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Fermi-gas approach to the problem of superfluidity in three- and two-dimensional solutions of ^{3}He in ^{4}He

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Contents

Abstract. New experimental approaches and theoretical results concerning the search for superfluidity in three- and two-dimensional solutions of ^{3}He in ^{4}He are reviewed. Estimates for the *s*-wave and *p*-wave pairing temperatures are given for both unpolarised and polarised cases. The role of monolayers and submonolayers of ^{3}He as an ideal-purity two-dimensional system for experimental verification of various current theories of high-temperature superconductivity is emphasised.

1. Introduction

One of the most interesting and still experimentally unresolved problems in low-temperature physics is the search for superfluidity in three-dimensional and particularly in two-dimensional (thin films, submonolayers) solutions of ^{3}He in ^{4}He. In the present review I shall concentrate on new experimental approaches and theoretical results that have been published in the last few years. I shall stress particularly the role of thin ^{3}He films and submonolayers as ideal two-dimensional systems for experimental checking of various theories which are actual in connection with the problem of high-temperature superconductivity.

It is known that a solution of ^{3}He in ^{4}He is the simplest low-density Fermi system of ^{3}He atoms in an inert superfluid ^{4}He condensate, which makes a solution of this kind an ideal object for the development and testing of methods belonging to the realm of Fermi-liquid theory. These methods have been used successfully in describing the normal properties of solu-tions (thermodynamic characteristics, transport coefficients) [1] and in prediction of possible superfluidity of the ^{3}He subsystem in such solutions [2 – 4]. The first classical theory of superfluidity of three-dimensional solutions was proposed by Bardeen, Baym, and Pines (BBP) in 1967 [2]: they established an elegant analogy between pairing of two ^{3}He atoms in a solution via the polarisation of the ^{4}He background (exchange of virtual phonons) and the electron – phonon interaction in the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity (Fig. 1). In accordance with the ideas of Bardeen, Baym, and Pines, the total interaction between two ^{3}He particles in a solution consists of two components, direct and exchange:

$$V(r) = V_{\rm dir}(r) + V_{\rm exch}(r) \ . \qquad (1)$$

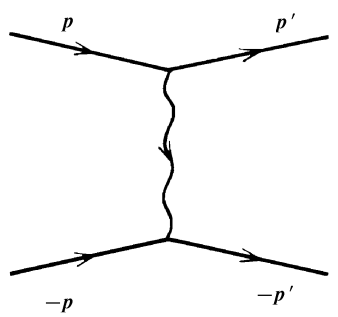

Figure 1. Interaction of two ^{3}He atoms via the polarisation of the superfluid ^{4}He background.

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The direct interaction includes the contributions of hard-core repulsion at short distances (V_1) and of the van der Waals attraction (V_2) at large distances:

$$V_{\rm dir}(r) = V_1(r) + V_2(r) \ . \tag{2}$$

The exchange interaction $V_{\rm exch}(r)$ represents the interaction of two ^{3}He atoms via a local change in the density of ^{4}He. This is an analogue of the deformation potential in the BCS theory.

The corresponding expression in the momentum space is

$$V(q) = V_{\rm dir}(q) + V_{\rm exch}(q) \ , \tag{3}$$

where $V_{\rm exch}(q)$ is due to the exchange of a virtual phonon in the three-dimensional case and the exchange of a quantum of third sound in the two-dimensional situation.

At low temperatures and concentrations the subsystem of ^{3}He atoms is a low-density Fermi liquid, i.e. it is effectively a Fermi gas. Therefore, the superfluid transition in this liquid is described by the BCS theory and it depends decisively on the amplitude and sign of the total interaction $V(q)$ on the Fermi surface. More rigorously, we have $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$, where $\boldsymbol{p}$ and $\boldsymbol{p}'$ are the momenta of the incoming and outgoing particles in the Cooper channel, $|\boldsymbol{p}| = |\boldsymbol{p}'| = p_{\rm F}$, and $q^2 = 2p_{\rm F}^2(1-\cos\theta)$; $\theta = \widehat{\boldsymbol{p}\boldsymbol{p}'}$ and the only quantity which must be known when dealing with the Cooper problem is the value of the s-wave harmonic of the potential $V(q)$ on the Fermi surface:

$$V_{l=0} = \int_{-1}^{1} V\big(q(\cos\theta)\big)\frac{{\rm d}\cos\theta}{2} \ . \tag{4}$$

2. Three-dimensional case

The deformation potential has the following form in the momentum space:

$$V_{\rm exch}(q) = g_q^2 \frac{\omega_q}{(\varepsilon_{p+q} - \varepsilon_p)^2 - \omega_q^2} \ , \tag{5}$$

where g_q is the coupling constant and ω_q is the frequency of the phonon spectrum of ^{4}He. If $|\varepsilon_{p+q} - \varepsilon_p| < \omega_q < \omega_{\rm D}$, we find that $V_{\rm exch}(q) = -g_q^2/\omega_q < 0$.

In complete analogy with the BCS theory we have $g_q^2 \propto q$, $\omega_q = sq$, where s is the velocity of sound in ^{4}He, so that the final result is $V_{\rm exch}(q \to 0) =$ const. In the case of solutions this constant is $-(1+\alpha)^2 m_4 s^2/n_4 < 0$, where $\alpha \cong 0.28$ is the relative increase in the volume of the solution owing to the replacement of a ^{4}He atom with a ^{3}He atom; n_4 and m_4 are the density and mass, respectively, of ^{4}He. It should be noted that in the low-density case we have $\omega_{\rm D} > \varepsilon_{\rm F}$ and the whole volume of the Fermi sphere (and not only the Debye shell) participates, as in the standard BCS theory, in the superconducting pairing.

The direct interaction of ^{3}He atoms in the momentum space is found from the thermodynamic identity describing the derivative of the chemical potential with respect to the density and has the following form:

$$V_{\rm dir}(q=0) = \frac{\partial\mu_{3\uparrow}}{\partial n_{3\downarrow}} = (1+2\alpha)\frac{m_4 s^2}{n_4} > 0 \ , \tag{6}$$

where $\mu_{3\uparrow}$ and $n_{3\downarrow}$ represent, respectively, the chemical potential of ^{3}He atoms with 'up' spin and the density of ^{3}He atoms with 'down' spin. The result is

$$V(q=0) \equiv V_{l=0} = V_{\rm exch}(q=0) + V_{\rm dir}(q=0) = -\alpha^2\frac{m_4 s^2}{n_4} < 0 \ . \tag{7}$$

We can therefore conclude that at very low ^{3}He concentrations (when $p_{\rm F} \to 0$ and, consequently, $q \to 0$) the total interaction is attractive and we can expect the spherically symmetric singlet s-wave pairing which is standard in the BCS theory.

However, spin diffusion experiments show that the situation is far from trivial (see Ref. [3] and the papers cited there). In these experiments the dependence of $\mathcal{D}T^2$ ($\mathcal{D}$ is the spin diffusion coefficient and T is the temperature) on the ^{3}He concentration is determined. The experimental curves are strongly nonmonotonic and exhibit a maximum at a certain concentration x_0 approximately equal to 4% (Fig. 2). They are described approximately by the expression

$$\mathcal{D}T^2 \propto \frac{x^{2/3}}{V_{l=0}^2 - \frac{2}{5}V_{l=0}V_{l=1} + \frac{11}{35}V_{l=1}^2} \ . \tag{8}$$

A theoretical analysis of these experimental curves shows that the absolute value of the s-wave harmonic of the total potential $V_{l=0}$ decreases with an increase in the concentration x, then vanishes at $x = x_0$, and at higher concentrations becomes repulsive. On the other hand, for $x \gtrsim x_0$ the p-wave harmonic of the total potential $V_{l=0}$ is significant and attractive (although smaller than $V_{l=0}$ at $x = 0$). These circumstances lead to two possible approaches to the superfluidity in solutions.

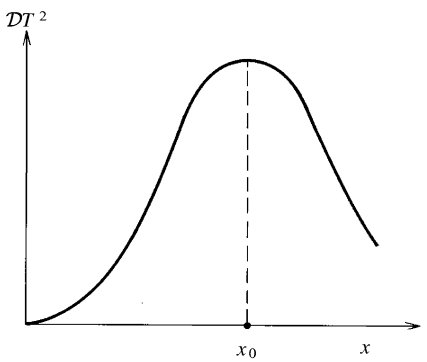

Figure 2. Approximate experimental dependence of the product $\mathcal{D}T^2$, representing the spin diffusion in a solution, on the concentration x of ^{3}He.

In the first approach it is assumed that the total interaction of two ^{3}He atoms described by $V(q)$ exhibits a significant momentum dependence and, moreover, its sign is reversed at the values of the vector q of the order of the Fermi momentum when the concentration is x_0, i.e. when it is $k_{\rm F}(x_0)$. This hypothesis leads to the model potential of the BBP theory:

$$V(q) = V(q=0)\cos\frac{q}{k_{\rm s}} \ ; \quad k_{\rm s} \sim k_{\rm F}(x_0) \ . \tag{9}$$

The BBP model potential was improved in 1989 by van de Haar, Frossati, and Bedell [3]. They introduced the concentration dependence of the amplitude of the potential $V(q=0)$:

$$V(q=0) = -\frac{m_4 s^2}{n_4}\alpha^2\left(1 + \gamma\frac{x}{x_{\rm max}}\right) \ , \tag{10}$$

where $x_{\rm max}$ is the solubility limit of ^{3}He at a given pressure P and $\gamma(P)$ is a fitting parameter. In both theories the s-wave harmonic of the total interaction is maximal and attractive at

low concentrations and then it begins to fall, changing sign to become a repulsive one at concentrations corresponding to $k_F \sim k_s$. At higher concentrations the p-wave harmonic of $V(q)$ becomes attractive. Therefore, van de Haar, Frossati, and Bedell predict singlet s-wave pairing in a solution at low concentrations and triplet p-wave pairing at high concentrations. It should be pointed out that two fitting parameters $k_s = k_s(P)$ and $\gamma(P)$, deduced from the experiments on spin diffusion and magnetostriction, are used in the improved model potential of van de Haar, Frossati, and Bedell.

The second approach, adopted by the author of the present review and by others [1, 4, 5], does not rely on any model potential. In this approach the only microscopic parameter of the system is the s-wave scattering length $a_0 = (m/4\pi)V_{l=0}$, which is independent of the pressure and concentration. It is assumed that its sign is reversed at a concentration corresponding to the maximum on the $\mathcal{D}T^2$ curve (Fig. 3).

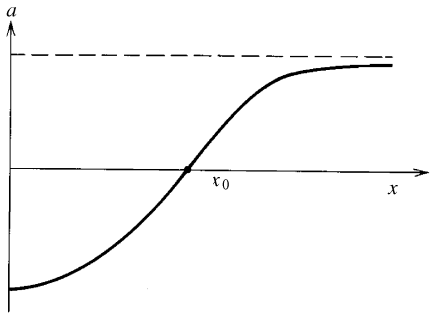

Figure 3. Qualitative dependence of the scattering length in a solution on the concentration of ^{3}He. At $x = 100\%$ the value of $a(x)$ tends to the scattering length of pure ^{3}He and is approximately equal to $2/k_{F0}$ at zero pressure (here, k_{F0} is the Fermi momentum of pure ^{3}He).

It should be pointed out that the higher harmonics ($V_{l=1}$, $V_{l=2}$, ...) appear in the second order but not because of the momentum dependence of the total interaction $V(q)$: they originate from the scattering length a_0 because of the effective interaction of two ^{3}He particles via the Fermionic background of their own ^{3}He subsystem.

The relationship between these two approaches is approximately the following. Let us assume, for the sake of simplicity, that the direct interaction of two ^{3}He particles in a solution is described by

$$V_{\rm dir}(r) = \begin{cases} V_1, & r < r_1 , \\ -V_2, & r_1 < r < r_2 , \end{cases} \tag{11}$$

where the first term is responsible for the hard-core repulsion at short distances and the second term is due to the van der Waals attraction at long distances (Fig. 4).

At low ^{3}He concentrations in a solution, i.e. in the case when $k_F r_1 \ll k_F r_2 \ll 1$, the s-wave harmonic of the direct interaction is $V_{\rm dir}^{l=0} = V_1^{l=0} - V_2^{l=0}$. However, at higher concentrations when $k_F r_2 \geqslant 1$, but with $k_F r_1$ still much less than unity, the van der Waals attraction becomes ineffective and we have $V_{\rm dir}^{l=0} = V_1^{l=0}$. Then, if

$$V_1^{l=0} - V_2^{l=0} - V_{\rm exch}^{l=0} < 0 , \tag{12}$$

but

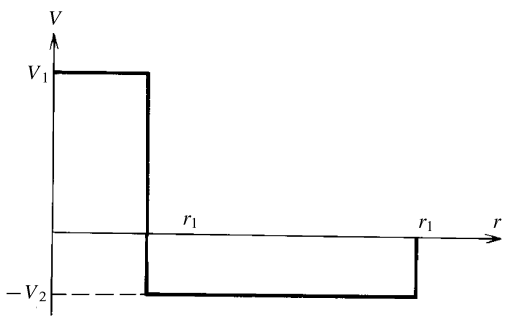

Figure 4. Model representation of the direct interaction of two particles in a solution as a function of the distance r between them.

$$V_1^{l=0} - V_{\rm exch}^{l=0} > 0 , \tag{13}$$

we have a low-density Fermi gas with the gas parameter $k_F r_1 \ll 1$ and with a scattering length which changes its sign at $k_F \sim 1/r_2$. Naturally, this approach ignores the p-wave harmonic of the van der Waals interaction, which need not be small in the transition region $k_F r_2 \sim 1$. It should be pointed out that at high concentrations when $k_F r_2 \gg 1$ we find that $V_2^{l=1}$ is small and of the same order as $V_2^{l=0}$. In this review the second (Fermi-gas) approach to the problem of superfluidity in solutions will be mainly used.

3. Three-dimensional Fermi gas with attraction

The expression for the temperature of the superfluid transition in a Fermi gas with attraction was first obtained by Gor'kov and Melik-Barkhudarov in 1961 [6], soon after creation of the BCS theory. Bashkin and Meyerovich [1] used this expression to describe the superfluidity of solutions at very low concentrations. For the concentrations in the range $x < x_0$ and an attractive s-wave scattering length $a_0 < 0$ the expression for this temperature is

$$T_{c0} = 0.1\varepsilon_{F0} x^{2/3} \exp\left(-\frac{\pi}{2|a_0|k_{F0}x^{1/3}}\right) , \tag{14}$$

where ε_{F0} and k_{F0} are the Fermi energy and momentum of pure ^{3}He. It is worth noting that the preexponential factor in this expression is proportional to ε_F and not ω_D, as in the case of the phonon model.

According to the estimates of Østgaard and Bashkin [7], the maximum value of T_{c0} is $T_{c0}(1\%) \sim 10^{-4}$ K. Frossati and his colleagues [3] proposed a lower critical temperature $\max T_{c0} = T_{c0}(2\%) \sim 4 \times 10^{-6} - 10^{-5}$ K. They obtained the larger value of T_{c0} of the order of 10^{-5} K by deducing the fitting parameters from the magnetostriction experiments, and 4×10^{-6} from the spin diffusion experiments. At a given concentration x the gas parameter of the theory is $a_0 k_{F0} x^{1/3}$ and it depends weakly on the gas pressure.

4. Three-dimensional Fermi gas with repulsion

At higher concentrations ($x > x_0$) the scattering length changes its sign, $a_0 > 0$, and s-wave pairing becomes impossible. Nevertheless, even in this case the subsystem of ^{3}He atoms in a solution may become a superfluid, but this is now due to an instability with respect to the triplet p-wave pairing. The mechanism of realisation of the triplet pairing in

the Fermi gas with a purely repulsive interaction was first considered by Fay and Layzer [8] and Kagan and Chubukov [9]. This mechanism is related to the presence of a Kohn singularity [10] (or of the Friedel oscillations [11]) in the effective interaction of Fermi particles via the polarisation of the Fermionic background. More specifically, this effective interaction is described by the following expression if we use the first two orders of perturbation theory with respect to the gas parameter:

$$V_{\mathrm{eff}}(q) = a_0 p_{\mathrm{F}} + (a_0 p_{\mathrm{F}})^2 \Pi(q) , \tag{15}$$

where $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{p}'$, and $\boldsymbol{p}$ and $\boldsymbol{p}'$ are the momenta of the incoming and outgoing particles in the Cooper channel, and

$$\Pi(q) = \int \frac{\Theta(\varepsilon_p) - \Theta(\varepsilon_{p+q})}{\varepsilon_p - \varepsilon_{p+q}} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \tag{16}$$

is the standard polarisation operator which is responsible for charge screening in the case of an electron gas in a metal. It should be pointed out that $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{p}'$, but not $\boldsymbol{p} - \boldsymbol{p}'$. This demonstrates that in reality the second order of perturbation theory includes contributions not only from a particle–hole loop diagram, but from all the other diagrams (there are three of these) that are irreducible in the Cooper channel (Fig. 5). In the case of a short-range potential the first three diagrams in Fig. 5 cancel each other and only the fourth exchange diagram survives: its magnitude is the same as that of the polarisation operator if we replace $\boldsymbol{p} - \boldsymbol{p}'$ with $\boldsymbol{p} + \mathbf{p}'$.

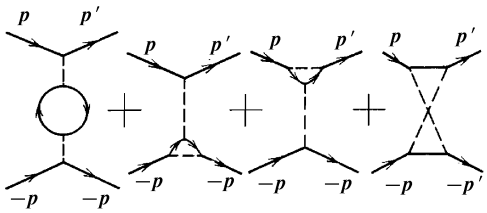

Figure 5. Diagrams of the second-order terms of the gas parameter $a_0 p_{\mathrm{F}}$, which contribute to the effective interaction.

The polarisation operator $\Pi(q)$ is described by the standard Lindhard function and contains not only a regular part, but also a singular part of the type $(q - 2p_{\mathrm{F}}) \ln |q - 2p_{\mathrm{F}}|$. In the coordinate space the singular part of $\Pi(q)$ leads to the Friedel oscillations in the effective interaction $V_{\mathrm{eff}}(r) \propto \cos(2k_{\mathrm{F}} r)/r^3$.

Therefore, the purely repulsive short-range potential between two particles in vacuum gives rise to an effective interaction in matter and this interaction is characterised by competition between attraction and repulsion. A rigorous calculation shows that for all the harmonics of the effective potential (except the s-wave harmonic) the attraction wins in this competition and the p-wave harmonic is the most attractive. Consequently, a three-dimensional Fermi gas with repulsion is unstable with respect to the superfluid transition with the triplet p-wave pairing below the critical temperature

$$T_{\mathrm{c1}} \propto \varepsilon_{\mathrm{F0}} x^{2/3} \exp\left[-\frac{5\pi^2}{4(2\ln 2 - 1) a_0^2 p_{\mathrm{F0}}^2 x^{2/3}}\right] , \tag{17}$$

where $\max T_{\mathrm{c1}} = T_{\mathrm{c1}}(P = 10\ \mathrm{bar}$, when the maximum solubility of ^{3}He is 9.5%$) \sim 10^{-10} - 10^{-9}$ K. The triplet pairing temperature of this order of magnitude was predicted also by Bardeen, Baym, and Pines [2].

Frossati and others give a more optimistic estimate for the triplet pairing case. At the maximum concentration $x = 9.5\%$ the value of T_{c1} lies between 10^{-6} and 10^{-4} K; the lower temperature (10^{-6} K) is obtained when the fitting parameters are deduced from transport experiments and the higher temperature follows from magnetostriction experiments.

5. Two-dimensional case

A solution of ^{3}He in ^{4}He is also very interesting because it can be made purely two-dimensional. In superconducting electron systems a film is regarded as two-dimensional if its thickness d is much less than the coherence length $\xi_0 \sim 1000$ A. In ^{3}He films on grafoil (exfoliated graphite) and in monolayers and submonolayers of ^{3}He in the solutions the radius of localisation of ^{3}He atoms in the third dimension (which is the film thickness) is indeed of the order of the distance between atoms. Therefore, by analogy with inversed layers in heterostructures, we are also dealing here with a purely two-dimensional system and, moreover, our system is free of impurities. In this sense a two-dimensional solution of ^{3}He in ^{4}He can be regarded as a bridge between superfluidity and superconductivity, particularly high-temperature superconductivity. In fact, the majority of the current theories of high-temperature superconductivity rely on two-dimensional or quasi-two-dimensional behaviour to account for the unusual normal properties (resistivity, susceptibility, small jumps in the distribution function on the Fermi surface, etc.) of these materials, as well as to account for the high temperature of their superconducting transition. Two-dimensional helium films and particularly monolayers with a low two-dimensional ^{3}He density are ideal objects for the experimental verification of the currently fashionable theories of high-temperature superconductivity, such as the theory of a marginal Fermi liquid proposed by Varma et al. [12] or a somewhat similar theory of the Luttinger Fermi liquid proposed by Anderson [13]. This topic will be discussed again at the end of the review. At this stage a brief review will be given of the history of the experimental discovery and theoretical prediction of the existence of two-dimensional solution. The first experiments were carried out by Esel'son and Bereznyak [14] and by Atkins and Narahara [15]. These experiments revealed a nontrivial temperature dependence of the surface tension (in fact, the surface free energy) of a weak solution of ^{3}He in ^{4}He. The experiments were interpreted by Andreev [16] who postulated the existence of surface impurity levels on the free surface of superfluid ^{4}He. This idea was subsequently confirmed by detailed experiments of Zinov'eva and Boldarev [17] and of Edwards et al. [18], as well as by variational calculations (cf. the review of Edwards and Saam [19] and the literature cited there). The correct interpretation of the experimental results yields the following parameters representing the surface state: $\varepsilon = -\Delta - \varepsilon_0 + (p_{\parallel}^2/2m^*)$, where $\Delta = 2.8$ K is the binding energy of a ^{3}He quasiparticle in the bulk (Andreev [16]; Bashkin and Meyerovich [1]); $\varepsilon_0 = 2.2$ K is the difference between the binding energies of a ^{3}He quasiparticle in the bulk and on the surface; $m^* = 1.5 m_3$ is the hydrodynamic effective mass governing the motion of ^{3}He quasiparticles along the surface. It should be pointed out that, according to

the variational calculations of Lekner [20] and Saam [21], the appearance of the Andreev levels is due to a combination of the effects associated with the van der Waals interaction between ^{3}He and the ^{4}He density profile (which varies when we approach the free surface) and with the difference between the energies of the zero-point motion of ^{3}He and ^{4}He. Such effects lead to the localisation of ^{3}He atoms near the free surface. These ^{3}He atoms can nevertheless move freely along the surface of ^{4}He, which is almost equipotential because the hydrodynamic condition $\mu_4 =$ const is satisfied on this surface. The wavefunction of the Andreev state is $\Psi = \Psi(z)\exp(i\boldsymbol{p}_{\|}\boldsymbol{r})$; $\Psi(z) \propto \exp(-z/a)$, where a is the radius of localisation along the normal to the surface.

The first experiments on thin ^{4}He films, of the same kind as the experiments of Zinov'eva and Boldarev [17] and of Edwards et al. [18], were carried out by Gasparini, Bhattacharyya, and DiPirro [22]. Gasparini and others determined the contribution of the surface states of ^{3}He to the specific heat of thin films. They also proposed the first theoretical interpretation of the results [23]. Subsequently several experimental papers were published by Hallock et al. [24–26], who measured the magnetisation and the spin–lattice relaxation time of ^{3}He submonolayers on the surfaces of thin ^{4}He films.

The theoretical interpretation of the experiments of Hallock et al. proposed by Dalfovo and Stringari [27], Pavloff and Treiner [28], Krotscheck, Saarela, and Epstein [29] require the assumption that not one but two Andreev levels exist on the surface of a thin ^{4}He film. The energy of the first Andreev level, $E_1 = -\Delta - \varepsilon_1 + (p_{\|}^2/2m_1)$, is practically identical with the energy of the Andreev level ($\varepsilon_1 \cong \varepsilon_0$) on a bulk surface, differing only in respect of the effective mass $m_1 \cong 1.35m_3$. The energy of the second Andreev level is still lower than the energy of ^{3}He in the bulk and is given by the expression $E_2 = -\Delta - \varepsilon_2 + (p_{\|}^2/2m_2)$, where in the limit of zero concentration of ^{3}He and not too thin films we have $m_2 \cong 1.6m_3$ and $\varepsilon_2 \cong 0.4$ K; consequently, $\varepsilon_2 - \varepsilon_1 = 1.8$ K.

The wavefunction of the first Andreev level is localised mainly near the free surface and has a significant tail (~ 3 A) above the surface. At the same time the wavefunction of the second Andreev level penetrates partly into the film. According to the authors of these theoretical treatments, two Andreev levels appear (instead of one) because of the competition between the size effect [vanishing of the Ψ-function of ^{3}He near the substrate and consequent increase in the kinetic energy $E_{\text{kin}} \propto (\nabla_z \Psi)^2 \propto 1/d^2$ of ^{3}He] and the van der Waals attraction by the substrate (which is proportional to $1/d^3$ and tends to reduce the energy). In the case of moderately thick films the van der Waals attraction is stronger than the size-effect repulsion and, therefore, the energy of the second Andreev level is still lower than the energy of ^{3}He in the bulk.

In the case of very thick films the van der Waals attraction of the substrate, proportional to $1/d^3$, may become unimportant compared with the kinetic energy, and the energy of the second Andreev level may prove higher than the energy of ^{3}He in the bulk. In this case the second level evidently vanishes by escaping into the bulk. At a fixed film thickness, the van der Waals attraction of the substrate depends on whether the substrate is 'strong' or 'weak'. On a weak substrate (Cs, Rb, K, Na, Li, Mg, H_2) it is found that ^{4}He is in the liquid phase. On a strong substrate (Ag, Au, Cu, Al) one or two solid ^{4}He layers form and ^{4}He becomes liquid only in the third and following layers. The presence of one or two solid layers reduces the van der Waals attraction of the substrate and increases the kinetic energy, leading to a possible disappearance of the second Andreev level at lower thicknesses of the film.

I shall conclude this section by noting that the topic is not yet fully understood. There is an alternative point of view according to which the second Andreev level can exist not only in thin films, but also in the bulk.

It is thus clear that in the case of not very thin and not very thick films there are definitely two Andreev levels whose energies differ by $\varepsilon_1 - \varepsilon_2 = 1.8$ K. Their existence is manifested in the Hallock experiments by the presence of a step in the dependence of the magnetisation on the surface density of ^{3}He. This step appears when the density of ^{3}He is equal to 0.85 of a monolayer. At lower densities the second Andreev level is not important and we are dealing with a purely two-dimensional one-level system whose spectrum is $E = -\Delta - \varepsilon_1 + (p_{\|}^2/2m_1)$ and the wave function is $\Psi = \Psi(z)\exp(i\boldsymbol{p}_{\|}\boldsymbol{r})$. Another important result reported by Hallock is an analysis of the temperature dependence of the suscepti-bility. At low temperatures ($T \ll T_{\text{F}}$) this susceptibility depends weakly on temperature and for surface densities from 0.03 to 0.3 of a monolayer it is well described by an expression for a two-dimensional Fermi gas with a weak repulsive interaction g between the particles:

$$\chi = \chi_0 \frac{1 + \frac{1}{2}F_1^{\text{s}}}{1 + F_0^{\text{a}}} \propto \chi_0(1 + g) , \tag{18}$$

where $F_1^{\text{s}} \propto g^2$ and $F_0^{\text{a}} \propto g$ are two-dimensional harmonics of the Landau function representing the interaction between quasiparticles; $g \propto 1/2(\ln p_{\text{F}} r_0)$ is a two-dimensional coupling constant [30]; r_0 is the radius of action of the potential.

At densities from 0.005 to 0.03 we have $\chi < \chi_0$, which supports the sign of the coupling constant corresponding to attraction [this is also true of the coupling constant $a(x)p_{\text{F}0}x^{1/3}$ in the case of three-dimensional solutions]. However, the exact densities at which the coupling constant changes its sign can be determined from measurements at lower temperatures, since $T_{\text{F}} \sim T_{\text{F}0}x$ is small and the transition from the Fermi-gas behaviour of the susceptibility to the Curie law occurs very early.

In conclusion, I must mention that there is also one other purely two-dimensional system: ^{3}He on the surface of grafoil which has very similar properties at a low surface density of ^{3}He (cf. experiments carried out by the Saunders group [31, 32].) The rest of this discussion can be also applied to such a system, subject only to small modifications.

6. Possibility of the superfluid transition in the two-dimensional solutions

We shall now consider the possibility of the superfluid transition in a two-dimensional ^{3}He submonolayer on the surface of ^{4}He.

By analogy with the three-dimensional case, the total interaction between two ^{3}He particles on the surface is given by the expression

$$V(\boldsymbol{r}, z) = V_{\text{dir}}(\boldsymbol{r}, z) + V_{\text{exch}}(\boldsymbol{r}, z) , \tag{19}$$

where the exchange interaction $V_{\text{exch}}(\boldsymbol{r}, z)$ is governed by the sum of the residual parts (not used to form an Andreev level) of the deformation potential of the interaction between two ^{3}He particles via the polarisation of ^{4}He and of the

van der Waals attraction of the substrate. These residual parts of the exchange interaction are related primarily to the interaction of ^{3}He particles with the curved surface of superfluid ^{4}He in the field of surface waves of third sound. The spectrum of third-sound waves is of the form $\omega^2 = (\alpha/\rho)(\varkappa^2 + q^2)q\tanh(qd)$, where the first term in the second parentheses describes the contribution of the van der Waals potential of the substrate and the second represents the local surface change in the density of ^{4}He. In the case of thin films the contribution of the first term predominates, i.e. the dynamic part of the van der Waals potential of the substrate is more important than 'surface phonons' (ripplons). Consequently, a reduction in the film thickness changes the spectrum from the purely ripplon type $\omega^2 = (\alpha/\rho)q^3$, where α is the surface tension, to an acoustic spectrum with a linear dispersion law $\omega^2 = (\alpha d\varkappa^2/\rho)q^2$, where $\varkappa$ is the capillary constant of the van der Waals potential and d is the film thickness: $qd \ll 1$.

In the two-dimensional problem it is important, as always, to determine the two-dimensional projection of the three-dimensional potential $V(\boldsymbol{r}, z)$. In full analogy with the two-dimensional projection of the Coulomb interaction, it is given by the expression

$$V(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \iint V(\boldsymbol{r}_1 - \boldsymbol{r}_2, z_1 - z_2) \times |\Psi(z_1)|^2 |\Psi(z_2)|^2 \, dz_1 \, dz_2 \,, \tag{20}$$

where $\Psi(z)$ is the wave function of an Andreev level. The two-dimensional projection of the total interaction can be represented in the form

$$V(r) = V_{\rm dir}(r) + V_{\rm exch}(r) \tag{21}$$

and correspondingly in the momentum space

$$V(q_\parallel) = V_{\rm dir}(q_\parallel) + V_{\rm exch}(q_\parallel) \,, \tag{22}$$

where $V_{\rm exch}(q_\parallel = 0)$ is due to the exchange of a virtual quantum of third sound and, in the case of thin films, is given by $V_{\rm exch}(q_\parallel = 0) = -m_4 c_{\rm III}^2$ (Kurihara [33], Monarkha and Sokolov [34]), where

$$c_{\rm III}^2 = \frac{3V_{\rm sub}\, h_4}{m_4 n_4 (d + h_4)^4} \,, \tag{23}$$

$c_{\rm III}$ is the velocity of third sound, $V_{\rm sub}$ is the van der Waals potential of the substrate, d and h_4 are the thicknesses of the solid and superfluid ^{4}He layers. (We recall that in the case of well-wetted substrates such as Au, Ag, Cu, etc, the first ^{3}He layer solidifies.)

It should be pointed out that, as in the three-dimensional problem, the limiting frequency of surface waves $\omega_{\rm D} \propto m_4 c_{\rm III}^2$ is much higher than the Fermi energy $\varepsilon_{\rm F}$. Therefore, we are again dealing with an antiadiabatic situation, when the whole volume of the two-dimensional Fermi sphere (and not only its Debye shell) is important in the problem of the superfluid transition.

We shall now consider the direct part of the total interaction. By analogy with the three-dimensional case, we have

$$V_{\rm dir}(q_\parallel = 0) = V_1(q_\parallel = 0) + V_2(q_\parallel = 0) \,, \tag{24}$$

where V_1 is determined by the hard-core repulsion between two ^{3}He particles at short distances, whereas V_2 is due to the attractive interaction of two ^{3}He particles at large distances.

As pointed out at the end of the preceding section, the experiments of Hallock et al. on the dependence of the magnetisation of a submonolayer on the surface density of ^{3}He demonstrate that the total interaction of two ^{3}He particles on the surface of a thin film is attractive when the ^{3}He concentration is $x < 3\%$ and repulsive when the concentration is higher. Therefore, the direct part of the total interaction can be represented again in a model form shown in Fig. 4, i.e.

$$V_{\rm dir}(r) = \begin{cases} V_1, & r < r_1 \,, \\ -V_2, & r_1 < r < r_2 \,, \end{cases} \tag{25}$$

where $1/r_2 \propto k_{\rm F}$ ($x = 3\%$). If the submonolayer density is such that $k_{\rm F}^2 < r_1^{-2}$, we can try to use the Fermi-gas approach again. It should be pointed out that the experiments of Hallock et al. demonstrate that at ^{3}He densities less than 0.3 of a monolayer we are indeed dealing with a weakly interacting low-density two-dimensional Fermi gas.

7. Two-dimensional Fermi gas with attraction

A special feature of the two-dimensional case is that even if the attraction is infinitesimally weak, we are dealing with the coexistence of two phenomena: pairing of two particles in vacuum in the coordinate space and the Cooper pairing of two particles in matter in the momentum space in the presence of a filled Fermi sphere. In the case of a purely attractive potential the energy of a bound state in vacuum is

$$E_{\rm b} = -\frac{1}{m r_0^2} \exp\left(-\frac{4\pi}{m|U_0|}\right) \,, \tag{26}$$

where U_0 is the s-wave harmonic of the potential and r_0 is its radius. In the case of the potential with a repulsive core and an attractive tail (Fig. 4), we have for $V_{20} > V_{10}$

$$E_{\rm b} = -\frac{1}{m r_2^2} \exp\left(-\frac{4\pi}{m(V_{20} - V_{10})}\right) \,. \tag{27}$$

The temperature of the superfluid transition (Cooper pairing in matter) is described by an expression standard in the BCS theory:

$$T_{\rm c0} \propto \varepsilon_{\rm F} \exp\left(-\frac{1}{|f_0|}\right) \,, \tag{28}$$

where the two-dimensional gas parameter f_0 is determined by the real part of the scatterring amplitude and is given by the following expression in the case of attraction:

$$f_0 = {\rm Re}\left[-\frac{m|U_0|/2\pi}{1 - (m|U_0|/4\pi)\ln(1/p_{\rm F}^2 r_0^2) + (m|U_0|/4\pi){\rm i}\pi}\right] , \tag{29}$$

where Re stands for the real part. Substituting the expression for f_0 into the formula for the critical temperature, we obtain everywhere apart from a narrow region close to the resonance $[|E_{\rm b}| = 2\varepsilon_{\rm F}$ or $1 = (m|U_0|/4\pi)\ln(1/p_{\rm F}^2 r_0^2)]$

$$T_{\rm c0} \propto \frac{\varepsilon_{\rm F}}{p_{\rm F} r_0} \exp\left(-\frac{2\pi}{m|U_0|}\right) = \left(2\varepsilon_{\rm F}|E_{\rm b}|\right)^{1/2} \,. \tag{30}$$

This formula was derived by Miyake [35] for two-dimensional solutions and was rederived by Randeria, Duan, and Shieh [36] and by Schmitt-Rink, Varma, and Ruckenstein [37] in connection with a possible marginal Fermi-liquid behaviour of high-temperature superconductors.

It is interesting to note that, from the formal point of view, the expressions for the critical temperature in the three-dimensional and two-dimensional cases differ only by the preexponential factor $\varepsilon_F \to \varepsilon_F/p_F r_0$.

However, in reality the two-dimensional situation is much more interesting. This is because in the case of a weak interaction when $(m|U_0|/2\pi)\ln(1/p_F r_0) \ll 1$ the bound-state energy $|E_b|$ is much less than ε_F and, consequently, $T_{c0} \ll \varepsilon_F$, i.e. we are dealing with ordinary Cooper pairs characterised by a large coherence length. The chemical potential of a superfluid system found by Miyake from the self-consistent Leggett approach [38] is given by the expression

$$\mu = \varepsilon_F - \tfrac{1}{2}|E_b| \,. \tag{31}$$

We recall that in the Leggett approach the chemical potential is deduced from the condition of conservation of the total number of particles:

$$n = \frac{p_F^2}{2\pi} = \int \frac{d^2k}{4\pi^2}\,\frac{E_k - \varepsilon_k}{E_k} \,, \tag{32}$$

where $E_k = (\varepsilon_k^2 + \Delta_0^2)^{1/2}$ and $\varepsilon_k = (k^2/2m) - \mu$ are, respectively, the energies of the superfluid and normal states; Δ_0 is the superfluid gap related, as in the usual BCS theory, to the critical temperature by $\Delta_0 = (\pi/\gamma)T_{c0}$ ($\ln\gamma = 0.57$ is the Euler constant). It is evident from Eqn (31) that if $|E_b| \ll \varepsilon_F$ we have $\mu \to \varepsilon_F$ and pairing occurs on the Fermi surface. In the opposite limiting case of a strong interaction, when $(m|U_0|/2\pi)\ln(1/p_F r_0) \gg 1$, we have $|E_b| \gg \varepsilon_F$ and $\varepsilon_F \ll T_{c0} \ll |E_b|$. In this case the chemical potential is $\mu \cong -|E_b|/2 < 0$ and we have a situation with strongly bound Cooper pairs (composite bosons) characterised by a short coherence length [39]. In Bashkin's terminology they are called dimers and are helium analogues of bipolarons, introduced into the theory of superconductivity by Alexandrov and Ranninger [40, 41]. In fact, in this case, the pairing process does not occur in the momentum space but in the coordinate space. It should be pointed out that in the case of dimers, as in the bipolaron model, there is not one but two critical temperatures: the temperature T_{c0} of preconditioning of local pairs and the temperature $T_c^* \propto n/m^* \propto \varepsilon_F$ of the Bose condensation of such pairs. It should also be pointed out that in the case of weakly bound pairs the Kosterlitz–Thouless (KT) fluctuation corrections to the result obtained above from the mean-field theory are small and proportional to $(T^{KT} - T^{BCS})/T^{BCS} \sim T^{BCS}/\varepsilon_F \ll 1$ [42], but in the case of dimers such corrections are very important and can play a decisive role.

It should be stressed that in the mean-field theory the transition from the BCS to the bipolaron regime is smooth in the two-dimensional case not only with respect to the chemical potential μ, but also in the case of the total energy of the system. In fact, the difference between the energies of the normal and superfluid states at $T = 0$ is

$$\frac{E_s^{BCS} - E_n}{N} = -\frac{\Delta^2}{\varepsilon_F} \sim -\frac{T_{c0}^2}{\varepsilon_F} = -\frac{[(\varepsilon_F|E_b|)^{1/2}]^2}{\varepsilon_F}$$

$$= -|E_b| = \frac{E_s^{bip} - E_n}{N} \,, \tag{33}$$

where N is the number of particles in the system.

We shall conclude this section with an estimate of the temperature of the two-dimensional superfluid transition in the attractive case: according to Bashkin, Kurihara, and Miyake, we can expect T_{c0} of the order of 1 mK when the surface density of ^{3}He is of the order of 0.01 of a monolayer.

8. Two-dimensional Fermi gas with repulsion

When the ^{3}He density exceeds 0.03 of a monolayer, the total interaction between ^{3}He particles changes sign and the s-wave pairing becomes impossible. The possibility of the p-wave pairing in two-dimensional solution depends by analogy with the three-dimensional case on the sign of the p-wave harmonic of the effective interaction. The effective interaction obtained within the first two orders of perturbation theory is

$$V_{eff}(q) = f_0 + f_0^2\,\Pi(q) \,, \tag{34}$$

where $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{p}'$, $\Pi(q)$ is a two-dimensional polarisation operator, and $f_0 = 1/2\ln(p_F r_0)$ is the gas parameter for the repulsive case.

In the coordinate space $V_{eff}(r) \propto f_0^2 \cos(2k_F r)/r^2$ exhibits oscillations much stronger (proportional to $1/r^2$) than in the three-dimensional case. However, a two-dimensional Kohn singularity induced by such oscillations in the momentum space is of one-sided nature: $\Pi_{sing}(q) \propto \mathrm{Re}\,(q - 2p_F)^{1/2}$ [43]. In the case of the Cooper problem we have $|\boldsymbol{p}| = |\boldsymbol{p}'| = p_F$ and $q^2 = 2p_F^2(1 + \cos\theta)$. Therefore, $q < 2p_F$ and $\mathrm{Re}\,(q - 2p_F)^{1/2} = 0$. Therefore, a strong two-dimensional Kohn singularity is ineffective in the problem of the Cooper pairing and the two-dimensional Fermi gas with repulsion remains in the normal state at least in the first two orders of perturbation theory. Recently, Chubukov [44] showed that inclusion of the vertex corrections to the expression for the effective interaction restores the superfluid p-wave pairing in two dimensions in the third order of perturbation theory $T_{c1} \propto \varepsilon_F \exp(-1/f_0^3)$. An estimate of the superfluid transition temperature obtained by Chubukov is 10^{-4} K for densities ~ 0.3 of a monolayer corresponding to the limit of validity of the Fermi-gas description.

It should be pointed out that allowance for nonquadratic corrections to the spectrum of ^{3}He quasiparticles, $\varepsilon = (p^2/2m)[1 - \gamma(p^2/p_c^2)]$, results in restoration of the p-wave pairing already in the second order of perturbation theory. This yields $T_{c1} \propto \varepsilon_F \exp(-1/\gamma f_0^2)$, as demonstrated by Baranov and Kagan [45]. However, the superfluid transition temperature now depends exponentially on the small constant representing the nonquadratic nature of the spectrum and, therefore, it is very small.

9. Superfluidity in polarised solutions

We shall now consider briefly the situation in strongly polarised solutions.

It is well known that the singlet s-wave pairing in a strongly polarised solution is suppressed by a paramagnetic effect. This means that in magnetic fields obeying $\mu_B H > T_{c0}$ (μ_B is the nuclear Bohr magneton) the direction of one of the spins (antiparallel to the field) of a Cooper pair is reversed and, therefore, the superfluid state is destroyed. The influence of an external magnetic field (or of the polarisation) on the triplet pairing temperature is less trivial. Chubukov and Kagan [46] showed that the p-wave pairing temperature of a three-dimensional polarised gas with repulsion depends strongly and nonmonotonically on the degree of polarisation α: it rises strongly at low and inter-

mediate polarisations, passes through a maximum when the polarisation is 48%, and falls on further increase in the polarisation. A strongly nonmonotonic dependence of T_{c1} on α is due to the competition between two effects: enhancement of a Kohn singularity when the polarisation is increased and reduction in the number of spins which are antiparallel to the field.

In the language of the diagrammatic technique these effects are caused by the following circumstance: in the polarised case, the effective interaction of two 'up' spins (parallel to the field) in a Cooper channel occurs via a zero-sound polarisation loop composed of a particle and a hole with 'down' spins (Fig. 6). This polarisation loop again contains a Kohn singularity, but it is now of the type

$$\Pi_{\text{sing}}(q) = (q_\uparrow - 2p_{F\downarrow}) \ln (q_\uparrow - 2p_{F\downarrow}) \,, \tag{35}$$

where $\boldsymbol{q}_\uparrow = \boldsymbol{p}_\uparrow - \boldsymbol{p}'_\uparrow$, $q_\uparrow^2 = 2p_{F\uparrow}^2(1-\cos\theta)$, $\theta = \widehat{\boldsymbol{p}\boldsymbol{p}'}$, and $p_{F\uparrow}$ and $p_{F\downarrow}$ are the radii of the Fermi spheres of the 'up' and 'down' spins, respectively. In a nonzero magnetic field we have $p_{F\uparrow} \neq p_{F\downarrow}$. Therefore, a Kohn singularity (i.e. vanishing of the difference $q_\uparrow - 2p_{F\downarrow}$) occurs not at the angle $\theta = \pi$ between $\boldsymbol{p}$ and $\boldsymbol{p}'$, but at the angle $\theta = \theta_c$, which deviates from π on increase in the degree of polarisation. In other words, if $\Pi_{\text{sing}}(\theta) \propto (\pi-\theta)^2 \ln(\pi-\theta)$ in the unpolarised case, then in the presence of polarisation we have $\Pi_{\text{sing}}(\theta) \propto (\theta_c-\theta)\ln(\theta_c-\theta)$. We can see that in the polarised case a Kohn singularity is stronger because already the first derivative (and not the second one, as in the absence of a field) of $\Pi_{\text{sing}}(\theta)$ diverges. Naturally, the enhancement of the Kohn singularity increases the critical temperature. This effect is counteracted by a reduction in the density of states of the particles with 'down' spin, which reduces the absolute magnitude of the zero-sound loop and, consequently, reduces the critical temperature.

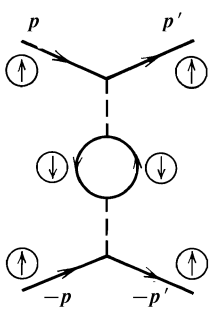

Figure 6. Effective interaction of two spins directed parallel to the field in the polarised case.

The competition between these two effects gives rise to a maximum in the dependence of T_{c1} on the degree of polarisation α (Fig. 7). The maximal temperature corresponds to $\alpha = 48\%$. At a pressure of 10 bar the maximum possible concentration is $x = 9.5\%$ and the temperature at the maximum corresponds to $10^{-6}-10^{-5}$ K, which is much higher than $T_{c1} \sim 10^{-10}-10^{-9}$ K corresponding to $\alpha = 0$. A qualitatively similar dependence of T_{c1} on α with a maximum at $\alpha = 32\%$ was also predicted by van de Haar, Frossati, and Bedell [3]. The temperature at the maximum predicted by these authors for the same values of pressure and concentration is somewhat higher and amounts to $10^{-5}-10^{-4}$ K.

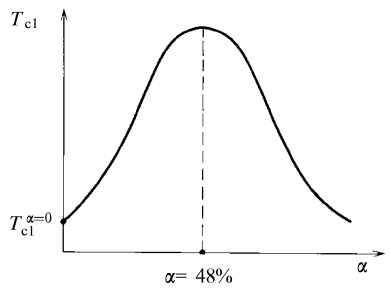

Figure 7. Theoretical dependence of the critical temperature T_{c1} on the degree of polarisation α of a three-dimensional solution of ^{3}He in ^{4}He.

The hope for experimental creation of strongly polarised solutions is based primarily on the elegant idea of Castaing and Nozieres [47]. In their classical paper they proposed to create a strong polarisation in a liquid solution by fast melting of a solid solution. The idea is that a solid solution (and pure crystalline ^{3}He) does not have a kinetic energy of the degeneracy of ^{3}He atoms associated with the Pauli principle. Therefore, the application of a magnetic field of the order of the Curie temperature

$$\mu_B H \sim T_c \sim T \sim 1\ \text{mK} \quad \text{and} \quad H \sim 1\ \text{T} \tag{36}$$

leads to an almost 100% polarisation of the solid solution. (It should be pointed out that a significant polarisation can be achieved in a liquid solution only by applying much stronger and experimentally unattainable fields such that $\mu_B H \sim \varepsilon_F \sim 0.1$ K and $H \sim 100$ T.) Fast melting of a strongly polarised ($\alpha \sim 90\%$) solid solution should, according to the estimates of Castaing and Nozieres [47], produce a liquid solution with $\alpha \sim 30\%$. Naturally, this polarisation is of a nonequilibrium nature, but its lifetime is very long ($t \sim 30$ min) because of the long relaxation time in the liquid phase.

Another very important idea for increasing the critical temperature is the suggestion of Meyerovich [35, 48, 49], according to which the maximum solubility of a strongly polarised solution may be 3–4 times higher than the maximum solubility in the absence of polarisation ($x^{\max} \sim 30\%$, instead of $x_{\alpha=0}^{\max} = 9.5\%$). A combination of the ideas of Castaing, Nozieres, and Meyerovich may produce an even greater increase (to $10^{-4}-10^{-3}$ K) in the superfluid transition temperature of a strongly polarised solution.

10. Two-dimensional polarised solutions

The situation in two-dimensional polarised submonolayers at ^{3}He densities from 0.03 to 0.3 of a monolayer is even more favourable from the point of view of the superfluid transition temperature. This is because, as mentioned above, a strong two-dimensional Kohn singularity of the type $\mathrm{Re}\,(q-2p_F)^{1/2}$ is ineffective in the absence of a magnetic field when the Cooper pairing occurs. In the presence of a polarisation the situation is fundamentally different: $\Pi_{\text{sing}}(\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}') \propto \mathrm{Re}\,(q_\uparrow - 2p_{F\downarrow})^{1/2}$ [46] and there is a range of angles θ between $\boldsymbol{p}$ and $\boldsymbol{p}'$ [$q_\uparrow^2$ is again equal to $2p_{F\uparrow}^2(1+\cos\theta)^{1/2}$], in which the radicand is positive. Therefore, the application of a magnetic field suppresses the one-sided nature of a two-dimensional Kohn singularity and makes it effective in the pairing problem.

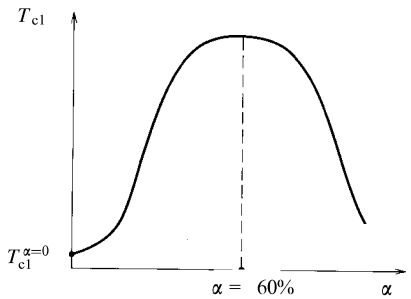

Figure 8. Theoretical dependence of the critical temperature T_{c1} on the degree of polarisation α of a two-dimensional solution of ^{3}He in ^{4}He.

The competition between a Kohn singularity and a reduction in the number of spins antiparallel to the field again gives rise, in the two-dimensional case, to a nonmonotonic dependence of T_{c1} on the degree of polarisation α, with a very strong maximum at $\alpha = 60\%$ (Fig. 8). It should be pointed out that this maximum is very broad and extends from 20% to 90% of the polarisation.

Estimates indicate that the critical temperature is now fairly high in experimentally attainable fields of ~ 15 T. In fact, the application of such a magnetic field leads, for example, to a 10% polarisation of a two-dimensional solution with the ^{3}He density n_3 amounting to ~ 0.05 of a monolayer and with $\varepsilon_F \sim 0.13$ K. In this case the triplet pairing temperature is easily experimentally attainable and amounts to 1 mK.

11. Experimental situation

It is pointed out in the Introduction that the search for superfluidity in three-dimensional and two-dimensional solutions has not yet resulted in experimental success. The published experimental results demonstrating the absence of superfluidity at certain pressures and concentrations impose limits on the various theoretical estimates of the superfluid transition temperature. They are forcing both theoreticians and experimentalists to concentrate on those ranges of the parameters where measurements have not yet been carried out.

The review of Østgaard and Bashkin [7] contains the experimental results obtained by the groups of Pobell [49] and Ogawa [50]. They demonstrate the absence of the superfluid transition in three-dimensional solutions right down to 0.2 mK for ^{3}He concentrations of 1%, 5%, and 6.4%. They show that the temperature of the singlet s-wave pairing and the temperature of the triplet p-wave pairing (we recall that the s-pairing is impossible at concentrations in excess of 4%) both most probably lie below 0.2 mK. The estimates of Østgaard and Bashkin on the subject of the singlet pairing show that the most promising avenue is to seek the singlet superfluidity at ^{3}He concentrations amounting to $\sim 0.5\% - 1\%$. Van de Haar, Frossati, and Bedell [3] assume that the optimal concentration lies in the interval 1.5% – 2.5%. The corresponding temperature T_{c0} is of the order of 0.1 mK for the results of both groups. According to the estimates of Frossati, Bedell, Meyerovich, and the author of the present paper, the triplet superfluidity is most likely to occur at the maximum concentration of ^{3}He amounting to $x \cong 9.5\%$, which corresponds to 10 bar, under strong polarisation conditions (in strong effective magnetic fields). As pointed out above, when the polarisation is very strong, it may be possible to reach ^{3}He concentrations in excess of 9.5%. The most realistic estimates once again predict a triplet pairing temperature only of the order of 0.1 mK or lower. Therefore, we obviously can expect that both the singlet and the polarisation-enhanced triplet pairing temperatures have values of the order of 0.1 mK.

It seems that the situation in two-dimensional solutions is more favourable from the experimental point of view. The most important experimental results imposing limits on the theoretical estimates are those reported by Pobell and Shirama [51]. They tried to observe superfluidity in a ^{3}He submonolayer on the surface of a very thin ^{4}He film. Their measurement method involved determination of the viscous penetration depth with the aid of torsional oscillations. The scheme of their experiment is shown in Fig. 9. Pobell and Shirama went down to 0.85 mK and did not observe the superfluid transition in the range of surface concentrations from 0.1 to 1 monolayers. Their experiment suggests that the superfluidity of two-dimensional solutions should be searched either at lower ^{3}He densities ($n_3 < 0.03$ of a monolayer, when the total interaction corresponds to attraction) or at approximately the same densities $n_3 \sim 0.05 - 0.1$ of a monolayer but in a strong magnetic fields of $\sim 15 - 30$ T. In both cases we can expect the singlet superfluidity in the absence of a magnetic field and the field-enhanced triplet superfluidity at temperatures ~ 1 mK. On the one hand, these temperatures are an order of magnitude higher than those in the case of three-dimensional solutions and, on the other, they are attainable from the experimental point of view.

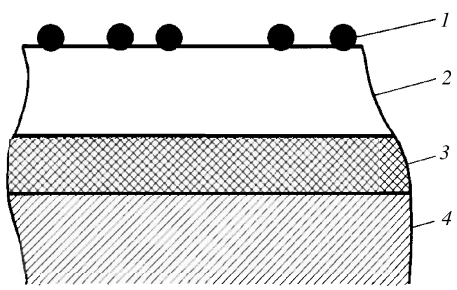

Figure 9. Basic layout of the experiments of Pobell and Shirama in which the authors attempted to detect superfluidity in two-dimensional solutions: (*1*) ^{3}He submonolayers; (*2*) superfluid ^{4}He (one monolayer); (*3*) solid ^{4}He films ~ 3 A thick; (*4*) silver substrate.

12. Two-dimensional monolayers as a bridge between superfluidity and superconductivity

In conclusion of this review it is necessary to stress that two-dimensional ^{3}He submonolayers on the surface of thin ^{4}He films and on grafoil are ideal two-dimensional systems for experimental checking of many currently popular theories of the normal and superconducting state of quasi-two-dimensional high-temperature superconductors. Many leading theoreticians attacking the problem of high-temperature superconductivity — including Anderson, Varma, and others — have stressed that the anomalously high values of T_c of high-temperature superconducting systems are closely related to their unusual properties in the normal state. Such unusual normal properties include, as pointed out in

Section 5, primarily the smallness of the jump in the distribution function of the interacting particles on the Fermi surface and the linear temperature dependence of the resistivity well below the Debye temperature.

The linear temperature dependence of the resistivity was explained some years ago by Anderson, who postulated the existence of a Luttinger Fermi liquid with zero jump in the distribution function on the Fermi surface [13]. A similar idea of a marginal Fermi liquid (which is an important special case of the Luttinger liquid) was put forward by Varma et al. [12].

Anderson later put forward an even less trivial hypothesis that not only a strongly interacting high-density two-dimensional Fermi system but even a weakly interacting low-density Fermi gas, is described not by the Fermi-liquid theory of Landau with a finite jump of the distribution function, but by the Luttinger Fermi-liquid theory [52]. In other words, Anderson insisted that a Fermi gas in the classical meaning of Galitskii [53] and Bloom [54] does not exist in two dimen-sions. Many theoreticians took part in the discussion following these suggestions of Anderson. The majority of them (Engelbrecht and Randeria [55]; Fukuyama et al. [56]; Fabrizio, Tossatti, and Parola [57]; Prokof'ev and Stamp [58]; Baranov, Kagan, and Mar'enko [59]) support the Fermi-gas idea and demonstrate that it is internally self-consistent in two dimensions if the ladder and parquet approximations are used in the diagrammatic technique. Anderson is still insisting that his view is correct and postulates that such diagrammatic technique is invalid in two dimensions (even at the level of summation of an infinite series of parquet diagrams). In other words, Anderson rejects the existence of any perturbation theory in two dimensions. One of the most serious arguments put forward by Anderson against perturbation theory is a strong singularity he proposes for the two-dimensional Landau function describing the interaction between quasi-particles $f(\boldsymbol{p}, \boldsymbol{p}')$. In accordance with Anderson's qualitative ideas, in the case of almost parallel momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$ and opposite spins of the colliding particles, the singular part of the function $f(\boldsymbol{p}, \boldsymbol{p}')$ is described by the following expression, valid in the case of small deviations from the Fermi surface:

$$f_{\text{sing}}(\boldsymbol{p},\boldsymbol{p}') \sim \frac{1}{|\boldsymbol{p}-\boldsymbol{p}'|} \,. \tag{37}$$

The existence of such a strong singularity leads to a logarithmic divergence of all Landau harmonics f_0, f_1, ... and, therefore, to complete collapse of a Fermi liquid. Calculations of $f(\boldsymbol{p}, \boldsymbol{p}')$ in the second order of perturbation theory, first carried out by Prokof'ev and independently by Baranov and Kagan, however, predict a much weaker singularity of the f function and this singularity is of the $(\boldsymbol{p}-\boldsymbol{p}')^{-1/2}$ type. Moreover, they predict this singularity only for a narrow angular interval $\varphi \propto |\boldsymbol{p}-\boldsymbol{p}'|^{3/2}$. Therefore, it results only in nontrivial temperature corrections to the f function, and not in the collapse of the Fermi-liquid scenario as a whole.

In this connection it would be very interesting to extend to low temperatures the experiments of the Hallock and Saunders groups, involving measurements of the susceptibility of ^{3}He submonolayers on the surfaces of thin ^{4}He films and on the surface of grafoil. The susceptibility of a degenerate two-dimensional Fermi gas,

$$\chi = \chi_0 \frac{1+\frac{1}{2}F_1^{\text{s}}}{1-F_0^{\text{a}}} \,, \tag{38}$$

determined in these experiments carries, at temperatures much lower than the Curie temperature, information on the concentration and temperature dependences of the Landau harmonics F_0^{a} and F_1^{s}. Information on these harmonics obtained from such experiments would help in obtaining an unambiguous answer to the following question: does a two-dimensional Fermi gas exist and if it exists, what nontrivial corrections missing in the three-dimensional case should be applied in the two-dimensional situation?

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