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Standard quantum limits of measurement errors and methods of overcoming them

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Contents

Abstract. The so-called standard quantum limits (SQL) of measurement errors of coordinate, momentum, amplitude of oscillations, energy, force, etc are due to back action of the meter on the system under test, whenever the meter responds to the coordinate of the system. These SQL are not fundamental and can be surmounted by various methods. In particular, in a coordinate measurement the SQL can be overcome by means of an appropriate correlation of conjugate meter variables. Conditions of quantum nonperturbing (nondemolition) and quasi-nonperturbing measurements of the energy of electromagnetic waves are discussed. Possible methods of these measurements are reviewed. Conditions for overcoming the SQL of wave energy measurement by the optical Kerr effect are analysed. The quantum limit of error of this measurement is discussed. The effects of dissipation, dispersion, and generation of combination waves are considered. Results of experiments reported in the literature are discussed. The dependence of the quantum limit of detection of an external action upon a system on the initial state of the system is considered. The relation between the measurement error of an observable A and a perturbation of an observable B, when $[A, B]$ is an operator, is examined.

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1. Introduction

1.1 Standard quantum limits (SQL) of measurement errors

The quantum limits of the errors in the measurement of physical quantities have stimulated the interest of theoreticians since the days when the latter had effectively no contact with experimenters [1, 2]. Resolution of this problem had a philosophical rather than a practical motivation. If it is suddenly realised that no physical quantity can in principle be measured exactly, what is the meaning of its exact value in theory? If the error of the measurement of energy is inversely proportional to the duration of the measurement, as is implied by the Heisenberg – Bohr relationship then the law of conservation of energy cannot be tested accurately in a finite time, and there may be grounds for doubting its fundamental nature. The development of lasers and of optical systems of information transfer provided a stimulus for the further development of the quantum theory of measurements and of the theories of estimation and testing hypotheses [3–6]. A specially strong impetus for the development of new principles and methods of quantum measurements was provided by the search for methods of detecting gravitational waves from extraterrestrial sources. This is because according to the most optimistic predictions by astrophysicists the relative change in size of bodies on the Earth induced by gravitational waves cannot be greater than $10^{-19}-10^{-21}$. Solid-state gravitational antennae of metric size were calculated for a wave frequency of a few kilohertz. For a frequency $\omega = 10^4\ \mathrm{s}^{-1}$ and a mass $m = 10^3$ kg the uncertainty in the amplitude of the natural oscillations of the antenna in the coherent state is $\Delta A = (\hbar/2m\omega)^{1/2} \approx 2.3\times10^{-19}$ cm, i.e. of the same order as (or greater than) the expected signal.

An analysis of the sensitivity limits of gravitation detectors by the traditional methods of observation carried out by Braginskii in 1967 identified the limiting sensitivity towards the force action on a harmonic oscillator [7–9]:

$$F_0\tilde{\tau} \geqslant (2\hbar m\omega)^{1/2}\,, \tag{1}$$

where F_0 and $\tilde{\tau}$ are the amplitude and the duration of the action of the force on an oscillator of frequency ω and mass m. This limit arises because in a continuous measurement of the coordinate the amplitude of a harmonic oscillator cannot be measured more accurately than its uncertainty in the coherent state.

Similar studies aimed at a free body showed that the uncertainty of its coordinate at time t after its measurement satisfies the inequality

$$\Delta x(t) \geqslant \left(\frac{\hbar t}{m}\right)^{1/2}\,, \tag{2}$$

which also applies to the limit of sensitivity to the momentum of the force

$$F_0\tilde{\tau} \geqslant \left(\frac{\hbar m}{\tilde{\tau}}\right)^{1/2}\,. \tag{3}$$

Expressions (1)–(3) have become known as the standard quantum limits (SQL) of the measurement errors of these physical quantities. The same name is given to an expression similar to (2) which defines the error limit of the determination of the instantaneous value of the coordinate of a free particle by the method of continuous measurement of the coordinate over a time τ [10–12].

The group of SQL includes also the error limits of evaluations of the amplitude and energy of a harmonic oscillator, equal to

$$\Delta\tilde{A} \geqslant \left(\frac{\hbar}{2m\omega}\right)^{1/2}, \quad \Delta\tilde{W} \geqslant \langle n\rangle^{1/2}\hbar\omega\,, \tag{4}$$

where $\langle n\rangle \gg 1$ is the average number of energy quanta.

None of the SQL [except (2)] is fundamental: they are a consequence of the fact that the measurement procedure which had been used to define them is not optimal. The aim of this review is to identify the source of the SQL, to suggest ways of eliminating them, and to comment on the results of attempts to overcome some of the SQL experimentally. We shall begin by considering the fundamental propositions of the quantum theory of measurements.

1.2 General scheme of indirect measurements

Measurements are classified as indirect or direct according to the nature of the effect of the apparatus on the evolution of the system. Indirect measurements are those after which the law of the evolution of the system is preserved. Under these conditions the system interacts with the first stage of the apparatus for a finite time. Information on the quantity being observed (the 'observable') remains in this stage, which is called the quantum readout system (QRS) or quantum transducer, in the form of a change in its state. The value of the observable being studied is determined indirectly through a measurement on the QRS. The indirect measurements are of interest because they are used in all the experiments which rely on the so-called test bodies.

Measuring apparatus consists of a number of interconnected stages. The quantum theory of measurements states that the first stages can be quantum, but the last must be classical. There is no generally accepted definition of the term classical. The mathematical conditions of classicality were formulated most precisely by Stratonovich [13]. The principal classicality conditions were formulated as follows [14]: a stage of the apparatus can be treated as classical if the quantum-mechanical uncertainties in the subsequent stages do not significantly affect the overall error of the measurement. The first stages (QRS), which behave as quantum-mechanical links, interact reversibly with the system under study. The interaction of the QRS with the classical part of the apparatus is irreversible. This destroys the correlation between the states of the system and the QRS. In a classical stage, 'dequantisation' of the signal takes place: microscopic changes in the QRS create macroscopic changes in the classical stage of the apparatus. We shall call the classical part of the apparatus the monitor. A schematic illustration of the indirect measurement is shown in Fig. 1, where $\hat{A}$ is the measured observable of the system and $\hat{Q}(\hat{A})$ is the observable of the QRS, which varies, as a result of interactions with the system, as a function of $\hat{A}$.

Example. A circuit for measuring the charge q on the capacitor of an LC circuit with the aid of an electron beam (Fig. 2). Here the LC circuit is the system under study,

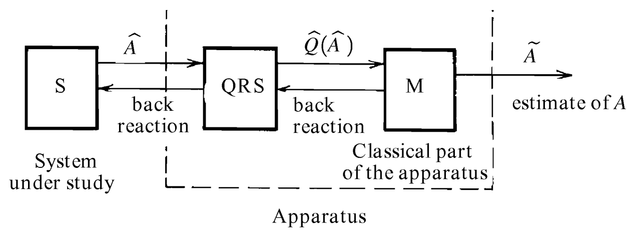

Figure 1.

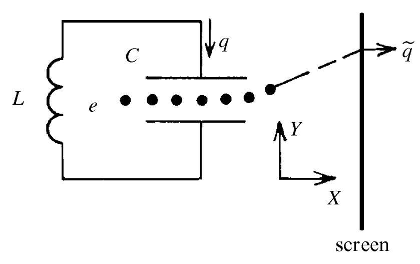

Figure 2.

the electron beam is the QRS, and the screen is the monitor. The momentum of the electrons after their flight between the plates depends on the charge on the plates. By measuring the coordinate of the scintillation of the electrons on the screen we obtain an estimate of the charge $\tilde{q}$.

The QRS interacts with the system and with the monitor successively in time. It is important to stress that at least one of the stages of the measuring circuit interacts with the others through an impulse.

1.3 Evolution of the state of the system during the measurement

The treatments by physicists of the change in state of a system brought about by the measurement differ as markedly as their views on the fundamentals of the quantum-mechanical laws. On the grounds that only what can be tested experimentally is physically significant, and that the presence of a living observer is not essential to a measurement, the stages in the change of state can be envisaged as shown in Fig. 3.

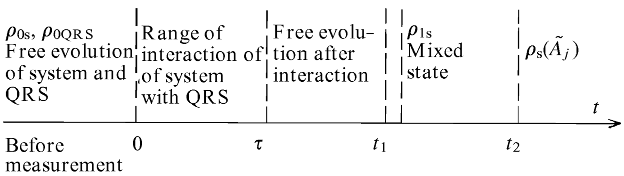

Figure 3.

Here t_1 is the time of the irreversible interaction of QRS with the monitor. At this instant the correlation between the QRS state and the state of the system is destroyed, and the system becomes a mixture of states $\rho_s(\tilde{A}_j)$ where $\tilde{A}_j$ are probable values of the assessment of the quantity being measured [15]. This means that during measurements in an ensemble of identical systems we shall have (after time t_1) an ensemble of systems each of which is in one of the $\rho_s(\tilde{A}_j)$ states. The instant t_2 is the time of separation of the mixture. Once the results of the measurements of $\tilde{A}_j$ are available the mixture can be separated into subensembles with a definite value of $\tilde{A}_j$. The system is then transformed from the initial ρ_{0s} to the $\rho_s(\tilde{A}_j)$ state, i.e. a reduction of the state takes place. In the case of a precise measurement, the $\rho_s(\tilde{A}_j)$ state of the system is the eigen state of the observable A (at the instant of measurement). In the case of measurements in a unique system the same statistical characteristics manifest themselves as a result of the long repetition of the measurement with the system reverting each time to its initial state.

The separation of the system according to the results of the measurement requires some classical actions, which can be performed either by the experimenter or automatically.

In the mixed state the uncertainty of the measured observable $\hat{A}$ is not smaller than it would have been for the free evolution of the system, because of the fluctuating back reaction of the apparatus on the system. The difference between the dispersions of some observable $\hat{B}$ in the mixed state and in the unperturbed state can be used as a measure of the perturbation introduced by the apparatus into that observable:

$$(\Delta\hat{B})^2_{\text{pert}} = (\Delta\hat{B})^2_1 - (\Delta\hat{B})^2_0 \ . \tag{5}$$

(The dispersion of the observable $\hat{B}$ in the $\rho(\tilde{A}_j)$ state can be smaller than in the initial state if $\hat{B}$ is correlated with $\hat{A}$ in the initial state.) It is usually assumed that the uncertainty of the perturbation of the observable $\hat{B}$ is related to the error in the measurement of the observable $\hat{A}$ through an expression identical to that corresponding to the ratio of the uncertainties. But this is not so [10]. The relationship between the measurement error and the perturbation is discussed in the next section of this review.

2. SQL of the error in the measurement of a coordinate and methods of overcoming them

2.1 Aim of the measurement and initial state of the apparatus

The aim of the measurement could be the value of the observable $\hat{A}$ referred to: (1) the unperturbed state of the system, or (2) a state which is perturbed by the measurement process. In each case we may be interested in the value of the observable at the instant of switching on the apparatus ($t = 0$), or at some time during the interaction with the apparatus, or after the measurement. The aim of the measurement could also be to prepare a new state of the system. In each case an apparatus in a specially chosen initial state must be assembled in order to minimise the error of the measurement. We shall illustrate these propositions by considering the simplest example.

Example. The measurement of the coordinate of a free body. To measure the coordinate we only require that the QRS interacts with the body according to the Hamiltonian $\hat{H}_i = \alpha_i(t)\hat{x}\hat{Y}$ for a certain time τ. [$\hat{H}_i = \alpha_i(t)f(\hat{x})\hat{Y}$ is also possible.] Here $\hat{x}$ is an operator for the coordinate of the body, $\hat{Y}$ is an operator for the QRS, $\alpha_i(t)$ is a linking function. We assume that $\alpha_i(t) = \alpha_0$ for $0 \leqslant t \leqslant \tau$ and $\alpha_i = 0$ outside this time interval. The QRS can be represented by a free particle with a mass M. In the simplest case the full Hamiltonian is expressed [16] as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \alpha_i(t)\hat{x}\hat{Y} + \frac{\hat{P}^2}{2M} \ ,$$

where $\hat{P}$ and $\hat{Y}$ are, respectively, the momentum and the coordinate operator of the QRS. The theory allows for the existence of this Hamiltonian, but a real object described by such a Hamiltonian should contain, in addition to the two bodies, something capable of generating negative rigidity between the bodies to compensate for the positive rigidity which arises during the elastic interaction of the bodies.

Using Heisenberg's picture we can write

$$\text{(a)}\ \frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{\hat{p}}{m} \ , \quad \text{(b)}\ \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\alpha_0\hat{Y} \ ,$$

$$\text{(c)}\ \frac{\mathrm{d}\hat{Y}}{\mathrm{d}t} = \frac{\hat{P}}{M} \ , \quad \text{(d)}\ \frac{\mathrm{d}\hat{P}}{\mathrm{d}t} = -\alpha_0\hat{x} \ . \tag{6}$$

If the mass M is large enough, the operator $\hat{Y}$ can be assumed to be constant ($\hat{Y}_0$) during a finite interaction time τ. Then it follows from (6) that during the period $0 < t < \tau$

$$\text{(a)}\ \hat{x}(t) = \hat{x}_0(t) - \frac{\alpha_0\hat{Y}_0 t^2}{2m} \ ,$$

(b) $\hat{p}(t) = \hat{p}_0 - \alpha_0 \hat{Y}_0 t$,

$$\begin{aligned}(c)\ \hat{P}(\tau) &= \hat{P}_0 - \alpha_0 \int_0^{\tau} \hat{x}(t)\,\mathrm{d}t \\ &= \hat{P}_0 - \alpha_0 \hat{x}_0(\tau/2)\tau - \frac{\alpha_0{}^2 \hat{Y}_0 \tau^3}{6m} \\ &= \hat{P}_0 - \alpha_0 \hat{x}(\tau/2)\tau - \frac{7\alpha_0{}^2 \hat{Y}_0 \tau^3}{24m}\,. \end{aligned} \tag{7}$$

The relation (7c) can be rewritten as

$$\text{(a)}\ \hat{x}_0(\tau/2) = \frac{\hat{P}_0 - \hat{P}(\tau)}{\alpha_0 \tau} - \frac{\alpha_0 \hat{Y}_0 \tau^2}{6m}\,,$$

$$\text{(b)}\ \hat{x}(\tau/2) = \frac{\hat{P}_0 - \hat{P}(\tau)}{\alpha_0 \tau} - \frac{7\alpha_0 \hat{Y}_0 \tau^2}{24m}\,, \tag{8}$$

where

$$\text{(a)}\ \hat{x}_0(\tau/2) = \hat{x}(0) + \frac{\hat{p}(0)\tau}{2m} \tag{9}$$

is the operator of the coordinate in the unperturbed state at time $t = \tau/2$, and

$$\text{(b)}\ \hat{x}(\tau/2) = \hat{x}(0) + \frac{\hat{p}(0)\tau}{2m} - \frac{\alpha_0 \hat{Y}_0^2 \tau^2}{8m}$$

in a state perturbed by the interaction with QRS.

By measuring $P(\tau)$ we can evaluate $x_0(\tau/2)$ and $x(\tau/2)$. If $\hat{P}_0$ is not correlated with $\hat{Y}_0$ the dispersion of the estimate of the coordinate $x_0(\tau/2)$ is given by

$$(\Delta\tilde{x}_0)^2 = \left(\frac{1}{\alpha_0 \tau}\right)^2 [(\Delta\tilde{P})^2 + (\Delta P_0)^2] + \left(\frac{\alpha_0 \tau^2}{6m}\right)^2 (\Delta Y_0)^2\,, \tag{10}$$

where $(\Delta\tilde{P})^2$ is the dispersion of the error in the measurement of the momentum $P(\tau)$, and $(\Delta P_0)^2$ are the dispersions of the coordinate and of the momentum of the QRS in the initial state. Since $(\Delta P_0)^2 (\Delta Y_0)^2 \geqslant \hbar^2/4$ by minimising the right-hand side of Eqn (10) we obtain

$$\Delta\tilde{x}_0 \geqslant \left(\frac{\hbar\tau}{3m}\right)^{1/2}\,. \tag{11}$$

This is one of the standard quantum limits of error in the measurement of the coordinate of a free body. The reason for the appearance of the SQL (11) is the at the directly measured momentum $P(\tau)$ includes a component depending on the Y_0 coordinate. There are several ways of surmounting this SQL. Aharonov and Safko [16] suggested eliminating the effect of Y_0 on $P(\tau)$ by a spring of stiffness $-\alpha_0^2\tau^2/6m$, connected to the QRS for the duration of the measurement. There are even more elegant ways of overcoming this limit. For example, $x_0(\tau/2)$ can be estimated by measuring directly the $P(\tau) + \alpha_0^2\tau^3 Y_0/6m$ combination rather than $P(\tau)$. This observable can be formed, for example, by using a field of the divergent-lens type. Another way of overcoming the SQL (11) is to prepare the QRS in an initial state such that the momentum is appropriately correlated with the coordinate. Let

$$\hat{P}(0) = \hat{P}^0 + \left(\frac{\alpha_0 \tau^2 \hat{Y}_0}{6m}\right)\,, \tag{12}$$

where $\hat{P}^0$ is not correlated with $\hat{Y}_0$. In this case (7a) leads to

$$(\Delta\tilde{x}_0)^2 = \frac{(\Delta\tilde{P})^2 + (\Delta P^0)^2}{(\alpha_0\tau)^2} \to 0 \tag{13}$$

if $\alpha_0\tau \to \infty$. A similar state of affairs exists in the determination of $x(\tau/2)$ if

$$\hat{P}(0) = \hat{P}^0 + \frac{7\alpha_0\tau^2\hat{Y}_0}{24m}\,.$$

2.2 Uncertainty of the coordinate after the measurement

The correlation of the coordinate and of the momentum of the QRS allows the effect of the back reaction of the apparatus on the system to be eliminated from the determination of the coordinate at a given instant. However, this back reaction of the apparatus is not excluded from the coordinate itself. It follows from Eqn (7a) that the mean square (m.s.) perturbation of the coordinate at time $\tau/2$ is

$$\tilde{\Delta}x(\tau/2) = \frac{\alpha_0\tau^2\Delta Y_0}{8m}\,. \tag{14}$$

The perturbation of the momentum during the measurement is

$$\tilde{\Delta}p(\tau) = \alpha_0\tau\Delta Y_0\,. \tag{15}$$

The perturbation of the momentum creates a perturbation of the coordinate also after the interaction ($t > \tau$), and for $t \gg \tau$ this perturbation is

$$\tilde{\Delta}x(t) = \frac{\alpha_0\tau t\Delta Y_0}{m}\,. \tag{16}$$

Not allowing for the initial uncertainty of the momentum $p(0)$ we find from (13) and (16) that the total dispersion of the coordinate at time $t \gg \tau$ is

$$\begin{aligned}[\tilde{\Delta}x(t)]^2 &= (\Delta\tilde{x}_0)^2 + [\tilde{\Delta}x(t)]^2 \\ &\geqslant 2\Delta P^0 \frac{\Delta Y_0 t}{m} \geqslant \frac{\hbar t}{m}\,. \end{aligned} \tag{17}$$

Therefore the result of a repeated measurement of the coordinate of a free body at time t after the first measurement cannot be predicted to better than within a m.s. error of $(ht/m)^{1/2}$ [9]. This quantity is called the SQL of the uncertainty of the coordinate of the body at time t after its measurement. It has been the object of wide-ranging discussions among physicists in connection with the problem of detecting gravity waves because it leads to Eqn (3) as the limit of sensitivity to the force [17–21].

It should be stressed that in the derivation of expression (17) it was assumed that information on the value of the coordinate in the time t is given only by measurements of the coordinate in the time interval $0-\tau$. The theory does not preclude preparation of the body in a state that at time t the uncertainty of its coordinate is as small as desired. However, such preparation cannot be accomplished with the aid of the apparatus used to measure the coordinate without first establishing the required correlation between the coordinate and the momentum of the body after the measurement.

According to the expression

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar(t_2 - t_1)}{m} \tag{18}$$

the uncertainties of the coordinate of a free body at times $t_1 = 0$ and $t_2 = t$ are related to each other by the expression

$$\Delta x(0)\Delta x(t) \geqslant \frac{\hbar t}{2m} \,. \tag{19}$$

Therefore the following inequality is possible:

$$\Delta x(t) < \left(\frac{\hbar t}{m}\right)^{1/2} , \tag{20a}$$

if

$$\Delta x(0) > \left(\frac{\hbar t}{4m}\right)^{1/2} . \tag{20b}$$

However, (20b) is not a sufficient condition. In general (for $\langle x\rangle = \langle p\rangle = 0$) we have

$$[\Delta x(t)]^2 = [\Delta x(0)]^2 + \left[\Delta p(0)\frac{t}{m}\right]^2 + \langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle\frac{t}{m} \,. \tag{21}$$

If the last term in the right-hand side of (21) is zero, we shall have $\Delta x(t) \geqslant (\hbar t/m)^{1/2}$. We shall assume that $\hat{p}(0) = \hat{p}^0 - \beta\hat{x}(0)$, where β is a number and $\hat{p}^0$ is not correlated with $\hat{x}(0)$. In this case we obtain

$$[\Delta x(t)]^2 = [\Delta x(0)]^2\left(1 - \frac{\beta t}{m}\right)^2 + \left(\Delta p^0\frac{t}{m}\right)^2$$

$$\geqslant \frac{\hbar t}{m}\left|1 - \frac{\beta t}{m}\right| . \tag{22}$$

The minimum value of $\Delta x(t)$ is reached for $\Delta x(0)\Delta p^0 = \hbar/2$ and $\Delta x(0) = (\hbar t/4m)|1 - (\beta t/m)|$.

2.3 Errors in the continuous measurement of the coordinate

In the measurement process discussed above only one particle was used as the QRS. Such measurements have no practical significance. In real measurements fluxes of particles or quasi-particles are used (electrons, photons, etc.). In this case the force responsible for the back reaction of the apparatus on the system used to measure the coordinate can be expressed as the sum

$$F_{ba}(t) = \sum_j F_j(t - t_j) \,, \tag{23}$$

where $F_j(t-t_j)$ is the back reaction force of one of the particles. The force $F_j(t-t_j)$ acts during the time interval $-\tau_j < t - t_j < 0$, where τ_j is the duration of the interaction of this particle with the system. (The theory of continuous quantum measurements has been discussed in Refs [10, 12, 22–24] amongst others.)

The result of the approximate measurement of the observable $\hat{A}(t)$ can be expressed as the result of the exact measurement of the sum $\hat{A}(t) + \hat{A}_a(t)$, where $\hat{A}_a(t)$ is an operator of the apparatus. In the example discussed here it is represented by the operator $\hat{P}_0(t)/\alpha_0\tau$.

The error of the measurement of the coordinate can be calculated by using the equivalent circuit shown in Fig. 4. In the case of stationary measurements the spectral densities of the random function $F_{ba}(t)$ and $x_a(t)$ satisfy the condition

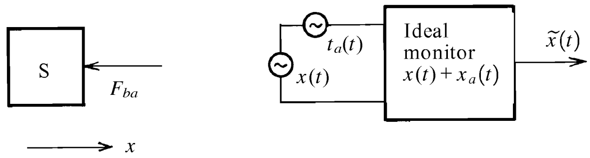

Figure 4.

$$S_F(\omega)S_x(\omega) - |S_{Fx}(\omega)|$$

$$\geqslant (\hbar^2/4) + \hbar\omega|\mathrm{Im}\, S_{Fx}(\omega)| \,, \tag{24}$$

where $S_{Fx}(\omega)$ is the spectral density of the cross correlation function for $F_{ba}(t)$ and $x_a(t)$, and $\mathrm{Im}S_{Fx}(\omega)$ is its imaginary part [10, 25]. The formula $S_F S_x \geqslant \hbar^2/4$ was obtained by Giffard in 1976.

Since the output signal of the apparatus is classical, the physical quantity inferred from the results of a continuous measurement of the coordinate can be calculated by the roles of the classical theory of estimate optimisation. The quantum limit of the estimation error (in the case of linear systems) is determined in this case with the use of expression (24). This problem has been considered in detail [10]. Instantaneous values of the coordinate can be estimated more accurately than SQL, but the mean square perturbation (the error of the estimation of the average coordinate of the free particle during the observation time τ) satisfies the condition

$$\Delta\bar{x}^\tau \geqslant \left[(S_F S_x)^{1/2}\tau/m\right]^{1/2} \geqslant \left(\frac{\hbar\tau}{2m}\right)^{1/2} , \tag{25}$$

if $S_{Fx} \equiv 0$, $S_F(\omega) = S_F$, $S_x(\omega) = S_x$.

The error in the estimate of the average momentum (in time τ) of the free particle under the same conditions is not less than

$$\Delta\bar{p}^\tau \geqslant \left[\frac{(S_F S_x)^{1/2} m}{\tau}\right]^2 \geqslant \left(\frac{\hbar m}{2\tau}\right)^{1/2} . \tag{26}$$

The amplitude A and the real (X_1) and the imaginary (X_2) parts of the complex amplitude of a harmonic oscillator can be calculated in this way with errors not smaller than

$$\left[\frac{(S_F S_x)^{1/2}}{m\omega_0}\right]^{1/2} \geqslant \left(\frac{\hbar}{2m\omega_0}\right)^{1/2} , \tag{27}$$

and an estimate of the energy is provided by Eqn (4).

Eqn (24) applies also in the case of an ideal apparatus, i.e. an apparatus not affected by the nature of the motion in the system as an average. An ideal apparatus does not contribute dissipation, stiffness, or inertia to the motion. In radio technology a voltmeter with an infinite input impedance and an ammeter with zero resistance are good examples of an ideal apparatus. In an apparatus with a finite input conductance Y_{11} the spectral densities of the fluctuations are mutually related by the expression

$$S_F(\omega)S_x(\omega) - |S_{Fx}|^2 \geqslant (\hbar^2/4) + \hbar\left|\mathrm{Re}\left(\frac{i\omega S_{Fx} Y_{11}^*}{Y_{11}}\right)\right.$$

$$\left. - \frac{\omega S_x \mathrm{Re} Y_{11}}{|Y_{11}|^2}\right| . \tag{28}$$

Expressions allowing for all four Y parameters of a real apparatus have been obtained [10, 25].

SQL of errors of measurements in distributed systems have been discussed in Ref. [26].

The SQL (4), (26), (27) can be overcome by using nonperturbing or quasi-nonperturbing measurements.

3. Quantum nonpertubing (nondemolition) measurements (QNDM)

The term quantum nonperturbing measurement refers to a measurement of the observable $\hat{N}$ in which the back reaction of the apparatus on the system under study does not affect the results of the first and of subsequent measurements of this observable. Such measurements are also called measurements free from fluctuational back reaction of the apparatus (back action evading measurements). A stimulus towards work of this type was provided by Braginskii and Vorontsov [9]. The basic aspects of the theory and practice of QNDM methods have been fully described, e.g. in Refs [10, 11, 12, 14, 23, 24, 27–28].

An observable which can, in principle, be measured without perturbation is called a nonperturbing (or QND) observable. Only an observable $\hat{N}$ which satisfies (in the Heisenberg picture) the following commutation condition during the free evolution of the system can be a QND observable:

$$\left[\hat{N}(t_j),\, \hat{N}(t_i)\right] = 0\ . \tag{29}$$

In particular, all the integrals of motion satisfy this condition. In a free particle the momentum and the energy are the QND observables. In a harmonic oscillator the observables include the energy and the real (X_1) and the imaginary (X_2) parts of the complex amplitude

$$\text{(a)}\ \hat{X}_1 = \hat{x}(t)\cos\omega_0 t - \left[\frac{\hat{p}(t)}{m\omega_0}\sin\omega_0 t\right]\ , \tag{30}$$

$$\text{(b)}\ \hat{X}_2 = \hat{x}(t)\sin\omega_0 t + \left[\frac{\hat{p}(t)}{m\omega_0}\cos\omega_0 t\right]\ . \tag{31}$$

QND observables are not necessarily integrals of motion [10, 11].

3.1 Evolution of the QND observable in the measurement process

There are two types of QND observables. Some observables can be free from fluctuational back reaction even during the interaction of the system with the QRS. Others are unpredictably perturbed during the interaction with the apparatus, but revert to their unperturbed value as soon as the link to the apparatus is cut off. In the latter case some observable $\hat{N}_{\mathrm{I}}(t)$, which is equal to the observable $\hat{N}(t)$ for the free evolution of the system, is retained during the measurement. Noncanonical observables such as velocity, kinetic energy, and other functions of the generalised velocity must vary randomly during their measurement [10, 16]. Were it not so, it would be possible to prepare a state of the system in which the ratio of the uncertainties was altered. Indeed, a perturbation of the coordinate must occur during measurements of the momentum. A random perturbation of the coordinate can be produced by motion with an indeterminate velocity over a definite time or by motion with a velocity known *a posteriori*, during indeterminate time. But if the velocity could be continuously monitored, its value at any instant would be known.

Example. Consider measurement associated with the following Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \alpha_i\hat{p}\hat{Y} + \frac{\hat{P}^2}{2M}\ . \tag{32}$$

In this case we have

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = 0\ , \tag{33}$$

i.e. the generalised momentum p is conserved in this interaction of the free body with the apparatus. However, the velocity

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{\hat{p}}{m} + \alpha_i\hat{Y}(t) \tag{34}$$

is perturbed during the same time. Nevertheless it reverts to its initial value as soon as the interaction is discontinued ($\alpha_i = 0$).

However, the same motion of the system and of the QRS can be described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_0^2}{2m} - \hat{x}\hat{Y}\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} + \frac{\alpha_i{}^2\hat{x}^2}{2M} - \frac{\alpha_i\hat{x}\hat{P}_1}{M} + \frac{\hat{P}_1^2}{2M}\ . \tag{35}$$

(This only requires adding to the appropriate Lagrangian the total derivative of the function $\alpha_i m x Y$.) In this case we have

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{\hat{p}_0}{m}\ , \qquad \frac{\mathrm{d}\hat{p}_0}{\mathrm{d}t} = m(\dot{\alpha}_i\hat{Y} + \alpha_i\dot{\hat{Y}})\ ,\ \text{i.e.}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\hat{x}} - m\alpha_i\hat{Y}) = 0\ . \tag{36}$$

Now the generalised momentum p_0 is equal to the kinetic momentum, and therefore it is not conserved during the interaction. But the same combination as in the previous case ($m\dot{x} - m\alpha_i Y$) is conserved.

3.2 Conditions for the realisation of a QNDM

The general condition for a measurement of the QND type can be formulated as follows. For a QND measurement of the QND observable $\hat{N}$ it is necessary and sufficient that after the interaction with the system the QRS carries information on the values of N but not on the values of the observables which do not commute with $\hat{N}$.

This condition is satisfied, in particular, for the following interaction:

$$\hat{H} = \hat{H}_{\mathrm{S}} + \alpha_i\hat{N}\hat{Y} + \hat{H}_{\mathrm{a}}\ ; \tag{37}$$

where $\hat{H}_{\mathrm{S}}$ and $\hat{H}_{\mathrm{a}}$ are the Hamiltonians of the system and the apparatus, respectively.

Many workers treat this as a necessary condition. However, strictly speaking, it is sufficient but not necessary. For example, in order to measure X_2 we can use the interaction corresponding to $\hat{H}_i = \alpha_i\hat{x}(t)\hat{Y}$ if the duration of the interaction is equal to one half of the period of the oscill-ator [10, 29]. In this case the total change in the momentum of the QRS is

$$\int_0^{\pi/\omega} x(t)\,\mathrm{d}t = \int_0^{\pi/\omega}(X_1\cos\omega t + X_2\sin\omega t)\,\mathrm{d}t = \frac{X_2^2}{\omega}\ , \tag{38}$$

i.e. as a result of the interaction the QRS acquires information only on X_2.

3.3 Nonperturbing measurement of the energy of a harmonic oscillator

Let us consider the LC circuit in Fig. 5 as a model of the system under study. A mechanical oscillator (M, k) attached to the mobile parts of the inductor and capacitor plays the role of the QRS. The systems can be constructed so as to ensure that

$$\frac{1}{L(Y)} = \frac{1+\alpha_i Y}{L_0}, \qquad \frac{1}{C(Y)} = \frac{(1+\alpha_i Y)}{C_0}, \tag{39}$$

where L_0 and C_0 are the unperturbed values of the circuit parameters. This circuit is represented by the Hamiltonian

$$\hat{H} = \left(\frac{\hat{p}^2}{2L_0}+\frac{\hat{q}^2}{2C_0}\right)(1+\alpha_i\hat{Y})+\hat{H}_\mathrm{a}$$

$$= \left(\hat{n}+\tfrac{1}{2}\right)\hbar\omega_0(1+\alpha_i\hat{Y})+\hat{H}_\mathrm{a}, \tag{40}$$

where $\hat{n}$ is an operator for the number of quanta, and $\omega_0 = (1/L_0C_0)^{1/2}$. In this case we have

$$\frac{\mathrm{d}\hat{n}}{\mathrm{d}t} = 0,$$

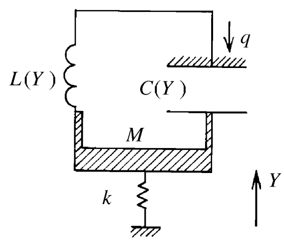

Figure 5.

i.e. the number of quanta is conserved even during the motion of the QRS. However, the frequency depends on the operator of the coordinate of the QRS, i.e. it is not a number but an operator:

$$\hat{\omega} = \omega_0(1+\alpha_i\hat{Y}). \tag{41}$$

Therefore the number of quanta n, but not the energy of the oscillator, stays constant during interaction with the QRS.

We also have

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{p}_y}{\mathrm{d}t} = [\hat{p}_y, \hat{H}_\mathrm{a}]+\hat{H}_0[\hat{p}_y, \alpha_i\hat{Y}], \tag{42}$$

where $\hat{H}_0$, equal to $(\hat{n}+\frac{1}{2})\hbar\omega_0$, is an operator for the energy of the unperturbed oscillator. By measuring p_y we can evaluate H_0 and n. After the measurement the mobile parts of the circuit can be fixed in the state L_0, C_0. Under these conditions the frequency reverts to its initial value ω_0, and thus the initial value of the energy of the circuit is restored. An analysis of this system [30] showed that the possibility of measuring the energy of a conservative lumped circuit with a mean-square perturbation of the error $\Delta H_0 < \hbar/\tau$ (where τ is the duration of the measurement) is not inconsistent with the fundamental propositions of quantum mechanics.

The uncertainty of the frequency ω during the measurement time increases the uncertainty of the phase by the amount

$$\tilde{\Delta}\varphi = \Delta\int_0^\tau \omega(t)\mathrm{d}t \geqslant \tfrac{1}{2}\Delta\tilde{n}. \tag{43}$$

The following relation also applies:

$$\Delta\tilde{H}\frac{\tilde{\Delta}H}{H_0} \geqslant \frac{\hbar}{2\tau}, \tag{44}$$

where $\Delta\tilde{H} = \Delta\tilde{n}\hbar\omega_0$ is the mean-square perturbation of the measurement error of the energy, $H_0 = (\langle n\rangle+\frac{1}{2})\hbar\omega_0$, and $\tilde{\Delta}H$ is the uncertainty of the perturbation of the energy during the measurement.

Expression (44) can be re-written

$$\Delta\tilde{H} \geqslant \frac{\hbar}{2\tau}\frac{\omega_0}{\tilde{\Delta}\bar{\omega}}, \tag{45}$$

where $\tilde{\Delta}\bar{\omega} = \tilde{\Delta}\varphi/\tau$ is the uncertainty in the perturbation of the frequency averaged over time τ. Therefore the necessary condition for the measurement of the energy of the oscillator with an error of $\Delta\tilde{H} < \hbar/\tau$ is an initial state of the QRS such that the relative uncertainty of the frequency of the oscillator is greater than 0.5.

During the QND measurement of the energy of the nonconservative oscillator a change in the relaxation time takes place. This makes it impossible to measure the energy to within an error smaller than $\hbar/\tau_0^*$, where τ_0^* is the relaxation time of the free oscillator [10, 31].

A general theory of continuous QND measurement of the number of photons has bee proposed by Mensky [23] and Veda et al. [32]. The problem of the change in the evolution of the system by a continuous measurement of its energy ('Zeno's quantum effect') has also been discussed [10, 12, 22a, 23]. The most recent thoughts on the energy–time relationships have been put forward by Mensky [23, 34] and by Busch [33].

3.4 Nonperturbing measurement of the energy of electromagnetic waves

The energy of an electromagnetic wave in volume V of a nondispersing medium is

$$H = \int\frac{1}{8\pi}(\varepsilon E^2+\mu H^2)\mathrm{d}V$$

$$= \int\frac{\varepsilon\mu}{8\pi}\left[\frac{E^2}{\rho}+\rho H^2\right]\mathrm{d}V, \tag{46}$$

where $\rho = (\mu/\varepsilon)^{1/2}$ and V is the volume of the wave. It follows from (46) that for a nonperturbing measurement of the energy we require an interaction of the QRS with the electromagnetic field such that the simultaneous changes in the dielectric (ε) and the magnetic permittivity (μ) leave ρ unchanged. In this case the velocity of the wave [$v = 1/(\varepsilon\mu)^{1/2}$] and its frequency ($\omega = 2\pi v/\lambda$) may vary during the measurement, but the number of quanta (n) and the wavelength (λ) stay constant. A suitable method of realising this measurement scheme has not so far been proposed. However, in some suggested measurement methods the error limits can be much lower than the SQL, though the conditions for nonperturbing energy measurement are not strictly fulfilled. These methods are usually called nonperturbing, though it would be more correct to call them quasi-nonperturbing.

Before proceeding any further we should stress the importance of the results obtained in an analysis of the quantum limits of error in the measurement of the energy of an electromagnetic wave. The uncertainty of the change in the frequency of the wave in a nonperturbing measurement (due to random changes in velocity) is

$$\Delta\omega = \frac{2\pi\Delta v}{\lambda} < \frac{2\pi c_0}{2\lambda} = \frac{\omega_0}{2}\,, \qquad (47)$$

i.e. $\Delta\omega/\omega_0 < \frac{1}{2}$. Therefore the error in the measurement of the energy of the electromagnetic wave cannot be less than $\hbar/\tau$ [10].

We can arrive at the same conclusion by a different path. The measurement of momentum should be accompanied by a perturbation of the coordinate such that $\tilde{\Delta}x \geqslant \hbar/2\Delta\tilde{p}$. Since in our $\tilde{\Delta}x = \omega\tilde{\Delta}v < \tau c_0/2$, and the energy of the waves is $H = pc_0$, we have

$$\Delta\tilde{H} \geqslant \frac{\hbar}{\tau}\,. \qquad (48)$$

This result is incompatible with that obtained in the analysis of the measurement with a lumped circuit. The difference in the results is due to the difference in the links between momentum and energy in these circuits.

4. Quasi-nonperturbing measurements of the energy of e.m. waves

4.1 Principles of quasi-nonperturbing measurements

The infringement of the conditions for a nonperturbing measurement causes the apparatus to accept information not only from the observable of interest ($\hat{N}_1$) but also from the observable $\hat{N}_2$ which does not commute with $\hat{N}_1$. This produces a fundamental limitation of the errors in the estimation of both these observables. The limit of the error in the estimate of N_1 depends on the precision of the estimate of N_2 from the response of the QRS. If their commutator $[\hat{N}_1, \hat{N}_2] = \mathrm{i}2\hbar\gamma$, these mean-square perturbations of the estimates are related to each other by the expression [10, 14]

$$\Delta\tilde{N}_1 \geqslant (\gamma\beta\hbar)^{1/2}\,, \quad \Delta\tilde{N}_2 \geqslant \left(\frac{\gamma\hbar}{\beta}\right)^{1/2}\,, \qquad (49)$$

where $\beta = \Delta\tilde{N}_1/\Delta\tilde{N}_2$. Changes in β can be produced, for example, by exploiting the frequency selectivity of the QRS.

Example. If we stipulate that in the circuit of Fig. 5 the inductance is independent of Y [10, 12, 35], then instead of the Hamiltonian (42) we obtain

$$\hat{H} = \frac{\hat{p}^2}{2L_0} + \frac{\hat{q}^2(1+\alpha_i\hat{Y})}{2C_0} + \hat{H}_{\mathrm{a}}\,, \qquad (50)$$

for which

$$\frac{\mathrm{d}\hat{n}}{\mathrm{d}t} \neq 0\,,$$

and the force acting on the QRS (the body M on a spring k) is $F_{\mathrm{b}} = \alpha_i q^2/2C_0$. This force will have not only a constant component (as in the circuit of Fig. 5) but also a variable component which contains information on the phase of the electrical oscillations. By observing the motion of the QRS we can obtain information on both the energy and the phase of the oscillations simultaneously. However, information on the energy can be obtained from the constant component of the displacement of the body M, but information on the phase can be obtained only from the high-frequency component of frequency 2ω. If the frequency of the characteristic oscillations of the body M is $\Omega \ll 2\omega$ the amplitude of its high-frequency oscillations will be $(\Omega/2\omega)^2$ times smaller than the constant component of the displacement. The quantity β also undergoes similar changes. The error limit in the measurement of the number of quanta is [10, 14]

$$\Delta\tilde{n} = \langle n\rangle^{1/2}\frac{\Omega}{2\omega}\,. \qquad (51)$$

In order to make clear which of the proposed schemes are quasi-nonperturbing (though their proposers call them nonperturbing) we shall list some specific properties of the circuits for nonperturbing energy measurements. In the scheme of Fig. 5 the displacement of the body M under the influence of the field form the electric circuit varies the characteristic frequency of the circuit but leaves its characteristic resistance $\rho_{\mathrm{c}} = (L/C)^{1/2}$ unchanged. Under these conditions the quantity ω is independent of the phase of the electrical oscillations, i.e. the oscillations in the circuit remain linear. The characteristic motion of the QRS also has no effect on ρ_{c}.

4.2 Quasi-nonperturbing measurement of the energy of waves by using the Kerr nonlinearity

From the electromechanical scheme for the quasi-nonperturbing measurement of the energy of a circuit we can easily pass to a wholly electrical scheme. The mechanical oscillator in the scheme plays the part of a link in which information on the phase of the electrical oscillations is suppressed by the inertia. But the same filtration can be achieved by using a low-frequency oscillatory circuit whose nonlinear capacitance (which depends on the square of the strength of the electric field) is simultaneously included in the high-frequency circuit [36a]. By measuring the characteristic frequency of the low-frequency circuit we can in principle evaluate the energy of the high-frequency oscillations with an error smaller than SQL.

The concept of using a quadratic dependence of the dielectric permittivity (cubic nonlinear polarisability) on the field strength has been proposed as a basis for a number of quasi-nonperturbing measurement schemes for the energy of electromagnetic waves. An optical waveguide with cubic polarisability has been placed in the capacitor gap of a UHF resonator [36b]. With some simplifications the dielectric permittivity of the waveguide can be expressed as

$$\varepsilon(x,y,z,t) = \varepsilon_1[1+\alpha E^2(x,y,z,t)]$$

The capacitance of a condenser filled with the optical waveguide will depend on the energy of the field for a given spatial distribution of the field. By measuring the capacitance we can determine the energy of the electromagnetic wave. In this case the electric field used to measure the capacitance of the condenser plays the role of the QRS.

The use of specific properties of the interaction of waves in a medium with cubic nonlinearity has been suggested [37, 38]. We know that when two harmonic waves are propagating in such a medium, one of which we shall call the signal wave (SW)

$$E_{\mathrm{s}}(x,t) = A_{\mathrm{s}}\exp\left[\mathrm{i}(\omega_{\mathrm{s}}t - k_{\mathrm{s}}x)\right] + \text{compl. conj.}$$

and the other the probe wave (PW)

$$E_{\mathrm{p}}(x,t) = A_{\mathrm{p}}\exp\left[\mathrm{i}(\omega_{\mathrm{p}}t - k_{\mathrm{p}}x)\right] + \text{compl. conj.}$$

their wavenumbers are

$$k_s = k_{s0}\Big[1 + (\alpha/4)(|A_s|^2 + 2|A_p|^2)\Big],$$

$$k_p = k_{p0}\Big[1 + (\alpha/4)(|A_p|^2 + 2|A_s|^2)\Big]. \quad (52)$$

The phase shift of the PW depends on the square of the amplitude of the SW. Therefore by measuring the phase of the PW after the interaction of the waves we can calculate $|A_s|^2$ and the energy of the SW.

In the condenser circuit the capacitance is independent of the phase of the SW, i.e. the information on the phase (position) of the wave is not transferred to the QRS. In the second circuit information on the phase of the signal wave can be carried by combination waves.

The first condition for QNDM — constancy of the measured quantity as a result of its interaction with the apparatus — can be assumed to be fulfilled in these systems only to the extent that the formation of harmonics and combination waves can be ignored. This assumption can be justified by considering the effect of the dispersion of the medium, but in strict calculations it should not be forgotten that dispersion is always accompanied by dissipation.

The second condition for QNDM is that the change in state of the QRS should be determined by a measurable quantity whose value is conserved. In our system the effect of the signal on the QRS is determined by the parameter $|A_s|^2$, which is independent of the amplitude of the PW according to the solutions (52). However, it should be noted that the solutions (52) are valid under the condition that the field strengths (rather than the energy fluxes) are given at the boundary ($x = 0$). However, if the wave passes from one medium into another, with a different wave resistance $\rho = (\mu/\varepsilon)^{1/2}$, the amplitude of the wave is changed even in the absence of reflection. In both the schemes examined ε is affected by the QRS, whereas μ remains unchanged. Therefore ρ also changes, and so does the amplitude of the SW field. Because of the uncertainty in the amplitude of the probe fields the change in ρ (and therefore also the change in E_s) becomes indeterminate. Hence, even if the quantity $|E_s|^2$ is accurately measured during interaction of the waves, its value before and after the interaction can be evaluated only approximately. It has been shown [39] that the mean-square perturbation of the error in the estimate of the energy of the wave in this case cannot be less than

$$\Delta W \geqslant \left(\frac{\langle n\rangle}{\omega_s \tau}\right)^{1/2} \hbar\omega_s\,, \quad (53)$$

where τ is the duration of the interaction of the SW with the probe field.

The error of the measurement in real circuits will be affected by dissipation, by the transfer of energy to the harmonics and beat frequency waves, and by self-action effects in the probe wave.

The effect of dissipation in a nonlinear medium on the error of the measurement of the energy of a wave has been analysed [40, 41]. It was shown that for a low absorption coefficient (q_d) the number of quanta in the SW can be calculated with an error of

$$\Delta\tilde{n}_s \geqslant (\langle n\rangle q_d)^{1/2}. \quad (54)$$

The lowest absorption we can find in modern quartz fibres is 0.2 dB km^{-1}, i.e. $q_d = 10^{-2}$ for a length of 1 km.

To my knowledge, no analysis of the effect of harmonics and combination waves has been published so far. Combination waves probably play the most important role if their frequencies are close to ω_s and ω_p, i.e. waves with frequencies $\omega_3 = 2\omega_s - \omega_p$ and $\omega_4 = 2\omega_p - \omega_s$, since they are closer to synchronism with the SW and the PW. It can be shown that, if the dispersion is such that the amplitudes of the combination waves are always much smaller than the amplitudes of the fundamental waves, the spatial period of the beats in the amplitudes of the combination waves will be equal to $2\pi/6\beta_\omega\omega_s(\omega_s - \omega_p)^2$ and the maxima of the amplitudes of the waves with frequencies ω_3 and ω_4 will be

$$|A_3|_m = \frac{\alpha\omega_3|A_s|^2|A_p|^2}{24\beta_\omega v_0\omega_s(\omega_s - \omega_p)^2}\,,$$

$$|A_4|_m = \frac{\alpha\omega_4|A_s|^2|A_p|^2}{24\beta_\omega v_0\omega_p(\omega_p - \omega_s)^2}\,. \quad (55)$$

These expressions were obtained for conditions such that the linear dispersion has the same dependence on frequency as in the Korteweg–de Vries equation, i.e. $k(\omega) = \omega/v + \beta_\omega\omega^3$. If this formula is applied to quartz we can use the value $\beta_\omega = 10^{-41}$ s^3 m^{-1} at wavelengths close to 1 μm. In the optical region quartz has $\chi^{(3)} = 5 \times 10^{-15}$ c.g.s. units (0.6×10^{-33} SI units). Accordingly $\alpha = \chi^{(3)}/\varepsilon_0 \approx 10^{-22}$ m^2 V^{-2}, i.e. $|A_{3,4}|^2 \ll |A_s|^2/\langle n_s\rangle^{1/2}$ will apply at energy fluxes of the SW (P_s) and of the PW (P_p) for which

$$P_s P_p \ll \frac{(10^4\ \mathrm{W\ \mu m^{-2}})^2}{\langle n_s\rangle^{1/2}}\,, \quad (56)$$

if $(\omega_s - \omega_p)/\omega_s \approx 5 \times 10^{-3}$, as has been reported [42].

4.3 Effect of the self-action of the probe wave on the error of the measurement of the signal wave

The phase shift of the PW after interaction with the SW for a length l is

$$\delta\phi_{pl} = D_{pp} n_p + D_{sp} n_s, \quad (57)$$

where $D_{pp} = \alpha k_{p0} l \rho\hbar\omega_p/2\tau_p S$ is the self-action coefficient of the PW, $D_{sp} = \alpha k_{p0} l\rho\hbar\omega_s/\tau_s S$ is the interaction coefficient of the PW with the SW, $n_{p,s} = |A_{p,s}|^2\tau_{p,s}S/2\rho\hbar\omega_{p,s}$ is the number of photons in a length $\tau_{p,s}$, and S is the effective cross-sectional area of the waves.

If the number of quanta in the SW is estimated by measurements on the phase φ_{pl}, the dispersion of the results is

$$(\Delta\tilde{n}_s)^2 = D_{sp}^{-2}\Big[(\Delta\tilde{\phi}_{pl})^2 + (\Delta D_{pp} n_p)^2\Big], \quad (58)$$

where $(\Delta\tilde{\phi}_{pl})^2$ is the dispersion of the error in the measurement of the phase shift ϕ_{pl}.

If the PW at the inlet is in a coherent state, then

$$\begin{aligned}(\Delta\tilde{n}_s)^2 &\geqslant D_{sp}^{-2}\left(\frac{\langle n_p\rangle}{4} + D_{pp}^2\langle n_p\rangle\right)\\ &= D_{sp}^{-2}\left(\frac{\hbar\omega_p}{\tau_p}\right)\left[\frac{1}{4W_p} + D_{pp}^2\left(\frac{\tau_p}{\hbar\omega_p}\right)^2 W_p\right]\\ &\geqslant D_{sp}^{-2} D_{pp}\,. \end{aligned} \quad (59)$$

Here W_p is the power of the test wave. The smallest error will be obtained for the optimum value of the power of the TW:

$$W_{\mathrm{p,opt}} = \frac{vS}{\alpha\omega_p l\rho} . \quad (60)$$

If the signal wave and the probe wave are pulsed, and their rates of propagation are different, the time of their interaction τ_{sp} can be shorter than their propagation time in the nonlinear medium (the self-action time) τ_{pp}. In this case Eqn (59) should be replaced by

$$(\Delta\tilde{n}_s)^2 = \left(\frac{\tau_s S}{\tau_{sp}\,4\alpha\omega_p\rho\hbar\omega_s}\right)^2 \frac{\hbar\omega_p}{\tau_p} \times\left[\frac{1}{4W_p} + \left(\frac{\alpha\omega_p\rho\tau_{pp}}{2S}\right)^2 W_p\right] \geqslant \left(\frac{\tau_{pp}}{\tau_{sp}}\right)^2 \frac{\tau_s^2 S}{4\alpha\rho\hbar\omega_s^2\tau_p\tau_{pp}} . \quad (61)$$

Eqns (59) and (61) are valid in the regions within which they are not inconsistent with the expressions (53) and (54). For what power of the SW can we expect to find $\Delta\tilde{n}_s \leqslant (\langle n_s\rangle q_1)^{1/2}$, where $q_1 > q_d$? From (59) we obtain

$$W_s \geqslant \frac{vS}{2\alpha\omega_s l\rho q_1} \approx 2.5\times 10^{12}\frac{S}{lq_1}\ \mathrm{W} . \quad (62)$$

Under these conditions the optimum power of the probe wave should be

$$W_{\mathrm{p,opt}} = \frac{n_{\mathrm{opt}}\hbar\omega_p}{\tau_s} = \frac{vS}{\alpha l\rho\omega_p} \approx 5\times 10^{12}\frac{S}{l}\ \mathrm{W} . \quad (63)$$

Therefore

$$\frac{W_{\mathrm{s,min}}}{W_{\mathrm{p,opt}}} = \frac{\omega_p}{2\omega_s q_1} . \quad (64)$$

In single-mode optical fibres $S \approx 25\ \mu\mathrm{m}^2$. Therefore for an interaction length $l \approx 10^3$ m we find $q_1 \approx 0.1$ if $W_s \geqslant 0.6$ W and $W_p \approx 0.1$ W.

The limits (59) and (61), due to the self-action of the PW, are not fundamental. In principle the self-action effect can be completely suppressed. This is done simply by introducing, before or after the interaction of the waves, a correlation between an initial phase of PW ϕ_{p0} and $|A_p|^2$ such that $\phi_{p0} = \phi_0 - D_{pp}n_p$ (ϕ_0 is a constant). This can be done, for example, by passing the PW through another medium having a nonlinearity, with a sign opposite to that of the medium in which the interaction takes place. This procedure was used, in particular to compensate the self-modulation of the radiating phase of a Nd/YAG laser. The compensation was applied in a cell containing caesium vapour. The elimination of the self-action of the PW through the dispersion of the nonlinearity of the medium has been suggested [43, 44]. The self-action is due to the part of the nonlinear polarisation whose frequency arises as a result of the following combination: $\omega_p - \omega_p + \omega_p$. The effect of the SW on the PW is caused by the combination of frequencies $\omega_p - \omega_s + \omega_s$. In order to suppress the effect of the self-action of the PW we must use a medium in which $\chi^{(3)}(\omega_p - \omega_p + \omega_p) \ll \chi^{(3)}(\omega_p - \omega_s + \omega_s)$. However, it should not be forgotten that the dispersion of the nonlinearity is associated with dissipation, and in an ideally transparent medium the nonlinearity coefficients are independent of frequency.

The influence of the self-action of the PW on the error of the measurement of the energy of the SW can be substantially lowered by an appropriate choice of the operating regime of the phase detector. The shift in the phase of the PW relative to the reference wave (provided by a local oscillator) is usually measured with a homodyne detector. The output current of this detector has a component proportional to $A_p A_{L0}\cos(\phi_{L0} - \phi_{pl})$, where A_{L0} and ϕ_{L0} are the amplitude and the phase of the reference wave. If the reference wave in the coherent state is strong enough, the uncertainties of its amplitude and phase can be ignored. Under these conditions the current variation associated with the change in the quantities A_p, ϕ_{p0}, and A_s is equal, to a first approximation, to

$$\delta i_p \approx \eta_e A_{L0}\Big\{\delta A_p\Big[\cos(\phi_{L0} - \bar\phi_{pl}) + \bar A_p\frac{\partial\phi_{pl}}{\partial A_p}\sin(\phi_{L0} - \bar\phi_{pl})\Big] + A_p\delta A_s\frac{\partial\phi_{pl}}{\partial A_s}\sin(\phi_{L0} - \bar\phi_{pl})\Big\} , \quad (65)$$

where η_e is the quantum efficiency of the detector and $\bar A_p$, $\bar\phi_{pl}$ are the average values of the amplitude and phase of the PW.

For measurements of the phase shift the regime which gives $\cos(\phi_{L0} - \bar\phi_{pl}) = 0$ is thought to be the best. However, in this case δi_p is affected by the phase shift caused by the self-action. Obviously, the effect of fluctuations in the amplitude of the PW on the current δi_p can be eliminated (in the linear approximation) by choosing a regime of the detector in which

$$\cos(\phi_{L0} - \bar\phi_{pl}) + \bar A_p\frac{\partial\phi_{pl}}{\partial A_p}\sin(\phi_{L0} - \bar\phi_{pl}) = 0 . \quad (66)$$

In practice this method of excluding the self-action effect of the PW is meaningful only for $\bar A_p\partial\phi_{pl}/\partial A_p \leqslant 1$, since in the opposite case the dependence of δi_p on δA_s would be much weaker.

5. Results of experiments on the QNDM of the energy of optical waves

5.1 Measurement of the energy of travelling waves

The first attempt to measure the energy of optical travelling waves by the interaction of the waves in a quartz fibre was reported [37] in 1986. The aim of the experiment was to demonstrate the link between the phase of the PW and the quantum fluctuations of the amplitude of the SW. A block diagram of the apparatus is shown in Fig. 6. The radiation of a frequency-stabilised krypton-ion laser working on two independent transitions at 647 and 675 nm limited by

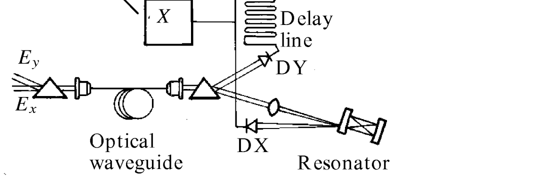

Figure 6.

quantum noise was introduced into an optical fibre. After interacting in a single-mode fibre 114 m long the waves had an additional phase modulation associated with self-action and interaction. The phase modulation of the wave E_x carried information on the modulation of the amplitude of the wave E_y. After separating the waves with a prism the SW was led to the photodetector DX. The current fluctuations of this detector for a sufficiently high power of the SW are proportional to the amplitude fluctuations of the SW and independent of its phase. The PW is directed to a confocal resonator adjusted so that the phase of the reflected wave at the carrier frequency (the average frequency of the PW) changes by $\theta = -\pi/3$. Under these conditions a phase shift is generated between the carrier and the side components which is sufficient to transform a phase modulation of the wave into amplitude modulation of the photodetector current.

The variable component of the current i_y, delayed for a time τ_d in a coaxial cable, was added to the variable component of the current i_x, and the sum was examined in a spectrum analyser. Correlation of the currents was revealed as a periodic dependence of the current spectral density on the frequency.

In the 54–58 MHz range the spectrum varied with a period of 2 MHz. The quantum fluctuations of the amplitude of the SW accounted for 37% of the mean-square perturbation of the total current (i_x) fluctuations. The remaining noise was associated with the amplitude fluctuations of the PW and with its phase fluctuations, arising during the generation of the wave in the laser and as a result of the self-action effect in the nonlinear medium. Some of the noise was caused by the nonideality of the photodiode DX (which had a quantum efficiency of 0.4). This noise level corresponds to $S_s/(S_s + S_x) = (0.37)^2$, where S_s, S_x are the components of the spectral fluctuation density of the current i_x, generated by the signal noise (S_s) and by other noises (S_x). Therefore the signal-to-noise ratio is $S_s/S_x \approx 0.2$. Accordingly the mean square perturbation of the error in the measurement of the energy of the SW in the same circuit is $5\langle n_s\rangle^{1/2}\hbar\omega_s$, i.e. it is 5 times greater than the corresponding SQL.

Let us compare this value of the error with the value predicted by Eqn (59). The power of the PW (60 mW) is much less than the optimum value. Therefore the self-action of the PW is irrelevant, and we can write

$$(\Delta\tilde{n}_s)^2 \cong \frac{D_{sp}^{-2}\hbar\omega_p}{4\tau_p W_P},$$

i.e.

$$\frac{(\Delta\tilde{n}_s)^2}{\langle n_s\rangle} \cong \frac{v^2 S^2}{4\alpha^2\rho^2 l^2\omega_p\omega_s W_p W_s} \cong 3\times10^{-22}\left(\frac{S}{\alpha}\right)^2. \tag{67}$$

For a single-mode quartz fibre in this range of wavelengths we can tentatively assume that $S/\alpha \approx 25\times10^{10}$ (SI units). In this case we obtain from (67) $(\Delta\tilde{n}_s)^2/\langle n_s\rangle \approx 20$, which agrees with experimental results.

The results of an experiment on the interaction of waves in an optical wave guide 500 m long have been reported [44]. For a SW power of 12.6 mW the phase shift of the PW was 1.38×10^{-2}. According to Eqn (62) the error of the measurement for this SW power level and length of fibre cannot be less than the SQL.

In order to overcome the SQL of the energy for a given length and nonlinearity of the optical fibre we must increase the power of both the SW and the PW. Sakai et al. [42] tried to solve the problem by this method. They measured the energy of a single optical soliton and observed the shift in another soliton interacting with the first. This was done by admitting into an optical fibre with a negative dispersion of the group velocity three solitons, produced with the aid of a special optical scheme from a single laser pulse. The first (reference) and the third (probe) had identical wavelengths of the carrier (1455 nm). In the middle (signal) soliton the wavelength was slightly greater (1460.7 nm). The velocity of the signal soliton was slightly less than the velocity of the probe soliton, producing an overtaking during which the velocities of the solitons were altered by their nonlinear interaction. As a result, the distance between the reference and the probe soliton changes in proportion to the energy of the signal soliton. The reference and the probe soliton are then passed through a Mach–Zender interferometer to a photodetector. The output photocurrent depends on the phase shift of the probe soliton relative to the reference. The total phase shift of the probe soliton, resulting from the interaction with the signal soliton, was 1.22 rad for a signal energy of 15 pJ (1.1×10^8 photons) and duration of 2.6 ps. Like Levenson et al. [37] these workers limited their study to a demonstration of the correlation between the output signal of the interferometer and the quantum fluctuations of the energy of the signal soliton. They used the method already fully described [37]. Groups of three solitons were generated at a frequency of 100 MHz. The signal solitons from the fibre were led to a photodetector, whose current was added to the current from an interferometer (retarded by a delay line). The total current was amplified and fed to a spectrum analyser. The required correlation was observed as a periodic frequency dependence of the intensity of the spectrum. According to the authors approximately 60% of the mean-square perturbation of the phase noises at the output of the phase detector consisted of noises associated with the shot noise of the signal soliton. This corresponds to a signal-to-noise ratio of $\sim 1/1.8$, i.e. $(\tilde{\Delta}n_s)^2/\langle n_s\rangle \approx 1.8$.

If there had been no noises associated with imperfections in the elements of the system (losses in the fibre, 0.1 dB; in the connectors, 0.2 dB; in the diffraction gratings, 0.6 dB; in the photodiodes, 1.7 dB) the estimated energy of the solitons could have been close to the SQL, but not less than it. Thus, it follows from the parameters of the scheme [$l = 400$ m, dispersion of the group velocity 12 ps km^{-1}, duration of signal 2.6 ps, duration of probe soliton 3.6 ps, energy of probe soliton 6 pJ (4.4×10^7 photons), difference in wavelengths between the signal and the probe soliton 5.7 nm] that the duration of the interaction of the solitons (overtaking time) due to dispersion is approximately $\tau_{sp} \approx 2.5\times10^{-7}$ s, whereas the duration of the self-action is $\tau_{pp} \approx 2\times10^{-6}$ s. From Eqn (61) we obtain $(\tilde{\Delta}n_s)^2 \geqslant 1.6\times10^8 \approx \langle n_s\rangle$. (The effective cross-sectional area was assumed to be 25 μm^2.) Unfortunately, even this elegant experiment failed to overcome the SQL of error for the energy.

5.2 Methods of increasing the effective nonlinearity

In order to overcome the SQL of the measurement error of the energy of a wave at reasonable signal power levels we need to establish an interaction between the SW and the apparatus such that the effective nonlinearity is much greater than in quartz, but the losses remain equally small. In practice it is found that in transparent media the greater the nonlinearity, the greater is the dissipation, though in principle this relationship is not inescapable. The search for more effective methods of measurement is proceeding along various paths. The exploitation of wave interactions in exciton semiconductors (CdS, GaAs), avoiding losses in the materials by the self-induced transparency effect, has been suggested [45]. Other directions include the use of the interaction between streams of particles and a standing or a travelling wave in a region of nonuniform spatial distribution of the wave. For example, it is hoped to make use of the diffraction of a stream of electrons travelling along a dielectric waveguide in the nonuniform field of a wave [46, 47]. Because of the radial nonuniformity of the field of the wave the electron experiences, in addition to the variable component, a constant component of the force (the Miller force [48]), proportional to the square of the amplitude of the electric field strength of the wave. The stronger this effect the smaller is the difference between the velocity of the electrons v_e and the velocity of the wave v. Calculations have shown that detection of the effect of an i.r. pulse with an energy of about one photon on an electron is theoretically possible under the following conditions: (a) the relative velocity difference between the electron and the wave should not be greater than 10^{-3}, (b) the interaction length should not be shorter than 10 cm, (c) the duration of the pulse should be approximately 1 ps.

An effective method of strengthening the nonlinear interaction at relatively low SW powers is to store the energy in microresonators with a high Q factor [49]. For example, Braginskiĭ and Il'chenko prepared spherical sapphire optical resonators having a Q of 10^8-10^9 for a diameter of 40 – 400 μm. The effective volume of the field in these resonators for modes of the 'whispering gallery' type is of the order of 10^{-9} cm^3. The volume of the soliton in this experiment [42] was an order of magnitude greater.

5.3 Measurements by wave interaction in the resonator

The possibility of measuring the energy of a wave by using the nonlinear interaction of waves in a resonator has been examined [50, 51, 51a]. Theoretical studies with relatively rigid limitations showed that the quasi-nonperturbing measurement of the energy of a wave by using a resonator is possible in principle. However, an experimental test of this conclusion by using a ring resonator showed that overcoming the SQL in a system of this type is not straightforward [51]. Very strict, almost unrealisable limitations on the classical noises associated with the scattering of the SW and of the PW in the resonator must be imposed. Nevertheless it is claimed [51] that the resonator scheme gives better results than the travelling-wave scheme.

Dianov et al. [52] reported the results of an experiment on the interaction of waves in a nonlinear resonator (the external resonator of a laser). The experimental scheme is shown in Fig. 7. Radiation from the single-frequency semiconductor laser *1* ($\lambda = 1.28$ μm) passes through the acoustooptical modulator *2* (working in the Bragg regime at 65 MHz) and is split into two beams corresponding to zeroth and first diffraction. The zero beam is then introduced into the single-mode optical fibre waveguide *3*, 5.7 km long. The principal feedback mechanism between the fibre and the laser was the Rayleigh scattering in the fibre. The zero beam was then combined with the first diffraction beam in the optical shunt *4*, in which one of the outlets was connected to the germanium avalanche photodiode *5*. The beat signal was amplified and fed to the radio-frequency spectrum analyser *6*. The SW was introduced into the interferometer by using the free outlet of the shunt, by means of a second semiconductor laser *7* ($\lambda = 1.5$ μm) as the source. The intensity of its emission was modulated sinusoidally by the generator *8*. The average power of the SW in the optical waveguide did not exceed 100 μW. When the modulation frequency was a multiple of the inter-mode distance for the longitudinal modes of the external resonator, a large increase in the effectiveness of the interaction between the PW and SW was observed. In this regime an incursion of 5×10^{-2} into the PW phase was obtained for a SW power of 70 μW. Under these conditions the PW passed nonlinearly through the medium not less than 50 times.

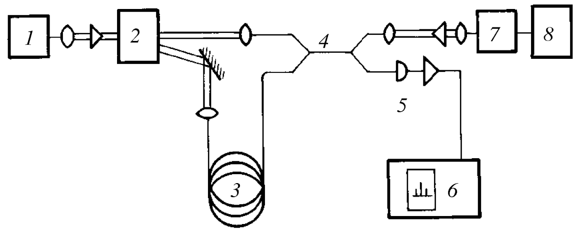

Figure 7.

This experiment produced a phase shift of the PW per unit power of the SW not less than three orders of magnitude greater than in the other experiments described above. However, even this success does not prove that the SQL can be overcome in this experiment. With fibre losses of 0.3 dB km^{-1} the attenuation of the SW for a single passage along the fibre was about 1.7 dB. Furthermore the increase in the effectiveness of the interaction at certain modulation frequencies of the SW suggests that a substantial amount of energy pumping is taking place through the combination waves.

According to Eqn (62) the SQL can be overcome at a SW power of 70 μW only if the effective interaction length is not less than 10^6 m.

It has been shown [53] that not only classical but also quantum noises can prevent the overcoming of the SQL in a resonator. If the resonator is operated in the standing-wave regime, a random reflection of the SW associated with the random change in frequency of the resonator during the measurement cannot be avoided. In order to surmount the SQL we must use the travelling-wave regime. Such a regime can be established in ideal ring resonators and spherical resonators. On the other hand, in real resonators energy can be transferred between different modes as a result of various nonuniformities, including those associated with the spontaneous or induced excitation of acoustic waves.

5.4 Other methods of QNDM of the energy of waves

The search for possible schemes of QNDM of the energy of electromagnetic waves has included all the phenomena known to produce low-frequency macroscopic effects in electromagnetic waves. The optical rectification effect has been examines [54], but found to be unattractive as a measurement method because of the vanishingly small value of the transformation coefficient. The possibility of QNDM of energy by generating the second harmonic in a resonator with quadratic nonlinearity has been demonstrated [55, 56]. The use of two-photon transitions in atoms has been suggested [57]. The possible use for the QNDM of the self-induced transparency effect in two-level systems has been discussed [58]. It has been stated [59] that a field containing a given number of photons can be produced and continuously maintained by the interaction (in a resonator) of a quasi-resonant beam of Rydberg atoms followed by the measurement of the phase shift of their wave function. Holland et al. [60] have obtained a proof that if the frequency of the light is substantially different from the frequency of the atomic resonance, and if the resonator in which the atoms are interacting with the optical field has a high Q factor, the atomic beam does not affect the number of photons in the resonator and the number of photons can be established by monitoring the deflections of the beam. An analysis of a measurement scheme which uses coherently prepared three-level atoms interacting with the field of the resonator has been reported [61, 62].

In the articles mentioned above the proposed methods of measurement are claimed to be quantum-nonperturbing. Formally this claim is justified by the fact that with the specified Hamiltonian for the interaction the measured observable is an integral of motion even during its interaction with the apparatus. In reality, however, approximate Hamiltonians are used, which fail to allow for various processes of relevance in the calculation of the quantum limits of error in the measurements. For example, in the analysis of the interaction of waves in a Kerr medium no allowance is made for the generation of harmonics and combination waves, or for changes in the wave resistance of the medium. The influence of these effects on the precision of the measurement has been discussed above. In the analysis of the interaction of a beam of atoms with an electromagnetic field in a resonator the usual interaction (linearly proportional to the electric field) is ignored, and so is the interaction of the field with the translational motion of the atoms in the regions of nonuniformity of the field.

6. Quantum limit of the detection of action on a system

It was pointed out above that one of the strongest stimuli for the development of the theory and methods of nonperturbing quantum measurements was the hope of overcoming SQL of the error in the detection of the interaction of a force with the system. In general the detection error depends on the initial state of the system and on which of the observables has been chosen as the object of the measurement. The relationship between the quantum error limit for the detection and the initial state of the system has been discussed [10, 63].

The process aiming to discover an interaction can be represented as a sequence of three stages. Some state of the system is first created. The system then undergoes an evolution. In the third stage a measurement is carried out in order to establish whether an interesting interaction has taken place in the system during the evolution. In order to minimise the detection error the measurement should be of the optimum type, i.e. it should be an accurate measurement of an observable of the system which allows minimisation of the detection error. In general the optimum measurement proce-dure is not the same as the optimum procedure for preparing the initial state of the system.

We shall try to clarify the quantum limit of detection of an external action on the system by discussing pure initial states. After the evolution (but before the measurement) the system can be found in two probable states:

$$|\Psi_0(t)\rangle = \hat{U}_0(t)|\Psi(0)\rangle,$$

or

$$|\Psi_1(t)\rangle = \hat{U}_1(t)|\Psi(0)\rangle, \tag{68}$$

where $|\Psi(0)\rangle$ is the vector of the initial state, and $\hat{U}_0(t)$, $\hat{U}_1(t)$ are operators of the evolution for the nonperturbed and the perturbed motion, respectively. The quantum limit of the average probability of the error in distinguishing between two pure states is [3]

$$P_{\mathrm{wd}} = 1 - (1 - 4\zeta_0\zeta_1|\gamma|^2)^{1/2}, \tag{69}$$

where ζ_0, ζ_1 are the *a priori* probabilities of the states $|\Psi_0\rangle$, $|\Psi_1\rangle$, and $|\gamma| = |\langle\Psi_0|\Psi_1\rangle|$. In the present case

$$|\gamma| = |\langle\Psi(0)|\hat{R}(t)|\Psi(0)\rangle|. \tag{70}$$

The operator $\hat{R} = \hat{U}_0^{+}\hat{U}_1$ satisfies the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{R}}{\mathrm{d}t} = \hat{H}_i^0\hat{R} ,$$

where $\hat{H}_i^0 = \hat{U}_0^{+}(\hat{H}_1 - \hat{H}_0)\hat{U}_0$, $\hat{H}_1$, $\hat{H}_0$ are the Hamiltonians of the perturbed and of the nonperturbed motions. The operator $\hat{R}(t)$ in the form $\hat{R}(t) = \mathrm{e}^{\mathrm{i}\hat{\varphi}(t)}$ gives

$$|\gamma|^2 = \left|\int_{\infty}^{\infty}\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}\Phi(\varphi)\right|^2 ,$$

where $\Phi(\varphi)$ is the distribution function for the probabilities of the eigen values of the operator $\hat{\varphi}$. Various $\Phi(\varphi)$ functions can give $|\gamma| = 0$.

For an optimum initial state we have [10, 63]

$$|\gamma| = \begin{cases} \cos^2\Delta\varphi, & \text{if } \Delta\varphi < \pi/2 , \\ 0, & \text{if } \Delta\varphi > \pi/2 . \end{cases} \tag{71}$$

The state in which $|\gamma| = 0$ for a minimum value of $\Delta\varphi$ is considered to be the optimum. It is characterised by the following density distribution of the probabilities φ:

$$W(\varphi) = \tfrac{1}{2}\Big[\delta(\Delta\varphi - \pi/2) + \delta(\Delta\varphi + \pi/2)\Big] . \tag{72}$$

Let us consider the important special case in which the operator $\hat{H}_i^0(t)$ satisfied the condition

$$\Big[[\hat{H}_i^0(t_1),\, \hat{H}_i^0(t_2)],\, \hat{Q}\Big] = 0 , \tag{73}$$

where $\hat{Q}$ is an arbitrary operator. This condition is fulfilled when a classical force $F(t)$ is acting on a linear system. In this case we have

$$\hat{\varphi}(t) = \frac{1}{\hbar}\int_0^t H_i^0(t')\mathrm{d}t' . \tag{74}$$

If the action of the classical force is observed over a period $\tilde{\tau}$ we obtain

$$\hat{H}_i^0(t) = F(t)\hat{x}^0(t) \tag{75}$$

and therefore

$$\hat{\varphi}(t) = \frac{1}{\hbar}\int_0^t F(t)\hat{x}^0(t)\mathrm{d}t , \tag{76}$$

where

$$\hat{x}^0(t) = \hat{U}_0^+(t)\hat{x}(0)\hat{U}_0(t) . \quad (77)$$

Formula (76) suggests a path for obtaining a maximum value of $\Delta\varphi$ when $F(t)$ is known. Unfortunately, there is no known way of preparing the optimum state.

Let us consider the initial states for which

$$W(\varphi) = \alpha_\varphi \exp\left(-\frac{\varphi}{2\sigma}\right)[1 + f(\varphi)] , \quad (78)$$

where $f(\varphi)$ is a polynomial of finite degree and σ and α_φ are parameters. This distribution occurs when a classical force acts on an oscillator in a coherent state [$f(\varphi) \equiv 0$], in a squeezed state [$f(\varphi) \equiv 0$], or in a state with a given energy [$1 + f(\varphi)$, the Hermitian polynomial].

The limit of the sensitivity of the detection corresponds to $\Delta\varphi < 1$ and $\sigma < 1$. In this case we obtain from (76) and (78)

$$|\gamma|^2 \cong 1 - (\Delta\varphi)^2 . \quad (79)$$

Therefore if $\zeta_0 = \zeta_1 = \frac{1}{2}$ we obtain

$$P_{\mathrm{wd}} \approx \tfrac{1}{2}(1 - \Delta\varphi) . \quad (80)$$

We shall take $P_{\mathrm{wd}} \approx 0.25$ as the detection threshold. This value of P_{wd} is found when the signal is equal to the mean-square perturbation of the fluctuations.

Example. If the force is applied as a rectangular pulse of amplitude F_0 and length $\tilde{\tau}$ we find from (76) and (80) that

$$F_0\tilde{\tau} = \frac{\hbar}{2\Delta\bar{x}} , \quad (81)$$

where

$$\Delta\bar{x} = \Delta\left[\frac{1}{\tilde{\tau}}\int_0^{\tilde{\tau}} \hat{x}^0(t)\,\mathrm{d}t)\right] . \quad (82)$$

(a) If the force is acting on a free mass m, we obtain

$$\Delta\bar{x} = \Delta x(\tilde{\tau}/2), \quad (83)$$

i.e. the detection threshold in this case is lower, the greater the uncertainty of the coordinate of the mass at the instant when the force is applied. (We should note that we are discussing pure initial states.)

(b) This force is acting on a harmonic oscillator of mass m and frequency ω_0.

Let $\omega_0\tilde{\tau} \ll 1$. Then, for a coherent initial state we can write

$$F_0\tilde{\tau} = \left(\frac{\hbar m\omega_0}{2}\right)^{1/2} . \quad (84)$$

This is the SQL of the detection of the action of a force on an oscillator. If the initial state has a given number of quanta n, and $n \gg 1$, we have

$$F_0\tilde{\tau} = \left(\frac{\hbar m\omega_0}{2n}\right)^{1/2} . \quad (85)$$

The expressions (84) and (85) were first obtained by Braginskii by analysing specific methods of detecting the action of a force.

If the force is in the form of a sinusoidal train of length $\tilde{\tau}$, frequency Ω, amplitude F_0, and $\Omega\tilde{\tau} = 2k\pi$ (where k is an integer), we can detect from the response of the free body (with an error probability of 0.25) a force in which

$$F_0\tilde{\tau} = \frac{\hbar m\Omega}{2\Delta p} , \quad (86)$$

where Δp is the uncertainty in the momentum of the body during the action of the force. From the response of the harmonic oscillator at $\omega_0 = \Omega$ we can detect (with the same probability of error) a force which obeys the condition

$$F_0\tilde{\tau} = \frac{\hbar}{\Delta X_2} , \quad (87)$$

where ΔX_2 is the uncertainty of the imaginary part of the complex amplitude. In a coherent state $\Delta X_{2(\mathrm{c})} = (\hbar/2m\omega_0)^{1/2}$. In the n-state $\Delta X_{2(n)} = (\hbar n/2m\omega_0)^{1/2} \gg \Delta X_{2(\mathrm{c})}$. This inequality can also be applied to the squeezed state.

We should stress the fact that the state of the system having the product $\Delta x \Delta p = \hbar/2$ does not offer any advantage over other pure Gaussian states. This is true under conditions of optimum measurements, i.e. when the observable being measured corresponds to the initial state and form of the force. This observable is defined by the expression

$$\hat{A} = \hat{U}_1|\psi(0)\rangle\langle\psi(0)|U_1^+ - U_0^+|\psi(0)\rangle\langle\psi(0)|U_0^+. \quad (88)$$

An analysis shows that the procedure for preparing known states is not the same as the procedure which gives optimum measurements. The optimum observable could be a combination of canonical variables which has no recommended measurement method. The experimental workers try to discover paths which allow the SQL of detection to be surmounted by using the same procedure for the preparation and for the measurement, even if this makes the detection threshold higher than the quantum limit. The nonperturbing and quasi-nonperturbing measurements whose theory and methods were discussed above offer this possibility.

QRS can interact with a test body in such a manner that it obtains information about the variation of the state of the test body produced by action of the external force rather than about the state itself [29]. Equations (76) and (80) are valid in this case also.

V B Braginskii (1967) was the first to draw attention to the problem of the quantum-mechanical limitations in the sensitivity of the gravity wave experiment. The problem was further analysed by Yu I Vorontsov, F Ya Khalili, K S Thorne, C M Caves, W G Unruh, V V Dadonov, V I Man'ko, V N Rudenko, A V Gusev [64–66], S P Vyatchanin [67], R Onofrio, F Bordoni, and others. None of these SQL has so far been surmounted experimentally.

7. Relationship between the measurement error and the perturbation

The interaction of the system with the apparatus used to measure the observable A produces a random perturbation of all the observables which are not commutative with A. How is the error of the measurement of A related to the perturbation of the observable B? We know that the dispersions of any observables $\hat{A}$ and $\hat{B}$ of a system in any of its states are interrelated by the uncertainty relation

$$(\Delta A)^2(\Delta B)^2 \geqslant \frac{\hbar^2|\langle\hat{C}\rangle|^2}{4(1 - r^2)} , \quad (89)$$

where r is the correlation coefficient between $\hat{A}$ and $\hat{B}$ in the given system state, and $\hat{C} = [\hat{A}, \hat{B}]/\mathrm{i}\hbar$.

The quantities ΔA and ΔB are not related to the measurement error, and they can be obtained experimentally by a statistical treatment of the results of accurate measurements of the observables $\hat{A}$ and $\hat{B}$ for different elements of the ensemble in systems, in a given state, or for multiple measurements in one system, returning the system to its initial state after each measurement. In the scientific literature and in textbooks we meet assertions that the uncertainty relation applies also to the dispersion of the measurements of the observable $\hat{A}$ $[(\Delta A)^2_{\text{meas}}]$ and of the change in the dispersion of the observable $\hat{B}$ $[(\Delta B)^2_{\text{pert}}]$ produced by the apparatus during the measurement. The evidence quoted to justify this belief is that the momentum is perturbed by the measurements of the coordinate according to the expression

$$(\Delta x)^2_{\text{meas}}(\Delta p)^2_{\text{pert}} \geqslant \frac{\hbar^2}{4} .$$

An analysis of this problem [10] showed that the uncertainty relation and the relation between the measurement error and the perturbation have the same form only when C is not an operator and $r = 0$. When $[[\hat{A}(t_1), \hat{A}(t_2)], \hat{A}(t_3)] = 0$ and $[[\hat{B}, \hat{A}], \hat{A}] = 0$ we have

$$(\Delta A)^2_{\text{meas}}(\Delta B)^2_{\text{pert}} \geqslant \frac{\hbar^2}{4}\langle|\hat{C}|^2\rangle \neq \frac{\hbar^2}{4}|\langle\hat{C}\rangle|^2 . \quad (90)$$

Here $(\Delta B)^2_{\text{pert}}$ refers to the quantity (5), defined with respect to a mixed state of the system. For example, in the measurement of the energy of free particles we have

$$(\Delta H)^2_{\text{meas}}(\Delta x)^2_{\text{pert}} \geqslant \left(\frac{\hbar}{2m}\right)^2 \langle p^2\rangle .$$

On the other hand, if

$$[[\hat{B}, \hat{A}], \hat{A}] \neq [[\hat{A}, \hat{B}], \hat{B}] , \quad (91)$$

we have

$$(\Delta A)^2_{\text{meas}}(\Delta B)^2_{\text{pert}} \neq (\Delta A)^2_{\text{pert}}(\Delta B)^2_{\text{meas}}. \quad (92)$$

In particular,

$$(\Delta x)^2_{\text{meas}}(\Delta H)^2_{\text{pert}} \neq (\Delta H)^2_{\text{meas}}(\Delta x)^2_{\text{pert}}. \quad (93)$$

A general expression similar to (90) applicable to arbitrary observables has not yet been obtained.

In this review no attempt has been made to examine the problems of the nonperturbing measurement of the momentum of a free body and of the quadrature amplitude of the oscillator and of the wave in quadrature. Work in those areas has been reviewed elsewhere [10, 12, 23, 28, 34, 68, 69–71].

A probable resolution of the controversy [1, 2] on the limits of measurement of the strength of an electromagnetic field has recently been proposed [72].

8. Conclusions

The standard quantum limits of measurement errors [1, 3, 4, 11] are not fundamental: they are associated with certain measurement procedures. Various methods of overcoming them are available. Observables which are integrals of motion and other nonperturbable observables can be measured, in principle, with an error lower than the corresponding SQL by using nonperturbing and quasi-nonperturbing measurement techniques. Attempts to surmount some of these SQL experimentally have been unsuccessful so far because of various technical difficulties. However, scientists working on this problem in many laboratories throughout the world are still hopeful of overcoming some of SQL of the measurement errors discussed above. Should they be successful they will present experimental physics with a range of new possibilities in fundamental research and new methods of transferring and processing information.

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