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George Green: his life and works (on the occasion of the bicentenary of his birthday)

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Abstract. A biography of George Green (1793 – 1841), the celebrated English mathematician and physicist, is presented, together with an analysis of all his published works. Green introduced new concepts and principles in his studies, which not only stimulated the progress of physics in the XIXth century, but re-emerged in the middle of the XXth century and have been found very effective in expressing recent advances in physics.

1. The life of George Green

The works and careers of outstanding scientists have always attracted attention. Such attention has increased when the circumstances of the life and works of the investigators were not entirely usual. This has also been the fate of George Green the English physicist, mathematician, and mechanician. There is probably no other country which, like England, has produced so many self-taught scientists: Hooke, Bull, Faraday, Joule, and Heaviside are only some of the best known. This phenomenon is particularly surprising in fundamental sciences — mathematics, and physics. The foregoing applies to a considerable extent to Green: he achieved his most important results long before joining Cambridge University — the application of the theory of the potential in electricity and magnetism, the familiar 'Green's formulae' and 'Green's function'.

His studies in the field of mathematical physics (primarily on the development of the theory of the potential) have long been recognised: none of the serious courses on higher mathematics can do without Green's formulae (relating the volume and surface integrals).

The circumstances surrounding the life and works of this outstanding English scientist have been little known hitherto and many still remain obscure. Green's most authoritative biographer † noted that 'the obscurities in his biography and historical gaps are natural consequences of the circumstances of his life' ([1], p. 550).

† Fortuitously the biographer has the same surname: H Gwynnedd Green.

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The Green family lived for a long time (approximately since the XIVth century) in the village of Saxondale six miles from the city of Nottingham; they were tenant farmers. The farm could not feed three brothers and the youngest of them, George (subsequently the father of the future scientist), had to become a baker's apprentice in Nottingham at the age of 15.‡ The apprenticeship lasted seven years, after which George Green became a burgess of the city of Nottingham enjoying full rights including the right to vote in parliamentary elections. The mother of the future physicist and mathematician, Sarah Butler, was born in 1770; she was the daughter of baker William Butler and his wife Mary (née Brewster); his parents came from a farming family living in the village Radcliffe-on-Trent, also in the vicinity of Nottingham.

George Green was born in Nottingham in all probability not earlier than the beginning of June and undoubtedly not later than the 14th June 1793. Unfortunately there are no objective data indicating the exact date of his birth; the date of birth usually quoted in the literature ([2], p. 124; [3], p. 150), namely 14th July represents the date of his christening. Admittedly Green's biographer, already mentioned above, notes ([1], p. 552) that one cannot rule out the probability that the christening took place immediately after birth: this was sometimes practised if there were serious doubts about the likelihood of the survival of the newly born baby. It is remarkable that the exact place of birth cannot be established either: George was probably born in the house of his parents on the corner of Meynell Row and Millstone Lane, although one cannot rule out that he first saw the light of day in the house of his mother's parents in Wheatsheaf Yard.

George Green's father, also George, married Sarah Butler in 1791. Apart from George, the future mathematician and physicist, a daughter was also born to this marriage in 1795; § subsequently Ann married William Tomlin, who played a role in George Green's life.

When George Green Senior moved to Nottingham in 1774, the latter was a quiet and picturesque provincial town. This is how one of his contemporaries described it: ''The situation is not exceeded by any in England and in the principal streets are many fine houses with lofty columns in the fronts, which makes them extremely grand. The streets are broad, open and well paved ... many gentlemen of great fortune reside here, which is not to be wondered at, as the prospects from the streets over the

‡ Evidently the baker's trade was extremely prestigious at the time because the Master was paid the high introductory fee of 10 guineas for George's training [4].
§ Another daughter died at the age of 8 months.

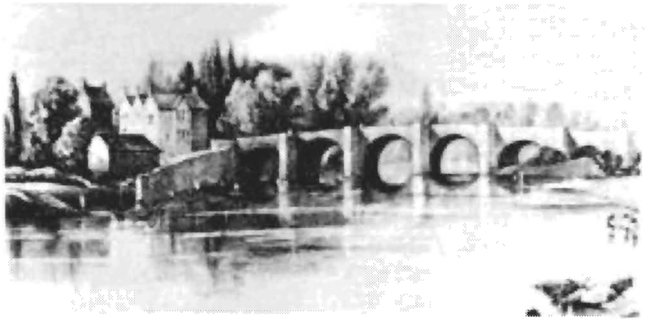

Figure 1. A view of the River Trent.

fields, and the windings of the Trent are so delightful that it even exceeds imagination.''

At the same time an intense growth of industry began in England; Nottingham got by with the development of the knitted stocking and glove industry. The need for working hands led to an influx of people, so that from 1741 to 1801 the population increased almost by a factor of 3 and amounted to 30 000 persons; by 1841 the population already exceeded 53 000. It is significant that, for social reasons, the size of the city territory hardly altered and Nottingham was transformed from a green city into a conglomerate of factory buildings, slaughter houses, scrap heaps, largely uncomfortable residential houses, and simply slums. The disappearance of orchards and greenery was accompanied by a considerable pollution of the air and an impairment of the quality of drinking water. Naturally the growth of population stimulated the expansion of trade in grain and the development of flour grinding and baking industries. The prosperity of millers and bakers arose both from the expansion of their industries and from the increase in the price of bread, which rose particularly during the Napoleonic Wars and the Continental blockade. The increase in the cost of living led to social strains in the lower strata of the city population, manifested by unrest and riots. Thus in 1800, after a sharp increase in the cost of bread, crowds of paupers devastated the granaries on the wharves along the River Trent and a number of baker's shops; among others, the house belonging to the Green family also suffered. Later Nottingham did not escape also the disturbances by Luddites ...

However, the life of young George was fairly troublefree, being spent between his parental home and the residence of his grandfather W Butler and also in the open air along the banks of the Trent. But, this idyll soon came to an end: in March 1801, George became a pupil at one of the secondary schools, where he studied until the summer of 1802.† Surprisingly, this brief education of the 8–9 year old boy proved extremely fruitful thanks to Robert Goodacre. This 25 year old headmaster was able to improve sharply the level of teaching. It is known [4] that he was able to interest the very young George in mathematics and natural science and to maintain the interest which he had already shown in these subjects. Goodacre wrote a number of school textbooks on arithmetic and educational questions. Unfortunately, school work soon ceased because Green Senior decided that his son should become his assistant in the bakery. Admittedly, on W Tomlin's evidence, by virtue of his mathematical abilities Green Junior 'quickly transcended his teachers and this was why his school attendance ceased at an early age' ([1], p. 554). Everything pointed to the prosperity of the family business because Green's father soon took on a further two apprentices.

In 1807, Green's father acquired a parcel of land with a windmill in the small town of Sneinton (several miles from Nottingham) and hence changed his trade.‡ From a still earlier period and almost until the death of his father (at the beginning of 1829), George Green had to help him initially in the bakery and then in the mill. According to certain indirect but fairly reliable information, he did not find this easy. It is known that during this period George no longer attended school. He had to educate himself. During evenings, days, and during gaps in his work in the mill George persistently expanded his knowledge. Evidently his interests were directed primarily to mathematics, mechanics, and also physics. What did he study? There exists a list of books which he used; they include Laplace's *World System* and the volumes of *Celestial Mechanics* by the same author (especially the first volume—*Analytical Mechanics*), the works of Lagrange, textbooks on mathematics and mech-anics by English authors, and also a complete collection of the annual sets of *Proceedings of the Royal Society of London* from the day of its foundation until the end of the XVIIIth century. Green used the English translations of the works of Laplace, Lagrange, and Biot, but everything points to the fact that subsequently he read the works of Coulomb and Poisson in the original. It is not clear whether he acquired an adequate knowledge of the French language already in the grammar school. In view of the brief period of his education, it may be that he did not. Perhaps he had to learn French independently as well.

Some authors writing about Green constantly emphasise the provincial character of his home city. In terms of the scale of XIXth century England, Nottingham is indeed remote from the capital. Cambridge and Oxford are nearer to it than London. Provinciality is not in fact a geographical concept. In Nottingham during that period, there were many secondary schools and there were numerous architectural monuments including ancient churches. A series of public lectures were delivered systematically in the city's Royal Theatre.

As a result of the efforts of a large group of enthusiasts, a public subscription library was opened in Nottingham in 1816 and its collection included also the best scientific literature of the time. Later this library played a major role in the life of Green himself. He not only used its books but the same group of enthusiasts, whose association promoted the opening of the library, helped Green to publish his first, largest, and most important scientific work: ''An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism''. This happened in 1828. It is difficult to establish at present the total number of subscribers: in the list quoted ([1], pp. 589–591) there are about 40 names; however, at a number of places in the list the surname of the succeeding subscriber is followed by the words 'and many others'. Those who supported Green by their participation in this subscription included

† A list of the pupils of this school during the period 1801–1802 has been preserved, No. 255 corresponding to George Green.

‡ According to Wilkins-Jones [4], Green's father continued in the baking business until 1817.

businessmen, doctors, teachers, clergymen, and persons belonging to other professions. Clearly only a very few could have had a professional interest in the 'Essay' itself; the vast majority of the subscribers simply endeavoured to help Green; they had faith in their fellow countryman. Everything points to the fact that it was not easy to publish this book: the necessary mathematical symbols, the appropriate typeset, could be found only in London. The edition was small — of the order of hundreds of copies. Most of them were scattered among the homes of his subscriber friends. When W Thomson (subsequently Lord Kelvin) attempted to obtain this edition less than 20 years later, he succeeded only with much difficulty.†

At the beginning of 1829, George Green's father died at the age of 70; Green lost his mother even earlier — in 1825. It became difficult to carry on the family business (milling) earlier still — during the last years of his life, Green's father began to fail in health seriously. After his father's death, Green decided to stop working as a miller and to cash in part of his inheritance (we may note that, contrary to claims sometimes encountered, he did not sell the mill, which remained in the possession of the Green family until 1919). We have no information about the course of Green's life in subsequent years. The settlement of the questions of property took some time and also there was his personal life which was not free of complications ... However, there was also another aspect: Green decided to enter a university. This decision was taken on the advice of the Baronet Sir Edward Ffrench Bromhead, who actively supported him.‡

In October 1833, Green entered the Gonville and Caius College, which is one of the oldest in Cambridge (founded in 1348). It is evident that the choice of both the university and of the College itself was made on the recommendation of Sir Edward who was at one time a student at this College.§ In order to enter the university, Green had to become intensely engaged in the humanities (in particular in Latin and ancient Greek), to which he could not previously devote due attention. However, Green continued, in addition to his scientific studies, preparing three articles, which were subsequently published in the *Proceedings of the Cambridge Philosophical Society*. Without stopping to analyse them, we shall only mention that the first two articles presented an extremely general analysis of the mutual attraction of bodies, on the one hand, when the attraction forces depend on the distance between them in inverse proportions to some power of the latter (not necessarily the square) and, on the other hand, for any dimensionality of the space itself (where the above forces act). The third article was devoted to the oscillations of a pendulum in a fluid; it turned out later that it began a series of studies on wave processes in acoustics and optics.

A record of registration, showing that George Green entered the Gonville and Caius College as a student paying for tuition and board and lodgings and that he paid an entrance fee of three shillings and four pence, has been preserved. His tutor was Mr Hanson. After four years, in 1837, Green passed brilliantly a complex examination in mathematics (tripos) and became the fourth wrangler in Cambridge. At the same time, he prepared his next paper 'On the Motion of Waves in a Variable Canal of Small Depth and Width' and he read it at a session of the Cambridge Philosophical Society with subsequent publication in the Society's *Proceedings*.¶

After half a year, in December of the same year 1837, Green presented the next two papers, the first of which was devoted to the reflection and refraction of sound waves and the second to similar phenomena for light. Despite the outward change in topic, he did not alter the actual method used in his studies, which is largely based on the use of the concept of the potential ('potential function' according to Green).

Formally his aim was attained: he graduated brilliantly from the university, which was fully reflected in the way that he passed the tripos.† However, his satisfaction derived from success was evidently incomplete... Green was already 44 years old, when a scientist, especially a theoretician, was usually not only fully established but had often acquired a definite name and position. Unfortunately, his first, most fundamental work on electromagnetism

Figure 2. The mill belonging to George Green's father in Sneinton (from an early XIXth century engraving).

† W Thomson was able to obtain three copies of the 'Essay' from Hopkins, a tutor at Cambridge University.

‡ He became acquainted with Sir Edward Bromhead during the period of the publication of the 'Essay' and the acquaintance was maintained in subsequent years.

§ Sir Edward Ffrench Bromhead evidently graduated from the Gonville and Caius College in 1813.

¶ An obligatory condition for the publication of papers in the *Proceedings of the Cambridge Philosophical Society* was that they be first presented at the Society's meetings.

† He was surpassed by W N Griffin, J J Sylvester, and E Brummel; among them only Sylvester made an appreciable mark in mathematics.

Figure 3. The reading room of the public [subscription] library in Nottingham opened in 1916. The publication of "An Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism" by George Green (1828) [9] was financed by a subscription from members of the library.

(1828) remained almost unknown. Nevertheless, Green continued to work. The next year (1838) was spent in the study for and acquisition of the degree of Bachelor of Arts. His research work continued: he reported and published in print a further two papers: "Note on the Motion of Waves in Canals" (February 1839) and "On the Propagation of Light in Crystallized Media" (in May of the same year). On 31st October 1839 George Green was elected Fellow of his alma mater — Gonville and Caius.† There is evidence that Green participated in the teaching at his college, apparently as an examiner. It might have been hoped that, albeit with much delay, his life and career would have become settled in some way...

Unfortunately this did not happen. There was a considerable impairment in Green's health. He set out for Sneinton never to return to Cambridge... Nothing definite can be said about the causes of Green's illness and its nature. Admittedly it is claimed in an English publication ([4], p. 54) that "...he died on 31st May 1841 in Sneinton of influenza".‡ He was buried next to his parents on 4th June against the walls of St Stephen's Church; other members of his family were laid to rest in neighbouring areas. Several months before his death, Green made his will, carefully dividing his property between the children and Jane Smith (see below), not forgetting also several other persons close to him.

Announcements about the death of George Green were published in three local Nottingham newspapers: *The Nottingham Journal*, *Nottingham Mercury*, and *Nottingham Review*. The texts stated: "in Sneinton on Monday evening on 31st ult., George Green, Esq., Bachelor of Arts, Fellow of the Gonville and Caius College, Cambridge". The *Nottingham Review* inserted also an additional communication: "In our obituary of last week, the death of Mr Green, a mathematician, was announced; we believe he was the son of a miller, residing near to Nottingham, but having a taste for study, he applied his gifted mind to the science of mathematics, in which he made a rapid success. In Sir Edward Ffrench Bromhead, Bart., he found a warm friend, and to his influence he owed much, while studying at Cambridge. Had his life been prolonged, he might have stood eminently high as a mathematician". The publisher of the *Nottingham Review* was then Robert Goodacre, the son of the headmaster of the school where Green had been a pupil. It has been suggested [4] that he in fact wrote the above announcement. Unfortunately, one has to conclude that the scientific world did not react in any way to George Green's death. The time of interest in him and the time of appreciation of his works lay still in the future.

We repeat that during Green's life the works which followed the 'Essay' did not arouse much interest either and did not receive their due appreciation.

Many years later, recalling Green's work on electromagnetism, F Klein chose to make the following remarks: "in 1828, Green laid the foundations of the theory

† Such rapid election was the traditional recognition of exceptional successes in passing the tripos.

‡ The word death was absent from the newspaper announcements since they appeared in the appropriate [deaths] column. These announcements were published at the beginning of June and they therefore referred to the month of May.

of the potential; however, for a long time this remained in complete obscurity, since, as the son of a poor Nottingham baker, he did not at first have means of exerting [personal] influence. Unfortunately, the fact that his talent was later discovered and brought to light did not benefit him; having been invited to Cambridge, he became a victim of alcohol'' ([5], p. 31). In general Klein writes about Green with a strange mixture of respect and even deference as regards his scientific achievements and almost with an undue familiarity about his biography: ''we have already spoken of the self-educated Green [...], who published in 1828 his innovatory but initially little noted work [...] 'An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism'. He entered Cambridge at the age of 40 and published there a number of important studies...'' ([5], p. 257).

Not only the style but also certain factual information reported by Klein cannot be regarded as entirely acceptable. It is incorrect to refer to Green as 'the son of a poor Nottingham baker', because, first, long before the publication of the 'Essay', his father changed the trade of a baker to that of a miller; second, towards the time of his death, Green's father's assets amounted to approximately £20 000 sterling (at the rate of exchange current in the middle of the XXth century) according to H G Green ([1], p. 559); apart from the mill in Sneinton, Green's family owned several parcels of land with houses and orchards (a certain part of Nottingham is called to this day 'Greens' Orchard'). Likewise Green can be regarded as a 'self-educated person' only partly, as regards his first work (the 'Essay', 1828) and the two subsequent works, because later, as already noted, he was educated in Cambridge. Finally, nobody invited him to Cambridge: he entered the university in the generally accepted manner.

We must now consider yet another delicate aspect of his personal life. Formally Green was not married; however, he had seven children by his common-law wife Jane Smith (daughter of a miller's assistant). They were born in the period between 1824 and 1840: two sons and five daughters. Among them, his son also called George (born in 1829 and died in 1870) was a mathematician in Cambridge. Green acknowledged his paternity and its official confirmation may be found in an announcement shortly before his death. It is not clear why in the existing situation Green did not in fact formalise officially his relations with Jane Smith. There exist two versions of the answer to this question. According to one, Green's father was categorically against the marriage; according to the other, Green did not want to marry at all because he proposed to become a Fellow of the College and the statutes of the Colleges in English universities did not permit this for persons who entered into marriage. Neither of these explanations is fully convincing because Green's father died at the beginning of 1829 and the wish to enter the university arose only several years later. It is known that, after Green's last arrival in Sneinton, Jane Smith lived in his house and was present at his death. After George Green's death, his widow Jane (née Smith) assumed the double-barrelled surname Green-Smith. Jane Smith was born in 1802 and died at the age of 75. Green's youngest and much loved daughter Clara departed this life in 1919 leaving no issue; the Green family died out with her.

As already mentioned, very little is known about Green's life, his character, tastes, and habits; neither his portrait nor even a verbal description of his outward appearance have been discovered hitherto. According to statements by his contemporaries, Green was a modest and courteous man. Together with his father he was for a long time a trustee in a charitable society in Sneinton; father and son not only helped the needy but also strove to achieve the most rational use of the collected means. In 1838, Green donated a considerable sum for the building of St Stephen's Church (in Sneinton). The Green family traditionally voted for the Tory Party in parliamentary elections.† Up to the middle of the previous century, nobody troubled to preserve the memory of George Green; ‡ later a memorial museum has been created in the mill. The facts concerning Green's biography were retrieved with difficulty, in an extremely fragmentary form, from the archives of the City of Nottingham [1, 4]. Fortunately, the archives in that city, as in general in Great Britain, are in an ideal order. Much information can be found in the archives and libraries of Cambridge University and its Colleges. Data concerning Green's life and works which it has been possible to find (and may still be found) among the correspondence of a number of scientists are also significant. Apart from the archive data, the letters of Green himself and those written about him are an important source of information; some of them are quoted in Refs [1] and [4].

Life did not allow the English scientist all that much time for a full creative scientific development. During adolescence and during the years of his youth, Green was unable to receive a serious systematic education; almost his entire career (apart from the years when he studied in Cambridge) took place in a provincial city (Nottingham), which virtually ruled out the possibility of social contact with scientific circles. By virtue of certain circumstances, the legend of 'the miller's son who studied Laplace during his leisure hours' has occasionally gained currency ([8], p. 236). Alas reality was not quite so idyllic. With bitterness and contrary to tradition, Green wrote in the foreword to his 'Essay' the following words: ''...the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded'' ([7], p. 8).

2. ''An Essay on the Application of Mathematical Analysis...''

In the newspapers known to us (*Nottingham Review* for 14th December 1827 and *Nottingham Journal* for 15th December of the same year), it was reported that the 'Essay' was already being printed and readers were invited to subscribe to it; the subscription price was seven shillings and six pence. The cost was not inconsiderable for a brochure of little more than 100 pages (bearing in mind the value of money at the time). Indeed the scientific value of this work would be duly appreciated many years later. About half the subscribers were members of the public library already mentioned (which from 1822 was called the Bromley House library after the name of the owner of the house). It was quite evident that Green counted on a

† The parliamentary elections were not secret.

‡ Green's grave remained neglected for a long time.

definite interest in his work; alas no response from the scientific world of Great Britain followed. His disappointment was so great that George Green seriously thought of altogether giving up science.

In retrospect, one can see that nothing else was likely to have happened. Indeed, we have an unknown man, who did not even study at a university, who publishes a work. Who will then be likely to read it? Do try to recall, dear reader, if you ever glanced at brochures and books published at the expense of the authors themselves claiming to have solved some kind of important scientific problems. Did they arouse your interest? This obviously happened in Green's case. Then too an appreciable 'activation energy' was required in order to overcome a threshold of disbelief and prejudice. So far as is known, no efforts were made to achieve a 'breakthrough' for the 'Essay' by either the author himself or anybody else.

We believe that the very idea of the publication of the 'Essay' was reasonable. However, the publication should have been regarded as the beginning of the promotion of the work and not its end. An attempt should have been made to bring its contents to the notice of the scientific elite of the country. Incidentally, this is by no means simple: we may recall that Cauchy lost (showed no interest in) the work of his fellow countryman Galois and that only many years later, through the efforts of Liouville, did mathematicians assimilate the legacy of the young genius.

However, let us return to 1828, to Nottingham, and to George Green. At that time the only but extremely significant result of the publication of the 'Essay' was that it drew the attention (and this was most important for its author) of Sir Edward Ffrench Bromhead, a baronet from Lincolnshire, with whom we have already become acquainted. He graduated from Cambridge University 15 years earlier and entered the scientific circle whose members were Charles Babbage (the future inventor of the computing machine), John Herschell (subsequently a famous astronomer), the mathematician George Peacock, and William Whewell, who later became a major natural scientist and historian of science. Babbage and his colleagues founded, probably with the support of Sir Edward, the Analytical Society in Cambridge and in 1820 published a book on mathematical analysis and also a translation of a brief exposition of this subject by Lacroix. Subsequently these books became university textbooks; they promoted the enrichment of differential and integral calculus in England in mathematical ideas from the Continent and also the introduction into current use of the system of notation adopted there.

Phillips [8] claims that "notice of Green's 'Essay' could not have fallen into better hands than Bromhead's. He understood its quality..". It is unquestionable that Bromhead himself offered active support to the author of the 'Essay'. It is known that E F Bromhead published in the *Philosophical Transactions* (1816) a paper entitled "Fluents† of Irrational Functions", but we have no information about his other works. Bowley et al. [9] believe that "Sir Edward did not appreciate the significance of the 'Essay', but recognised that Green was a man of considerable talents". It may be that Bromhead's praise was to a large extent a compliment. On the other hand, it would have been difficult to expect a true appreciation of the 'Essay' at the time; it would have required an exceptional insight. It is to Bromhead's undoubted credit that he showed a constant interest in Green, that he encouraged him in difficult times, and that he assisted in his entry into Cambridge (see above).

As already mentioned, "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" (1828) was the first, largest, and most important work by George Green. Those who have taken the trouble to become acquainted with this work in the original cannot fail to experience a feeling of surprise: a young man in the provinces, who received virtually no education and had no instructor published a work the ideas of which were far ahead of the contemporary studies both in England and on the Continent. In essence this is the first and fairly successful attempt to construct a unique theory of electromagnetism applicable to various specific problems. Green selected a 'peculiar function', which he called 'potential function' as a 'universal instrument' for this purpose. The introduction of this function into mechanics and hydrodynamics dates back to the second half of the XVIIIth century ‡ (admittedly it had no special name). This was done in the works of Euler (1755–1756), Lagrange (1773), and Laplace (1782). Later this function was also used by Poisson in studies on electrostatics and magnetism [11]. However, George Green transformed this function (later in science it was given the brief name 'potential', introduced in 1840 by Gauss) into a powerful and universal method; in his first work, he already put forward the idea that the potential function could be used fruitfully also in the description of wave processes in acoustics and optics.

As already mentioned, the potential function was first applied to the study of electromagnetism by Poisson (1811). He introduced, in terms of the contemporary language, the concept of the Newtonian potential of a simple layer. In his work of 1828, Green, starting from Poisson's studies, went much further. Green formulated the potential function V at the point (x, y, z) in terms of the rectangular coordinate system:

$$V(x, y, z) = \int \frac{\rho' \mathrm{d}x' \mathrm{d}y' \mathrm{d}z'}{r'} ;$$

where ρ' is the density of electricity at the point $\rho'(x', y', z')$; r' is the distance from the point $\rho(x, y, z)$ to the point $\rho'(x', y', z')$. As pointed out by Sologub [10], Green's approach is more rigorous than that of Coulomb and Poisson. By the density of surface distribution, Green understands the amount of electricity per unit area of the surface of the conductor. After this, Green derives the Poisson equation which Poisson obtained in 1811. Next, Green justifies mathematically the distribution of electric charge solely on the surface of ideally conducting bodies, which had largely been already investigated by Coulomb and Poisson.

The results presented subsequently are, on the contrary, completely original and evidently of the greatest importance for everything achieved by Green. He turns to the general case of several bodies: then the Poisson

† Newton's term (Yu L).

‡ The history of the introduction of the potential (potential function) has been described in detail, in particular by Sologub [10].

equation "...will give the value of ρ the density of the electricity in the interior of any of the bodies, when these are not perfect conductors, provided we can ascertain the value of the potential function V in their interior" ([7], p. 23). The general theorem for two continuous functions U and V of the [rectangular] coordinates, having finite derivatives at any point within a body of arbitrary shape, is formulated in the first place. It is proved (in terms of contemporary notation) that, if two functions, U and V, are specified in a volume υ bounded by a surface s, then

$$\int_{\upsilon} (V\Delta U - U\Delta V)\,\mathrm{d}\upsilon = \int_{s} (V\,\mathrm{grad}U - U\,\mathrm{grad}V)\,\mathrm{d}s, \qquad (1)$$

where Δ is the Laplace operator and $\mathrm{d}s$ is a vector element of the surface s. This in fact represents the content of Green's second formula. Incidentally Green's first formula is also obtained:

$$\int_{s} V\,\mathrm{grad}U\,\mathrm{d}s = \int_{\upsilon} (V\Delta U + \mathrm{grad}\,V \cdot \mathrm{grad}\,U)\,\mathrm{d}\upsilon. \qquad (2)$$

Next, Green establishes an extremely important relation, which is now called the 'fundamental integral formula for harmonic functions' ([12], p. 286) (sometimes also called 'Green's third formula'—see, for example Ref. [13], p. 219 and further):

$$U(P) = -\frac{1}{4\pi}\int_{\upsilon} \frac{\Delta U}{r}\,\mathrm{d}\upsilon - \int_{s} \left[U(M)\,\mathrm{grad}\,\frac{1}{r_{PM}} - \frac{1}{r_{PM}}\,\mathrm{grad}\,U\right]\mathrm{d}s, \qquad (3)$$

where P and M are points within a volume υ and r_{PM} is the distance between these points. This formula is obtained from Green's second formula (1) on the hypothesis that $V = 1/r$; in its derivation, Green assumed that the point P is surrounded by an infinitesimal sphere, naturally without going to the limit in view of the level of mathematical rigour adopted in his time. Here too Green introduced for the first time into common use the term 'singularity' of a function (at a certain point).

Eqns (1) and (2) together with Green's function considered below evidently represent his greatest achievement. We emphasise yet again that Green's approach is more general from the very beginning than that of Poisson because the English investigator does not usually impose any limitations on the geometry of the bodies considered. Green points out that real bodies can hardly be regarded as ideally conducting and therefore a certain charge will remain also within them; for these conditions, it is useful to apply the potential function.

In the first place, Green uses the potential function method to construct a theory of the Leiden jar (i.e. an electric condenser) of arbitrary configuration, only the thickness of the dielectric layer between the surfaces of the jar and the radii of curvature of the latter being specified. The author also shows that series connection of Leiden jars does not lead to a gain in the accumulation of the electric charge compared with the use of a single jar. Analysis of the shielding effect of a very thin ideally conducting sphere with a circular opening is of definite interest. It was found that, even for a fairly large size of the opening, an extremely small charge density would be present on the inner surface of the sphere.

The application of the general results obtained to magnetism is less original: the author initially describes Coulomb's views on the nature of magnetism. Then Green proposes that this treatment be extended to the case where the particles of the body 'do not conduct the magnetic fluid ideally'. Green showed that here too there is a shielding effect established by the hollow iron sphere. Green completes his 'Essay' by a theoretical analysis of the magnetic state of long rods magnetised up to saturation; he shows that his theoretical conclusions are confirmed by Coulomb's experimental data (for further details, see Lyubimov [14]).

As already mentioned, the appearance of the 'Essay' in 1828 did not attract any attention and it became known in the scientific world only after its publication by W Thomson in 1850–1854. The publisher emphasised that although definite results had been obtained in the same field during the time which had elapsed, "...the Essay as a whole, in the way that it is now presented, will apparently still be found interesting". At that time the field later called the 'potential theory' was still in its development stage. Gauss's studies were the most significant [15]. It is frequently believed that it is in fact Green's 'Essay' and Gauss's work which "...were of fundamental importance for the further development of the theory" ([16], p. 419). Thus, although the English and German scientists developed their studies independently, their approaches were similar, which was formally reflected also in the similarity of the terms which they introduced: Green's 'potential function' and Gauss's 'potential'; in view of its brevity, the latter term was adopted in science. The relation between Green's results and the studies of other investigators is now clear. Thus Green's first [Eqn (2)] and second [Eqn (1)] formulae are direct corollaries of Ostrogradskii's formula and can be readily derived from it; J C Maxwell was the first to draw attention to this.

3. Green's function and its importance for modern physics

The function, later called 'Green's function' (B Riemann), was introduced by its author in the 'Essay' when he considered the relationship between the density of surface electric charges and the potential at an arbitrary point. After a time, it began to be used widely in a wide variety of branches of theoretical physics. We may note that the introduction of this important function was not described by Green himself with complete clarity ([7], pp 30–39); we shall quote the treatment [16] closest to the original. Green's second formula (1) was initially formulated for $V = 1/r$ (where r is the distance from the pole P, within the body, to the point M at which the potential function is defined) and then, assuming that V is a harmonic function, for $V = w(M)$. When the two expressions obtained are added together term by term, we obtain †

$$\int_{\upsilon} \Delta U(1/r + w)\,\mathrm{d}\upsilon - \int_{s} (1/r + w)\frac{\partial U}{\partial n}\,\mathrm{d}s + \int_{s} U\frac{\partial}{\partial n}(1/r + w)\,\mathrm{d}s = 4\pi U(P).$$

† The partial derivative corresponds to the external normal.

The expression $G(P, M) = 1/r + w(M)$ is in fact referred to as Green's function.

Mathematically, the relation $G(P, M) = (1/r) + w(M)$ constitutes the complete solution of the equation $\Delta G = -4\pi\delta(r - r')$. The first term in G represents the field of a single charge in a vacuum, while the second term w represents the general solution of the homogeneous equation $\Delta w = 0$. Here it is assumed that Green's function is independent of time. Later the concept of Green's function was extended also to the case involving time dependence.

The extremely clear, purely physical method of 'proving' or more precisely justifying the existence of the function G used by Green himself is of interest. The surface of the body is regarded as an ideal conductor connected to the earth and a single charge of positive electricity is concentrated at the point P. The total potential function for the point M, arising from the electricity at P and the electricity induced on the surface by the point charge at P, is defined by the function G. In Green's electrostatic treatment, $1/r$ corresponds to the potential of a single point charge in free space and w is the potential of the field of the charges induced by the same charge on the ideally conducted surface mentioned above. Naturally, the condition that the surface is earthed ensures that G becomes zero at all points on the surface.

One should also recall yet again that the G function introduced by Green (subsequently referred to by his name) was to him to a large extent a 'physically tangible' concept linked directly to the mechanical, force interactions in the system of electrified bodies under consideration; a purely physical approach to the problem was natural to the English scientist. For example, he was not in principle concerned with the question of the existence of the G function itself. This later led to complaints by mathematicians: "Green himself argued that such a function existed from physical evidence. Of course the static charge on S exists! We have here an excellent example of the value and danger of intuitive reasoning. On the credit side is the fact that it led Green to a series of important discoveries, since well established. On the debit side, is its unreliability, for there are, in fact, regions for which Green's function does *not* exist" ([13], pp 237–238).

This assessment appears excessively rigorous. First, in securing new results of fundamental importance, it is fairly difficult to ensure immediately a high level of mathematical rigour (let alone the fact that the very requirement of such rigour has become more marked, evolving with time). Second, the author of the 'Essay' paid attention to the presence of singular points in the function considered, questions of uniqueness, etc., although naturally certain postulates later required corrections and justification. It is in fact their profound physical content that has made George Green's most important results so suitable for gradual generalisation and extension to virtually all the fields of modern theoretical physics and mechanics.

Green's function has sometimes been referred to as the function of source and also, when a time dependence is considered, a 'function of an instantaneous source', emphasising thereby the more general significance of this function compared with that attributed to it by its author. It can, for example, reproduce the distribution of temperature in a certain body (temperature field) after heat has been evolved instantaneously at a specified point; naturally, many other treatments of Green's function are also encountered. We shall return later to the use of Green's function at the present time.

4. Subsequent studies on the electrical potential theory

Green returned twice more to the study of electromagnetism. His article "Mathematical Investigations concerning the Laws of the Equilibrium of Fluids analogous to the Electric Fluid, with Other Similar Researches" ([7], p. 119) represents an attempt to extend the potential function to the case where the force interaction (attraction and repulsion) between point bodies is inversely proportional not to the square of the distance between them but to the distance raised to another arbitrary power $n \neq 2$, which can also be a fractional or even an irrational number. Admittedly in the entire subsequent treatment the author begins with the hypothesis that n is a rational number. Certain other limitations (which are incidentally extremely weak) are also imposed on the choice of this number. Green writes: "When we conceive, moreover, the law of the mutual action of the particles to be such that the forces which emanate from them may become insensible at sensible distances, the researches to which the consideration of these forces lead will be greatly simplified by the limitation thus introduced ... " ([7], p. 119].

This paper by Green is constructed similarly to his previous work, i.e. the 'Essay', the results of which can naturally be regarded as a special case with $n = 2$. Here too Green emphasises that "... it is always advantageous to avoid the direct consideration of the various forces acting upon any particle p of the fluid in the system, by introducing a particular function V of the coordinates of this particle, from the differentials of which the values of all these forces may be immediately deduced". The nature of the exposition is deductive; after the analysis of the general method and the derivation of relations between the density of a hypothetical fluid ρ and the potential function $V = \int \rho\, \mathrm{d}v/r^{n-1}$ which depends on it, the applications of the results are analysed: the establishment of relations between the density of the hypothetical fluid ρ and the potential function V which depends on it and also the applications of the resulting expressions to different special cases. The author solved also the problem of the density of a fluid for a conducting sphere acted upon by the fluid concentrated in some kind of external bodies.

It is remarkable that the question of the existence of the forces of gravity, decreasing in inverse proportion to the nth power of the distance between the bodies, was already considered by Laplace at the end of the XVIIIth century ([17], pp 30–32). Then, soon after the publication of Green's second article, in 1839 Dirichlet also turned to the consideration of the problem where n is not necessarily -2: "furthermore, the method is not restricted to the hypothesis that the attraction is inversely proportional to the square of the distance but remains applicable also to any integral or fractional power of the distance" ([17], p. 100). Admittedly, the author soon explains that it will be assumed that n lies between 2 and 3. Unfortunately, Green's work ([7], p. 119) remained completely unnoticed by his contemporaries and successors and did not influence the development of the potential theory.

Green's third and last work, associated with electromagnetism, is entitled "On the Determination of the Exterior and Interior Attractions of Ellipsoids of Variable Densities" ([7], p. 187) (admittedly, it is clear from the introductory remarks and other considerations that Green proposed to continue the exposition in at least yet another article, but his intention remained unfulfilled). Green posed the problem of the attraction of an ellipsoid in the most general form: he not only considers jointly both possible cases of the position of a point at which the attraction is to be determined (outside and within the ellipsoid), but also specifies in a general form the dimensionality (S) of the space in which the density of the fluid and the potential function which depends on it are determined. According to the English author, this general approach constitutes in fact a simpler solution of the problem of the attraction of the ellipsoid as a whole, making it possible to obtain readily, for example, the solutions for a conducting sphere and for a circular infinitely thin conducting plate as special cases, assuming respectively that in the first case $n = 2$ and $S = 3$, while in the second, with $n = 2$, $S = 2$, a solution valid only in the plane of the plate is obtained. Unfortunately, as already mentioned, this study was not completed and its author stopped with the derivation of extremely general but fairly complex relations which are difficult to comprehend.

Like the previous one, this study by Green was not appreciated either during his lifetime or afterwards. Admittedly, brief mentions of this investigation are sometimes encountered in the literature ([7], p. VI). Much later, the last of two studies by Green on electromagnetism were noted by F Klein, who stated that the English scientist "...published [...] a number of important studies of which we shall mention only the investigation of the attraction of ellipsoids (1835); it is of particular mathematical interest compared with his other important contributions in the field of acoustics and optics, since it was carried out for n dimensions at once—long before the development of n-dimensional geometry began [...] in Germany" ([5], p. 273). The history of physics has shown that profound and original studies, significantly ahead of their time, frequently find an unexpected fruitful application and development many years later. One cannot rule out that such a fate may await also Green's studies on the generalisation of the potential function in relation to n-dimensional space.

The development of mathematical and theoretical physics confirmed the great importance of Green's studies on electromagnetism. The English author himself understood the value of the method which he developed: "Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, and reflecting on the advantages that arise in the solution of many difficult problems, from dispensing altogether with a particular examination of each of the forces which actuate the various bodies in any system, by confining the attention solely to that peculiar function on whose differentials they all depend, I was induced to try whether it would be possible to discover any general relations, existing between this function and the quantities of electricity in the bodies producing it" ([7], p. 6).

5. Green's studies on hydrodynamics

In terms of subject, the remaining seven studies by Green can be divided into two parts: those devoted to the motion of a liquid and those referring to the propagation, reflection, and refraction of acoustic and optical waves in condensed media. The first part comprises three articles. The earliest of these is "Researches on the Vibrations of Pendulums in Fluid Media" (1836).† The periodic motion of a pendulum in the form of an ellipsoid in a fluid under the influence of an external force is considered here; the displacements are assumed to be small compared with the dimensions of the body; the effect of the fluid medium on the moving ellipsoid is also analysed. It is found that the equation of motion of the ellipsoid within the fluid is equivalent to its equation of motion *in vacuo* if it is assumed that the density of the ellipsoid increases by a certain amount, which depends, in particular, on the size and orientation of the axes of the ellipsoid. When the ellipsoid degenerates to a sphere, the above apparent density increment is half of its initial value. Alas, as regards this study we have to remark that it was not noticed during the author's lifetime or appreciated subsequently. This is particularly regrettable, since the problem of the phenomena associated with the flow of fluid around a body of a fairly complex configuration was first posed in the study described.‡

Green's remaining two studies in the field of hydrodynamics, "On the Motion of Waves in a Variable Canal of Small Depth and Width" (1838) and "Note on the Motion of Waves in Canals" (1839), deal with the propagation of single low-amplitude waves in an incompressible fluid filling an infinitely long channel. Green was apparently the first to consider theoretically single (or 'unified') waves, which are now called 'solitons'. He was able to show that the rate of propagation of a wave is proportional to the square root of the product of the depth of the fluid in the channel and the acceleration due to gravity, while the height (amplitude) of the wave is inversely proportional to the product of the square root of the channel width and the fourth root of its depth.

In the second article (1839), Green notes a very good agreement between his calculations of the rate of propagation of a wave with the experimental results of J S Russell for different depths of the fluid (when the amplitude of the wave is much smaller than the depth of the fluid in the channel). Green's achievement is, in particular, allowance for the finiteness of the depth of the fluid. For an infinitely deep fluid, Newton already obtained the wave velocity $v = (g\lambda/2\pi)^{1/2}$ (λ is the wavelength); Green's more rigorous theory, taking into account the depth of the layer of fluid, yields an approximately 25% increase in v, in agreement with experiment. In conclusion, considering the motion of a particle of the fluid in a channel during the passage of a wave, Green shows that the particle executes circular movements in the vertical plane, since it moves forward when the fluid rises and backwards when it falls, remaining ultimately in its previous position after the passage of the wave.

† Reported on 16th December and published in the *Transactions of the Royal Society of Edinburgh.*

‡ The problem of the motion of a cylinder under the surface of a heavy liquid was posed by Kelvin in 1904 and stimulated a number of subsequent investigations (for further details, see Sretenskii [18]).

The results of these studies by Green have left a definite trace ([18], p. 224). L N Sretenskii analysed this question by purely mathematical means, considering the integrability "of the equation for long waves in variable cross-section channels"; the relationship between the wave amplitude, the cross-section of the channel, and its depth, mentioned above, is referred to by a number of authors as 'Green's law' ([18], p. 224–227). Green's results were extended in the studies by Airy (1845) and later by Boussinesq and Rayleigh; however, the exact solution of the problem was not obtained until the 1940s by M A Lavrent'ev.

6. Studies in the field of acoustics and optics

We shall now turn to Green's studies on the propagation of sound and light accompanied by a transition from one medium to another. The article "On the Reflexion and Refraction of Sound" (1838) was clearly inspired by Poisson's work "Memoir on Motion in Two Elastic Liquids One on Top of the Other" (1831) and evidently the article by the same author "Memoir on the Propagation of Motion in Elastic Media" of the same year, which is a continuation of the first study. Although the English investigator considered the same problem as his French colleague and actually used the same notation, their approaches were different. At the beginning of his study, Green notes that some of Poisson's results are erroneous, proposing his own analytical procedure by means of the potential function; furthermore, the English scientist's approach is broader, since the case of both elastic and non-elastic media is considered. It is difficult not to agree with Green when he notes that the risk of errors in the course of long computations "induced me to think, that by employing a more simple method, we should possibly be led to some useful result, which might easily be overlooked in a more complicated investigation" ([7], p. 233).

Green considers the propagation of a planar wave with transition from one infinite medium to another, also infinite, which is naturally accompanied by the appearance of reflected and refracted waves at the interface. Here an important aspect is the analysis of the total internal reflection of sound waves (by analogy with optics); a similar study had already been carried out by Poisson. A fully original aspect of Green's investigation is the hypothesis, confirmed by calculation, that, even for total internal reflection, the sound wave penetrates the second medium but soon decays.

N M Ferrers, the publisher of Green's collected works ([7], p. VII), suggests that the study on the reflection and refraction of sound should be analysed jointly with that "On the Laws of the Reflexion and Refraction of Light at the Common Surface of Two Non-crystallized Media" (1838), which was also devoted to wave processes in isotropic media (for example, liquids), but this time optical processes are considered. In order to explain the propagation of transverse vibrations through the light-bearing ether, the author found it necessary to investigate the equations of motion in an elastic solid.

Green began with Cauchy's study: "M Cauchy seems to have been the first who saw fully the utility of applying to the Theory of Light those formulae which represent the motions of a system of molecules acting on each other by mutually attractive and repulsive forces; supposing always that in the mutual action of any two particles, the particles may be regarded as points animated by forces directed along the right line which joins them" ([7], p. 245). However, Green prefers not to relate his reasoning to any specific ideas and models: "... it would seem a safer method to take some general physical principle as the basis of our reasoning, rather than assume certain modes of action, which, after all, may be widely different from the mechanism employed by nature ..." ([7], p. 245).

Green adopted d'Alembert's principle and the principle of virtual velocity, which were well known to him (from Lagrange's *Analytical Mechanics*), as a general approach. At the same time, the potential function (used by Green in all his works) is also applied: "The principle selected as the basis of the reasoning contained in the following paper is this: In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass [system] will always be the exact differential of some function" ([7], p. 245). Green justifies the existence of the exact differential $d\varphi$ of a certain function φ on the basis of the impossibility of perpetual motion: "Indeed, if $d\varphi$ were not an exact differential, a perpetual motion would be possible, and we have every reason to think, that the forces in nature are so disposed as to render this a natural impossibility" ([7], pp 248–249).

Green considers a particle of the ether and writes the corresponding equation of motion (the overall equation for the entire system considered). In the general case, an arbitrary volume element of the medium (represented initially by a rectangular parallelepiped) undergoes a change in size and shape during motion; in his study, Green uses the fact that any deformations of the medium can be regarded as a combination of compression (extension) and twisting. The required potential function φ then depends on six variables (three arising from the components of extension and three arising from the directing cosines defining the twisting of the volume element). Further calculation shows that φ can be represented as a homogeneous second-order form containing 21 coefficients. This is already a significant achievement by Green, because mathematical analysis of this situation by other workers led to 36 parameters. The real values (and number) of these coefficients are determined by the properties of the medium considered. Thus the number of coefficients decreases to nine for a medium symmetrical relative to three mutually perpendicular planes and to five for a medium with one axis of symmetry, while for an isotropic (or noncrystalline) medium it is two. This simplest case Green actually considers subsequently in this study.

Green does not deny the similarity of his method to that proposed earlier by Cauchy, but emphasises the advantages of his own approach: "The method given in the text, however, and which is very similar to the one used by M Cauchy, is not only more simple, but has the advantage of furnishing two intermediate results, which may possibly be of use on some future occasion" ([7], p. 253). Next, Green obtains the boundary conditions on the surface of contact between two isotropic media in the derivation of equation of motion for the propagation of a perturbation on crossing the surface of contact mentioned above. In short, the English investigator established that the tension (like the displacement) on the contact surface should be continuous. An equation similar to the wave equation can be derived ultimately from the relations obtained by Green (it is quoted in Ref [19], p. 237, but Green himself 'did not reach' this stage).

Poisson showed earlier (1828) that a perturbation in an elastic medium is propagated by waves of two kinds: transverse and longitudinal. In the transverse waves, the density of the medium remains constant and only some parts of the latter are displaced relative to others; on the other hand, in the longitudinal kind a compression–rarefaction wave is propagated. However, Fresnel showed that light waves are determined solely by transverse vibrations. Hence it was necessary to account for the absence of longitudinal waves in optics.

An attempt at such an explanation was made in Green's study under consideration. He began with the fact that the equation of motion in the ether contains two constants, A and B, the velocity of the longitudinal wave being proportional to $\sqrt{A}$ and that of the transverse wave to $\sqrt{B}$. Green was faced by two possibilties of excluding the longitudinal wave: by postulating a zero or infinite velocity ratio (i.e. $\sqrt{A}/\sqrt{B}$). In view of the condition that the medium is stable, it is necessary to choose the second version. This conclusion does not deny the very existence of longitudinal waves; it merely means that the ratio of the resistance to compression and the resistance to bending is extremely large. This conclusion imparted some likelihood to the ether hypothesis. Since Newton's time, physicists endeavoured to imagine the ether clearly, to 'model' it as a kind of construction, perhaps artificial, but having features of reality.

Green's concept made it possible to attain a definite likelihood and clarity in the ether theory: it was found that, assuming that the resistance of the ether to twisting is exceptionally small, one can adopt a fairly small resistance to its compression (as in a rarefied gas) in order to obtain a very large value of $\sqrt{A}/\sqrt{B}$. This explained the fact that the ether does not hinder appreciably the motion of planets, for example. Nevertheless some scientists could not imagine a medium sufficiently rigid for transverse vibrations and at the same time not offering a resistance to the slow motion of planets.

A significant advantage of Green's study was that, in contrast to Cauchy, Green did not endeavour to obtain Fresnel's relations (for the refracted and reflected waves) from the differential equations of motion; the boundary conditions were then simply 'adjusted to the required result'. On the contrary, Green obtained the conditions at the interface between two media in the course of the derivation of the equation of motion for the propagation of a perturbation in the ether on crossing the boundary between two media. Assuming the continuity of the displacement of the ether at the interface between the media, Green found naturally that the tensions at the interface should also be continuous. After this, having first established the correct boundary conditions, Green investigated the laws of the reflection and refraction of light waves and derived the Fresnel laws for plane waves and calculated certain other optical effects (in particular the change in phase on reflection).†

† We shall not dwell on the 'Supplement to a Memoir on the Reflexion and Refraction of Light' (1839) ([7], p. 281), which deals with certain special problems.

Thus Green was the first to propose a 'really working' theory of the ether, which, on the one hand, was capable of explaining a number of important optical phenomena, and, on the other, was found later to serve as the starting point for the construction of numerous theories of the ether which played an important role in the development of physics in the XIXth century and at the beginning of the XXth century.

In the concluding study on the propagation of waves (the last published by Green) "On the Propagation of Light in Crystallized Media" (1839), the author proceeds to the optics of anisotropic media. The principle declared in the previous study ("On the Laws of the Reflexion and Refraction of Light at the Common Surface of Two Non-crystallized Media") is recalled initially. As already mentioned (see p. 9), it constitutes the principle of the conservation of energy. It was proposed to consider here also extraneous pressures which had not been taken into account earlier. This significantly complicated the solution of the problem: "But with these pressures we are obliged to introduce six additional coefficients; so that without some limitation, it appears quite hopeless thence to deduce any consequences which could have the least chance of a physical application" ([7], p. 293).

Endeavouring to minimise the necessary limitations, the author confines himself to the consideration of media in which the directions of transverse vibrations are always located exactly in the wave front.

Here Green begins with his idea of an elastic (more precisely quasi-elastic) ether with specially selected elasticity coefficients. He succeeded in selecting the properties of the anisotropic ether in such a way that the results agreed with those yielded by Fresnel's crystal optics.

At first sight, this work contains fewer concrete, practical results than the previous one. However, one must not forget that Green's work, continuing the studies of Poisson and Cauchy, in essence laid the foundations of crystal optics, having placed it on the firm base of the theory of elasticity. The attempts on these lines were undertaken (simultaneously with Green) by MacCullogh. G S Landsberg made the following statement (in the foreword to Fresnel's works): "It is known that MacCullogh attempted (1839) to find a solution by considering the ether as a medium of a special kind, the potential energy of which depends only on the rotation of the volume elements. The formulae which he obtained for such a quasi-elastic medium formally agree with those of electromagnetic optics. However, such analysis proved to be possible only after Cauchy and Green developed a detailed elastic theory and, in order to eliminate the longitudinal wave, it was necessary to introduce additional conditions which did not follow from the boundary conditions" ([19], p. 50).

On solving the purely optical problems concerning the propagation of light in isotropic and crystalline bodies, Green obtained results which were incomparably more general, having taken an important step towards the development of the theory of elasticity. "The revolution which Green carried out in the elements of the theory is comparable in importance with Navier's discovery of general equations ... " ([20], p. 11). In Green's works, these results are next in importance to his 'Essay'; we repeat that significant progress in the theory of elasticity is associated with Green.

7. George Green's influence on the development of physics

What has George Green meant for the natural science of his time and the future? There exist two points of view. One of them has been expressed, in particular, by F Klein: "However, both Green and MacCullogh had an importance only as isolated phenomena. Mathematical physics in England began its uninterrupted and brilliant rise only when Stokes and William Thomson came to the fore among the young talents in Cambridge at the beginning of the 1840s " ([5], p. 259). On the other hand, E Whittaker suggests that "... it is no exaggeration to describe Green as the real founder of that 'Cambridge School' of natural philosophers, of which Kelvin, Stokes, Rayleigh, Clerk Maxwell, Lamb, J J Thomson, Larmor, and Love were the most illustrious members in the latter half of the nineteenth century" ([21], p. 153). D J Struik, a historian of mathematics, also regards Green's studies as 'the beginning of modern mathematical physics in England' ([22], p. 234). We believe that the second assessment is much more objective.

The mentions of Stokes's name just made are not fortuitous: Stokes continued directly Green's studies in optics, acoustics, and hydrodynamics. Stokes studied Green's works with very great care and highly appreciated them. In comparing the works of Poisson and Green on acoustics and optics, Stokes wrote "... This problem had been previously considered by Poisson in an elaborate memoir. Poisson treats the subject with extreme generality, and his analysis is consequently very complicated. Mr Green, on the contrary, restricts himself to the case of plane waves, a case evidently comprising nearly all the phenomena connected with this subject which are of interest in a physical point of view, and thus is enabled to obtain his results by a very simple analysis. Indeed, Mr Green's memoirs are very remarkable, both for elegance and rigour of the analysis, and for the ease with which he arrives at most important results'' ([23], p. 178). Stokes also notes that Green was the first to turn to the analysis of single waves (subsequently called solitons), observed in 1834 by J S Russel.

Green had a considerable influence on Rayleigh's works (who can be regarded partly as Stokes' pupil) both as regards the choice of the subjects of study and the approach to the latter. Both Stokes and Rayleigh frequently referred to Green in their works. Rayleigh regarded Green's theory of the ether as the best of all those proposed. Green made the remarkably brilliant prediction that the range of action of intermolecular forces is much smaller than the length of the light wave. Subsequently this was demonstrated when the theory of Rayleigh scattering was created: this is the so called Ornstein–Zernike direct correlation radius (1914 — for further details see Artamonov and Lyubimov [24]). Green's works date back to the time when mechanistic views predominated. He did much to develop and establish the theory of the light bearing ether; his studies were continued by Stokes, Rayleigh, W Thomson, and others.

Green's results became an organic part of the current usage of the next period of the development of physics, associated with Maxwellian electromagnetism and the electromagnetic theory of light, to a large extent, precisely through the studies of W Thomson and J Rayleigh. Here too, Green's formulae and Green's function found an extensive application.

The latter acquired particular importance only in our time. As found subsequently, Green introduced into current usage a completely new and universal mathematical construction. Indeed, such a construction is capable of taking into account the influence of any change (perturbation) at any (arbitrarily fixed) point of the system in a specified instant on the situation at other points of the same system in the subsequent instants. Many years later, it led to the development of the theory of the response of a physical system to an external influence (Kubo, Tomita, 1957). As already mentioned, even earlier Ornstein and Zernike (1914) introduced into physics the concepts of the direct and indirect correlation functions; although they are determined by random, fluctuation phenomena, the mathematical construction and the essential physical nature of these functions are in essence similar to those of Green's function. All these ideas concerning the interrelation of the changes in various points in space were subsequently transferred to the time dependence of processes in any kind of system. Thus the idea of Green's time functions (and also time correlation functions) was born, which greatly extended the possibilties of these methods.

We may note that the evolution of the concept of Green's function is not restricted to this. After the introduction of Green's function methods into classical physics, the time came to introduce the function into the quantum field theory and into the quantum theory of solids. Naturally, even a cursory review of these problems requires a separate description, which is far outside the framework of the present communcation.

It is remarkable that in the *Encyclopedic Physical Dictionary* there are two articles devoted to separate descriptions of Green's function for classical physics and in relation to quantum theory [25]. Without going into details, we may note that the physical sense of Green's function is significantly modified (and complicated) compared with its sense generally adopted in those quantum-theoretical problems where the systems considered contain a variable number of particles, which can lead to an ambiguity in the definition of Green's function. The so called 'method of temperature quantum Green's functions', which constitutes a synthesis of the ideas of statistical physics and the quantum field theory, proved to be very fruitful: it is used in the theory of the solid state, in ferromagnetism, etc. [26].

Nowadays Green is usually remembered as the author of the 'Essay'; we have tried to show that the range of his studies and achievements was much wider and his influence on the subsequent development of physics, mathematics, and mechanics was much deeper.

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