_z}\cos(\Delta q_x x)\sin\phi \\
& + 2\frac{\Delta E}{\hbar\omega}\,\frac{mc^2}{\mathscr{E}}\,\xi_x a_y \sin(\Delta q_x x)\sin\phi \biggr\}\ , \qquad (68) \\
I_{1z} = \rho_0\mu\biggl\{ & \frac{\Delta E}{\mathscr{E}}\,\xi_y \sin(\Delta q_x x)\sin\phi \\
& + \frac{\Delta E}{\mathscr{E}}\,\frac{mc^2}{\mathscr{E}}\,\frac{1}{\beta_x\beta_z}\cos(\Delta q_x x)(a_x\xi_x\cos\phi - a_y\xi_y\sin\phi) \\
& + 2\frac{\Delta E}{\hbar\omega}\,\frac{mc^2}{\mathscr{E}}\,\xi_x a_z \sin(\Delta q_x x)\sin\phi \biggr\}\ ,
\end{aligned}$$

by virtue of Eqns (53)–(56). Here $\mu = e\hbar/2mc$ is the magnetic moment of the electron, the vector $\boldsymbol{I}_0$ is related to the initial polarisation of the electron beam $\boldsymbol{\zeta}$ by the equation

$$\boldsymbol{I}_0 = \rho_0\,\mu\,\frac{mc^2}{\mathscr{E}}\left[\boldsymbol{\zeta} + \frac{\boldsymbol{p}(\boldsymbol{\zeta}\cdot\boldsymbol{p})}{(\mathscr{E}+mc^2)m}\right]\ . \tag{69}$$

By comparing Eqns (69) and (58) we have

$$\boldsymbol{I}_0 = \rho_0\left(\mu\frac{mc^2}{\mathscr{E}}\right)\boldsymbol{a}\ .$$

The vectors $\boldsymbol{q}$, $\Delta\boldsymbol{q}$, and the quantity ΔE are specified by the expressions (45) and (19); the dimensionless parameters $\xi_{x,y}$ are $\xi_{x,y} = eA_{0x,y}/mc^2$.

We will now analyse the above formulae. At first we suppose that the electron beam is not polarised before the interaction: $|\boldsymbol{\zeta}| = 0$. Analysis of the first term in Eqn (68) shows that the asymmetric part of the loss Δq_x causes the induced orientation of magnetic moments of electrons along the magnetic field of the wave [Eqn (11)], as well as along the magnetic field $\boldsymbol{H} = [v, \partial\boldsymbol{A}/\partial t]/c^2$, which appears when we pass to the frame of reference moving together with the initial electron.

If the initial polarisation of electrons is not zero, $|\boldsymbol{\zeta}| \neq 0$, then, after the interaction with the laser radiation, the magnetisation at a fixed point of observation oscillates at the frequency ω, first, due to oscillation of the magnetic moment about the magnetic field of the wave [the second terms in Eqn (68)] and, second, due to the modulation of the density of the electron beam [the third term in Eqn (68)]. The expressions (68) are true when the angular and energy spreads of the electron beam obey inequalities (82). The spin current ($\boldsymbol{j}_{\text{sp}} = c\,\text{curl}\,\boldsymbol{I}$) and the magnetic field ($\boldsymbol{H} = 4\pi\boldsymbol{I}$) which occur in the electron beam can readily be determined by means of the formulae for magnetisation. Note that the terms responsible for the spin current arise automatically when expressions (60) and (63) are determined. However, they are omitted since their contribution is small under the condition (6). Several effects associated with the spin current will be considered in Sections 2.11–2.13.

2.7 Accounting for spreads in frequencies, energies, and angles

We shall now examine the influence of the angular, energy, and frequency spreads of electron and photon beams on the effects we considered in Sections 2.3–2.6. Let the central axis of an electron beam (the z' axis) lie in the xz plane at an angle θ to the z axis, and let its momentum distribution be Gaussian:

$$f(\boldsymbol{p}) = \left(\frac{4\ln 2}{\pi}\right)^{3/2}\frac{1}{\Delta_\perp^2\Delta_\parallel} \times \exp\left[-4\ln 2\,\frac{(p_{z'}-p_0)^2}{\Delta_\parallel^2} - 4\ln 2\,\frac{p_{x'}^2+p_y^2}{\Delta_\perp^2}\right] \tag{70}$$

(the axes y and y' of both coordinate systems coincide). The widths of the energy and angular spreads of such an electron beam are $\Delta = v_0\Delta_\parallel$ and $\delta = \Delta_\perp/p_0$, respectively. If the duration of the electromagnetic wave is τ, its spread in frequencies is specified by means of the formula

$$g(\omega) = \frac{\tau}{2\sqrt{\pi}}\exp\left[-\frac{(\omega-\omega_0)^2\tau^2}{4}\right]\ . \tag{71}$$

Since the photon wave vector distribution is specified by the expression (12), the widths of spreads of photons in angles and frequencies are $\delta_\phi = \lambda\sqrt{2\ln 2}/\pi nd$ (see Section 2.2) and $\Delta\omega = \sqrt{8\ln 2}/\tau$, respectively. To simplify the analysis we will average the expressions for the density of the particle beam [Eqns (29), (46)] over the spread [Eqn (70)]. Let the average momentum p_0 satisfy the synchronism condition $\omega - q_{z'}v_0 = 0$. Taking into account the fact that $|p_{z'} - p_0|$, $|p_{x'}|$, $|p_y| \ll p_0$, we can expand the projections [Eqn (17)] of the wave vector $\boldsymbol{q}$ into the Taylor series such that

$$q_x = \frac{\omega n}{cp_0} p_{x'} - \frac{\omega}{v_0 p_0 \sin\theta} \left(\frac{mc^2}{\mathscr{E}_0}\right)^2 (p_{z'} - p_0) ,$$

$$q_z = \frac{\omega}{v_0 \cos\theta} \tag{72}$$

[it is assumed that $q_x(0, 0, p_0) = 0$].

By substituting the expressions (72) into (27), (43), and averaging over momenta we have

$$\rho_{\text{кл}} = \rho_0 \left\{ 1 + R \left[x \frac{\Delta\mathscr{E}}{\mathscr{E}} \frac{\omega}{v_x} (n^2 - 1) \sin(q_z z - \omega t) + \frac{\Delta\mathscr{E}}{\mathscr{E}} \cos(q_z z - \omega t) \right] \right\} , \tag{73}$$

$$\rho_{\text{кв}} = \rho_0 \left\{ 1 + R \left[\frac{\Delta\mathscr{E}}{\hbar\omega} \sin(\Delta q_x x) \sin(q_z z - \omega t) + \frac{\Delta\mathscr{E}}{\mathscr{E}} \cos(\Delta q_x x) \cos(q_z z - \omega t) \right] \right\} . \tag{74}$$

Here the quantities $\Delta\mathscr{E}$, Δq_x, q_z are specified by means of formulae (14), (19), (45), where the quantities v, $\boldsymbol{p}$, $\mathscr{E}$ are changed for v_0, $\boldsymbol{p}_0$, $\mathscr{E}_0$. The factor R is given by

$$R = \sqrt{2}\, \delta_\phi \frac{p_0}{D_{\text{pe}}} \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 n\beta_0 \sin\theta \exp\left(-\frac{x^2}{d^2} \frac{D_{\text{e}}^2}{D_{\text{pe}}^2} \right) , \tag{75}$$

where

$$D_{\text{e}} = \left[\varDelta_{\|}^2 + \varDelta_{\perp}^2 n^2 \beta_0^2 \sin^2\theta \left(\frac{\mathscr{E}_0}{mc^2}\right)^4 \right]^{1/2} \tag{76}$$

is the effective width of the electron beam, and

$$D_{\text{pe}} = \left[D_{\text{e}}^2 + \delta_\phi^2 2n^2 p_0^2 \beta_0^2 \sin^2\theta \left(\frac{\mathscr{E}_0}{mc^2}\right)^4 \right]^{1/2} \tag{77}$$

is the combined effective width of the electron and photon beams.

If the width of the angular spread of the photon beam is small,

$$\delta_\phi \ll \min\left\{ \frac{1}{\sqrt{2}} \frac{\varDelta_\perp}{p_0} , \ \frac{1}{\sqrt{2}} \frac{1}{n\beta_0 \sin\theta} \left(\frac{mc^2}{\mathscr{E}_0}\right)^2 \frac{\varDelta_\|}{p_0} \right\} , \tag{78}$$

then $D_{\text{pe}} \approx D_{\text{e}}$ and $R \sim \exp(-x^2/d^2)$. Since the drift distance is $x \gg d$, the modulation effect is exponentially low in this case. If

$$\delta_\phi \gg \min\left\{ \frac{1}{\sqrt{2}} \frac{\varDelta_\perp}{p_0} , \ \frac{1}{\sqrt{2}} \frac{1}{n\beta_0 \sin\theta} \left(\frac{mc^2}{\mathscr{E}_0}\right)^2 \frac{\varDelta_\|}{p_0} \right\} , \tag{79}$$

the factor R is

$$R = \exp\left[-\frac{x^2}{\lambda^2} \frac{\pi^2}{4\ln 2} \frac{\varDelta_\|^2 + \varDelta_\perp^2 n^2 \beta_0^2 \sin^2\theta (\mathscr{E}_0/mc^2)^4}{p_0^2 \beta_0^2 \sin^2\theta (\mathscr{E}_0/mc^2)^4} \right] . \tag{80}$$

Clearly, the modulation effect is not small when the index of the exponent (80) is less than or of the order of unity. We can now determine the constraints on the energy and angular spreads of the electron beam:

$$\frac{\varDelta}{\mathscr{E}_0} \lesssim \frac{2\sqrt{\ln 2}}{\pi} \frac{\lambda}{x} \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 \beta_0^3 \sin\theta ,$$

$$\delta \lesssim \frac{2\sqrt{\ln 2}}{\pi} \frac{\lambda}{x} \frac{1}{n} . \tag{81}$$

The larger the drift distance x, the progressively greater will be the requirements on the quality of the electron beam. Since the quantum modulation occurs over distances $x \sim 1/\Delta q_x$, we obtain very severe constraints on the spreads for the effect to be observed:

$$\frac{\varDelta}{\mathscr{E}_0} \lesssim 2\sqrt{\ln 2}\, \beta_0^2 (n^2 - 1) \frac{\hbar\omega}{\mathscr{E}_0} \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 ,$$

$$\delta \lesssim 2\sqrt{\ln 2} \frac{n^2 - 1}{n\beta_0 \sin\theta} \frac{\hbar\omega}{\mathscr{E}_0} . \tag{82}$$

Let the interaction between the electron beam and light occur in a gaseous atmosphere ($n = 1 + \Delta n$, where $\Delta n \ll 1$). In this case $\mathscr{E}_0 \gg mc^2$, $\theta \approx mc^2/\mathscr{E}_0$, and the angular spread of the electron beam has a dominant role in the cutting factor [Eqn (80)]. The factor R can suitably be rewritten for the further analysis as

$$R = \exp(-\omega^2 T^2) , \qquad \frac{T = xn\delta}{4\sqrt{\ln 2}\, c} . \tag{83}$$

After the interaction, electrons reach the point of observation x at different instances because of the angular spread of the electrons. The time T is proportional to the maximal difference between these times. The factor R is not small if

$$T \leqslant T_0 , \tag{84}$$

where $T_0 = 2\pi c/\omega_0$ is the average period of oscillation of the electromagnetic wave. Since the duration of the wave is $\tau \gg T_0$,

$$T \ll \tau . \tag{85}$$

By averaging the expressions (73) and (74) over the spread [Eqn (71)] under the condition (85), we obtain the expression

$$R = \exp\left[-\omega_0^2 T^2 - \frac{(t - n_0 z/c)^2}{\tau^2} \right] \tag{86}$$

[for simplicity we do not take into account the dispersion of the gaseous atmosphere $n(\omega) = n_0$]. Clearly, under condition (84) the cutting factor coincides with the envelope of the electromagnetic wave $R = \exp[-(t - n_0 z/c)^2/\tau^2]$. If the wave duration is greater than the time for which the modulation effect is observed ($\tau \gg \Delta t$) and if condition (81) is satisfied, then $R = 1$ and the depth of modulation is maximum.

2.8 The classical and quantum theories of the Cherenkov klystron

We shall consider the possibility of transmitting the kinetic energy of an electron beam to a wave on the basis of the SCE. The change in the energy of a single particle after the interaction with the field [Eqn (11)] is determined in the linear approximation with respect to the field in Section 2.2. Here we extend this result to the electron beam.

First we consider the case of a spatially homogeneous particle beam before the interaction. It follows that the same number of electrons falls within each phase of the field. Since the sign of expression (13) depends on the phase ϕ, half the particles of the electron beam are accelerated ($\Delta\mathscr{E} > 0$), and the other half are decelerated ($\Delta\mathscr{E} < 0$). This presents significant problems as regards accelerating the electrons so as to amplify the electromagnetic wave. In the first case only a small fraction of electrons which fall within a proper phase of the wave is maximally accelerated. In the second case the effect is absent since the number of decelerated electrons is equal to the number of accelerated electrons.

The last result can be illustrated vividly by determining the energy loss (W) for the electron beam,

$$W = \int \boldsymbol{j}\cdot\boldsymbol{E}\,\mathrm{d}v\,. \tag{87}$$

Integration is performed over the whole space where the interaction between the field and electrons occurs. If we substitute the expression for the homogeneous electron beam $\boldsymbol{j} = e\rho_0 v$ and take into account the fact that the electric field strength is $\boldsymbol{E} = -c^{-1}\partial\boldsymbol{A}/\partial t$ [see Eqn (11)], then we obtain $W = 0$. The calculation shows that the electron beam must be inhomogeneous, $\rho = \rho(\boldsymbol{r}, t)$, for an exchange of energies between the wave and electrons to occur.

If the period of its spatial inhomogeneity is of the order of the emission wavelength λ, then the balance between the electrons falling within the decelerating and accelerating phases is violated. In order to amplify the electromagnetic wave the initial conditions are to be chosen so that the number of decelerated electrons is greater than the number of accelerated electrons. The scheme of such an amplifier is well known in radiophysics and bears the name klystron [39]. It was considered for the first time in the optical range of frequencies by A N Skrinskii and N A Vinokurov for the undulator version of the amplifier [45]. An interesting method for increasing the efficiency of the klystron was proposed in Ref. [46].

We shall now consider the feasibility of the Cherenkov klystron. It is shown in Sections 2.3 and 2.4 that the current and density of an electron beam are modulated by the SCE. Since the oscillating terms of the currents [Eqns (30) and (48)] are proportional to the outer field, the exchange of energies between electrons and the field [Eqn (11)] occurs in the second approximation with respect to the wave only. Since the modulation effect can be both classical ($x \ll x_1$) and quantum ($x \gg x_1$) in nature, we shall find the gain in both cases.

To determine the gain of the classical Cherenkov klystron we use the closed consistent system of the Maxwell–Vlasov equations,

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0\,, \tag{88}$$

$$\boldsymbol{j} = e\rho_0\int (vf)\cdot\mathrm{d}\boldsymbol{p}\,, \tag{89}$$

$$\operatorname{curl}\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} + \frac{n^2}{c}\frac{\partial \boldsymbol{E}}{\partial t}\,, \qquad \operatorname{curl}\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \tag{90}$$

(it is assumed that the magnetic permeability of the medium is $\mu = 1$).

In what follows we also use the system of equations which results from combining Eqn (88) with the wave equation

$$\nabla^2\boldsymbol{A} - \frac{n^2}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j}\,, \tag{91}$$

where $\boldsymbol{A}$ is the vector potential of the electromagnetic wave. As a rule the amplitude of the field $\boldsymbol{A}_0$ [Eqn (2)] depends weakly on the $\boldsymbol{r}$ coordinate in the amplifying medium. In this case the left-hand side of Eqn (91) can be accelerated and written as

$$\nabla^2\boldsymbol{A} - \frac{n^2}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} \approx -\mathrm{i}\exp[\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})](\boldsymbol{k}\cdot\vec{\nabla})\boldsymbol{A}_0 + \mathrm{c.c.}$$

On solving Eqn (91) we can find the gain for the amplitude of the electromagnetic wave Γ_A. In what follows we shall always determine the gain for the intensity of the electromagnetic wave Γ, which is related to Γ_A by $\Gamma = 2\Gamma_A$.

In the analysis of different versions of the free electron laser we restrict ourselves to the linear approximations of the amplified wave and the gain (the latter implies that we neglect the dependence of the amplitude on distance and time). The general remarks having been made, we can return to calculations of the Cherenkov klystron.

Let us study the scheme in Fig. 1 for $\boldsymbol{H}_0 = 0$. An electron beam crosses an electromagnetic wave propagating along the z axis at an angle θ and then moves in the drift region $x > d$. In this region the current of the electron beam is specified by the expression (26). At the distance $x = x_0$ the same beam of light is again directed at the electron beam by means of two mirrors R_1 and R_2 and is amplified or absorbed.

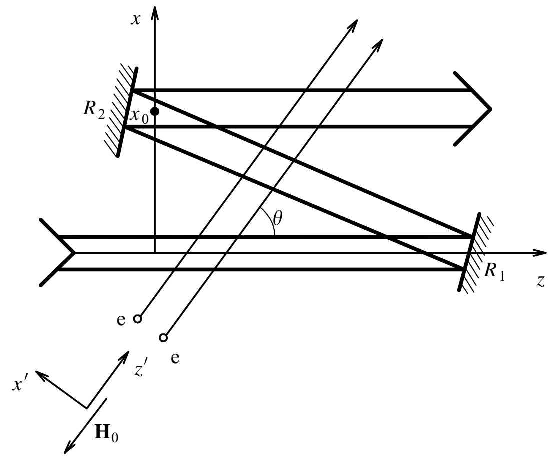

Figure 1.

The vector potential of the amplified wave has the form

$$A_x = \int A(k_x)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t - \mathrm{i}k_x x_0 + \mathrm{i}\phi)\,\mathrm{d}k_x + \text{c.c.}\,,$$

$$A_z \approx 0\,, \tag{92}$$

$$A(k_x) = -\frac{\mathrm{i}}{4\sqrt{\pi}}A_{1x}d\exp\left(-\frac{k_x^2 d^2}{4}\right),$$

$$k_z = \left(\frac{\omega^2}{c^2}n^2 - k_x^2\right)^{1/2}, \qquad k_y = 0\,,$$

where the phase ϕ is a function of the distance R_1R_2 between the mirrors, and the amplitude A_{1x} varies slowly with the z coordinate. As in the case of Eqn (11) the Fourier transform is chosen so that the beam of light has the Gaussian envelope of width d in the plane $z = 0$. (The diffraction divergence of the light beam is assumed to be small and its width is assumed to be close to a constant).

The simplest way to find the gain for a spatially inhomogeneous wave is to use the energy relations. By multiplying Eqn (90) by $\boldsymbol{H}$ and $\boldsymbol{E}$ and summing them up we have

$$\frac{1}{8\pi}\frac{\partial}{\partial t}(n^2|\boldsymbol{H}|^2 + |\boldsymbol{H}|^2) = -\boldsymbol{j}\cdot\boldsymbol{E} - \frac{c}{4\pi}\,\mathrm{div}(\boldsymbol{E}\times\boldsymbol{H})\,. \tag{93}$$

If we integrate both sides of this inequality over the space between two parallel planes at a distance dz from each other and neglect the rapidly oscillating terms, we have

$$\mathrm{d}P = P(z_2) - P(z_1) = -\,\mathrm{d}z\int_{-b/2}^{b/2}\mathrm{d}y\int_{-\infty}^{+\infty}\mathrm{d}x\,\overline{\boldsymbol{j}\cdot\boldsymbol{E}}\,, \tag{94}$$

where,

$$P = \frac{c}{4\pi}\int_{-b/2}^{b/2}\mathrm{d}y\int_{-\infty}^{+\infty}\mathrm{d}x\,(\boldsymbol{E}\times\boldsymbol{H})\cdot\boldsymbol{n} \tag{95}$$

is the flux of the energy of the electromagnetic wave, the vector $\boldsymbol{n}$ is directed along the z axis. Since there is no feedback between the amplified wave [Eqn (92)] and the electron beam [Eqn (26)], on passing through the interval $[0, z]$ the time averaged flux of the electromagnetic wave takes the form

$$P = P_0\left(1 + \frac{1}{2}\Gamma z\right)^2, \tag{96}$$

where P_0 is the energy flux through the plain $z = 0$ and

$$\Gamma = -\frac{1}{\sqrt{P_0P}}\int_{-b/2}^{b/2}\mathrm{d}y\int_{-\infty}^{+\infty}\mathrm{d}x\,\overline{\boldsymbol{j}\cdot\boldsymbol{E}}\,. \tag{97}$$

If $\Gamma z \ll 1$, then the wave is amplified linearly: $P = P_0(1 + \Gamma z)$, where the amplification factor is specified by Eqn (97). On substituting the expression for the current [Eqn (28)] and for the energy flux

$$P_0 = c\,\frac{\sqrt{2\pi}}{16\pi}\left(\frac{\omega}{c}\right)^2 A_{0x}^2 nbd \tag{98}$$

into Eqn (97), we have

$$\Gamma = 4\pi^2\sqrt{2\pi}\,\rho_0 e^2\,\frac{d}{\lambda}\left(\frac{c}{\omega}\right)^2\frac{1}{nc^2}\,\mathrm{e}^{\mathrm{i}\phi}$$
$$\times\int v_x\left(q_x\frac{\partial f_0}{\partial P_x} + q_z\frac{\partial f_0}{\partial P_z}\right)\exp\left(\mathrm{i}q_xx_0 - \frac{q_x^2d^2}{2}\right)\mathrm{d}\boldsymbol{p} + \mathrm{c.c.} \tag{99}$$

Since the amplitude A_{1x} of the field [Eqn (92)] varies slowly with the coordinate z, the flux P is specified by means of formula (98) with A_{1x} in place of A_{0x}. The projection of the wave vector $\boldsymbol{q}$ is specified by Eqn (17).

The analysis of the previous section shows that the integral in Eqn (99) is not small when inequality (79) holds. On substituting the distribution function (70) and the expansion (72) into Eqn (99), and integrating with respect to the variables $P_{x'}$, P_y, $P_{z'}$ with regard to Eqn (79) we get

$$\Gamma = 8\pi^2\sqrt{2\pi}\,\rho_0r_0x_0\,\frac{d}{\lambda n}\,\frac{mc^2}{\mathscr{E}_0}$$
$$\times(n^2-1)\sin\phi\exp\left\{-\frac{\pi^2}{4\ln 2}\left(\frac{x_0}{\lambda}\right)^2\right.$$
$$\left.\times\left[\left(\frac{\Delta_\perp}{P_0}\right)^2 n^2\left(\frac{\Delta_\parallel}{P_0}\right)^2\left(\frac{mc^2}{\mathscr{E}_0}\right)^4\frac{1}{\beta_0^2\sin^2\theta}\right]\right\}. \tag{100}$$

Here $r_0 = e^2/mc^2$ is the classical electron radius.

Clearly, the gain is not small if the phase $\phi = \pi/2$, and

$$x_0 < \min\left\{\frac{2\sqrt{\ln 2}}{\pi}\,\lambda\beta_0\cos\theta\,\frac{P_0}{\Delta_\perp}\,,\right.$$
$$\left.\frac{2\sqrt{\ln 2}}{\pi}\,\lambda\sin\theta\left(\frac{\mathscr{E}_0}{mc^2}\right)^2\frac{P_0}{\Delta_\parallel}\right\}. \tag{101}$$

The angular spread of the electron beam gives rise to the principal constraint on the drift distance x_0 for relativistic particles ($\mathscr{E}_0 \gg mc^2$). In Section 2.9 we will discuss how the effect of the angular spread can be neutralised by a constant magnetic field. If $x_0 = \lambda\sqrt{2\ln 2}\,P_0/\pi\Delta_\perp n$, then the gain is maximum:

$$\Gamma = 45\rho_0r_0\lambda\,\frac{d}{\lambda}\,\frac{P_0}{\Delta_\perp}\,\frac{mc^2}{\mathscr{E}_0}\,\frac{n^2-1}{n}\,. \tag{102}$$

We shall now determine the gain of the quantum klystron. If we substitute the expressions for the current [Eqn (48)] into Eqn (97) and average over the spread [Eqn (70)] with regard to Eqn (79), then

$$\Gamma = 8\pi\sqrt{2\pi}\,\rho_0r_0\beta_0d\,\frac{mc^2}{\hbar\omega}\sin\theta$$
$$\times\frac{1}{n}\sin(\Delta q_xx_0)\sin\phi\exp\left\{-\frac{\pi^2}{4\ln 2}\left(\frac{x_0}{\lambda}\right)^2\right.$$
$$\left.\times\left[\left(\frac{\Delta_\perp}{P_0}\right)^2 n^2 + \left(\frac{\Delta_\parallel}{P_0}\right)^2\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\frac{1}{\beta_0^2\sin^2\theta}\right]\right\}. \tag{103}$$

Here $\Delta q_x = \hbar\omega(n^2-1)/(2\mathscr{E}_0v_0\sin\theta)$. Since the quantum modulation of the particle beam occurs over distances $x_0 \sim 1/\Delta q_x$, very severe constraints on the quality of the electron beam [Eqn (82)] result.

Note also that the perturbation theory we use to determine the currents [Eqns (26), (48)] is true for field inequalities (31), (51) specified. In the classical limit ($\hbar \to 0$), formula (103) goes into Eqn (100).

2.9 Theory of the Cherenkov klystron in a constant magnetic field

The negative role of the angular spread of an electron beam in the Cherenkov klystron can be neutralised by a constant magnetic field, which is applied in the direction of motion of the electrons (see Fig. 1). The gain for the electromagnetic wave is determined using the system of equations (88), (90). In the case considered, the Lorentz force is

$$\boldsymbol{F} = e\boldsymbol{E} + \frac{e}{c}[v\cdot(\boldsymbol{H} + \boldsymbol{H}_0)]\,, \tag{104}$$

where the strength of the constant magnetic field $\boldsymbol{H}_0(0, 0, -H_0)$ is opposite to the z axis.

In determining the electron distribution function, we take into account the constant magnetic field exactly and the modulating wave [Eqn (11)] in the first approximation.

In a constant magnetic field the solution to Eqn (88) takes the form

$$f_1 = f_0(p_{0x'}, p_{0y}, p_{0z'}) . \tag{105}$$

Here the function f_0 is specified by the expression (70). The equations

$$\begin{aligned} p_{0x'} &= p_{x'} \cos \Omega t - p_y \sin \Omega t , \\ p_{0y} &= p_y \cos \Omega t + p_{x'} \sin \Omega t , \\ p_{0z'} &= p_z \end{aligned} \tag{106}$$

are the characteristics of Eqn (88); $\Omega = \Omega_0 mc^2/\mathscr{E}$; $\Omega_0 = |e| H_0/mc$ is the Larmor frequency; and

$$\begin{aligned} \mathscr{E} &= [(mc^2)^2 + c^2(p_{0x'}^2 + p_{0y}^2 + p_{0z'}^2)]^{1/2} \\ &= [(mc^2)^2 + c^2|\boldsymbol{p}|^2]^{1/2}. \end{aligned}$$

In the first approximation with respect to the field [Eqn (11)] we have

$$f = f_1 + f_2 ,$$

where

$$\begin{aligned} f_2 = & -2\pi e \sin\theta \frac{\partial f_0}{\partial p_{z'}} \int J_0\left(\frac{v_\perp q'_{x'}}{\Omega}\right) q'_z \\ & \times \delta\left(\frac{\omega^2}{c^2} n^2 - |\boldsymbol{q}'|^2\right) \delta(\omega - q'_{z'} v_{z'}) \\ & \times E_{q'} \exp\left(\mathrm{i}\boldsymbol{q}' \boldsymbol{\cdot} \boldsymbol{r} - \mathrm{i}\omega t - \mathrm{i} q'_{x'} \frac{v_\perp}{\Omega} \sin\varphi\right) \mathrm{d}\boldsymbol{q}' + \text{c.c.} \end{aligned} \tag{107}$$

is the Fourier transform of the electric field $E_{q'} = \mathrm{i}\omega A_x(\boldsymbol{q}')/c$ [see Eqn (12)]; θ is the angle between the axes z' and z; $\tan\varphi = p_y/p_x$; $J_0(a)$ is the Bessel function of zero order. In Eqn (107) only terms responsible for the stimulated Cherenkov effect are taken into account:

$$\omega - q_{z'} v_{z'} = 0 . \tag{108}$$

Clearly, the difference between Eqns (108) and Eqn (15), which we used to derive (102), is that the former involves only one projection of the velocity $v_{z'}$. The calculations show that this fact is crucial in neutralising the angular spread of electrons in a constant magnetic field.

Let us determine x — the projection of the current of the electron beam in the region $x \gg d$. Remembering that $v_x = v_{x'} \cos\theta + v_{z'} \sin\theta$ and retaining only the terms oscillating at the frequency ω, we have

$$\begin{aligned} j_x = & -\pi e^2 \rho_0 \sin\theta \int \left(1 + \frac{v_{x'}}{v_{z'}} \cot\theta\right) \frac{\partial f_0}{\partial p_{z'}} J_0\left(\frac{v_\perp q_{x'}}{\Omega}\right) \\ & \times E_q \exp\left(\mathrm{i}\boldsymbol{q} \boldsymbol{\cdot} \boldsymbol{r} - \mathrm{i}\omega t - \mathrm{i} q_{x'} \frac{v_\perp}{\Omega} \sin\varphi\right) \mathrm{d}\boldsymbol{p} + \text{c.c.} , \end{aligned} \tag{109}$$

where

$$\begin{aligned} & q_{z'} = \frac{\omega}{v_{z'}} , \quad q_{x'} = \frac{\omega}{v_{z'}} \sqrt{n^2 \beta_{z'}^2 - 1} , \\ & v_\perp = \frac{p_\perp c^2}{\mathscr{E}} , \quad p_\perp = \sqrt{p_{x'}^2 + p_y^2} . \end{aligned} \tag{110}$$

By substituting these expressions into Eqn (97) we find the gain of the Cherenkov klystron,

$$\begin{aligned} \Gamma = & -2\mathrm{i}\pi\sqrt{\pi}\, \rho_0 e^2 x_0 d \, \frac{\sin\theta}{nc} \, \mathrm{e}^{-\mathrm{i}\phi} \\ & \times \int \left(1 + \frac{v_{x'}}{v_{z'}} \operatorname{ctg}\theta\right) \frac{\partial q_x}{\partial p_{z'}} J_0\left(\frac{v_\perp q_{x'}}{\Omega}\right) \\ & \times f_0 \exp\left(\mathrm{i} q_x x_0 - \mathrm{i} q_{x'} \frac{v_\perp}{\Omega} \sin\varphi\right) \\ & \times \exp\left(-\frac{q_x^2 d^2}{2}\right) \mathrm{d}\boldsymbol{p} + \text{c.c.} , \end{aligned} \tag{111}$$

where $q_x = q_{x'} \cos\theta + q_{z'} \sin\theta$, and the phase ϕ is a function of the distance $R_1 R_2$ between the mirrors (see Fig. 1).

If the average momentum p_0 of the electron beam is chosen such that $q_x(p_0) = 0$, then the wave vector is

$$q_x \approx \frac{\omega}{p_0 v_0 \sin\theta} \left(\frac{mc^2}{\mathscr{E}_0}\right)^2 (p_{z'} - p_0) + \frac{\omega c^2}{\mathscr{E}_0^2 v_0 \sin\theta} p_\perp^2 .$$

By substituting this expansion into Eqn (111) and integrating with respect to the variables $p_{z'}$, $p_\perp$, φ we get

$$\begin{aligned} \Gamma = & \, 8\pi^2 \sqrt{2\pi}\, \rho_0 r_0 x_0 \frac{d}{\lambda} \frac{1}{n\beta_0^2} \\ & \times \left(\frac{mc^2}{\mathscr{E}_0}\right)^3 \sin\phi \, I_0\left[\frac{1}{8\ln 2} \left(\frac{\omega \Delta_\perp n \sin\theta}{\Omega_0 mc}\right)^2\right] \\ & \times \exp\left\{-\left(\frac{x_0}{\lambda}\right)^2 \frac{\pi^2}{4\ln 2} \left[\frac{\Delta_\parallel}{p_0} \left(\frac{mc^2}{\mathscr{E}_0}\right)^2 \frac{1}{\beta_0 \sin\theta}\right]^2 \right. \\ & \left. - \frac{1}{8\ln 2} \left(\frac{\omega \Delta_\perp n \sin\theta}{\Omega_0 mc}\right)^2 \right\} . \end{aligned} \tag{112}$$

Here $I_0(a)$ is the modified Bessel function; $v_0 = p_0 c^2/\mathscr{E}_0$; $\mathscr{E}_0 = [(mc^2)^2 + c^2 p_0^2]^{1/2}$.

Note also that the term proportional to $(p_{z'} - p_0)^2$ is omitted in the Taylor series expansion of q_x since its contribution is negligible.

Expression (112) is valid for

$$x_0 < \lambda \frac{2\ln 2}{\pi\beta_0} \left(\frac{p_0}{\Delta_\perp}\right)^2 \sin\theta . \tag{113}$$

Clearly, this constraint on x_0, associated with the angular spread of the electron beam $\delta = \Delta_\perp/p_0$, is considerably weaker than that defined by Eqn (101). If the strength of the constant magnetic field is

$$H_0 \gg \frac{1}{\sqrt{8\ln 2}} \frac{\omega \Delta_\perp n \sin\theta}{|e|} , \tag{114}$$

the phase is $\phi = \pi/2$, and the distance is

$$x_0 = \frac{\lambda}{\pi} \sqrt{2\ln 2} \frac{p_0}{\Delta_\parallel} \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 \beta_0 \sin\theta ,$$

then the gain is maximum,

$$\Gamma = 45 \rho_0 r_0 \lambda \frac{d}{\lambda} \frac{p_0}{\Delta_\parallel} \frac{mc^2}{\mathscr{E}_0} \frac{1}{n\beta_0} \sin\theta . \tag{115}$$

By substituting the expression for x_0 into Eqn (113) we find the range of energies in which the constant magnetic field neutralises the angular spread of the particle beam:

$$mc^2 < \mathscr{E}_0 < mc^2 \frac{\sqrt[4]{2\ln 2}}{\beta_0} \sqrt{\frac{p_0}{\Delta_\perp}} \sqrt{\frac{\Delta_\parallel}{\Delta_\perp}} . \tag{116}$$

Note in conclusion that in Ref. [47] the gain of the Cherenkov klystron is examined in the case of a constant magnetic field which is perpendicular to the velocity of particles in the drift region.

2.10 The classical theory of the Cherenkov laser

We shall discuss the interaction between an electron beam and a plane monochromatic wave,

$$A_x = \tfrac{1}{2} A_0 \exp[\mathrm{i}(\omega t - kz)] + \text{c.c.} , \qquad (117)$$

propagating in a dielectric medium with the index of refraction n. The current of the electron beam can be expanded into a series of the perturbation theory in terms of the field

$$\boldsymbol{j} = \boldsymbol{j}_0 + \boldsymbol{j}_1 \exp[\mathrm{i}(\omega t - kz)] + \boldsymbol{j}_2 \exp[2\mathrm{i}(\omega t - kz)] + \ldots . \qquad (118)$$

The first term $\boldsymbol{j}_0 = e\rho_0 v$ corresponds to the zero approximation and, as is noted in Section 2.8, does not contribute to the exchange of energies between the electron beam and the wave ($W = 0$). The current $\boldsymbol{j}_1 \exp[\mathrm{i}(\omega t - kz)]$ is linear with respect to the field. Clearly, the integral in Eqn (87) is not zero in this approximation. Thus the energy of electrons can be transmitted to the wave, i.e., the Cherenkov laser is feasible. The terms of the second and higher orders in Eqn (118) are responsible for harmonics and are not considered here.

The amplification of an electromagnetic wave in the Cherenkov laser has the following important distinctions from that in the Cherenkov klystron.

1. In the klystron the difference between the phases (ϕ) of the current and the amplified wave is a function of the distance R_1R_2 between the mirrors. Thus it may be chosen so that the number of electrons falling within the decelerating phase is greater than the number of accelerated electrons ($W < 0$). In the Cherenkov laser the difference between the phases ϕ of the current $\boldsymbol{j}_1 \exp[\mathrm{i}(\omega t - kz)]$ and the amplified wave [Eqn (117)] is a function of the coefficient $\boldsymbol{j}_1$ which is proportional to the derivative of the electron distribution function [Eqn (120)]. Clearly, if the distribution functions have extrema, then the sign of the derivative and, consequently, the difference between the phases ϕ can be chosen so that the electron beam transmits its energy to the wave ($W < 0$).

2. An important distinction between the Cherenkov laser and the Cherenkov klystron lies in the nature of amplification. In the klystron scheme the velocities of the electrons are modulated near the $x = 0$ plane, whereas the electromagnetic wave is amplified near the $x = x_0$ plane (see Fig. 1); there is no feedback between the current and the wave. Hence the intensity of the wave grows linearly ($\Gamma z \ll 1$) or quadratically ($\Gamma z \gg 1$) with z [see Eqn (96)]. In the Cherenkov laser, the field and the current are specified at the same point. Therefore the electromagnetic wave is amplified exponentially [see Eqn (124)].

We shall determine the gain of the Cherenkov laser using the closed consistent system of equations (88), (90). We suppose that the width of the beam (d) of the amplified radiation is large ($\lambda/d \ll \delta$, $\Delta/\mathscr{E}$) and the beam is approximated by the plane wave [Eqn (117)]. We suppose also that the amplitude of the wave A_0 varies slowly with the z coordinate.

On solving Eqn (81) in the linear approximation with respect to the field we obtain the electron distribution function in the form

$$f = f_0 + f_1 . \qquad (119)$$

Here f_0 is the initial electron distribution function,

$$f_1 = \frac{\mathrm{i}}{\omega - kv_z} \boldsymbol{F} \cdot \frac{\partial f_0}{\partial \boldsymbol{p}} + \text{c.c.} , \qquad (120)$$

the force,

$$\boldsymbol{F} = \frac{1}{2} e \left[\boldsymbol{E}_0 \left(1 - \frac{kv_z}{\omega} \right) + \frac{\boldsymbol{k}}{\omega} (v \cdot \boldsymbol{E}) \right] \exp[\mathrm{i}(\omega t - kz)]$$

[$\boldsymbol{E}_0 = -\mathrm{i}(\omega/c)\boldsymbol{A}_0$ is the amplitude of the electric field strength]. The field [Eqn (117)] is assumed to be turned on adiabatically slowly at $t = -\infty$.

By substituting Eqn (120) into Eqn (89) and integrating by parts we arrive at the expression for the x-projection of the current:

$$j_x = -\frac{1}{2} e^2 \rho_0 c A_0 \exp[\mathrm{i}(\omega t - kz)]$$
$$\times \left[\int \frac{f_0(\boldsymbol{p})}{\mathscr{E}} \, \mathrm{d}\boldsymbol{p} + (n^2 - 1) \int \frac{v_x p_x f_0(\boldsymbol{p})}{(\mathscr{E} - ncp_z)^2} \, \mathrm{d}\boldsymbol{p} \right] + \text{c.c.} \quad (121)$$

The first term in brackets yields the index of refraction of the electron beam, and the second term is responsible for amplification or absorption of the electromagnetic wave.

By substituting the current [Eqn (121)] into Eqn (93) and integrating over the space between the $z = z_1$ and $z = z_2$ planes, we have

$$\mathrm{d}P = -\mathrm{d}z \int_{-b/2}^{b/2} \mathrm{d}y \int_{-a/2}^{a/2} \mathrm{d}x \, \overline{j_x E_x} . \qquad (122)$$

Here,

$$P = \frac{1}{8\pi} \frac{n\omega^2}{c} abA_0^2 \qquad (123)$$

is the flux of the energy of the wave through the area $S = ab$, and $\mathrm{d}P = P(z_2) - P(z_1)$ is the variation of the flux over the interval $\mathrm{d}z = z_2 - z_1$.

The solution to Eqn (122) has the form

$$P = P_0 \mathrm{e}^{\Gamma z} , \qquad (124)$$

where P_0 is the flux of the energy of the wave through the $z = 0$ plane,

$$\Gamma = -2\rho_0 r_0 \lambda \frac{n^2 - 1}{n} \operatorname{Im} \int \frac{mc^2 v_x p_x}{(\mathscr{E} - ncp_z)^2} f(\boldsymbol{p}) \, \mathrm{d}\boldsymbol{p} . \qquad (125)$$

If the distribution function has a maximum at $\boldsymbol{p} = \boldsymbol{p}_0$, then

$$\Gamma = -2\rho_0 r_0 \lambda \frac{n^2 - 1}{n} mc^2 v_{0x} p_{0x} J . \qquad (126)$$

Here the factor

$$J = -\operatorname{Im} \int \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} \right) f_0(\boldsymbol{p}) \, \mathrm{d}\boldsymbol{p} ; \qquad (127)$$

$x = \mathscr{E} - ncp_z$; $r_0 = e^2/mc^2$ is the classical radius of an electron; λ is the wavelength of the amplified radiation. The notation for the gain [Eqn (126)] is convenient for various models of the electron distribution function. To establish the dependence of the amplification factor for the Cherenkov laser on the energy spread of electrons, $f_0(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}$ must be replaced by $g(\mathscr{E})\,\mathrm{d}\mathscr{E}$. The change of $f_0(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}$ for $g(\theta,\varphi)\,\mathrm{d}\theta\,\mathrm{d}y$ makes it possible to examine the role of the angular spread of electrons.

We consider the general case of Gaussian spreads of an electron beam in energies and angles,

$$f(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = g(\mathscr{E},\theta)\,\mathrm{d}\mathscr{E}\,\mathrm{d}\theta\,\frac{\mathrm{d}\varphi}{2\pi}\,, \tag{128}$$

$$g(\mathscr{E},\theta) = \frac{8\ln 2}{\pi}\,\frac{1}{\delta\varDelta}\exp\left\{-4\ln 2\left[\frac{(\mathscr{E}-\mathscr{E}_0)^2}{\varDelta^2}+\frac{\theta^2}{\delta^2}\right]\right\}\,.$$

In this case it is convenient to determine the quantity

$$J = -\mathrm{Im}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\mathscr{E}\int_0^{\infty}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\,g(\mathscr{E},\theta)\left(\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{1}{x}\right)\,, \tag{129}$$

in the system of coordinates associated with the beam of particles.

We direct the z' axis along the central axis of the particle beam and pass the $x'z'$ plane through the z' axis and the vector $\boldsymbol{A}$ (Fig. 2). The angle θ_0 is the angle between the wave vector $\boldsymbol{k}$ and the z' axis, θ is the angle between the z' axis and the velocity vector v, and θ' is the angle between the wave vector $\boldsymbol{k}$ and the velocity vector v. Since the integrand damps rapidly for $\mathscr{E}\neq\mathscr{E}_0$ and $\theta\neq 0$, the limits of integration for these variables are chosen to be $[-\infty, +\infty]$ and $[0, +\infty]$, respectively.

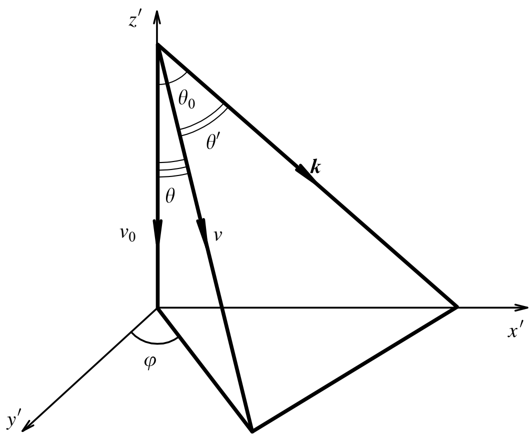

Figure 2.

We can now write the new variables as

$$u = \mathscr{E} - ncp\cos\theta'\,,\qquad \upsilon = \theta\,,\qquad v = \varphi\,, \tag{130}$$

on the interval [Eqn (129)]. Then

$$J = -\mathrm{Im}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}u\int_0^{\infty}\mathrm{d}\upsilon\int_0^{2\pi}\mathrm{d}v\,g(u,\upsilon,v)\left|\frac{\partial\mathscr{E}}{\partial u}\right|\left(\frac{\mathrm{d}}{\mathrm{d}u}\,\frac{1}{u}\right)\,. \tag{131}$$

Integrating this expression by parts and using the rule,

$$\left.\frac{1}{x-\mathrm{i}\eta}\right|_{\eta\to+0} = \mathrm{i}\pi\delta(x) + P\,\frac{1}{x}\,, \tag{132}$$

we have

$$J = \frac{2(4\ln 2)^2}{\pi\varDelta^3\delta}\int_0^{\infty}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\left(\frac{p}{mc}\right)^4(\mathscr{E}-\mathscr{E}_0)$$
$$\times\exp\left[-4\ln 2\,\frac{(\mathscr{E}-\mathscr{E}_0)^2}{\varDelta^2} - 4\ln 2\,\frac{\theta^2}{\delta^2}\right]\,, \tag{133}$$

where $\mathscr{E}$ is found from the equation

$$\mathscr{E} - ncp\cos\theta' = 0\,. \tag{134}$$

In formula (133) the contribution from electrons of all possible directions of the velocity v is taken into account. The greatest contribution is made by particles moving along the axis z'. We find from Eqn (134) that the energy $\mathscr{E}_{\mathrm{m}}$ is

$$\mathscr{E}_{\mathrm{m}} = \frac{mc^2}{[1-n^{-2}(\cos\theta_0)^{-2}]^{1/2}}\,. \tag{135}$$

It must be close enough to $\mathscr{E}_0$ for the gain to be exponentially low.

For velocities which make the angle θ' with the z axis the energy is specified by the expression

$$\mathscr{E} = \frac{mc^2}{[1-n^{-2}(\cos\theta')^{-2}]^{1/2}}\,. \tag{136}$$

Since only particles with energies close to $\mathscr{E}_{\mathrm{m}}$ make nonzero contributions, the denominator in Eqn (136) can be expanded into the Taylor series

$$\mathscr{E} = \mathscr{E}_{\mathrm{m}}\left[1 - \theta\tan\theta_0\sin\varphi\left(\frac{p_{\mathrm{m}}}{mc}\right)^2\right]\,. \tag{137}$$

Here we take into account the relation between the angle θ' and the angles θ and φ (see Fig. 2):

$$\cos\theta' = \cos\theta\cos\theta_0(1+\tan\theta\tan\theta_0\sin\varphi)$$
$$\approx \cos\theta_0(1+\theta\tan\theta_0\sin\varphi)\,. \tag{138}$$

(Analysis shows that it suffices to consider the linear approximation in the expansion in terms of a small angle θ.)

The expansion [Eqn (137)] is true for beams with angular spreads

$$\delta < \left(\frac{mc}{p_0}\right)^2\cot\theta_0\,. \tag{139}$$

By substituting the expansion [Eqn (137)] into formula (133) and factoring the slowly varying functions of angle outside the integral sign, we have

$$J = 2\sqrt{\pi}\,(4\ln 2)^{3/2}\left(\frac{p_0}{mc}\right)^4\frac{\mathscr{E}_{\mathrm{m}}-\mathscr{E}_0}{D^3}$$
$$\times\exp\left[-4\ln 2\,\frac{(\mathscr{E}_{\mathrm{m}}-\mathscr{E}_0)^2}{D^2}\right]\,, \tag{140}$$

where the effective width is

$$D = \left[\varDelta^2 + \delta^2\mathscr{E}_0^2\left(\frac{p_0}{mc}\right)^4\tan^2\theta_0\,\overline{\sin^2\varphi}\right]^{1/2}\,. \tag{141}$$

Since the dependence of the integrand on the angle φ is complicated, it is integrated approximately with respect to this variable. In what follows we assume that $\sin^2\varphi$ takes the largest value — equal to unity.

Note that the same result can be obtained exactly if the initial electron distribution function is chosen in the form of

Eqn (70). The gain [Eqn (126)] peaks for the energy $\mathscr{E}_{\mathrm{m}}$, where $\mathscr{E}_{\mathrm{m}} = \mathscr{E}_0 - D/\sqrt{8\ln 2}$, and is given by

$$\Gamma = 16\ln 2\sqrt{\frac{\pi}{2\mathrm{e}}}\rho_0 r_0 \lambda \frac{n^2-1}{n} \frac{\mathscr{E}_0 m v_0^2 (p_0/mc)^4 \sin^2\theta_0}{\Delta^2 + \delta^2 \mathscr{E}_0^2 (p_0/mc)^4 \tan^2\theta_0}, \quad (142)$$

where $\mathrm{e} = 2.718\ldots$ is the base of the natural logarithm. The frequency corresponding to the energy $\mathscr{E}_{\mathrm{m}}$ is found from Eqn (134) for $\theta' = \theta_0$.

If the angular spread of an electron beam is

$$\delta \ll \frac{\Delta}{\mathscr{E}_0}\left(\frac{mc}{p_0}\right)^2 |\cot\theta_0|, \quad (143)$$

then the gain depends solely on the energy spread,

$$\Gamma = 8.4 \rho_0 r_0 \lambda \beta_0^3 \frac{n^2-1}{n}\left(\frac{p_0}{mc}\right)^3\left(\frac{\mathscr{E}_0}{\Delta}\right)^2 \sin^2\theta_0 . \quad (144)$$

Note that in this case Γ depends strongly on the average energy of the particles ($\Gamma \propto (\mathscr{E}_0/mc^2)^3$ if $\Delta/\mathscr{E}_0 = \mathrm{const}$).

For ordinary relativistic beams the inverse inequality

$$\delta \gg \frac{\Delta}{\mathscr{E}_0}\left(\frac{mc}{p_0}\right)^2 |\cot\theta_0|, \quad (145)$$

holds. In this case the gain depends solely on the angular spread of the electron beam and is given by

$$\Gamma = 8.4\, \rho_0 r_0 \lambda \frac{mc^2}{\mathscr{E}_0}\frac{n^2-1}{n^3}\frac{1}{\delta^2} . \quad (146)$$

Clearly, the dependence of the gain on the energy of a particle is totally different when the angular spread is taken into account: the gain starts to decrease on increase in the average energy of the electron beam as $mc^2/\mathscr{E}_0$.

It is shown in Section 2.14 that the negative role of the angular spread can be neutralised by a constant magnetic field. Note also that the computational technique we have developed in this section, for the analysis of the operation of the Cherenkov laser, can be applied for gains for which the inequality

$$\Gamma < k\frac{\Delta v}{v}, \quad (147)$$

holds. Here $\Delta v/v$ is the relative spread of the electron beam in velocities in the direction in which the wave propagates.

The exponential growth of the amplitude of the plane transverse wave [Eqn (117)] and, consequently, corrections to the initial distribution function (120) mean that the electron beam and the decelerating medium are an unstable system, and the instability is of the Cherenkov nature. The issues of absorption and growth of perturbations as a result of the stimulated Cherenkov effect are well known and studied in the theory of plasmas: Landau damping and beam instability [48, 49].

In contrast to the cited schemes, the longitudinal wave is decelerated in a plasma. Therefore, a plasma can be used as an active medium only if there are converters of the transverse wave into a longitudinal wave, and vice versa, at the entry and at the exit from the plasma. The mechanism of amplification and the computational technique for the increment of instability are alike in both cases.

Note that according to the terminology adopted in plasma physics there are two types of instability, namely, the hydrodynamic and the kinetic instability [49]. In the first type, the temperature of the plasma and the energy spread of the electron beam can be neglected, i.e., the increment of instability is a function of the density of particles and their velocity. In the second type both spreads are essential and, therefore, the increment of instability is defined by distribution functions both for the particle beam and for the plasma.

In our work we assume that inequality (147) holds, i.e. the kinetic Cherenkov instability of the system is studied. In Ref. [50] both kinetic and hydrodynamic instabilities were considered (see also Section 2.14).

2.11 The quantum theory of the Cherenkov laser

If the angular spread of an electron beam is $\delta \to 0$, then by substituting $f(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = g(\mathscr{E})\,\mathrm{d}\mathscr{E}$ into Eqn (127) and integrating by parts, we have that the sign of the gain of the Cherenkov laser depends on the derivative of the electron energy distribution function. The physical meaning of this fact can be made clear by the analysis of the SCE with the aid of the laws of conservation of energy and momentum:

$$\mathscr{E}^{\mp} \mp \hbar\omega = \mathscr{E}_2, \quad \boldsymbol{p}^{\mp} \mp \hbar\boldsymbol{k} = \boldsymbol{p}_2 . \quad (148)$$

Given the angle θ_0 between the wave vector $\boldsymbol{k}$ and the momentum of a particle $\boldsymbol{p}$, the system of equations (148) governs the energy and momentum of electrons involved in emission and absorption of a photon,

$$\mathscr{E}^{-} = \mathscr{E}_1 + \Delta\mathscr{E}, \quad \mathscr{E}^{+} = \mathscr{E}_1 - \Delta\mathscr{E} . \quad (149)$$

Here the energy $\mathscr{E}_1$ is found from Eqn (134) for $\theta = \theta_0$ and

$$\Delta\mathscr{E} = \frac{\hbar\omega}{2}(n^2-1)\left(\frac{p_1}{mc}\right)^2 . \quad (150)$$

Since different electrons are involved in emission and absorption, the radiation can be amplified if the number of emitting electrons $N(\mathscr{E}^-)$ is greater than the number of absorbing electrons $N(\mathscr{E}^+)$. Taking into account that $\Delta N = N(\mathscr{E}^-) - N(\mathscr{E}^+) \propto g(\mathscr{E}^-) - g(\mathscr{E}^+)$, and expanding the distribution function into the Taylor series in terms of the small parameter $\Delta\mathscr{E}$, we establish that the overpopula-tion is $\Delta N \propto 2(\mathrm{d}g/\mathrm{d}\mathscr{E}_1)\Delta\mathscr{E} > 0$ on the left-hand wing of the electron energy distribution function. From the classical standpoint (see Section 2.10) it follows that for $\mathrm{d}g/\mathrm{d}\mathscr{E}_1 > 0$ the number of particles in the decelerating phase is larger than the number of accelerated particles.

We shall now examine the role of the spin and polarisation effects in the Cherenkov laser. We shall determine the gain for the elliptically polarised monochromatic electromagnetic wave,

$$A_x = A_1\cos(kz - \omega t), \quad A_y = -A_2\sin(kz - \omega t), \quad (151)$$

using the closed consistent system of the Dirac and Maxwell equations,

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left[c\boldsymbol{\alpha}\left(\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right) + mc^2\beta\right]\psi, \quad (152)$$

$$\nabla^2\boldsymbol{A} - \frac{n^2}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j}, \quad (153)$$

$$\boldsymbol{j} = ec\psi^{+}\boldsymbol{\alpha}\psi . \quad (154)$$

If the wave amplitudes $A_{1,2}$ vary slightly with the z coordinate, then the solution to the Dirac equation in the linear approximation with respect to the field is

$$\psi = \psi_0 + \psi_+ + \psi_- \,. \tag{155}$$

Here ψ_0 is the initial wave function of the particle beam (55); the terms

$$\psi_\pm = \pm \frac{ec(\hat{p}_\pm + mc)\hat{A}_\pm \exp[\pm \mathrm{i}(kz - \omega t)]}{4\hbar\omega\mathscr{E}[1 - n\beta_z \mp (\hbar\omega/2\mathscr{E})(n^2 - 1)]}\,\psi_0 \,, \tag{156}$$

describe the processes of emitting and absorbing a photon; the four-vectors $p^\mu_\pm = p^\mu \pm \hbar k^\mu$, $k^\mu(\omega/c,\, 0,\, 0,\, K)$; the operators $\hat{p} = p_\mu \gamma^\mu$, $\hat{A}_\pm = A^\pm_\mu \gamma^\mu$; the vectors $\boldsymbol{A}^\pm = \hat{\mathrm{i}} A_1 \pm \hat{\jmath}\mathrm{i} A_2$.

The polarisation matrix of the density is assumed to be of the form Eqn (57) before the interaction. Then the projections of the current of an energy homogeneous electron beam are:

$$\begin{aligned} j_x = {} & \frac{e^2 \rho_0 c}{2\mathscr{E}} \left[(1 - n\beta_z)^2 - \left(\frac{\hbar\omega}{2\mathscr{E}}\right)^2 (n^2 - 1)^2 \right]^{-2} \\ & \times \Bigg\{ [(1 - n^2)\beta_x^2 - (1 - n\beta_z)^2] A_1 \\ & + \mathrm{i}(1 - n^2) + \left[\beta_x \beta_y + \mathrm{i}\,\frac{\hbar\omega}{2\mathscr{E}}\,\frac{mc^2}{\mathscr{E}}\,(a_3 - na_0) \right] A_2 \Bigg\} \\ & \times \exp[\mathrm{i}(kz - \omega t)] + \text{c.c.} \,, \end{aligned} \tag{157}$$

$$\begin{aligned} j_y = {} & \frac{e^2 \rho_0 c}{2\mathscr{E}} \left[(1 - n\beta_z)^2 - \left(\frac{\hbar\omega}{2\mathscr{E}}\right)^2 (n^2 - 1)^2 \right]^{-2} \\ & \times \Bigg\{ (1 - n^2) \left[\beta_x \beta_y - \mathrm{i}\,\frac{\hbar\omega}{2\mathscr{E}}\,\frac{mc^2}{\mathscr{E}}\,(a_3 - na_0) \right] A_1 \\ & + \mathrm{i}\left[(1 - n^2)\beta_y^2 - (1 - n\beta_z)^2\right] A_2 \Bigg\} \\ & \times \exp[\mathrm{i}(kz - \omega t)] + \text{c.c.} \end{aligned} \tag{158}$$

We shall compare the x-projection of the current [Eqn (157)] with the classical expression [Eqn (121)]. First of all, note that the energies of particles involved in emission and absorption are split in quantum calculations. According to the pole of the expression (157),

$$1 - n\beta_z = \mp \frac{\hbar\omega}{2\mathscr{E}}\,(n^2 - 1) \,. \tag{159}$$

Here $-$ corresponds to emission of a photon and $+$ to absorption. The solutions to these equations are specified by the expressions (149) and (150).

We shall compare the coefficient of A_1 [Eqn (157)] and the numerator in the second term of Eqn (121). Clearly, the quantum calculation introduces an additional term proportional to $(1 - n\beta_z)^2$. Analysis of the expression for the current in the nonrelativistic limit [Eqn (64)] shows that the term is associated with the induced magnetisation of the electron beam (a similar effect was considered in Sections 2.5, 2.6).

We shall now consider the coefficient of A_2. The term proportional to $\beta_x \beta_y$ is absent in expression (121) because it is assumed that $A_y = 0$. The term proportional to $a_3 - na_0$ is associated with the initial polarisation of the electron beam (a similar effect was considered in Section 2.5). Clearly, the polarised electron beam is an anisotropic medium, having hydrotropy [44], with the coefficient

$$g_3 = \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2 \frac{\hbar\omega}{\mathscr{E}} \left(\frac{mc^2}{\mathscr{E}}\right)^2 \frac{(n^2 - 1)(a_3 - na_0)}{(1 - n\beta_z)^2 - (\hbar\omega/2\mathscr{E})^2 (n^2 - 1)^2} \,. \tag{160}$$

If we substitute Eqns (157) and (158) into Eqn (153), and take into account the fact that the amplitudes of the field [Eqn (151)] vary weakly with the z coordinate, we arrive at the system of truncated differential equations for $A_{1,2}$:

$$\frac{\mathrm{d}A_1}{\mathrm{d}z} = aA_1 + bA_2 \,, \tag{161}$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}z} = qA_1 + dA_2 \,. \tag{162}$$

Here the coefficients are

$$\begin{aligned} a &= \mathrm{i}R[(1 - n^2)\beta_x^2 - (1 - n\beta_z)^2] \,, \\ b &= -R[(1 - n^2)\beta_x \beta_y - \mathrm{i}v] \,, \\ d &= \mathrm{i}R[(1 - n^2)\beta_y^2 - (1 - n\beta_z)^2] \,, \\ q &= R[(1 - n^2)\beta_x \beta_y + \mathrm{i}v] \,, \\ R &= \frac{\omega_p^2}{2n\omega c}\,\frac{mc^2}{\mathscr{E}} \left[(1 - n\beta_z)^2 - \left(\frac{\hbar\omega}{2\mathscr{E}}\right)^2 (n^2 - 1)^2 \right]^{-1} , \\ \omega_p^2 &= \frac{4\pi e^2 \rho_0}{m} \,, \end{aligned}$$

and

$$v = \frac{\hbar\omega}{2\mathscr{E}}\,\frac{mc^2}{\mathscr{E}}\,(a_3 - na_0)(n^2 - 1) \,. \tag{163}$$

First we consider the amplification of an electromagnetic wave, which is linearly polarised along the x axis, neglecting spin effects. The electron beam is assumed to move in the xz plane and have a Gaussian energy spread

$$g(\mathscr{E}) = \left(\frac{4\ln 2}{\pi}\right)^{1/2} \frac{1}{\Delta} \exp\left[-4\ln 2\,\frac{(\mathscr{E} - \mathscr{E}_0)^2}{\Delta^2} \right] . \tag{164}$$

By presenting the factor R [Eqn (163)] as a difference between the amplitudes of emission and absorption of a photon,

$$\begin{aligned} R \sim {} & \left[(1 - n\beta_z)^2 - \left(\frac{\hbar\omega}{2\mathscr{E}}\right)^2 (n^2 - 1)^2 \right]^{-1} \\ = {} & -\left[\frac{\hbar\omega}{\mathscr{E}}\,(n^2 - 1) \right]^{-1} \Bigg\{ \left[1 - n\beta_z + \frac{\hbar\omega}{2\mathscr{E}}\,(n^2 - 1) + \mathrm{i}\eta \right]^{-1} \\ & - \left[1 - n\beta_z - \frac{\hbar\omega}{2\mathscr{E}}\,(n^2 - 1) + \mathrm{i}\eta \right]^{-1} \Bigg\}_{\eta \to 0} \,, \end{aligned} \tag{165}$$

and averaging the right-hand side of Eqn (161) over energies by means of the rule [Eqn (163)], we obtain the gain for the intensity of the electromagnetic wave,

$$\begin{aligned} \varGamma = {} & 2\pi \rho_0 r_0 \lambda \sin^2\theta_0\,\frac{mv_0^2}{n\hbar\omega} \left(\frac{p_0}{mc}\right)^2 \\ & \times \mathscr{E}_0 [g(\mathscr{E}_1 + \Delta\mathscr{E}) - g(\mathscr{E}_1 - \Delta\mathscr{E})] \,. \end{aligned} \tag{166}$$

Clearly, the gain depends on the extent to which the processes of emitting and absorbing a photon are split. If the quantity

$$\Delta\mathscr{E} = \frac{\hbar\omega}{2}(n^2-1)\left(\frac{p_1}{mc}\right)^2 \ll \varDelta\,, \tag{167}$$

then, expanding the function into the Taylor series, we come back to formula (144). If the quantity $\Delta\mathscr{E} \gg \varDelta$, then for relativistic particles the initial conditions can be chosen $[1 - n\beta_z - (\hbar\omega/2\mathscr{E})(n^2-1) = 0]$ such that the contribution of absorption is exponentially low. In this case the gain is purely quantum in nature:

$$\varGamma = 5.9\,\rho_0 r_0 \lambda\,\frac{mv_0^2}{n\hbar\omega}\left(\frac{p_0}{mc}\right)^2\frac{\mathscr{E}_0}{\varDelta}\sin^2\theta_0\,. \tag{168}$$

However, for large energies the angular spread of the particle beam [Eqn (145)] comes into play. The feasibility of this limiting case is up against the complicated problem of generating particle beams with a negligible angular spread [Eqn (143)]. If we average the poles of Eqn (165) over the energy and angular spreads [Eqn (128)], then we return to formula (142) as $\hbar \to 0$.

We shall analyze the contribution of spin to the amplification of the electromagnetic wave. In order to extract the pure spin interaction between an electron and the wave, we suppose that the electron moves in parallel with the z axis ($\beta_x = \beta_y = 0$). Amplification of an electromagnetic wave by a polarised electron beam is analyzed in the next section. Here we consider amplification of an electromagnetic wave by a nonpolarised electron beam ($|\boldsymbol{\zeta}| = 0$) due to its induced magnetisation. If the electron beam has the Gaussian spread in energies [Eqn (116)], then it follows from Eqns (161) and (162) that

$$\varGamma = 2.1\,\rho_0 r_0 \lambda \beta_0\,\frac{(n^2-1)^2}{n}\left(\frac{\hbar\omega}{\mathscr{E}_0}\right)^2\left(\frac{p_0}{mc}\right)^3\left(\frac{\mathscr{E}_0}{\varDelta}\right)^2 \tag{169}$$

[the contribution of the angular spread is not essential for $\theta_0 = 0$; see Eqn (141)] The gain of the spin Cherenkov laser is not large since $\hbar\omega/\mathscr{E}_0 \ll 1$.

2.12 Rules of selection in the Cherenkov laser

It is possible to suppress completely photon absorption by amplifying circularly polarised electromagnetic radiation by a polarised electron beam ($|\boldsymbol{\zeta}| \neq 0$). We shall show, using the law of conservation of moment of momentum for the photon–electron system, that suppression is possible when all photons are polarised in a clockwise manner and move along the z axis. The moment of momentum of such photons is $\hbar$. Let the electron beam have the same direction as the wave vector of photons. Electrons emit and absorb photons according to the laws of conservation of energy and momentum [Eqn (148)], as well as by the law of conservation of moment of momentum,

$$S_{1z} \pm I_z = S_{2z}\,. \tag{170}$$

Here the z-projection of the moment of momentum of a photon I_z runs the values $\hbar$, 0, $-\hbar$; the z-projection of the moment of momentum of an electron S_z takes the values $\hbar/2$ and $-\hbar/2$.

If $I_z = \hbar$, $S_{1z} = \hbar/2$ (photons are clockwise polarised, electrons are polarised along the z axis), then electrons can emit photons by flipping the spin: $S_{2z} = -\hbar/2$. If electrons are polarised in the direction opposite to the z axis ($S_{1z} = -\hbar/2$), then the conservation law [Eqn (170)] permits absorption of photons only. Thus an electron beam polarised in the direction of motion is a totally overpopulated medium for a right-hand circularly polarised light beam which propagates in the same direction.

In this case the gain for an electromagnetic wave can be determined as follows. Let

$$A_x = A_0\cos(\omega t - kz)\,,$$
$$A_y = A_0\sin(\omega t - kz)\,, \tag{171}$$

be the vector potential of the wave. On solving the Dirac equation (152) in the linear approximation with respect to field, we have

$$\psi = \psi_0 + \psi_+ + \psi_-\,, \tag{172}$$

where ψ_0 is specified by Eqn (55),

$$\psi_\pm = \mp\frac{ec(\hat{p}_\pm + mc)\hat{A}_\pm\exp[\mp\mathrm{i}(\omega t - kz)]}{4\mathscr{E}\hbar[\omega - kv \mp (\hbar\omega^2/2\mathscr{E})(n^2-1) \pm \mathrm{i}\eta]_{\eta\to 0}}\,\psi_0\,, \tag{173}$$

$\boldsymbol{A}_\pm = \hat{\imath}A_0 \pm \hat{\jmath}\mathrm{i}A_0$. The other symbols are the same as in Eqn (156).

If the polarisation matrix of the density of electrons has the form of Eqn (57) before the interaction, then the x-projection of the current is

$$j_x = \frac{e^2\rho_0 c}{4\mathscr{E}}A_0\exp[\mathrm{i}(\omega t - kz)]$$
$$\times\left[\frac{(\omega - kv)(1+\zeta_z)}{\omega - kv + (\hbar\omega^2/2\mathscr{E})(n^2-1) - \mathrm{i}\eta} + \frac{(\omega - kv)(1-\zeta_z)}{\omega - kv - (\hbar\omega^2/2\mathscr{E})(n^2-1) - \mathrm{i}\eta}\right]_{\eta\to 0}\,. \tag{174}$$

The y-projection of current has a similar form.

The first term in curly brackets in Eqn (174) describes the emission of a photon, the second describes the absorption of a photon. If $\zeta_z = 1$, i.e., the particle beam is fully polarised along the z axis, then the absorption amplitude is zero and the gain is maximum.

If $\zeta_z = 0$, then the only cause of amplification is the induced magnetisation (see Section 2.6) and the amplification is a function of the difference of amplitudes of emission and absorption of a photon [see Eqn (168)]. By averaging the current [Eqn (174)] over the spread [Eqn (164)] for $\zeta_z = 1$ and substituting the result into the truncated equation (153), we have

$$\varGamma = 18.5\,\rho_0 r_0 \lambda_{\mathrm{C}}\beta_0\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\frac{\mathscr{E}_0}{\varDelta}\,. \tag{175}$$

Here $\lambda_{\mathrm{C}} = \hbar/mc$ is the Compton wavelength of an electron.

Comparison of gains for the intensity of an electromagnetic wave [Eqns (169) and (175)] shows that the efficiency of the Cherenkov laser is greater by a factor of $\varDelta/\hbar\omega$ when absorption is suppressed. However, the absolute value of $\varGamma$ [Eqn (175)] is not large in comparison with Eqn (142) since the spin interaction between an electron and the wave ($H_1 = -\boldsymbol{\mu}\cdot\boldsymbol{H}$) is significantly weaker than the orbital interaction ($H_2 = -ec\boldsymbol{A}\cdot\boldsymbol{\beta}_0$).

Note also that, in the spin laser, electrons and the amplified wave travel in the same direction with almost the same speed. Therefore they interact over a prolonged period of time so that the effect can be observed.

Note in conclusion that the classical and quantum theories of spontaneous emission of the magnetic moment (including forward direction) were considered in detail in Refs [52–54]. According to the results of Refs [52–53], on threshold $[1-(nv/c)=0$ or $\mathscr{E}^{-}=\mathscr{E}_1$, where $\mathscr{E}_1$ is the energy of an initial electron moving with the speed of the wave given by Eqn (149)] both spontaneous and stimulated Cherenkov effects are absent in the forward direction. The spontaneous effect is absent in the forward direction since the equation

$$1-n\frac{v}{c}=-\frac{\hbar\omega}{2\mathscr{E}_1}(n^2-1)$$

yields the zero frequency of the emitted photon.

In the case of the stimulated effect, the factor $1-(nv/c)$ nullifies both terms in the current [Eqn (174)]. If the velocity of an electron is greater than the velocity of a photon

$$v=\frac{c}{n}\left[1+\frac{\hbar\omega}{2\mathscr{E}_1}(n^2-1)\right]$$

or $\mathscr{E}^{-}=\mathscr{E}_1+\Delta\mathscr{E}$ [see Eqn (149)], then the spontaneous and stimulated effects are possible and are of the order of the loss [22, 54]. It is worthwhile to evaluate the velocity of an electron which has emitted a photon. Since $v_2=c^2p_2/\mathscr{E}_2$, we have

$$\frac{v}{2}=\frac{c}{n}\left[1-\frac{\hbar\omega}{2\mathscr{E}_1}(n^2-1)\right],$$

i.e., the velocity of the electron is less than the velocity of the photon. Thus the electron lags behind the emitted field†.

2.13 Rotation of the polarisation plane

The polarisation characteristics of an electromagnetic wave travelling within an electron beam can be examined with the aid of the system of equations (161), (162). Since a polarised electron beam is an anisotropic medium, deformation and rotation of the ellipse of the electromagnetic wave [Eqn (151)] result. We shall dwell on the last effect since it is useful in the analysis of the structure and polarisation of an electron beam.

To simplify the problem it is convenient to separate polarisation effects from amplification effects. It suffices to assume that the inequality

$$|\mathscr{E}_1-\mathscr{E}_0|>\Delta \tag{176}$$

holds for the average energy of the electron beam. Here $\mathscr{E}_1$ is found from Eqn (134), Δ is the width of the energy spread of the electron beam. In this case the right-hand sides of Eqns (161), (162) may be averaged over the spread of the electron beam [Eqn (128)] by simply replacing all the parameters of the beam with the average quantities: $\mathscr{E}\to\bar{\mathscr{E}}$, etc. In what follows we assume that $\bar\beta_x=0$, $\bar\beta_y=\bar\beta\sin\theta$, $\bar\beta_z=\bar\beta\cos\theta$, and the sign of the average is omitted.

The solution for the amplitude A_1 is sought in the form

$$A_1(z)=f\exp(\mathrm{i}\chi_1 z)+l\exp(\mathrm{i}\chi_2 z). \tag{177}$$

†The authors are grateful to D M Sedrakyan and V O Papanyan for valuable comments on the issues.

The field [Eqn (151)] is assumed to be elliptically polarised at the point $z=0$:

$$A_1(z=0)=A_{0x}, \quad A_2(z=0)=A_{0y}. \tag{178}$$

The principal axes of the ellipse are directed along x and y.

At an arbitrary point z we have

$$A_1=a_1\cos(k_1z-\omega t)+a_2\cos(k_2z-\omega t),$$
$$A_2=a_3\sin(k_1z-\omega t)+a_4\sin(k_2z-\omega t). \tag{179}$$

Here

$$a_1=\frac{1}{\Delta n}\left\{vA_{0y}-[n_2+(1-n\beta_z)^2]A_{0x}\right\},$$
$$a_2=\frac{1}{\Delta n}\left\{-vA_{0y}+[n_1+(1-n\beta_z)^2]A_{0x}\right\},$$
$$a_3=-\frac{1}{\Delta n}\left\{vA_{0x}+[n_1+(1-n\beta_z)^2]A_{0y}\right\},$$
$$a_4=\frac{1}{\Delta n}\left\{vA_{0x}+[n_2+(1-n\beta_z)^2]A_{0y}\right\}, \tag{180}$$

where

$$n_{1,2}=-(1-n\beta_z)^2+\frac{1}{2}(1-n^2)\beta_y^2\times\left(1\pm\left\{1+\left[\frac{\hbar\omega mc^2}{\mathscr{E}^2\beta_y^2}(a_3-na_0)\right]^2\right\}^{1/2}\right),$$
$$\Delta n=n_1-n_2, \quad k_{1,2}=k+Rn_{1,2}=k+\chi_{1,2}.$$

The quantities R and v are specified by the expressions (163). Let the system of coordinates (x', y', z) be rotated through the angle,

$$\varphi=\frac{1}{2}\arctan\left\{2\sqrt{1+4\chi^2}\left(\chi\frac{r^2-1}{r}-1\right)\sin(\Delta kz)\times\left[(1+4\chi^2)\frac{r^2-1}{r}-8\chi\left(\chi\frac{r^2-1}{r}-1\right)\sin^2\frac{\Delta kz}{2}\right]^{-1}\right\}, \tag{181}$$

about the system (x,y,z). Then the vector potential [Eqn (179)] describes an ellipse whose principal axes are directed along x' and y'. The quantity $\Delta k=k_1-k_2$, and $r=A_{0x}/A_{0y}$ is the axial ratio of the ellipse at the point $z=0$; the parameter $\chi=\hbar\omega mc^2(a_3-na_0)/(2\mathscr{E}^2\beta_y^2)$.

If $|\chi|\ll 1$ [or $\theta\gg(\hbar\omega mc^2|a_3-na_0|)^{1/2}/2cp$], then rotation of the ellipse is due mainly to the fact that the electron beam moves at an angle θ to the direction in which the wave propagates:

$$\varphi_\theta=\frac{1}{2}\arctan\left\{\frac{2r}{1-r^2}\sin\left[\pi\left(\frac{\omega_p}{\omega}\right)^2\beta^2\sin^2\theta\times\frac{1-n^2}{n}\frac{mc^2}{\mathscr{E}}\frac{1}{(1-n\beta_z)^2}\frac{z}{\lambda}\right]\right\}. \tag{182}$$

If $|\chi|\gg 1$ [or $\theta\ll(\hbar\omega mc^2|a_3-na_0|)^{1/2}/2cp$], then the angle of rotation is a function of the polarisation of the electron beam

$$\varphi_g=\frac{\pi}{n}g_3\frac{z}{\lambda}. \tag{183}$$

Here the factor g_3 is specified by expression (160).

We shall consider constraints on the angular and energy spreads of the electron beam in more detail. When averaging the factor R [Eqn (163)] the angular and energy spreads can be neglected if

$$1 - n\beta + \frac{\theta^2}{2} > \theta\delta \,,$$

$$1 - n\beta + \frac{\theta^2}{2} > \frac{\Delta}{\mathscr{E}}\left(\frac{mc^2}{\mathscr{E}}\right)^2 \frac{n}{\beta} \,. \tag{184}$$

Let $\theta \neq 0$. For definiteness we assume that

$$1 - n\beta + \frac{\theta^2}{2} = 3\theta\delta \,. \tag{185}$$

Then the average energy of the electron beam is limited by the inequality

$$\mathscr{E} > mc^2 \,\frac{1}{\beta}\sqrt{\frac{\Delta}{\mathscr{E}}\,\frac{1}{\theta\delta}} \,. \tag{186}$$

We assume also that

$$1 - n\beta = 3\,\frac{\Delta}{\mathscr{E}}\left(\frac{mc^2}{\mathscr{E}}\right)^2 \frac{n}{\beta} \,, \tag{187}$$

for $\theta = 0$.

Using the relations (185) and (187), we can conveniently transform formulae (182) and (183) for numerical evaluation:

$$\varphi_\theta = 0.1\,\rho_0 r_0 \lambda z \beta^2\,\frac{mc^2}{\mathscr{E}}\,\frac{1}{\delta^2}\,\frac{r}{1-r^2}\,\frac{1-n^2}{n} \,, \tag{188}$$

$$\varphi_g = \zeta_z \rho_0 r_0 \lambda_C z \beta\,\frac{\mathscr{E}}{\Delta}\,\frac{n^2-1}{n^2} \,. \tag{189}$$

Here λ_C is the Compton wavelength of an electron, ζ_z is the rate of polarisation of the electron beam along the z axis, and r_0 is the classical radius of an electron.

2.14 Theory of the Cherenkov laser in a constant magnetic field

The analysis of operation of the Cherenkov laser in Section 2.10 shows that the efficiency of the laser is limited by the angular spread of the electron beam [Eqn (146)]. The negative role of the angular spread can be neutralised by applying a constant magnetic field along the particle beam. We assume that an electron beam propagates along the axis and has the Gaussian spread in momenta:

$$f_0(\boldsymbol{p}) = \left(\frac{4\ln 2}{\pi}\right)^{3/2} \frac{1}{\Delta_\perp^2 \Delta_\parallel} \times \exp\left[-4\ln 2\,\frac{(p_z - p_0)^2}{\Delta_\parallel^2} - 4\ln 2\,\frac{p_x^2 + p_y^2}{\Delta_\perp^2}\right] \,, \tag{190}$$

and the strength of the constant magnetic field is $H_z = -H_0$. Let the amplified wave be linearly polarised in the xz plane and let it propagate at the angle θ to the z axis.

$$A_{x'} = \tfrac{1}{2} A_0 \exp(\mathrm{i}\omega t - \mathrm{i}kz') + \mathrm{c.c.} \tag{191}$$

The vector potential given by Eqn (191) is written in the system of coordinates x', y, z' associated with the wave.

We shall determine the gain for the electromagnetic wave using the system of Eqns (88) and (91). If we solve the kinetic equation for the constant magnetic field exactly and for the amplified wave [Eqn (91)] in the first approximation, then

$$f(\boldsymbol{p}) = f_0(\boldsymbol{p}_0) + f_1(\boldsymbol{p}) \,. \tag{192}$$

Here the function f_0 is specified by the expression (190), the vector $\boldsymbol{p}_0$ has the projections

$$\begin{aligned} p_{0x} &= p_x \cos\Omega t - p_y \sin\Omega t \,, \\ p_{0y} &= p_y \cos\Omega t + p_x \cos\Omega t \,, \\ p_{0z} &= p_z \,, \\ \Omega &= |e| H_0\,\frac{c}{\mathscr{E}} \,, \\ \mathscr{E} &= [(mc^2)^2 + c^2|\boldsymbol{p}_0|^2]^{1/2} = [(mc^2)^2 + c^2|\boldsymbol{p}|^2]^{1/2} \,. \end{aligned} \tag{193}$$

The second term is

$$\begin{aligned} f_1 = \frac{\mathrm{i}}{\omega - k_z v_{0z}} \Bigg\{ & F_{1,x}\left(\frac{\partial f_0}{\partial p_{0x}} - \mathrm{i}\,\frac{\partial f_0}{\partial p_{0y}}\right) \mathrm{e}^{\mathrm{i}\varphi'} \\ & + F_{-1,x}\left(\frac{\partial f_0}{\partial p_{0x}} + \mathrm{i}\,\frac{\partial f_0}{\partial p_{0y}}\right) \mathrm{e}^{-\mathrm{i}\varphi'} + F_{0,z}\,\frac{\partial f_0}{\partial p_{0z}} \Bigg\} \\ & \times \exp \mathrm{i}\left[\omega t - k_z z - k_x x - k_x\,\frac{v_{0\perp}}{\Omega}\sin(\Omega t - \varphi')\right] \,, \end{aligned} \tag{194}$$

where

$$\begin{aligned} F_{r,x} &= \frac{eE_{x'}}{2\mathrm{i}}(\cos\theta - n\beta_{0z}) J_r\!\left(k_x\,\frac{v_{0\perp}}{\Omega}\right) \,, \\ F_{r,z} &= \frac{eE_{x'}}{2\mathrm{i}}\left(-\sin\theta + n\beta_{0\perp}\,\frac{r\Omega}{k_x v_{0\perp}}\right) J_r\!\left(k_x\,\frac{v_{0\perp}}{\Omega}\right) \,, \\ v_{0\perp} &= \frac{c^2 p_{0\perp}}{\mathscr{E}} \,, \qquad p_{0\perp} = (p_{0\perp}^2 + p_{0y}^2)^{1/2} \,, \\ \beta_{0\perp} &= \frac{v_{0\perp}}{c} \,, \qquad \tan\varphi' = \frac{p_{0y}}{p_{0x}} \,. \end{aligned} \tag{195}$$

Note that f_1 is in fact a function of all harmonics of the frequency Ω, given by

$$f_1 = \sum_{r=-\infty}^{+\infty} \frac{Q_r}{\omega - k_z v_{0z} + r\Omega} \,.$$

In Eqn (194) we retained only the zero harmonic ($r = 0$) which is responsible for the SCE. The terms with $r \neq 0$ describe the cyclotron radiation of electrons in a dielectric medium. The gain of the cyclotron laser for $r = -1$ is examined in the next section.

By substituting Eqn (192) into Eqn (89), changing the variables $\boldsymbol{p}$ to $\boldsymbol{p}_0$, and singling out the terms proportional to $\exp[\mathrm{i}(\omega t - kz')]$, we obtain the expression for the x'-projection of the current which is responsible for the amplification of the electromagnetic wave [Eqn (191)]:

$$j_{x'} = \pi e^2 \rho_0 \sin^2(\theta E_{x'}) \exp \mathrm{i}(\omega t - kz') \times \int_{-\infty}^{+\infty} \mathrm{d}p_{0z} \int_0^\infty \mathrm{d}p_{0\perp} p_{0\perp}\,\frac{v_{0z}}{\omega - k_z v_{0z}}\,\frac{\partial f_0}{\partial p_{0z}}\,J_0^2\!\left(k_x\,\frac{v_\perp}{\Omega}\right) . \tag{196}$$

We integrate with respect to the variable p_{0z} in accordance with the rule given by Eqn (132). By substituting the resultant expression into Eqn (91) and taking into account the fact that the distribution function [Eqn (190)] is maximum for $p_{0x} = p_{0y} = 0$ and $p_{0z} = p_0$, we obtain the gain of the Cherenkov laser in a constant magnetic field,

$$\Gamma = -256\sqrt{\pi}(\ln 2)^{5/2}\rho_0 r_0 \lambda \sin^2\theta\cos\theta$$
$$\times\left(\frac{p_0}{mc}\right)^2 \frac{p_0 mc}{\Delta_\perp^2\Delta_{||}^3}\int_0^\infty \mathrm{d}p_{0\perp}(p_{0z}-p_0)J_0^2\left(\frac{k_x p_{0\perp}}{m\Omega_0}\right)p_{0\perp}$$
$$\times\exp\left[-4\ln 2\,\frac{p_{0\perp}^2}{\Delta_\perp^2}-4\ln 2\,\frac{(p_{0z}-p_0)^2}{\Delta_{||}^2}\right]. \qquad (197)$$

Here $\Omega_0 = |e|H_0/mc$, and the quantity p_{0z} is found from the equation

$$\omega - k_z v_{0z} = 0\,. \qquad (198)$$

To simplify the subsequent analysis we extract the explicit dependence of p_{0z} on $p_{0\perp}$. Let the solution to Eqn (198) have the form $p_{0\perp}=0$ for $p_{0z}=b$. If $p_{0\perp}\neq 0$, then the quantity p_{0z} can be presented in the form

$$p_{0z} = b + q_1 p_{0\perp} + q_2 p_{0\perp}^2\,. \qquad (199)$$

If we substitute the last expression into Eqn (198), then $q_1=0$, and $q_2=(b/mc)^2/2b$. The expansion of Eqn (199) is true for $q_2\Delta_\perp^2 \ll b$. Since $b_0\approx p_0$ the average momentum — or the average energy $\mathscr{E}_0=[(mc^2)^2+c^2p_0^2]^{1/2}$ — of the electron beam is limited by the inequalities

$$0 < p_0 \ll 1.4\,\frac{mc}{\delta}\,, \qquad (200)$$

where $\delta=\Delta_\perp/p_0$ is the angular spread of the electron beam. The explicit dependence of the integrand on $p_{0\perp}$ can be extracted by substituting Eqn (199) into Eqn (197). However, the integration cannot be performed exactly.

We consider the two limiting cases:
(1) the magnetic field H_0 is arbitrary but the momentum of electrons is limited by the inequalities

$$q_2\Delta_\perp^2 \ll |b-p_0|$$

or

$$0 < p_0 \ll 1.4\,\frac{mc}{\sqrt{\delta}}\left(\frac{\Delta_{||}}{\Delta_\perp}\right)^{1/2} \qquad (201)$$

[the latter is a more severe constraint than Eqn (200)];
(2) the magnetic field is large [see Eqn (206)] and the momentum of electrons belongs to the interval [Eqn (200)].

In the first case it follows from the inequality (201) that $p_{0z}-p_0\approx b-p_0$. Taking into account this last fact and integrating Eqn (197) with respect to the variable $p_{0\perp}$, we have

$$\Gamma = -32\sqrt{\pi}(\ln 2)^{3/2}\rho_0 r_0\lambda\sin^2\theta\cos\theta\left(\frac{p_0}{\Delta_{||}}\right)^2\frac{p_0}{mc}I_0(R)$$
$$\times\frac{b-p_0}{\Delta_{||}}\exp\left[-4\ln 2\,\frac{(b-p_0)^2}{\Delta_{||}^2}-R\right]. \qquad (202)$$

Here,

$$R=\frac{\Delta_\perp^2\omega^2 n^2\sin^2\theta}{8\ln 2(mc)^2\Omega_0^2} \qquad (203)$$

is the argument of the modified Bessel function of zero order.

Clearly, in the region defined by Eqn (201) the gain [Eqn (202)] grows proportionally to the average momentum (or to the average energy $\mathscr{E}_0=p_0c^2/v_0$) of the electron beam. The gain is maximum if the detuning is

$$b-p_0=-\frac{\Delta_{||}}{(8\ln 2)^{1/2}}\,, \qquad (204)$$

and the parameter $R\ll 1$:

$$\Gamma = 8.4\rho_0 r_0\lambda\beta_0\sin^2\theta\cos\theta\left(\frac{p_0}{\Delta_{||}}\right)^2\frac{\mathscr{E}_0}{mc^2}\,. \qquad (205)$$

Taking into account the definition of R [Eqn (203)], we establish that the constant magnetic field is limited by the inequality:

$$H_0 \gg 0.4\,\frac{\Delta_\perp}{|e|}\,\omega n\sin\theta\,. \qquad (206)$$

We shall now examine the gain given by Eqn (197) in a wider range of average momenta [Eqn (200)] [case (2)]. We suppose that the magnetic field strength satisfies the condition (206). Since $p_{0\perp}\lesssim\Delta_\perp$, we have in this case that $J_0^2(k_x p_{0\perp}/m\Omega_0)\approx 1$ in the integrand of Eqn (197). Integrating with respect to the variable $p_{0\perp}$, we get

$$\Gamma = 32\sqrt{\pi}(\ln 2)^2\rho_0 r_0\lambda\sin^2\theta\cos\theta$$
$$\times\left(\frac{p_0}{mc}\right)^2\frac{p_0 mc}{\Delta_\perp^2\Delta_{||}^2}\exp\left[-4\ln 2\,\frac{(b-p_0)^2}{\Delta_{||}^2}\right]$$
$$\times\frac{1}{a}\left\{\frac{\sqrt{\pi\ln 2}}{q_2}\,\frac{\Delta_{||}}{\Delta_\perp^2}\exp\left(\frac{d^2}{4a^2}\right)\left[1-\Phi\left(\frac{d}{2a}\right)\right]-1\right\}. \qquad (207)$$

Here the quantity $a=2\sqrt{\ln 2}\,q_2/\Delta_{||}$, the parameter

$$\frac{d}{2a}=\sqrt{\ln 2}\left(1+2q_2\Delta_\perp\frac{\Delta_\perp}{\Delta_{||}}\,\frac{b-p_0}{\Delta_{||}}\right)\left(q_2\Delta_\perp\frac{\Delta_\perp}{\Delta_{||}}\right)^{-1}, \qquad (208)$$

and the function $\Phi(x)$ is the probability integral [55].

We suppose that the longitudinal and transverse spreads of the electron beam are of the same order: $\Delta_{||}\sim\Delta_\perp$, and the detuning is $b-p_0\sim\Delta_{||}$. In this case the argument of the probability integral [Eqn (208)] is a function of the parameter

$$q_2\Delta_\perp=\frac{1}{2}\left(\frac{b}{mc}\right)^2\frac{\Delta_\perp}{b}\,.$$

If $q_2\Delta_\perp\ll 1$, then the quantity $d/2a\gg 1$. Taking into account the asymptotic expansion of the function $\Phi(x)$ for $x\gg 1$, given by

$$\Phi(x)=1-\frac{1}{\sqrt{\pi}x}\left(1-\frac{1}{2x^2}\right)\exp(-x^2) \qquad (209)$$

and the inequality $R\ll 1$, we have that the expression (207) coincides with Eqn (202) in this limit and with Eqn (205) under the condition (205).

We use this fact to sharpen the upper bound for the momentum p_0. We suppose that the detuning $b-p_0$ is specified by Eqn (204) for all values of p_0. The expansion (209) is true for $1/2x^2\ll 1$. For definiteness we set $1/2x^2=0.1$. Then, by use of Eqn (208), we find that the gain of the Cherenkov laser has the form of Eqn (205) in the region

$$0<p_0\leqslant 0.9mc\left(\frac{p_0}{\Delta_\perp}\,\frac{\Delta_{||}}{\Delta_\perp}\right)^{1/2}. \qquad (210)$$

The argument of the probability integral, $x=d/2a$, first decreases to zero ($q_2\Delta_\perp\sim 1$) and then to -1 ($q_2\Delta_\perp\gg 1$) on further increase in the momentum, p_0. The gain is inversely proportional to the average energy of the electron beam within the region: $\Gamma\propto mc^2/\mathscr{E}_0$.

Note also that in this range of momenta the expression in curly brackets in Eqn (207) first decreases to zero and then becomes negative. This effect is due to the fact that the momentum $p_{0z} = b + q_2 \Delta_\perp^2$ [see Eqn (199)] becomes larger than p_o and the electron beam absorbs the energy of the electromagnetic wave on the right-hand wing of the distribution function (199) on increase in p_o. Thus the region where the gain grows linearly on increase in the average momentum of the electron beam is limited by the inequality (210).

We shall show in conclusion that linear, not cubic [see Eqn (144)], dependence of the gain [Eqn (205)] on the average energy of electrons can be interpreted simply by using the law of conservation of energy and momentum (see Section 2.15). On solving the system of equations (212) for $r = l$, we find that the z-projections of momenta of electrons involved in emission and absorption of a photon differ in Δp, where

$$\Delta p = \frac{\hbar\omega}{2v_0} \, . \tag{211}$$

Clearly, in this case the asymmetric part of the momentum and, consequently, of the gain does not contain the factor $(p_0/mc)^2(n^2-1)$ characteristic of the Cherenkov effect [see Eqn (150)].

Note that the theory of the Cherenkov laser was also developed by Walsh [50]. Our investigations show that one of the principal parameters on which the operation of the Cherenkov laser depends is the angular spread of the electron beam, δ. The principal distinction between Walsh's work and our work is that he does not consider the angular spread of the particle beam. To justify his model, Walsh supposes that an infinitely large magnetic field is applied along the electron beam. Clearly, the magnetic field does not eliminate the angular spread of an electron beam. Let us consider a spatially homogeneous electron beam be described at the point (x, y, z) by the distribution function (190). If a constant magnetic field is applied along the beam (the z axis), then electrons start to rotate in the xy plane. If the electron beam is homogeneous, then the number of electrons which have a momentum $p_\perp$ and start from the point is exactly equal to the number of electrons which have the same momentum and come to the same point. Therefore the angular spread of electrons is the same. (Here we have not accounted for effects such as the magnetobraking radiation and scattering of electrons by molecules of the medium). This conclusion is verified by exact calculations with the use of Eqns (192), (193). Therefore the energy and the angular spreads are to be taken into account if one wants to describe operation of the Cherenkov laser in the region [Eqn (147)].

2.15 Stimulated cyclotron radiation near the Cherenkov cone

The analysis of Sections 2.9 and 2.14 shows that a constant magnetic field has a significant effect on the SCE. We shall now study the influence of a dielectric medium on the stimulated cyclotron radiation of electrons. The stimulated radiation of electrons moving in a constant magnetic field in vacuum (the cyclotron resonance laser) has been well studied theoretically and experimentally [56–58].

We will see here how the feasible region of such generators can be extended to the regions of the infrared and visible spectrum of electromagnetic waves. The central difficulty of this problem, i.e., how to create high-intensity magnetic fields and high-precision high-energy electron beams, can be bypassed by considering the cyclotron radiation in a dielectric medium near the Cherenkov cone [Eqn (1)].

We shall examine the operational characteristics of the cyclotron resonance laser using the laws of conservation of energy and momentum together with the dispersion equation:

$$\mathscr{E}_r \pm \hbar\omega = \mathscr{E}_l \, , \quad p_z^\pm \pm \hbar k_z = p \, , \quad \frac{\omega^2}{c^2} n^2 = |\boldsymbol{k}|^2 \, . \tag{212}$$

Here the energy $\mathscr{E}_r = [(mc^2)^2 + c^2p_z^2 + 2mc^2\hbar\Omega_0(r+1/2)]^{1/2}$, $r = 0, 1, 2, \ldots$ are arbitrary numbers, n is the index of refraction of the gaseous atmosphere, $\Omega_0 = |e|H_0/mc$ is the Larmor frequency, H_0 is the strength of the constant magnetic field which is applied in the direction opposite to the z axis. Emission or absorption of a photon is accompanied by electron transition from the level r to the level l.

If $r = l$, then Eqns (212) describe the SCE (see Sections 2.9, 2.14). Other transitions $(r \neq l)$ describe the stimulated cyclotron radiation on all harmonics of the Larmor frequency. We dwell on the first harmonic: $r - l = \pm 1$. Let the photon beam be directed along the z axis $(k_x = 0)$. Once the quantum numbers r and l are determined and the frequencies Ω_0 and ω are fixed, the system of equations (212) specifies the z-projections of the momenta of electrons which are involved in emission $(-)$ and absorption $(+)$,

$$p_z^\mp = p_0 \pm \Delta p \, . \tag{213}$$

The quantity p_0 is found from the equation

$$\omega = \frac{\Omega_0}{1 - n\beta_r} \frac{mc^2}{\mathscr{E}_r} \tag{214}$$

and the asymmetric part of the momentum from

$$\Delta p = \frac{\hbar k}{2} \frac{n^2 - 1}{n(n - \beta_r)} \, . \tag{215}$$

Here $\beta_r = p_0 c/\mathscr{E}_r$. If $n > 1$ then $\Delta p > 0$, and amplification, as is the case with Eqn (150), occurs on the left-hand wing of the Gaussian spread of the electron beam in the z-projections of momentum.

Note that in the adopted geometry (the wave vector of photons and the constant magnetic field are directed along the z axis):

(a) the asymmetric part of the momentum and, consequently, the gain of the cyclotron laser $(\Gamma \sim \Delta p\, \partial f_0/\partial p_0)$ are nonzero in a dielectric medium $(n \neq 1)$ only;

(b) the Cherenkov factor $(1 - n\beta_r)$ in Eqn (214) can be made arbitrarily small; thus amplification can be obtained in the region of the optical frequencies even for weak magnetic fields (clearly, $n < 1$ for x-ray frequencies and the cited consideration is no longer true).

We shall find the gain for the electromagnetic wave described by

$$A_x = A_1 \sin(\omega t - kz) \, , \quad A_y = A_2 \cos(\omega t - kz) \, , \tag{216}$$

by using system of equations (88), (91). The constant magnetic field is assumed to be applied along the z axis. The electron propagation function is found from Eqn (88) by taking account of the constant magnetic field exactly and the amplified wave in the first approximation:

$$f(\boldsymbol{p}) = f_0(\boldsymbol{p}_0) + f_1 \, . \tag{217}$$

Here the characteristics $\boldsymbol{p}_0$ are specified by the expressions (193) and the linear addition with respect to the field [Eqn (216)] by

$$f_1 = -e\{G_1 \exp \mathrm{i}[(\omega+\Omega)t - kz] + B_1 \exp \mathrm{i}[(\omega-\Omega)t-kz]\} + \text{c.c.}\,, \tag{218}$$

where

$$G_1 = -\mathrm{i}\frac{G_2E_+}{\Omega+\omega(1-n\beta_{0z})} + \frac{B_2\Omega E_+}{[\Omega+\omega(1-n\beta_z)]^2} - \mathrm{i}t\,\frac{B_2\Omega E_+}{\Omega+\omega(1-n\beta_{0z})}\,,$$

$$B_1 = \mathrm{i}\,\frac{G_2^*E_-}{\Omega-\omega(1-n\beta_{0z})} + \frac{B_2^*\Omega E_-}{[\Omega-\omega(1-n\beta_z)]^2} + \mathrm{i}t\frac{B_2^*\Omega E_-}{\Omega-\omega(1-n\beta_{0z})}\,,$$

$$G_2 = \frac{1-n\beta_{0z}}{2}\left(\frac{\partial f_0}{\partial p_{0x}} - \mathrm{i}\,\frac{\partial f_0}{\partial p_{0y}}\right) + \frac{n\beta_-}{2}\,\frac{\partial f_0}{\partial p_{0z}}\,,$$

$$B_2 = \frac{1}{2}\beta_-\left(\beta_{0y}\,\frac{\partial f_0}{\partial p_{0x}} - \beta_{0x}\,\frac{\partial f_0}{\partial p_{0y}}\right),$$

$$E_\pm = \frac{1}{2}\,\frac{\omega}{c}(-A_1 \pm A_2)\,, \qquad \beta_\pm = (\beta_{0x} \pm \mathrm{i}\beta_{0y})\,,$$

$$\Omega = \Omega_0\frac{mc^2}{\mathscr{E}}\,, \qquad \mathscr{E} = [(mc^2)^2 + c^2\boldsymbol{p}_0^2]^{1/2}\,. \tag{219}$$

The terms proportional to B_1 describe the amplification at the normal frequency $(1-n\beta_{0z} > 0)$, and the terms proportional to G_1 the amplification at the abnormal frequency $(1-n\beta_{0z} < 0)$. In what follows we consider only the first case and assume that the amplified wave is circularly polarised $(A_1 = A_2 = A_0)$:

$$E_+ = 0\,, \qquad E_- = -\frac{\omega}{c}A_0\,. \tag{220}$$

Retaining only the time-dependent terms [such as $\exp(\mathrm{i}\omega t)$] in expression (89) for the current, we have

$$j_x = \frac{\mathrm{i}}{4}\rho_0 e^2\frac{\omega}{c}A_0 \exp \mathrm{i}(\omega t - kz) \times \int_0^\infty \mathrm{d}p_{0\perp}\int_{-\infty}^{+\infty}\mathrm{d}p_{0z}\int_0^{2\pi}\mathrm{d}\varphi\,\frac{v_{0\perp}p_{0\perp}(1-n\beta_{0z})}{\Omega-\omega(1-n\beta_{0z})+\mathrm{i}\eta} \times\left[\left(\frac{\partial f_0}{\partial p_{0\perp}} + \frac{\mathrm{i}}{p_{0\perp}}\,\frac{\partial f_0}{\partial\varphi}\right) + n\beta_{0\perp}\frac{\partial f_0}{\partial p_{0z}}\right]\Bigg|_{\eta\to 0} + \text{c.c.} \tag{221}$$

Expression (221) is written in the cylindrical coordinate system $(p_{0\perp}, p_{0z}, \varphi)$. If the amplitude, A_0, varies slowly with the z coordinate, then the gain of the cyclotron laser can be found:

$$\Gamma = \mathrm{Re}\left\{\mathrm{i}\frac{2\pi}{n}\,e^2\rho_0\int_0^\infty \mathrm{d}p_{0\perp}\int_{-\infty}^{+\infty}\mathrm{d}p_{0z}\int_0^{2\pi}\mathrm{d}\varphi \times\frac{\beta_{0\perp}p_{0\perp}(1-n\beta_{0z})}{\Omega-\omega(1-n\beta_{0z})+\mathrm{i}\eta} \times\left[\left(\frac{\partial f_0}{\partial p_{0\perp}} + \frac{\mathrm{i}}{p_{0\perp}}\,\frac{\partial f_0}{\partial\varphi}\right) + n\beta_{0\perp}\frac{\partial f_0}{\partial p_{0z}}\right]\right\}\Bigg|_{\eta\to 0}\,. \tag{222}$$

We choose the initial electron distribution function, with regard for the cylindrical symmetry of the problem, in the form:

$$g(\mathscr{E},\theta) = \frac{4\ln 2}{\pi}\,\frac{1}{\Delta\delta} \times \exp\left\{-4\ln 2\,\frac{(\mathscr{E}-\mathscr{E}_0)^2}{\Delta^2} - 4\ln 2\,\frac{(\theta-\theta_0)^2}{\delta^2}\right\}, \tag{223}$$

where θ is the angle between the velocity of the electron and the z axis. We are now going from the electron distribution function in the momentum space $f_0(\boldsymbol{p})$ to the energy and angle distribution functions given by Eqn (223). From rule (132) we have that

$$\Gamma = -16\sqrt{\pi}(\ln 2)^{3/2}\rho_0 r_0\lambda\,\frac{n^2-1}{n}\,\frac{\beta_0^4\sin^2\theta_0}{(\beta_0-n\cos\theta_0)^2} \times\frac{mc^2\mathscr{E}_0}{D^2}\,\frac{\mathscr{E}_1-\mathscr{E}_0}{D}\exp\left[-4\ln 2\,\frac{(\mathscr{E}_1-\mathscr{E}_0)^2}{D^2}\right], \tag{224}$$

where $r_0 = e^2/mc^2$ is the classical radius of an electron,

$$\beta_0 = \frac{cp_0}{\mathscr{E}_0}\,, \qquad p_0 = \frac{[\mathscr{E}_0^2 - (mc^2)^2]^{1/2}}{c}\,,$$

$$D = \left[\Delta^2 + \delta^2\mathscr{E}_0^2 n^2\,\frac{\beta_0^4\sin^2\theta_0}{(\beta_0-n\cos\theta_0)^2}\right]^{1/2} \tag{225}$$

is the effective width, the energy $\mathscr{E}_1$ is found from the equation

$$\omega = \frac{\Omega_0 mc^2/\mathscr{E}_1}{1-n\beta_1\cos\theta_0}\,, \tag{226}$$

$$\beta_1 = \frac{p_1 c}{\mathscr{E}_1}\,, \quad \text{and} \quad \mathscr{E}_1 = [(mc^2)^2 + c^2p_1^2]^{1/2}\,.$$

When integrating we use the inequality

$$|\beta_0 - n\cos\theta_0| \gg \delta n\beta_0^2\sin\theta_0\,. \tag{227}$$

If the frequencies ω and $\Omega_0 = |e|H_0/mc$ are chosen such that $\mathscr{E}_1 = \mathscr{E}_0 - (D/\sqrt{8\ln 2})$, then the gain [Eqn (224)] is maximum:

$$\Gamma = 4.2\rho_0 r_0\lambda\,\frac{n^2-1}{n}\,\frac{\beta_0^4\sin^2\theta_0}{(\beta_0-n\cos\theta_0)^2} \times\mathscr{E}_0 mc^2\left(\Delta^2 + \delta^2\mathscr{E}_0^2 n^2\,\frac{\beta_0^4\sin^2\theta_0}{(\beta_0-n\cos\theta_0)^2}\right)^{-1}. \tag{228}$$

If

$$\delta n\,\frac{\beta_0^2\sin\theta_0}{|\beta_0-n\cos\theta_0|} \gg \frac{\Delta}{\mathscr{E}_0}\,,$$

then the amplification depends solely on the angular spread of the electron beam and is independent of the angle θ_0:

$$\Gamma = 4.2\rho_0 r_0\lambda\,\frac{n^2-1}{n^3}\,\frac{mc^2}{\mathscr{E}_0}\,\frac{1}{\delta^2}\,. \tag{229}$$

Clearly, the amplification factors for the cyclotron laser and the Cherenkov laser are practically the same.

Note in conclusion that the related issues of the cyclotron laser were considered in Ref. [59].

2.16 Experimental observation of the stimulated Cherenkov effect

In this section we will determine when the above effects can be observed experimentally. Clearly, it is worthwhile to use a gaseous atmosphere for the spatial SCE. Thus the negative role of multiple scatterings of electrons by atoms can be reduced. Since the index of refraction of a gaseous atmosphere is close to unity ($n = 1 + \Delta n$, where $\Delta n \sim 10^{-4}$), it follows from the condition $1 - n\beta\cos\theta = 0$ that the energy $\mathscr{E}$ and angle θ of the particle are limited by the inequalities

$$\theta^2 < 2\Delta n\,, \quad \left(\frac{mc^2}{\mathscr{E}}\right)^2 < 2\Delta n\,. \tag{230}$$

If $\Delta n = 0.5 \times 10^{-4}$, then $\theta < 10^{-2}$ rad and $\mathscr{E} > 50$ MeV.

Consequently, the SCE is only possible for relativistic electron beams in the usual gaseous atmospheres. If a resonance gas is used, then Δn can be increased up to a value of order of 10^{-2}. Thus the energy of the particles is diminished from 3.5 – 1.1 MeV and the angle θ is increased from 0.1 – 0.4 rad. In this case, however, the resonance condition imposes a severe constraint on the frequency of the electromagnetic radiation: $\omega \approx \omega_0$, where ω_0 is the frequency of the resonance transition of an atom or molecule.

We shall determine when dynamic effects can be observed. For estimates the following relations may be useful:

$$\xi_1 = 0.85 \times 10^{-9}\lambda\sqrt{P_1}\,, \quad \xi_2 = 0.61 \times 10^{-9}\lambda\sqrt{P_2}\,, \tag{231}$$

where P_1 and P_2 represent the wave power (measured in watts per square centimetre), λ is its wavelength (in micrometres), $\xi = |e|A_0/mc^2$ is a dimensionless parameter, and the indices 1 and 2 correspond to linear and circular polarisation of radiation.

If the laser radiating power is $P = 4.5 \times 10^{10}$ W cm^{-2}, $\lambda = 1.06$ μm, and the width of the beam is $2d = 2$ mm, the amplitude $\Delta\mathscr{E}$ [Eqn (14)] is equal to 10^6 eV. Since the energy of a photon is $\hbar\omega = 1.2$ eV, an electron can emit or absorb up to 10^6 photons due to the SCE. Clearly, multiple repetition of this process is necessary to achieve a significant acceleration.

We shall now evaluate the fields which are required for quantum modulation of the current and density of the electron beam [Eqns (46), (48)]. The depth of modulation runs to 10% for the laser radiating power $P = 1.6 \times 10^{-4}$ W cm^{-2}, $\lambda = 1.06$ μm, $d = 1$ mm. If the refractive index of the gaseous atmosphere is $n = 1.021$, the energy of the particles is $\mathscr{E} = 2.5$ MeV, and the angle is $\theta = 2.77 \times 10^{-2}$, the gap $x_1 = 1/\Delta q_x$ between the regions of quantum and classical modulation is equal to 45.4 cm.

Let us now assume that the mechanism of emission and absorption of photons is based on the interaction between the magnetic moment of electrons and the radiation (see Section 2.5). We suppose that the particle beam with the above parameters is fully polarised along the x axis ($\zeta_x = 1$). In this case the laser radiating power must be $P = 4.9 \times 10^7$ W cm^{-2} for the depth of modulation to run to 10 % [see Eqns (59), (60)]. We shall assess the possibility of magnetisation of a particle beam on the basis of the SCE. If an electron beam is not polarised before the interaction ($|\boldsymbol{\zeta}| = 0$), then after the interaction the magnetisation level $\eta_x = \max(I_{1x}/\rho_0\mu)$ along the x axis [see Eqn (68)] is 31.6 % for $P = 1.2 \times 10^{10}$ W cm^{-2}, $\lambda = 1.06$ μm, $n = 1.021$. The depth of magnetisation modulation of a polarised electron beam associated with the modulation of its density, given by Eqn (68), runs to 10 % for $P = 1.6 \times 10^{-4}$ W cm^{-2}.

In all cited cases the angular and energy spreads of an electron beam must satisfy the conditions (82). For the above parameters $\varDelta/\mathscr{E} \approx 0.7 \times 10^{-6}$, $\delta \approx 1.02 \times 10^{-6}$, and the constraints on the angular and energy spreads can be made weak by means of a constant magnetic field and the use of a resonance medium with $n = 1.1 - 1.01$, respectively. Note that such severe constraints on the quality of an electron beam are the main obstacle to the experimental observation of quantum effects.

We are coming to the analysis of several schemes of how to amplify the electromagnetic radiation on the basis of the SCE. Although the gain of the quantum klystron is large [see Eqn (103)], unique particle beams, as is noted above, are required for its implementation. Let us compare the gain for the Cherenkov klystron [Eqn (102)], $\Gamma = \Gamma_{\text{kl}}$, with that for the Cherenkov klystron in a constant magnetic field [Eqn (115)], $\Gamma = \Gamma^H_{\text{kl}}$. Their ratio is

$$\frac{\Gamma^H_{\text{kl}}}{\Gamma_{\text{kl}}} = \frac{n\sin\theta}{\beta_0(n^2-1)}\,\frac{\varDelta_\perp}{\varDelta_{||}}\,. \tag{232}$$

If the angle θ and the energy of particles $\mathscr{E}_0$ is related to the refractive index $n = 1 + \Delta n$ by the equation $\theta = mc^2/\mathscr{E}_0 = \sqrt{\Delta n}$, and the longitudinal and transverse spreads are of the same order $\varDelta_{||} \approx \varDelta_\perp$, then the ratio

$$\frac{\Gamma^H_{\text{kl}}}{\Gamma_{\text{kl}}} = \frac{\mathscr{E}_0}{2mc^2} \gg 1$$

for relativistic particles.

Let us compare the gains for the Cherenkov laser [Eqn (146)], $\Gamma = \Gamma_{\text{l}}$, and for the Cherenkov laser in a constant magnetic field (215): $\Gamma = \Gamma^H_{\text{l}}$. Their ratio is

$$\frac{\Gamma^H_{\text{l}}}{\Gamma_{\text{l}}} = \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 \frac{n^2\sin^2\theta}{n^2-1}\left(\frac{\varDelta_\perp}{\varDelta_{||}}\right)^2. \tag{233}$$

If $\theta = \sqrt{2\Delta n}$, $\varDelta_\perp \simeq \varDelta_{||}$, $mc^2/\mathscr{E} < \theta$, then

$$\frac{\Gamma^H_{\text{l}}}{\Gamma_{\text{l}}} = \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 \gg 1$$

in the range of momenta the condition (210) specifies. Clearly, a magnetic field enables the efficiency of amplification to be increased greatly both for the klystron and for the laser.

Let us compare the gains of the Cherenkov laser, given by Eqn (205), and of the Cherenkov klystron [Eqn (115)]:

$$\frac{\Gamma^H_{\text{l}}}{\Gamma^H_{\text{kl}}} = 0.2\,\frac{\lambda}{d}\,\frac{p_0}{\varDelta_{||}}\left(\frac{\mathscr{E}_0}{mc^2}\right)^2\beta_0\sin\theta\,. \tag{234}$$

If, for example, $d/\lambda = p_0/\varDelta_{||}$, $\theta = mc^2/\mathscr{E}_0$, then the ratio is

$$\frac{\Gamma^H_{\text{l}}}{\Gamma^H_{\text{kl}}} = 0.2\,\frac{\mathscr{E}_0}{mc^2}\,.$$

Clearly, in this case $\Gamma^H_{\text{l}} < \Gamma^H_{\text{kl}}$ if $\mathscr{E}_0 < 5mc^2$; and $\Gamma^H_{\text{l}} > \Gamma^H_{\text{kl}}$ if $\mathscr{E}_0 > 5mc^2$. Let the average energy of an electron beam be $\mathscr{E} = 12.6$ MeV; $\varDelta_{||}/p_0 = 10^{-3}$; $\rho_0 = 2 \times 10^{10}$ cm^{-3}; $n = 1.0016$; $\theta = 3.97 \times 10^{-2}$ rad; and the strength of the constant magnetic field $H = 100$ kG. Then the gain [Eqn (205)] is $\Gamma^H_{\text{l}} = 0.2$ cm^{-1} over the wavelength $\lambda = 1$ μm.

In conclusion we will examine optical polarisation effects in a system of the Cherenkov laser type (see Section 2.13). If an electron beam moves at an angle to the direction of a propagating wave and it is polarised, then a rotation of the polarisation ellipse of the test signal [Eqn (181)] results. Taking account of Eqn (188), we find that the contribution of magnetisation of the electron beam in the effect is negligible. If the density of particles is $\rho_0 = 10^{10}$ cm^{-3}, their average energy is $\mathscr{E} = 6.65$ MeV; $\delta = \Delta/\mathscr{E} = 10^{-3}$; the wavelength of the test signal is $\lambda = 0.67$ μm; the angle is $\theta = 6.9 \times 10^{-2}$ rad; the coefficient is $r = 1.6$. This implies that $\varphi_0 = 1.4 \times 10^{-5}$ rad for the region of interaction $L = 1$ cm.

Let a particle beam and a wave propagate in the same direction. In this case, rotation is due to the spins of electrons [Eqn (189)]. If the particles are fully polarised along the z axis ($\zeta_z = 1$), $\mathscr{E} = 4.98$ MeV, $\rho_0 = 10^{12}$ cm^{-3}, $L = 10$ m, then the angle is $\varphi_\zeta = 1.1 \times 10^{-7}$ rad. The refractive index (n) of the gaseous atmosphere is assumed to be equal to 1.005 in both cases.

3. The stimulated surface Cherenkov effect (SCCE)

In the Cherenkov laser and Cherenkov klystron, the electron beam propagates in a dielectric medium at an angle to the amplified radiation. Clearly, this presents problems in the transportation of the electron beam and restricts the region of its interaction with the wave. The region of interaction between electrons and the wave can be chosen arbitrarily if the amplified wave propagates in a waveguide and the electron beam moves over its surface.

As is known from Ref. [14], electrons moving over a dielectric medium can spontaneously emit electromagnetic radiation if the electron velocity, the frequency, and the wave vector of the wave are related by the equation,

$$\omega - k_z v = 0 \ . \tag{235}$$

Here the z axis is directed along the surface of the medium, v is the velocity of an electron along the same axis, and k_z is the z-projection of the wave vector of the wave. Equation (235) coincides with the condition for the spontaneous emission of an electromagnetic wave in an unbounded dielectric medium, given by Eqn (1). Note that in both cases the spontaneous emission intensities are quantities of the same order provided that the distance from the electron to the surface of the dielectric medium is not large.

The situation is totally different in the case of the SCE. In the first case (an unbounded medium) the asymmetric part of the loss is always nonzero [see Eqns (42), (150)] and, thus, the electromagnetic wave can be amplified (see Sections 2.8, 2.10); in the second case it is often impossible to separate the emission and the radiation of photons. We shall show this using the law of conservation of energy of momentum.

Let an electromagnetic wave propagate in a plane waveguide placed in a dielectric (gaseous) medium. The expressions for the projections of the electric field within the waveguide and outside it, and the dispersion equation are all given in Section 3.1 [see Eqns (247)–(250)]. Note that unlike in an unbounded medium, the x-projection of the wave vector over the surface of the dielectric is a purely imaginary quantity: $k_x = \mathrm{i}q_x$.

To establish the conditions for emission and absorption of a photon we use the Klein–Gordon equations (33). The principal problem is how to choose the initial wave function of the electron. From the classical standpoint the electron moves only over the surface of the dielectric. To obtain a quantum mechanical description adequate to the classical description we suppose that the wave function of the electron has the form of the de Broglie wave over the surface of the waveguide and is zero inside the waveguide:

$$\psi = \sqrt{\frac{\rho_0}{2\mathscr{E}}} \exp\left(-\mathrm{i}\,\frac{\mathscr{E}}{\hbar}\,t + \mathrm{i}\,\frac{p_z}{\hbar}\,z\right), \qquad x > a \ ,$$

$$\psi = 0 \, , \qquad x < a \ .$$

(The field of the crystal is approximated by an infinitely high potential barrier.) This choice is in good agreement with experimental results according to which slow electrons penetrate a crystal thickness of several atomic layers [Maradulin A in *Defekty i Kolebatel'nyi Spektr Kristallov* (Defects and Vibrational Spectrum of Crystals) (Moscow: Nauka, 1968), p. 359].

We assume that the field, described by Eqn (247), is adiabatically slow, and turned on and off at $z = \mp\infty$. The z-projection of the vector potential is presented as

$$A_z = -\frac{\mathrm{i}c}{2\omega}\,E_{1z} \exp\left[\mathrm{i}(k_z z - \omega t) - q_x x\right] + \text{c.c.}$$

In the linear approximation with respect to the field the wave function of an electron takes the form

$$\psi = \psi_0 + \psi_- + \psi_+ \ ,$$

where ψ_0 is the initial wave function of the electron, and the terms $\psi_\mp$ describe emission and absorption of a photon.

We shall seek the function $\psi_\mp$ in the form

$$\psi_\mp = \varphi_\mp(z) \exp\left[-\frac{\mathrm{i}}{\hbar}(\mathscr{E} \mp \hbar\omega) - q_x x\right] .$$

By substituting this expression into Eqn (33) and integrating the second-order equation as it is done in Section 2.4, we find the wave functions after the interaction:

$$\psi_\mp = \frac{2\pi e E_z p_z c^2}{\omega} \sqrt{\frac{\rho_0}{2\mathscr{E}}}$$
$$\times \exp\left(-\mathrm{i}\,\frac{\mathscr{E} \mp \hbar\omega}{\hbar}\,t - q_x x + \mathrm{i}\frac{p_z \mp \hbar k_z}{\hbar}\,z\right)$$
$$\times \delta\left[(\mathscr{E}^\mp \mp \hbar\omega)^2 + (\hbar q_x c)^2 - (mc^2)^2 - (p_z \mp \hbar k_z)^2 c^2\right] .$$

Taking into account the relation (248) we establish that an electron can emit or absorb a photon when its initial energy ($\mathscr{E} \to \mathscr{E}^\mp$) and momentum ($p_z \to p_z^\mp$) satisfy the conditions,

$$(\mathscr{E}^\mp \mp \hbar\omega)^2 + (\hbar q_x c)^2 - (p_z^\mp \mp \hbar k_z)^2 c^2 = (mc^2)^2 \ ,$$
$$\frac{\omega^2}{c^2}\,n_1^2 = k_z^2 - q_x^2 \ . \tag{236}$$

Since the electron moves in the region $x > a$, and the field [Eqn (247)] does not have a determinate value of the x-projection of the wave vector, there is no point in suggesting a determinate value of the x-projection both for the initial and for the final state of the electron in the case of the SCCE in contrast to the SCE.

Formally we assume that the momentum of the particle can take imaginary values. In this case conditions (236) can be found with the aid of the system of equations

$$\mathscr{E}^{\mp} \mp \hbar\omega = \mathscr{E}_2 , \quad p_z^{\mp} \mp \hbar k_z = p_{2z} ,$$
$$\mp\hbar k_x^{\mp} = p_{2x} , \quad \frac{\omega^2}{c^2} n_1^2 = k_x^2 + k_z^2 . \tag{237}$$

Here $k_x^+ = \mathrm{i}q_x$, $k_x^- = (k_x^+)^*$, where $+$ corresponds to absorption of a photon and $-$ to emission of a photon. Although only the corollary [Eqn (236)] to this system of equations is physically meaningful [in calculations $\mathscr{E}_2^2 = (mc^2)^2 + (c|\boldsymbol{p}_2|)^2$, $(p_{2x}^{\mp})^2 = -|p_{2x}^{\mp}|^2$ are to be taken into account], it is convenient for analysis since it is analogous to the ordinary system of equations (148).

According to Eqn (236), in vacuum ($n_1 = 1$) the same electron is involved in both processes. Its velocity can be found from Eqn (235). Since the absolute values of the amplitudes of both processes also coincide, the depth of the klystron modulation (see Section 2.8), the overpopulation of the electron beam (see Section 2.11), and the gains for the Cherenkov laser and Cherenkov klystron are zero. This peculiar degeneration can be removed in three ways: (1) by placing the waveguide in a gaseous atmosphere; (2) by applying a constant magnetic field along the waveguide; and (3) by considering amplification by particles whose velocities lie outside the Cherenkov cone.

(1) If the waveguide is placed in a gaseous atmosphere ($n_1 \neq 1$), then it follows from system of equations (236) that the energies of particles involved in emission and absorption of a photon are different:

$$\mathscr{E}^{\mp} = \mathscr{E} \pm \Delta\mathscr{E} . \tag{238}$$

Here the quantity $\mathscr{E}$ is found from Eqn (235), and the asymmetric part of the energy is

$$\Delta\mathscr{E} = \frac{1}{2}\,\hbar\omega(n_1^2 - 1)\left(\frac{p}{mc}\right)^2 . \tag{239}$$

(2) Let a constant magnetic field be applied along the surface of a waveguide placed in vacuum. The energies and momenta of electrons involved in emission and absorption are found from the equations

$$\mathscr{E}_r^{\mp} \mp \hbar\omega = \mathscr{E}_l , \quad p^{\mp} \mp \hbar k_z = p_z , \quad \frac{\omega^2}{c^2} = k_x^2 + k_z^2 . \tag{240}$$

Here $\mathscr{E}_r$, where

$$\mathscr{E}_r = \left[(mc^2)^2 + c^2 p_z^2 + 2mc^2\hbar\Omega_0\left(r + \frac{1}{2}\right) \right]^{1/2}$$

is the energy of an electron in the constant magnetic field; $r = 1, 2, \ldots$; $\Omega_0 = |e|H_0/mc$ is the Larmor frequency; k_z is the z-projection of the wave vector of the field described by Eqn (247).

If $r = l$, and the frequency ω and the wave vector k_z of the photon are given, then the system of equations (240) specifies the z-projections of the momenta of particles involved in emission ($-$) and absorption ($+$) of the photon:

$$p_z^{\mp} = p \pm \Delta p . \tag{241}$$

The quantity p is found from Eqn (235) and

$$\Delta p = \frac{\hbar\omega}{2v} . \tag{242}$$

Clearly, the degeneration is removed because the law of conservation of momentum is violated along the x axis (the direction perpendicular to the magnetic field).

(3) Let an electron, with an energy $\mathscr{E}$ and momentum $\boldsymbol{p}(0, 0, p)$ before the interaction, interact with the field [Eqn (247)] within a finite portion of the waveguide, $L \geqslant z \geqslant 0$. The calculation of the amplitudes of the probabilities of emission and absorption of a photon shows that, in this case, the law of conservation of momentum along the z axis,

$$p_z \mp \hbar k_z = p_z^{\mp} , \tag{243}$$

can be violated. Therefore emission and absorption of a photon [combined with the dispersion equation (248)] are governed by the equations

$$\mathscr{E} \mp \hbar\omega = \mathscr{E}^{\mp} , \quad \mp\hbar k_x^{\mp} = p_x^{\mp} , \quad \frac{\omega^2}{c^2} = k_x^2 + k_z^2 . \tag{244}$$

Since the waveguide is placed in vacuum, $n_1 = 1$.

The z-projection of the momentum of the electron after the interaction can be determined as follows. Taking into account

$$p_z^{\mp} = \frac{1}{c}\,[(\mathscr{E}^{\mp})^2 - c^2(p_x^{\mp})^2 - (mc^2)^2]^{1/2} ,$$

and using Eqn (244), we have

$$p_z^{\mp} = p \mp \hbar\left(k_z + \frac{\omega - k_z v}{v}\right) - \hbar k_z\,\frac{\hbar(\omega - k_z v)}{pv} . \tag{245}$$

The comparison between this inequality and Eqn (243) shows that in the field [Eqn (247)] an electron whose velocity lies outside the Cherenkov cone absorbs and emits photons which have the z-projections of the wave vectors $Q_z^{\pm}$ given by

$$Q_z^{\pm} = k_z + \frac{\omega - k_z v}{v} \mp \Delta Q \neq k_z .$$

Here the term

$$\Delta Q = k_z\,\frac{\hbar(\omega - k_z v)}{pv} . \tag{246}$$

Clearly it introduces an asymmetry in emission and absorption. Since in this case the synchronism condition (235) is not satisfied, it is not, strictly speaking, the Cherenkov effect.

A somewhat different approach to the analysis of stimulated processes in fields with singularities at certain points [in our case this corresponds to turning on and off the field at the points $z = (0, L)$] was considered by V M Arutyunyan and S G Oganesyan [*Zh. Eksp. Teor. Fiz.* **72** 466 (1977)].

All three possibilities for removing the degeneracy are used in Sections 3.2 and 3.3 to obtain a modulated electron beam. In Section 3.3 the role of the angular and energy spreads of an electron beam is examined. In Sections 3.4–3.7 we show that the SSCE can be used to develop the Cherenkov laser and the Cherenkov klystron [60–64]. In Section 3.1 the dynamics of a particle in the field of a surface wave is studied.

Since the quantum effects are thoroughly studied in Section 2 in the case of the SCE, the analysis of this section is based on the classical equations only.

3.1 Motion of an electron in the presence of a surface wave

Let a monochromatic electromagnetic wave propagate in a dielectric waveguide of thickness $2a$ and of infinite length and width (Fig. 3). The waveguide is symmetric about the yz plane. The strength of the electric field of an electromagnetic TM wave over the waveguide ($x > a$) is given by

$$E_z = E_{1z} \exp \mathrm{i}(k_x x + k_z z - \omega t) , \quad E_x = -\frac{k_z E_z}{k_x} , \tag{247}$$

where k_x, $k_y = 0$, and k_z are the projections of the wave vector $\boldsymbol{k}$ [65]. Assuming total internal reflection, we find that k_x is a pure imaginary quantity, $k_x = \mathrm{i} q_x$, where

$$q_x^2 = k_z^2 - \frac{\omega^2}{c^2} n_1^2 . \tag{248}$$

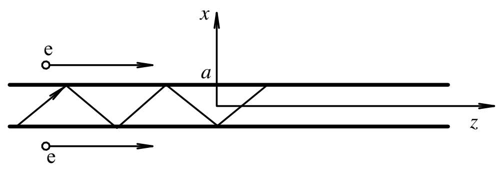

Figure 3.

(It is assumed that the waveguide is placed in a gaseous atmosphere of refractive index n_1). In the waveguide,

$$E_z = E_{2z} \sin(k'_x x) \exp \mathrm{i}(k'_z z - \omega t) ,$$

$$E_x = \mathrm{i} \frac{k'_z}{k'_x} E_{2z} \cos(k'_x x) \exp \mathrm{i}(k'_z z - \omega t) , \tag{249}$$

and the dispersion equation takes the form

$$\frac{\omega^2}{c^2} n_2^2 = (k'_x)^2 + (k'_z)^2 , \tag{250}$$

where n_2 is the refractive index of the waveguide.

Since the fields are continuous on the boundary, the z-projections of the wave vectors of the fields [Eqns (247), (249)] are related by the equations

$$k_z = k'_z , \quad \tan k'_x a = \frac{n_2^2 q_x}{n_1^2 k'_x} . \tag{251}$$

The amplitudes of the fields, given by Eqns (247), (249), are related by the equation,

$$E_{1z} = E_{2z} \sin(k'_x a) \exp(q_x a) . \tag{252}$$

The solutions to Eqn (251) give the set of natural modes of the waveguide.

Let an electron move in parallel with the surface of the waveguide before the interaction:

$$x = x_0 , \quad y = 0 , \quad z = z_0 + v_0 t . \tag{253}$$

We shall find the changes in its momentum and energy in two cases: (1) the field is turned on adiabatically slowly, and (2) the electron interacts with the wave within a finite portion of the waveguide ($L \geqslant z \geqslant 0$).

In the first case, by substituting the expressions for electric [see Eqn (247)] and magnetic [$\boldsymbol{H} = c(\boldsymbol{k} \times \boldsymbol{E})/\omega$] fields into Eqn (3) we have

$$\boldsymbol{p} = \boldsymbol{p}_0 + \Delta \boldsymbol{p} , \quad \mathscr{E} = \mathscr{E}_0 + \Delta \mathscr{E} .$$

Here

$$\Delta p_x = \frac{e E_{1z}(\omega - k_z c)}{c q_x(\omega - k_z v_0)} \exp(-q_x x_0) \cos[k_z z_0 + (k_z v_0 - \omega)t] ,$$

$$\Delta p_y = 0 ,$$

$$\Delta p_z = -\frac{e E_{1z}}{\omega - k_z v_0} \exp(-q_x x_0) \sin[k_z z_0 + (k_z v_0 - \omega)t] ,$$

$$\Delta \mathscr{E} = -v_0 \frac{e E_{1z}}{\omega - k_z v_0} \exp(-q_x x_0) \sin[k_z z_0 + (k_z v_0 - \omega)t] , \tag{254}$$

where the velocity $v_0 = p_0 c^2/\mathscr{E}_0$. Note that the expressions (254) differ from Eqns (6), (7) in the cutting factor, $\exp(-q_x x_0)$, only. Therefore all explanations to Eqn (5) remain valid here.

Let the particle and the wave move synchronously,

$$\omega - k_z v_0 = 0 , \tag{255}$$

and the field is turned on for a time 2τ by the law

$$E_{1z} = \frac{1}{2} E'_{1z} \left(1 + \frac{\tanh t}{\tau} \right) .$$

By integrating Eqn (3) with respect to time, we find that the increment of energy increases proportionally to the duration of interaction between the electron and wave for $t \gg \tau$:

$$\Delta \mathscr{E} = e E'_{1z} v_0 t \exp(-q_x x_0) \cos(k_z z_0) . \tag{256}$$

We assume now that the electron interacts with the wave within a finite portion of the waveguide $L \geqslant z \geqslant 0$. Equation (3) integrates to

$$\Delta \mathscr{E} = e E_{1z} L \frac{\sin \alpha}{\alpha} \exp(-q_x x_0) \cos\left(\alpha - \frac{\omega}{v_0} z_0 \right) , \tag{257}$$

where the detuning is $\alpha = (\omega - k_z v_0)L/2v_0$. Depending on the phase $\phi = \alpha - \omega z_0/v_0$, the particle is either accelerated ($\Delta \mathscr{E} > 0$) or decelerated ($\Delta \mathscr{E} < 0$). If the electron beam crosses the interval $[0, L]$, then the interaction results in modulation of its energy.

Since the amplitudes of the fields E_{1z} and E_{2z} are related by Eqn (252), the interaction between electrons and the wave is efficient when the aiming parameter is

$$x_0 - a < \frac{1}{q_x} . \tag{258}$$

Taking account of Eqns (248), (255) we find that

$$x_0 - a < \frac{\lambda v_0}{2\pi c} \left(1 - n_1^2 \frac{v_0^2}{c^2} \right)^{1/2} ,$$

in a gaseous atmosphere ($n_1 \neq 1$), and

$$x_0 - a < \frac{\lambda v_0}{2\pi c} \frac{\mathscr{E}_0}{mc^2} ,$$

in vacuum ($n_1 = 1$). Note also that formulae (254), (256), (257) are true when

$$|\Delta \boldsymbol{p}| < p_0 , \quad |\Delta \mathscr{E}| < \mathscr{E}_0 . \tag{259}$$

3.2 Modulation of the density and current of an electron beam

Modulation of the energy of an electron beam due to the SSCE [Eqn (256)] results in the modulation of its velocity. However, it is not sufficient, as is noted at the beginning of Section 3, to obtain the klystron modulation for the current and density of the electron beam. Note yet another fact which hampers the analysis of the modulation effect on the basis of the SSCE. Actual electron beams always have angular spreads. Clearly, if electrons move at an angle to the surface of the waveguide, they either leave the region of interaction with the field [Eqn (247)] or penetrate the waveguide and are rapidly scattered there. The number of such electrons can be diminished by applying a constant magnetic field along the central axis of the particle beam (see Section 3.3).

We will begin the analysis of the modulation effect with a simple model. We suppose that all particles of the beam move along the surface of the waveguide and have the same momentum $\boldsymbol{p}_0(0, 0, p_0)$. We determine the density and current of the electron beam after its interaction with the field given by Eqn (217), within the interval $L \geqslant z \geqslant 0$, using the kinetic equation (23). On solving it in the linear approximation with respect to the field, we find that the function

$$f = f_0 + f_1 ,$$

in the region $z > L$. Here $f_0 = \delta(p_z - p_0)\,\delta(p_x)\,\delta(p_y)$ is the initial electron distribution function,

$$f_1 = -\mathrm{Re}\left\{ L\boldsymbol{F}_1 \frac{\partial f_0}{\partial \boldsymbol{p}} \frac{\sin\alpha'}{\alpha' v_z} \times \exp \mathrm{i}\left[k_x x + \frac{\omega - k_x v_x}{v_z} z - \omega t - \alpha' \right] \right\} , \quad (260)$$

the force is

$$\boldsymbol{F}_1 = e\left[\boldsymbol{E}_1 \left(1 - \frac{\boldsymbol{k}\cdot v}{\omega} \right) + \frac{\boldsymbol{k}}{\omega} (v\cdot\boldsymbol{E}_1) \right], \quad E_{1x} = -\frac{k_z E_{1z}}{k_x} ,$$

and the detuning is

$$\alpha' = \frac{\omega - \boldsymbol{k}\cdot v}{2 v_z} L .$$

By determining the density and current of the electron beam and retaining only the terms of the klystron type we have

$$j_x = j_y = 0 , \quad (261)$$

$$j_z = j_0 \left[1 - 4\pi^2 \xi_z \frac{mc^2}{\mathscr{E}_0 \beta_0} \frac{zL}{\lambda^2} \left(n_1^2 - 1 + \frac{\omega - k_z v_0}{\omega \beta_0^2} \right) \times \frac{\sin\alpha}{\alpha} \exp(-q_x x) \sin\left(\omega t - \frac{\omega}{v_0} z + \alpha \right) \right] , \quad (262)$$

$$\rho = \rho_0 \frac{j_z}{j_0} . \quad (263)$$

Here $j_0 = e\rho_0 v_0$ is the initial electron current, $\xi_z = eE_{1z}/mc\omega$, $\beta_0 = v_0/c$, and the detuning

$$\alpha = \frac{\omega - k_z v_0}{2 v_0} L . \quad (264)$$

The notation in Eqns (261)–(263) for the current and density of electrons is convenient for different limiting cases. If the waveguide is placed in vacuum ($n_1 = 1$), and the synchronism condition (255) is satisfied, then the second term in Eqn (262) vanishes and we get

$$\boldsymbol{j} = \boldsymbol{j}_0 , \quad \rho = \rho_0 ,$$

according to the analysis at the beginning of Section 3.

If the waveguide is placed in a gaseous atmosphere ($n_1 > 1$) and the synchronism condition (255) is satisfied, then

$$j_z = j_0 \left[1 - 4\pi^2 \xi_z \frac{mc^2}{\mathscr{E}_0 \beta_0} \frac{zL}{\lambda^2} (n_1^2 - 1) \times \exp(-q_x x) \sin\left(\omega t - \frac{\omega}{v_0} z \right) \right] ,$$

$$\rho = \rho_0 \frac{j_z}{j_0} . \quad (265)$$

If the waveguide is placed in vacuum ($n_1 = 1$) and the velocities of electrons lie outside the Cherenkov cone, then $\alpha \neq 0$ and

$$j_z = j_0 \left[1 - 4\pi \xi_z \frac{mc}{p_0 \beta_0} \frac{z}{\lambda} \sin\alpha \times \exp(-q_x x) \sin\left(\omega t - \frac{\omega}{v_0} z + \alpha \right) \right] ,$$

$$\rho = \rho_0 \frac{j_z}{j_0} . \quad (266)$$

Clearly, in the last two cases the current and density of the electron beam oscillate at the frequency ω, and the depth of modulation increases in direct proportionality to the drift distance z (the klystron modulation).

Formulae (265), (266) are applicable when

$$4\pi^2 \xi_z \frac{mc^2}{\mathscr{E}_0 \beta_0} \frac{zL}{\lambda^2} (n_1^2 - 1) \ll 1 ,$$

$$4\pi \xi_z \frac{mc}{p_0 \beta_0} \frac{z}{\lambda} \sin\alpha \ll 1 . \quad (267)$$

3.3 Accounting for spreads of electrons in energies and angles

Let us consider the modulation of an electron beam which has the Gaussian spread in momenta before the interaction

$$f(\boldsymbol{p}) = \frac{(4\ln 2)^{3/2}}{\Delta_\perp^2 \Delta_{||} \pi^{3/2}} \exp\left[-4\ln 2 \frac{(p_z - p_0)^2}{\Delta_{||}^2} - 4\ln 2 \frac{p_x^2 + p_y^2}{\Delta_\perp^2} \right] . \quad (268)$$

The widths of the energy and angular spreads of this beam are specified by the expressions $\Delta = v_0 \Delta_{||}$, $\delta = \Delta_\perp / p_0$. Let electrons interact with the field [Eqn (247)] within the interval $L \geqslant z \geqslant 0$. We shall determine their density and current in the region $z > L$ when a constant magnetic field $H_z = -H_0$ is applied along the z axis. Since our concern is only with electrons which move over the surface of the waveguide, the kinetic equation must be added by the inequality

$$x > a . \quad (269)$$

On solving Eqn (88) for the constant magnetic field exactly and for the field [Eqn (247)] in the first approximation, we have

$$f = f_0 + f_1 . \quad (270)$$

Here f_0 is the initial electron distribution function (268),

$$f_1 = -\mathrm{Re}\left[eE_{1z} \frac{L}{v_z} \frac{\sin\alpha}{\alpha} \frac{\partial f_0}{\partial p_z} I_0\left(q_x \frac{v_\perp}{\Omega} \right) \times \exp\left(-q_x \frac{v_\perp}{\Omega} \sin\varphi - q_x x + \mathrm{i} \frac{\omega}{v_z} z - \mathrm{i}\omega t - \mathrm{i}\alpha \right) \right] , \quad (271)$$

$\Omega = |e| H_0 c / \mathscr{E}$, the velocity $v_\perp = c^2 (p_x^2 + p_y^2)^{1/2} / \mathscr{E}$, the detuning

$$\alpha = \frac{\omega - k_z v_z}{2v_z} L \ , \tag{272}$$

the angle φ is found from the equation, $\tan\varphi = p_y/p_x$. Note that in Eqn (271) only the term responsible for the SCE is taken into account.

We will now analyse inequality (269). When solving Eqn (88) in a constant magnetic field, we need to use the characteristics (106), in which the primes must be omitted. Simultaneously the expressions

$$x = x_0 + \frac{v_{0\perp}}{\Omega} \sin(\Omega t - \varphi') \ , \quad \tan\varphi' = \frac{p_{0y}}{p_{0x}} \ ,$$

must be substituted in inequality (269). Since this inequality must hold at any moment of time, $x_0 > a + v_{0\perp}/\Omega$. Returning to the variables $\boldsymbol{k}, v$, and considering $\Omega t - \varphi' = -\varphi$, we establish constraints on the angle φ:

$$\pi - \varphi_1 > \varphi > \varphi_1 \ . \tag{273}$$

Here

$$\varphi_1 = \frac{\pi}{2} - \alpha_1 = \arcsin\frac{a + v_\perp/\Omega - x}{v_\perp/\Omega} \ , \tag{274}$$

for $a < x < a + 2v_\perp/\Omega$; and $\varphi_1 = -\pi/2$ for $x \geqslant a + 2v_\perp/\Omega$.

Inequalities have a simple physical meaning. Since $v_\perp$ is the projection of the velocity of a particle onto the xy plane, $v_\perp/\Omega = r_l$ is its Larmor radius. If we let a particle go through a point x in the region $a < x < a + 2r_l$, its trajectory does not intersect the plain $x = a$ if the angle between the momentum $\boldsymbol{p}_\perp$ and the x axis lies in the interval defined by Eqn (273) (Fig. 4). If $a \geqslant a + 2r_l$, then the angle φ can be chosen arbitrarily. Thus inequalities (273) take up particles from the assemblage of particles going through the point x such that their trajectories pass over the surface of the waveguide.

On converting to the cylindrical coordinates in momentum space ($p_x, p_y, p_z \to \varphi, p_\perp, p_z$), we find the expression for the z-projection of the current of the electron beam,

$$j_z = j_{0z} + j_{1z} \ , \tag{275}$$

where j_{0z} is the z-projection of the current of the electron beam before the interaction,

$$j_{1z} = -\mathrm{Re}\Bigg[e^2 \rho_0 L E_{1z} \exp(-q_x x - \mathrm{i}\omega t) \times \int_{\varphi_1}^{\pi-\varphi_\perp} \mathrm{d}\varphi \exp\left(-q_x \frac{v_\perp}{\Omega}\sin\varphi\right) \int_{-\infty}^{+\infty} \mathrm{d}p_z \int_0^\infty p_\perp \, \mathrm{d}p_\perp \times I_0\left(q_x \frac{v_\perp}{\Omega}\right) \frac{\sin\alpha}{\alpha} \frac{\partial f_0}{\partial p_z} \exp \mathrm{i}\left(\frac{\omega}{v_\perp} z - \alpha\right)\Bigg] \ , \tag{276}$$

for $a < x < a + 2v_\perp/\Omega$, and

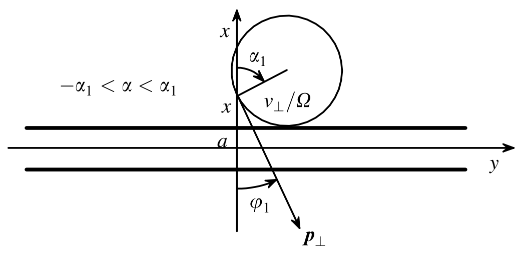

Figure 4.

$$j_{1z} = -\mathrm{Re}\Bigg[2\pi e^2 \rho_0 L E_{1z} \exp(-q_x x - \mathrm{i}\omega t) \times \int_{-\infty}^{+\infty} \mathrm{d}p_z \int_0^\infty p_\perp \, \mathrm{d}p_\perp \times I_0^2\left(q_x \frac{v_\perp}{\Omega}\right) \frac{\sin\alpha}{\alpha} \frac{\partial f_0}{\partial p_z} \exp \mathrm{i}\left(\frac{\omega}{v_z} z - \alpha\right)\Bigg] \ , \tag{277}$$

for $x \geqslant a + 2v_\perp/\Omega$.

We assume that the average momentum of the electron beam satisfies the synchronism condition $\omega - k_z v_0 = 0$. To simplify further calculations, we also assume that the strength of the constant magnetic field is

$$H_0 > \frac{mc\omega}{|e|} \delta \ . \tag{278}$$

In this case the relative contribution of the term defined in Eqn (276) to the strength of the current

$$I_z = \int_a^\infty \mathrm{d}x \int_{-l/2}^{l/2} \mathrm{d}y \, j_z \ ,$$

is of the order of $\delta mc\omega/|e|H_0 \ll 1$ and can be neglected.

Taking into account $p_\perp \leqslant \Delta_\perp$, $|p_z - p_0| \leqslant \Delta_\|$, the quantity $1/v_z$ can be expanded into the Taylor series about the point $1/v_0$ as

$$\frac{1}{v_z} = \frac{1}{v_0} - \frac{1}{p_0 v_0}\left(\frac{mc^2}{\mathscr{E}_0}\right)^2 (p_z - p_0) + \left(\frac{mc^2}{\mathscr{E}_0}\right)^2 \frac{(p_z - p_0)^2}{p_0 \mathscr{E}_0} + \frac{p_\perp^2}{2p_0 \mathscr{E}_0} \ . \tag{279}$$

By substituting this expansion into Eqn (277) and applying definition (268) we have

$$j_{1z} = j_{0z} 4\pi^2 \xi_z \left(\frac{mc}{p_0}\right)^3 \frac{zL}{\lambda^2} \times \exp\left[-\left(\frac{z}{z_1}\right)^2 - q_x x\right] \cos\left(\omega t - \frac{\omega}{v_0} z\right) \ . \tag{280}$$

Here $\xi_z = eE_{1z}/mc\omega$, and

$$z_1 = \frac{2\sqrt{\ln 2}}{\pi} \lambda \left(\frac{\mathscr{E}_0}{mc^2}\right)^2 \beta_0 \frac{p_0}{\Delta_\|} \ . \tag{281}$$

In calculations it is assumed that the distance z is given by

$$z \ll \frac{2\ln 2}{\pi} \lambda \left(\frac{p_0}{\Delta_\perp}\right)^2 \frac{1}{\beta_0} \ , \tag{282}$$

and the average momentum of the electron beam is

$$p_0 \ll 1.4 mc \frac{p_0}{\Delta_\perp} \left(\frac{\Delta_\|}{p_0}\right)^{1/2} \ . \tag{283}$$

Note that under conditions (278) and (282) the current [Eqn (280)] is independent of the angular spread of the electron beam. The energy spread defines the drift region for electrons, $z < z_1$, where z_1 is specified by expression (281).

The perturbation theory used to determine the expression (280) is true for

$$|j_{1z}| \ll j_{0z} \ . \tag{284}$$

3.4 Theory of the Cherenkov laser (a plane waveguide)

We shall consider the feasibility of amplification of an electromagnetic wave on the basis of the SSCE in the simplest case, i.e. when the waveguide is placed in a gaseous atmosphere and an electron beam moves in parallel with the waveguide and has a spread in energies only. The analysis at the beginning of Section 3 shows that electrons involved in emission and absorption have different energies [see Eqn (258)]. Since $\Delta\mathscr{E} > 0$, the number of electrons involved in emission $[N(\mathscr{E}^-) \sim g(\mathscr{E} + \Delta\mathscr{E})]$ is greater than the number of electrons involved in absorption $[N(\mathscr{E}^+) \sim g(\mathscr{E} - \Delta\mathscr{E})]$ on the left-hand wing of the Gaussian electron energy distribution; the gain $\Gamma > 0$.

We determine the gain of the Cherenkov laser using the system of equations (88), (93). The electron distribution function is determined from the kinetic equation (88) in the linear approximation with respect to the field [Eqn (247)], as:

$$f = f_0 + f_1 \,, \tag{285}$$

where f_0 is the electron distribution function before the interaction,

$$f_1 = \mathrm{i}\boldsymbol{F}\,\frac{\partial f_0}{\partial \boldsymbol{p}}\,\frac{1}{\boldsymbol{k}\cdot v - \omega + \mathrm{i}\eta}\bigg|_{\eta\to 0} \,, \tag{286}$$

where $\boldsymbol{F} = e\boldsymbol{E} + (e/c)[v \times \boldsymbol{H}]$ is the Lorentz force.

The gain Γ can be found from Eqn (93) under the assumption that electrons move on either side of the waveguide (see Fig. 3):

$$\Gamma = -2\mathrm{Re}\,\frac{1}{P}\int_a^{\infty}\mathrm{d}x\int_{-l/2}^{l/2}\mathrm{d}y\,\boldsymbol{j}\cdot\boldsymbol{E}^* \,. \tag{287}$$

Here,

$$\begin{aligned} P &= \frac{c}{4\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-l/2}^{l/2}\mathrm{d}y\,(\boldsymbol{E}^*\times\boldsymbol{H})\cdot\boldsymbol{n} \\ &= \frac{c}{4\pi}\,l\,\frac{k_z\omega}{\varepsilon_2 c}\,|E_{1z}|^2\,\frac{\exp(-3q_x a)}{k_x'^2 q_x^2} \\ &\times\left[a(\varepsilon_2^2 q_x^2 + \varepsilon_1^2 k_x'^2) + \varepsilon_1\varepsilon_2\,\frac{q_x^2 + k_x'^2}{q_x}\right] \end{aligned} \tag{288}$$

is the energy flux of the wave [Eqns (247), (249)] along the x axis; l is an arbitrary width along the y axis; $\varepsilon_{1,2} = n_{1,2}^2$ are the dielectric constants of the gaseous atmosphere and waveguide.

By substituting Eqns (89), (286), (288) into Eqn (287) and assuming that the energy spread of the electron beam is Gaussian in nature [Eqn (154)], we obtain

$$\begin{aligned} \Gamma &= 2.7\,\rho_0 r_0 \lambda\beta_0^3\left(\frac{\mathscr{E}_0}{mc^2}\right)^3\left(\frac{\mathscr{E}_0}{\varDelta}\right)^2 \\ &\times(\varepsilon_1 - 1)(\varepsilon_2\beta_0^2 - 1)(1 - \varepsilon_1\beta_0^2)\,\varepsilon_2 - \varepsilon_1 \\ &\times\left\{1 + 2\pi a\left[\varepsilon_1 + \varepsilon_2(1 - \varepsilon_1\beta_0^2)\right]\frac{(1 - \varepsilon_1\beta_0^2)^{1/2}}{\lambda\beta_0^3\varepsilon_1\varepsilon_2}\right\}^{-1} , \end{aligned} \tag{289}$$

where $\beta_0 = v_0/c = p_0 c/\mathscr{E}_0$, and r_0 is the classical radius of an electron. Clearly, in vacuum ($\varepsilon_1 = 1$), $\Gamma = 0$ according to the laws of conservation of energy and momentum.

Let us now analyse the case when electrons interact with the wave in vacuum ($\varepsilon_1 = 1$) within a bounded portion of the waveguide L (Fig. 5). The solution to the kinetic equation (88) for this scheme has the form given by Eqn (285)], where

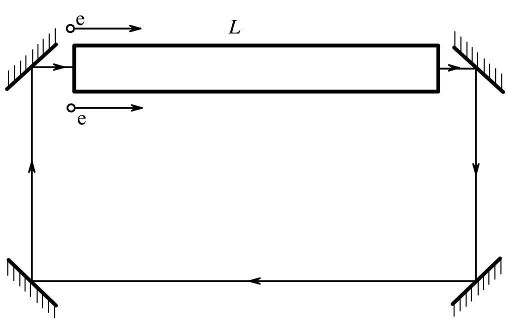

Figure 5.

$$f_1 = \mathrm{i}\boldsymbol{F}\,\frac{\partial f_0}{\partial \boldsymbol{p}}\left\{\frac{1}{\boldsymbol{k}\cdot v - \omega} + \frac{\exp[\mathrm{i}(\omega - \boldsymbol{k}\cdot v)z/v_z]}{\omega - \boldsymbol{k}\cdot v}\right\} . \tag{290}$$

The amplification, for which the first term in brackets is responsible, has been analysed above. The second term in the distribution function (290) is responsible for turning on the interaction at the point $z = 0$. The function brings about modulation of the density of electrons, the velocities of which do not coincide with the wave velocity,

$$\omega - k_z v_z \neq 0 \,. \tag{291}$$

The analysis at the beginning of Section 3 shows that this condition is satisfied for those particles of the beam for which the law of conservation of momentum is not satisfied along the z axis. We shall find conditions under which the electron beam amplifies the field [Eqns (247), (249)] and determine the gain in the case when the waveguide is a part of a circular resonator (see Fig. 5). We assume zero losses, and first integrate Eqn (93) over the space between the planes at a distance L from each other:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,W = -2\mathrm{Re}\int_0^L\mathrm{d}z\int_a^{\infty}\mathrm{d}x\int_{-l/2}^{l/2}\mathrm{d}y\,\boldsymbol{j}\cdot\boldsymbol{E}^* \,. \tag{292}$$

Here,

$$W = \frac{1}{8\pi}\int_0^L\mathrm{d}z\int_{-\infty}^{\infty}\mathrm{d}x\int_{-l/2}^{l/2}\mathrm{d}y\left[\varepsilon|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\right] , \tag{293}$$

is the energy of the electromagnetic field.

By integrating Eqn (292) with respect to time, we find the gain for the electromagnetic wave:

$$\Gamma = -2\mathrm{Re}\,\frac{1}{cW}\int_0^L\mathrm{d}z\int_a^{\infty}\mathrm{d}x\int_{-l/2}^{l/2}\mathrm{d}y\,\boldsymbol{j}\cdot\boldsymbol{E}^* \,. \tag{294}$$

Taking account of Eqns (247), (248), (89), (290), and (293) we find that the gain

$$\begin{aligned} \Gamma &= -2\pi\rho_0 r_0 L\left(\frac{mc^2}{\mathscr{E}_0}\right)^3\alpha\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\sin^2\alpha}{\alpha^2}\right) \\ &\times\frac{\varepsilon_2\beta_0^2 - 1}{\beta_0^3(\varepsilon_2 - 1)}\left\{1 + 2\pi\frac{a}{\lambda}\frac{mc^2}{\mathscr{E}_0}\left[1 + \varepsilon_2\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\right]\right\}^{-1} , \end{aligned} \tag{295}$$

where $\alpha = (\omega - k_z v_0)L/2v_0$. In calculations it is assumed that the electron beam is monoenergetic: $g(\mathscr{E}) = \delta(\mathscr{E} - \mathscr{E}_0)$. This approximation is valid for

$$\frac{\varDelta}{\mathscr{E}_0} \ll \frac{\lambda}{2\pi L}\left(\frac{\mathscr{E}_0}{mc^2}\right)^2\beta_0^3 \,. \tag{296}$$

The maximal value of the factor

$$v = -\alpha \frac{d}{d\alpha}\left(\frac{\sin^2\alpha}{\alpha^2}\right),$$

is $v = 0.8$ for $\alpha = -1.75$.

Note that this waveguide version of the Cherenkov laser was considered for the first time in Ref. [66]. The authors supposed that the electron beam could move in one direction only (see Section 2.14) and considered the z-projection of the electric field only. These suppositions result in an increase of the gain, given by Eqn (295), by the factor $(1 - k_z v_0/\omega)^{-1}(mc^2/\mathscr{E}_0)^2$.

3.5 Theory of the Cherenkov laser in a constant magnetic field

Let an electromagnetic wave propagate in the plane waveguide [Eqns (247), (249)]. Let us direct an electron beam with the Gaussian spread in momenta along the surface. The constant magnetic field $H_z = -H_0$ is applied in the direction opposite to the z axis. The gain of the Cherenkov laser is determined with the use of system of equations (88), (93).

On solving the kinetic equation (88) for the constant magnetic field exactly, and for the field, given by Eqn (247), in the first approximation, and considering the inequality $x > a$, we have

$$f = f_0 + f_1 \,. \tag{297}$$

Here f_0 is specified by the expression (268), and

$$f_1 = -ieE_{1z} I_0\left(q_x \frac{v_\perp}{\Omega}\right)\frac{\partial f_0}{\partial p_z}\frac{1}{\omega - k_z v_z} \times \exp\left(ik_z z - i\omega t - q_x x - q_x \frac{v_\perp}{\Omega}\sin\varphi\right). \tag{298}$$

In the calculation of f_1, only terms responsible for the stimulated Cherenkov effect are retained; $I_0(R)$ is the modified Bessel function; the angle φ is specified by the relation $\tan\varphi = p_y/p_x$, $p_\perp = (p_x^2 + p_y^2)^{1/2}$; and $v_\perp = p_\perp c^2/\mathscr{E}$. The inequality $x > a$ imposes constraints on the angle φ:

$$\pi - \varphi_1 > \varphi > \varphi_1 \,, \tag{299}$$

where

$$\varphi_1 = \frac{\arcsin a + v_\perp/\Omega - x}{v_\perp/\Omega},$$

for $a < x < a + 2v_\perp/\Omega$ and $\varphi_1 = -\pi/2$ for $x \geqslant a + 2v_\perp/\Omega$. The analysis of Section 3.3 shows that only electrons moving over the surface of the waveguide are taken into account in this case.

By substituting Eqn (298) into Eqn (89), we find the z-projection of the current of the electron beam:

$$j_{1z} = -ie^2\rho_0 E_{1z}\exp(ik_z z - i\omega t - q_x x) \times \int_{\varphi_1}^{\pi-\varphi_1} d\varphi \int_{-\infty}^{\infty} dp_z \int_0^\infty p_\perp\, dp_\perp\, I_0\left(q_x\frac{v_\perp}{\Omega}\right)\frac{\partial f_0}{\partial p_z}\frac{v_z}{\omega - k_z v_z} \times \exp\left(-q_x\frac{v_\perp}{\Omega}\sin\varphi\right), \tag{300}$$

for $a < x < a + 2v_\perp/\Omega$, and

$$j_{2z} = -2i\pi e^2\rho_0 E_{1z}\exp(ik_z z - i\omega t - q_x x) \times \int_{-\infty}^{\infty} dp_z \int_0^\infty p_\perp\, dp_\perp\, I_0^2\left(q_x\frac{v_\perp}{\Omega}\right)\frac{\partial f_0}{\partial p_z}\frac{v_z}{\omega - k_z v_z}, \tag{301}$$

for $x \geqslant a + 2v_\perp/\Omega$.

Let electrons move on either side of the waveguide. Then the gain is found from Eqn (93):

$$\Gamma = -2\,\mathrm{Re}\frac{1}{P}\int_a^{a+2v_\perp/\Omega} dx \int_{-l/2}^{l/2} dy\, j_{1z}E_z^* + \int_{a+2v_\perp/\Omega}^{\infty} dx \int_{-l/2}^{l/2} dy\, j_{2z}E_z^*\,. \tag{302}$$

Here the quantities j_{1z}, j_{2z}, and the flux P are specified by Eqns (300), (301), and (288). Integration with respect to the variables x, y, p_z yields

$$\begin{aligned}\Gamma = {} & -32\pi\ln 2\left(\frac{4\ln 2}{\pi}\right)^{3/2}\rho_0 r_0\lambda\,\frac{\mathscr{E}_0 mc}{\Delta_\perp^2\Delta_{||}^3}\left(\frac{p_0}{mc}\right)^2 \\ & \times \frac{k_x'^2 q_x^2}{k_z\omega[a(\varepsilon^2 q_x^2 + k_x'^2) + \varepsilon(k_x'^2 + q_x^2)/q_x]} \\ & \times\left\{\int_a^{a+2v_\perp/\Omega} dx\, \exp(-2q_x x)\int_{\varphi_1}^{\pi-\varphi_1} dy\, \exp\left(-q_x\frac{v_\perp}{\Omega}\sin\varphi\right)\right. \\ & \times\int_0^\infty p_\perp\, dp_\perp (p_z - p_0)\, I_0\left(q_x\frac{v_\perp}{\Omega}\right) \\ & \times\exp\left[-4\ln 2\,\frac{p_\perp^2}{\Delta_\perp^2} - 4\ln 2\,\frac{(p_z - p_0)^2}{\Delta_{||}^2}\right] \\ & + \frac{\pi}{q_x}\exp(-2q_x a)\int_0^\infty p_\perp\, dp_\perp (p_z - p_0)\, I_0^2\left(q_x\frac{v_\perp}{\Omega}\right) \\ & \left.\times\exp\left[-4\ln 2\,\frac{p_\perp^2}{\Delta_\perp^2} - 4\ln 2\,\frac{(p_z - p_0)^2}{\Delta_{||}^2} - 4q_x\frac{v_\perp}{\Omega}\right]\right\}.\end{aligned} \tag{303}$$

The momentum p_z is determined from the equation

$$\omega - k_z v_z = 0\,. \tag{304}$$

If we assume that $p_z = b$ for $p_x = p_y = 0$, then p_z can explicitly be expressed as a function of the projection of the momentum onto the xy plane,

$$p_z = b + q_2 p_\perp^2\,. \tag{305}$$

Here the quantity $q_2 = (1/2b)(b/mc)^2$. The expansion (305) is valid provided that $q_2\Delta_\perp^2 \ll b$ or, taking into account $b \approx p_0$, $\delta = \Delta_\perp/p_0$, that

$$p_0 \ll 1.4\,\frac{mc}{\delta}\,. \tag{306}$$

The explicit dependence of the integrand in Eqn (303) on the variable $p_\perp$ can be extracted by substituting Eqn (305) into Eqn (303). However, the integrals in Eqn (303) cannot be taken in general form. To simplify the calculations we assume that the parameter $q_x v_\perp/\Omega \ll 1$. Since the maximal value of the velocity is $v_\perp = c^2 p_\perp/\mathscr{E} \lesssim c^2\Delta_\perp/\mathscr{E}_0$, the constraint on the constant magnetic field follows the inequality

$$H_0 \gg \frac{\omega mc}{|e|}\,\delta\,, \tag{307}$$

where $\delta = \Delta_\perp/p_0$ is the angular spread of the electron beam. In this case the ratio of the first term in curly brackets in Eqn (303) to the second term is of the order of $q_x v_\perp/\Omega \ll 1$, and it can be neglected.

The analysis of Section 2.14 shows that the second integral in Eqn (303) is maximum for $q_2\Delta_\perp^2 \ll |b - p_0|$. Since $q_2 = (1/2b)(b/mc)^2$ and $|b - p_0| \sim \Delta_\|$, the average momentum of the electron beam is

$$p_0 \ll 1.4\,\frac{mc}{\sqrt{\delta}}\left(\frac{\Delta_\|}{\Delta_\perp}\right)^{1/2}. \tag{308}$$

If the detuning is $b - p_0 = -\Delta_\|/(\delta \ln 2)^{1/2}$ (see Section 2.14) and inequalities (307), (308) hold, then the final expression for the gain of the Cherenkov laser in a constant magnetic field is

$$\Gamma = 8.4\,\rho_0 r_0 \lambda\left(\frac{p_0}{\Delta_\|}\right)^2 \frac{mc}{p_0}\,\frac{(n\beta_0)^2 - 1}{\beta_0^2(n^2-1)}$$
$$\times\left\{1 + \frac{2\pi}{(n\beta_0)^2}\,\frac{amc}{\lambda p_0}\left[1 + n^2\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\right]\right\}^{-1}. \tag{309}$$

3.6 Theory of the Cherenkov laser (a tubular hollow waveguide)

We consider the amplification of an electromagnetic wave propagating in a hollow tubular waveguide (Fig. 6). The axis of the tubular waveguide, of inner and outer radii d and b, is the z axis. The projections of the electric and magnetic field strengths of the TM_0 mode onto the z axis,

$$E_z = \exp(\mathrm{i}k_\| z - \mathrm{i}\omega t)\times\begin{Bmatrix} AI_0(k_\perp r), & r < d \\ BJ_0(k'_\perp r) + CN_0(k'_\perp r), & d \leqslant r \leqslant b \\ DK_0(k_\perp r), & r > b \end{Bmatrix} + \text{c.c.},$$
$$H_z = 0, \tag{310}$$

define the other components of the wave,

$$E_r = \mathrm{i}\,\frac{k_\|}{k_\perp}\exp(\mathrm{i}k_\| z - \mathrm{i}\omega t)\times\begin{Bmatrix} -AI_1(k_\perp r), & r < d \\ -\dfrac{k_\perp}{k'_\perp}\left[BJ_1(k'_\perp r) - CN_1(k'_\perp r)\right], & d \leqslant r \leqslant b \\ DK_1(k_\perp r), & r > b \end{Bmatrix} + \text{c.c.},$$
$$E_\theta = 0, \tag{311}$$
$$H_\theta = \mathrm{i}\,\frac{\omega}{ck_\perp}\exp(\mathrm{i}k_\| z - \mathrm{i}\omega t)\times\begin{Bmatrix} -AI_1(k_\perp r), & r < d \\ -n^2\dfrac{k_\perp}{k'_\perp}\left[BJ_1(k'_\perp r) + CN_1(k'_\perp r)\right], & d \leqslant r \leqslant b \\ DK_1(k_\perp r), & r > b \end{Bmatrix} + \text{c.c.},$$
$$H_r = 0.$$

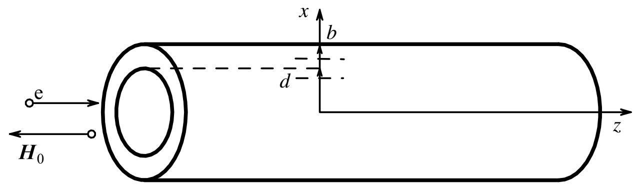

Figure 6.

Formulae (310), (311) are written in the cylindrical coordinates r, θ, z; the constants A, B, C, D are given below; J_m, N_m, I_m, K_m, where $m = 0, 1$, are common and modified Bessel functions. The projections of the wave vector onto the z axis and onto the xy plane inside and outside the waveguide are related by the equations

$$k'_\perp = \left[\frac{\omega^2}{c^2}\,n^2 - k_\|^2\right]^{1/2}, \qquad k_\perp = \left[k_\|^2 - \frac{\omega^2}{c^2}\right]^{1/2}.$$

By the sewing condition for the components of the electromagnetic field [Eqns (310), (311)] on the boundaries $r = d$ and $r = b$, we obtain the existence conditions for the modes,

$$\frac{J_0(u_2)I_1(u_1)/I_0(u_1) - n^2(u_1/u_2)J_1(u_2)}{N_0(u_2)I_1(u_1)/I_0(u_1) - n^2u_1/u_2N_1(u_2)}$$
$$= \frac{J_0(u_3)K_1(u_4)/K_0(u_4) + n^2(u_4/u_3)J_1(u_3)}{N_0(u_3)K_1(u_4)/K_0(u_4) + n^2(u_4/u_3)N_1(u_3)}, \tag{312}$$

and expressions for the dimensionless quantities B/A, C/A, and D/A:

$$\frac{B}{A} = \frac{B}{C}\,\frac{C}{A},$$
$$\frac{B}{C} = \frac{n^2(u_1/u_2)N_1(u_2) - N_0(u_2)I_1(u_1)/I_0(u_1)}{J_0(u_2)I_1(u_1)/I_0(u_1) - n^2(u_1/u_2)J_1(u_2)},$$
$$\frac{C}{A} = \frac{I_0(u_1)}{N_0(u_2) + (B/C)J_0(u_2)},$$
$$\frac{D}{A} = \frac{(B/A)J_0(u_3) + (C/A)N_0(u_3)}{K_0(u_4)}. \tag{313}$$

The parameters u_i (where $i = 1, 2, 3, 4$) have the form

$$u_1 = k_\perp d, \quad u_2 = k'_\perp d, \quad u_3 = k'_\perp b, \quad u_4 = k_\perp b. \tag{314}$$

Let a homogeneous electron beam with the Gaussian spread in momenta [Eqn (268)] enter the hollow of the waveguide and interact with the field [Eqns (310), (311)] and the constant magnetic field of strength $H_z = -H_0$. We determine the gain of the Cherenkov laser using the system of equations (88), (93). On solving the kinetic equation in the region $r < d$, we find the electron distribution function

$$f = f_0 + f_1.$$

The first term f_0 is specified by the expression (268), the linear component with respect to the field [Eqns (310), (311)] is

$$f_1 = \frac{eA}{\mathrm{i}(\omega - k_\| v_z)}\left[I_0(k_\perp r) + \frac{k_\perp v_\perp}{\Omega}\sin(\varphi - \theta)I_1(k_\perp r)\right]$$
$$\times\frac{\partial f_0}{\partial p_z}\exp(\mathrm{i}k_\| z - \mathrm{i}\omega t) + \text{c.c.}, \tag{315}$$

where $\Omega = \Omega_0 mc^2/\mathscr{E}$, $\Omega_0 = |e|H_0/mc$ is the Larmor frequency, and the angles θ and φ are related by the equations $\tan\theta = y/x$, $\tan\varphi = p_y/p_x$.

In calculations the magnetic field is assumed to be sufficiently large, i.e.

$$H_0 \gg \max\left\{\frac{c\Delta_\perp}{|e|d}\,;\;\; \frac{ck_\perp\Delta_\perp}{|e|}\right\}, \tag{316}$$

and only terms which contain the Cherenkov pole are taken into account. Given the coordinates of the point of observation $\boldsymbol{r}(r,\theta,z)$, the inequality $r<d$ implies that the constraints on the angle φ are

$$-\pi - v + \theta < \varphi < v + \theta\,, \qquad d - 2v_\perp/\Omega < r < d\,,$$
$$0 < \varphi < 2\pi\,, \qquad 0 < r < d - 2v_\perp/\Omega\,. \tag{317}$$

In this case,

$$v = \arcsin\frac{d^2 - 2dv_\perp/\Omega - r^2}{2rv_\perp/\Omega}\,. \tag{318}$$

The constraints (317) take up electrons which go through the point $\boldsymbol{r}$ and do not penetrate the walls of the waveguide.

The gain of the Cherenkov laser is found from Eqn (93):

$$\Gamma = -\frac{1}{P}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}y\,\boldsymbol{j}\cdot\boldsymbol{E}\,. \tag{319}$$

Here P is the flux of the energy of the electromagnetic wave, described by Eqns (310), (311), through the xy plane as specified by the expressions (325), (322), (323). In Eqn (325) the numerator and denumerator are time averaged.

Taking Eqns (88), (315), and (317) into account, we get the gain,

$$\Gamma = 8.4\,\rho_0 r_0\lambda\,\frac{mc}{p_0}\left(\frac{p_0}{\Delta_{||}}\right)^2(1-\gamma)G\,, \tag{320}$$

where ρ_0 is the density of the initial electron beam; r_0 is the classical radius of an electron; $\mathscr{E}_0 = p_0c^2/v_0$ is the average energy of the electron beam; and $\beta_0 = v_0/c$. The parameters γ and G are specified by the expressions

$$\gamma = \frac{I_0^2(u_1)}{I_0^2(u_1) - I_1^2(u_1)}\,\frac{c\Delta_\perp}{|e|H_0 d}\,, \tag{321}$$

$$G = \left[I_0^2(u_1) - I_1^2(u_1)\right]\left\{I_1^2(u_1) - I_0^2(u_1) + \frac{2}{u_1}\,I_0(u_1)\,I_1(u_1)\right.$$
$$+\left(\frac{Du_4}{Au_1}\right)^2\left[K_0^2(u_4) - K_1^2(u_4) + \frac{2}{u_4}\,K_0(u_4)\,K_1(u_4)\right]$$
$$\left. + \frac{n^2}{u_2^2}\left[u_4^2\,\psi(u_3) - u_1^2\,\psi(u_2)\right]\right\}^{-1}, \tag{322}$$

where the quantity

$$\psi = \left(\frac{B}{A}\right)^2(J_1^2 - J_0J_2) + \left(\frac{C}{A}\right)^2(N_1^2 - N_0N_2)$$
$$+\frac{BC}{A^2}(2J_1N_1 - N_2J_0 - N_0J_2)\,. \tag{323}$$

Formula (320) is valid under the conditions

$$p_0 \ll 1.4\,mc\sqrt{\frac{p_0}{\Delta_\perp}}\sqrt{\frac{\Delta_{||}}{\Delta_\perp}}\,, \qquad \gamma \ll 1\,. \tag{324}$$

The factor G is related to the flux of the field energy, given by Eqns (310), (311), by the equation

$$P = A^2\,\frac{d^2\omega k_z}{4k_\perp^2 G}\left[I_0^2(u_1) - I_1^2(u_1)\right]. \tag{325}$$

We shall consider the amplification of an electromagnetic wave by an electron beam of moderate energy ($\mathscr{E}_0 \geqslant mc^2$). If the parameters of the waveguide are chosen so that $d > \lambda$, the expressions (312) and (320)–(323) can be simplified. Considering that $u_i \gg 1$ and using the asymptotic representation for the Bessel function [55], we obtain

$$\tan\frac{k'_\perp(b-d)}{2} = \frac{n^2k_\perp}{k'_\perp}\,, \tag{326}$$

$$G = \left\{2 + 2\pi\,\frac{(b-d)mc}{\lambda p_0}\left[\frac{1}{n^2} + \frac{n^2}{n^2\beta_0^2 - 1}\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\right]\right\}^{-1}, \tag{327}$$

$$\gamma = 1.1\,\frac{\omega\Delta_\perp}{\Omega_0 p_0}\,. \tag{328}$$

Note that only the first (principal) terms are taken into account in the asymptotic representations of the Bessel and Neumann functions when we estimate the energy flux in the waveguide ($d > r > b$). We also cite the expression for G in the case where the second terms are retained:

$$G = \frac{n^2\beta_0^2 - 1}{2\beta_0^2(n^2-1)}\left\{1 + \pi\,\frac{(b-d)mc}{\lambda p_0}\,\frac{1}{n^2\beta_0^2}\right.$$
$$\left.\times\left[1 + n^2\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\right]\right\}^{-1}. \tag{329}$$

The quantity γ is associated with the first correction in the parameter $\omega mc\Delta_\perp/|e|H_0p_0$ (it is omitted when determining the gain in the plane waveguide). Clearly, in line with the analysis at the beginning of Section 3, the gain diminishes on decrease in the magnetic field H_0.

The feasibility of amplification of electromagnetic radiation in plane and tubular waveguides was also considered by Walsh et al. in Refs [61–64] in case of a one-dimensional electron beam ($\delta = 0$) that moves in parallel with the waveguide (we examined this model at the end of Section 2.14). In calculations of the amplification the authors assumed that the electron beam is monoenergetic ($\Delta = 0$) and they studied only the case when a large gain is achieved. Clearly, in the limit $\Delta = 0$ the mechanism of amplification is different from the one described in this paper. The role of the magnetic field is also different. Walsh et al. consider it as the leading field (see [Ref. 50], page 303), whereas we introduce it to create an asymmetry in emission and absorption of photons by electrons, i.e. to create the mechanism of amplification.

Large gains and the related issues of nonlinear effects (saturation, generation of harmonics, etc.) are not considered in this review.

Note that, strictly speaking, the magnetic field strength does not enter the original equations given by Walsh et al. as a parameter: they simply postulate that one-dimensionality of an electron beam is equivalent to a very strong magnetic field. Therefore, the results obtained on the basis of this model can be considered as qualitative.

In Refs [71–73] the experimental research of the SSCE was initiated.

3.7 Theory of the Cherenkov klystron in a constant magnetic field

The scheme in which the amplified wave moves in a waveguide and electrons move over its surface can be used for developing the Cherenkov klystron. Let electrons move in a constant magnetic field $H_z = -H_0$ and interact with the field [Eqn (247)] within two portions of the plane waveguide: $(0, L)$ is the modulating interval and $(z_0, z_0 + L_1)$ is the amplifying interval; the drift distance $z_0 \gg L, L_1$. To determine the gain of the klystron we use system of equations (88), (93).

On solving Eqn (88) for the constant magnetic field exactly and for the field [Eqn (247)] in the first approximation, we obtain the electron distribution function in the region $z > L$, given by Eqns (270), (271). By substituting Eqn (271) into Eqn (89), we find the z-projection of the current, given by Eqns (276), (277).

We shall find the gain of the Cherenkov klystron from Eqn (93) in the case where particles move on either side of the waveguide. After the interaction with electrons within the interval $[0, L_1]$ the radiation flux is given by

$$P = P_0\left(1 + \tfrac{1}{2}\,\Gamma L_1\right)^2 ,$$

where

$$\Gamma = -\frac{2}{L_1}\,\mathrm{Re}\int_a^{\infty} \mathrm{d}x \int_{-l/2}^{l/2} \mathrm{d}y \int_{z_0}^{z_0+L_1} \mathrm{d}z\,\frac{\boldsymbol{j}\cdot\boldsymbol{E}^*}{\sqrt{PP_0}} . \tag{330}$$

Here P_0 is the flux of the energy of the electromagnetic wave [Eqn (288)].

To simplify further calculations we assume that the magnetic field strength is large enough [see Eqn (278)]. In this case the current [Eqn (276)] can be neglected. We also assume that the relation,

$$\omega - k_z v_0 = 0 , \tag{331}$$

is obeyed for the average momentum of the electron beam. By substituting Eqns (277), (279) into Eqn (330) and integrating with respect to the variables $x, y, z, p_\perp, p_z$ we have

$$\Gamma = 4\pi\rho_0 r_0 L\left(\frac{mc}{p_0}\right)^5 \frac{\omega}{c}\, z_0 \exp\left(-\frac{z_0^2}{z_1^2}\right)\frac{n^2\beta_0^2 - 1}{n^2 - 1}$$
$$\times \sin(\phi + \pi)\left\{1 + \frac{2\pi}{n^2\beta_0^2}\,\frac{amc}{\lambda p_0}\left[1 + n^2\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\right]\right\}^{-1} . \tag{332}$$

Here the distance

$$z_1 = \frac{2\sqrt{\ln 2}}{\pi}\,\lambda\left(\frac{\mathscr{E}_0}{mc^2}\right)^2 \beta_0\,\frac{p_0}{\Delta_{\|}} , \tag{333}$$

and the phase difference ϕ between the current [Eqn (277)] and the amplified wave [Eqns (247), (249)] depends on the drift distance z_0. When determining Eqn (332) we assumed that the average momentum of the particle beam is

$$p_0 < 1.4mc\,\frac{p_0}{\Delta_\perp}\sqrt{\frac{\Delta_{\|}}{p_0}} . \tag{334}$$

If $z_0 = z_1/\sqrt{2}$, and the phase $\phi = 3\pi/2$, then the gain is maximum

$$\Gamma = 17.9\,\rho_0 r_0 L\,\frac{p_0}{\Delta_{\|}}\left(\frac{mc}{p_0}\right)^3 \frac{n^2\beta_0^2 - 1}{\beta_0(n^2 - 1)}$$
$$\times\left\{1 + \frac{2\pi}{n^2\beta_0^2}\,\frac{amc}{\lambda p_0}\left[1 + n^2\left(\frac{mc^2}{\mathscr{E}_0}\right)^2\right]\right\}^{-1} . \tag{335}$$

3.8 On the experimental observation of the stimulated surface Cherenkov effect

We shall determine when the SSCE can be observed. Since the refractive index of a waveguide n reaches the value of 1.5 – 2, the synchronism condition (255) can be satisfied for high-precision low-energy electrons. If $\mathscr{E}_0 \gg mc^2$, then the interaction between electrons and the wave is efficient for the aiming parameter $x_0 - a$ of the order of the wavelength λ [see Eqn (258)].

Let us consider the amplification of an electromagnetic radiation in a plane waveguide (see Sections 3.5, 3.7). The comparison of the gains for the Cherenkov laser Γ_1^H [Eqn (309)] and Cherenkov klystron Γ_{kl}^H [Eqn (335)] shows that

$$\frac{\Gamma_1^H}{\Gamma_{\mathrm{kl}}^H} = 0.5\,\frac{\lambda}{L}\,\frac{p_0}{\Delta_{\|}}\left(\frac{p_0}{mc}\right)^2 \frac{1}{n^2\beta_0} . \tag{336}$$

Let $\Delta_{\|}/p_0 = \lambda/L$. Then

$$\frac{\Gamma_1^H}{\Gamma_{\mathrm{kl}}^H} = 0.5\left(\frac{p_0}{mc}\right)^2 \frac{1}{n^2\beta_0} .$$

Clearly, in this case $\Gamma_1^H \ll \Gamma_{\mathrm{kl}}^H$ when $\beta_0 \ll 1$ and $\Gamma_1^H \gg \Gamma_{\mathrm{kl}}^H$ when $p_0 \gg 1.4nmc$. If the current of the electron beam is $1\,\mathrm{kA\ cm^{-2}}$ and its average energy is $\mathscr{E}_0 = 1$ MeV, then it follows from Eqn (309) that the optimal region of amplification for spreads $\delta = \Delta/\mathscr{E}_0 = 10^{-1}$ is the millimetre range of wavelengths. If the thickness of the waveguide is $2a = 1$ mm, its refractive index $n = 1.5$, the strength of the constant magnetic field $H \gg 22$ kG, then the gain given by Eqn (309) is $\Gamma = 0.1\ \mathrm{cm^{-1}}$ over the wavelength $\lambda = 5$ μm.

If the quality of the electron beam is one order of magnitude better ($\delta = \Delta/\mathscr{E} = 10^{-2}$), the gain $\Gamma = 0.1\ \mathrm{cm^{-1}}$ can be achieved over the wavelength $\lambda = 5$ μm. In this case the strength of the constant magnetic field is $H_0 \gg 220$ kG. (The ratio a/λ and the refractive index n are assumed to be the same in either case.)

We shall now consider amplification of an electromagnetic wave in a tubular hollow waveguide made from quartz. We assume that the wavelength of the amplified radiation is $\lambda = 4$ mm. The dielectric constant of quartz is $\varepsilon = 3.8$ in the millimetre region of wavelength. The inner and outer radii of the waveguide are chosen to be equal to 5 and 10 mm. Let the average energy of the electron beam be $\mathscr{E}_0 = 150$ keV; its density is $\rho_0 = 0.5 \times 10^9\ \mathrm{cm^{-3}}$; and its angular and energy spreads are $\delta = 10^{-2}$ and $\Delta/\mathscr{E} = 0.5 \times 10^{-2}$. If the strength of the constant magnetic field is $H_0 = 4$ kG, then the gain [Eqn (320)] is $\Gamma = 0.1\ \mathrm{cm^{-1}}$.

The exponential growth of the radiating power for the Cherenkov laser is obtained in the linear approximation with respect to the field. Clearly, this result is true in the region in which nonlinear effects are negligible. In this review we have not considered these issues. In this regard we note only that the limits of the region where the linear theory is applicable can be estimated by considering a cubic addition, with respect to the field, to the current of the electron beam and requiring that it is small in comparison with the linear part. For electron beams of moderate energies ($\mathscr{E} \sim mc^2$) this condition implies that the inequality $\xi < (\Delta/\mathscr{E})^2$ (where $\xi = |e|E_0/mc\omega$ is a dimensionless parameter). Hence, for the above parameters of the tubular waveguide the nonlinear effects become significant for a power of about 1 kW.

4. Conclusions

In this review the interaction between electrons and a monochromatic electromagnetic wave is studied in detail in an unbounded dielectric medium and over the surface of a dielectric waveguide. The dynamics of particles and also the feasibilities of modulation and polarisation of an electron beam by a laser radiation are examined. The feasibility of the Cherenkov laser and Cherenkov klystron are considered. The studies are performed in the linear approximation with respect to the field on the basis of classical and quantum approaches. The latter is especially fruitful for analysis of mechanisms underlying the cited effects, as well as in the search of possible ways to affect their course. It is established that a constant magnetic field provides extensive means for affecting the SCE.

The theory of the SCE is far from being complete. We list below the issues which are partly clear by now or are under development. On the theoretical side, there is the analysis of the quantum modulation effect when individual particles are described by a wave packet [9]. Together with A M Akopyan we conducted the theoretical work on amplification of electromagnetic momentum in the Cherenkov waveguide laser. The nonlinear theory of the SCE is in its completion stage. Note that several results in this field were obtained in Ref. [84].

There are a number of outstanding issues as regards the experimental realisation of the SCE. There is the negative role of multiple scattering of an electron beam in a medium. This difficulty can be bypassed for the SSCE. In this case, however, another problem arises — deposition of electrons on the surface of the waveguide. The electrostatic charge perturbs the trajectories of particles and thereby upsets the normal operation of the laser, while the breakdown destroys the walls of the waveguide. In this connec-tion we, together with R A Akopov and E M Lasarev, have developed the Cherenkov waveguide laser with conducting surfaces. The optimal shape of the waveguide is yet to be found. The SCE can find a very important application in the acceleration of particles.

Finally, note in conclusion of the review that the experimental and theoretical research of the SCE is fundamental in nature because the stimulated transient, diffraction, and Compton effects are reduced to it in the end.

References

1. Varfolomeev A A *Lazery na Svobodnykh Elektronakh i Perspektivy Ikh Razvitiya*: *Obzor* (Free Electron Lasers and Their Perspectives: A Review) (Moscow: Institut Atomnoi Energii, 1980)
2. Fedorov M V *Usp. Fiz. Nauk* **135** 213 (1981) [*Sov. Phys. Uspekhi* **24** 801 (1981)]
3. Karbushev N I, Rukhadze A A, Fedotov A V, et al., Preprint Phys. Inst. Acad. Sci. USSR, No. 84 (Moscow, 1982)
4. Didenko A N, Kozhevnikov A V *Izv. Vyssh. Uchebn. Zaved., Ser. Fiz.* **26** (3) 12 (1983)
5. Varfolomeev A A *Eksperimental'nye Issledovaniya LSE*: *Obzor* (Investigations of Free Electron Lasers: A Review) (Moscow: Institut Atomnoi Energii, 1987)
6. Buzzari U, Ciocci F, Dattoli C, et al. *Nuovo Cimento* **10** (5) 1 (1987)
7. *Generatory Kogerentnogo Izlucheniya na Svobodnykh Elektronakh* (Free Electron Generators of Coherent Radiation) Collection of papers translated from English, Ed. A A Rukhadze (Moscow: Mir, 1983)
8. Marshall T *Free Electron Lasers* (translated into Russian) (Moscow: Mir, 1987)
9. *Voprosy Atomnoi Nauki i Tekhniki. Seriya Yaderno-Fiz. Issled.* (Problems of Atomic Science and Engineering. Nuclear Physics Research Series) Nauchno-Tekh. Sbornik, No. 6 (14), 7 (15) (Erevan, 1990).
10. Amatuni A Ts, Lasiev E M, Nagorskii G A, et al., *Razvitie Novykh Metodov Uskoreniya Zaryazhennykh Chastits in ErFI*: *Obzor* (New Methods of Accelerating Charged Particles Developed in the Erevan Physical Institute: A Review) (Erevan: TsNIIatominform, 1989)
11. Cherenkov P A *Dokl. Akad. Nauk SSSR* **2** 451 (1934)
12. Vavilov S I *Dokl. Akad. Nauk SSSR* **2** 457 (1934)
13. Tamm E I, Frank I M *Dokl. Akad. Nauk SSSR* **14** 107 (1937)
14. Bolotovskii B M *Usp. Fiz. Nauk* **62** 201; **75** 295 (1961) [*Sov. Phys. Uspekhi* **4** 781 (1962)]
15. Jelley J V *Cerenkov Radiation and Its Applications* (Oxford: Pergamon Press, 1958)
16. Oganesyan S G, Sargsyan N A *Izv. Akad. Nauk Arm. SSR, Ser. Fiz.* **25** 201 (1990)
17. Oganesyan S G, Sargsyan N A *Izv. Akad Nauk Arm. SSR, Ser. Fiz.* **23** 112 (1988)
18. Oganesyan S G, Sargsyan N A Abadjan S V *Polyarizatsiya Elektronnykh Puchkov Lazernym Izlucheniem v Srede* (Polarization of Electron Beams by Laser Radiation in a medium) (Deposited with the Arm. NIINTI 10-Ar-87, 1987)
19. Oganesyan S G, Abadjan S V *Opt. Commun.* **73** 380 (1989)
20. Oganesyan S G, Sargsyan N A *Zh. Tekh. Fiz.* **59** 138 (1989) [*Sov. Phys. Tech. Phys.* **34** 80 (1989)]
21. Arutyunyan V M, Oganesyan S G *Pis'ma Zh. Tekh. Fiz.* **7** 539 (1981) [*Sov. Tech. Phys. Lett.* **7** 230 (1981)]
22. Oganesyan S G, Abadjan S V *Izv. Akad. Nauk Arm. SSR, Ser. Fiz.* **22** 133 (1987)
23. Oganesyan S G, Sargsyan N A *Phys. Lett. A* **140** 249 (1989)
24. Oganesyan S G, Sargsyan N A *Zh. Tekh. Fiz.* **61** 205 (1991) [*Sov. Phys. Tech. Phys.* **36** 844 (1991)]
25. Arutyunyan V M, Oganesyan S G, in *Tez. Dokl. XII Vsesoyuz. Konf. po Kogerentnoi i Nelineinoi Optike* Ch. I (Abstracts of Papers Presented at the XII All-Union Conference on Coherent and Nonlinear Optics. Part I) (Moscow: 1985) p. 351
26. Cherenkov P A, Tamm I E, Frank I M *Nobelevskie Lektsii* (Nobel Lectures) (Moscow: Fizmatgiz, 1960)
27. Frank I M *Izv. Akad. Nauk SSSR, Ser. Fiz.* **6** (1-2) 3 (1937)
28. Ter-Mikaelyan M L *Vliyanie Sredy na Elektromagnitnye Protsessy pri Vysokikh Energiyakh* (Influence of the Medium on High-Energy Electromagnetic Processes) (Erevan, 1969)
29. Ginzburg V L, Tsytovich V N *Perekhodnoe Izluchenie i Perekhodnoe Rasseyanie* (Transient Radiation and Transient Scattering) (Moscow: Nauka, 1984)
30. Arutyunyan V M, Avetisyan G K *Kvantovaya Elektron. (Moscow)* (7) 54 (1972) [*Sov. J. Quantum Electron.* **2** 39 (1972)]
31. Piesrup M A, Rothbard G B, Fleming R H, Pantell R H *J. Appl. Phys.* **46** 132 (1975)
32. Edinghoffer J A, Kimura W D, Pantell R H, et al. *IEEE J. Quantum Electron.* **17** 1507 (1981)
33. Edinghoffer J A, Kimura W D, Pantell R H, et al. *Phys. Rev. A* **23** 1948 (1981)
34. Kimura W D, Wang D Y, Piestrup M A, et al. *IEEE J. Quantum Electron.* **QE-18** 239 (1982)
35. Kimura W D, Wang D Y, Piestrup M A, et al. *Appl. Phys. Lett.* **40** 102 (1982)
36. Kimura W D, *J. Appl. Phys.* **53** 5433 (1982)
37. Mussakhanyan V V *Phys. Lett. A* **70** 313 (1979)
38. Babakhanyan E A, Musakhanyan V V *Izv. Akad. Nauk Arm. SSR, Ser. Fiz.* **16** 186 (1981)
39. Gaiduk V I, Palatov K I, Petrov D V *Fizicheskie Osnovy Elektroniki Sverkhvysokikh Chastot* (Physical Principles of Ultrahigh-Frequency Electronics) (Moscow: Sov. Radio, 1971)

40. Schwartz H, Hora H *Appl. Phys. Lett.* **15** 349 (1969)
41. Varshalovich D A, D'yakov M I *Zh. Eksp. Teor. Fiz.* **60** 90 (1971) [*Sov. Phys. JETP* **33** 51 (1971)]
42. Hora H, Handel P H *Adv. Electron. Electron Phys.* **69** 55 (1987)
43. Berestetskii V B, Lifshitz E M Pitaevskii L P *Relyativistkaya Kvantovaya Teoriya*, Ch. I (Relativistic Quantum Theory) Part I (Moscow: Nauka, 1974)
44. Landau L D, Lifshitz E M *Quantum Mechanics* (*Non-Relativistic Theory*) 3rd edition (translated from Russian; Oxford: Pergamon Press, 1977)
45. Vinokurov N A, Skrinskii A N, Preprint Inst. Nuclear Phys. 77-59 (Novosibirsk, 1977)
46. Varfolomeev A A *Zh. Eksp. Teor. Fiz.* **85** 41 (1983) [*Sov. Phys. JETP* **58** 24 (1983)
47. Wang D Y, Fauchet A.-M., Piestrup M A, Pantell R H *IEEE J. Quantum Electron.* **19** 389 (1983)
48. Lifshitz E M, Pitaevskii L P *Fizicheskaya Kinetika* (Physical Kinetics) (Moscow: Nauka, 1979)
49. Aleksandrov A F, Bogdankevich L S, Rukhadze A A *Osnovy Elektrodinamiki Plasmy* (The Elements of Plasma Electrodynamics) (Moscow: Vysshaya Shkola, 1988)
50. Walsh J E *Adv. Electron. Electron Phys.* **58** 271 (1982)
51. Landau L D, Lifshitz E M *Electrodynamics of Continuous Media* 2nd edition (translated from Russian; Oxford: Pergamon Press, 1984)
52. Ginzburg V L *Zh. Eksp. Teor. Fiz.* **10** 589 (1940)
53. Ginzburg V L *Teoreticheskaya Fizika i Astrofizika* (Theoretical Physics and Astrophysics) (Moscow: Nauka, 1987)
54. Sokolov A A, Loskutov Yu M *Zh. Eksp. Teor. Fiz.* **32** 630 (1957) [*Sov. Phys. JETP* **5** 523 (1957)]
55. Gradshtein I S, Ryzhik I M *Tablitsy Integralov, Summ, Ryadov i Proizvedenii (Tables of Integrals, Sums, Series, and Products) (Moscow: Nauka, 1971)*
56. Gaponov A V, Petelin M I, Yulpatov V K *Izv. Vyssh. Uchebn. Zaved., Ser. Radiofiz.* **10** 1414 (1967)
57. Hirshfield J L, Bernstein J B, Wachtel J M IEEE *J. Quantum Electron.* **1** 237 (1965)
58. Petelin M I *Izv. Vyssh. Uchebn. Zaved., Ser. Radiofiz.* **17** 902 (1974)
59. Avetisyan G K, Atsagortyan K Z *Zh. Tekh. Fiz.* **54** 2347 (1984) [*Sov. Phys. Tech. Phys.* **29** 1381 (1984)
60. Oganesyan S G *Kvantovaya Elektron. (Moscow)* **12** 1058 (1985) [*Sov. J. Quantum Electron.* **15** 695 (1985)]
61. Oganesyan S G, Sargsyan N A *Kvantovaya Elektron.* **16** 2183 (1989) [*Sov. J. Quantum Electron.* **19** 1404 (1989)]
62. Oganesyan S G, Abadjan S V *Opt. Commun.* **75** 197 (1990)
63. Oganesyan S G, Abadjan S V *Zh. Tekh. Fiz.* **60** (2) 187 (1990) [*Sov. Phys. Tech. Phys.* **35** 250 (1990)]
64. Oganesyan S G, Sargsyan N A *Opt. Spektroskop.* **69** 1157 (1990)
65. Vainshtein L A *Elektromagnitnye Volny* (Electromagnetic waves) (Moscow: Sov. radio, 1957)
66. Walsh J E, Murphy J B *IEEE J. Quantum Electron.* **QE-18** 1259 (1982)
67. Garate E P, Shaughnessy C H, Walsh J E *IEEE J. Quantum Electron.* **QE-23** 1267 (1987)
68. Walsh J E, Marshall T C, Schlesinger S P *Phys. Fluids* **20** 709 (1977)
69. Walsh J E, Marshall T C, Mross M R, Schlesinger S P *IEEE Trans. Microwave Theory Tech.* **25** 561 (1977)
70. Garate E P, Walsh J E *IEEE Trans. Plasma Sci.* **PS-13** (6) 524 (1985)
71. Van Laven S, Branscum J, Colub J, et al. *Appl. Phys. Lett.* **41** 408 (1982)
72. Garate E, Cook R, Heim P, et al. *J. Appl. Phys.* **58** 627 (1985)
73. Garate E P, Walsh J, Shaughnessy C, et al. *Nucl. Instrum. Methods Phys. Res. A* **259** (1-2) 125 (1987)
74. Saakyan G S *Dokl. Akad. Nauk Arm. SSR* **28** 121 (1959)
75. Davydovskii V Ya, Yakushev E M *Zh. Eksp. Teor. Fiz.* **52** 1068 (1967) [*Sov. Phys. JETP* **25** 709 (1967)]
76. Bonch-Osmolovskii A G *Zh. Tekh. Fiz.* **35** 1757 (1965) [*Sov. Phys. Tech. Phys.* **10** 1360 (1966)
77. Nagorskii G A, Orlov Yu F *Zh. Eksp. Teor. Fiz.* **64** 761 (1973) [*Sov. Phys. JETP* **37** 387 (1973)]
78. Babakhanyan E A, Musakhanyan V V, Preprint Erevan Phys. Inst. EFI-549 (36)-82 (Erevan, 1982)
79. Babakhanyan E A, Musakhanyan V V *Voprosy Atomnoi Nauki i Tekhniki. Ser. Yaderno-Fiz. Issled.* (Problems of Atomic Science and Engineering. Nuclear Physics Series) Nauchno-Tekh. Sbornik No. 6 (14) (Erevan, 1990) p. 35
80. Fontana J R, Pantell R H *J. Appl. Phys.* **54** 4285 (1983)
81. Zhuang Jie Jia *Acta Phys. Sin.* **33** 1255 (1984)
82. Harutunian V M, Avetissian H K *Phys. Lett.* **44** 281 (1973)
83. Oganesyan S G, Avetisyan G K *Izv. Akad. Nauk Arm. SSR, Ser. Fiz.* **8** 395 (1973)
84. Arutyunyan V M, Avetisyan G K *Zh. Eksp. Teor. Fiz.* **62** 1639 (1972) [*Sov. Phys. JETP* **35** 854 (1972)]
85. Musakhanyan V V, Nikishov A I *Zh. Eksp. Teor. Fiz.* **66** 1258 (1972) [*Sov. Phys. JETP* **39** 615 (1974)]
86. Ivanter I G, Lomonosov V V *Zh. Eksp. Teor. Fiz.* **80** 879 (1981) [*Sov. Phys. JETP* **53** 447 (1981)]
87. Gover A, Livni Z *Opt. Commun.* **26** 375 (1978)
88. Becker W, Mc. Iver J C *Phys. Rev. A* **25** 956 (1982)
89. Piestrup M A *IEEE J. Quantum Electron.* **QE-19** 1827 (1983)
90. Soln J *J. Appl. Phys.* **54** 2036 (1983)
91. Sargsyan N A *Pis'ma Zh. Tekh. Fiz.* **16** (3) 59 (1990) [*Sov. Tech. Phys. Lett.* **16** 109 (1990)]
92. Walsh J, Johnson B, Dattoli G, Renieri A *Phys. Rev. Lett.* **53** 779 (1984)
93. Johnson B, Walsh J *Nucl. Instrum. Methods Phys. Res. A* **237** 239 (1985)
94. Walsh J E, Johnson B, Shaughnessy C, et al. *Nucl. Instrum. Methods Phys. Rev. A* **250** 308 (1986)
95. Johnson B, Walsh J E *Phys. Rev. A* 33 3199 (1986)
96. Ciocci F, Dattoli G, De Angellis A, et al. *Nucl. Instrum. Methods Phys. Res. A* **259** 128 (1987)
97. Dattoli G, Doria A, Gallerano G P, et al. *Nuovo Cimento B* **103** (3) 281 (1989)