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Integrability and matrix models

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Contents

Abstract. The theory of matrix models is reviewed from the point of view of its relation to integrable hierarchies. Discrete 1-matrix, 2-matrix, 'conformal' (multicomponent), and Kontsevich models are considered in some detail, together with the Ward identities ('$\boldsymbol{W}$-constraints'), determinantal formulas, and continuum limits, taking one kind of model into another. Subtle points and directions for future research are also discussed.

1. Introduction

The purpose of these notes is to review one of the branches of modern string theory: the theory of matrix models, with the emphasis on their intrinsic integrable structure. I begin with a brief description of the field and its place within string theory.

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The main content of string theory (see my earlier article [1] for a general review) is the study of symmetries, in the broadest possible sense of the word, by the methods of quantum field theory. The usual scheme is to start from some symmetry and construct a field-theoretical model [usually 2-dimensional (2d), for a reason not discussed here], which possesses this symmetry in some simple sense (e.g. as Noether symmetry or as a chiral algebra). The main idea at this stage is to find a model which is exactly solvable (if nothing but the symmetry is given, this is a nice principle to restrict dynamics). The next step is to study the hidden symmetries of the model, which are somehow responsible for its exact solvability and are usually much larger than the original symmetry.

This 'inverse' step, *model* → *symmetry*, can be made with at least three different ideas in mind.

One can look for some hidden local (gauge) symmetry which is fixed or spontaneously broken, i.e. identify it with some other model which has more fields: *auxiliary* with respect to the smaller model, and *gauge* with respect to the larger one. [Examples include the gauged Wess – Zumino – Novikov – Witten (WZNW) model and topological theories in the Becchi – Rouet – Stora – Tyutin (BRST) formalism.]

One can take for a new (full) symmetry of the model just its operator algebra (algebra of observables) (see [2, 3, 4] for the first results in this direction). It deserves mentioning that the *gauging* of the entire algebra of observables gives rise to a 'string field theory' associated with the original model (considered as a string model).

One can construct the effective action of the theory by exact evaluation of the functional integral.

As to the direct step, *symmetry* → *model*, one can take as an example the best understood case, in which the original symmetry is just a Lie algebra. Then the quantum-mechanical model can be constructed by the geometric quantisation technique (see [5] for the most important example of a Kac–Moody algebra and the WZNW model).

Mathematically, the two elements of the above scheme appear to be *algebra* (in the theory of symmetries), and *analysis and geometry* (in the field-theoretical models). The idea of constructing models with a given symmetry (and nothing else relevant to the dynamics) can be identified with the mathematical concept of 'universal objects'.

The sequence of iterations of the two arrows in Fig. 1 leads to a deeper understanding, enlarging and generalising all the notions involved — symmetry, exact solvability, field theory, geometrical structures, quantisation, etc — thus stimulating considerable progress both in physics and in mathematics. If this iterative process can somehow converge, the limit point will deserve to be called the *theory of everything*, which will indeed unify all the possible field-theoretical models by embedding them into a huge but well structured theory, which will also be exactly solvable in some yet unspecified sense of the word. I refer the reader to my earlier review [1] for more details on this semi-philosophical programme, known as (modern) *string theory*, and now turn to a more narrow subject: the theory of matrix models.

symmetry	=	algebra
	↑↓	
model with this symmetry	=	analysis and geometry

Figure 1. Theory of everything.

At the moment this subject is mainly associated with the theory of effective actions; so far this is where the main results of the modern theory of matrix models find their applications. This technique is especially suited for the study of effective actions, obtained after integration over 2d geometries (including the sum over genera), and produces nonperturba-tive (exact) partition functions of particular string models. The main result of these studies indicates that these partition functions exhibit two remarkable (though expected [6]) properties.

First, the effective action for a given model is essentially the *same* as for any other model. In fact the effective action is a function of the coupling constants ('sources' in old-fashioned terms), which are nothing but coordinates in the *space of various models* (the configuration space of the entire string theory); variation of couplings change one model for another.

Second, effective action possesses a huge additional symmetry, which is somewhat similar to the general covariance in the space of all models (the above-mentioned configuration space) and, in the simplest examples which have been studied so far, can be expressed in terms of integrable hierarchies. (This 'general covariance' in the configuration space can, after all, turn into the main dynamical principle of the string theory.)

Both these features seem to be very general, arising whenever the largest possible Lagrangian with a given symmetry is considered (without restrictions on the possible counter-terms, imposed by requirements of renormalisability or by locality–minimality 'principles'— this is why this phenomenon is not widely known to field theorists). An example of highly nontrivial calculations leading to similar conclusions can be found in [7].

It is hoped that these remarks will become clearer after some specific examples have been considered below. Still, they deserve to be formulated in full generality, not only to intrigue the reader but also because they can serve to aid better understanding of the ideas and outcomes of generic string theory.

The 'corner' of the string theory associated with matrix models can be described as follows (see Fig. 2).

The big blocks within the body of string theory, which are directly related to matrix models, are the theory of conformal models, the theory of $\mathcal{N} = 2$ supersymmetry, and the (loop equation version of) Yang–Mills (YM) theory (in any number of dimensions). Also, Einstein gravity should be related to the subject in a way similar to YM theory, but these links are yet not clarified.

Both conformal theory and $\mathcal{N} = 2$ supersymmetry are sources of the concept of 'topological models' [8–11]. These arise after the gauging of all continuous symmetries of the WZNW models and/or as models with BRST-exact stress tensors, naturally appearing in the context of $\mathcal{N} = 2$ supersymmetry. If formulated in a self-consistent way in the 'universal module space' (unification of module spaces of all finite-genus Riemann surfaces and bundles over them) these models turn into those of 'topological gravity'. Generating functionals of topological-gravity models in fact generate infinite sequences of topological invariants of certain spaces {an inverse definition is also possible in some cases [8], though the universal (generic) algorithm for the operation *topology of some space* → *topological gravity* has not yet been formulated].

Alternative models of 2d quantum gravity arise straight-forwardly from conformal models through the procedure of 'summation over geometries'. There are essentially two different approaches to the problem. One (the 'Polyakov approach') is to make use of the complex structure, intrinsic for conformal theory [12], and sum over *Riemann surfaces*, which involves integration over module spaces and the sum over genera. The main techniques used in this approach are the theory of free fields on Riemann surfaces [13, 14] and the bosonisation formalism for conformal field theories [15, 16]. This approach requires solution of Liouville theory, which is still a problem under intensive investigation (and is in turn related to conformal field theory). Further progress in this direction should be related to (or can be expressed in terms of) the adequate theory of the *universal module space*, *handle-gluing operators*, etc. Similar objects arise in the field-theoretical approach to topological gravity (see [17] for a recent review).

An alternative approach to summation over geometries does not refer at all to the complex structure but instead

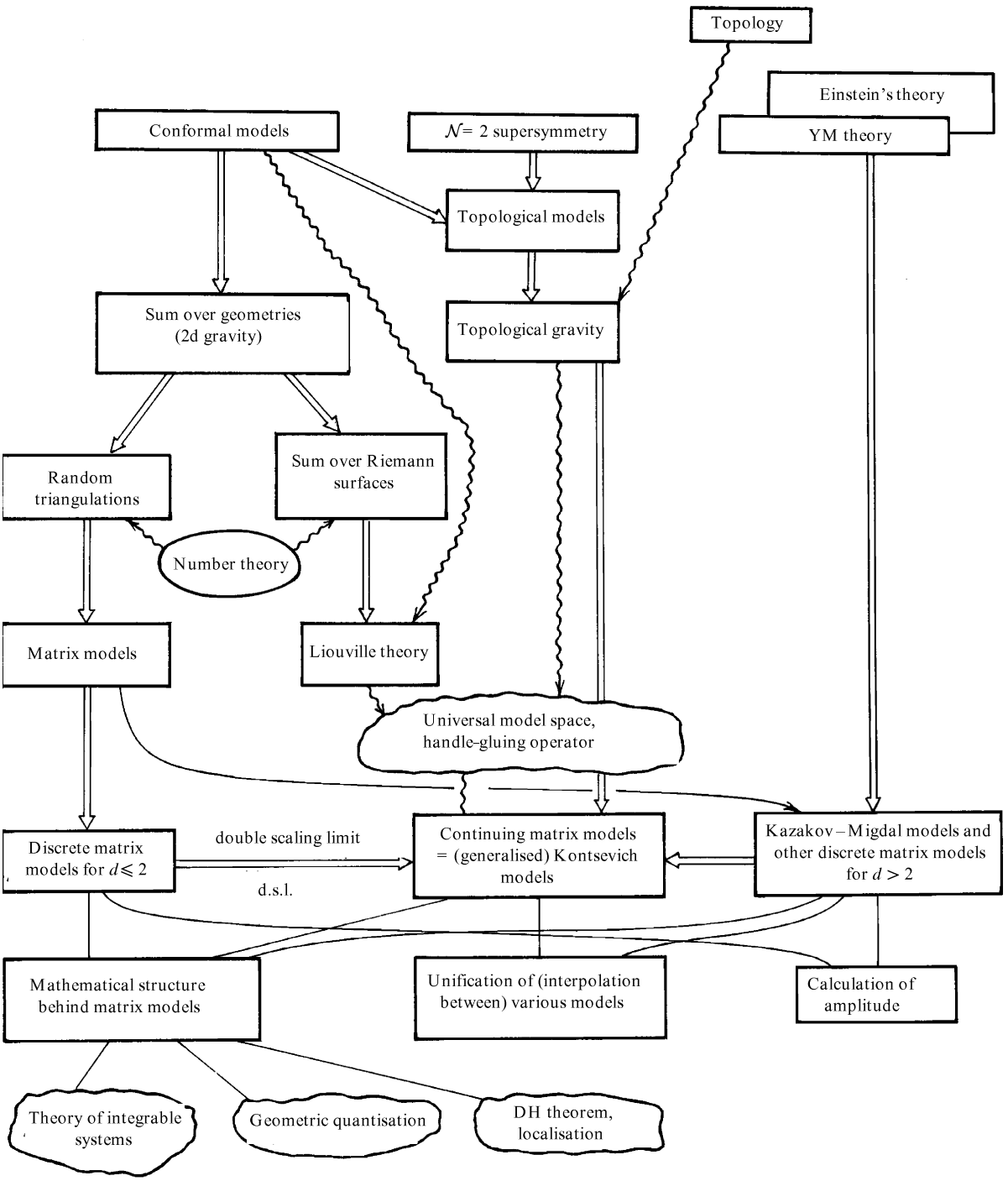

Figure 2.

involves a sum over random *equilateral* triangulations [18–20].† This is the place where matrix models first appear in the context of string theory. The random triangulation approach is by no means specific to conformal models (since it ignores the complex structure) and can be applied in many other situations — for example, to YM theories in any number of dimensions (where, instead of summation over geometries, one needs 'simply' to sum over ordinary Feynman diagrams).

Applications of the matrix-model method usually involve two steps: formulating and studying the 'discrete' model, and then taking its 'continuum limit', giving rise to a new 'continuous matrix model', which sometimes can again be represented in a form of some matrix integral.

One of the main discoveries in the field of matrix models is that *continuous* models arising finally from the random-equilateral-triangulation description of the simplest

† Its relation to the Polyakov approach is a separate very interesting, important, and badly understood problem, which allows a nontrivial reformulation in terms of number theory [21]. The main puzzle here is that equilateral triangulations are in fact *arithmetic* Riemann surfaces — a dense discrete subset in the entire module space, with interesting and deep algebraic properties. Equivalence of the two approaches to 2d quantum gravity should imply the existence of some number-theoretical background behind the scenes, which would be very nice to discover in full purity.

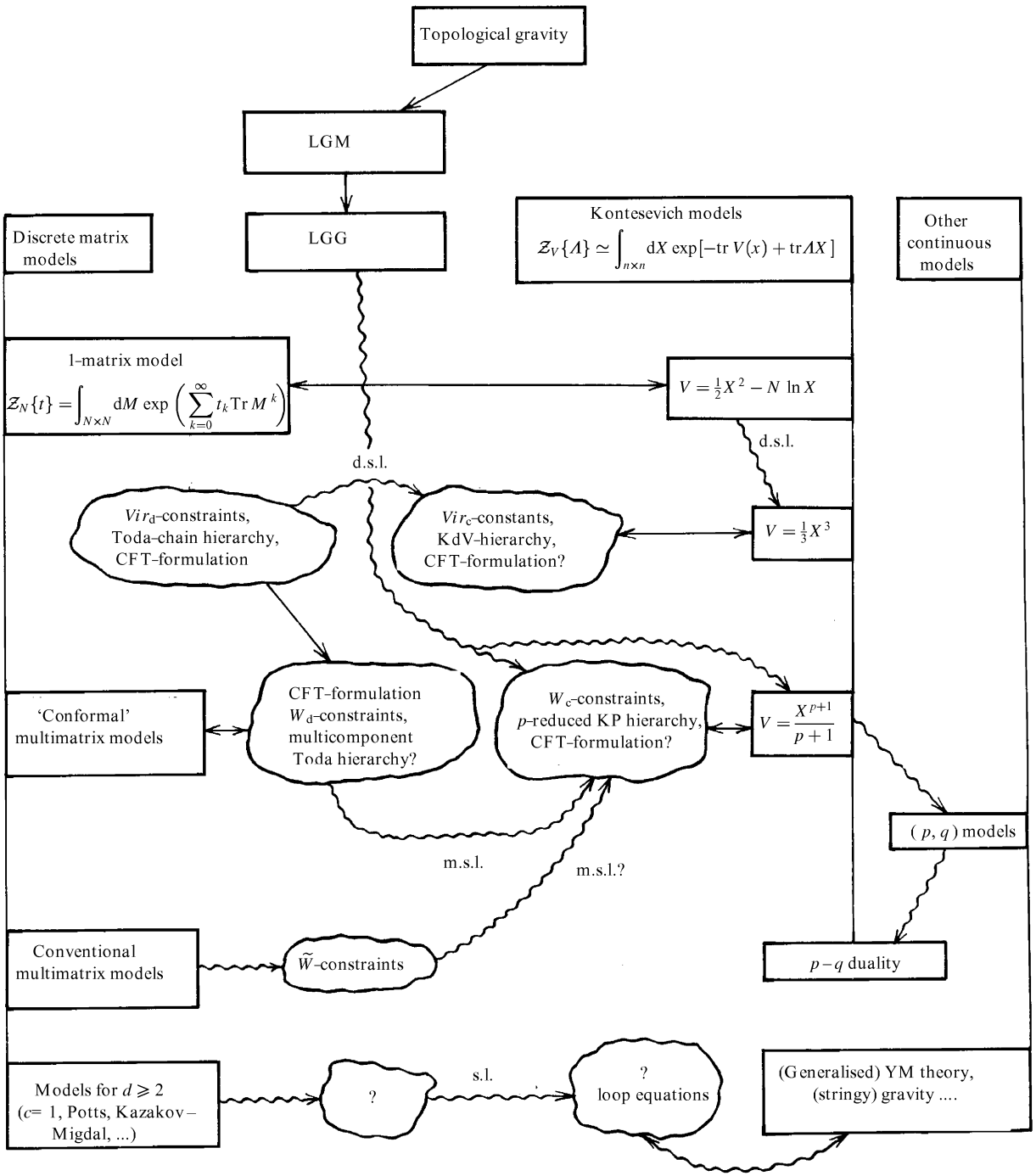

Figure 3.

(minimal, with $c < 1$) string models coincide with the simplest (CP^1 Landau–Ginzburg) models of topological gravity [9, 22–24]: two (classes of) theories are identical (this is not yet proved in full detail, but is more than plausible).

So far, *continuous* models are actually found and somehow understood only for string models, based on the $c < 1$ minimal conformal theories [moreover, only for $q = 1$ in the (p, q) series. Conformal models with $c \geqslant 1$ — which are relevant for description of gauge theories in spacetime dimension, $d \geqslant 2$ (which possess particles, rather than only topological degrees of freedom) — should give rise to the discrete matrix models with 'nonfactorisable' integration over 'angular variables', of which the simplest (solvable) example is the Kazakov–Migdal model [25]. The issue of the continuum limit for such models is not yet understood (at least in terms of integrable structures, which should probably generalise the familiar theory of Toda hierarchies).

The goal of the study of matrix models is threefold. First of all, one can look for the nonperturbative (exact) answers for the physical amplitudes in the given model. This is the subject which attracts most attention in the literature (for several obvious reasons). However, it is equally (and, perhaps, even more) important to understand the mathematical

structure behind the matrix models [which involves topics like the general theory of integrable hierarchies, geometrical quantisation, the Duistermaat–Heckman theorem ('localisation theory'), etc]. Also important for the purposes of string theory is to use the results of the study of matrix models in order to unify *a priori* different models (according to the above-mentioned principle, nonperturbative partition func-tions for different models differ by a change of variables in the space of coupling constants). Matrix models have already played an important role in making this principle clearer and more acceptable to many string theorists.

Let us take the next step and look even closer at the field of matrix models, especially at its most intensively studied domain, associated with the $d < 2$ string models. Then the structure shown in Fig. 3 will be seen.

A sample example of a matrix model is that of the 1-matrix integral†

$$Z_N\{t\} \equiv c_N \int_{N\times N} \mathrm{d}H \, \exp\left(\sum_{k=0}^{\infty} t_k \operatorname{Tr} H^k\right) , \tag{1.1}$$

where the integral is over the $N \times N$ Hermitian matrix H, and $\mathrm{d}H = \prod_{i,j} \mathrm{d}H_{ij}$. There are three directions in which one can proceed starting from Eqn (1.1).

The first [26] is to look for an invariant formulation of properties of the functional $Z_N\{t\}$. It appears that $Z_N\{t\}$ satisfies the following infinite set of differential equations {in fact these are just Ward identities (WIs) for the functional integral (1.1) [27]}:

$$L_n Z_N\{t\} = 0, \quad n \geqslant -1 ,$$

$$L_n \equiv \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}} ,$$

$$\frac{\partial}{\partial t_0} Z_N = N Z_N . \tag{1.2}$$

These equations are known as the 'discrete Virasoro constraints'. $Z_N\{t\}$ can be represented as a correlator of screening operators in some auxiliary conformal model (of one free field on the 'spectral surface'), and the Virasoro constraints (1.2) are of course related to the Virasoro algebra in that conformal model. Also $Z_N\{t\}$ is some τ-function of an integrable 'Toda-chain' hierarchy (in fact this statement should be a corollary of the Virasoro constraints, but this relation is still not very well understood).

The most straightforward approach to further development [26, 28] is to take the continuum limit of the Toda-chain hierarchy. In the specially adjusted 'double-scaling' (d.s.) limit [20] it gives rise to the Korteveg de Vries (KdV) hierarchy, and the corresponding τ-function appears to be subject to the slightly different constraints [28, 29] (which again form a Borel subalgebra of some other 'continuous Virasoro algebra')

$$\mathcal{L}_{2n} \mathcal{Z}^{\mathrm{cont}}\{T\} = 0, \quad n \geqslant -1 ,$$

where

$$\mathcal{L}_{2n} \equiv \frac{1}{2} \sum_{\mathrm{odd}\, k=1}^{\infty} k(T_k + r_k) \frac{\partial}{\partial T_{k+2n}}$$

$$+ \frac{1}{4} \sum_{\mathrm{odd}\, k=1}^{2n-1} \frac{\partial^2}{\partial T_k \partial T_{2n-k}} + \frac{1}{16} \delta_{n,0} + \frac{1}{4} (T_1 + r_1)^2 \delta_{n,-1} , \tag{1.3}$$

† In this review the operators det and tr apply to $n \times n$ matrices; Det and Tr apply to $N \times N$ matrices and *Det* applies to $(N+n) \times (N+m)$ matrices.

and $r_k = -\frac{2}{3}\delta_{k,3}$. In fact,

$$\mathcal{Z}^{\mathrm{cont}}\{T\} \sim \lim_{\mathrm{d.s.}\{N\to\infty\}} \sqrt{Z_N\{t\}}\Big|_{t_{2k+1}=0} , \tag{1.4}$$

and the T values are related to t by linear transformation [19, 28]:

$$T_k = \frac{1}{2} \sum_{m \geqslant \frac{1}{2}(k-1)} \frac{g_m}{[m - \frac{1}{2}(k-1)]!} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2}k+1)} , \quad k \text{ odd} ;$$

$$g_m = m t_{2m} , \quad m \geqslant 1 ; \quad g_0 = 2N . \tag{1.5}$$

This $\mathcal{Z}^{\mathrm{cont}}\{T\}$ can again be represented in the form of a matrix integral (over $n \times n$ Hermitian matrices) [22, 30–33]:

$$\mathcal{Z}^{\mathrm{cont}}\{T\} = \mathcal{Z}_V\{T\} , \tag{1.6}$$

with $V(X) = \frac{1}{3}X^3$, where

$$\mathcal{Z}_V\{T\} \sim \mathcal{F}_{V,n}\{L\} \equiv \int_{n\times n} \mathrm{d}X \exp[-\operatorname{tr} V(x) + \operatorname{tr} LX] , \tag{1.7}$$

and

$$T_k = \frac{1}{k} \operatorname{tr} L^{-k/2} , \quad k \text{ odd} . \tag{1.8}$$

The function $\mathcal{Z}_V\{T\}$ (but not $\mathcal{F}_{V,n}\{L\}$) is in fact independent of n: the only thing that happens for finite values of n is that the right-hand side (r.h.s.) of Eqn (1.7) cannot describe $\mathcal{Z}_V\{T\}$ at *arbitrary* points in the T-space, in accordance with Eqn (1.8). The continuous Virasoro constraints (1.3) are in fact equivalent to the trivial matrix-valued WI

$$\left[V'\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right) - L\right] \mathcal{F}_{V,n}\{L\} = 0 . \tag{1.9}$$

Another direction in which to proceed from the discrete 1-matrix model is to rewrite it identically in the form of a Kontsevich model: this time with $V(X) = X^2$ and with an additional factor of $(\det X)^N$ under the integral in $\mathcal{F}_{V,n}\{L\}$ [56]. Then the double-scaling limit can be studied in internal terms of Kontsevich models [36].

The third direction is towards multimatrix models. In the *continuous* version they should provide τ-functions of reduced Kadomtsev–Petviashvili (KP)-hierarchies [37] (KdV is the $p = 2$ reduction), which are subjected to 'continuous W-constraints' [29]. Matrix models of such τ-functions are Kontsevich models with $V(X) \sim X^{p+1}$ [30–33]. At the *discrete* level, however, things are not so simple. The most popular discrete multimatrix models [34] are defined as multiple matrix integrals of the form

$$Z_N\{t^{(\alpha)}\} \equiv c_N^{p-1} \int_{N\times N} \mathrm{d}H^{(1)} \ldots \mathrm{d}H^{(p-1)}$$

$$\times \prod_{\alpha=1}^{p-1} \exp\left(\sum_{k=0}^{\infty} t_k^{(\alpha)} \operatorname{Tr} H_{(\alpha)}^k\right) \prod_{\alpha=1}^{p-2} \exp(\operatorname{Tr} H^{(\alpha)} H^{(\alpha+1)}) \tag{1.10}$$

(the form of the 'interaction term' $\operatorname{Tr} H^{(\alpha)} H^{(\alpha+1)}$ is restricted by the 'solvability' principle, but not unambiguously). In fact these models are particular examples of 'scalar-product eigenvalue models' and are not really distinguished except for the 1-matrix ($p = 2$) and 2-matrix ($p = 3$) cases. This is

reflected in the absence of any reasonable WIs and integrable structures for these models, which would somehow involve their dependence on the variables $t^{(\alpha)}$ with $2 \leqslant \alpha \leqslant p-2$. Therefore, the 'multi-scaling continuum limit' of these models can hardly be investigated with any degree of rigour. (It is not very important for 'physical' applications to have discrete models associated with the continuum ones, but this is an interesting problem for the 'science of science'.) For the 2-matrix ($p = 3$) case the WIs can be expressed in the from of '$\widetilde{W}$-constraints' [38] and look like [30]

$$\widetilde{W}^{(m+1)}_{n-m}\{t\}Z_N\{t,\bar{t}\} = (-1)^{m+n}\widetilde{W}^{(n+1)}_{m-n}\{\bar{t}\}Z_N\{t,\bar{t}\} \quad (1.11)$$

(here t and $\bar{t}$ stand for $t^{(1)}$ and $t^{(2)}$; and m and n are any nonnegative integers).

The really interesting set of discrete multimatrix models does exist, but it is somewhat different from Eqn (1.10). These theories will be referred to as 'conformal matrix models', since they arise straightforwardly as a generalisation of the 'conformal field theory (CFT)-formulation' of the 1-matrix model [39]; it is enough to substitute discrete Virasoro constraints in the theory of one free field by the W_p-constraints in the theory of $p-1$ free fields. The matrix-integral formulation then involves an 'interaction term' $Det\,(H^{(\alpha)}\otimes \mathrm{I} - \mathrm{I}\otimes H^{(\alpha+1)})$ instead of $\exp(\mathrm{Tr}\, H^{(\alpha)}H^{(\alpha+1)})$, which is not very easy to guess a priori, but the models so defined and their continuum limits can be examined in a manner quite parallel to the 1-matrix case (though there is more to be done in this direction). Also, this approach provides the possibility of formulating discrete models for any set of constraints, e.g. associated with the more exotic W-algebras and with quantum groups (i.e. they can help to solve the inverse problem: *constraints* $\rightarrow$ *discrete matrix model*). This is an option which also deserves further investigation. Another natural name for this set of theories is 'multicomponent eigenvalue models'.

Kontsevich models should also be related to topological models of Landau–Ginzburg gravity (LGG), though this relation has not yet been clarified in full detail (see, however, [17, 40]).

Among the main unresolved puzzles in this whole field is the description of generic (p,q)-models. Formally, the generalised Kontsevich model (1.7) provides this description, but in fact the partition function (τ-function) gets singular when the 'phase transition' point where q changes is approached, and the Kontsevich model with $V(X)$ equal to a *polynomial of degree* $p+1$ provides a nice description only of $(p,1)$-models. Generically, the Kontsevich integral describes a duality transformation between (p,q)- and (q,p)-models: $(p,q)\rightarrow(q,p)$ [41], but not any of these models separately. [The only exceptions are $(p,1)$-models because they are related by Kontsevich transformation to the $(1,p)$-models, which are completely trivial.]

In fact continuous models have two *different* sets of 'time-variables'. Thus far I have introduced T-values, which are essentially expansion parameters of the generating functional for correlation functions. More precisely, these parameters $\hat{T}$ depend on the particular model (vacuum) around which the perturbation expansion is performed, and they differ slightly from the model-independent T. Another set of 'times', r_k†, parametrises the shape of the polynomial 'potential' $V_p(X)$ (of degree $p+1$) and describes the coordinates in the space of (matrix) models. These two types of variable — parameters of the generating functional and those labelling the shape of the Lagrangian — are almost the same [in fact they would be exactly the same if there were no loop (quantum) effects]. This similarity between Ts and rs is reflected in the remarkable property of the partition function of the $(p,1)$-model — essentially it depends only on the *sum* of 'times' $\hat{T}$ and r [40]:

$$\mathcal{Z}_{V_p}\{T\} = f_p(r|\hat{T}_k + r_k)\,\tau_p\{\hat{T}_k + r_k\} \quad (1.12)$$

with some simple (and explicitly known) function f_p. [In Eqn (1.8), for the *monomial* cubic potential $V_3(x) = \frac{1}{3}x^3$, $\hat{T}_k = T_k = (1/k)\,\mathrm{tr}L^{-k/2}$, while $r_k = -\frac{2}{3}\delta_{k,3}$.]

The last thing to be mentioned in this general description of matrix model theory is its relation to group theory. The generalised Kontsevich model (1.7) is intimately connected to the 'integrable nature' of group characters and the coadjoint orbit integrals {the characters of all the irreducible representations of $\mathrm{U}(N)$ are usually KP τ-functions [101]}. In fact a 'discrete (or quantum) version' of the Kontsevich integral is the sum over all unitary irreducible representations of $\mathrm{U}(n)$ [the 'integral' over a *model* of $\mathrm{U}(n)$, or over the set of all coadjoint orbits]

$$\mathcal{F}^{\mathrm{qu}}_V\{G\} \equiv \sum_R d_R\chi_R(G)\exp\left[-\sum_{k=0}^{\infty} v_k C_k(R)\right], \quad (1.13)$$

where d_R, χ_R, and $C_k(R)$ are the dimension, character, and the kth Casimir operator of the irreducible representation, R, of $\mathrm{U}(n)$. The time variables $T_k \sim (1/k)\,\mathrm{tr}\,G^k$, while the potential $V(X) = \sum_{k=0}^{\infty} s_k X^k$. This expression can be further generalised to

$$\mathcal{F}^{\mathrm{qu}}_V\{G\} \equiv \sum_R \chi_R(\bar{G})\,\chi_R(G)\exp\left[-\sum_{k=0}^{\infty} v_k C_k(R)\right]. \quad (1.14)$$

Properties of these 'quantum' Kontsevich models deserve further investigation [objects like Eqn (1.13) are also known to arise in the localisation theory; in particular, in the study of 2d YM theory — see, for example, [43, 44].

These notes are essentially a review of the views and results of the group working in Moscow (and Kiev). Since references will not be given every time, I present here the list of people involved in these investigations: L Chekhov, A Gerasimov, A Losev, S Kharchev, Yu Makeenko, A Marshakov, A Mikhailov, A Mironov, A Orlov, S Pakulyak, I Polyubin, A Zabrodin.

I also apologise for the somewhat sporadic references to the works of other groups.

2. Ward identities for the simplest matrix models

2.1 Ward identitites versus equations of motion

I begin systematic consideration of matrix models from their simplest, and at the same time most basic, property: the Ward identities (WIs) for partition functions. A partition function is, by definition, a functional of the coupling constants in the Lagrangian, and WIs will be understood here as (differential or finite-difference) equations, imposed on this functional. If the partition function is represented in

† where $r_k = [p/k(p-k)]\,\mathrm{Res}[V'(\mu)]^{1-k/p}\mathrm{d}\mu$.

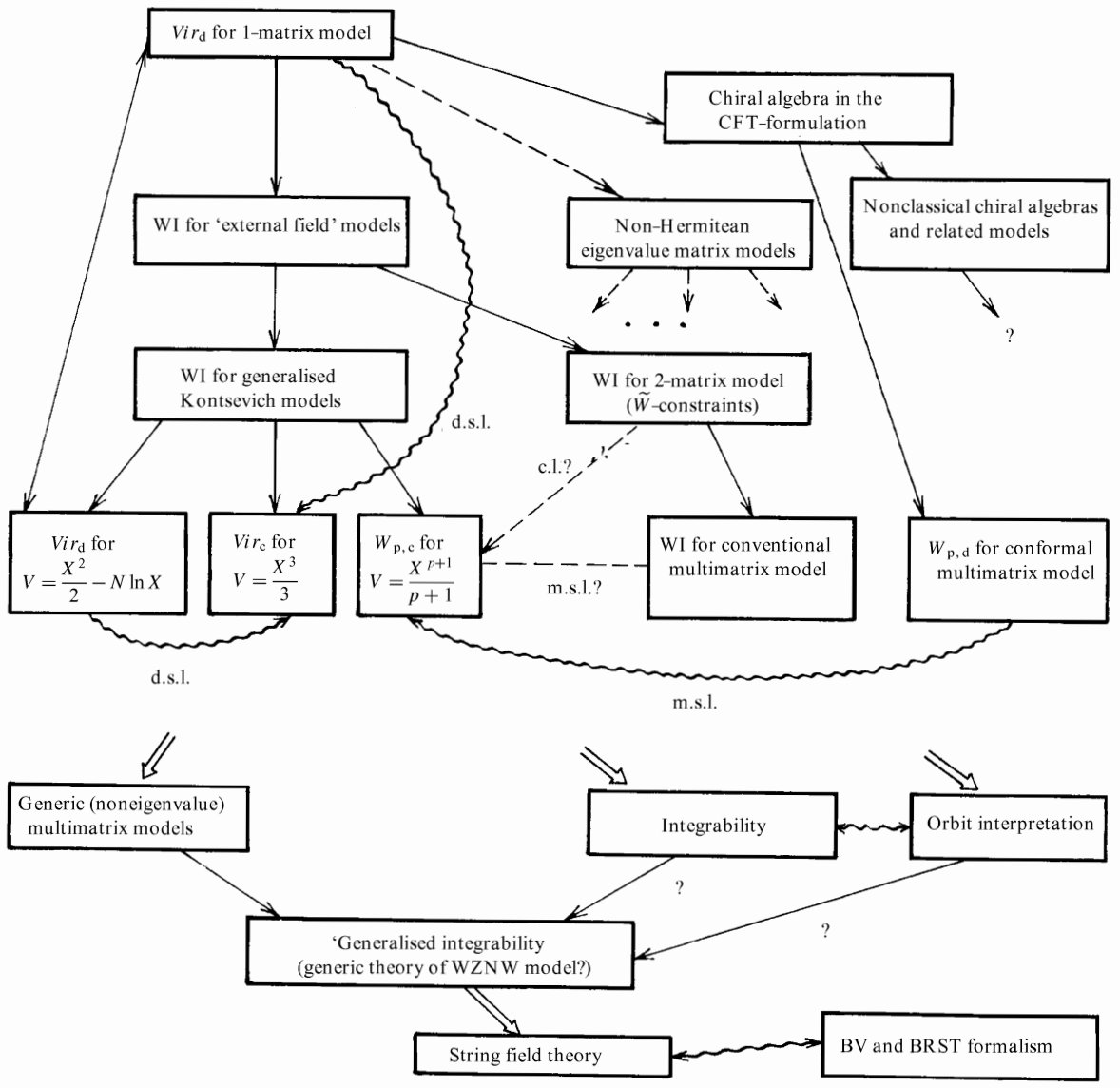

Figure 4.

the form of a matrix integral,† the WIs are usually implied by its covariance under the change of the integration variables (thus the name 'WI').

In ordinary field theory, one is usually dealing with models where the WIs either do not exist at all, or at most there is a finite number of them — then they are interpreted as reflecting the *symmetry* of the theory. However, by no means does the finite set of these WIs prove a *complete* description of the dynamics of the theory: the number of (quantum) equations of motion (EqMs) is usually infinite and their solutions are never fixed by the WIs. In fact this difference between WI and EqM arises because the Lagrangians, considered in the ordinary field theory, are not of the most general form: they are usually severely restricted by 'principles' like renormalisability or minimality. Because of this there is simply not enough coupling constants in the Lagrangian to describe the result of *any* variation of integration variables as that of the variation of coupling constants, and thus not every EqM can be represented as a (differential) equation for the partition function. In other words, by restricting the shape of the Lagrangian for 'nonsymmetric' reasons one breaks the model's original huge 'symmetry' (covariance), which was enough to describe all the dynamics (all EqMs) as dictated by symmetry, and a broader view is necessary in order to recognise EqM as the WI associated with that original high symmetry. This symmetry (which is not a Noether symmetry, of course) is a peculiar property of all the *quantum-mechanical* partition functions, since these usually arise from the procedure of functional integration.

†To avoid confusion I should emphasise that such a representation does *not* need to exist, at least in any simple form. The more the theory of matrix models develops, the less it has to do with *matrices* and matrix integrals. However, (as in the case of entire *string theory*) the original name has a tendency to survive. Anyhow, the main content of the theory of matrix models (at least of the branch, analysed in these notes) is the search for *invariant* formulations of the properties of partition functions, while matrix integrals (if existing at all) are considered as their particular realisations (representations). Moreover, there can exist very different matrix integral representations of the same partition function, the simplest example being just the basic discrete 1-matrix model, which can also be represented in the form of a Kontsevich integral (see below).

Matrix models appeared to be the first class of quantum-mechanical systems (functional integrals) for which the identity

all EqMs $\equiv$ all WIs

was not simply observed as a curious phenomenon, but became a subject of intensive investigation and is identified as the source of exact solvability (integrability) of the theory. Of course, the significance of this observation (and its implications) is quite universal and is by no means restricted to the field of matrix models themselves; however, it is not yet appreciated enough by the experts in other fields. In any case, I am going to deal only with matrix models in these notes.

I proceed to the consideration of the WI according to the plan illustrated in Fig. 4 (not all the arrows will be discussed).

2.2 Virasoro constraints for the discrete 1-matrix model

The basic example [26, 27] which illustrates the arguments from the previous subsection is provided by the 1-matrix model

$$Z_N\{t\} \equiv c_N \int_{N\times N} \mathrm{d}H \exp\left(\sum_{k=0}^{\infty} T_k \operatorname{Tr} H^k\right) . \tag{2.1}$$

This integral is invariant under any change of variables $H \to \mathrm{f}(H)$. It is convenient to choose the following special basis in the space of such transformations:

$$\delta H = \varepsilon_n H^{n+1} . \tag{2.2}$$

Here ε_n is some infinitesimal matrix and, of course, $n \geqslant -1$. The value of the integral cannot change under the change of integration variable, and the following identity is obtained:

$$\int_{N\times N} \mathrm{d}H \exp\left(\sum_{k=0}^{\infty} t_k \operatorname{Tr} H^k\right) = \int \mathrm{d}(H+\varepsilon_n H^{n+1}) \exp\left[\sum_{k=0}^{\infty} t_k \operatorname{Tr}(H+\varepsilon_n H^{n+1})^k\right] ,$$

that is,

$$\int \mathrm{d}H \exp\left(\sum_{k=0}^{\infty} t_k \operatorname{Tr} H^k\right)\left(\sum_{k=0}^{\infty} k t_k \operatorname{Tr} H^{k+n} + \operatorname{Tr}\frac{\delta H^{n+1}}{\delta H}\right) \equiv 0 . \tag{2.3}$$

In order to evaluate the Jacobian $\operatorname{Tr}(\delta H^{n+1}/\delta H)$, the matrix indices should be restored:

$$(\delta H^{n+1})_{ij} = \sum_{k=0}^{n} (H^k \delta H H^{n-k})_{ij} = \sum_{k=0}^{n} (H^k)_{il} (\delta H)_{lm} (H^{n-k})_{mj} .$$

In $\operatorname{Tr}(\delta H^{n+1}/\delta H)$ one should take $l = i$ and $m = j$, so that

$$\operatorname{Tr}\frac{\delta H^{n+1}}{\delta H} = \sum_{k=0}^{n} \operatorname{Tr} H^k \operatorname{Tr} H^{n-k} . \tag{2.4}$$

Now note that because one started from a Lagrangian of the most general form (consistent with the symmetry $H \to UHU^{\dagger}$), any correlation function can be obtained as a variation of the coupling constants (all possible *sources* are included as counter-terms). In my particular example this is just a trivial remark:

$$\begin{aligned}\langle \operatorname{Tr} H^{a_1} \ldots \operatorname{Tr} H^{a_n}\rangle &= \int \mathrm{d}H \exp\left(\sum_{k=0}^{\infty} t_k \operatorname{Tr} H^k\right) \operatorname{Tr} H^{a_1} \ldots \operatorname{Tr} H^{a_n} \\ &= \frac{\partial^n}{\partial t_{a_1} \ldots \partial t_{a_n}} Z_N\{t\} .\end{aligned} \tag{2.5}$$

This relation can be used together with Eqn (2.4) in order to rewrite Eqn (2.3) as

$$L_n Z_N\{t\} = 0 , \quad n \geqslant -1 , \tag{2.6}$$

with

$$L_n \equiv \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}} . \tag{2.7}$$

Note that, according to the definition (2.1),

$$\frac{\partial}{\partial t_0} Z_N = N Z_N .$$

Several remarks are now in order.

First, the expression in brackets in Eqn (2.3) represents *all* the EqMs for the model (2.1), and Eqn (2.6) is nothing but another way to represent the same set of equations. This is an example of the above-mentioned identification of EqM and WI.

Second, the commutator of any two operators L_n appearing in Eqn (2.6) should also annihilate $Z_N\{t\}$. Another indication (not a convincing one, however) that we already have a *complete* set of constraints is that L_n forms a closed (Virasoro) algebra:

$$[L_n, L_m] = (n-m) L_{n+m} , \quad n, m \geqslant -1 . \tag{2.8}$$

Third, Eqn (2.6) can be considered as an invariant formulation of the definition of Z_N: Z_n is a solution of this set of compatible differential equations. From this point of view Eqn (2.1) is a particular representation of Z_N and it is sensible to look for other representations as well (I shall later discuss two of them: one in terms of conformal field theory (CFT), another in terms of Kontsevich integrals).

Fourth, one can try to analyse the uniqueness of the solutions of Eqn (2.6). If there are not too many of them the set of constraints can be considered complete. A natural approach to the classification of solutions of the algebra of constraints is in terms of the orbits of the corresponding group [45]. Let us consider an oversimplified example, which can still be useful in understanding the implications of the complete set of WIs as well as in clarifying the meaning of classes of universality and of integrability.

Imagine that instead of Eqn (2.6) with L_n defined in Eqn (2.7), we would obtain the somewhat simpler equations†

$$l_n Z = 0 , \quad n \geqslant 0 ; \text{ with } \quad l_n = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} .$$

Then operator l_1 can be interpreted as generating the shifts

$$t_2 \to t_2 + \varepsilon_1 t_1 ,$$
$$t_3 \to t_3 + 2\varepsilon_1 t_2 ,$$
$$\vdots$$

† One can call them the 'classical' approximation to Eqn (2.6), since they would arise if the variation of measure (i.e. a 'quantum effect') was not taken into account in the derivation of Eqn (2.6). Though this concept is often used in physics it does not make much sense in the present context, where I am analysing *exact* properties of functional (matrix) integrals.

It can be used to shift t_2 to zero, and the equation $l_1 Z = 0$ then implies that

$$Z(t_1, t_2, t_3, \ldots) = Z(t_1, 0, \bar{t}_3, \ldots)$$

$$[\bar{t}_k = t_k - (k-1)t_2 t_{k-1}/t_1 , \quad k \geqslant 3] .$$

Next, the operator l_2 generates the shifts

$$t_3 \to t_3 + \varepsilon_2 t_1 ,$$

$$t_4 \to t_4 + \varepsilon_2 t_2 ,$$

$$\vdots$$

and does *not* affect t_2. One can now use the equation $l_2 Z = 0$ to argue that

$$Z(t_1, t_2, t_3, t_4, \ldots) = Z(t_1, 0, \tilde{t}_3, \tilde{t}_4, \ldots) = Z(t_1, 0, 0, \tilde{\tilde{t}}_4, \ldots) ,$$

etc. Assuming that Z is not very dependent on t_k with $t_k \to \infty$†, it is possible to conclude that

$$Z(t_1, t_2, t_3, \ldots) = Z(t_1, 0, 0, \ldots) = Z(1, 0, 0, \ldots)$$

(in the last step I also used the equation $l_0 Z = 0$ to rescale t_1 to unity).

All this reasoning is correct provided $t_1 \neq 0$. Otherwise one would get $Z(0, 1, 0, 0, \ldots)$, if $t_1 = 0$ and $t_2 \neq 0$; $Z(0, 0, 1, 0, \ldots)$ if $t_1 = t_2 = 0$, $t_3 \neq 0$, etc. In other words, one obtains classes of universality (such that the value of the partition function is the same in the whole class), which in this oversimplified example are labelled just by the first nonvanishing time-variable. Analysis of the orbit structure for the actually important realisations of groups, like that connected to Eqn (2.7), has never been performed in the context of matrix model theory. It may deserve emphasising that the constraints, as we saw, can actually allow one to eliminate (solve exactly) all the dependence on the time-variables. In less trivial examples they somehow imply the integrability structure, which is just a slightly more complicated version of the same solvability phenomenon.

2.3 Conformal-field-theory formulation of matrix models

Given a complete set of the constraints on a partition function of infinitely many variables which form some closed algebra one can now ask an inverse question: how these equations can be solved or what the integral representation of the partition function is. One approach to this problem is the analysis of the orbits, briefly mentioned at the end of the previous section. Now I turn to another technique [39], which makes use of knowledge from CFT. These constructions can have some meaning from the 'physical' point of view, which implies certain duality between the 2d world surfaces and the spectral surfaces, associated with the configuration space of the string theory. However, the goal now is more formal: to use the methods of CFT to solve the constraint equations.

This is very natural in the case when the algebra of constraints is a Virasoro algebra, as in the case of the 1-matrix model, or some other algebra if it is known to arise naturally as a chiral algebra in some simple conformal models. In fact, the approach which will now be discussed is rather general and can be applied to the construction of matrix models associated with many different algebraic structures.

I begin from the set of Eqn (2.6) which shall be referred to as 'discrete Virasoro constraints'. The CFT formulation of interest should provide the solution to these equations in the form of some correlation function in some CFT. Of course, it becomes natural to somehow identify the operators L_n, which form a Virasoro algebra, with the harmonics, T_n, of the stress tensor, which satisfy the same algebra, and manage to relate the constraint that the L_n operators annihilate the correlator to the statement that the T_n values annihilate the vacuum state. Thus, the procedure is naturally split into two steps: first one should find a t-dependent operator ('Hamiltonian'), $H(t)$, such that

$$L_n(t)\langle \exp[H(t)] \ldots = \langle \exp[H(t)] t_n \ldots . \tag{2.9}$$

This will relate the differential operators L_n to the t_n values expressed through the fields of the conformal model. Second one needs to enumerate the states that are annihilated by the operators t_n with $n \geqslant -1$, i.e. solve the equation

$$T_n |G\rangle = 0 \tag{2.10}$$

for the ket states; this is an internal problem of CFT. If both ingredients, $H(t)$ and $|G\rangle$, are found, the solution to the problem is given by

$$\langle \exp[H(t)] \,|\, G \rangle . \tag{2.11}$$

To be more explicit, for the case of the discrete Virasoro constraints one can look just for solutions in terms of the simplest possible conformal model: that of a 1-holomorphic scalar field:

$$\phi(z) = \hat{q} + \hat{p} \ln z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^k ;$$

$$[J_n, J_m] = n\delta_{n+m,0} , \quad [\hat{q}, \hat{p}] = 1 . \tag{2.12}$$

Then the procedure is as follows: define vacuum states

$$J_k |0\rangle = 0 , \quad \langle N| J_{-k} = 0 \quad (k > 0) ,$$

$$\hat{p}|0\rangle = 0 , \quad \langle N|\hat{p} = N\langle N| ; \tag{2.13}$$

the stress tensor

$$T(z) = \frac{1}{2}[\partial\phi(z)]^2 = \sum T_n z^{-n-2} ,$$

$$T_n = \sum_{k>0} J_{-k} J_{k+n} + \frac{1}{2} \sum_{\substack{a+b=n \\ a,b \geqslant 0}} J_a J_b ; \tag{2.14}$$

and the Hamiltonian

$$H(t) = \frac{1}{\sqrt{2}} \sum_{k>0} t_k J_k = \frac{1}{\sqrt{2}} \oint_{C_0} U(z) J(z) ,$$

$$U(z) = \sum_{k>0} t_k z^k , \quad J(z) = \partial\phi(z) . \tag{2.15}$$

It can now easily be checked that

$$L_n \langle N| \exp[H(t)] \ldots = \langle N| \exp[H(t)] T_n \ldots \tag{2.16}$$

and

$$T_n |0\rangle = 0 , \quad n \geqslant -1 . \tag{2.17}$$

As an immediate consequence, any correlator of the form

$$Z_N\{t|G\} = \langle N \,|\, \exp[H(t)] G|0\rangle \tag{2.18}$$

gives a solution to Eqn (2.6), provided

$$[T_n, G] = 0 , \quad n \geqslant -1 . \tag{2.19}$$

† This, by the way, is hardly correct in this particular example, when the group has no compact orbits.

In fact, operators, G, that commute with the stress tensor are well known: these are just any functions of the 'screening charges'† $Q_\pm$, where

$$Q_\pm = \oint J_\pm = \oint \exp\left(\pm\sqrt{2}\phi\right) . \tag{2.20}$$

The correlator (2.18) will be nonvanishing only if the matching condition for zero-modes of ϕ is satisfied. If one demands that the operator depend only on Q_+, this implies that only one term of the expansion in powers of Q_+ will contribute to Eqn (2.18), so that the result is essentially independent of the choice of the function $G(Q_+)$; one can, for example, take $G(Q_+) = \exp Q_+$ and obtain:

$$Z_N\{t\} \sim \frac{1}{N!}\langle N|\exp[H(t)](Q_+)^N|0\rangle . \tag{2.21}$$

This correlator is easy to evaluate by means of the Wick theorem and the propagator $\phi(z)\,\phi(z') \sim \ln(z-z')$ and finally one gets

$$\begin{aligned} Z_N\{t\} &= \frac{1}{N!}\langle N|{:}\exp\left[\frac{1}{\sqrt{2}}\oint_{C_0} U(z)\partial\phi(z)\right]{:} \\ &\qquad \times \prod_{i=1}^{N}\oint_{C_i} dz_i\, {:}\exp\left[\sqrt{2}\phi(z_i)\right]{:}\,|0\rangle \\ &= \frac{1}{N!}\prod_{i=1}^{N}\oint_{C_i} dz_i \exp[U(z_i)]\prod_{i<j}^{N}(z_i-z_j)^2 \end{aligned} \tag{2.22}$$

in the form of a multiple integral, which can in fact be directly related to the matrix integral in Eqn (2.1); see [46] and the next section.

Thus, in the simplest case the inverse problem has been resolved: the integral representation has been reconstructed from the set of discrete Virasoro constraints. However, this answer seem to be a little more general than Eqn (2.1): the r.h.s. of Eqn (2.22) still depends on the contours of integration. Moreover, one can also recall that the operator G, above, could depend not only on Q_+, but also on Q_-. The most general formula is a little more complicated than Eqn (2.22):

$$\begin{aligned} Z_N\{t\,|C_i, C_r\} &\sim \frac{1}{(N+M)!M!}\langle N|\exp[H(t)](Q_+)^{N+M}(Q_-)^M|0\rangle \\ &= \frac{1}{(N+M)!M!}\prod_{i=1}^{N+M}\oint_{C_i} dz_i \exp[U(z_i)] \\ &\times \prod_{r=1}^{M}\oint_{C'_r} dz'_r \exp[U(z'_r)]\frac{\prod_{i<j}^{N+M}(z_i-z_j)^2\prod_{r<s}^{N}(z'_r-z'_s)^2}{\prod_{i=1}^{N+M}\prod_{r=1}^{M}(z_i-z_r)^2} . \end{aligned} \tag{2.23}$$

See [39] for a discussion of the issue of contour dependence. In a certain sense, all these different integrals can be considered as branches of the same analytical function, $Z_N\{t\}$. Dependence on M is essentially eliminated by Cauchy integration around the poles in the denominator in Eqn (2.23).

The above construction can be applied straightforwardly to other algebras of constraints, provided:

(i) The free-field representation of the algebra is known in the CFT framework, such that the generators are *polynomials* in the fields ϕ (only in such a case is it straightforward to construct a Hamiltonian H, which relates the CFT realisation of the algebra to that in terms of the differential operators with respect to the t-variables; in fact, under this condition, H is usually linear in the ts and ϕs). There are examples (like the Frenkel–Kac representation of level $k = 1$ simply-laced Kac–Moody algebras [47] or generic reductions of the WZNW model [16, 48–51] in which the generators are *exponents* of the free fields; in these cases the construction should be slightly modified.

(ii) It is easy to find the vacuum state, annihilated by the relevant generators (here, for example, is the problem with the application of this approach to the case of 'continuous' Virasoro and W-constraints). The resolution of this problem involves consideration of correlations on Riemann surfaces with nontrivial topologies, often of infinite genus.

(iii) The free-field representation of the 'screening charges' (i.e. operators that commute with the generators of the group within the conformal model) is explicitly known.

These conditions are fulfilled in many cases in CFT, including conventional **W**-algebras [52] and $\mathcal{N} = 1$ supersymmetric models [53].‡

For illustration purposes, I present here several formulas from the last paper of [39] for the case of the $\mathbf{W}_{r+1}$-constraints, associated with the simply-laced algebras $\mathcal{A}$ of rank r.

The partition function in such a 'conformal multimatrix model' is a function of 'time variables' $t_k^{(\lambda)}$, $k = 0\ldots\infty$, $\lambda = 1,\ldots,r =$ rank $\mathcal{A}$, and also depends on the integer-valued
r-vector $\boldsymbol{N} = \{N_1,\ldots,n_r\}$. The $\mathbf{W}_{r+1}$-constraints imposed on the partition function are

$$W_n^{(a)}(t)Z_N^{\mathcal{A}}\{t\} = 0 , \quad n \geqslant 1-a , \quad a = 2,\ldots,r+1 . \tag{2.24}$$

The form of the W-operators is somewhat complicated; for example, in the case $r+1 = 3$ {i.e. for $\mathcal{A} = A_2$ [SL (3)]},

$$\begin{aligned} W_n^{(2)} &= \sum_{k=0}^{\infty}\left(kt_k\frac{\partial}{\partial t_{k+n}} + k\bar{t}_k\frac{\partial}{\partial \bar{t}_{k+n}}\right) \\ &\qquad + \sum_{a+b=n}\left(\frac{\partial^2}{\partial t_a\partial t_b} + \frac{\partial}{\partial \bar{t}_a\partial \bar{t}_b}\right) , \end{aligned} \tag{2.25}$$

$$\begin{aligned} W_n^{(3)} &= \sum_{k,l>0}\left(kt_k lt_l\frac{\partial}{\partial t_{k+n+l}} - k\bar{t}_k l\bar{t}_l\frac{\partial}{\partial t_{k+n+l}} - 2kt_k l\bar{t}_l\frac{\partial}{\partial \bar{t}_{k+n+l}}\right) \\ &\quad + 2\sum_{k>0}\sum_{a+b=n+k}\left(kt_k\frac{\partial^2}{\partial t_a\partial t_b} - kt_k\frac{\partial^2}{\partial \bar{t}_a\partial \bar{t}_b} - 2k\bar{t}_k\frac{\partial^2}{\partial t_a\partial \bar{t}_b}\right) \\ &\quad + \frac{4}{3}\sum_{a+b+c=n}\left(\frac{\partial^3}{\partial t_a\partial t_b\partial t_c} - \frac{\partial^3}{\partial t_a\partial \bar{t}_b\partial \bar{t}_c}\right) , \end{aligned} \tag{2.26}$$

† For notational simplicity I omit the normal ordering signs; in fact the operators involved are $:\exp H:$ and $:\exp\left(\pm\sqrt{2}\varphi\right):$

‡ In the case of $\mathcal{N} = 2$ supersymmetry a problem arises because of the lack of reasonable screening charges. At the most naive level the relevant operator to be integrated over superspace (over $dz\,d^{\mathcal{N}}\theta$) in order to produce screening charge has dimension $1-\frac{1}{2}\mathcal{N}$, which *vanishes* when $\mathcal{N} = 2$.

and two types of time-variables, denoted by t_k and $\bar{t}_k$ are associated with two *orthogonal* directions in the Cartan plane of A_2:

$$\boldsymbol{e} = \frac{\alpha_1}{\sqrt{2}} , \qquad \bar{\boldsymbol{e}} = \frac{\sqrt{3}\boldsymbol{v}_2}{\sqrt{2}} \quad .\dagger$$

All other formulas, however, are very simple: the conformal model is usually that of the r free fields, where $S \sim \int \bar{\partial}\boldsymbol{\phi}\partial\boldsymbol{\phi}\,\mathrm{d}^2z$, which is used to describe representation of the level-one Kac–Moody algebra associated with $\mathcal{A}$. The Hamiltonian

$$H(t^{(1)}, \ldots, t^{(r+1)}) = \sum_{\lambda=1}^{r+1} \sum_{k>0} t_k^{(\lambda)} \boldsymbol{\mu}_\lambda \boldsymbol{J}_k \ , \tag{2.27}$$

where the set $\{\boldsymbol{\mu}_\lambda\}$ is associated with the 'fundamental weight' vectors $\boldsymbol{v}_\lambda$ in the Cartan hyperplane and, in the simplest case of $\mathcal{A} = A_r\,[\mathrm{SL}(r+1)]$, satisfy

$$\boldsymbol{\mu}_\lambda \cdot \boldsymbol{\mu}_{\lambda'} = \delta_{\lambda\lambda'} - \frac{1}{r+1} \ , \qquad \sum_{\lambda=1}^{r+1} \boldsymbol{\mu}_\lambda = 0 \ ;$$

thus, only r of the time variables $t^{(1)}, \ldots, t^{(r+1)}$ are linearly independent. The relation between the differential operators $W_n^{(a)}(t)$ and the operators $\mathrm{W}_n^{(a)}$ in CFT is now defined by

$$W_n^{(a)} \langle \boldsymbol{N} \,|\, \exp[H(t)] \ldots = \langle \boldsymbol{N} \,|\, \exp[H(t)] \mathrm{W}_i^{(a)} \ldots ,$$
$$a = 2, \ldots, p \ ; \qquad i \geqslant 1 - a \ , \tag{2.28}$$

where

$$\mathrm{W}_n^{(a)} = \oint z^{a+n-1} \mathrm{W}^{(a)}(z) \ ,$$
$$\mathrm{W}^{(a)}(z) = \sum_\lambda [\boldsymbol{\mu}_\lambda \partial\boldsymbol{\phi}(z)]^a + \ldots , \tag{2.29}$$

are spin-a generators of the $\mathbf{W}_{r+1}^{\mathcal{A}}$-algebra. The screening charges, which commute with all the $\mathrm{W}^{(a)}(z)$, are given by

$$Q^{(\alpha)} = \oint J^{(\alpha)} = \oint \exp \boldsymbol{\alpha}\boldsymbol{\phi} \ , \tag{2.30}$$

$\{\boldsymbol{\alpha}\}$ being the roots of the finite-dimensional simply laced Lie algebra $\mathcal{A}$.

Thus, the partition function arises in the form

$$Z_N^{\mathcal{A}}\{t\} = \langle \boldsymbol{N} \,|\, \exp[H(t)] G\{Q^{(\alpha)}\} \,|\, 0 \rangle \ , \tag{2.31}$$

where G is an exponential function of screening charges. Evaluation of the free-field correlator gives

$$Z_N^{\mathcal{A}}\{t\} \sim \int \prod_\alpha \left[\prod_{i=1}^{N_\alpha} \mathrm{d}z_i^{(\alpha)} \exp\left(\sum_{\lambda;\,k>0} t_k^{(\lambda)} (\boldsymbol{\mu}_\lambda \boldsymbol{\alpha}) (z_i^{(\alpha)})^k \right) \right]$$
$$\times \prod_{(\alpha,\beta)} \prod_{i=1}^{N_\alpha} \prod_{j=1}^{N_\beta} (z_i^{(\alpha)} - z_j^{(\beta)})^{\boldsymbol{\alpha}\boldsymbol{\beta}} \quad . \tag{2.32}$$

In fact this expression can be rewritten in terms of an r-matrix integral — a 'conformal multimatrix model':

$$Z_N^{\mathcal{A}}\{t^{(\alpha)}\} = c_N^{p-1} \int_{N\times N} \mathrm{d}H^{(1)} \ldots \mathrm{d}H^{(p-1)}$$
$$\times \prod_{\alpha=1}^{p-1} \exp\left(\sum_{k=0}^{\infty} t_k^{(\alpha)} \operatorname{Tr} H_{(\alpha)}^k \right)$$
$$\times \prod_{(\alpha,\beta)} \operatorname{Det}(H^{(\alpha)} \otimes \mathrm{I} - \mathrm{I} \otimes H^{(\alpha+1)})^{\boldsymbol{\alpha}\boldsymbol{\beta}} \quad . \tag{2.33}$$

In the simplest case of the $\mathbf{W}_3$-algebra, Eqn (2.32), with the insertion of only two (of the six) screening charges Q_{α_1} and Q_{α_2}, turns into

$$Z_{N_1,N_2}^{A_2}(t,\bar{t}) = \frac{1}{N_1!N_2!} \langle N_1, N_2 \,|\, \exp[H(t,\bar{t})] (Q^{(\alpha_1)})^{N_1} (Q^{(\alpha_2)})^{N_2} \,|\, 0 \rangle$$
$$= \frac{1}{N_1!N_2!} \prod_i \int \mathrm{d}x_i \exp[U(x_i)]$$
$$\times \prod_j \int \mathrm{d}y_j \exp[\bar{U}(y_i)] \Delta(x) \Delta(x,y) \Delta(y) \ , \tag{2.34}$$

where $\Delta(x,y) \equiv \Delta(x)\Delta(y) \prod_{i,j} (x_i - y_i)$. This model is associated with the algebra $\mathcal{A} = A_2\,[\mathrm{SL}(3)]$, while the original 1-matrix model (2.21)–(2.23) is associated with $\mathcal{A} = A_1\,[\mathrm{SL}(2)]$.

The whole series of models (2.32)–(2.33) for $\mathcal{A} = A_r[\mathrm{SL}(r+1)]$ is distinguished by its relation to the level $k = 1$ simply-laced Kac–Moody algebras. In this particular situation the underlying conformal model has integer central charge $c (= r = \operatorname{rank}\mathcal{A})$ and can be 'fermionised'.‡ The main feature of this formulation is that the Kac–Moody currents (which, after integration, turn into 'screening charges' in the above construction) are quadratic in fermionic fields, while they are represented by exponents in the free-boson formulation.

In fact the fermionic (spinor) model naturally possesses $\mathrm{GL}(r+1)$ rather than $\mathrm{SL}(r+1)$ symmetry (other simply-laced algebras can be embedded into larger GL-algebras and this provides a fermionic description for them in the case of $k = 1$). The model contains $r+1$ spin-$\frac{1}{2}$ fields ψ_i and their conjugates $\tilde{\psi}_i$ (b, c-systems):

$$S = \sum_{j=1}^{r+1} \int \tilde{\psi}_j \bar{\partial} \psi_j \,\mathrm{d}^2z \ ,$$

where the central charge is given by $c = r + 1$, and the operator algebra is

$$\tilde{\psi}_j(z)\psi_k(z') = \frac{\delta_{jk}}{z - z'} + :\tilde{\psi}_j(z)\psi_k(z'):$$
$$\psi_j(z)\psi_k(z') = (z - z')\delta_{jk} \ :\psi_j(z)\psi_k(z'): \ + (1 - \delta_{jk}) :\psi_j(z)\psi_k(z'):$$
$$\tilde{\psi}_j(z)\tilde{\psi}_k(z') = (z - z')\delta_{jk} \ :\tilde{\psi}_j(z)\tilde{\psi}_k(z'): \ + (1 - \delta_{jk}) :\tilde{\psi}_j(z)\tilde{\psi}_k(z'):$$

† Such an orthogonal basis is especially convenient for the discussion of integrability properties of the model; these t and $\bar{t}$ are linear combinations of the time-variables t_k^λ appearing in Eqns (2.27) and (2.32).

‡ This is possible only for very special Kac–Moody algebras, and such a formulation is important in order to deal with the *conventional* formulation of integrability, which usually involves *commuting* Hamiltonian flows (not just a closed algebra of flows) and the fermionic realisation of the universal module space (universal Grassmannian). In fact these restrictions are quite arbitrary and can be removed (though this has not yet been done in full detail); see Section 4 below for a more detailed discussion.

The Kac–Moody currents of level $k = 1$ GL$(r+1)$ are just $J_{jk} = :\widetilde{\psi}_j\psi_k:$ $(j, k = 1, \ldots, r+1)$, and the screening charges are given by $Q^{(\alpha)} = \mathrm{i}E^{(\alpha)}_{jk}\oint :\widetilde{\psi}_j\psi_k:$, where $E^{(\alpha)}_{jk}$ are representations of the roots $\boldsymbol{\alpha}$ in the matrix representation of GL$(r+1)$. The Cartan subalgebra is represented by J_{jj}, while positive and negative Borel subalgebras are represented by J_{jk} with $j < k$ and $j > k$, respectively. In Eqn (2.23),

$$Q_+ = \mathrm{i}\oint\widetilde{\psi}_1\psi_2, \qquad Q_- = \mathrm{i}\oint\widetilde{\psi}_2\psi_1 ;$$

while in Eqn (2.34),

$$Q^{(\alpha_1)} = \mathrm{i}\oint\widetilde{\psi}_1\psi_2, \qquad Q^{(\alpha_2)} = \mathrm{i}\oint\widetilde{\psi}_1\psi_3 ,$$

$$Q^{(\alpha_3)} = \mathrm{i}\oint\widetilde{\psi}_2\psi_3, \qquad Q^{(\alpha_4)} = \mathrm{i}\oint\widetilde{\psi}_2\psi_1 ,$$

$$Q^{(\alpha_5)} = \mathrm{i}\oint\widetilde{\psi}_3\psi_1, \qquad Q^{(\alpha_6)} = \mathrm{i}\oint\widetilde{\psi}_3\psi_2 .$$

$Q^{(\alpha_6)}$ can be substituted for $Q^{(\alpha_2)}$ in Eqn (2.34) without changing the answer. For generic r the similar choice of 'adjacent' (not simple!) roots (such that their scalar products are $+1$ or 0) leads to the selection of the following r screening operators

$$Q^{(1)} = \mathrm{i}\oint\widetilde{\psi}_1\psi_2, \quad Q^{(2)} = -\mathrm{i}\oint\psi_2\widetilde{\psi}_3, \quad Q^{(3)} = \mathrm{i}\oint\widetilde{\psi}_3\psi_4, \ldots ,$$

i.e. $Q^{(j)} = \mathrm{i}\oint\widetilde{\psi}_j\psi_{j+1}$ for odd j and $Q^{(j)} = -\mathrm{i}\oint\psi_j\widetilde{\psi}_{j+1}$ for even j.

2.4 Gross–Newman equation

I now consider the WIs for another sort of matrix model. This subject concerns at least two important classes: the conventional discrete 2-matrix models and the Kontsevich models. As was explained in the introduction, theories of the second type arise from consideration of the $(p, 1)$ continuous matrix models, as well as from the study of topological Landau–Ginzburg theories; while the 2-matrix model is believed to exhibit a rich pattern of continuous limits and is capable of providing representations of all the (p, q) universality classes (this line of reasoning, however, has never been fully developed and I shall not discuss it in these notes).

The starting point and the basic example is provided by the integral

$$\mathcal{F}_{V,n}\{L\} \equiv \int_{n\times n} \mathrm{d}X \exp[-\mathrm{tr}\, V(X) + \mathrm{tr}\, LX] \tag{2.35}$$

over $n \times n$ Hermitian matrices, which I shall further refer to as the 'Kontsevich integral', in order to keep in mind its most important application (though, this obvious quantity has, of course, been considered by many other people). It may seem that the action in this integral is *not* of the most general type and one can no longer perform an *arbitrary* change of variables $X \to \mathrm{f}(X)$, without changing the functional form of the integral. In fact this is incorrect, because the 'external field' L is matrix valued and is coupled linearly to X, and therefore *any* correlator of X fields can be represented through L derivatives. Consider again the shift $X \to X + \varepsilon_n X^{n+1}$, $n \geqslant -1$. Invariance of the integral implies

$$\int \mathrm{d}X \exp[-\mathrm{tr}\, V(X) + \mathrm{tr}\, LX]\, \mathrm{tr}\, \varepsilon_n \Big(-X^{n+1}V'(X) + LX^{n+1} + \sum_{k=0}^{n} X^k\, \mathrm{tr}\, X^{n-k}\Big) = 0 ,$$

which can be written as†

$$\mathrm{tr}\, \varepsilon_n \left[\left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{n+1} V'\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right) + L\left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{n+1} + \sum_{k=0}^{n}\left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{k} \mathrm{tr}\left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{n-k}\right]\mathcal{F}_V\{L\}$$

$$= \mathrm{tr}\, \varepsilon_n \left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{n+1}\left[V'\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right) - L\right]\mathcal{F}_V\{L\} = 0 . \tag{2.36}$$

This system is in fact equivalent to a single matrix-valued equation:

$$\left[V'\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right) - L\right]\mathcal{F}_V\{L\} = 0 . \tag{2.37}$$

As far as I know, this equation was first written down by Gross and Newman [54]; therefore, it will be referred to as the Gross–Newman (GN) equation. It was rediscovered and its implications for the theory of matrix models were investigated in [24, 30, 38].

There are essentially two types of corollary, which will be discussed in the next two subsections. First, the GN equation can be used to characterise the function $\mathcal{F}_V\{L\}$ itself. This will lead to the consideration of Kontsevich models. Second, it can be used to derive equations for the 2-matrix model, which arises after $\mathcal{F}_V\{L\}$ is further integrated with some weight over L.

2.5 Ward identities for the generalised Kontsevich model

Being just the complete set of EqMs, the GN equation (2.37) provides complete information about the function $\mathcal{F}_V\{L\}$. However, this statement needs to be formulated more carefully. A reason for this comes, for example, from the observation that the operators

$$\mathrm{tr}\, L^m\left[V'\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right) - L\right] \tag{2.38}$$

do not form a closed algebra: their commutators have some different functional form. One of the reasons for these complications is that Eqn (2.37) does not account explicitly for a very important property of $\mathcal{F}_V\{L\}$: that this function in fact depends only on the eigenvalues of L. This information should be added somehow to the GN equation. I shall analyse this issue of eigenvalue dependence in more detail in the next sections. For my current purposes this argument implies that one should try to express Eqn (2.37) in terms of eigenvalues. Here, however, one should be careful again. Clearly, $\mathcal{F}_V\{L\}$ depends not only on eigenvalues, it depends also on their 'symmetric' (Weyl-group invariant) combinations, i.e. it depends more on quantities like $\mathrm{tr}\, L^a$ than on particular eigenvalues. Moreover, powers a here should be negative and fractional.

Indeed, integrals like that in Eqn (2.35) are usually understood as the analytical continuation from some values of parameters in the potential V, when the integral is

† The obvious relation is used here: $X_{\gamma\delta}\exp(\mathrm{tr}LX) = (\partial/\partial L_{\delta\gamma})\exp(\mathrm{tr}LX)$. Note that the order of the matrix indices $\gamma\delta$ is reversed on the r.h.s. as compared to that on the left hand side (l.h.s), i.e. derivatives are in fact taken with respect to the *transposed* matrix L: $\mathrm{f}(X)\exp(\mathrm{tr}LX) = f(\partial/\partial l_{\mathrm{tr}})\exp(\mathrm{tr}\, LX)$ [at least for any function $\mathrm{f}(x)$ which can be represented as a formal series in integer powers of X].

convergent. They can also be related to the formal (perturbation) series arising when the integrand is expanded around a stationary point. To begin with, it is reasonable to take $n = 1$ i.e. to consider just an ordinary integral. For the sake of simplicity, one may also take $V(x)$ to be given by $V(x) = -x^{p+1}/(p+1)$. Then, the stationary point is at $x = \lambda^{1/p}$ and

$$\int \mathrm{d}x \exp\left(-\frac{x^{p+1}}{p+1} + lx\right)$$

$$\sim l^{-\frac{1}{2}(p-1)} \exp\left[\frac{p}{p+1} l^{(p+1)/p}\right] \sum_{k \geqslant 0} c_k l^{-k/p} . \quad (2.39)$$

It is now easy to understand what should be done in the general situation with matrices and arbitrary potentials. First of all, one needs to solve the equation for the stationary point: $V'(X) = L$. For this purpose it is most convenient to introduce a new matrix variable Λ instead of L, which by definition satisfies $V'(\Lambda) = L$. Then, the stationary point is just $X = \Lambda$. Second, one should separate the analogue of the complicated prefactor (quasiclassical contribution) $\mathcal{C}_V\{\Lambda\}$, where

$$\mathcal{C}_V\{\Lambda\} = (2\pi)^{n^2/2} \frac{\exp\{\mathrm{tr}[\Lambda V'(\Lambda) - V(\Lambda)]\}}{\sqrt{\det V''(\Lambda)}} . \quad (2.40)$$

Then, the function that describes the pure 'quantum' contribution,†

$$\mathcal{Z}_V\{T\} \equiv \mathcal{C}_V\{\Lambda\}^{-1} \mathcal{F}_V\{V'(\Lambda)\} , \quad (2.41)$$

to be referred to as the partition function of the generalised Kontsevich model (GKM) [30], can be represented as a formal (perturbation) series expansion in the variable, where

$$T_k = \frac{1}{k} \mathrm{tr} \Lambda^{-k} . \quad (2.42)$$

The GN equations (2.37) can be now rewritten as a set of differential Eqns for $\mathcal{Z}_V\{T\}$. Indeed, we already have

$$\mathcal{C}_V^{-1}\left[V'\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right) - L\right] \mathcal{C}_V \mathcal{Z}_V\{T\} = 0 , \quad (2.43)$$

but it is still necessary to express the operator on the l.h.s. in terms of T. This is in fact possible by means of the relation

$$\frac{\partial}{\partial L_{\mathrm{tr}}} \mathcal{Z}_V\{T\} = \sum_k \frac{\partial T_k}{\partial L_{\mathrm{tr}}} \frac{\partial Z}{\partial T_k} , \quad (2.44)$$

and substituting the *traces* of the Λ matrices, which can arise in the process of calculation, by Ts. It is important only that the Λs usually appear in negative powers: this is achieved by the choice of a proper normalisation factor $\mathcal{C}_V\{\Lambda\}$. For the monomial potential

$$V_p(X) = \frac{X^{p+1}}{p+1}$$

this is especially simple:

$$L = \Lambda^p \quad \text{and} \quad \frac{\partial T_k}{\partial L_{\mathrm{tr}}} = -\frac{1}{p} \Lambda^{-p-k} .$$

This reasoning allows one to rewrite Eqn (2.43) identically in the form

$$\sum_l \Lambda^{-l} \mathcal{O}_l(T) \mathcal{Z}_V\{T\} = 0 , \quad (2.45)$$

where the $\mathcal{O}_l$ are differential operators that are dependent on the shape of V, but are independent of the size, n, of the matrix (because none of the above reasoning referred to particular values of n, except for an *example* at the very beginning). It remains to use the fact that matrix L can be arbitrarily large and have arbitrarily many independent entries, in order to deduce a set of constraints on $\mathcal{Z}_V$ in the form

$$\mathcal{O}_l(T) \mathcal{Z}_V\{T\} = 0 . \quad (2.46)$$

For a potential V of degree $p + 1$ these appear to be exactly the 'continuous Virasoro constraints'. See [24, 30] for a detailed analysis of the Virasoro case $p = 2$ (associated with pure topological gravity and with the double-scaling limit of the 1-matrix model), and [55] for an exhaustive presentation of the case of $p = 3$.

2.6 Discrete Virasoro constraints for the gaussian Kontsevich model

As a simple illustration of the technique described in the previous subsection, I now derive the constraints for the gaussian Kontsevich model [56] with potential $V(X) = \frac{1}{2}X^2$:

$$\mathcal{Z}_{\frac{1}{2}X^2}\{N, T\} = \frac{\exp(-\mathrm{tr}\, \frac{1}{2}L^2)}{(\det L)^N} \times \int \mathrm{d}X (\det X)^N \exp(-\mathrm{tr}\, \tfrac{1}{2}X^2 + LX) . \quad (2.47)$$

In this case, $L = V'(\Lambda) = \Lambda$, and the time-variables are just

$$T_k = \frac{1}{k} \mathrm{tr}\, \Lambda^{-k} = \frac{1}{k} \mathrm{tr}\, L^{-k} . \quad (2.48)$$

To make the model nontrivial an extra 'zero-time' variable [36], N, is introduced, which was not included in the previous definition (2.41). Now note that the N dependence of the Kontsevich integral (2.35) can be described simply as an extra term in the potential: $V(X) \to \hat{V}(X) = V(X) - N \ln X$ (though, this can be done neither in the quasiclassical factor $\mathcal{C}_V$ nor in the definition of the time variables T). Since the GN equation depends only on the Kontsevich equation, one can use it with V substituted by $\hat{V}$. Then one has the following instead of Eqn (2.43):

$$\frac{\exp(-\mathrm{tr}\, \frac{1}{2}L^2)}{(\det L)^N} \left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{n+1} \left[\frac{\partial}{\partial L_{\mathrm{tr}}} - N\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{-1} - L\right] (\det L)^N \times \exp(\mathrm{tr}\, \tfrac{1}{2}L^2) \mathcal{Z}'_{\frac{1}{2}X^2}\{N, T\} = 0 . \quad (2.49)$$

† The 'classical action' in Eqn (2.40) can also be represented as $\mathrm{tr}\,[\Lambda V'(\Lambda) - V(\Lambda)] = \mathrm{tr} \int \Lambda\, \mathrm{d}V'(\Lambda)$. The determinant of the quadratic fluctuations is defined as

$$(2\pi)^{n^3/2} [\det V''(\Lambda)]^{-1/2} \sim \int \mathrm{d}Y \exp[-\mathrm{tr}\, V_2(\Lambda, Y)] ,$$

where $V_2(\Lambda, Y) \equiv \lim_{\varepsilon \to 0} [V(\Lambda + \varepsilon Y) - V(\Lambda) - \varepsilon V'(\Lambda) Y]/\varepsilon^2$. For $V(\Lambda) = \Lambda^{p+1}/(p+1)$ we have $V''(\Lambda) = \sum_{k=0}^{p-1} \Lambda^k \otimes \Lambda^{p-k-1}$. One could easily choose an 'opposite' parametrisation in Eqn (2.42): $T_k = -(1/k)\, \mathrm{tr}\, \Lambda^{-k}$. Though not quite obvious, this never influences any results (see Section 2.10 for an example). The choice of signs is motivated by a simplification of formulas for the GKM including the relations between L and Λ. Instead, some sign factors appear in the formulas, and are related to Toda-like representations of partition functions and those involving $\widetilde{W}$-operators.

In order to get rid of the integral operator $(\partial/\partial L)^{-1}$ one should take here $n \geqslant 0$ rather than $n \geqslant -1$. In fact all the equations with $n > 0$ follow from the one with $n = 0$, and I restrict consideration to the last one. For $n = 0$ one obtains from Eqn (2.49):

$$\left[\left(\frac{\partial}{\partial L_{\mathrm{tr}}}+\frac{N}{L}+L\right)^2-2N-L\left(\frac{\partial}{\partial L_{\mathrm{tr}}}+\frac{N}{L}+L\right)\right]\mathcal{Z}=0\,,$$

or

$$\left[\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^2+\left(L+\frac{2N}{L}\right)\frac{\partial}{\partial L_{\mathrm{tr}}}+\frac{N^2}{L^2}-\frac{N}{L}\,\mathrm{tr}\,\frac{1}{L}\right]\mathcal{Z}=0\,, \tag{2.50}$$

and it remains to substitute

$$\frac{\partial\mathcal{Z}}{\partial L_{\mathrm{tr}}}=-\sum_{k=0}^{\infty}\frac{1}{L^{k+1}}\frac{\partial\mathcal{Z}}{\partial T_k}$$

and

$$\begin{aligned}\frac{\partial^2\mathcal{Z}}{\partial L_{\mathrm{tr}}^2}&=\sum_{k=1}^{\infty}\left(\sum_{a=1}^{k+1}\frac{1}{L^{k+2-a}}\,\mathrm{tr}\,\frac{1}{L^a}\right)\frac{\partial\mathcal{Z}}{\partial T_k}+\sum_{k,l=1}^{\infty}\frac{1}{L^{k+l+2}}\frac{\partial^2\mathcal{Z}}{\partial T_k\partial T_l}\\&=\sum_{m=-1}^{\infty}\frac{1}{L^{m+2}}\left[\sum_{k>\max(m,0)}\left(\mathrm{tr}\,\frac{1}{L^{k-m}}\right)\frac{\partial\mathcal{Z}}{\partial T_k}\right.\\&\qquad\left.+\sum_{k=1}^{m-1}\frac{\partial^2\mathcal{Z}}{\partial T_k\partial T_{m-k}}\right]\,,\end{aligned}$$

and finally obtain

$$\begin{aligned}&\sum_{m=-1}^{\infty}\frac{1}{L^{m+2}}\left[\sum_{k=1+\delta_{m,-1}}^{\infty}\left(\mathrm{tr}\,\frac{1}{L^k}\right)\frac{\partial}{\partial T_{k+m}}+\sum_{k=1}^{m-1}\frac{\partial^2}{\partial T_k\partial T_{m+k}}\right.\\&\qquad\left.-\frac{\partial}{\partial T_{m+2}}-2N\frac{\partial}{\partial T_m}+N^2\delta_{m,0}-N\left(\mathrm{tr}\,\frac{1}{L}\right)\delta_{m,-1}\right]\mathcal{Z}\\&=\sum_{m=-1}^{\infty}\frac{1}{L^{m+2}}\exp(NT_0)L_m(T+r)\exp(-NT_0)\mathcal{Z}=0\,.\end{aligned} \tag{2.51}$$

Here, $L_m(t)$ are just the generators (2.7) of the discrete Virasoro algebra (2.6):

$$\begin{aligned}&\exp(Nt_0)L_m(t)\exp(-Nt_0)\\&\quad=\exp(Nt_0)\left(\sum_{k=1}^{\infty}kt_k\frac{\partial}{\partial t_{k+m}}+\sum_{k=0}^{m}\frac{\partial^2}{\partial t_k\partial t_{m-k}}\right)\exp(-Nt_0)\,,\end{aligned} \tag{2.52}$$

and, on the r.h.s. of Eqn (2.51), $r_k = -\frac{1}{2}\delta_{k,2}$.†

Thus, it was found that the WIs of the gaussian Kontsevich model (2.47) coincide with those of the ordinary 1-matrix model; moreover, the size of the matrix N in the latter model is associated with the 'zero time' in the former one. This result [56], of course, implies that the two models are identical:

$$\exp(-NT_0)\mathcal{Z}_{\frac{1}{2}X^2}\{N,T_1,T_2,\ldots\}\sim Z_N\{T_0,T_1,T_2,\ldots\}\,. \tag{2.53}$$

I shall discuss the direct connection between the two matrix integrals (2.1) and (2.47) in the next section, after some more details have been presented about the structure of 'eigenvalue' matrix models.

2.7 Continuous Virasoro constraints for the $V = \frac{1}{3}X^3$ Kontsevich model

This example is a little more complicated than that in the previous subsection, and I do not present the calculations in full detail (see [24, 30]). My goal is to demonstrate that the constraints which arise in this model, though they still form (a Borel subalgebra of) some Virasoro algebra, are *different* from Eqn (2.6). From the point of view of the CFT-formulation the relevant model is that of the *twisted* (in this particular case, antiperiodic) free field. These so called 'continuous Virasoro constraints' give the simplest illustration of the difference between discrete and continuous matrix models: this is essentially the difference between 'homogeneous' (Frenkel–Kac) and 'principal' (soliton vertex operator) representation of the level $k = 1$ Kac–Moody algebra. In terms of integrable hierarchies, this is the difference between Toda-chain-like and KP-like hierarchies. I shall come back to a more detailed discussion of this difference later, when the 'multiscaling continuum limit' will be considered.

Another (historical) aspect of the same relation also deserves mentioning, since it also illustrates the interrelation between different models. The discrete 1-matrix model arises naturally in the description of quantum 2d gravity as the sum over 2-geometries in the formalism of random equilateral triangulations. The model, however, describes only lattice approximation to 2d gravity and the (double-scaling) continuum limit should be taken in order to obtain the real (continuous) theory of 2d gravity. This limit was originally formulated in terms of the constraint algebra (EqMs, or 'loop' or 'Schwinger–Dyson' equations — the terminology is dependent on taste), leaving open the problem of what the form of the partition function, $\mathcal{Z}^{\mathrm{cont}}\{T\}$, of the continuous theory, is. Since the relevant algebra appeared to be just the WIs for the Kontsevich model [with $V(X) = \frac{1}{3}X^3$], this proves that the latter one is exactly the continuous theory of pure 2d gravity. At the same time, the Kontsevich model itself can be naturally introduced as a theory of *topological* gravity (in fact this is how the model was originally discovered in [22]). From this point of view the constraint algebra, to be discussed below, plays a central role in the proof of the equivalence between pure 2d quantum gravity and pure topological gravity (in both cases 'pure' means that 'matter' fields are not introduced).

After these introductory remarks, I now proceed to the calculations. Actually they just repeat those for the gaussian model, performed in the previous subsection, but the formulas get somewhat more complicated. This time I do not include zero-time N and just use Eqn (2.37) with $V(X) = \frac{1}{3}X^3$. Now it is also much more tricky (though possible) to work in matrix notation (because fractional powers of L will be involved) and I rewrite everything in terms of the eigenvalues of L.

† This small correction is a manifestation of a very general phenomenon: in terms of symmetries (WIs) it is more natural to consider Z_V not as a function of T-variables, but of some more complicated combination, $\hat{T}_k + r_k$, depending on the shape of potential V. If V is a polynomial of degree $p+1$, then $\hat{T}_k = \mathrm{tr}\,[V'(\lambda)]^{-k/p}/k$, while r_k is given by

$$r_k = \frac{p}{k(p-k)}\,\mathrm{Res}\,[V'(\mu)]^{1-k/p}\,\mathrm{d}\mu\,.$$

For monomial potentials these expressions become very simple: $\hat{T}_k = T_k$ and $r_k = -[p/(p+1)]\delta_{k,p+1}$. See [39] and Section 4.9 below for more details. In most places in these notes I prefer to use invariant potential-independent times T_k instead of $\hat{T}_k$, but then the WIs acquire some extra terms with r_k (in fact, these terms will be very simple in my examples, which are all given for monomial potentials).

The following substitutions are made:

$$\mathcal{C}_{\frac{1}{3}X^3} = \frac{\prod_\delta \exp\left(\frac{2}{3}\lambda_\delta^{3/2}\right)}{\sqrt{\prod_{\gamma,\delta}\left(\sqrt{\lambda_\delta}+\sqrt{\lambda_\gamma}\right)}} ,$$

$$\left(\frac{\partial^2}{\partial L_{\mathrm{tr}}^2}\right)_{\gamma\gamma} = \frac{\partial^2}{\partial\lambda_\gamma^2} + \sum_{\delta\neq\gamma}\frac{1}{\lambda_\gamma-\lambda_\delta}\left(\frac{\partial}{\partial\lambda_\gamma}-\frac{\partial}{\partial\lambda_\delta}\right) ,$$

introducing the notation

$$\frac{\mathcal{D}}{\mathcal{D}\lambda_\gamma} \equiv \mathcal{C}_{\frac{1}{3}X^3}^{-1}\frac{\partial}{\partial\lambda_\gamma}\mathcal{C}_{\frac{1}{3}X^3}$$
$$= \frac{\partial}{\partial\lambda_\gamma} + \sqrt{\lambda_\gamma} - \frac{1}{4\lambda_\gamma} - \frac{1}{2}\sum_{\delta\neq\gamma}\frac{1}{\sqrt{\lambda_\gamma}\left(\sqrt{\lambda_\delta}+\sqrt{\lambda_\gamma}\right)} .$$

Then, Eqn (2.37) becomes

$$\left[\left(\frac{\mathcal{D}}{\mathcal{D}\lambda_\gamma}\right)^2 + \sum_{\delta\neq\gamma}\frac{1}{\lambda_\gamma-\lambda_\delta}\left(\frac{\mathcal{D}}{\mathcal{D}\lambda_\gamma}-\frac{\mathcal{D}}{\mathcal{D}\lambda_\delta}\right)\right]\mathcal{Z}_{\frac{1}{3}X^3}\{T\} = 0 . \tag{2.54}$$

Now an explicit expression for T is needed:

$$T_k = \frac{1}{k}L^{-k} , \tag{2.55}$$

and, as is already known from the previous subsections,

$$r_k = -\frac{2}{3}\delta_{k,3} . \tag{2.56}$$

is also needed. Although the fact will not be explained until I turn to consider the integrable structure of the Kontsevich model in the following sections, $\mathcal{Z}_{\frac{1}{3}X^3}\{T\}$ is independent of all time-variables with *even*-numbered subscripts. Therefore, one can take only $k = 2a+1$ in Eqn (2.55):

$$T_{2a+1} = \frac{1}{2a+1}\sum_\delta \lambda_\delta^{-a-\frac{1}{2}} , \qquad r_{2a+1} = -\frac{2}{3}\delta_{a,1} , \tag{2.57}$$

and

$$\frac{\partial}{\partial\lambda_\gamma}\mathcal{Z}_{\frac{1}{3}X^3}\{T\}$$
$$= \sum_{a=0}^{\infty}\frac{\partial T_{2a+1}}{\partial\lambda_\gamma}\frac{\partial\mathcal{Z}}{\partial T_{2a+1}} = -\frac{1}{2}\sum_{a=0}^{\infty}\lambda_\gamma^{-a-\frac{3}{2}}\frac{\partial\mathcal{Z}}{\partial T_{2a+1}} ,$$

$$\frac{\partial^2}{\partial\lambda_\gamma^2}\mathcal{Z}_{\frac{1}{3}X^3}\{T\} = \frac{1}{4}\sum_{a,b=0}^{\infty}\lambda_\gamma^{-a-b-3}\frac{\partial^2\mathcal{Z}}{\partial T_{2a+1}\partial T_{2b+1}}$$
$$+\frac{1}{2}\sum_{a=0}^{\infty}\left(a+\frac{3}{2}\right)\lambda_\gamma^{-a-\frac{5}{2}}\frac{\partial\mathcal{Z}}{\partial T_{2a+1}} .$$

When these expressions are substituted into Eqn (2.54), one obtains

$$\frac{1}{4}\sum_{a,b=0}^{\infty}\lambda_\gamma^{-a-b-3}\frac{\partial^2\mathcal{Z}}{\partial T_{2a+1}\partial T_{2b+1}}$$
$$+\sum_{a=0}^{\infty}\left[\frac{1}{2}\sum_{a=0}^{\infty}\left(a+\frac{3}{2}\right)\lambda_\gamma^{-a-\frac{5}{2}}\right.$$
$$-\frac{1}{2}\sum_{\delta\neq\gamma}\frac{1}{\lambda_\gamma-\lambda_\delta}\left(\lambda_\gamma^{-a-\frac{3}{2}}-\lambda_\delta^{-a-\frac{3}{2}}\right)$$
$$\left.-\left(\sqrt{\lambda_\gamma}-\frac{1}{4\lambda_\gamma}-\frac{1}{2}\sum_{\delta\neq\gamma}\frac{1}{\sqrt{\lambda_\gamma}\left(\sqrt{\lambda_\delta}+\sqrt{\lambda_\gamma}\right)}\right)\lambda_\gamma^{-a-\frac{3}{2}}\right]\frac{\partial\mathcal{Z}}{\partial T_{2a+1}}$$
$$+[\ldots]\,\mathcal{Z} = \sum_{n=-1}^{\infty}\frac{1}{\lambda_\gamma^{n+2}}\mathcal{L}_n\mathcal{Z} , \tag{2.58}$$

with

$$\mathcal{L}_{2n} = \sum_{n=0}^{\infty}\left(a+\frac{1}{2}\right)(T_{2a+1}+r_{2a+1}) + \frac{\partial}{\partial T_{2a+2n+1}}$$
$$+\frac{1}{4}\sum_{\substack{a+b=n-1\\ a,\,b\geqslant 0}}\frac{\partial^2}{\partial T_{2a+1}\partial T_{2b+1}} + \frac{1}{16}\delta_{n,0} + \frac{1}{4}T_1^2\delta_{n,-1}$$
$$= \frac{1}{2}\sum_{\mathrm{odd}\,k=1}^{\infty}k(T_k+r_k)\frac{\partial}{\partial T_{k+2n}} + \frac{1}{4}\sum_{\mathrm{odd}\,k=1}^{2n-1}\frac{\partial^2}{\partial T_k\partial T_{2n-k}}$$
$$+\frac{1}{16}\delta_{n,0} + \frac{1}{4}T_1^2\delta_{n,-1} . \tag{2.59}$$

The factor of $\frac{1}{2}$ in front of the first term at the r.h.s. of Eqn (2.59) is important for $\mathcal{L}_{2n}$ to satisfy the properly normalised Virasoro algebra:†

$$[\mathcal{L}_{2n},\mathcal{L}_{2m}] = (n-m)\mathcal{L}_{2n+2m} .$$

The coefficient $\frac{1}{4}$ in front of the second term in Eqn (2.59) can be eliminated by rescaling the time variables: $T \to \frac{1}{2}T$. Then, the last term turns into $\frac{1}{16}T_1^2\delta_{n,-1}$.

I shall not actually discuss evaluation of the coefficient in front of $\mathcal{Z}$ (with no derivatives), which is denoted by [...] in Eqn (2.58) (see [24, 30]). In fact, almost all the terms in the original complicated expression cancel, giving finally

$$[\ldots] = \frac{1}{16\lambda_\gamma^2} + \frac{T_1^2}{4\lambda_\gamma} ,$$

and this is represented by the terms with $\delta_{n,0}$ and $\delta_{n,-1}$ in expressions (2.59) for the Virasoro generators $\mathcal{L}_{2n}$.

The term with the double T-derivative in Eqn (2.58) is already of the necessary form. Of intermediate complexity is the evaluation of the coefficient in front of $\partial\mathcal{Z}/\partial T_{2a+1}$ in Eqn (2.58), which I shall briefly describe now. First of all, rewrite this coefficient, reordering the items, as follows:

$$\frac{1}{2}\left[\left(a+\frac{3}{2}\right)\lambda_\gamma^{-a-\frac{5}{2}} - \sum_{\delta\neq\gamma}\frac{1}{\lambda_\gamma-\lambda_\delta}\left(\lambda_\gamma^{-a-\frac{3}{2}}-\lambda_\delta^{-a-\frac{3}{2}}\right)\right]$$
$$+\left[\frac{1}{4}\lambda_\gamma^{-a-\frac{5}{2}} + \frac{1}{2}\sum_{\delta\neq\gamma}\frac{\lambda_\gamma^{-a-2}}{\sqrt{\lambda_\delta}+\sqrt{\lambda_\gamma}}\right] - \lambda_\gamma^{-a-1} . \tag{2.60}$$

The first two terms together are equal to the sum over *all* j (including $j = i$):

$$-\frac{1}{2}\sum_\delta\frac{1}{\lambda_\gamma-\lambda_\delta}\left(\lambda_\gamma^{-a-\frac{3}{2}}-\lambda_\delta^{-a-\frac{3}{2}}\right)$$
$$= \frac{1}{2}\sum_\delta\frac{\lambda_\gamma^{a+\frac{3}{2}}-\lambda_\delta^{a+\frac{3}{2}}}{\lambda_\gamma-\lambda_\delta}\frac{1}{\lambda_\gamma^{a+\frac{3}{2}}\lambda_\delta^{a+\frac{3}{2}}}$$
$$= \frac{1}{2\lambda_\gamma^{a+2}}\sum_\delta\frac{\lambda_\gamma^{a+2}-\lambda_\gamma^{\frac{1}{2}}\lambda_\delta^{a+\frac{3}{2}}}{\lambda_\gamma-\lambda_\delta}\frac{1}{\lambda_\delta^{a+\frac{3}{2}}} .$$

† Therefore, it could be reasonable to use a different notation: $\mathcal{L}_n$ instead of $\mathcal{L}_{2n}$. I prefer $\mathcal{L}_{2n}$, because it emphasises the property of the model to be a 2-reduction of the KP hierarchy (to KdV); see Section 4 below.

Similarly, the next two terms can be rewritten as

$$\frac{1}{2}\sum_\delta \frac{\lambda_\gamma^{-a-2}}{\sqrt{\lambda_\gamma}+\sqrt{\lambda_\delta}} = \frac{1}{2\lambda_\gamma^{a+2}}\sum_\delta \frac{\sqrt{\lambda_\gamma}-\sqrt{\lambda_\delta}}{\lambda_\gamma-\lambda_\delta}$$
$$= \frac{1}{2\lambda_\gamma^{a+2}}\sum_\delta \frac{\lambda_\gamma^{\frac{1}{2}}\lambda_\delta^{a+\frac{3}{2}}-\lambda_\delta^{a+2}}{\lambda_\gamma-\lambda_\delta}\frac{1}{\lambda_\delta^{a+\frac{3}{2}}} .$$

The sum of these two expressions is equal to

$$\frac{1}{2\lambda_\gamma^{a+2}}\sum_\delta \frac{\lambda_\gamma^{a+2}-\lambda_\delta^{a+2}}{\lambda_\gamma-\lambda_\delta}\frac{1}{\lambda_\delta^{a+\frac{3}{2}}} .$$

Note that the powers $a+2$ are already integers and the remaining ratio can be represented as a sum of $a+2$ terms. Adding also the last term of the coefficient (2.60), one finally obtains

$$-\frac{1}{\lambda_\gamma^{a+1}}+\frac{1}{2}\sum_{n=-1}^{a}\frac{1}{\lambda_\gamma^{n+2}}\sum_\delta\frac{1}{\lambda_\delta^{a-n+\frac{1}{2}}}$$
$$=\frac{1}{2}\sum_{n=-1}^{a}\frac{1}{\lambda_\gamma^{n+2}}(2a-2n+1)(T+r)_{2a-2n+1} ,$$

in accordance with Eqns (2.58) and (2.59).

2.8 $\widetilde{W}$-constraints for the asymmetric 2-matrix model

I turn now to a very different application [38] of the GN equation (2.37). Namely, I shall now consider $\mathcal{F}_{V,n}\{L\}$ as a building block in the construction of the conventional discrete 2-matrix model:

$$Z_N\{t,\bar{t}\}$$
$$\equiv c_N^2\int \mathrm{d}\bar{H}\,\mathrm{d}H\exp\left[\sum_k\left(t_k\mathrm{Tr}\,H^k+\bar{t}_k\mathrm{Tr}\,\hat{H}^k\right)+\mathrm{Tr}\,H\bar{H}\right]$$
$$=\int \mathrm{d}L\exp\left(\sum_k t_k\mathrm{Tr}\,L^k\right)\mathcal{F}_{\bar{U},N}\{L\} . \tag{2.61}$$

Now L plays the role of H, and $\bar{U}(\bar{H}) = \sum_k \bar{t}_k\bar{H}^k$.

The GN equation may also be used to derive a relation for $Z_N\{t,\bar{t}\}$. Take Eqn (2.37):

$$\left(\bar{U}\frac{\partial}{\partial L_{\mathrm{tr}}}+L\right)\mathcal{F}_{\bar{U},N}\{L\}=0 , \tag{2.62}$$

multiply by $\exp[\mathrm{Tr}U(L)]$ [which is equal to $\exp(\sum_k t_k\mathrm{Tr}L^k)$] and integrate over L. In order to express this relation in terms of t-derivatives of z it is necessary to have scalar rather than *matrix* equations; therefore, it will be necessary to take the trace of Eqn (2.62). However, in order not to lose any information, one must first multiply Eqn (2.62) by L^n and *then* take the trace. In this way one obtains

$$\int \mathrm{d}L\exp\left(\sum_k t_k\mathrm{Tr}\,L^k\right)\mathrm{Tr}\,L^n\left(\bar{U}\frac{\partial}{\partial L_{\mathrm{tr}}}+L\right)\mathcal{F}_{\bar{U}}\{L\}=0 .$$

Integration by parts gives

$$\int \mathrm{d}L\mathcal{F}_{\bar{U}}\{L\}\,\mathrm{Tr}\left[\bar{U}\left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)+L\right]L^n\exp\sum_k t_k\mathrm{Tr}\,L^k . \tag{2.63}$$

Now it is necessary to introduce a new class of operators [38]. Consider the action of $\mathrm{Tr}\,[(\partial^m/\partial L^m_{\mathrm{tr}})L^n]$ on $\exp[\mathrm{Tr}\,U(L)]$. It gives a linear combination of terms like

$$\mathrm{tr}\,L^{a_1}\ldots\mathrm{tr}\,L^{a_l}\exp[\mathrm{tr}\,U(L)]=\frac{\partial^l}{\partial t_{a_1}\ldots\partial t_{a_l}}\exp[-\mathrm{tr}\,U(L)] ,$$

i.e. one obtains a combination of differential operators with t-derivatives, to be denoted $\widetilde{W}(t)$:

$$\widetilde{W}^{(m+1)}_{n-m}(t)\exp[\mathrm{tr}\,U(L)]\equiv\mathrm{Tr}\frac{\partial^m}{\partial L^m_{\mathrm{tr}}}L^n\exp[\mathrm{tr}\,U(L)] , \quad m,n\geqslant 0 . \tag{2.64}$$

For example,

$$\widetilde{W}^{(1)}_n=\frac{\partial}{\partial t_n} , \quad n\geqslant 0 ;$$

$$\widetilde{W}^{(2)}_n=\sum_{k=0}^{\infty}kt_k\frac{\partial}{\partial t_{k+n}}+\sum_{k=0}^{n}\frac{\partial^2}{\partial t_k\partial t_{n-k}} , \quad n\geqslant -1 ;$$

and

$$\widetilde{W}^{(3)}_n=\sum_{k,l=1}^{\infty}kt_klt_l\frac{\partial}{\partial t_{k+l+n}}+\sum_{k=1}^{\infty}kt_k\sum_{a+b=k+n}\frac{\partial^2}{\partial t_a\partial t_b}$$
$$+\sum_{k=l}^{\infty}kt_k\sum_{a+b=n+1}\frac{\partial^2}{\partial t_a\partial t_{b+k-1}}$$
$$+\sum_{a+b+c=n}\frac{\partial^3}{\partial t_a\partial t_b\partial t_c}+\frac{(n+1)(n+2)}{2}\frac{\partial}{\partial t_n} ; \tag{2.65}$$

Note that while $\widetilde{W}^{(1)}_n$ and $\widetilde{W}^{(2)}_n$ are just the ordinary U(1) Kac–Moody and Virasoro operators respectively, the higher $\widetilde{W}^{(m)}$-operators do *not* coincide with the generators of the **W**-algebras: intact

$$\widetilde{W}^{(3)}_n\neq W^{(3)}=\sum_{k,l=1}^{\infty}kt_klt_l\frac{\partial}{\partial t_{k+l+n}}$$
$$+2\sum_{k=1}^{\infty}kt_k\sum_{a+b=k+n}\frac{\partial^2}{\partial t_a\partial t_b}+\frac{4}{3}\sum_{a+b+c=n}\frac{\partial^3}{\partial t_a\partial t_b\partial t_c} .$$

The $\widetilde{W}$-operators (in contrast with ordinary W-operators) satisfy the recurrence relation

$$\widetilde{W}^{(m+1)}_n=\sum_{k=1}^{\infty}kt_k\widetilde{W}^{(m)}_{n+k}+\sum_{k=0}^{m+n-1}\frac{\partial}{\partial t_k}\widetilde{W}^{(m)}_{n-k} , \quad n\geqslant -m . \tag{2.66}$$

Actually, not much is yet known about the $\widetilde{W}$-operators and the structure of $\widetilde{\mathbf{W}}$-algebras (in particular it remains unclear whether the negative harmonics $\widetilde{W}^{(m+1)}_n$ with $n<-m$ can be introduced in any reasonable way), see [38] for some preliminary results.

Eqn (2.63) can now be presented in terms of the $\widetilde{W}$-operators as follows:

$$\int \mathrm{d}L\mathcal{F}_{\bar{U}}\{L\}\left[\sum_{k\geqslant 1}k\bar{t}_k\left(-\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^{k-1}+L\right]L^n\exp[\mathrm{Tr}\,U(t)]$$
$$=\left[\sum_{k\geqslant 1}(-1)^{k-1}k\bar{t}_k\widetilde{W}^{(k)}_{n+1-k}+\widetilde{W}^{(1)}_{n+1}\right]Z_N\{t,\bar{t}\}=0 . \tag{2.67}$$

This relation is highly asymmetric in t and $\bar{t}$, and in fact it provides a suitable description of the WI only in the somewhat peculiar case when the potential $\bar{U}(\bar{H})$ is a polynomial of *finite* degree. See [57, 38] for a discussion of such asymmetric models.

2.9 $\widetilde{W}$-constraints for the generic 2-matrix model

When both potentials U and $\bar{U}$ in Eqn (2.61) are generic formal series, Eqns (2.67) represent only a 1-parameter subset of the 2-parameter family of WIs. Before I describe the whole set, let me emphasise that the 2-matrix model (2.61) is the one (where the action is *not* of the most general form) consistent with some symmetry. Therefore, it is not covariant under the arbitrary change of variables $H, \bar{H} \to \mathrm{f}(H, \bar{H})$, $\bar{\mathrm{f}}(H, \bar{H})$, and the usual method of deriving the WIs does not work. The reason why the generic 2-matrix model with action containing all the possible combinations $\mathrm{Tr}\,(H^{a_1}\bar{H}^{b_1}H^{a_2}\bar{H}^{b_2}\ldots)$ is never considered seriously is essentially our poor understanding of the 1-matrix integrals for 'noneigenvalue' theories, to which class such a generic model belongs. For reasons to be explained in the next section, such problems do not arise for models of the form (2.61) or (2.33), and this is why they have attracted most attention. Hopefully the problems with the unitary-matrix integrals are temporary and this restricted class of multimatrix models will be enlarged; this should be especially easy to do in the part of the theory dealing with constraint algebras, but this subject is beyond the scope of the present notes.

In order to derive the complete set of WIs for the model (2.61), I apply the following semi-artificial trick. Note that the exponential $\exp(\mathrm{Tr}\,H\bar{H})$ satisfies

$$\left(\mathrm{Tr}\,H^n\frac{\partial^m}{\partial H^m_{\mathrm{tr}}} - \mathrm{Tr}\,\bar{H}^m\frac{\partial^n}{\partial \bar{H}^n_{\mathrm{tr}}}\right)\exp(\mathrm{Tr}\,H\bar{H}) = 0\,. \quad (2.68)$$

If one integrates this identity over H and $\bar{H}$ with the weight $\exp[\mathrm{Tr}\,U(H) + \mathrm{Tr}\,\bar{U}(\bar{H})]$ and then integrates by parts, one obtains an identity:

$$\int \mathrm{d}H\,\mathrm{d}\bar{H}\,\exp(\mathrm{Tr}\,H\bar{H})\left[\mathrm{Tr}\left(\frac{\partial}{\partial H_{\mathrm{tr}}}\right)^m H^n - \mathrm{Tr}\left(-\frac{\partial}{\partial \bar{H}_{\mathrm{tr}}}\right)^n \bar{H}^m\right]$$
$$\times\exp[\mathrm{Tr}\,U(H) + \mathrm{Tr}\,\bar{U}(\bar{H})] = 0\,, \quad (2.69)$$

which can be represented in terms of $\widetilde{W}$ operators [30]:†

$$\widetilde{W}^{(m+1)}_{n-m}(t)Z\{t,\bar{t}\} = (-1)^{m-n}\widetilde{W}^{(n+1)}_{m-n}(\bar{t})Z\{t,\bar{t}\}\,, \quad (2.70)$$

for all $m, n \geqslant 0$.

This is the complete (?) set of WIs for the 2-matrix model. When one of the potentials [say, $U(t)$] is a polynomial of finite degree, most of this symmetry is 'spontaneously broken', the surviving part being described by Eqn (2.67).

† Relations (2.68), and thus Eqn (2.70), are in the obvious sense associated with $\mathrm{Tr}\,H^n\bar{H}^m$. Of course, there are similar relations, in the same sense associated with any object like $\mathrm{Tr}\,(H^{a_1}\bar{H}^{b_1}H^{a_2}\bar{H}^{b_2}\ldots)$ and with products of such traces: it is enough to substitute all $\bar{H}\to\partial/\partial H_{\mathrm{tr}}$ to obtain the l.h.s. of the equation, and to substitute all of $H\to\partial/\partial\bar{H}_{\mathrm{tr}}$ to obtain its r.h.s. (one should remember that such a substitution is possible, say on the l.h.s., if all the $\bar{H}$ are put to the right of all H; in order to restore the matrix-product form of the relation, one should carefully take into account all the commutators arising when $\partial/\partial H_{\mathrm{tr}}$ is carried back to the original position of the corresponding $\bar{H}$). All such relations can appear to be just implications of Eqn (2.70).

Among other things, Eqn (2.70) reveals an amusing automorphism of the $\widetilde{\mathbf{W}}_\infty$-algebra:

$$\widetilde{W}^{(m+1)}_{n-m} \leftrightarrow \widetilde{W}^{(n+1)}_{m-n}\,, \qquad m, n \geqslant 0\,. \quad (2.71)$$

For example, Virasoro's Borel subalgebra is formed not only by the operators $\widetilde{W}^{(2)}_n$, but also by $\widetilde{W}^{(n+2)}_{-n}$, $n \geqslant -1$ [while the $\widehat{\mathrm{u}(1)}$ Borel subalgebra is formed not only by $\widetilde{W}^{(1)}_n = \partial/\partial t_n$, but also by $\widetilde{W}^{(n+1)}_n$, $n \geqslant 0$].

One can attempt to apply the same procedure and derive $\widetilde{W}$-identities for the conventional $(p-1)$-matrix models with $p-1 > 2$. In principle, this is possible, but unfortunately the equations arising neither have a nice form nor are there enough of them. However, for illustrational purposes I shall sketch some relevant formulas in the rest of this subsection.

Consider the multimatrix integral

$$Z = \int \mathrm{d}H_1\ldots\mathrm{d}H_{p-1}\exp[\mathrm{Tr}\,U_1(H_1) + \ldots + \mathrm{Tr}\,U_{p-1}(H_{p-1})]\ldots$$
$$\times\exp[\mathrm{Tr}\,(H_1H_2 + H_2H_3 + \ldots + H_{p-2}H_{p-1})]\,. \quad (2.72)$$

Acting on Z, the operator $\widetilde{W}^{(m+1)}_{n-m}(t^{(1)})$ produces the term $\mathrm{Tr}\,H^n_1(\overleftarrow{\partial}/\partial H_{1,\mathrm{tr}})^m$ at the position denoted by $\ldots$ in (2.72). Integration by parts gives

$$\mathrm{Tr}\,H^n_1\left(-\frac{\overrightarrow{\partial}}{\partial H_{1,\mathrm{tr}}}\right)^m \to (-1)^m\,\mathrm{Tr}\,H^n_1H^m_2 = (-1)^m\,\mathrm{Tr}\,H^m_2H^n_1\,.$$

In the case of $p-1 = 2$, discussed above, this can be rewritten as

$$(-1)^m\,\mathrm{Tr}\,H^m_1\left(\frac{\overrightarrow{\partial}}{\partial H_{2,\mathrm{tr}}}\right)^n,$$

and integration by parts gives

$$(-1)^m\,\mathrm{Tr}\,H^m_2\left(-\frac{\overrightarrow{\partial}}{\partial H_{2,\mathrm{tr}}}\right)^n,$$

which is equivalent to the action of $(-1)^{m+n}\widetilde{W}^{(n+1)}_{m-n}(t^{(2)})$ on Z: we have thus reproduced Eqn (2.70).

However, for $p-1 > 2$ things are more complicated. Insertion of $\mathrm{Tr}\,H^m_2H^n_1$ is equivalent to that of

$$\mathrm{Tr}\,H^m_2\left(\frac{\overrightarrow{\partial}}{\partial H_{2,\mathrm{tr}}} - H_3\right)^n,$$

which after integration by parts and operation on $\exp[U_2(H_2)]$ gives:

$$\mathrm{Tr}\,H^m_2\left(-\frac{\overleftarrow{\partial}}{\partial H_{2,\mathrm{tr}}} - H_3\right)^n \sim \mathrm{Tr}\,H^m_2\left(\sum_k kt^{(2)}_kH^{k-1}_2 - H_3\right)$$
$$\times\left(-\frac{\overleftarrow{\partial}}{\partial H_{2,\mathrm{tr}}} - H_3\right)^{n-1} \sim \ldots\,. \quad (2.73)$$

Derivatives remaining at the r.h.s. should be carried through the first bracket and then act on $\exp[U_2(H_2)]$ etc. The end result is some linear combination of terms like $\mathrm{Tr}\,H^{b_1}_2H^{c_1}_3H^{b_2}_2H^{c_2}_3\ldots$ with $t^{(2)}$-dependent coefficients.

Now, if we are dealing with the $p-1 = 3$ matrix model, every H_2 standing to the right of H_3s can be substituted by $\partial/\partial H_{3,\mathrm{tr}}$; otherwise one should also include terms with commutators when this $\partial/\partial H_{3,\mathrm{tr}}$ is carried back to the place

where H_2 was standing. This leads to a combination of insertions of the form

$$\mathrm{Tr}\left(\frac{\vec{\partial}}{\partial H_{3,\mathrm{tr}}}\right)^{b_1} H_3^{c_1}\left(\frac{\vec{\partial}}{\partial H_{3,\mathrm{tr}}}\right)^{b_2} H_3^{c_2}\cdots$$
$$\sim \mathrm{Tr}\left(\frac{-\overleftarrow{\partial}}{\partial H_{3,\mathrm{tr}}}\right)^{b_1} H_3^{c_1}\left(\frac{\overleftarrow{\partial}}{\partial H_{3,\mathrm{tr}}}\right)^{b_2} H_3^{c_2}\cdots . \quad (2.74)$$

The resulting operator can be expressed in terms of $\widetilde{W}(t^{(3)})$ resulting in an identity that states that some algebraic combination of $\widetilde{W}(t^{(1)})$ and $\widetilde{W}(t^{(3)})$ with $t^{(2)}$-dependent coefficients annihilates the partition function.

For $p-1>3$ insertion of H_2 is equivalent to that of $\partial/\partial H_{3,\mathrm{tr}}-H_4$ rather than $\partial/\partial H_{3,\mathrm{tr}}$, and the procedure should be repeated again and again. Finally one arrives at constraints where the operators are algebraic combinations of $\widetilde{W}(t^{(1)})$ and $\widetilde{W}(t^{(p-1)})$ with coefficients which depend on $t^{(2)},\ldots,t^{(p-2)}$ (moreover these are *infinite* series in the $\widetilde{W}$-operators, unless all the intermediate potentials $U_2,\ldots,U_{p-2}$ are polynomials of *finite* degree).

This is, of course, not a very illuminating procedure and in fact it has never been possible to obtain with its use concrete identities in any nice form. Instead it can serve to illustrate the problems peculiar for the class of conventional multimatrix models (at least for $p-1>2$). It can also emphasise the beauty of *conformal* multimatrix models, which have clear advantages at the level of the WIs.

2.10 $\widetilde{W}$-operators in the Kontsevich model

One can rewrite the GN equation (2.43) for Kontsevich models in terms of $\widetilde{W}$s. Namely, I shall prove the following identity [38]:

$$\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\right)^{m+1}\mathcal{Z}\{T_k\}=(\pm1)^{m+1}\sum_{l\geqslant0}\Lambda^{-l-1}\widetilde{W}^{(m+1)}_{l-m}(T)\mathcal{Z}\{T_k\}\ , \quad (2.75)$$

valid for *any* function $\mathcal{Z}$ which depends on $T_k=\mp(1/k)\mathrm{tr}\Lambda^{-k}(k\geqslant1)$ and $T_0=\pm\mathrm{tr}\ln\Lambda$, where Λ is an $n\times n$ matrix. Application of the identity (2.75) is most straightforward in the gaussian model (2.47), e.g. for the transformation of Eqn (2.50) into Eqn (2.51) (recall that $L=\Lambda$ in this case). In other cases, calculations with the use of identity (2.75), accounting for the quasiclassical factor $C_V\{L\}$ and the difference between $L=V'(\Lambda)$ and Λ, become somewhat more involved, though they still seem fairly straightforward. Also, for particular potentials $V(X)$ the partition function $\mathcal{Z}_V\{T\}$ is actually independent of certain (combinations of) time variables [for example, if $V(X)=X^{p+1}/(p+1)$ it is independent of all T_{pk}, $k\in Z^+$], and this is important for the appearance of the constraints in standard form, like Eqns (2.58) and (2.59), i.e. for a certain *reduction* of the $\widetilde{W}$-constraints to the ordinary W-constraints. This relation between $\widetilde{W}$- and W-operators deserves further investigation.

The proof of Eqn (2.75) is provided by the following ploy. Let us make a sort of Fourier transformation:

$$\mathcal{Z}\{T\}=\int \mathrm{d}H\,\mathcal{G}\{H\}\exp\left(\sum_{k=0}^{\infty}T_k\mathrm{Tr}H^k\right)\ , \quad (2.76)$$

where the integral is over $N\times N$ Hermitian matrix H.† Then it is clear that once identity (2.75) is established for $\mathcal{Z}\{T\}$ substituted by $\exp[\mathrm{Tr}\,U(H)]$, $U(H)=\sum_{k=0}^{\infty}T_k\mathrm{Tr}H^k$, with any matrix H, it is valid for *any* function $\mathcal{Z}\{T\}$. The advantage of such a substitution is that use can be made of the definition (2.64) of the $\widetilde{W}$-operators in order to rewrite Eqn (2.75) in a very explicit form:

$$\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\right)^{m+1}\exp[\mathrm{Tr}\,U(H)]$$
$$=(\pm1)^{m+1}\sum_{l\geqslant0}^{\infty}\Lambda^{-l-1}\widetilde{W}^{(m+1)}_{l-m}(T)\exp[\mathrm{Tr}\,U(H)]$$
$$=(\pm1)^{m+1}\sum_{l\geqslant0}^{\infty}\Lambda^{-l-1}\mathrm{Tr}\left(\frac{\partial}{\partial H_{\mathrm{tr}}}\right)^{m}H^l\exp[\mathrm{Tr}\,U(H)]$$
$$=(\pm1)^{m+1}\mathrm{Tr}\left(\frac{\partial}{\partial H_{\mathrm{tr}}}\right)^{m}\frac{1}{\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H}\exp[\mathrm{Tr}\,U(H)]\ . \quad (2.77)$$

Now the expression for Ts in terms of Λ should be used. Then

$$\exp[\mathrm{Tr}\,U(H)]=Det^{\pm1}(\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H)\ ,$$

and substituting this into Eqn (2.77) we see that Eqn (2.75) is equivalent to

$$\left[\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\right)^{m+1}-(\pm1)^{m+1}\mathrm{I}\cdot\mathrm{Tr}\left(\frac{\partial}{\partial H_{\mathrm{tr}}}\right)^{m}\frac{1}{\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H}\right]$$
$$\times Det^{\pm1}(\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H)=0\ .$$

Here 'Tr' stands for the trace in the H-space only, while $Det=\mathrm{Det}\otimes\det$ stands for the determinant in both H and Λ spaces. One Λ-derivative gives explicitly:

$$(\mathrm{I}\otimes\mathrm{Tr})\left[\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\right)^{m}\otimes\mathrm{I}-\mathrm{I}\otimes\left(\pm\frac{\partial}{\partial H_{\mathrm{tr}}}\right)^{m}\right]$$
$$\times\frac{Det^{\pm1}(\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H)}{\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H}=0\ . \quad (2.78)$$

This is actually a matrix identity, valid for any Λ and H of the sizes $n\times n$ and $N\times N$ respectively. For example, if $m=0$ ($\widetilde{W}^{(1)}$-case), it is obviously satisfied. If both $n=N=1$, it is also trivially true, though for different reasons for different choice of signs: for the upper signs, the ratio on the l.h.s. is just unity and all derivatives vanish; for the lower signs one has:

$$\left(\frac{\partial}{\partial\lambda}\right)^{m}-\left(-\frac{\partial}{\partial h}\right)^{m}$$
$$=\left[\sum_{\substack{a+b=m-1\\ a,b\geqslant0}}\left(\frac{\partial}{\partial\lambda}\right)^{a}\left(-\frac{\partial}{\partial h}\right)^{b}\right]\left(\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial h}\right)\ ,$$

† It is here that we encounter for the first time an important idea: that matrix models — the ordinary 1-matrix model (2.1) in this case — can be considered as defining integral transformations. This view on matrix models can to a large extent define their role in the future development of string theory.

and this obviously vanishes since

$$\left(\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial h}\right)\mathrm{f}(\lambda-h)\equiv 0$$

for any $\mathrm{f}(x)$.

If $m>0$ and Λ, H are indeed *matrices*, direct evaluation becomes much more sophisticated. I present the first two nontrivial examples: $m=1$ and $m=2$. The following relations will be useful. Let $Q\equiv 1/(\Lambda\otimes\mathrm{I}-\mathrm{I}\otimes H)$. Then

$$Det^{\pm 1}Q\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}Det^{\mp 1}Q=\pm[(\mathrm{I}\otimes\mathrm{tr})Q]\ ;$$

$$Det^{\pm 1}Q\frac{\partial}{\partial H_{\mathrm{tr}}}Det^{mp1}Q=\mp[(\mathrm{tr}\otimes\mathrm{I})Q]\ ;$$

$$\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\otimes\mathrm{I}\right)Q=-[(\mathrm{tr}\otimes\mathrm{I})Q]Q\ ;$$

$$\left(\mathrm{I}\otimes\frac{\partial}{\partial H_{\mathrm{tr}}}\right)Q=[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\ . \tag{2.79}$$

This is enough for the proof in the case of $m=1$. Indeed,

$$\begin{aligned}Det^{\pm 1}Q&\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\otimes\mathrm{I}\mp\mathrm{I}\otimes\frac{\partial}{\partial H_{\mathrm{tr}}}\right)Q\,Det^{\mp 1}Q\\&=\{-[(\mathrm{tr}\otimes\mathrm{I})Q]Q\pm[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\}\\&\quad\mp\{[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\mp[(\mathrm{tr}\otimes\mathrm{I})Q]Q\}=0\ .\end{aligned}$$

The first two terms on the r.h.s. arise from Λ-derivatives, while the last two arise from H-derivatives.

In the case of $m=2$ one should take derivatives once again. This is a little more tricky, and the same compact notation is not sufficient. In addition to relations (2.79), one now needs:

$$\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\otimes\mathrm{I}\right)[(\mathrm{tr}\otimes\mathrm{I})Q]Q=-[(\mathrm{tr}\otimes\mathrm{I})Q]^2Q-\mathcal{B}\ . \tag{2.80}$$

Here,

$$[(\mathrm{tr}\otimes\mathrm{I})Q]^2=\{(\mathrm{tr}\otimes\mathrm{I})[(\mathrm{tr}\otimes\mathrm{I})Q]Q\}\ , \tag{2.81}$$

while, in order to write $\mathcal{B}$ explicitly, we need to restore matrix indices (Greek for Λ-space and Latin for H-space). The $(\alpha i,\gamma k)$-component of Eqn (2.80) looks like

$$\left(\frac{\partial}{\partial\Lambda_{\beta\alpha}}\delta^{im}\right)Q^{\ mj}_{\delta\delta}Q^{\ jk}_{\beta\gamma}=-Q^{\ ij}_{\delta\delta}Q^{\ jl}_{\beta\beta}Q^{\ lk}_{\alpha\gamma}-Q^{\ il}_{\delta\beta}Q^{\ lj}_{\alpha\delta}Q^{\ jk}_{\beta\gamma}\ , \tag{2.82}$$

and the appearance of the second term on the r.h.s. implies that $\mathcal{B}^{\ ik}_{\alpha\gamma}=Q^{\ il}_{\delta\beta}Q^{\ lj}_{\alpha\delta}Q^{\ jk}_{\beta\gamma}$. Further,

$$\begin{aligned}&\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\otimes\mathrm{I}\right)[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\\&=-\{(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{tr}\otimes\mathrm{I})Q]Q\}Q-\{(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\}Q\ ;\end{aligned}$$

$$\begin{aligned}&\left(\mathrm{I}\otimes\frac{\partial}{\partial H_{\mathrm{tr}}}\right)[(\mathrm{tr}\otimes\mathrm{I})Q]Q\\&=+\{(\mathrm{tr}\otimes\mathrm{I})[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\}Q+\{(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{tr}\otimes\mathrm{I})Q]Q\}Q\ ;\end{aligned}$$

$$\begin{aligned}&\left(\mathrm{I}\otimes\frac{\partial}{\partial H_{\mathrm{tr}}}\right)[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\\&=+\{(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{I}\otimes\mathrm{Tr})Q]Q\}Q+\mathcal{B}\ .\end{aligned} \tag{2.83}$$

It is important that the $\mathcal{B}$ that appears in the last relation in the form of $\mathcal{B}^{\ ik}_{\alpha\gamma}=Q^{\ lj}_{\alpha\delta}Q^{\ il}_{\delta\beta}Q^{\ jk}_{\beta\gamma}$ is exactly the same $\mathcal{B}$ as in Eqn (2.80).

Now Eqn (2.78) can be proved for $m=2$:

$$\begin{aligned}&Det^{\pm 1}Q\left[\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\right)^2\otimes\mathrm{I}-\mathrm{I}\otimes\left(\frac{\partial}{\partial H_{\mathrm{tr}}}\right)^2\right]Q\,Det^{\mp 1}Q\\&\quad=\{\pm[(\mathrm{I}\otimes\mathrm{Tr})Q](-[(\mathrm{tr}\otimes\mathrm{I})Q]Q\pm[(\mathrm{I}\otimes\mathrm{Tr})Q]Q)\\&\qquad\qquad-(-[(\mathrm{tr}\otimes\mathrm{I})[(\mathrm{tr}\otimes\mathrm{I})Q]Q]Q-\mathcal{B})\\&\qquad\pm(-[(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{tr}\otimes\mathrm{I})Q]Q]Q-[(\mathrm{tr}\otimes\mathrm{I})[(\mathrm{I}\otimes\mathrm{Tr})Q]Q]Q)\}\\&\qquad-\{\mp[(\mathrm{tr}\otimes\mathrm{I})Q]([(\mathrm{I}\otimes\mathrm{Tr})Q]Q\mp[(\mathrm{tr}\otimes I)Q]Q)\\&\qquad\qquad+([(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{I}\otimes\mathrm{Tr})Q]Q]Q+\mathcal{B})\\&\qquad\mp([(\mathrm{tr}\otimes\mathrm{I})[(\mathrm{I}\otimes\mathrm{Tr})Q]Q]Q+[(\mathrm{I}\otimes\mathrm{Tr})[(\mathrm{tr}\otimes\mathrm{I})Q]Q]Q)\}\ ,\end{aligned} \tag{2.84}$$

where terms 1, 2, 3, 4, 5, 6 in the first pair of curly braces cancel the terms 1, 3, 2, 4, 6, 5 in the second pair of curly braces and the identity (2.81) and its counterpart with $(\mathrm{tr}\otimes\mathrm{I})\to(\mathrm{I}\otimes\mathrm{Tr})$ has been used.

An explicit proof of Eqn (2.78) for general m is unknown.

3. Eigenvalue models

3.1 What are eigenvalue models?

Given the present state of knowledge, we need to consider in most cases only the narrow class of the 'eigenvalue' models. These models have the property of being associated with conventional integrable hierarchies [of (multicomponent) Kadomtsev–Petviashvili (KP) and Toda type], where integrable flows just commute (instead of forming less trivial closed algebras), and thus with the level-1 Kac–Moody algebras (by artificial tricks, familiar from the bosonisation formalism in conformal field theory [58] these can sometimes be generalised to particular other levels like $k=2$). This means that the models are essentially associated with abelian Cartan subalgebras rather than with full matrix algebras.† In the conformal-field-theory (CFT) formulation (see below) this means that the eigenvalue models can be represented in terms of the free fields, which bosonise the Cartan subalgebra of the whole group in the Wess–Novikov–Witten (WZNW) model [the remaining (β,γ)-fields [16] being (almost) neglected — their remnants are observed in the form of 'cocycle' factors in the Frenkel–Kac formulas [47], see [58]. In the matrix–integral representations the integrals for the eigenvalue models are in fact reduced to those over diagonal matrices (consisting of eigenvalues of original matrices, thus the name 'eigenvalue models').

Most important, from the physical point of view eigenvalue models describe only *topological* (discrete)

† Groups arising in the theory of matrix models and integrable hierarchies are not just those of matrices appearing in the integral representations: the latter are at best related to the zero-modes of the former. Moreover, even this relation is not usually simple to reveal. This remark is important to avoid confusion in the following paragraphs.

degrees of freedom, but not any propagating *particles*.† This can be understood if one notes that matrix models usually possess gauge symmetry, associated with the unitary rotation of matrices, $M_\alpha \to U_\alpha^\dagger M_\alpha M_\alpha$; i.e. matrix models are usually *gauge* theories. In the case of eigenvalue models this symmetry is realised without 'gauge fields' $V_{\alpha\beta}$, which would depend on pairs of indices α, β and transform like $V_{\alpha\beta} \to H_\alpha^\dagger V_{\alpha\beta} U_\beta$. In other words, eigenvalue models are gauge theories without gauge fields, i.e. are purely topological. Thus, it is not a surprise that they usually live in the spacetime of dimension $d < 2$,‡ since for $d > 2$ there *should* be particles, associated with the gauge fields. At the 'boundary' lies the model of the '$d = 2$ ($c = 1$) string', which has one particle-like degree of freedom (dilaton, which becomes tachyon in $d > 2$ models). This very interesting model is much worse understood than the $d < 2$ models, at least its properties are somewhat different from other eigenvalue models (especially in the most interesting 'compactified' case), and will not be discussed in these notes. Later I shall return to the subject of noneigenvalue ($d > 2$) theories, though not very much is yet known about them; now I am going to concentrate on the eigenvalue models.

3.2 1-matrix model

Hermitian matrix integrals are usually transformed to the eigenvalue form by separation of angular and eigenvalue variables. As usual, the simplest is the case of the 1-matrix model

$$Z_N\{t\} \equiv c_N \int_{N\times N} \mathrm{d}H \exp\left(\sum_{k=0}^{\infty} t_k \operatorname{Tr} H^k\right) , \tag{3.1}$$

where this separation does not involve any information about unitary-matrix integrals. Take

$$H = U^\dagger D U , \tag{3.2}$$

where U is a unitary matrix whose diagonal matrix, D [$= \operatorname{diag}(h_1 \ldots h_N)$], has eigenvalues of H as its entries. Then the integration measure

$$\mathrm{d}H = \prod_{i,j=1}^{N} \mathrm{d}H_{i,j} = \frac{[\mathrm{d}U]}{[\mathrm{d}U_{\mathrm{Cartan}}]} \prod_{i-1}^{N} \mathrm{d}h_i \Delta^2(h) , \tag{3.3}$$

where the 'Van der Monde determinant' $\Delta(h) \equiv \det_{(ij)} h_i^{j-1} = \prod_{i>j}^{N}(h_i - h_j)$, and $[\mathrm{d}U]$ is the Haar measure of integration over unitary matrices.

† Particles are always related to the 'angular (unitary-) matrix' integrals (as is well known from the example of Wilson-lattice quantum chromodynamics) which are far less trivial to deal with, though these are also integrable in some broader sense of the word — within the (as yet nonexistent) generalisation of integrable hierarchies from the fields in the Cartan subalgebra to the entire WZNW model.

‡ Recall that in the Polyakov formulation, which is the least counterintuitive formulation for interpreting what happens in the spacetime (target space), string models usually involve the Liouville field, identified as a time-variable in the target-space formalism. (Note that for this reason there is usually (at least one) *time* in the string theory, while space can be of any dimension (at least between 0 and 25), not necessarily integer.) Because of this extra Liouville field, the spacetime dimension, d, usually differs by 1 from the central charge of the CFT model, which is coupled to two-dimensional gravity to form a string model: $d = c + 1$ and $d < 2$ is the same as $c < 1$.

The way to derive Eqn (3.3) is to consider the norm of the infinitesimal variation

$$\begin{aligned}\|\delta H\|^2 &\equiv \sum_{i,j=1}^{N} |\delta H_{ij}|^2 = \sum_{i,j=1}^{N} \delta H_{ij}\, \delta H_{ji} = \operatorname{Tr}(\delta H)^2 \\ &= \operatorname{Tr}(-U^\dagger \delta U\, U^\dagger D U + U^\dagger D\, \delta U + U^\dagger \delta D\, U)^2 \\ &= \operatorname{Tr}(\delta D)^2 + 2\,\mathrm{i}\operatorname{Tr} \delta u[\delta D, D] + 2\operatorname{Tr}[-\delta u\, D\, \delta u\, D \\ &\qquad + (\delta u)^2 D^2] ,\end{aligned}$$

where $\delta u \equiv (1/\mathrm{i})\delta U\, U^\dagger = \delta u^\dagger$ and $\delta D = \operatorname{diag}(\delta h_1, \ldots, \delta h_N)$. The second term on the r.h.s. vanishes because both D and δD are diagonal and commute. Therefore,

$$\|\delta H\|^2 = \sum_{i=1}^{N} (\delta h_i)^2 + \sum_{i,j=1}^{N} (\delta u)_{ij} (\delta u)_{ji} (h_i - h_j)^2 .$$

Now it remains to recall the basic relation between the infinitesimal norm and the measure: if $\|\delta l\|^2 = G_{ab}\, \delta l^a\, \delta l^b$ then $[\mathrm{d}l] = \sqrt{\det_{ab} G_{ab}} \prod_a \mathrm{d}l^a$, and we obtain Eqn (3.3) with Haar measure $[\mathrm{d}U] = \prod_{ij}^{N} \mathrm{d}u_{ij}$ being associated with the infinitesimal norm

$$\|\delta u\|^2 = \operatorname{Tr}(\delta u)^2 = \sum_{i,j=1}^{N} \delta u_{ij}\, \delta u_{ji} = \sum_{i,j=1}^{N} |\delta u_{ij}|^2 ,$$

and $[\mathrm{d}U_{\mathrm{Cartan}}] \equiv \prod_{i=1}^{N} \mathrm{d}u_{ii}$.

Coming back to the 1-matrix model, it remains to note that the 'action' $\operatorname{Tr} U(H) \equiv \sum_{k=0}^{\infty} t_k \operatorname{Tr} H^k$ with H substituted in the form (3.2) is independent of U:

$$\operatorname{Tr} U(H) = \sum_{i=l}^{N} U(h_i) .$$

Thus

$$\begin{aligned} Z_N\{t\} &= \frac{1}{N!} \prod_{i=1}^{N} \int \mathrm{d}h_i \exp[U(h_i)] \prod_{i>j}^{N} (h_i - h_j)^2 \\ &= \frac{1}{N!} \prod_{i=1}^{N} \int \mathrm{d}h_i \exp[U(h_i)] \Delta^2(h) , \end{aligned} \tag{3.4}$$

provided c_N is chosen such that

$$c_N^{-1} = N! \frac{\mathrm{Vol}_{\mathrm{U}(N)}}{(\mathrm{Vol}_{\mathrm{U}(1)})^N} , \tag{3.5}$$

where the volume of the unitary group in the Haar measure is given by

$$\mathrm{Vol}_{\mathrm{U}(N)} = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{\prod_{k=1}^{N} k!} . \tag{3.6}$$

A simple way to derive Eqn (3.6) will be described at the end of this section, as an example of the application of the orthogonal-polynomials technique.

3.3 Itzykson–Zuber and Kontsevich integrals

Let us proceed now to the Kontsevich integral,

$$\mathcal{F}_{V,n}\{L\} = \int_{n\times n} \mathrm{d}X \exp[-\operatorname{tr} V(X) + \operatorname{tr} LX] . \tag{3.7}$$

We shall see shortly that it in fact depends only on the eigenvalues of the matrix L (this fact has already been used

in the previous section); however, this time somewhat more sophisticated unitary-matrix integrals will be involved.

Substitute $X = U_X^\dagger D_X U_X$, and $L = U_L^\dagger D_L U_L$ in Eqn (3.7), and let $U \equiv U_X U_L^\dagger$. Then,

$$\mathcal{F}_{V,n}\{L\} = \prod_{i=1}^{n} \int \mathrm{d}x_i \exp[-V(x_i)]\Delta^2(x) \int_{n\times n} \frac{[\mathrm{d}U]}{[\mathrm{d}U_{\mathrm{Cartan}}]} \times \exp\left(\sum_{\gamma,\delta=1}^{n} x_\gamma l_\delta |U_{\gamma\delta}|^2\right) . \tag{3.8}$$

In order to proceed further we need to evaluate the integral over unitary matrices which appear on the r.h.s.

This integral can actually be presented in two different ways:

$$I_n\{X,L\} \equiv \int_{n\times n} \frac{[\mathrm{d}U]}{[\mathrm{d}U_{\mathrm{Cartan}}]} \exp(\mathrm{tr}\, XULU^\dagger) \tag{3.9}$$

$$= \int_{n\times n} \frac{[\mathrm{d}U]}{[\mathrm{d}U_{\mathrm{Cartan}}]} \exp\left(\sum_{\gamma,\delta=1}^{n} x_\gamma l_\delta |U_{\gamma\delta}|^2\right) , \tag{3.10}$$

(the Us in the two integrals are related by the transformation $U \to U_X U U_L^\dagger$ and the Haar measure is both left and right invariant). Formula (3.9) implies that $I_n\{X,L\}$ satisfies a set of simple equations [59]:

$$\left[\mathrm{tr}\left(\frac{\partial}{\partial X_{\mathrm{tr}}}\right)^k - \mathrm{tr}\, L^k\right] I_n\{X,L\} = 0 , \quad k \geqslant 0 ;$$

$$\left[\mathrm{tr}\left(\frac{\partial}{\partial L_{\mathrm{tr}}}\right)^k - \mathrm{tr}\, X^k\right] I_n\{X,L\} = 0 , \quad k \geqslant 0 ; \tag{3.11}$$

which by themselves are not very restrictive. However, another formula, Eqn (3.10), implies that $I_n\{X,L\}$ in fact depends only on the eigenvalues of X and L, and, for such $I_n\{X,L\} = \hat{I}\{x_\gamma, l_\delta\}$, Eqns (3.11) become very restrictive† and allow one to determine $\hat{I}\{x_\gamma, l_\delta\}$ unambiguously (at least if $\hat{I}\{x_\gamma, l_\delta\}$ is expandable in a formal power series in x_γ and l_δ). The final solution is

$$I_n\{X,L\} = \frac{(2\pi)^{\frac{1}{2}n(n-1)}}{n!} \frac{\det_{\gamma\delta} \exp(x_\gamma l_\delta)}{\Delta(x)\Delta(l)} . \tag{3.12}$$

One can define the normalisation constant by taking $L = 0$, whence

$$I_n\{X, L=0\} = \frac{\mathrm{Vol}_{\mathrm{U}(n)}}{(\mathrm{Vol}_{\mathrm{U}(1)})^n} = \frac{(2\pi)^{\frac{1}{2}n(n-1)}}{\prod_{k=1}^{n} k!} ,$$

† When acting on $\hat{I}$, which depends only on eigenvalues, matrix derivatives become

$$\mathrm{tr}\frac{\partial}{\partial X_{\mathrm{tr}}}\hat{I} = \sum_\gamma \frac{\partial}{\partial x_\gamma}\hat{I} ,$$

$$\mathrm{tr}\frac{\partial^2}{\partial X_{\mathrm{tr}}^2}\hat{I} = \sum_\gamma \frac{\partial^2}{\partial x_\gamma^2}\hat{I} + \sum_{\gamma\neq\delta} \frac{1}{x_\gamma - x_\delta}\left(\frac{\partial}{\partial x_\gamma} - \frac{\partial}{\partial x_\delta}\right)\hat{I} ,$$

etc.

and by using the fact that

$$\left.\frac{\det_{\gamma\delta} f_\gamma(l_\delta)}{\Delta(l)}\right|_{\{l_\delta=0\}} = \left(\prod_{k=0}^{n-1}\frac{1}{k!}\right) \det_{\gamma\delta} \partial^{\delta-1} f_\gamma(0) .$$

Eqn (3.12) is usually referred to as the Itzykson–Zuber formula [60]. In mathematical literature it was earlier derived by Harish-Chandra [61], and in fact the integral (3.9) is a basic example of coadjoint orbit integrals [62–65], which can be evaluated exactly with the help of the Duistermaat–Heckmann theorem [43, 44, 64, 65]. This calculation is the simplest example of the very important technique of *exact* evaluation of *nongaussian* unitary-matrix integrals, which is now at an early stage (see [66–68]) and will be discussed at the end of these notes.

Now we turn back to the eigenvalue formulation of the generalised Kontesevich model (GKM). Substitution of Eqn (3.12) into Eqn (3.8) gives:

$$\mathcal{F}_{V,n}\{L\} = \frac{(2\pi)^{\frac{1}{2}n(n-1)}}{\Delta(l)} \times \prod_{\delta=1}^{n} \int \mathrm{d}x_\delta \exp[-V(x_\delta)]\Delta(x) \frac{1}{n!} \det_{\gamma\delta} \exp(x_\gamma l_\delta)$$

$$= \frac{(2\pi)^{\frac{1}{2}n(n-1)}}{\Delta(l)} \prod_{\delta=1}^{n} \int \mathrm{d}x_\delta \exp[-V(x_\delta) + x_\delta l_\delta]\Delta(x) , \tag{3.13}$$

where I used the antisymmetry of $\Delta(x)$ under permutations of x_γs in order to change $(1/n!)\det_{\gamma\delta}\exp(x_\gamma l_\delta)$ for $\exp(\sum_\delta x_\delta l_\delta)$ under the sign of the x_δ integration.

One can now use the fact that $\Delta(x) = \det_{\gamma\delta} x_\delta^{\gamma-1}$ in order to rewrite the r.h.s. of Eqn (3.13):

$$\mathcal{F}_{V,n}\{L\} = (2\pi)^{\frac{1}{2}n(n-1)} \frac{\det_{\gamma\delta} \hat{\varphi}_\gamma(l_\delta)}{\Delta(l)} , \tag{3.14}$$

where

$$\hat{\varphi}_\gamma(l) \equiv \int \mathrm{d}x\, x^{\gamma-1} \exp[-V(x) + lx] , \quad \gamma \geqslant 1 . \tag{3.15}$$

These functions $\hat{\varphi}(l)$ satisfy a simple recurrence relation:

$$\hat{\varphi}_\gamma = \frac{\partial\hat{\varphi}_{\gamma-1}}{\partial l} = \left(\frac{\partial}{\partial l}\right)^{\gamma-1} \hat{\Phi} \tag{3.16}$$

with

$$\hat{\Phi}(l) \equiv \hat{\varphi}_1(l) = \int \mathrm{d}x \exp[-V(x) + lx] . \tag{3.17}$$

Note also that if the 'zero-time' N is introduced (see Subsection 2.6 and [36]), then

$$\mathcal{F}_{V,n}\{N\,|L\} \equiv \mathcal{F}_{V(X)-N\ln X,n}\{L\} = (2\pi)^{\frac{1}{2}n(n-1)} \frac{\det_{\gamma\delta} \hat{\varphi}_{\gamma+N}(l_\delta)}{\Delta(l)} , \tag{3.18}$$

with just the *same* $\hat{\varphi}_\gamma(l)$ and $\gamma, \delta = 1\ldots n$. If one divides by the quasiclassical factor $\mathcal{C}_V\{\Lambda\}(\det\Lambda)^N$ [with $L = V'(\Lambda)$] in order to transform the Kontsevich integral into the Kontsevich model (see Section 2.5), one obtains

$$\mathcal{Z}_V\{N,T\} = \frac{1}{(\det\Lambda)^N} \frac{\det_{\gamma\delta} \varphi_{\gamma+N}(\lambda_\delta)}{\Delta(\lambda)} . \tag{3.19}$$

The role of $\mathcal{C}_V\{\Delta\}$ is to convert $\hat{\varphi}(l)$ into properly normalised expansions in negative integer powers of λ:

$$\varphi_\gamma(\lambda) = \frac{\exp[-\lambda V'(\lambda) + V(\lambda)]\sqrt{V''(\lambda)}}{\sqrt{2\pi}}\hat{\varphi}_\gamma[V'(\lambda)]$$

$$= \lambda^{\gamma-1}[1 + \mathrm{O}(\lambda^{-1})] \,, \tag{3.20}$$

and to change $\Delta(l) = \Delta[V'(\lambda)]$ in the denominator of Eqn (3.18) for $\Delta(\lambda)$ in Eqn (3.19). Instead of the simple recurrence relations (3.16) for $\hat{\varphi}$, the normalised functions φ satisfy

$$\varphi_\gamma(\lambda) = \mathcal{A}\varphi_{\gamma-1}(\lambda) = \mathcal{A}^{\gamma-1}\Phi(\lambda) \,, \tag{3.21}$$

where $\Phi(\lambda) = \varphi_1(\lambda)$ and the operator $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{1}{V''(\lambda)}\frac{\partial}{\partial\lambda} - \frac{1}{2}\frac{V'''(\lambda)}{[V''(\lambda)]^2} + \lambda \,, \tag{3.22}$$

and now depends on the potential $V(x)$.

3.4 Conventional multimatrix models

The multimatrix integrals of the form

$$Z_N\{t^{(\alpha)}\}$$

$$\equiv c_N^{p-1}\int_{N\times N} \mathrm{d}H^{(1)}\ldots\mathrm{d}H^{(p-1)}\prod_{\alpha=1}^{p-1}\exp\left(\sum_{k=0}^{\infty} t_k^{(\alpha)}\,\mathrm{Tr}\,H_{(\alpha)}^k\right)$$

$$\times\prod_{\alpha=1}^{p-2}\exp\left(\mathrm{Tr}\,H^{(\alpha)}H^{(\alpha+1)}\right) \tag{3.23}$$

can be rewritten in the eigenvalue form by means of the same Itzykson–Zuber formula (3.12). Indeed, substituting $H^{(\alpha)} = U^{(\alpha)\dagger}D^{(\alpha)}U^{(\alpha)}$ and then defining $U^{(\alpha)}U^{(\alpha+1)\dagger} \equiv \hat{U}^{(\alpha)}$, one obtains

$$Z_N\{t^{(\alpha)}\} = \frac{1}{N!}\prod_{\alpha=1}^{p-1}\prod_{i=1}^{N}\int \mathrm{d}h_i^{(\alpha)}\,\exp[-V(h_i^{(\alpha)})]$$

$$\times\Delta^2(h^{(\alpha)})\prod_{\alpha=1}^{p-2} I_N\{H^{(\alpha)}, H^{(\alpha+1)}\}$$

$$= \frac{1}{N!}\prod_{\alpha=1}^{p-1}\prod_{i=1}^{N}\int \mathrm{d}h_i^{(\alpha)}\exp[-V(h_i^{(\alpha)})]$$

$$\times\prod_{\alpha=1}^{p-2}\exp(h_i^{(\alpha)}h_i^{(\alpha+1)})\Delta(h^{(1)})\Delta(h^{(2)}) \,, \tag{3.24}$$

where the same trick is done with the substitution of $(1/N!)\det_{ij}\exp(h_i^{(\alpha)}h_j^{(\alpha+1)})$ for $\exp(\sum_{i=1}^N h_i^{(\alpha)}h_i^{(\alpha+1)})$ under the sign of the $h_i^{(\alpha)}$ integration (step by step: first for $\alpha = 1$, then for $\alpha = 2$, and so on). Note that all the Van der Monde determinants disappeared from the final formula on the r.h.s. of Eqn (3.24), except for those at the ends of the matrix chain (at $\alpha = 1$ and $\alpha = p-1$).

If the chain was closed rather than open, i.e. with an additional factor of $\exp(\mathrm{Tr}\,H^{(p-1)}H^{(1)})$ under the integral in Eqn (3.23), then the trick with separation of all angular-variable (unitary-matrix) integrations would not work so simply: in addition to the Itzykson–Zuber integral, much more involved quantities would be required, like

$$I_n\{X_1, X_2; L\} \equiv c_n\int_{n\times n}\frac{[\mathrm{d}U_1]}{[\mathrm{d}U_{1,\,\mathrm{Cartan}}]}\frac{[\mathrm{d}U_2]}{[\mathrm{d}U_{2,\,\mathrm{Cartan}}]}$$

$$\times\exp[\mathrm{tr}\,X_1U_1LU_1^\dagger + \mathrm{tr}\,X_2U_2LU_2^\dagger$$

$$+\mathrm{tr}\,X_1(U_1U_2^\dagger)X_2(U_2U_1^\dagger)] \,. \tag{3.25}$$

This (so far unresolved) closed-chain model (lattice Potts model) is an example of a noneigenvalue model, in the $p = \infty$ case it turns into a 'compactified' $c = 1$ model. This theory is more complicated than what so far is the simplest class of noneigenvalue models of 'induced Yang–Mills theory', known as Kazakov–Migdal models.

3.5 Determinant formulas for eigenvalue models

We are now prepared to make the crucial step towards understanding the mathematical structure behind eigenvalue models, which distinguishes their partition functions in the entire variety of arbitrary N-fold integrals. This structure expresses itself in the form of determinantal formulas, which I am now going to discuss. In Section 4 these formulas will be identified as examples of τ-functions of KP and Toda hierarchies.

Looking at the relevant integrals — Eqns (3.4) and (3.24) — one can notice that integrals over different eigenvalues with nontrivial measures which depend on the shape of potentials U or V are almost separated, the only 'interaction' between different eigenvalues being defined by *universal* (potential-independent) quantities made from the Van der Monde determinants. This feature is intimately related both to its origin (decoupling of angular variables in the original matrix integral) and to its most important implication (integrability). The main property of the Van der Monde determinant is that it is at the same time a *Pfaffian* (it is in this quality that it arises from matrix integrals) and a *determinant* (this is the feature that implies integrability):

$$\prod_{i>j}(h_i - h_j) = \Delta(h) = \det_{ij} h_i^{j-1} \,. \tag{3.26}$$

This property was used above, when going from Eqn (3.13) to Eqn (3.14), which as we shall see later is the crucial step in the proof of integrability of the Kontsevich model. In that case the determinantal formula (3.14) for the partition function was trivial to derive, because the integrand was *linear* in Van der Monde determinants. Now I turn to slightly more complicated situations, involving products of Van der Monde determinants.

Consider an eigenvalue model of the form

$$Z_N = \frac{1}{N!}\prod_{k=1}^{N}\int \mathrm{d}\mu_{h_k,\bar{h}_k}\Delta(h)\Delta(\bar{h}) \,, \tag{3.27}$$

to be referred to as the 'scalar-product' model. All conventional multimatrix models (3.23) belong to this class. In the case of the 1-matrix model (3.4)

$$\mathrm{d}\mu_{h,\bar{h}} = \mathrm{d}h\,\mathrm{d}\bar{h}\exp[U(h)]\,\delta(h-\bar{h}) \,, \tag{3.28}$$

while for conventional multimatrix models (3.24)

$$\mathrm{d}\mu_{h^{(1)},h^{(p-1)}} = \mathrm{d}h^{(1)}\mathrm{d}h^{(p-1)}\prod_{\alpha=2}^{p-2}\int \mathrm{d}h^{(\alpha)}\prod_{\alpha=1}^{p-1}\exp[U_\alpha(h^{(\alpha)})]$$

$$\times\prod_{\alpha=1}^{p-2}\exp(h^{(\alpha)}h^{(\alpha+1)}) \,. \tag{3.29}$$

If $d\mu_{h,\bar{h}} = \delta(h-\bar{h})d\bar{h}\,d\mu_h$ we call this measure local. The main feature of a local measure is that the operation of multiplication by H (or any function of h) is Hermitian. Thus, the measure is local in the 1-matrix model, but is nonlocal for all $p-1>1$. In the latter case the measure is defined to depend only on $h = h^{(1)}$ and $\bar{h} = h^{(p-1)}$, all other $h^{(\alpha)}$ ($\alpha = 2,\ldots,p-2$) being integrated out; this makes the 'interaction' between h and $\bar{h}$ more complicated than just $\delta(h-\bar{h})$ in the 1-matrix ($p = 2$) and $\exp(h\bar{h})$ in the 2-matrix ($p = 3$) cases. In no sense is the set of particular formulas (3.29) for $p>3$ distinguished among other scalar-product models, and from now on we shall not consider conventional multimatrix models with $p-1>2$ as a separate class of theories.

Eqns (3.26) and (3.27) together imply that

$$Z_N = \frac{1}{N!}\prod_{k=1}^{N}\int d\mu_{h_k,\bar{h}_k}\,\mathrm{Det}_{ik}h_k^{i-1}\mathrm{Det}_{jk}\bar{h}_k^{j-1}$$
$$= \mathrm{Det}_{ij}\int d\mu_{h,\bar{h}}\,h^{i-1}\bar{h}^{j-1} = \mathrm{Det}_{ij}\langle h^{i-1}|\bar{h}^{j-1}\rangle\ , \quad (3.30)$$

where an obvious notation has been introduced for the scalar product:

$$\langle \mathrm{f}(h)|\mathrm{g}(\bar{h})\rangle \equiv \int d\mu_{h,\bar{h}}\,\mathrm{f}(h)\mathrm{g}(\bar{h})\ .$$

We can now be a little more specific and introduce time variables t_k and $\bar{t}_k$, so that

$$d\mu_{h,\bar{h}} = \exp[U(h)+\bar{U}(\bar{h})]d\hat{\mu}_{h,\bar{h}}\ ,$$
$$U(h) = \sum_{k=-\infty}^{\infty} t_k h^k\ ,\quad \bar{U}(\bar{h}) = \sum_{k=-\infty}^{\infty}\bar{t}_k\bar{h}^k\ , \quad (3.31)$$

and $d\mu_{h,\bar{h}}$ is already independent of h and $\bar{h}$. If we now define $\mathcal{H}^{\mathrm{f}}(t,\bar{t}) \equiv \langle 1|1\rangle$, then

$$\mathcal{H}^{\mathrm{f}}_{ij} \equiv \langle h^i|\bar{h}^j\rangle = \frac{\partial^2}{\partial t_i\partial\bar{t}_j}\mathcal{H}^{\mathrm{f}}(t,\bar{t})$$
$$\overset{\text{if } i,j\geqslant 0}{=} \left(\frac{\partial}{\partial t_1}\right)^i\left(\frac{\partial}{\partial\bar{t}_1}\right)^j\mathcal{H}^{\mathrm{f}}(t,\bar{t})\ , \quad (3.32)$$

and

$$Z_N = \mathrm{Det}_N\mathcal{H}^{\mathrm{f}}_{ij}\ , \quad (3.33)$$

where Det_N stands for determinant of the $N\times N$ matrix $\mathcal{H}_{i-1,j-1}$ (which is itself defined for *any* integers i,j) with $i,j = 0,\ldots,N-1$. A characteristic property of $\mathcal{H}^{\mathrm{f}}_{ij}$ is its peculiar time dependence:

$$\frac{\mathcal{H}^{\mathrm{f}}_{ij}}{\partial t_k} = \mathcal{H}^{\mathrm{f}}_{i+k,j}\ ,\qquad \frac{\mathcal{H}^{\mathrm{f}}_{ij}}{\partial\bar{t}_k} = \mathcal{H}^{\mathrm{f}}_{i,j+k}\ . \quad (3.34)$$

Eqn (3.33) provides the determinantal formula for all scalar-product models. The case of the local measure — for the 1-matrix model — is a little special. In this case $U(h)$ contains full information about the measure: $d\mu_{h,\bar{h}} = \delta(h-\bar{h})\,d\mu_h$, $d\mu_h = \exp[U(h)]\,dh$, and there is no $\bar{U}(\bar{h})$ (or $\bar{t}$ simply coincides with t). Then Eqn (3.33) is still valid but

$$\mathcal{H}^{\mathrm{f}}_{ij} = \langle h^i|\bar{h}^j\rangle\big|_{d\mu_{h,\bar{h}}} = \langle h^{i+j}\rangle\big|_{d\mu_h} = \frac{\partial}{\partial t_{i+j}}\mathcal{H}^{\mathrm{f}}(t)$$
$$\overset{\text{if } i,j\geqslant 0}{=} \left(\frac{\partial}{\partial t_1}\right)^{i+j}\mathcal{H}^{\mathrm{f}}(t)\ . \quad (3.35)$$

The same formula (3.35) can also be derived as a limit of Eqn (3.14) for the Kontsevich integral. Indeed,

$$Z_N\{t\} = c_N\int_{N\times N} dH\exp[\mathrm{Tr}U(H)] = \lim_{L\to 0}\mathcal{F}_{U,N}\{L\}$$
$$= \lim_{\{l_j\}\to 0}\frac{\mathrm{Det}_{ij}\hat{\varphi}_i^{\{U\}}(l_j)}{\Delta(l)} = \mathrm{Det}_{ij}\frac{\partial^{j-1}\hat{\varphi}_i^{\{U\}}(l_j)}{\partial l^{j-1}}(0)$$
$$= \mathrm{Det}_{ij}\mathcal{H}^{\mathrm{f}}_{i-1,j-1}\ , \quad (3.36)$$

where this time

$$\mathcal{H}^{\mathrm{f}}_{i-1,j-1} \overset{i,j>0}{=} \frac{\partial^{j-1}\hat{\varphi}_i^{\{U\}}(l_j)}{\partial l^{j-1}}(l=0) \overset{(3.14)}{=} \left(\frac{\partial}{\partial l}\right)^{i+j-2}\hat{\Phi}^{\{U\}}\big|_{l=0}. \quad (3.37)$$

Now we note, that the action of $\partial/\partial l$ on $\hat{\Phi}^{\{U\}}(l) = \int dx\exp[U(x)+lx]$ is equivalent to that of $(\partial/\partial t_1)$, since this is no longer a matrix integral, and thus

$$\mathcal{H}^{\mathrm{f}}_{ij} = \left(\frac{\partial}{\partial t_t}\right)^{i+j}\hat{\Phi}^{\{U\}}(0)\ , \quad (3.38)$$

i.e. $\mathcal{H}^{\mathrm{f}}(t) = \hat{\Phi}^{\{U\}}(0)$.

Conformal multimatrix models were introduced in Section 2.3 as eigenvalue models. For the A_{p-1} series, the partition functions are defined to be

$$Z^{A_{p-1}}_{N_1\ldots N_{p-1}}\{t^{(1)},\ldots,t^{(p-1)}\}$$
$$= \prod_{\alpha=1}^{p-1}c_{N_\alpha}\int_{N_\alpha\times N_\alpha} dH^{(\alpha)}\exp[\mathrm{Tr}\,U_\alpha(H^{(\alpha)})]$$
$$\times\prod_{\alpha=1}^{p-2}\mathrm{Det}(H^{(\alpha)}\otimes\mathrm{I}-\mathrm{I}\otimes H^{(\alpha+1)})$$
$$= \prod_{\alpha=1}^{p-1}\frac{1}{N_\alpha!}\prod_{i=1}^{N_\alpha}\int dh_i^{(\alpha)}\exp[U_\alpha(h_i^{(\alpha)})]\Delta^2(h^{(\alpha)})$$
$$\times\prod_{\alpha=1}^{p-2}\prod_{i,k}(h_i^{(\alpha)}-h_k^{(\alpha+1)})\ . \quad (3.39)$$

This expression does not have the form of Eqn (3.27); thus, conformal matrix models for $p-1>1$ are not of the 'scalar-product' type. I shall sometimes call them $(p-1)$-component models, because they are related to the multicomponent integrable hierarchies. The simplest way to proceed with their investigation is to use on the Kontsevich integral the same trick that was just applied in the 1-matrix case.

Let us start from a very general $(p-1)$-component model:

$$Z = \prod_{\alpha=1}^{p-1}\int_{N_\alpha\times N_\alpha} dH^{(\alpha)}\exp[\mathrm{Tr}\,U_\alpha(H^{(\alpha)})]K(H^{(1)},\ldots,H^{(p-1)})\ . \quad (3.40)$$

It can also be represented in terms of Kontsevich integrals:

$$Z = K\left(\frac{\partial}{\partial L^{(1)}_{\mathrm{tr}}},\ldots,\frac{\partial}{\partial L^{(p-1)}_{\mathrm{tr}}}\right)\prod_{\alpha=1}^{p-1}\mathcal{F}_{U_\alpha,N_\alpha}\{L^{(\alpha)}\}\Bigg|_{L^{(\alpha)}=0}\ . \quad (3.41)$$

This representation is not very useful, since the limit $L \to 0$ is not easy to take unless K is a polynomial in the eigenvalues of all its arguments. However, this is exactly the case for our conformal models (3.39). Indeed,

$$K^{A_{p-1}} = \prod_{\alpha=1}^{p-2} \mathrm{Det}\left(\frac{\partial}{\partial L_{\mathrm{tr}}^{(\alpha)}} \otimes \mathrm{I} - \mathrm{I} \otimes \frac{\partial}{\partial L_{\mathrm{tr}}^{(\alpha+1)}}\right) . \tag{3.42}$$

Still, this is not very convenient, because the representation (3.14) for $\mathcal{F}$ contains $\Delta(L)$ in the denominator, which is not very pleasant to differentiate. Simplification can be achieved if instead I rewrite the original expression on the r.h.s. of Eqn (3.39) as follows:

$$Z^{A_{p-1}}_{N_1 \ldots N_{p-1}}\{t^{(1)} \ldots t^{(p-1)}\}$$

$$= \Delta\left(\frac{\partial}{\partial l^{(1)}}\right) \prod_{\alpha=1}^{p-2} \Delta\left(\frac{\partial}{\partial l^{(\alpha)}}, \frac{\partial}{\partial l^{(\alpha+1)}}\right) \Delta\left(\frac{\partial}{\partial l^{(p-1)}}\right)$$

$$\times \prod_{\alpha=1}^{p-1} \left\{ \frac{1}{N_\alpha!} \prod_{i=1}^{N_\alpha} \int \mathrm{d}h_i^{(\alpha)} \exp[U_\alpha(h_i^{(\alpha)}) + l_i^{(\alpha)} h_i^{(\alpha)}] \right\}\Bigg|_{l^{(\alpha)}=0} , \tag{3.43}$$

where

$$\Delta(h, h') \equiv \prod_{i>j}^{N} (h_i - h_j) \prod_{k>l}^{N'} (h'_k - h'_l) \prod_{i=1}^{N} \prod_{k=1}^{N'} (h'_k - h_i) .$$

This formula takes the specific form of K into account. The product of integrals in brackets on the r.h.s. of Eqn (3.43) is equal (for every fixed α) to

$$\frac{1}{N_\alpha!} \prod_{j=1}^{N_\alpha} \hat{\Phi}^{\{U_\alpha\}}(l_j^{(\alpha)}) \tag{3.44}$$

[compare with Eqn (3.38)].

In order to simplify the notation I shall further denote

$$\hat{\Phi}^{\{U_\alpha\}}(l) \equiv \int \mathrm{d}x e^{U_\alpha(x) + lx}$$

by $\hat{\Phi}_\alpha(l)$, and

$$\left(\frac{\partial}{\partial t_1^{(\alpha)}}\right)^k \hat{\Phi}^{\{U_\alpha\}}(l^{(\alpha)}) = \left(\frac{\partial}{\partial l^{(\alpha)}}\right)^k \hat{\Phi}^{\{U_\alpha\}}(l^{(\alpha)}) .$$

by $\partial^k \hat{\Phi}_\alpha(l)$. Thus,

$$Z^{A_{p-1}}_{N_1 \ldots N_{p-1}}\{t^{(1)}, \ldots, t^{(p-1)}\}$$

$$= \Delta\left(\frac{\partial}{\partial l^{(1)}}\right) \prod_{\alpha=1}^{p-2} \Delta\left(\frac{\partial}{\partial l^{(\alpha)}}, \frac{\partial}{\partial l^{(\alpha+1)}}\right) \Delta\left(\frac{\partial}{\partial l^{(p-1)}}\right)$$

$$\times \prod_{\alpha=1}^{p-1} \left(\frac{1}{N_\alpha!} \prod_{j=1}^{N_\alpha} \hat{\Phi}_\alpha(l_j^{(\alpha)}) \right)\Bigg|_{l^{(\alpha)}=0} . \tag{3.45}$$

If $p-1 = 1$, the differential operator is just the square of the determinant $\Delta(\partial/\partial l)$ and one can use the relation

$$\Delta^2(h) = \sum_P \mathrm{Det}_{ij} h_{P(j)}^{i+j-2}$$

$$= \sum_P \mathrm{Det} \begin{bmatrix} 1 & h_{P(2)} & h^2_{P(3)} \cdots & h^{N_1-1}_{P(N_1)} \\ h_{P(1)} & h^2_{P(2)} & h^3_{P(3)} \cdots & h^{N_1}_{P(N_1)} \\ h^2_{P(1)} & h^3_{P(2)} & h^4_{P(3)} \cdots & h^{N_1+1}_{P(N_1)} \\ \vdots & & & \vdots \\ h^{N_1-1}_{P(1)} & h^{N_1}_{P(2)} & h^{N_1+1}_{P(3)} \cdots & h^{2N_1-2}_{P(N_1)} \end{bmatrix} , \tag{3.46}$$

where the sum is over all the $N!$ permutations P of N elements $1, \ldots, N$, in order to conclude that Eqn (3.45) reproduces our old formulas (3.33), and (3.38): $Z_N = \mathrm{Det}_{ij} \partial^{i+j-2} \hat{\Phi}$.

For $p-1 = 2$ one needs to use a more complicated analogue of (3.46):

$$\Delta(h)\Delta(h, h')\Delta(h')$$

$$= \sum_P \sum_{P'} \mathrm{Det} \begin{bmatrix} 1 & h_{P(2)} \cdots & h^{N_1-1}_{P(N_1)} & 1 & \bar{h}_{\bar{P}(2)} \cdots & \bar{h}^{N_2-1}_{\bar{P}(N_2)} \\ h_{P(1)} & h^2_{P(2)} \cdots & h^{N_1}_{P(N_1)} & \bar{h}_{\bar{P}(1)} & \bar{h}^2_{\bar{P}(2)} \cdots & \bar{h}^{N_2}_{\bar{P}(N_2)} \\ h^2_{P(1)} & h^3_{P(2)} \cdots & h^{N_1+1}_{P(N_1)} & \bar{h}^2_{\bar{P}(1)} & \bar{h}^3_{\bar{P}(2)} \cdots & \bar{h}^{N_2+1}_{\bar{P}(N_2)} \\ \vdots & & & & & \vdots \\ h^{\mathcal{N}_1-1}_{P(1)} & h^{\mathcal{N}_1}_{P(2)} \cdots & h^{\mathcal{N}+N_1-2}_{P(N_1)} & \bar{h}^{\mathcal{N}_2-1}_{\bar{P}(1)} & \bar{h}^{\mathcal{N}_2}_{\bar{P}(2)} \cdots & \bar{h}^{\mathcal{N}+N_2-2}_{P(N_2)} \end{bmatrix} , \tag{3.47}$$

where $\mathcal{N} = \sum_{\alpha=1}^{p-1} N_\alpha$. Making use of this formula, we conclude that the r.h.s. of Eqn (3.45) for $p-1 = 2$ is also representable in the form of a determinant:

$$\mathrm{Det} \begin{bmatrix} \hat{\Phi} & \partial\hat{\Phi} \ldots & \partial^{N_1-1}\hat{\Phi} & \hat{\bar{\Phi}} & \partial\hat{\bar{\Phi}} \ldots & \partial^{N_2-1}\hat{\bar{\Phi}} \\ \partial\hat{\Phi} & \partial^2\hat{\Phi} \ldots & \partial^{N_1}\hat{\Phi} & \partial\hat{\bar{\Phi}} & \partial^2\hat{\bar{\Phi}} \ldots & \partial^{N_2}\hat{\bar{\Phi}} \\ \vdots & & \cdots & & & \\ \partial^{\mathcal{N}-1}\hat{\Phi} & \partial^{\mathcal{N}}\hat{\Phi} \ldots & \partial^{\mathcal{N}+N_1-2}\hat{\Phi} & \partial^{\mathcal{N}-1}\hat{\bar{\Phi}} & \partial^{\mathcal{N}}\hat{\bar{\Phi}} \ldots & \partial^{\mathcal{N}+N_2-2}\hat{\bar{\Phi}} \end{bmatrix} ,$$

where $\hat{\Phi} = \hat{\Phi}_1$, $\hat{\bar{\Phi}} = \hat{\Phi}_2$, and $l^{(\alpha)} = 0$. It is especially easy to check formula (3.47) in the simplest case of $N_1 = N_2 = 1$. Then it just says that

$$\bar{h} - h = \mathrm{Det} \begin{bmatrix} 1 & 1 \\ h & \bar{h} \end{bmatrix} .$$

Analogous expressions for $p-1 > 2$ are more involved; they are no longer just determinants: this is already obvious from consideration of the simplest case of $N_1 = \ldots = N_{p-1} = 1$, when the product $\prod_{\alpha=1}^{p-2}(h^{(\alpha)} - h^{(\alpha+1)})$ is no longer the determinant of any nice matrix.

3.6 Orthogonal polynomials

The formalism of orthogonal polynomials was intensively used in the early days of the theory of matrix models. It is applicable to scalar-product eigenvalue models and allows one to further (diagonalise) transform the remaining determinants into products. In variance both with reduction from the original N^2-fold matrix integrals to the eigenvalue problem [which, when possible, reflects a physical phenomenon — decoupling of the angular (unitary-matrix) degrees of freedom (associated with d-dimensional gauge bosons)] and with the occurrence of the determinant formulas which reflect the integrability of the model, orthogonal polynomials appear more as a technical device. Essentially, orthogonal polynomials are necessary if one wants to explicitly separate the dependence on the size N of the matrix in the matrix integral ('zero-time') from the dependencies on all other time-variables and to explicitly construct variables which satisfy Toda-like equations. However, a modern description of integrable hierarchies in terms of τ-functions does not require explicit separation of the zero-time and treats it more or less on an equal footing to all other variables, thus making the use of orthogonal polynomials unnecessary. Still, this technique remains in the arsenal of matrix model theory† and we now briefly explain

† Of course, once can also use this link just with the aim of putting the rich and beautiful mathematical theory of orthogonal polynomials into the general context of string theory. Among interesting problems here is the matrix-model description of q-orthogonal polynomials.

what it is about. At the end of this section, two simple applications will also be described: the evaluation of the volume of the unitary group, and a direct proof of equivalence of the ordinary 1-matrix model and the gaussian Kontsevich model. Both these examples make use of explicitly known orthogonal Hermite polynomials and in this sense are not quite representative: usually orthogonal polynomials are not known explicitly. Some applications of such an 'abstract' theory of orthogonal polynomials to the study of matrix models will be mentioned in the following sections.

In the context of the theory of scalar-product matrix models, orthogonal polynomials naturally arise when one notes that after the partition functions appear in the simple determinantal form of Eqn (3.30), any linear change of bases, $h^i \to Q_i(h) = \sum_k A_{ik} h_k$, $\bar{h}^j \to \bar{Q}_j(\bar{h}) = \sum_l B_{jl}\bar{h}^l$, can be easily performed and $Z \to Z \det A \det B$. In particular, if A and B are triangular with units at diagonals, their determinants are just unity and Z does not change at all. This freedom is, however, enough to diagonalise the scalar product and to allow the choice of polynomials Q_i and $\bar{Q}_j$ so that

$$\langle Q_i(h)|\bar{Q}_j(\bar{h})\rangle = \exp(\phi_i)\delta_{ij} \quad . \tag{3.48}$$

Q_i and $\bar{Q}_j$ defined in this way up to normalisation are called orthogonal polynomials. (Note that $\bar{Q}$ does not need to be a *complex* conjugate of Q: the bar does not mean complex conjugation.) Because of the above restriction on the form of matrices A and B, these polynomials are normalised so that

$$Q_i(h) = h^i + \dots \ ; \quad \bar{Q}_j(\bar{h}) = \bar{h}^j + \dots ,$$

i.e. the leading power enters with a *unit* coefficient. From Eqns (3.30) and (3.48) it follows that

$$Z_N = \prod_{i=1}^{N} \exp(\phi_{i-1}). \tag{3.49}$$

This formula is essentially the main outcome of orthogonal polynomial theory for matrix models: it provides complete separation of the N-dependence of Z (on the size of the matrix) from that on all other parameters (which specify the shape of the potential, i.e. the measure $\mathrm{d}\mu_{h,\bar{h}}$); this information is encoded in a rather complicated fashion in ϕ_i. As was already mentioned, any feature of the matrix model can already be examined at the level of Eqn (3.30), which does not refer to orthogonal polynomials and thus they are not really relevant to the subject.

One can, however, reverse the problem and ask what it is that matrix models can provide *for* the theory of orthogonal polynomials.† The first question to ask in the theory of orthogonal polynomials is: given the measure $\mathrm{d}\mu_{h,\bar{h}}$, what are the corresponding orthogonal polynomials?

Usually the answer to this type of question is not at all straightforward. Its complexity, however, depends on what one agrees to accept as a suitable answer. Of particular interest for our purposes below would be integral representations. It would be very helpful to have just an integral transformation, converting the set of orthogonal polynomials for given $\mathrm{d}\mu_{h,\bar{h}}$ into some standard set, like $Q_i^{(0)} = x^i$. Unfortunately, such transformations are rarely available, though there are important examples: classical orthogonal polynomials and their q-analogues [expressed through the (q-)-hypergeometric functions, which usually possess integral representation of a simple form, see [69] for an introductory review of such integral formulas, which are in fact well known in CFT]. The simplest example of this kind, which will be used below, is the set of Hermite polynomials:

$$\begin{aligned}
\mathrm{He}_k(h) &= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}h^2\right)\int(\mathrm{i}x)^k \exp\left(-\frac{1}{2}x^2 - \mathrm{i}xh\right)\mathrm{d}x \\
&= \left(h - \frac{\mathrm{d}}{\mathrm{d}h}\right)^k \cdot 1 = \exp\left(\frac{1}{2}h^2\right)\left(-\frac{\mathrm{d}}{\mathrm{d}h}\right)^k \exp\left(-\frac{1}{2}h^2\right) \\
&= \frac{1}{2^k}\exp\left(\frac{1}{4}h^2\right)\left(h - 2\frac{\mathrm{d}}{\mathrm{d}h}\right)^k \exp\left(-\frac{1}{4}h^2\right) \\
&= h^k + \dots .
\end{aligned} \tag{3.50}$$

These polynomials are orthogonal with the *local* measure $\mathrm{d}\mu_h = \exp\left(-\frac{1}{2}h^2\right)$.

For a generic measure an answer of this type does not exist in any universal form. However, matrix models still provide a somewhat peculiar integral representation for *any* measure, with the number of integrations depending on the number of polynomials. In order to obtain this expression, let us consider a slight generalisation of formula (3.27):

$$Z_N\{\lambda_\gamma\} \equiv \frac{1}{N!}\prod_{k=1}^{N}\int \mathrm{d}\mu_{h_k,\bar{h}_k}\Delta(h)\Delta(\bar{h})\prod_{k,\gamma}(\lambda_\gamma - h_k). \tag{3.51}$$

Then, $\Delta(h)\prod_{k,\gamma}(\lambda_\gamma - h_k) = \Delta(h,\lambda)/\Delta(\lambda)$, and λ_γ can be considered just as $h_{N+\gamma}$, which are *not* integrated over in Eqn (3.51). Then it is clear that

$$\Delta(h,\lambda) = \mathrm{Det}\begin{bmatrix} Q_{i-1}(h_k) & Q_{N+\gamma-1}(h_k) \\ Q_{i-1}(\lambda_\delta) & Q_{N+\gamma-1}(\lambda_\delta) \end{bmatrix}, \tag{3.52}$$

while $\Delta(\bar{h}) = \mathrm{Det}_{jk}\bar{Q}_{j-1}(\bar{h}_k)$. Since all the $Q_{N+\gamma-1}(h_k)$ are orthogonal to all $\bar{Q}_{j-1}(\bar{h}_k)$ (because $N+\gamma-1 \neq j-1$), one obtains:

$$Z_N\{\lambda_\delta\} = \frac{\det_{\gamma\delta} Q_{N+\gamma-1}(\lambda_\delta)}{\Delta(\lambda)} Z_N. \tag{3.53}$$

In particular,

$$Q_N(\lambda) = \frac{Z_N\{\lambda\}}{Z_N}, \tag{3.54}$$

where both the numerator and the denominator can be represented by $N \times N$-matrix integrals.

The inverse of the 'main question' of the theory of orthogonal polynomials is: given a set of polynomials,

$$Q_i(h) = h_i + \dots ,$$

$$\bar{Q}_j(\bar{h}) = \bar{h}_j + \dots ,$$

what is the measure $\mathrm{d}\mu_{h,\bar{h}}$ with respect to which they form an orthogonal system?

† Of course, one can hardly get anything *new* for that theory, but the purpose is to see which features are immediate consequences of the 'physically inspired' approach. Usually this can help one to somehow organise the existing knowledge on the appropriate system. This is, however, my goal in these notes: only a very simple example will be mentioned, which will also be of use to us later.

I shall not discuss the complete answer to this question and consider only the case of the *local* measure, when $\bar{Q}_i = Q_i$. Then, usually the answer does not exist at all: not *every* system of polynomials is orthogonal with respect to some local measure. It is easy to find the necessary (and in fact sufficient) condition. As was mentioned above, the local measure is distinguished by the property that multiplication by (any function of) h is a Hermitian operation:

$$\langle h\mathrm{f}(h)|\mathrm{g}(\bar{h})\rangle = \langle \mathrm{f}(h)|\bar{h}\mathrm{g}(\bar{h})\rangle \ , \quad \text{if } \mathrm{d}\mu_{h,\bar{h}} \sim \delta(h-\bar{h}). \tag{3.55}$$

This property implies that the coefficients c_{ij} in the recurrence relation

$$hQ_i(h) = Q_{i+1}(h) + \sum_{j=0}^{i} c_{ij}Q_i(h) \tag{3.56}$$

are almost all vanishing. Indeed, for $j < i$

$$c_{ij} = \frac{\langle hQ_i(h)|Q_j(\bar{h})\rangle}{\langle Q_j(h)|Q_j(\bar{h})\rangle} = \frac{\langle Q_i(h)|\bar{h}Q_j(\bar{h})\rangle}{\langle Q_j(h)|Q_j(\bar{h})\rangle}$$

$$= \delta_{i,j+1}\frac{\langle Q_i(h)|Q_i(\bar{h})\rangle}{\langle Q_j(h)|Q_j(\bar{h})\rangle} = \delta_{j,i-1}\exp(\phi_i - \phi_{j-1}) \ . \tag{3.57}$$

In other words, polynomials orthogonal with respect to a local measure are obliged to satisfy the '3-term recurrence relation'

$$hQ_i(h) = Q_{i+1}(h) + C_iQ_i(h) + R_iQ_{i-1}(h) \tag{3.58}$$

(the coefficient of Q_{i+1} can, of course, be changed by a change of normalisation). Parameter C_i vanishes if the measure is even (symmetric under the change $h \to -h$); in this case the polynomials are split into two orthogonal subsets: even and odd in h. The partition function (3.49) of the 1-component model can be expressed through parameters $R_i = \exp(\phi_i - \phi_{i-1})$ of the 3-term relation:

$$Z_N = Z_1 \prod_{i=1}^{N-1} R_i^{N-i} \ , \tag{3.59}$$

thus defining a 1-component matrix model (i.e. the particular shape of potential) associated with any system of orthogonal polynomials.

Our 'inverse main question' in the case of the local measure should now be formulated as follows: given a set of orthogonal polynomials $Q_i(h) = h^i + \ldots$ *which satisfy the 3-term relation* (3.58), what is the measure $\mathrm{d}\mu_h$?

As with every complete orthogonal system of functions, orthogonal polynomials satisfy the completeness relation

$$\sum_{i=0}^{\infty} \exp(-\phi_i)\bar{Q}_i(\bar{h})Q_i(h) = \delta^{\{\mathrm{d}\mu\}}(\bar{h}, h) \ , \tag{3.60}$$

where the δ-function associated with the measure $\mathrm{d}\mu_{h,\bar{h}}$ is defined so that

$$\iint \mathrm{f}(h)\delta^{\{\mathrm{d}\mu\}}(\bar{h}, h')\mathrm{d}\mu_{h,\bar{h}} = \mathrm{f}(h') \tag{3.61}$$

for any function $\mathrm{f}(h)$. Since for the *local* measure $\mathrm{d}\mu_h = \exp[U(h)]\mathrm{d}h$ the δ-function is just $\delta^{\{\mathrm{d}\mu\}}(\bar{h},h) = \exp[-U(h)]\delta(\bar{h}-h)$, as an answer to our question we can take a representation of $U(h)$ in terms of the corresponding orthogonal polynomials:

$$\exp[-U(h)]\delta(\bar{h}-h) = \sum_{k=0}^{\infty} \frac{Q_k(\bar{h})Q_k(h)}{\langle Q_k|Q_k\rangle} \ . \tag{3.62}$$

As usual, this relation should be understood as an analytical continuation. The squared norms $\|Q_k\|^2$ in the denominator are expressed through the coefficients R_i of the 3-term relation (3.59) up to an overall constant as follows: $\|Q_k\|^2 = \prod_{i=1}^{k} R_i\|Q_0\|^2$.

For example, in the case of the Hermite polynomials (3.50) we have:

$$\mathrm{He}_{k+1}(h) = \left(h - \frac{\mathrm{d}}{\mathrm{d}h}\right)\mathrm{He}_k(h) = h\mathrm{He}_k(h) - \frac{\mathrm{d}}{\mathrm{d}h}\mathrm{He}_k(h)$$

$$= h\mathrm{He}_k(h) - k\mathrm{He}_{k-1}(h) \tag{3.63}$$

(the last equality holds because $\mathrm{d}/\mathrm{d}h$ and $h - \mathrm{d}/\mathrm{d}h$ play the role of annihilation and creation operators, respectively). This means that the 3-term relation is satisfied with $R_k = k$ and thus $\|\mathrm{He}_k\|^2 = \|\mathrm{He}_0\|^2 k!$. We shall use the normalisation condition $\|\mathrm{He}_0\|^2 = \sqrt{2\pi}$. Then, for $\exp[-U(h)]$, we get:

$$\exp[-U(h)]\delta(\bar{h}-h) = \sum_{k=0}^{\infty} \frac{\mathrm{He}_k(\bar{h})\,\mathrm{He}_k(h)}{\|\mathrm{He}_k\|^2}$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(h - \frac{\mathrm{d}}{\mathrm{d}h}\right)^k\left(\bar{h} - \frac{\mathrm{d}}{\mathrm{d}\bar{h}}\right)^k \cdot 1$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}h^2 + \frac{1}{2}\bar{h}^2\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{d}^2}{\mathrm{d}h\,\mathrm{d}\bar{h}}\right)^k$$

$$\times \exp\left(-\frac{1}{2}h^2 - \frac{1}{2}\bar{h}^2\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}h^2 + \frac{1}{2}\bar{h}^2\right)\exp\left(\frac{\mathrm{d}^2}{\mathrm{d}h\,\mathrm{d}\bar{h}}\right)\exp\left(-\frac{1}{2}h^2 - \frac{1}{2}\bar{h}^2\right)$$

$$= \frac{1}{\sqrt{2\pi}}\mathrm{Im}\iint\frac{\mathrm{d}\alpha\,\mathrm{d}\bar{\alpha}}{2\pi}\exp(-\alpha\bar{\alpha})\exp\left(\frac{1}{2}h^2 + \frac{1}{2}\bar{h}^2\right)$$

$$\times \exp\left(\alpha\frac{\mathrm{d}}{\mathrm{d}h} + \bar{\alpha}\frac{\mathrm{d}}{\mathrm{d}\bar{h}}\right)\exp\left(-\frac{1}{2}h^2 - \frac{1}{2}\bar{h}^2\right)$$

$$= \frac{1}{\sqrt{2\pi}}\mathrm{Im}\iint\frac{\mathrm{d}\alpha\,\mathrm{d}\bar{\alpha}}{2\pi}\exp\left[-\frac{1}{2}(\alpha+\bar{\alpha})^2\right]$$

$$\times \exp\left[-\frac{1}{2}(\alpha+\bar{\alpha})(h+\bar{h})\right]\exp\left[-\frac{1}{2}(\alpha-\bar{\alpha})(h-\bar{h})\right]$$

$$= \exp\left(\frac{1}{2}h^2\right)\delta(h-\bar{h}) \ .$$

3.7 Scalar-product models in the Miwa parametrisation

I shall now take the first step towards clarification of the interrelation between the scalar-product and Kontsevich models. We already know that in the latter case an important role is played by the representation of time variables in the form of

$$T_k = \frac{1}{k}\mathrm{tr}\Lambda^{-k} \ , \tag{3.64}$$

(with the $n \times n$ matrix Λ), which will be further referred to as the Miwa parametrisation (expressions of a similar form were first introduced in [70]). Let us now perform such a transformation in the case of the scalar-product model. Let us use Eqn (3.31) to define the time-dependence of the measure, but ignore the $\bar{t}$-variables. Namely, introduce $\mathrm{d}\mu_{h,\bar{h}} = \exp[U(h)]\mathrm{d}\hat{\nu}_{h,\bar{h}}$ {i.e. $\mathrm{d}\hat{\nu}_{h,\bar{h}} = \exp[\bar{U}(\bar{h})]\mathrm{d}\hat{\mu}_{h,\bar{h}}$}. Substitute

$$t_k = \mp\left(\frac{1}{k}\mathrm{tr}\Lambda^{-k} + r_k\right) , \tag{3.65}$$

and obtain

$$\begin{aligned}\exp[U(h)] &= \exp[-\hat{V}(h)]\exp\left[\mp\mathrm{tr}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{h}{\Lambda}\right)^k\right] \\ &= \exp\left[-\hat{V}(h)\frac{\det^{\pm 1}(\Lambda - h\mathrm{I})}{\det\Lambda}\right] \\ &= \frac{\exp[-\hat{V}(h)]}{\det\Lambda}\prod_{\gamma=1}^{n}(\lambda_\gamma - h)^{\pm 1} ,\end{aligned} \tag{3.66}$$

where $\hat{V}(h) \equiv \pm\sum_k r_k h^k$. Let us choose the *upper* signs in these formulas. Then we can use Eqns (3.51) and (3.53) to conclude that, in the Miwa parametrisation,

$$\begin{aligned}Z_N^{\{\mathrm{d}\mu\}} &= \frac{1}{(\det\Lambda)^N} Z_N^{\{\mathrm{d}\hat{\nu}\}}\{\lambda_\delta\} \\ &= Z_N^{\{\mathrm{d}\hat{\nu}\}}\frac{\det_{\gamma\delta}\hat{Q}_{N+\gamma-1}(\lambda_\delta)}{\Delta(\lambda)(\det\Lambda)^N} ,\end{aligned}$$

where $\mathrm{d}\hat{\nu}_{h,\bar{h}} \equiv \exp[-\hat{V}(\bar{h})]\mathrm{d}\nu_{h,\bar{h}}$ and $\hat{Q}_k$ are the corresponding orthogonal polynomials. In other words, the model with potential $U(h)$ has been reduced to another model, with potential $-\hat{V}(h)$, and the difference has been expressed in terms of orthogonal polynomials $\hat{Q}_k$:

$$\frac{Z_N^{\{\mathrm{d}\mu\}}}{Z_N^{\{\mathrm{d}\hat{\nu}\}}} = \frac{1}{(\det\Lambda)^N}\frac{\det_{\gamma\delta}\hat{Q}_{N+\gamma-1}(\lambda_\delta)}{\Delta(\lambda)} . \tag{3.67}$$

If $\hat{V}(h)$ is adjusted to give rise to some simple orthogonal polynomials (i.e. if the new model $Z_N^{\{\mathrm{d}\hat{\nu}\}}$ is easy to solve) this representation can considerably simplify the original model.

Another interpretation of this formula is that we obtained a GKM-like representation of the form of Eqn (3.19) for the *discrete* scalar-product model. The only difference is that $\varphi_\gamma^{\{V\}}$ in Eqn (3.19) are changed for $\hat{Q}_{\gamma-1}$ in Eqn (3.67). This is an important difference, because $\varphi_\gamma^{\{V\}}$ in GKM are defined by integral formulas like Eqn (3.15), $\varphi_\gamma^{\{V\}} = \langle\langle x^{\gamma-1}\rangle\rangle$ or, alternatively, satisfy recursive relations like Eqn (3.21). Moreover, generic $\varphi_\gamma^{\{V\}}$ are infinite formal series in λ^{-1}, while $Q_{\gamma-1}$ are orthogonal *polynomials*. This discrepancy is one of the important stimuli for further development of the concept of the GKM, as well as for the search for convenient integral representations for orthogonal polynomials.

There is, however, at least one interesting situation when the two formulas indeed coincide. This is the case of the gaussian potentials V and $\hat{V}$, when both $\varphi_\gamma^{\{V\}}$ and $Q_{\gamma-1}$ are represented by orthogonal Hermite polynomials, which possess integral representation, and are exactly adequate in the context of GKM. This is the subject of the next subsection.

3.8 Equivalence of the discrete 1-matrix and gaussian Kontsevich models

Let us take the ordinary 1-matrix model with the *local* measure $\mathrm{d}\mu_h = \exp[U(h)]\mathrm{d}h$ to be the scalar-product model, considered in the previous subsection, and take the Miwa parametrisation with upper signs and with $r_k = -\frac{1}{2}\delta_{k,2}$ (as in Section 2.6). Then $\hat{V}(h) = \sum_k r_k h^k = -\frac{1}{2}h^2 = \frac{1}{2}(\mathrm{i}h)^2$. The relevant orthogonal polynomials $\hat{Q}$ are just Hermite polynomials of *imaginary* argument:† $Q_k^{\{-\frac{1}{2}h^2\mathrm{d}h\}} = \mathrm{i}^{-k}\mathrm{He}_k(\mathrm{i}h) = h^k + \ldots$. These polynomials possess an integral representation (3.50):

$$\begin{aligned}\mathrm{i}^{1-k}\mathrm{He}_{k-1}(\mathrm{i}h) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}h^2\right) \\ &\times\int x^{k-1}\exp\left(-\frac{1}{2}x^2 + xh\right)\mathrm{d}x \overset{(3.20)}{=} \varphi_k^{\{\frac{1}{2}x^2\}}(h) ,\end{aligned} \tag{3.68}$$

Using Eqns (3.67) and (3.19) one obtains a remarkable relation between the two matrix models:

$$\begin{aligned}&\frac{Z_N\{t_0 = 0; t_k = -(1/k)\mathrm{tr}\,\Lambda^{-k} + \frac{1}{2}\delta_{k,2}\}}{Z_N\{t_k = \frac{1}{2}\delta_{k,2}\}} \\ &= \frac{\int_{N\times N}\mathrm{d}H\exp\left(\sum_{k=0}^{\infty}t_k\mathrm{Tr}\,H^k\right)}{\int_{N\times N}\mathrm{d}H\exp(\frac{1}{2}H^2)} \\ &= \frac{\exp\left(-\mathrm{tr}\frac{1}{2}\Lambda^2\right)}{(2\pi)^{\frac{1}{2}n^2}(\det\Lambda)^N}\int_{n\times n}\mathrm{d}X(\det X)^N\exp(-\mathrm{tr}\frac{1}{2}X^2 + \Lambda X) \\ &= \mathcal{Z}_{\frac{1}{2}X^2}\{N, t\} ,\end{aligned} \tag{3.69}$$

where $Z_N\{t_k = \frac{1}{2}\delta_{k,2}\} = (-2\pi)^{\frac{1}{2}N^2}c_N$. This relation can also be regarded as an identity:

$$\begin{aligned}&\frac{\int_{N\times N}\mathrm{d}H\exp\left(\frac{1}{2}\mathrm{Tr}\,H^2\right)\mathrm{Det}(\Lambda\otimes\mathrm{I} - \mathrm{I}\otimes H)}{\int_{N\times N}\mathrm{d}H\exp\left(\frac{1}{2}\mathrm{Tr}\,H^2\right)} \\ &= \frac{\int_{n\times n}\mathrm{d}X\exp(-\frac{1}{2}\mathrm{tr}X^2)\det^N(X + \Lambda)}{\int_{n\times n}\mathrm{d}X\exp(-\frac{1}{2}\mathrm{tr}X^2)} ,\end{aligned} \tag{3.70}$$

valid for any Λ. Note that the integrals are of different sizes: $N \times N$ on the l.h.s. and $n \times n$ at the r.h.s. While the N-dependence is explicit on both sides of the equation, the n-dependence on the l.h.s. enters only implicitly, through the allowed domain of variation of variables $t_k = -(1/k)\mathrm{tr}\Lambda^{-k} + \frac{1}{2}\delta_{k,2}$. (This can serve as an illustration to the general statement that the shape of the Kontsevich partition function $\mathcal{Z}_V$, considered as a function of Ts rather

† Note that this system of functions $\varphi_k = \mathrm{i}^{-k}\mathrm{He}_k(\mathrm{i}h)$ looks like $\varphi_0 = 1$, $\varphi_1 = h$, $\varphi_2 = h^2 + 1, \ldots$, and does not resemble any set of orthogonal polynomials with a local measure (for example the product $\varphi_0\varphi_2 = h^2 + 1$ may seem positive definite, this being inconsistent with the orthogonality requirement $\langle\varphi_0|\varphi_2\rangle = 0$). The thing is that integration on the l.h.s. of Eqn (3.69) is well defined only along the imaginary axis, while integrals along the real axis are understood as analytical continuations.

than of L or Λ, is independent of the matrix size n.) The identity (3.69) was anticipated from the study of the WIs for the gaussian Kontsevich model in [56] [see Eqn (2.53) in Section 2.6], and was derived in the present form in [36].

Eqn (3.69) can be used to perform analytical continuation in N and define Z_N for N, which are not positive integers. Since $c_N = 0$ for all *negative* integers [see Eqn (3.77) below], the same is true for Z_N. In Section 4 we shall see that it is a characteristic property of τ-functions of *forced* hierarchies.

3.9 Volume of the unitary group

The formalism of orthogonal polynomials provides a simple derivation of Eqn (3.6) for the volume of the unitary group. Consider Eqn (3.4) with $U(H) = H^2$. Then the gaussian matrix integral can be easily evaluated:

$$c_N \int_{N\times N} \mathrm{d}H \exp(-\tfrac{1}{2}\mathrm{Tr}\, H^2) = c_N \prod_{i=1}^{N} \int \mathrm{d}H_{ii} \exp(-\tfrac{1}{2}H_{ii}^2)$$

$$\times \prod_{i<j}^{N} \int \mathrm{d}^2 H_{ij} \exp\left(-|H_{ij}|^2\right) = (2\pi)^{\frac{1}{2}N^2} ,$$

while according to Eqns (3.48) and (3.49) the same integral is given by

$$\frac{1}{N!}\prod_{i=1}^{N}\int_{-\infty}^{+\infty} \exp(-\tfrac{1}{2}h_i^2)\prod_{i>j}^{N}(h_i-h_j)^2 = \prod_{j=1}^{N}\|\mathrm{He}_{j-1}\|^2 .$$

Here $\|\mathrm{He}_{j-1}\|$ stands for the norm of orthogonal Hermite polynomials (3.50), $\|\mathrm{He}_k\|^2 = \sqrt{2\pi}k!$. Comparing the two expressions for the same integral we get:

$$c_N^{-1} = (2\pi)^{\frac{1}{2}N^2}\prod_{k=0}^{N-1}\frac{1}{\sqrt{2\pi}k!} = \frac{(2\pi)^{\frac{1}{2}N(N-1)}}{\prod_{k=0}^{N-1}k!} . \tag{3.71}$$

According to Eqn (3.5),

$$c_N^{-1} = N!\frac{\mathrm{Vol}_{\mathrm{U}(N)}}{(\mathrm{Vol}_{\mathrm{U}(1)})^N} ,$$

and $\mathrm{Vol}_{\mathrm{U}(1)} = 2\pi$. Thus, we obtain Eqn (3.6):

$$\mathrm{Vol}_{\mathrm{U}(N)} = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{\prod_{k=0}^{N}k!} .$$

An example of a somewhat more sophisticated (quantum) group-theoretical quantity, arising from gaussian matrix models, is provided by the following formula for the q-factorial [71, 72]:

$$\frac{1}{(q,q)_N} \equiv \prod_{n=1}^{N}\frac{1}{1-q^n}$$

$$= \frac{\int\int_{N\times N} \mathrm{d}H[\mathrm{d}U]\exp\left(-m^2\,\mathrm{Tr}\,H^2 + \mathrm{Tr}\,HUHU^\dagger\right)}{\mathrm{Vol}_{\mathrm{U}(N)}\int_{N\times N}\mathrm{d}H\exp\left(-m^2\,\mathrm{Tr}\,H^2\right)} . \tag{3.72}$$

The integral in the numerator is over Hermitian (H) and unitary (U) $N\times N$ matrices, and $q \equiv m^2 - \sqrt{m^4-1}$.

The explicit expression (3.71) can be used to prove that $c_N = 0$ for all *negative* integers N [36]. Eqn (3.71) defines c_N only for positive integers N, as a finite product. There is an obvious prescription for analytical continuation of such products, provided continuation of the items is known (it can be considered as implied by the similar formula for integrals with varying upper limit): let

$$F(N) = \sum_{k=-\infty}^{N} f(k) , \tag{3.73}$$

then

$$S(N) \equiv \sum_{k=1}^{N} f(k) = F(N) - F(0) , \tag{3.74}$$

and, obviously $F(0) - F(-N) = \sum_{k=1-N}^{0} f(k)$, so that

$$S(-N) \equiv F(-N) - F(0) = -\sum_{k=0}^{N-1} f(-k) . \tag{3.75}$$

Exponentiation of this formula gives the rule for the products. In the case of c_N one can treat factorials in Eqn (3.71) as gamma functions,

$$(2\pi)^{\frac{1}{2}N(N-1)}c_N = \prod_{k=1}^{N}\Gamma(k) , \tag{3.76}$$

and obtain:

$$(2\pi)^{\frac{1}{2}N(N+1)}c_{-N} = \left[\prod_{k=0}^{N-1}\Gamma(-k)\right]^{-1} = 0 , \tag{3.77}$$

because of the poles of the gamma functions.

4. Integrable structure of eigenvalue models

4.1 The concept of integrability

The integrable structure of dynamical systems implies that all the dynamical characteristics — the solutions of the equations of motion (EqMs) for a classical system and functional integrals for a quantum one — can be found exactly. According to this description the notion of integrability is not very concrete, and in fact it evolves with time, including more and more classes of theories into the class of integrable systems. Nowadays we consider the following types of theories as clearly belonging to this class:

— free motion (classical or quantum) on group manifolds and homogeneous spaces;

— 2d conformal theories and their 'integrable massive deformations';

— integrable hierarchies of the (multicomponent) Kadomtsev – Petviashvili (KP) and Toda type, and their reductions;

— functional integrals, subjected to the conditions of the (generalised) Duistermaat – Heckman theorem;

— (eigenvalue) matrix models;

— topological theories;

— many supersymmetric models (at least those allowing for Nicolai transformations and/or a Duistermaat – Heckman-like description);

— systems with (infinitely) many local integrals of motion.

This list (which is in no particular order) is rather arbitrary. Also, different items are not really different and (as it should be) can be considered as different descriptions of

the same reality. Now I discuss very briefly at least some of the most important views on the concept of integrability.

Often the notion of integrability is related to the occurrence of 'sufficiently many' integrals of motion ('sufficiently' means equal to the number of degrees of freedom). This is, however, not as rigid a definition as one might think. In fact, in classical mechanics there is usually a complete set of integrals of motion available: just initial conditions in the phase space (or, to be more sophisticated, angle–action variables). The problem is, however, that:

—these obvious integrals are complicated (nonlocal and multivalued) functionals of the *current* coordinates; and

—in the general situation they are very 'unstable' under a small change of current coordinates ('divergence of trajectories').

In order to avoid these problems one usually imposes a 'locality' condition on EqMs. While this is a reasonable thing to do for particular classes of theories (e.g. possessing a well-defined kinetic term, which is quadratic in momenta), this is *not* a nice description in the general situation, since 'locality' is not invariant under arbitrary (including nonlocal) changes of variables. In practice, when approached from this side, integrability implies a kind of 'regular' behaviour of trajectories and some more-or-less nicely defined transformation from 'natural' (or, rather 'original') coordinates to the action–angle variables.

The situation becomes even less clear when quantum theory is considered, since 'chaotic behaviour' no longer implies anything really 'chaotic' for the quantum system. Again, very much depends on what kind of observables one wants to consider, and any notion of 'regularity' is not enough under an arbitrary change of variables.

This can be made even more transparent, if one recalls the idea of universality classes, so important in the modern theory. The idea is that even in the cases when the behaviour of the system seems absolutely chaotic from any naive point of view (as in the cases of turbulence or quantum gravity), one can and should introduce new variables (which can be very complicated functions of the original ones), which have smooth and well defined correlation functions. In most cases one is not attempting to find a *complete* set of such variables (and thus some information is lost), but this reflects nothing but the current state of knowledge, and in fact in studies of 2d quantum gravity the *goal* of a *complete* description is already clearly formulated.

Despite these comments, the 'definition' of integrability in terms of 'sufficiently' *local* integrals of motion should be given priority in this discussion because most of the systems which so far were considered as integrable, more or less naturally get into this class, allowing for some preferred choice of dynamical variables ('more or less' appears because some 'minor' nonlocality is usually present in any interesting examples, where angle–action variables are not obvious from the very beginning).

This 'definition' is so unclear because I attempted to look for a generic description of integrability. Most interesting approaches, however, are in another direction. One starts from some simple system and then performs a change of variables, which makes it look much more complicated (being still simple in its essence). This appears to be a much more fruitful view on the problem and in fact all the other items in my list above are describable in terms of this kind.

A trivial, but surprisingly representative example of this approach is provided by a free particle, moving in flat D-dimensional space. The eigenfunctions of the Laplace operator are just plain waves or, equivalently, spherical harmonics. The radial part of the jth harmonic is already a not very simple function, satisfying the equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{D-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{C_2(j)}{r^2}\right]\psi(r)=E\psi(r)\ . \tag{4.1}$$

This equation is of course less trivial than the original Laplace equation, but solutions are related in a simple way. In order to find a solution of Eqn (4.1), say, for $j=0$, one should take an angular average of a plane wave:

$$\phi_k(r)=\int \exp(\mathrm{i}k\vec{r}\vec{v})\mathrm{d}^{D-1}\vec{v}\ ;\qquad |\vec{v}|=1\ . \tag{4.2}$$

This integral representation expresses the solutions of Eqn (4.1) through Bessel functions, and this is in fact the proper way to derive the well-known formula

$$\phi_k(r)=2^{\frac{1}{2}D-1}\Gamma(\tfrac{1}{2}D)(kr)^{1-\frac{1}{2}D}J_{\frac{1}{2}D-1}(kr)\ . \tag{4.3}$$

If one expands the exponent in the integral in a series, the standard expansion for the Bessel function arises.

A slightly more involved example is the quantum mechanical model of a particle in the potential $\exp(-q)$, i.e. the theory of the equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+\exp(-q)\right]\psi(q)=0 \tag{4.4}$$

(one of course recognises a simplified version of Toda models). It can be solved by projection of the simple Schrodinger equation for a particle moving on the upper part of the hyperboloid $x_0^2-x_1^2-x_2^2=1$, $x_0>0$ [73]. If

$$x_0=\cosh(\tfrac{1}{2}q)+\tfrac{1}{2}z^2\exp(\tfrac{1}{2}q)\ ,$$
$$x_1=\sinh(\tfrac{1}{2}q)-\tfrac{1}{2}z^2\exp(\tfrac{1}{2}q)\ ,$$
$$x_2=z\exp(\tfrac{1}{2}q)\ ,$$

then $q=\ln(x_0+x_1)$. The Laplace operator on the hyperboloid is

$$\nabla^2=\frac{\partial^2}{\partial q^2}-\frac{1}{2}\frac{\partial}{\partial q}+\frac{1}{4}\exp(-q)\frac{\partial^2}{\partial x^2}\ , \tag{4.5}$$

and average of the wave function $\psi_\lambda(q,z)$ provides the following expression for solutions of Eqn (4.4):

$$\psi_\lambda(q)=\exp(\mathrm{i}\lambda q)\int_0^\infty t^{2\mathrm{i}\lambda-1}\exp\left[-\left(t+\frac{\exp q}{t}\right)\right]\mathrm{d}t\ . \tag{4.6}$$

This idea, which is sometimes referred to as the 'projection method' (see [73] for a broad review) reveals hidden symmetries of some complicated systems (which do not possess any symmetry at all in the usual, Noether-like, sense of the word) by considering them as embedded into wider theories with more degrees of freedom. Quantum mechanical examples of the applicability of the method are by no means exhausted by the two systems above; one can consider various projections, starting from (the exactly solvable problem of) the free motion on any group manifold, and in general this gives rise to the very important

theory of 'zonal spherical functions', which nowadays is increasingly attracting attention because of its obvious links to integrability theory and quantum geometry (see [74], for a discussion of the latter relation and [75], where relations with orthogonal poly-nomials and the generalised Kontsevich model are partly revealed). An extremely important example of free motion on a group manifold (in the infinite-dimensional Kac–Moody case) is provided by the 2d Witten–Zumino–Novikov–Wess (WZNW) model and the corresponding version of projection method is known as the Hamiltonian reduction in conformal field theory (CFT). Again the resulting theories (like minimal conformal models) do not possess any kind of symmetry in the usual sense of the word, but still they are very simple and exactly solvable because of their origin in the theory of free fields.

In principle, the theory which is reduced — i.e. complemented by constraints (initial conditions) — does not need to be absolutely symmetric — i.e. to have a Casimir operator of even zero (as in the WZNW case) as its Hamiltonian. In fact, it is possible to use the projection method to gain a lot of information about reductions of theories with more sophisticated Hamiltonians which are nontrivial functions of group generators. The simplest example is provided by the theory of quantum-mechanical 'quasi-exactly-solvable models' [76, 77] and its CFT generalisations [77, 78]. A more elaborate technique has the name 'localisation theory'† (also known as geometrical quantisation, Fourier analysis on group manifolds, and Duistermaat–Heckman theory) and provides a very wide generalisation of the above averaging procedure, which maps plain waves into Bessel functions. The classical example of a system illustrating all the aspects of integrability, starting from free motion and ending with anionic statistics, W_∞-algebras and 2d Yang–Mills theory, is the Calogero–Sutherland system, which can be associated in a uniform way with any simple Lie algebra and, in an 'intermediately involved' form, looks like a multiparticle theory in 1 + 1 dimensions with interaction potential $g^2[\sin\varepsilon(x_i - x_j)]^{-2}$, (see [73] for an introduction to the theory of Calogero-type models, and [80, 81] for the new developments).

This discussion was necessary to illustrate a very simple idea: that the theory of free particles, though trivial, is in fact inexhaustively deep. It is enough to impose sophisticated initial conditions or to perform a sophisticated change of variables in order to obtain very complicated dynamical systems, which, after they are studied, *per se* appear to be surprisingly systematic, the reason for this simplicity being that the real underlying dynamics is just trivial — that of the free particles — though it may be a very hard problem to reveal this simplicity when the system is given. It is an advantage of the *general* theory that one can begin from the proper side: from the theory of free particles and making it more and more complicated; by introducing a different kind of variables; by considering correlators of sophisticated operators; and so on. Everything that can be obtained in this way is by definition trivially integrable, though it may not be so simple to guess for somebody who did not know where the particular system at the end of this procedure appeared from.

I now proceed to a discussion of a particularly important realisation of this idea: the theory of $\bar\partial$-operators in 1 complex dimension (i.e. the theory of free holomorphic fields in 2 real dimensions). When considered as functions of moduli of bundles over Riemann surfaces (i.e. boundary conditions, imposed on 2d free fields), these simple objects (known as 'τ-functions') start looking a little involved and after all appear related to sophisticated nonlinear equations (but of course integrable) in 2 and 3 dimensions [like KdV or the KP equation]. I do not attempt to present an exhaustive theory of τ-functions and integrable hierarchies (besides being still uncompleted, this is a very big field), but instead concentrate on the very core of it, which consists of simple determinant formulas for the simplest τ-functions (namely, those associated with free-fermion theory and level $k = 1$ Kac–Moody algebras). This issue will be discussed in some detail, because besides being the basis of integrable hierarchy theory, it is also where the links with the matrix models are found.

4.2 The notion of τ-function

There are several different definitions of τ-functions, but all of them are particular realisations of the following idea: the τ-function is a generating functional of all the correlation functions in the theory of free particles in 1 + 1 dimensions. This basic quantity is a kind of '*det* D', where D is a time-evolution operator (continuous or discrete) and '*det* ' is a sort of product over eigenvalues of D, which is usually expressed in the form of a functional integral, associated with free particles (it is not *a priori* gaussian in the original variables). This quantity is the most general definition of the τ-function.

In practice one is usually more specific. The most well-studied version of τ-function arises if one thinks about free particles of a peculiar type: free fermions with quadratic Hamiltonian and continuous time evolution, i.e. the theory of the spin-$\frac{1}{2}$ b, c-system (fermions), $\tilde\psi(\tilde z, z)$, $\psi(\tilde z, z)$, described by the functional integral

$$\tau\{A\} \sim Det\,(\bar\partial + \mathcal{A})$$
$$\sim \int \mathrm{D}\tilde\psi \mathrm{D}\psi \exp\left(\int_{\mathrm{d}^2 z} \tilde\psi \bar\partial \psi\right)$$
$$\times \exp\left[\int_{\mathrm{d}^2 z}\int_{\mathrm{d}^2\tilde z} A(z,\tilde z)\delta(\tilde{\bar z} - \bar z)\psi(z)\tilde\psi(\tilde z)\right]$$

where $\bar z$ plays the role of time and $\mathcal{A} = A(z,\tilde z)\delta(\tilde{\bar z} - \bar z)\mathrm{d}\tilde{\bar z}\,\mathrm{d}\bar z$ is some $(\frac{1}{2}, 1; \frac{1}{2}, 1)$-bidifferential (i.e. contains a factor of $\mathrm{d}\tilde z^{1/2}\mathrm{d}\tilde{\bar z}\,\mathrm{d}z^{1/2}\mathrm{d}\bar z$).

Of course, one can think about more general τ-functions, involving many fermions (this is often done), and more general b-, c- and β-, γ-systems, in particular, arising in the context of the WZNW model associated with any Kac–Moody algebra of any level.‡ It is also of interest to consider discrete time evolution (described by difference equations rather than in differential equations), though, as usual in the 2d theories, this is not really a independent problem.

† For various views and approaches to this theory see [5, 43, 44, 61–65, 79]. (So far there are no connections with Andersson localisation in solid-state physics.)

‡ The main technical difference between the generic and the 'free-fermion' cases is that the Lagrangian of generic free-field theory is not just quadratic in the scalar fields φ, but can also contain particular combinations of exponents $\exp(\varphi)$. It is also worth noting that the most general expression, quadratic in scalar fields, if rewritten in terms of fermions is in fact quartic (but, of course, a generic quartic interaction does not arise in this way). The integrable nature of certain quartic-fermion interactions is well known from the theory of Thirring models (in this class of models interactions are usually local).

In the language of matrix models the restriction to *free-fermion* τ-functions is essentially equivalent to the restriction to *eigenvalue* models. Serious consideration of noneigenvalue models, aimed at revealing their integrable (solvable) structure will certainly involve the theory of generic τ-functions, but both these things are matters for future research, and I shall not go into details about them in these notes.

4.3 The τ-function, associated with the free fermions

Because of the specific form of the Lagrangian in Eqn (4.7) the functional integral can easily be represented in Hamiltonian form, provided the topology of the 2-surface on which $(\bar{z}, z)$ are coordinates, is trivial (genus 0: sphere or annulus). Namely, consider $\tilde{\psi}$ and ψ as operator-valued functions of z only (not of the time $\bar{z}$). Then the only thing reminiscent of a kinetic term $\int_{d^2z} \tilde{\psi}\bar{\partial}\psi$ is the canonical commutation relation

$$[\tilde{\psi}(\tilde{z}), \psi(z)]_+ = \delta(\tilde{z} - z) d\tilde{z}^{1/2} dz^{1/2} . \tag{4.7}$$

Then,

$$\tau\{A\} \sim \langle 0| \exp\left[\oint_{d\tilde{z}} \oint_{dz} A(z, \tilde{z})\, \psi(z)\, \tilde{\psi}(\tilde{z})\right] |0\rangle . \tag{4.8}$$

Now it is usual to expand around $z = 0$:

$$\psi(z) = \sum_{n\in Z} \psi_n z^n dz^{1/2} ; \quad \tilde{\psi}(z) = \sum_{n\in Z} \tilde{\psi}_n z^{-n-1} dz^{1/2} ;$$

$$[\tilde{\psi}_m, \psi_n]_+ = \delta_{m,n} ;$$

$$\psi_m|0\rangle = 0 \quad \text{for} \quad m < 0 ; \quad \tilde{\psi}_m|0\rangle = 0 \quad \text{for} \quad m \geqslant 0 ;$$

$$A(z, \tilde{z}) = \sum_{m,n\in Z} z^{-m-1} \tilde{z}^n A_{mn}\, dz^{1/2} d\tilde{z}^{1/2} ;$$

so that

$$\oint_{d\tilde{z}} \oint_{dz} A(z, \tilde{z})\, \psi(z)\, \tilde{\psi}(\tilde{z}) = \sum_{m,n\in Z} A_{mn} \psi_m \tilde{\psi}_n .$$

In fact, this expansion could be around *any* point z_0 and on a 2-surface of any topology: topological effects can be easily included as specific shifts of the functional $A(z, \tilde{z})$ by a combination of 'handle-gluing operators'. Analogous shifts can imitate the change of basic functions z^n for $z^{n+\alpha}$ and more complicated expressions (holomorphic $\frac{1}{2}$-differentials with various boundary conditions on surfaces of various topologies).

One can now consider whether *local* functionals $A(z, \tilde{z}) = U(z)\, \delta(\tilde{z} - z)\, dz^{1/2} d\tilde{z}^{1/2}$ play any special role. The corresponding contribution to the Hamiltonian looks like†

$$H_{\text{Cartan}} = \oint_{dz} U(z)\, \psi(z)\, \tilde{\psi}(z) = \oint_{dz} U(z)\, J(z) , \tag{4.9}$$

where

$$J(z) = \psi(z)\, \tilde{\psi}(z) = \sum_{n\in Z} J_n z^{-n-1} dz \tag{4.10}$$

is the $U(1)_{k=1}$ Kac–Moody current;

$$J_n = \sum_{m\in Z} \psi_m \tilde{\psi}_{m+n} ; \quad [J_m, J_n] = m\delta_{m+n,0} . \tag{4.11}$$

† Note that the normalisation factor here is different by a factor of $1/\sqrt{2}$ from that in the discussion of discrete models in Sections 2.3, 2.7, and 2.8. This is not just a change of *notation*, since the Miwa transformation can lead to different results when this normalisation is changed. See a footnote in Section 4.6 below for more detailed discussion.

If the scalar function (potential) $U(z)$ is expanded as $U(z) = \sum_{k\in Z} t_k z^k$, then

$$H_{\text{Cartan}} = \sum_{n\in Z} t_k J_k . \tag{4.12}$$

This contribution to the whole Hamiltonian can be considered to be special for the following reason. Let us return to the original expression (4.8) and try to consider it as a generating functional for all the correlation functions of $\tilde{\psi}$ and ψ. Naively, variation with respect to $A(z, \tilde{z})$ should produce the bilinear combination $\psi(z)\tilde{\psi}(\tilde{z})$ and this would solve the problem. However, things are not so trivial, because the operators involved do not commute (and in particular, the exponential operator in Eqn (4.8) should still be defined less symbolically, see the next subsection). Things would be much simpler if one were to consider a *commuting* set of operators: this is where the abelian $\widehat{U(1)}_{k=1}$ subgroup of the entire $GL(\infty)_{k=1}$ group (and even its purely commuting Borel subalgebra) enters the game. Remarkably, it is sufficient to deal with this abelian subgroup in order to reproduce all the correlation functions.‡ The crucial point is the identity for free fermions (generalisable to any b, c-systems):

$$:\psi(\lambda)\tilde{\psi}(\tilde{\lambda}): = :\exp\left(\int_{\lambda}^{\tilde{\lambda}} J\right): , \tag{4.13}$$

which is widely known in the form of bosonisation formulas:§ if $J(z) = \partial\phi(z)$,

$$\tilde{\psi}(\tilde{\lambda}) \sim :\exp[(\phi(\tilde{\lambda})]: \quad \{:\psi(\infty)\tilde{\psi}(\tilde{\lambda}): = :\exp[\phi(\tilde{\lambda}) - \phi(\infty)]:\},$$

$$\psi(\lambda) \sim :\exp[-\phi(\lambda): \quad \{:\psi(\lambda)\tilde{\psi}(\infty): = :\exp[\phi(\infty) - \phi(\lambda)]:\}.$$

This identity implies that one can generate any bilinear combinations of ψ-operators by variation of the potential $U(z)$ only; moreover, this variation should be of the specific form

$$\Delta\oint UJ = \Delta\left(\sum_{k\in Z} t_k J_k\right) = \int_z^{\tilde{z}} J = \sum_{k\in Z} \int_z^{\tilde{z}} z^{-k-1} dz$$

$$= \sum_{k\in Z} \frac{1}{k} J_k \left(\frac{1}{z^k} - \frac{1}{\tilde{z}^k}\right) ,$$

that is

$$\Delta t_k = \frac{1}{k}\left(\frac{1}{z^k} - \frac{1}{\tilde{z}^k}\right) . \tag{4.14}$$

‡ I once again emphasise that this trick is specific to free fermions and for the level $k = 1$ Kac–Moody algebras, which can be expressed entirely in terms of free fields associated with Cartan generators (modulo some unpleasant details, related to 'cocycle factors' in the Frenkel–Kac representations [47], which are in fact reminiscent of free fields associated with the non-Cartan generators (parafermions) [58]. These can, however, be put under the carpet and/or taken into account 'by hand' as unpleasant but nonessential(?) sophistications).

§ The formulas in brackets are indeed correct; before them the usual symbolic relations are written. Using these formulas we get

$$:\psi(\lambda)\tilde{\psi}(\tilde{\lambda}): = :\exp\left[\phi(\tilde{\lambda}) - \phi(\lambda)\right]: = :\exp\left(\int_{\lambda}^{\tilde{\lambda}} \partial\phi\right): = :\exp\left(\int_{\lambda}^{\tilde{\lambda}} J\right):$$

This identity can of course be obtained within fermionic theory; one should take into account only those ψ-operators that are nilpotent, so that the exponent of a single ψ-operator would be just the sume of two terms (polynomial).

Note that this is not an infinitesimal variation and that it has exactly the form consistent with the Miwa parametrisation used in Section 3.

Since any bilinear combination can be generated in this way from $U(z)$, it is clear that the entire Hamiltonian $\sum A_{mn}\widetilde{\psi}_m\psi_n$ can also be considered as resulting from some transformation of V (i.e. of 'time-variables' t_k). In other words,

$$\tau\{A\} = \mathcal{O}_A[t]\tau\{A = U\} .$$

These operators $\mathcal{O}_A$ are naturally interpreted as elements of the group GL(∞), acting on the 'universal Grassmannian' [82, 83, 84], parametrised by the matrices A_{mn} modulo changes of coordinates $z \to f(z)$. This representation for $\tau\{A\}$ is, however, not very convenient, and usually one considers an *infinitesimal* version of the transformation, which just shifts A:

$$\tau\{t|A + \delta A\} = \hat{\mathcal{O}}_{\delta A}[t]\tau\{t|A\} . \tag{4.15}$$

Note that this transformation clearly distinguishes between the dependencies of τ on t and on all other components of A. The possibility of using such a representation with the privileged role of Cartan generators is the origin of all the simplifications arising in the case of free-fermion τ-functions.†

Relation (4.15) is the basis of the orbit interpretation of τ-functions [83]. It is also important to understand the role of the 'string equation' and other constraints, imposed on τ-functions in the theory of matrix models. These arise as some particular subalgebras in the set of $\hat{\mathcal{O}}$-operators, and their role is to specify particular points, A, in the Grassmannian, of which this subalgebra is a stabiliser.‡ The simplest examples are in fact provided by formulas from Section 2.3 above, where combinations of the screening charges describe As which are stable points of *discrete* Virasoro and W-constraints (in the latter case the multi-fermion system is used).

The fact that the τ-function at all the points A of a Grassmannian can be obtained by the group action from $\tau\{0\}$, has an implication, known as the Hirota equation. The idea [83] is just that there are Casimir operators in the group, which commute with the group action and thus the eigenvalue of the Casimir operator is the same for $\tau\{A\}$ at all points A. In the free-fermion case the simplest example of a Casimir operator is given by

$$J_0 = \oint J = \oint \psi\widetilde{\psi} = \sum_{n\in Z}\psi_n\widetilde{\psi}_n . \tag{4.16}$$

The eigenvalue of this operator for the vacuum state $|0\rangle$ is an infinite subtraction constant, and this makes the equation $J_0\mathcal{O}_A|0\rangle = \mathcal{O}_A J_0|0\rangle = \text{const}\cdot\mathcal{O}_A|0\rangle$, or $J_0\tau\{A\} = \text{const}\cdot\tau\{A\}$, not very interesting. However, this operator is represented in bilinear form and in such cases the following trick is usually useful.

If the operator T^aT^a, which is bilinear in the generators of the algebra, commutes with the action of the group, so does $T^a\otimes T^a$ if the group action on the tensor product of representations is defined as $|\rangle\otimes|\rangle \to \mathcal{O}_A|\rangle\otimes\mathcal{O}_A|\rangle$. Indeed, $(T^a\otimes \mathrm{I} + \mathrm{I}\otimes T^a)^2$ then commutes with the group action and so does $T^a\otimes T^a = \frac{1}{2}[(T^a\otimes\mathrm{I}+\mathrm{I}\otimes T^a)^2 - T^aT^a\otimes\mathrm{I} - \mathrm{I}\otimes T^aT^a)]$. If, further, $T^a\otimes T^a$ annihilates the product of two vacuum states:

$$(T^a\otimes T^a)|0\rangle\otimes|0\rangle = 0 , \tag{4.17}$$

then the same equation holds for all A:

$$(T^a\otimes T^a)|\mathcal{O}_A\rangle\otimes|\mathcal{O}_A\rangle = 0 . \tag{4.18}$$

Condition (4.17) is trivially valid in our case:

$$\sum_{n\in Z}\psi_n|0\rangle\otimes\widetilde{\psi}_n|0\rangle = 0 , \tag{4.19}$$

since in every term in the sum, one of the vacuum states is annihilated: the first one if $n \geqslant 0$ and the second if $n < 0$.§ Thus, we obtain the relation

$$\sum_{n\in Z}\psi_n|\mathcal{O}_A\rangle\otimes\widetilde{\psi}_n|\mathcal{O}_A\rangle = 0 , \tag{4.20}$$

which can now be multiplied from the left by

$$\langle 0|\psi(\infty)\exp[H_{\text{Cartan}}(t)]\otimes\langle 0|\widetilde{\psi}(\infty)\exp[H_{\text{Cartan}}(t')]$$

(t'_k need not coincide with t_k) and after the ψ-operators are expressed as the time shifts, we obtain

$$\sum_{n\in Z} D_n^-\tau\{t|A\}\otimes D_n^+\tau\{t'|A\} = 0 , \tag{4.21}$$

where

$$\sum_{n\geqslant 0} D_n^{\pm} z^{-n} = \exp\left(\pm\sum_{k>0}\frac{1}{kz^k}\frac{\partial}{\partial t_k}\right) .$$

This is a particular form of the Hirota equation [86], which is often used to define τ-functions, associated with integrable hierarchies. If one takes Eqn (4.8) for the definition, as it is

† This is also the reason, why these are the free-fermion τ-functions that appear in the study of ordinary integrable hierarchies: the Hamiltonian flows, which describe evolution in different t-directions, just commute because the ts are associated with the commuting Cartan generators of GL(∞). In the more general situation the flows would form a closed, but *nonabelian*, algebra.

‡ This relation is straightforward in the case of Virasoro constraints, since Virasoro algebra is just a subalgebra of GL(∞) acting on τ-functions, and thus is a symmetry (covariance) of the associated integrable hierarchies [84]. W-constraints do not form a Lie-subalgebra of this GL(∞), they arise after a certain reduction, which in turn exists in a simple form *not* everywhere on the Grassmannian (in particular W is *not* a symmetry of the entire KP hierarchy [85]: here we deal with a more sophisticated self-consistency relation, which remains to be understood in full detail (e.g. it is unknown whether reduction exists at all at *any* Virasoro-stable point, which would significantly simplify this kind of consideration). In fact, the entire relation between the constraints and τ-functions is not exhaustively worked out: for example, there is still no clear and satisfactory proof that the full set of Virasoro and/or W-constraints implies that the partition function is a τ-function, which would be purely algebraic and not refer to the uniqueness of solutions to the constraints. The result, widely discussed in the literature (see [29]) is that the string equation (the lowest Virasoro constraint $L_{-1}Z = 0$), if imposed on Z, which is somehow known to be the properly reduced τ-function, implies the entire set of Virasoro and W-constraints (though even this proof can still have some loopholes).

§ It is easy to verify directly that $\sum_n \psi_n\otimes\widetilde{\psi}_n$ is indeed a Casimir operator in the tensor product

$$\left[\sum_n\psi_n\otimes\widetilde{\psi}_n,\ \mathrm{I}\otimes\sum_{l,m}A_{lm}\psi_l\widetilde{\psi}_m + \sum_{l,m}A_{lm}\psi_l\widetilde{\psi}_m\otimes\mathrm{I}\right]$$
$$= \sum_n\left(\psi_n\otimes\sum_m A_{nm}\widetilde{\psi}_m - \sum_l A_{ln}\psi_l\otimes\widetilde{\psi}_n\right)$$
$$= \sum_l\sum_m A_{lm}(\psi_l\otimes\widetilde{\psi}_m - \psi_l\otimes\widetilde{\psi}_m) = 0 .$$

more natural to do in the general 'theory of everything' and as we did above, Eqn (4.21) is the starting point for the path, leading to hierarchies in conventional form of differential equations; the Lax and pseudodifferential representations *naturally* appearing on the way. I do not go along this path in these notes.

The last remark to be made, before I proceed to more detailed formulas, is that τ-functions can be considered as the determinants $Det\,\bar\partial$ of the $\bar\partial$-operators acting on fields with some complicated boundary conditions [like $\psi(z) \sim \exp\left(\sum_{k>0} t_k z^{-k}\right)$ in the simplest cast of t-dependencies]. Entire A-dependence is usually described in this context as that on points in the 'universal module space', which once appeared in the study of string models on Riemann surfaces of arbitrary genus [87]. From this point of view, more general τ-functions are sections of the bundles over the universal module space associated with conformal models and is more sophisticated than just the theory of free fermions (and b-, c-systems). The WZNW model is, of course, the most important example to be studied in this context.

The crucial feature of all the quantities associated in this way to conformal models is the applicability of the Wick theorem, reducing multipoint correlation functions to pair correlators. In the free-fermion case this is just a consequence of the quadratic form of the Lagrangian; in the generic situation this follows from the existence of holomorphic operator algebra, which allows one to define the correlators by fixing the monodromy properties dictated by the pairwise collision of points. The Wick theorem is the concrete source of determinant formulas for τ-functions, which are used in order to establish their relations with matrix models and other branches of string theory.

After this discussion of the context where free-fermion τ-functions can and do appear, I turn now to more detailed and exact formulas that are relevant in this particular free-fermion case. The only sophisticated part of the work with these formulas is the accurate accounting for the normal ordering routine [88] (mostly due to the Japanese school [88], though many other people contributed to this field after it was established), which will be mostly unnecessary for our purposes. In the main, I shall follow the presentation of [30, 36, 89].

4.4 Basic determinant formula for the free-fermion correlator

Let us consider the following matrix element:

$$\tau_N\{t,\bar t|G\} = \langle N|\exp(H)G\exp(\bar H)|N\rangle\,, \tag{4.22}$$

where

$$\psi(z) = \sum_{n\in Z}\psi_n z^n \mathrm{d}z^{1/2}\,; \quad \tilde\psi(z) = \sum_{n\in Z}\tilde\psi_n z^{-n-1}\mathrm{d}z^{1/2}\,;$$

$$G = \exp\left(\sum_{m,n\in Z} A_{mn}\psi_m\tilde\psi_n\right)\,;$$

$$H = \sum_{k>0} t_k J_k\,, \quad \bar H = \sum_{k>0}\bar t_k J_{-k}\,;$$

$$J(z) = \psi(z)\tilde\psi(z) = \sum_{n\in Z} J_n z^{-n-1}\mathrm{d}z\,; \quad J_n = \sum_k \psi_k\tilde\psi_{k+n}\,;$$

$$[\tilde\psi_m,\psi_n]_+ = \delta_{m,n}\,; \quad [J_m,J_n] = m\delta_{m+n,0}\,;$$

$$\psi_m|N\rangle = 0\,, \quad m<N\,; \quad \langle N|\psi_m = 0\,, \quad m\geqslant N\,;$$

$$\tilde\psi_m|N\rangle = 0\,, \quad m\geqslant N\,; \quad \langle N|\tilde\psi_m = 0\,, \quad m<N\,;$$

$$J_m|N\rangle = 0\,, \quad m>0\,; \quad \langle N|J_m = 0\,, \quad m<0\,. \tag{4.23}$$

The 'Nth vacuum state' $|N\rangle$ is defined as the Dirac sea, filled up to level N:

$$|N\rangle = \prod_{i=N}^{\infty}\tilde\psi_i|\infty\rangle = \prod_{i=-\infty}^{N-1}\psi_i|-\infty\rangle\,;$$

$$\langle N| = \langle\infty|\prod_{i=N}^{\infty}\psi_i = \langle-\infty|\prod_{i=-\infty}^{N-1}\tilde\psi_i\,, \tag{4.24}$$

where the 'empty' (bare) and 'completely filled' vacua are defined so that:

$$\tilde\psi_m|-\infty\rangle = 0\,, \quad \langle-\infty|\psi_m = 0\,,$$

$$\psi_m|\infty\rangle = 0\,, \quad \langle\infty|\tilde\psi_m = 0\,, \tag{4.25}$$

for *any* $m\in Z$. For the same reason that operators J, H, $\bar H$, and G are defined so that they have usually $\tilde\psi$ at the very right and ψ at the very left, we have also:

$$J_m|-\infty\rangle = 0\,, \quad \langle-\infty|J_m = 0\,,$$

$$G^{\pm1}|-\infty\rangle = |-\infty\rangle\,; \quad \langle-\infty|G^{\pm1} = \langle-\infty|\,;$$

$$\exp(\pm\bar H)|-\infty\rangle = |-\infty\rangle\,; \quad \langle-\infty|\exp(\pm H) = \langle-\infty|\,. \tag{4.26}$$

Now one can use all these formulas to rewrite the original correlator Eqn (4.22) as:

$$\begin{aligned}
&\langle N|\exp(H)\,G\exp(\bar H)|N\rangle \\
&= \langle-\infty|\left(\prod_{i=-\infty}^{N-1}\tilde\psi_i\right)\exp(H)\;G\exp(\bar H)\left(\prod_{i=-\infty}^{N-1}\psi_i\right)|-\infty\rangle \\
&= \langle-\infty|\exp(-H)\left(\prod_{i=-\infty}^{N-1}\tilde\psi_i\right)\exp(H)\,G \\
&\qquad\times\exp(\bar H)\left(\prod_{i=-\infty}^{N-1}\psi_i\right)\exp(-\bar H)|-\infty\rangle \\
&= \langle-\infty|\prod_{i=-\infty}^{N-1}\tilde\Psi_i[t]\prod_{j=-\infty}^{N-1}\Psi_j^G[\bar t]|-\infty\rangle \\
&= \mathrm{Det}_{-\infty<i,j<N}\langle-\infty|\tilde\Psi_i[t]\Psi_j^G[\bar t]|-\infty\rangle \\
&= \mathrm{Det}_{i,j<0}\mathcal{H}_{i+N,j+N}\,.
\end{aligned} \tag{4.27}$$

The last two steps here introduce 'GL(∞)-rotated' fermions,

$$\tilde\Psi_i[t] \equiv \exp(-H)\psi_i\exp(H)\,;$$

$$\Psi_j[\bar t] \equiv \exp(\bar H)\psi_j\exp(-\bar H)\,;$$

$$\Psi_j^G[\bar t] \equiv G\bar\Psi_j[\bar t]G^{-1}\,; \tag{4.28}$$

and an application of the Wick theorem to express the multifermion correlation function through pair correlators:

$$\begin{aligned}
\mathcal{H}_{ij}(t,\bar t) &\equiv \langle-\infty|\tilde\Psi_i[t]\Psi_j^G[\bar t]|-\infty\rangle \\
&= \langle-\infty|\tilde\Psi_i[t]G\Psi_j[\bar t]|-\infty\rangle\,,
\end{aligned} \tag{4.29}$$

(once again the fact that $G^{-1}|-\infty\rangle = |-\infty\rangle$ was used). The only nontrivial dynamical information entered through the use of the Wick theorem, and for that it was crucial that all

the operators $\exp(H)$, $\exp(\bar{H})$, G are *quadratic* exponents, i.e. can only modify the shape of the propagator, but do not destroy the quadratic form of the action (fields remain *free*). This is exactly equivalent to the statement that 'Heisenberg' operators $\Psi[t]$ are just 'rotations' of ψ, i.e. that transformations (4.28) are *linear*.

I shall now describe these transformations in a little more detail. Namely, their entire time-dependence can be encoded in terms of Schur polynomials, $P_n(t)$. These are defined to have a very simple generating function (which we have already encountered many times in the theory of matrix models):

$$\sum_{n\geqslant 0} P_n(t)\, z^n = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \tag{4.30}$$

(i.e. $P_0 = 1$, $P_1 = t_1$, $P_2 = \frac{1}{2}t_1^2 + t^2$, etc.), and satisfy the relation

$$\frac{\partial P_n}{\partial t_k} = P_{n-k} \ . \tag{4.31}$$

Since

$$\exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \prod_{k>0}\left(\sum_{n_k\geqslant 0}\frac{1}{n_k!}t_k^{n_k} z^{kn_k}\right) ,$$

the Schur polynomials can also be represented as

$$P_n(t) = \sum_{\left\{n_k \,\middle|\, \sum_{k>0} kn_k = n\right\}}\left(\prod_{k>0}\frac{1}{n_k!}t_k^{n_k}\right) . \tag{4.32}$$

Now, since

$$\exp(-B)A\ \exp(B) = A + [A,B] + \frac{1}{2!}\left[[A,B],B\right] + \frac{1}{3!}\Big[\big[[A,B],B\big],B\Big] + \dots ,$$

and

$$[\widetilde{\psi}_i, J_k] = \widetilde{\psi}_{i+k} , \quad \left[[\widetilde{\psi}_i, J_{k_1}], J_{k_2}\right] = \widetilde{\psi}_{i+k_1+k_2}, \dots ,$$

we have, for every fixed k,

$$\exp(-t_k J_k)\widetilde{\psi}_i \exp(t_k J_k) = \sum_{n_k\geqslant 0}\frac{t_k^{n_k}}{n_k!}\widetilde{\psi}_{i+kn_k} .$$

It remains to note that all the harmonics of J in $H = \sum_{k>0} t_k J_k$ commute with each other, which yields:

$$\begin{aligned}\widetilde{\Psi}_i(t) &= \exp(-H)\,\widetilde{\psi}_i \exp(H) \\ &= \left[\prod_{k>0}\exp(-t_k J_k)\right]\widetilde{\psi}_i\left[\prod_{k>0}\exp(t_k J_k)\right] \\ &= \sum_{n\geqslant 0}\widetilde{\psi}_{i+n}\left[\sum_{\{n_k|\sum_{k>0}kn_k=n\}}\left(\prod_{k>0}\frac{1}{n_k!}t_k^{n_k}\right)\right] \\ &\overset{(4.32)}{=} \sum_{n\geqslant 0}\widetilde{\psi}_{i+n}P_n(t) = \sum_{l\geqslant i}\widetilde{\psi}_l P_{l-i}(t) \ .\end{aligned} \tag{4.33}$$

Similarly, the relation $[J_k, \psi_j] = \psi_{k+j}$ implies that

$$\begin{aligned}\Psi_j(\bar{t}) &= \exp(\bar{H})\psi_j \exp(-\bar{H}) \\ &= \sum_{n\geqslant 0}\psi_{j+n}P_n(\bar{t}) = \sum_{m\geqslant j}\psi_m P_{m-j}(\bar{t})\end{aligned} \tag{4.34}$$

and finally†

$$\begin{aligned}\mathcal{H}_{ij} &= \sum_{l\geqslant i,\, m\geqslant j}\langle -\infty|\widetilde{\psi}_l\, G\,\psi_m|-\infty\rangle P_{l-i}(t)P_{m-j}(\bar{t}) \\ &= \sum_{l\geqslant i,\, m\geqslant j} T_{lm}P_{l-i}(t)P_{m-j}(\bar{t}) ,\end{aligned} \tag{4.35}$$

which implies also that

$$\frac{\partial\mathcal{H}_{ij}}{\partial t_k} = \mathcal{H}_{i+k,j} , \quad \frac{\partial\mathcal{H}_{ij}}{\partial\bar{t}_k} = \mathcal{H}_{i,j+k} . \tag{4.36}$$

The matrix

$$T_{lm} \equiv \langle -\infty|\widetilde{\psi}_l\, G\,\psi_m|-\infty\rangle \tag{4.37}$$

is the one which defines fermion rotations under the action of the $\mathrm{GL}(\infty)$ group element G:

$$\begin{aligned}&G\psi_m G^{-1} = \sum_{l\in Z}\psi_l T_{lm} \ ; \\ &G^{-1}\widetilde{\psi}_l G = \sum_{m\in Z}T_{lm}\widetilde{\psi}_m , \quad \text{or} \quad G\widetilde{\psi}_l G^{-1} = \sum_{m\in Z}(T^{-1})_{lm}\widetilde{\psi}_m .\end{aligned} \tag{4.38}$$

If $G = 1$, $T_{lm} = \delta_{lm}$. If all $t_k = \bar{t}_k = 0$, then $\mathcal{H}_{ij} = T_{ij}$.

4.5 Toda-lattice τ-function and linear reductions of the Toda-lattice hierarchies

In the previous subsection I derived a formula,

$$\tau_N\{t,\bar{t}|G\} = \mathrm{Det}_{i,j<0}\mathcal{H}_{i+N,j+N} , \tag{4.39}$$

for the basic correlator, which defines the 'Toda-lattice τ-function'. For obvious reasons, $\bar{t}$ is often referred to as negative-time. The τ-function can be normalised by dividing by the same quantity for all vanishing time-variables, but this is not always convenient. Eqn (4.39) has generalisations when similar matrix elements in a multifermion system are considered — this leads to 'multicomponent Toda' (or AKNS) τ-functions. Generalisations to arbitrary conformal models should be considered as well. It has also particular 'reductions', of which the most important are: KP, forced (semi-infinite), and Toda-chain τ-functions. This is the subject to be discussed in this subsection.

The idea of linear reduction is that the form of the operator G or, equivalently, of the matric T_{lm} in Eqn (4.35), can be adjusted in such a way that $\tau_N\{t,\bar{t}|G\}$ becomes independent of some variables; i.e. equation(s)

$$\left(\sum_k \alpha\frac{\partial}{\partial t_k} + \sum_k \bar{\alpha}\frac{\partial}{\partial\bar{t}_k} + \sum_k \beta_k D_N(k) + \gamma\right)\tau_N\{t,\bar{t}|G\} = 0 \tag{4.40}$$

can be solved as equations for G for all the values of t, $\bar{t}$, and N at once. [In Eqn (4.40) $D_N(k)f_N \equiv f_{N+k} - f_N$.] In this case the system of integrable equations (hierarchy), arising from

†Eqns (4.34) can be also interpreted as representations of Schur polynomials in terms of fermionic correlators in the bare vacuum:

$$P_m(\bar{t}) = \langle -\infty|\widetilde{\psi}_{j+m}\exp(\bar{H})\psi_j|-\infty\rangle$$

$$P_m(t) = \langle -\infty|\widetilde{\psi}_i\exp(H)\psi_{i+m}|-\infty\rangle .$$

the Hirota equation for τ, gets reduced and one usually speaks about a 'reduced hierarchy'. Usually Eqn (4.40) is imposed directly on matrix $\mathcal{H}_{ij}$; of course, then Eqn (4.40) is just a corollary.

I shall refer to the situation when Eqn (4.40) is fulfilled for *any* $t, \bar{t}, N$ as a 'strong reduction'. It is often reasonable to consider also 'weak reductions', when Eqn (4.40) is satisfied on particular infinite-dimensional hyperplanes in the space of time-variables. Weak reduction is usually a property of the entire τ-function as well, but is not expressible in the form of a local linear equation, satisfied identically for *all* values of $t, \bar{t}, N$. Now I proceed to concrete examples.

Toda-chain hierarchy. This is a *strong* reduction. The corresponding constraint Eqn (4.40) is just

$$\frac{\partial \mathcal{H}_{ij}}{\partial t_k} = \frac{\partial \mathcal{H}_{ij}}{\partial \bar{t}_k}\,, \tag{4.41}$$

or, because of Eqn (4.36), $\mathcal{H}_{i+k,j} = \mathcal{H}_{i,j+k}$. It has an obvious solution:

$$\mathcal{H}_{i,j} = \hat{\mathcal{H}}_{i+j}\,, \tag{4.42}$$

i.e. $\mathcal{H}_{ij}$ is expressed in terms of a one-index quantity $\hat{\mathcal{H}}_i$. It is, however, not enough to ask what the restrictions on $\mathcal{H}_{ij}$ are — the equations should be satisfied for all t and $\bar{t}$ at once, i.e. should be resolvable as equations for T_{lm}. In the case under consideration this is simple: T_{lm} should be such that

$$T_{lm} = \hat{T}_{l+m}\,. \tag{4.43}$$

Indeed, then

$$\mathcal{H}_{ij} = \sum_{l,m} T_{lm} P_{l-i}(t) P_{m-j}(\bar{t}) = \sum_{l,m} \hat{T}_{l+m} P_{l-i}(t) P_{m-j}(\bar{t})$$
$$= \sum_{n \geqslant 0} \hat{T}_{n+i+j} \left[\sum_{k=0}^{n} P_k(t) P_{n-k}(\bar{t}) \right],$$

and

$$\hat{\mathcal{H}}_i = \sum_{n \geqslant 0} \hat{T}_{n+i} \left[\sum_{k=0}^{n} P_k(t) P_{n-k}(\bar{t}) \right]. \tag{4.44}$$

Volterra hierarchy. The Toda-chain τ-function can be further *weakly* reduced to satisfy the identity

$$\left. \frac{\partial \tau_{2N}}{\partial t_{2k+1}} \right|_{\{t_{2l+1}=0\}} = 0\,, \quad \text{for all } k\,, \tag{4.45}$$

i.e. τ_{2N} is required to be an even function of all odd-times t_{2l+1} (this is an example of 'global characterisation' of the weak reduction). Note that Eqn (4.45) is imposed only on *Toda-chain* τ-functions with *even* values of zero-time. Then Eqn (4.45) will hold whenever $\hat{\mathcal{H}}_i$ in Eqn (4.44) are even (odd) functions of t_{odd} for even (odd) values of i. Since Schur polynomials $P_k(t)$ are even (odd) functions of odd-times for even (odd) k, it is enough that the sum in Eqn (4.44) goes over even (odd) n when i is even (odd). In other words, the restriction on T_{lm} is that

$$T_{lm} = \hat{T}_{l+m}\,, \quad \text{and} \quad \hat{T}_{2k+1} = 0 \quad \text{for all } k\,. \tag{4.46}$$

Forced hierarchies. This is another important example of strong reduction. It also provides an example of *singular* τ-function, arising when

$$G = \exp\left(\sum A_{mn} \psi_m \tilde{\psi}_n \right)$$

blows up and normal ordering operators should be used to define regularised τ-functions. Forced hierarchy appears when G can be represented in the form [89] $G = G_0 P_+$, where projection operator P_+ is such that

$$P_+|N\rangle = |N\rangle \quad \text{for} \quad N \geqslant N_0$$
$$P_+|N\rangle = 0 \qquad \text{for} \quad N < N_0\,. \tag{4.47}$$

Explicit expression for this operator is†

$$P_+ = :\exp\left(-\sum_{l<N_0} \tilde{\psi}_l \psi_l \right): = \prod_{l<N_0} (1 - \tilde{\psi}_l \psi_l) = \prod_{l<N_0} \psi_l \tilde{\psi}_l\,.$$

Because of (4.47), $P_+|-\infty\rangle = 0$, and the identity $G|-\infty\rangle = |-\infty\rangle$, which played an essential role in the derivation of (4.27), can be satisfied only if G_0 is singular and $T_{lm} = \infty$. In order to avoid this problem one usually introduces in the vicinity of such singular points in the universal module space a sort of a normalised (forced) τ-function $\tau_N^{\mathrm{f}} \equiv \tau_N/\tau_{N_0}$. One can check that now $T_{lm}^{\mathrm{f}} = \infty$ for all $l, m < N_0$, and τ^{f} can be represented as determinant of a finite dimensional matrix [90, 89].

$$\tau_N^{\mathrm{f}} = \mathrm{Det}_{N_0 \leqslant i,\, j<N} \mathcal{H}_{ij}^{\mathrm{f}}, \quad \text{for} \quad N > N_0\,;$$
$$\tau_{N_0}^{\mathrm{f}} = 1\,; \tag{4.48}$$
$$\tau_N^{\mathrm{f}} = 0, \quad \text{for} \quad N < N_0\,.$$

For $N > N_0$ we have now a determinant of a *finite*-dimensional $(N-N_0) \times (N-N_0)$ matrix. The choice of N_0 is not really essential; therefore it is better to put $N_0 = 0$ in order to simplify the formulas, phrasing, and relations with the discrete matrix models (N_0 is easily restored if everywhere N is substituted by $N-N_0$). For forced hierarchies one can also represent $\hat{\tau}$ as

$$\tau_N^{\mathrm{f}} = \mathrm{Det}_{0 \leqslant i,j<N} \partial_1^i \bar{\partial}_1^j \mathcal{H}^{\mathrm{f}}\,, \tag{4.49}$$

where $\mathcal{H}^{\mathrm{f}} = \mathcal{H}_{00}^{\mathrm{f}}$ and $\partial_1 = \partial/\partial t_1$, $\bar{\partial}_1 = \partial/\partial \bar{t}_1$. For the forced *Toda-chain* hierarchy this turns into an even simpler expression:

$$\tau_N^{\mathrm{f}} = \mathrm{Det}_{0 \leqslant i,j<N}\, \partial_1^{i+j} \hat{\mathcal{H}}^{\mathrm{f}}\,, \tag{4.50}$$

while for the *forced Volterra* case we get a product of two Toda-chain τ-functions with a halvedvalue of N [91]:

$$\tau_{2N}^{\mathrm{f}} = \left(\mathrm{Det}_{0 \leqslant i,j<N}\, \partial_2^{i+j} \hat{\mathcal{H}}^{\mathrm{f}} \right) \left(\mathrm{Det}_{0 \leqslant i,j<N}\, \partial_2^{i+j} (\partial_2 \hat{\mathcal{H}}^{\mathrm{f}}) \right)$$
$$= \tau_N^{\mathrm{f}}[\hat{\mathcal{H}}^{\mathrm{f}}] \tau_N^{\mathrm{f}}[\partial_2 \hat{\mathcal{H}}^{\mathrm{f}}]. \tag{4.51}$$

Forced τ_N^{f} can *always* be represented in the form of a scalar-product matrix model. Indeed,

$$\mathcal{H}_{ij} = \sum T_{lm} P_{l-i}(t) P_{m-j}(\bar{t})$$
$$= \oint\!\!\oint \exp[U(h) + \bar{U}(\bar{h})]\, h^i \bar{h}^j T(h, \bar{h})\, \mathrm{d}h\, \mathrm{d}\bar{h}\,, \tag{4.52}$$

where $T(h, \bar{h}) \equiv \sum_{lm} T_{lm} h^{-l-1} \bar{h}^{-m-1}$, and

$$\exp[U(h)] = \exp\left(\sum_{k>n} t_k h^k \right) = \sum_{l \geqslant 0} h^l P_l(t)\,.$$

† Normal ordering sign : : means that all operators $\tilde{\psi}$ stand to the *left* of all operators ψ. The product at the r.h.s. obviously implies both the property (4.47) and the projection property $P_+^2 = P_+$.

Then, since

$$\mathrm{Det}_{0\leqslant i,j<N}\, h_j^i = \Delta_N(h)$$

(this is where it is essential that the hierarchy is forced),

$$\mathrm{Det}_{0\leqslant i,j<N}\mathcal{H}_{ij} = \prod_i \oint\oint \exp[U(h_i)+\bar U(\bar h_i)] \times T(h_i,\bar h_i)\,\mathrm{d}h_i\,\mathrm{d}\bar h_i\,\Delta_N(h)\Delta_N(\bar h)\ , \quad (4.53)$$

i.e. we obtain a scalar-product model with

$$\mathrm{d}\mu_{h,\bar h} = \exp[U(h)+\bar U(\bar h)]T(h,\bar h)\,\mathrm{d}h\,\mathrm{d}\bar h\ . \quad (4.54)$$

The inverse is also true: the partition function of every scalar-product model is a forced Toda-lattice τ-function — see Section 4.7 for more details.

KP hierarchy. In this case we just ignore the dependence of the τ-function on times $\bar t$. Every Toda-lattice τ-function can be considered also as a KP τ-function: the operator $G^{\mathrm{KP}} \equiv G\exp\bar H$ (a point of the Grassmannian) becomes $\bar t$ dependent. Usually N dependence is also eliminated — this can be considered as a little more sophisticated change of G. When N is fixed, extra changes of field-variables are allowed, including the transformation from the Ramond to the Neveu–Schwarz (NS) sector, etc. Often the KP hierarchy is formulated from the very beginning in terms of Neveu–Schwarz (antiperiodic) fermionic fields, ψ_{NS}, associated with principal representations of Kac–Moody algebras) i.e. expansions in the first line of Eqn (4.23) are in semi-integer powers of z: $\psi_{\mathrm{NS}}(z) = \sum_{n\in Z}\psi_n z^{n-\frac12}\,\mathrm{d}z^{1/2}$.

Given a KP τ-function one can usually construct a Toda-lattice one with the *same* G by introducing, in an appropriate way, dependences on $\bar t$ and N. For this purpose τ^{KP} should be represented in the form of Eqn (4.39):

$$\tau^{\mathrm{KP}}\{t\,|G\} = \mathrm{Det}_{i,j<0}\mathcal{H}_{ij}^{\mathrm{KP}}\ , \quad (4.55)$$

where $\mathcal{H}_{ij}^{\mathrm{KP}} = \sum_l T_{lj}P_{l-i}(t)$. Since T_{lm} is a function of G only, it does not change when one constructs a Toda-lattice τ-function:

$$\tau_N\{t,\bar t\,|G\} = \mathrm{Det}_{i,j<0}\mathcal{H}_{i+N,j+N}\ ,$$

$$\mathcal{H}_{ij} = \sum_{l,m} T_{lm}P_{l-i}(t)P_{m-j}(\bar t) = \sum_m \mathcal{H}_{im}^{\mathrm{KP}}P_{m-j}(\bar t)\ . \quad (4.56)$$

Then,

$$\tau^{\mathrm{KP}}\{t\,|G\} = \tau_0\{t,0|G\}\ . \quad (4.57)$$

If one goes in the opposite direction, when the Toda-lattice τ-function is considered as KP τ-function,

$$\tau_0\{t,\bar t\,|G\} = \tau^{\mathrm{KP}}\{t\,|\widetilde G(\bar t)\}\ ,$$

$$\widetilde{\mathcal{H}}_{ij}^{\mathrm{KP}} = \sum_m \mathcal{H}_{im}P_{m-j}(\bar t)\ ,$$

$$\widetilde T_{lj}\{\widetilde G(\bar t)\} = \sum_m T_{lm}\{G\}P_{m-j}(\bar t)\ . \quad (4.58)$$

The KP reduction in its turn has many further weak reductions (KdV and Boussinesq being the simplest examples). I shall mention them again in Section 4.9, after the Miwa transformation of representation Eqn (4.39) has been considered in the next subsection.

4.6 Fermion correlator in Miwa coordinates

Let me now return to the original correlator Eqn (4.22) and discuss in a little more detail the implications of the bosonisation identity Eqn (4.13). In order not to write down integrals of J, I introduce the scalar field†

$$\phi(z) = \sum_{\substack{k\neq 0\\ k\in Z-0}} \frac{J_{-k}}{k}z^k + \phi_0 + J_0\ln z\ , \quad (4.59)$$

such that $\partial\phi(z) = J(z)$. Then Eqn (4.13) states that

$$:\psi(\lambda)\widetilde\psi(\widetilde\lambda): = :\exp[\phi(\widetilde\lambda)-\phi(\lambda)]:\ . \quad (4.60)$$

'Normal ordering' here means nothing more than the requirement to neglect all mutual contractions (or correlators) of operators between the colons when the Wick theorem is applied to evaluate correlation functions. One can also get rid of the normal ordering sign on the l.h.s. of Eqn (4.60), then

$$\psi(\lambda)\widetilde\psi(\widetilde\lambda) = :\exp[\phi(\widetilde\lambda)]:\,:\exp[-\phi(\lambda)]:\ . \quad (4.61)$$

In distinguished coordinates on a sphere, when the free-field propagator is just $\ln(z-\widetilde z)$, one also has

$$\psi(z)\widetilde\psi(\widetilde z) = \frac{1}{z-\widetilde z}\ :\psi(z)\,\widetilde\psi(\widetilde z):\ .$$

My task now is to express operators $\exp(H)$ and $\exp(\bar H)$ in terms of the field ϕ. This is simple:

$$H = \oint_0 U(z)\,J(z) = \oint_0 U(z)\,\partial\phi(z) = -\oint_0 \phi(z)\,\partial U(z)\ . \quad (4.62)$$

Here, as usual, $U(z) = \sum_{k>0} t_k z^k$ and the integral is around $z = 0$. This is very similar to the generic linear functional of $\phi_-(\lambda) \equiv -\sum_{k>0}(1/k)J_k\lambda^{-k}$,

$$H = \int \phi_-(\lambda)f(\lambda)\mathrm{d}\lambda\ ; \quad (4.63)$$

one should require only that‡

$$\partial U(z) = \int \frac{f(\lambda)}{z-\lambda}\mathrm{d}\lambda\ ,$$

that is

$$U(z) = \int \ln\left(1-\frac{z}{\lambda}\right)f(\lambda)\,\mathrm{d}\lambda\ . \quad (4.64)$$

In terms of time-variables this means that

$$t_k = -\frac{1}{k}\int \lambda^{-k}f(\lambda)\,\mathrm{d}\lambda\ . \quad (4.65)$$

Here, one requires that $U(z=0) = 0$; sometimes it can be more natural to introduce also

$$t_0 = \int \ln(\lambda)f(\lambda)\,\mathrm{d}\lambda\ . \quad (4.66)$$

This change from time-variables to 'time density', $f(\lambda)$, is known as the Miwa transformation. In order to establish the

† One can consider ϕ as introduced for simplicity of notation, but it should be kept in mind that the scalar-field representation is in fact more fundamental for *generic* τ-functions not related to the level $k = 1$ Kac–Moody algebras (this phenomenon is well known in CFT, see [16] for more details).

‡ As is usual nowadays, a factor of $2\pi\mathrm{i}$ is assumed to be included in the definition of the contour integral $\oint$.

relation with fermionic representation and also with matrix models we shall need it in 'discretised' form:

$$t_k = \frac{\xi}{k}\left(\sum_\gamma \lambda_\gamma^{-k} - \sum_\gamma \widetilde{\lambda}_\gamma^{-k}\right) ,$$

$$t_0 = -\xi\left(\sum_\gamma \ln \lambda_\gamma - \sum_\gamma \ln \widetilde{\lambda}_\gamma\right) . \tag{4.67}$$

The integral over λ has been replaced by a discrete sum, i.e. the density function $f(\lambda)$ is a combination of δ-functions picked at some points λ_γ, $\widetilde{\lambda}_\gamma$. This is, of course, just another basis in the space of linear functionals, but the change from one basis to another is highly nontrivial. The thing is that the basis has been selected where the amplitudes of the different δ-functions are the *same*: the parameter ξ in Eqn (4.67) is *independent* of γ. Thus, the real parameters are just the positions of the points λ_γ, $\widetilde{\lambda}_\gamma$, while the amplitude is defined by the density of these points in the integration (summation) domain. This domain does not need to be specified *a priori*: it can be the real line, any other contour or — better still — some Riemann surface. The parameter ξ is also unnecessary because bases with different ξ are essentially equivalent. I shall soon put it equal to *one*, but not before the Miwa transformation has been discussed in a little more detail.

Substitution of Eqn (4.63) into Eqn (4.67) gives

$$H = -\xi \sum_\gamma \phi_-(\lambda_\gamma) + \xi \sum_\gamma \phi_-(\widetilde{\lambda}_\gamma) . \tag{4.68}$$

In fact, what is needed is not the operator H itself, but the state which is created when $\exp(H)$ acts on the vacuum state $\langle N|$. Then, since $\langle N|J_m = 0$ for $m < 0$, $\langle N|\exp[-\xi\phi_-(\lambda)]$ is essentially equivalent to $\langle N|\exp[-\xi\phi(\lambda)]$ with $\phi_-(\lambda)$ replaced by $\phi(\lambda)$. If $\xi = 1$, $\exp[-\phi(\lambda)]$ can be further changed for $\psi(\lambda)$ and one obtains an expression for the correlator (4.22), an expression where $\exp(H)$ is replaced by a product of operators $\psi(\lambda_\gamma)$. The same is, of course, true for $\exp(H)$. Then the Wick theorem can be applied and a new type of determinant formula arises, like, for example,

$$\tau \sim \frac{\Delta(\lambda,\widetilde{\lambda})}{\Delta^2(\lambda)\Delta^2(\widetilde{\lambda})} \det{}_{\gamma\delta}\langle N|\psi(\lambda_\gamma)\,\widetilde{\psi}(\widetilde{\lambda}_\delta)\,G\,|N\rangle . \tag{4.69}$$

It can also be obtained directly from Eqns (4.27), (4.29), and (4.35) by Miwa transformation. The rest of this subsection describes this derivation in somewhat more detail.

The first task is to replace ϕ_- by ϕ. For this purpose I introduce the operator

$$\sum_{k=-\infty}^{\infty} t_k J_k = H_+ + H_- , \tag{4.70}$$

where $H_+ = \sum_{k>0} t_k J_k$ is just our old H; $H_- = \sum_{k\geqslant 0} t_{-k} J_k$; and 'negative times' t_{-k} are defined by 'analytical continuation' of the same formulas (4.65) and (4.67):

$$t_{-k} = \frac{1}{k}\int \lambda^k f(\lambda)\,\mathrm{d}\lambda = -\frac{\xi}{k}\left(\sum_\gamma \lambda_\gamma^k - \sum_\gamma \widetilde{\lambda}_\lambda^k\right) . \tag{4.71}$$

Then

$$\sum_{k=-\infty}^{\infty} t_k J_k = H_+ + H_- = -\xi\left[\sum_\gamma \phi(\lambda_\gamma) - \sum_\gamma \phi(\widetilde{\lambda}_\gamma)\right] . \tag{4.72}$$

Further,

$$\exp(H_+ + H_-) = \exp\left[-\tfrac{1}{2}s(t)\right]\exp(H_+)\exp(H_-)$$
$$= \exp\left[\tfrac{1}{2}s(t)\right]\exp(H_-)\exp(H_+) , \tag{4.73}$$

where

$$s(t) \equiv \sum_{k>0} k t_k t_{-k}$$
$$= -\xi^2 \sum_{k>0}\frac{1}{k}\left[\sum_\gamma\left(\lambda_\gamma^{-k} - \widetilde{\lambda}_\gamma^{-k}\right)\sum_\delta\left(\lambda_\delta^k - \widetilde{\lambda}_\delta^k\right)\right]$$
$$= \xi^2 \ln\left[\prod_{\gamma,\delta}\frac{(1-\lambda_\delta/\lambda_\gamma)(1-\widetilde{\lambda}_\delta/\widetilde{\lambda}_\gamma)}{(1-\widetilde{\lambda}_\delta/\lambda_\gamma)(1-\lambda_\delta/\widetilde{\lambda}_\gamma)}\right] + \text{const} , \tag{4.74}$$

where the prime means that the terms with $\gamma = \delta$ are excluded from the product in the numerator and accounted for in the infinite 'constant', added on the r.h.s. In other words,

$$\exp\left[\tfrac{1}{2}s(t)\right] = \text{const} \times \left[\frac{\prod_{\gamma>\delta}(\lambda_\gamma - \lambda_\delta)(\widetilde{\lambda}_\gamma - \widetilde{\lambda}_\delta)}{\prod_\gamma\prod_\delta(\lambda_\gamma - \widetilde{\lambda}_\delta)}\right]^{\xi^2}$$
$$= \text{const} \times \left[\frac{\Delta^2(\lambda)\Delta^2(\widetilde{\lambda})}{\Delta(\lambda,\widetilde{\lambda})}\right]^{\xi^2} . \tag{4.75}$$

Since $\langle N|J_m = 0$ for all $m < 0$, we have $\langle N|\exp(H_-) = \langle N|$, and therefore

$$\langle N|\exp(H) \equiv \langle N|\exp(H_+) = \langle N|\exp(H_-)\exp(H_+)$$
$$= \exp\left[-\tfrac{1}{2}s(t)\right]\langle N|\exp(H_+ + H_-) . \tag{4.76}$$

From Eqn (4.72),

$$\exp(H_+ + H_-)$$
$$= \text{const} \times \prod_\gamma :\exp[-\xi\phi(\lambda_\gamma)]: :\exp[\xi\phi(\widetilde{\lambda}_\gamma)]: , \tag{4.77}$$

where 'const' is exactly the same as in Eqn (4.75). If $\xi = 1$, Eqn (4.61) can be used to write†

$$\langle N|\exp(H) = \frac{\Delta(\lambda,\widetilde{\lambda})}{\Delta^2(\lambda)\Delta^2(\widetilde{\lambda})}\langle N|\prod_\gamma \psi(\lambda_\gamma)\prod_\gamma \widetilde{\psi}(\widetilde{\lambda}_\gamma) . \tag{4.78}$$

† The value of ξ can be chosen to suit particular purposes. Here I impose the requirement that the Miwa transform represents $\exp(H) = \exp(H_{\text{Cartan}})$ as a product of dimension-$\frac{1}{2}$ operators — this is most natural from the point of view of Hirota equations and simplifies the relation with integrable hierarchies. However, in Section 2.7 and 2.8 I used another requirement (and there $\xi = 1/\sqrt{2}$ rather than $\xi = 1$). There the 1-matrix model, which is characterised by an especially simple form of the *full* Hamiltonian (product of dimension-zero operators), was considered and it was more important to adjust operators which arise from $\exp(H_{\text{Cartan}})$ after Miwa transformation so that they have simple correlators with $\exp(A\psi\widetilde{\psi})$. When analysing the 1-matrix model from this point of view one should also keep in mind that it was actually represented in Section 2.3 in terms of *two* complex fermions. The screening charges are

$$Q^{(+)} = \oint \exp(\sqrt{2}\phi) = \oint \widetilde{\psi}_1\psi_2 = \oint \exp(\phi_1 - \phi_2) ,$$

$$Q^{(-)} = \oint \exp(-\sqrt{2}\phi) = \oint \widetilde{\psi}_2\psi_1 \oint \exp(\phi_2 - \phi_1) ,$$

while $\phi = (1/\sqrt{2})(\phi_1 - \phi_2)$. The Hamiltonian is

$$H_{\text{Cartan}} = \frac{1}{\sqrt{2}}\sum_k t_k J_k = \frac{1}{2}\sum_k t_k (J_k^1 - J_k^2) ,$$

and the Miwa transformation generators are operators $\chi_i\widetilde{\chi}_2$, where χ_1 and $\widetilde{\chi}_2$ have dimension $\frac{1}{8}$ [rather than $\frac{1}{2}$ as in the one (complex)-fermion system considered in *this* section].

Similarly,

$$\exp(\bar{H})|N\rangle = \prod_\delta \psi(\bar{\lambda}_\delta) \prod_\delta \widetilde{\psi}(\widetilde{\bar{\lambda}}_\delta)|N\rangle \frac{\Delta(\bar{\lambda}, \widetilde{\bar{\lambda}})}{\Delta^2(\bar{\lambda})\Delta^2(\widetilde{\bar{\lambda}})}, \quad (4.79)$$

where

$$\bar{t}_k = -\frac{1}{k}\sum_\delta \left(\bar{\lambda}_\delta^k - \widetilde{\bar{\lambda}}_\delta^k\right), \quad (4.80)$$

and I used the fact the $J_m|N\rangle = 0$ for all $m > 0$. Finally,

$$\tau_N\{t, \bar{t}|G\} = \langle N|\exp(H)\, G \exp(\bar{H})|N\rangle$$
$$= \frac{\Delta(\lambda, \widetilde{\lambda})}{\Delta^2(\lambda)\Delta^2(\widetilde{\lambda})} \frac{\Delta(\bar{\lambda}, \widetilde{\bar{\lambda}})}{\Delta^2(\bar{\lambda})\Delta^2(\widetilde{\bar{\lambda}})} \langle N| \prod_\gamma \psi(\lambda_\gamma) \prod_\gamma \widetilde{\psi}(\widetilde{\lambda}_\gamma)$$
$$\times G \prod_\delta \psi(\bar{\lambda}_\delta) \prod_\delta \widetilde{\psi}(\widetilde{\bar{\lambda}}_\delta)|N\rangle. \quad (4.81)$$

Singularities at the coinciding points are completely eliminated from this expression, since poles and zeroes of the correlator are cancelled by those coming from the Van der Monde determinants.

Let me now put $N = 0$ and define the normalised τ-function:

$$\hat{\tau}_0\{t, \bar{t}|G\} \equiv \frac{\tau_0\{t, \bar{t}|G\}}{\tau_0\{0, 0|G\}}, \quad (4.82)$$

i.e. divide the r.h.s. of Eqn (4.18) by $\langle 0|G|0\rangle$. The Wick theorem now allows one to rewrite the correlator on the r.h.s. as the determinant of the block matrix

$$\begin{bmatrix} \dfrac{\langle 0|\psi(\lambda_\gamma)\widetilde{\psi}(\widetilde{\lambda}_\delta)G|0\rangle}{\langle 0|G|0\rangle} & \dfrac{\langle 0|\psi(\lambda_\gamma)\, G\widetilde{\psi}(\widetilde{\bar{\lambda}}_\delta)|0\rangle}{\langle 0|G|0\rangle} \\ -\dfrac{\langle 0|\widetilde{\psi}(\widetilde{\lambda}_\delta)\, G\psi(\bar{\lambda}_\gamma)|0\rangle}{\langle 0|G|0\rangle} & \dfrac{\langle 0|G\psi(\bar{\lambda}_\gamma)\widetilde{\psi}(\widetilde{\bar{\lambda}}_\delta)|0\rangle}{\langle 0|G|0\rangle} \end{bmatrix}. \quad (4.83)$$

Special choices of points $\lambda_\gamma, \ldots, \widetilde{\bar{\lambda}}_\delta$ can lead to simpler formulas. If $\widetilde{\bar{\lambda}}_\gamma \to \bar{\lambda}_\gamma$, so that $\bar{t}_k \to 0$, the matrix elements at the lower right block in the matrix (4.83) blow up, so that the off-diagonal blocks can be neglected. Then

$$\tau_0\{t, \bar{t}|G\} \to \tau^{\mathrm{KP}}\{t|G\} = \frac{\langle 0|\exp(H)\, G|0\rangle}{\langle 0|G|0\rangle}$$
$$= \frac{\Delta(\lambda, \widetilde{\lambda})}{\Delta^2(\lambda)\Delta^2(\widetilde{\lambda})} \det{}_{\gamma\delta} \frac{\langle 0|\psi(\lambda_\gamma)\, \widetilde{\psi}(\widetilde{\lambda}_\delta)\, G|0\rangle}{\langle 0|G|0\rangle}. \quad (4.84)$$

This function no longer depends on $\bar{t}$-times and is just a KP τ-function.

The matrix element

$$\varphi(\lambda, \widetilde{\lambda}) = \frac{\langle 0|\psi(\lambda)\widetilde{\psi}(\widetilde{\lambda})G|0\rangle}{\langle 0|G|0\rangle} \quad (4.85)$$

is singular when $\lambda \to \widetilde{\lambda} : \varphi(\lambda, \widetilde{\lambda}) \to 1/(\lambda - \widetilde{\lambda})$. If now in Eqn (4.84) all $\widetilde{\lambda} \to \infty$,

$$\tau^{\mathrm{KP}}\{t|G\} = \frac{\det_{\gamma\delta} \varphi_\delta(\lambda_\gamma)}{\Delta(\lambda)}, \quad (4.86)$$

where

$$\varphi_\delta(\lambda) \equiv \langle 0|\psi(\lambda)\, (\partial^{\delta-1}\widetilde{\psi})(\infty)\, G|0\rangle \sim \lambda^{\delta-1}\left[1 + \mathrm{O}\left(\frac{1}{\lambda}\right)\right]. \quad (4.87)$$

This is the main determinant representation of the KP τ-function in the Miwa parametrisation.

Starting from representation (4.86) one can restore the corresponding matrix $\mathcal{H}_{ij}^{\mathrm{KP}}$ in Eqn (4.55) [36]:

$$\mathcal{H}_{ij}^{\mathrm{KP}}\{t\} = \oint z^i \varphi_{-j}(z) \exp\left(\sum_k t_k z^k\right) \mathrm{d}z, \quad (4.88)$$

that is

$$T_{ij}^{\mathrm{KP}} = \oint z^l \varphi_{-j}(z). \quad (4.89)$$

Then obviously

$$\frac{\partial \mathcal{H}_{ij}^{\mathrm{KP}}}{\partial t_k} = \mathcal{H}_{i+k,j}^{\mathrm{KP}}.$$

Now one needs to prove that the τ-function is given at once by $\det_{\varphi\gamma}(\lambda_\delta)/\Delta(\lambda)$ and $\mathrm{Det}\mathcal{H}_{ij}^{\mathrm{KP}}\{t\}$. In order to compare these two expressions one should take $t_k = (1/k)\sum_\gamma^n \lambda_\gamma^{-k}$, so that

$$\exp\left(\sum_{k>0} t_k z^k\right) = \prod_{\gamma=1}^n \frac{\lambda_\gamma}{\lambda_\gamma - z}$$
$$= \left(\prod_\gamma^n \lambda_\gamma\right) \sum_\gamma \frac{(-1)^\gamma}{z - \lambda_\gamma} \frac{\Delta_\gamma(\lambda)}{\Delta(\lambda)}, \quad (4.90)$$

where

$$\Delta_\gamma(\lambda) = \prod_{\substack{\alpha>\beta \\ \alpha,\beta\neq\gamma}} (\lambda_\alpha - \lambda_\beta) = \frac{\Delta(\lambda)}{\prod_{\alpha\neq\gamma}(\lambda_\alpha - \lambda_\gamma)}, \quad (4.91)$$

and

$$\mathcal{H}_{ij}^{\mathrm{KP}}\Big|_{t_k = \frac{1}{k}\sum_\gamma^n \lambda_\gamma^{-k}} = \left(\prod_\gamma^n \lambda_\gamma\right) \sum_\gamma \frac{(-1)^{\gamma+1}\Delta_\gamma(\lambda)}{\Delta(\lambda)} \lambda_\gamma^i \varphi_{-j}(\lambda_\gamma). \quad (4.92)$$

As long as n is kept finite,

$$\mathrm{Det}_{i,j<0}\mathcal{H}_{ij}^{\mathrm{KP}}\Big|_{t_k = \frac{1}{k}\sum_\gamma^n \lambda_\gamma^{-k}} = 0,$$

since it is obvious from Eqn (4.92) that the rank of the matrix is equal to n. Therefore, let us consider the maximal nonvanishing determinant,

$$\mathrm{Det}_{-N\leqslant i,j<0}\mathcal{H}_{ij}^{\mathrm{KP}}\Big|_{t_k = \frac{1}{k}\sum_\gamma^n \lambda_\gamma^{-k}}$$
$$= \left(\prod_\gamma^n \lambda_\gamma\right)^n \det{}_{i\gamma}\left[\frac{(-1)^{\gamma+1}\Delta_\gamma(\lambda)}{\lambda_\gamma^i \Delta(\lambda)}\right] \det{}_{\gamma j}\varphi_j(\lambda_\gamma)$$
$$= \frac{\det_{\gamma j}\, \varphi_j(\lambda_\gamma)}{\Delta(\lambda)}. \quad (4.93)$$

I used here the fact that determinant of a matrix is a product of determinants, and reversed the signs of i and j. Also used were some simple relations:

$$\prod_{\gamma=1}^n \frac{\Delta_\gamma(\lambda)}{\Delta(\lambda)} = \frac{1}{\Delta^2(\lambda)},$$

$$\det{}_{i\gamma}\frac{1}{\lambda_\gamma^i} = \left(\prod_\gamma^n \lambda_\gamma\right)^{-1} \Delta\left(\frac{1}{\lambda}\right),$$

$$\Delta\left(\frac{1}{\lambda}\right) = \prod_{\alpha>\beta}\left(\frac{1}{\lambda_\alpha} - \frac{1}{\lambda_\beta}\right) = (-1)^{\frac{1}{2}n(n-1)}\Delta(\lambda)\left(\prod_\gamma^n \lambda_\gamma\right)^{-(n-1)};$$

thus,

$$\left(\prod_{\gamma}^{n}\lambda_\gamma\right)(-1)^{\frac{1}{2}n(n-1)}\prod_{\gamma=1}^{n}\frac{\Delta_\gamma(\lambda)}{\Delta(\lambda)}\det{}_{i\gamma}\frac{1}{\lambda_\gamma^i}=\frac{1}{\Delta(\lambda)}\,.$$

Since Eqn (4.93) is true for any n, one can claim that in the limit $n\to\infty$ one recovers the statement that $\tau^{\mathrm{KP}}\{t\} = \mathrm{Det}_{i,j<0}\mathcal{H}_{ij}^{\mathrm{KP}}$ with $\mathcal{H}_{ij}^{\mathrm{KP}}$ given by Eqn (4.92) (that formula does not refer directly to Miwa parametrisation and is defined for any t and for any $j<0$ and i). This relation between the φ_γs and $\mathcal{H}_{ij}^{\mathrm{KP}}$ can now be used to introduce negative times $\bar{t}_k$ according to the rule (4.58). Especially simple is the prescription for zero-time: $\mathcal{H}_{ij}\to\mathcal{H}_{i+N,j+N}$, which, when expressed in terms of φ, implies just that

$$\frac{\det\varphi_\gamma(\lambda_\delta)}{\Delta(\lambda)}\to\frac{\det\varphi_{\gamma+N}(\lambda_\delta)}{(\det\Lambda)^N\Delta(\lambda)}\,. \tag{4.94}$$

Generalisations of Eqn (4.88), like

$$\mathcal{H}_{ij}\{t,\bar{t}\} = \oint\!\!\oint z^i\bar{z}^j\langle 0|\,\psi(z)\,G\,\widetilde{\psi}(\bar{z})\,|0\rangle\exp\left[\sum_k(t_kz^k+\bar{t}_k\bar{z}^k)\right]\mathrm{d}z\,\mathrm{d}\bar{z}\,, \tag{4.95}$$

also can be considered.

4.7 Matrix models versus τ-functions

I can now return to my main subject and discuss the integrability properties of eigenvalue matrix models. The claim is that the partition functions of all these models, when considered as functions of time-variables (parametrising the shapes of potentials) are in fact τ-functions of (perhaps multicomponent) Toda-lattice and/or KP type. (Interesting noneigenvalue models are believed to be related to integrable systems of more general type, not restricted to level $k=1$ Kac–Moody algebras).

Partition functions are, however, not generic Toda or KP τ-functions: first, they usually belong to some reduced hierarchies; second, the relevant operators G (points of a Grassmannian) are restricted to stay in particular domains of the universal module space, specified by 'string equations'. The string equations are in fact nothing but the set of Ward identities (WIs) (Virasoro or W-constraints, in the examples under investigation), which are now interpreted as equations on G. The very possibility of such interpretation is highly nontrivial and reflects some deep relation between the constraints and integrable structure. In the case of Virasoro constraints this is not a puzzle, because Virasoro algebra is a symmetry (covariance) of the hierarchy, the situation with other constraints is less clear (see the footnote in Section 4.3). In fact, when applied to a τ-function of appropriately reduced hierarchy, the infinitely many constraints usually become dependent and it is enough to impose only the lowest Virasoro constraint $L_{-1}\tau=0$ (or $\mathcal{L}_{-p}\tau=0$, where p is the degree of reduction), in order to recover the entire set [29]. It is this lowest constraint [or rather its t_1-derivative, $(\partial/\partial t_1)/(L_{-1}\tau)=0$] that traditionally carries the name 'string equation'. It is often much simpler to deduce than the entire set of identities, which is important in practical applications (especially because determinant formulas, which imply integrability, can also be simpler to find, in some situations, than the WIs).

In order to give a complete description of some sort of (matrix) model from the point of view of integrability theory it is enough to specify the hierarchy to which it belongs (if the partition function is interpreted as a τ-function),

$$Z_{\mathrm{model}}\{t\}=\tau\{t\,|G_{\mathrm{model}}\}\,, \tag{4.96}$$

and the string equation which serves to fix the operator G — the point in the universal module space.† After that, it becomes an internal (yet unsolved) problem of integrability theory to explain what is so special about the set of points $\{G_{\mathrm{model}}\}$ in this space. (I shall touch this problem in the next subsection, devoted to Kac–Schwarz operators.) Alternatively, if there is nothing special, it is an (unsolved) problem of matrix model theory to find models associated with *any* points G in the universal module space (or to explain what, if anything, is an obstacle).

I proceed now to a description of particular matrix models from this point of view. As everywhere in these notes I consider only the most important classes of scalar product, conformal (multicomponent) and generalised Kontsevich models (GKM). All other examples (like models of complex matrices, orthogonal matrices, unitary matrices, etc.) can be taken into consideration with more or less effort (see [28, 91] for cases of complex and unitary models, respectively), but they do not add much to the general theory that we are now considering. String equations will be discussed in the next subsection.

Scalar-product models. These were exhaustively discussed in Sections 3.5 — 3.7. Recall that all conventional multimatrix models [with intermatrix interaction of the form $\exp(\mathrm{Tr}\,H^{(\alpha)}H^{(\alpha+1)})$] belong to this class. The crucial formulas are:

$$\begin{aligned} Z_N &= \mathrm{Det}_N\mathcal{H}_{ij}^{\mathrm{f}} = \mathrm{Det}_{0\leqslant i,j\leqslant N-1}\mathcal{H}_{ij}^{\mathrm{f}} \\ &= \mathrm{Det}_{-N\leqslant i,j<0}\mathcal{H}_{i+N,j+N}^{\mathrm{f}}\,, \\ \mathcal{H}_{ij}^{\mathrm{f}} &= \frac{\partial^2}{\partial t_i\partial\bar{t}_j}\mathcal{H}^{\mathrm{f}} = \left(\frac{\partial}{\partial t_1}\right)^i\left(\frac{\partial}{\partial\bar{t}_1}\right)^j\mathcal{H}^{\mathrm{f}}\,. \end{aligned} \tag{4.97}$$

Here,

$$\mathcal{H}_{ij}^{\mathrm{f}} = \langle h^i\,|\bar{h}^j\rangle = \int\mathrm{d}\hat{\mu}_{h,\bar{h}}\exp[U(h)+\bar{U}(\bar{h})]h^i\bar{h}^j\,. \tag{4.98}$$

Further,

$$\begin{aligned} \exp[U(h)] &= \exp\left(\sum_{k\geqslant 0}t_kh^k\right) = \sum_l h^lP_l(t)\,, \\ \exp[\bar{U}(\bar{h})] &= \exp\left(\sum_{k\geqslant 0}\bar{t}_k\bar{h}^k\right) = \sum_m\bar{h}^mP_m(\bar{t})\,, \end{aligned} \tag{4.99}$$

and thus,

$$\begin{aligned} \mathcal{H}_{ij}^{\mathrm{f}} &= \sum_{l,m}\langle\langle h^{i+l}\,|\bar{h}^{j+m}\rangle\rangle P_l(t)P_m(\bar{t}) \\ &= \sum_{l,m}T_{lm}^{\mathrm{f}}P_{l-i}(t)\,P_{m-j}(\bar{t})\,, \\ T_{lm}^{\mathrm{f}} &= \langle\langle h^l\,|\bar{h}^m\rangle\rangle\,, \end{aligned} \tag{4.100}$$

† As argued in the Introduction and in Section 2.1, the word 'matrix' can probably be omitted if generic Lagrangians are considered in other models of quantum field theory. Also, the universal module space (- where moduli are of bundles over *spectral* Riemann surfaces) can (and should) be treated as a 'space of theories'. It is one of the great puzzles (and beauties) of string theory that Riemann surfaces appear both in the world-sheet and in the spectral 'dimensions'. See [6] for more discussion of this issue.

where the scalar product $\langle\langle\ |\ \rangle\rangle$ is with respect to the measure $\mathrm{d}\hat{\mu}_{h,\bar{h}}$ {while $\langle\ |\ \rangle$ is with respect to the measure $\mathrm{d}\mu_{h,\bar{h}} = \exp[U(h) + \bar{U}(\bar{h})]\mathrm{d}\hat{\mu}_{h,\bar{h}}$}.

One would immediately recognise in these formulas representation (4.39) of the Toda-lattice τ-function, were there no additional restriction that the determinant in Eqn (4.97) is over a finite-dimensional $N \times N$ matrix (indices are constrained: $i,j \geqslant -N$). This can be automatically taken into account if one requires that

$$T^{\mathrm{f}}_{lm} = \infty \quad \text{for all } l,m < 0\ , \tag{4.101}$$

and identify Z_N as a normalised τ-function, $\hat{\tau}$, of *forced* Toda-lattice hierarchy (thus the superscript f carried by $\mathcal{H}$ and T). One concludes that the partition function of any scalar-product model is a $\hat{\tau}$-function of the forced Toda-lattice hierarchy.

Let us now consider them as KP τ-functions. This means that the $\bar{t}$-dependence is simply ignored. However, N will be preserved explicitly as a parameter labelling the KP τ-function. After the Miwa transformation $t_k = -(1/k)\sum_\gamma \lambda_\gamma^{-k} - r_k$, described in Section 3.7, one gets:

$$Z_N = \hat{Z}_N \frac{\det_{\gamma\delta} \hat{Q}_{N+\gamma-1}(\lambda_\delta)}{\Delta(\lambda)}\ , \tag{4.102}$$

where the $\hat{Q}$s are orthogonal polynomials with respect to the measure $\mathrm{d}\hat{\nu}_{h,\bar{h}} = \exp(-\sum_k r_k h^k)\mathrm{d}\hat{\mu}_{h,\bar{h}}$.

One concludes that in the framework of the KP hierarchy the scalar-product models are distinguished by the fact that the corresponding $\varphi_\gamma(\lambda)$ in Eqn (4.86) are *polynomials* rather than infinite series in powers of λ^{-1}.

The 1-matrix model. This is a particular example of a scalar-product model with a *local* measure given by

$$\mathrm{d}\mu_{h,\bar{h}} = \exp[U(h) + \bar{U}(\bar{h})]\,\delta(h-\bar{h})\,\mathrm{d}h\,\mathrm{d}\bar{h}.$$

In this case,

$$\mathcal{H}^{\mathrm{f}}_{ij} = \langle h^i \,|\, \bar{h}^j \rangle = \langle h^{i+j} \rangle = \frac{\partial}{\partial t_{i+j}}\mathcal{H}^{\mathrm{f}} = \left(\frac{\partial}{\partial t_1}\right)^{i+j}\mathcal{H}^{\mathrm{f}}\ . \tag{4.103}$$

Thus, in this case one is dealing with the (forced) Toda-chain reduction of a Toda-lattice hierarchy. At the end of this section orthogonal polynomials are used to present a detailed description of 1-matrix models as Toda-chain τ-functions.

This model can alternatively be defined as a gaussian Kontsevich model: see Section 3.8. The fact that the partition function is a τ-function follows then from the general statement for the GKM, see below. The fact that it is a *forced* τ-function is related to the property $c_{-N} = 0$, mentioned at the end of Section 3.8 (and proved in Section 3.9). Also, the reduction to a Toda-chain hierarchy can be observed directly in terms of the GKM; see [36] for more details.

Multicomponent (conformal) matrix models. These are related to multicomponent hierarchies, with τ-functions representable as correlators in multifermion systems. An example of a determinant formula which substitutes Eqn (4.39) in the 2-component case is given at the end of Section 3.5, where it is derived from a consideration of the relevant matrix model [39]. For derivation of the same determinant formula in the theory of τ-functions see [92]. The generic theory of multicomponent hierarchies is now making its first steps and I do not review it in these notes. See [93] for the group-theory approach to the problem.

Generalised Kontsevich model (GKM). Determinant formulas for this case are derived in Section 3.3. The most important expression is

$$Z_V\{N,T\} = \frac{1}{(\det \Lambda)^N}\frac{\det_{\gamma\delta}\varphi_{\gamma+N}(\lambda_\delta)}{\Delta(\lambda)}\ , \tag{4.104}$$

where

$$\begin{aligned}\varphi_\gamma(\lambda) &= \frac{1}{\sqrt{2\pi}}\exp[-\lambda V'(\lambda) + V(\lambda)]\sqrt{V''(\lambda)} \\ &\qquad \times \int x^{\gamma-1}\exp[-V(x) + V'(\lambda)x]\,\mathrm{d}x \\ &= \lambda^{\gamma-1}[1 + \mathrm{O}(\lambda^{-1})]\ ,\end{aligned} \tag{4.105}$$

and

$$\varphi_\gamma(\lambda) = \mathcal{A}\varphi_{\gamma-1}(\lambda) = \mathcal{A}^{\gamma-1}\Phi(\lambda)\ . \tag{4.106}$$

For $N = 0$ this is just the representation, peculiar to the KP τ-function in the Miwa parametrisation, $T_k = (1/k)\mathrm{tr}\Lambda^{-k}$; see Eqn (4.86) above. Thus,

$$Z_V\{T\} = \tau^{\mathrm{KP}}\{T \mid G_V\}\ , \tag{4.107}$$

where it is the operator G (the point in the Grassmannian) which depends on the shape of potential $V(X)$. Also, recall that the only way in which Z depends on the size of the matrix n is through the *domain of variation* of the time variables T. If Eqn (4.104) is extended to the full Toda-lattice τ-function by the introduction of negative times, one obtains [36]

$$\begin{aligned}Z_V\{T,N,\bar{T}\} &= \frac{\mathcal{C}_V^{-1}(\Lambda)}{(\det\Lambda)^N}\exp\left(-\sum_{k>0}\bar{T}_k\,\mathrm{tr}\,\Lambda^{-k}\right) \\ &\times \int_{n\times n}\mathrm{d}X(\det X)^N\exp\left[-\mathrm{tr}\,V(X) + \mathrm{tr}\,\Lambda X + \sum_{k>0}\bar{T}_k\,\mathrm{tr}\,X^{-k}\right].\end{aligned} \tag{4.108}$$

When *this* extended partition function is considered as a KP τ-function we have, instead of Eqn (4.104),

$$Z_V\{T,N,\bar{T}\} = \frac{1}{(\det\Lambda)^N}\frac{\det_{\gamma\delta}\varphi^{\{\hat{V}\}}_{\gamma+N}(\lambda_\delta)}{\Delta(\lambda)}\ , \tag{4.109}$$

and the relevant φ-functions are

$$\begin{aligned}\varphi^{\{\hat{V}\}}_{\gamma+N}(\lambda) &= \frac{1}{\sqrt{2\pi}}\exp[-\lambda V'(\lambda) + \hat{V}(\lambda)]\sqrt{V''(\lambda)} \\ &\qquad \times \int x^{\gamma-1}\exp[-\hat{V}(x) + V'(\lambda)x]\,\mathrm{d}x \\ &= \lambda^{N+\gamma-1}[1 + \mathrm{O}(\lambda^{-1})]\ ,\end{aligned} \tag{4.110}$$

with

$$\begin{aligned}\hat{V}(x) &\equiv V(x) - N\ln x - \sum_{k>0}\bar{T}_k x^{-k}\ , \\ V(x) &= \hat{V}_+(x)\ ,\end{aligned} \tag{4.111}$$

where $\hat{V}_+(x)$ is the positive-power portion of the Laurent series $\hat{V}(x)$. Functions $\varphi_\gamma(\lambda)$ in Eqn (4.105) are equal to

$$\varphi^{\{\hat{V}\}}_\gamma(\lambda)\big|_{\bar{T}=0}\ .$$

4.8 String equations and the general concept of reduction

The role of the string equation is to fix the point G in the universal module space (UMS) associated with the particular matrix model, so that the partition function, considered as a function of time-variables, will appear as the corresponding τ-function of a fixed shape. In this sense the idea behind the string equation is exactly the same as the reduction of integrable hierarchies. The difference is that *linear* reductions, as defined in Section 4.5 above, are not enough to fix G unambiguously: they just specify certain subsets in the Grassmannian, which are still infinite-dimensional. The reason why these are usually linear reductions that are considered in the conventional theory of integrable hierarchies is that they are associated with the simplest possible — Kac–Moody — subalgebras in the entire $\mathrm{GL}(\infty)$. String equations, even their simplest examples, are usually fragments of more complicated Virasoro and W-algebras, and are in fact considerably more restrictive. Moreover, the string equation is usually a *distinguished* fragment, because it usually belongs to the Virasoro component of the WIs, and the Virasoro algebra is still a *Lie* subalgebra in $\mathrm{GL}(\infty)$. This is what makes the problem of string equations very similar to the 'classical' one with linear reduction.

More specifically, in order to take string equations (and in fact the entire set of Virasoro — but not W — constraints) into consideration of reduction it is enough to allow the coefficients in Eqn (4.40) to depend on t and $\bar{t}$, without changing the order of time-derivatives. Of course, there are no obvious reasons to think that *any* point G in the UMS can be selected by imposing these kind of linear-derivative constraints on τ-function, and further investigation may require essential generalisation of such a restricted notion of string equations. However, some of the eigenvalue matrix models are already known to possess string equations of such simple type, associated with Virasoro subalgebras of $\mathrm{GL}(\infty)$. I will not go into details of the general theory — it is far from completed yet — but instead present several examples of how string equations arise in particular matrix models. These examples can illustrate also the simplifications arising when only string equations and not the entire sets of WIs need to be derived. In particular, it is clear that in cases when τ is represented as $\mathrm{Det}_{ij}\mathcal{H}_{ij}$, a linear differential equation imposed on $\mathcal{H}_{ij}$ will give rise to a similar equation on τ itself. Most known string equations can be derived with the help of this technical idea. They are usually associated with the invariance of integrals under *constant* shifts of integration variables $\delta h =$ const in scalar-product and other discrete models, and with the action of the operator $\mathrm{tr}(\partial/\partial L_{\mathrm{tr}})$ in the GKM. For somewhat more involved ideas associated with string equations, see [94].

Scalar-product models. The string equation can be easily deduced for very specific types of measures $\mathrm{d}\hat{\mu}_{h,\bar{h}}$. Since the integral

$$\mathcal{H}_{ij} = \int h^i \bar{h}^j \exp[U(h) + \bar{U}(\bar{h})]\,\mathrm{d}\mu_{h,\bar{h}} \tag{4.112}$$

is invariant under the shift of integration variable δh = const,

$$\int h^i \bar{h}^j \exp[U(h) + \bar{U}(\bar{h})]\mathrm{d}\hat{\mu}_{h,\bar{h}} \times \left[\mathrm{i}h^{-1} + \frac{\partial U(h)}{\partial h} + \frac{\partial}{\partial h}\ln(\mathrm{d}\hat{\mu}_{h,\bar{h}})\right] = 0\,, \tag{4.113}$$

or

$$\mathrm{i}\mathcal{H}_{i-1,j} + \sum_{k>0} k t_k \frac{\partial}{\partial t_{k-1}}\mathcal{H}_{ij} + \left[S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}\right)\right]_{ij} = 0\,. \tag{4.114}$$

The string equation arises straightforwardly when the operator S is linear. This is true if $\ln(\mathrm{d}\hat{\mu}_{h,\bar{h}}) \sim h\,\mathrm{f}(\hat{h})$ with any function $\mathrm{f}(h)$. If the measure $\mathrm{d}\hat{\mu}_{h,\bar{h}}$ is also required to be symmetric in h and $\bar{h}$, one obtains the conventional 2-matrix model as the only example:

$$\mathrm{d}\hat{\mu}_{h,\bar{h}} = \exp(ch\bar{h})\,\mathrm{d}h\,\mathrm{d}\bar{h}\,. \tag{4.115}$$

The equation for $\mathcal{H}_{ij}$ is:

$$\left(\sum_{k>0} k t_k \frac{\partial}{\partial t_{k-1}} + c\frac{\partial}{\partial \bar{t}_1}\right)\mathcal{H}_{ij} = -\mathrm{i}\mathcal{H}_{i-1,j}\,. \tag{4.116}$$

Its implication for $\hat{\tau}_N$ is:

$$\left(\sum_{k>0} k t_k \frac{\partial}{\partial t_{k-1}} + c\frac{\partial}{\partial \bar{t}_1}\right)\hat{\tau}_N = 0\,, \tag{4.117}$$

since the r.h.s. of Eqn (4.116) does not contribute to the determinant (the entries in the ith row are proportional to those in row $i-1$).

In the particular case of the *1-matrix model*, $c = 0$, one recognises the lowest Virasoro constraint $L_{-1}\hat{\tau}_N = 0$. Traditionally the name string equation is given not to the L_{-1} constraint itself, but to its t_1 derivative: $(\partial/\partial t_1)(L_{-1}\hat{\tau}_N) = 0$. For the 2-matrix model, Eqn (4.117) is the lowest ($m = 1$, $n = 0$) component of the WIs

$$[\widetilde{W}^{(m+1)}_{n-m}(t) - (-1)^{m+n}c^{n+1}\widetilde{W}^{(n+1)}_{m-n}(\bar{t})]\hat{\tau}_N = 0\,.$$

Of course, there is also a similar equation with $t \leftrightarrow \bar{t}$.

Multicomponent (conformal) models. The crucial feature of these models is that the intermatrix interaction, when rewritten in terms of eigenvalues, usually contains only differences $h_i^{(\alpha)} - h_j^{(\beta)}$. Thus there is usually covariance under *simultaneous* shift of all eigenvalues $\delta h_i^{(\alpha)} =$ const by the *same* constant. This gives rise to a string equation of the form

$$\left(\sum_{\alpha} L_{-1}^{(\alpha)}\right)\tau_N = 0\,. \tag{4.118}$$

See [39] for details.

Generalised Kontsevich model. In order to derive the string equation, one should act on the partition function $Z_V\{T_k = (1/k)\mathrm{tr}\Lambda^{-k}\} = \mathcal{C}_V^{-1}\mathcal{F}_V\{L = V'(\Lambda)\}$ with the operator

$$\mathrm{tr}\frac{\partial}{\partial L_{\mathrm{tr}}} = \mathrm{tr}\frac{1}{V''(\Lambda)\partial\Lambda_{\mathrm{tr}}}\,.$$

One can rewrite the result of this action in terms of time-derivatives:

$$\mathrm{tr}\frac{\partial}{\partial L_{\mathrm{tr}}}\ln Z_V\{T\} = -\sum_{k>0}\left(\mathrm{tr}\frac{1}{V''(\Lambda)\Lambda^{k+1}}\right)\frac{\partial}{\partial T_k}\ln Z_V\{T\}. \tag{4.119}$$

Alternatively one can use the fact that

$$\mathrm{tr}\frac{\partial}{\partial L_{\mathrm{tr}}} = \sum_{\gamma}\frac{1}{V''(\lambda_\gamma)}\frac{\partial}{\partial\lambda_\gamma}, \qquad l = V'(\lambda)\,,$$

and obtain an explicit expression for Z_V in terms of eigenvalues (Miwa coordinates):

$$Z_V \sim \exp[\operatorname{tr} V(\Lambda) - \operatorname{tr} \Lambda V'(\Lambda)] \sqrt{\prod_\gamma V''(\lambda_\gamma)} \frac{\det \hat{\varphi}_\gamma(\lambda_\delta)}{\Delta(\lambda)}$$
$$\sim \frac{\det \varphi_\gamma(\lambda_\delta)}{\Delta(\lambda)}, \tag{4.120}$$

to get:

$$\left(\operatorname{tr} \frac{\partial}{\partial L_{\mathrm{tr}}}\right) \ln Z_V\{T\}$$
$$= \frac{1}{2} \operatorname{tr} \frac{V'''(\Lambda)}{[V''(\lambda)]^2} + \frac{1}{2} \sum_{\gamma>\delta} \frac{V''(\lambda_\gamma) - V''(\lambda_\delta)}{\lambda_\gamma - \lambda_\delta} \frac{1}{V''(\lambda_\gamma) V''(\lambda_\delta)}$$
$$- \operatorname{tr} \Lambda + \sum_\beta \frac{\partial}{\partial l_\beta} \ln \det{}_{\gamma\delta} \hat{\varphi}_\gamma(l_\delta) . \tag{4.121}$$

Comparison of these two expressions gives:

$$\frac{\mathcal{L}_{-1}^{(V)} Z_V}{Z_V} \equiv \frac{1}{Z_V} \left[\sum_{k>0} \left(\operatorname{tr} \frac{1}{V''(\Lambda) \Lambda^{k+1}} \right) \frac{\partial}{\partial T_k} \right.$$
$$\left. + \frac{1}{2} \sum_{\gamma>\delta} \frac{V''(\lambda_\gamma) - V''(\lambda_\delta)}{\lambda_\gamma - \lambda_\delta} \frac{1}{V''(\lambda_\gamma) V''(\lambda_\delta)} - \frac{\partial}{\partial T_1} \right] Z_V$$
$$= -\frac{\partial}{\partial T_1} \ln Z_V + \operatorname{tr} \Lambda - \sum_\beta \frac{\partial}{\partial l_\beta} \ln \det{}_{\gamma\delta} \hat{\varphi}_\gamma(l_\delta) . \tag{4.122}$$

One can show that the r.h.s. is equal to zero, and thus the string equation arises in the form

$$\mathcal{L}_{-1}^{(V)} Z_V = 0 . \tag{4.123}$$

If the potential is monomial, $V_p = X^{p+1}/(p+1)$, then $r_k = -[p/(p+1)]\delta_{k,p+1}$ and

$$\mathcal{L}_{-1}^{V_p} \to \mathcal{L}_{-p} \equiv \frac{1}{p} \left[\sum_{k>0} (k+p)(T_{k+p} + r_{k+p}) \frac{\partial}{\partial T_k} \right.$$
$$\left. + \frac{1}{2} \sum_{k=1}^{p-1} k(p-k) T_k T_{p-k} \right] . \tag{4.124}$$

The technical idea behind the proof [30] is to represent

$$\frac{\partial}{\partial T_1} \ln Z_V = \operatorname{Res} \frac{Z_V\{T_k + (1/k\lambda^k)\} \mathrm{d}\lambda}{Z_V\{T_k\}}, \tag{4.125}$$

and to make use of the second determinant representation in (4.120) in both the denominator and the numerator:

$$\frac{\partial}{\partial T_1} \ln Z_V = \operatorname{Res} \frac{\mathrm{d}\lambda}{\prod_{\gamma=1}^n (\lambda - \lambda_\gamma)} \frac{\det \begin{bmatrix} \varphi_\delta(\lambda_\gamma) & \varphi_{n+1}(\lambda_\gamma) \\ \varphi_\delta(\lambda) & \varphi_{n+1}(\lambda) \end{bmatrix}}{\det \varphi_\delta(\lambda_\gamma)} . \tag{4.126}$$

Now recall that

$$\varphi_\gamma(\lambda) \sim \lambda^{\gamma-1}[1 + \mathrm{O}(\lambda^{-1})] . \tag{4.127}$$

At some point we shall need even more: in fact,

$$\varphi_\gamma(\lambda) \sim \lambda^{\gamma-1}[1 + \mathrm{O}(\lambda^{-2})] ,$$

that is,

$$\varphi_\gamma(\lambda) = \lambda^{\gamma-1} + c_\gamma \lambda^{\gamma-2} + \ldots, \quad \text{and} \quad c_\gamma = 0 \text{ for any } \gamma . \tag{4.128}$$

This is a rather delicate property of the GKM. It follows from two facts: first, that

$$\varphi_1 = 1 + \mathrm{O}\left(\frac{V''''}{(V'')^2}, \frac{(V''')^2}{(V'')^3} \right),$$

thus $c_1 = 0$; and second, that the Kac–Schwarz operator $\mathcal{A}$, defined in Eqn (4.106), does not have contributions with zeroth powers of λ, thus $c_{\gamma+1} = c_\gamma$. (For example, if

$$V(x) = \tfrac{1}{2}x^2 + ax ,$$

then

$$\varphi_\gamma(x) = \frac{1}{\sqrt{2\pi}} \int x^{\gamma-1} \exp[-\tfrac{1}{2}(x - \lambda^2)] \, \mathrm{d}x = \lambda^{\gamma-1} + 0 \cdot \lambda^{\gamma-2} + \ldots;$$

the dangerous terms with a simply do not show up in the expression for φ_γ.)

After this comment I can come back to the evaluation of Eqn (4.126). The product in the denominator, which arose from the Van der Monde determinant, is already proportional to λ^2: $\prod_{\gamma=1}^n (\lambda - \lambda_\gamma) = \lambda^n[1 + \mathrm{O}(\lambda^{-1})]$. Because of this and the asymptotic formulas (4.127), it is clear that if determinant in the numerator of Eqn (4.126) is rewritten as a linear combination of $n \times n$ determinants with the coefficients $\varphi_\gamma(\lambda)$ from the last row, only terms with $\gamma \geqslant n$ can contribute. There are two such terms: with $\gamma = n$ and $\gamma = n+1$. In the expansion of the $(n+1) \times (n+1)$ determinant, $\varphi_{n+1}(\lambda)$ is multiplied by $\det \varphi_\gamma(\lambda_\delta)$, which exactly cancels with the determinant in the denominator, and the relevant contribution is

$$\operatorname{Res} \frac{\varphi_{n+1}(\lambda) \mathrm{d}\lambda}{\prod_{\gamma=1}^n (\lambda - \lambda_\gamma)} = c_{n+1} + \sum_\gamma \lambda_\gamma = c_{n+1} + \operatorname{tr} \Lambda . \tag{4.129}$$

The term with $\varphi_n(\lambda)$ is

$$\frac{\det[\varphi_1(\lambda_\gamma) \ldots \varphi_{n-1}(\lambda_\gamma) \, \varphi_{n+1}(\lambda_\gamma)]}{\det[\varphi_1(\lambda_\gamma) \ldots \varphi_{n-1}(\lambda_\gamma) \, \varphi_n(\lambda_\gamma)]} \operatorname{Res} \frac{\varphi_n(\lambda) \mathrm{d}\lambda}{\prod_{\gamma=1}^n (\lambda - \lambda_\gamma)} . \tag{4.130}$$

The remaining residue is just unity. The determinant in the numerator differs from the one in the denominator by a substitution of the column with entries $\varphi_n(\lambda_\gamma)$ for that with $\varphi_{n+1}(\lambda_\gamma)$.

At last we can return to Eqn (4.122) and recall that $(\partial/\partial l)\hat{\varphi}_{\gamma+1}(l) = \varphi_{\gamma+1}(l)$; thus

$$\sum_\beta \frac{\partial}{\partial l_\beta} \ln \det{}_{\gamma\delta} \hat{\varphi}_\delta(l_\gamma) = \frac{\det[\hat{\varphi}_1(l_\gamma) \ldots \hat{\varphi}_{n-1}(l_\gamma) \, \hat{\varphi}_{n+1}(l_\gamma)]}{\det[\hat{\varphi}_1(l_\gamma) \ldots \hat{\varphi}_{n-1}(l_\gamma) \, \hat{\varphi}_n(l_\gamma)]}, \tag{4.131}$$

the r.h.s. of which is just the same as the term (4.130), since the $\hat{\varphi}_\delta$ differ from the φ_δs by a δ-independent factor of $\exp[V(\lambda) - \lambda V'(\lambda)]\sqrt{V''(\lambda)}$. Thus, one concludes that the r.h.s. of Eqn (4.122) is equal to $-c_{n+1}$, which actually vanishes, as was explained several lines above.

Two things deserve attention in this derivation. First, it was absolutely crucial that we had $(\partial/\partial T_1) \ln Z_V$ on the r.h.s. of Eqn (4.122) to make it vanish, and therefore $\partial/\partial t_1$ immediately appears in the expression for the $\mathcal{L}_{-1}^{(V)}$ operator

on the l.h.s. [this is the origin of the r_k-corrections in Eqn (4.124)]. Second, the result is both simple and natural, but the proof is full of technical details and looks somewhat artificial. It becomes even more involved when the general formula (4.136) for the T_k-derivatives of Z_V with $1 \leqslant k \leqslant p$ (see [40]) is derived; this formula plays an important role in the theory of GKM and its applications to the theory of quantum gravity. The proof of the string equation is just a particular case of that formula, since using the integral representation of $\hat{\varphi}(l)$ one can represent the r.h.s. of Eqn (4.131) as $(1/Z_V)\langle \operatorname{tr} X \rangle$, where $\langle\ \rangle$ now stands for the average defined by the Kontsevich integral. Thus,

$$\mathcal{L}_{-1}^{(V)} Z_V \overset{(4.122)}{=} -\frac{\partial}{\partial T_1} Z_V + \langle \operatorname{tr} \Lambda - \operatorname{tr} X \rangle \overset{(4.136)}{=} 0 \,. \qquad (4.132)$$

4.9 On the theory of the generalised Kontsevich model

This theory is a naturally broad collection of topics for a separate big section in these notes. However, I decided not to include too detailed a presentation because the GKM theory seems to be incomplete. First, I believe that the natural invariant formulation — of which the existing matrix integral is only a specific realisation — is still lacking. Second, the GKM is not yet generalised enough to fulfil its main purpose of incorporating information about *all* the models of 2d gravity (in fact it should include even more: the entire theory of integrable hierarchies and geometrical quantisation). Third, though the whole approach is very conceptual and deep, many *proofs*, as available nowadays, are still very technical and long. All this implies that a proper view on the subject of GKM still needs to be found. At the moment I could describe two complementary approaches: one, starting from the integral representations, the other from the Duistermaat–Heckman (localisation) theory and Fourier analysis on group manifolds. Though intimately related, these two approaches are still technically different in too many respects. The second one is more fundamental (since ordinary integrals arise from discrete sums either in special limits or in the case of infinite-dimensional algebras, and is more important, since the integral representation is only one of many possible ways to define the quantities of interest). However, many of the most important results obtained in the first approach do not have their proper names and exact counterparts in the second one. I believe that this whole issue will be greatly clarified in the near future and have decided to postpone a detailed review till that time. What one cannot avoid in these notes is giving at least a *list* of topics already included in the theory of GKM, and this is the purpose of the present subsection.

The Kontsevich model with $V = \frac{1}{3}X^3$ was derived by Kontsevich [22] from the original definition of topological 2d gravity, given by Witten [9] in terms of the generating functional for Chern classes of certain bundles over Riemann surfaces. Generalisation of this reasoning (when more bundles are taken into consideration) leads to the theory of Landau–Ginzburg gravity (LGG), which is believed to be the same as the GKM, though not all the proofs are yet available.†

† Intermediate results include the study of the spherical approximation to LGG, which exhibits the structures peculiar to 'quasiclassical integrable hierarchies' (of which the Bateman hierarchy, to be briefly mentioned in Section 5.2, is an example), and which also arise in 'quasiclassical approximation' to the GKM. For some results in this direction see [17, 40, 41, 95–97], references therein.

The crucial feature of nonperturbative partition func-tions, as discussed at the beginning of Section 2, is their intrinsic integrability. For 2d gravity this general idea acquires a very concrete formulation: the partition functions are usually just τ-functions of conventional integrable hierarchies; moreover, for LGG, associated with minimal models, these are just ordinary multicomponent Toda hierarchies.‡

Kontsevich found a representation for the generating func-tional in the form of a matrix integral, i.e. he formulated a matrix model, which later allowed him to prove Witten's conjecture that the functional is in fact a τ-function. The concept of the GKM as a *universal* matrix model, including all the information about generic (eigenvalue?) matrix models and thus all the models of 2d(?) gravity, was introduced in [30], and the analogue of the Kontsevich model with arbitrary potential $V(X)$, i.e. the expression

$$Z_V\{T\}\Big|_{T_k=(1/k)\operatorname{tr}\Lambda^{-k}} = C_V(\Lambda)^{-1}\mathcal{F}_V[V'(\Lambda)]$$
$$\sim \frac{\sqrt{\det V''(\Lambda)}}{(2\pi)^{\frac{1}{2}n^2}\exp\operatorname{tr}[\Lambda V'(\Lambda) - V(\Lambda)]}$$
$$\times \int_{n\times n} \mathrm{d}X \exp[-\operatorname{tr} V(X) + \operatorname{tr} V'(\Lambda)X] \qquad (4.133)$$

was proposed as an intermediate step in this direction.§ This (still restricted) version of the GKM is already enough to unify all the $(p, 1)$-models of 2d gravity. In some sense, (p, q)-models with $q \neq 1$ are also included, but in a very nontransparent way (using analytical continuation), which does not even explicitly respect the $p \leftrightarrow q$ symmetry. The partition function of such a GKM, $Z_V\{T\}$, depends on two types of variables: time-variables, $\hat{T}_k$, and the potential, V. Formally these two types of variables are absolutely different, V being responsible for the choice of a particular LGG model or, what is essentially the same, of a particular reduction of the Toda-lattice or KP hierarchy; $\hat{T}_k$ are parameters of the generating functional of all correlation functions in this particular model. But of course, since one is dealing with an exact (nonperturbative) approach, there is almost no real difference between these types of dependencies — on the model (vacuum state) and on the $\hat{T}$s: the model can be changed by a noninfinitesimal shift of the $\hat{T}$ variables. Technically, in the GKM this is reflected in the identity of the form [40]:

$$\mathcal{Z}_{V_p}\{T\} = f_p(r\,|\,\hat{T}_k + r_k)\,\tau\{\hat{T}_k + r_k\,|\,G_p\}\,, \qquad (4.134)$$

where

$$r_k = \frac{p}{k(p-k)}\operatorname{Res}[V'(\mu)]^{1-k/p}\,\mathrm{d}\mu$$

provides a specific parametrisation of potentials V (which is here assumed to be any polynomial of degree p) and f_p is some simple function:

‡ One can say that this is natural: both such models and Toda hierarchies are associated with the level $k = 1$ Kac–Moody algebras and corresponding simplified versions of the WZNW model. However, too much still remains to be clarified about this 'obvious' connection.

§ I remind the reader that $\det V''(\Lambda)$ has a somewhat tricky definition, see Section 2.5. The same matrix integral (4.133) was also considered in [31–33, 98].

$$f_p(r|\hat{T}_k + r_k) = \exp\left[-\frac{1}{2}\sum_{i,j} A_{ij}(r)(\hat{T}_i + r_i)(\hat{T}_j + r_j)\right],$$

$$A_{ij} = \mathrm{Res}[V'(\mu)]^{i/p}\,\mathrm{d}[V'(\mu)]^{j/p} = \frac{\partial^2 \ln \tau_0^{(p)}}{\partial t_i \partial t_j}, \qquad (4.135)$$

and $\tau_0^{(p)}$ is a τ-function of the 'quasiclassical hierarchy'. The important thing to note is that G_p (which defines the shape of the τ-function as a function of $\hat{T} + r$) and f_p depend only on the degree p and not on the other details of the shape of the potential. This is a deep formula. It accounts for two phenomena at once. First, it says that Z depends on the *sum* of $\hat{T}$ and r.† Second, the dependence on V is not *quite* smooth: when the degree of the potential changes, the shape of the functions f and τ also change abruptly. Another side of the same phenomenon is that the partition function $Z_V\{T\}$, which in principle is well defined as a matrix integral for all choices of V and L (and thus $\hat{T}$) at once, is in fact singular at some points: there are phase transitions, manifesting themselves in the switch from one LGG model to another. After a phase transition the original integral expression becomes somewhat symbolic: it defines the partition function only in the sense of analytical continuation, and it is a separate problem to find an integral representation that is adequate in the new phases. In practice, what is nicely described by the GKM integral representation in the form of Eqn (4.133) are $(p, 1)$-models, with $p + 1$ being just the power of the potential $V(x)$. What has not yet been found is an analogous representation for (p, q)-models with $q \neq 1$ (it can involve multiple matrix integrals, and the universal model is supposed to be 'matrix quantum mechanics in external fields').

Derivation of the crucial formula (4.134) by any approach —starting from the GKM in the form of either LGG or matrix integrals — is still very tedious. In the matrix-model representation it relies upon the identities [40]

$$\frac{\partial Z_V}{\partial T_k} = \langle \mathrm{tr}\,\Lambda^k - \mathrm{tr}\,X^k \rangle \equiv \mathcal{C}_V^{-1}\int(\mathrm{tr}\,\Lambda^k - \mathrm{tr}\,X^k) \times \exp[-\mathrm{tr}\,V_p(X) + \mathrm{tr}\,V_p'(\Lambda)X]\,\mathrm{d}X \quad \text{for } 1 \leqslant k \leqslant p, \qquad (4.136)$$

which look trivial but are rather hard to derive. (A proof of the string equation in the GKM at the end of the previous subsection is the simplest example of this kind of exercise.) Certainly, some simple derivation 'in two lines' should exist, but has not yet been found. Formulas of this kind are very important for all aspects of GKM theory. Besides other things, they are necessary to evaluate the correlation functions in $(p, 1)$-models of 2d gravity, of which $Z_V\{T\}$ is a generating functional. If instead of these 'physical' questions, one asks about integrability theory, identities of this sort also play an important role. For example, looking at Eqn (4.136) for a special $k = p$ and special choice of potential — monomial $V_p(X) = X^{p+1}/(p+1)$ — one can note that the r.h.s. vanishes: this is just a WI, reflecting invariance under the shift of the integration variable, $\delta X =$ const. This is the simplest version of a more general statement:‡

$$\text{if } V_p(X) = \frac{X^{p+1}}{p+1}, \quad \text{then} \quad \frac{\partial Z_V}{\partial T_{pk}} = 0 \quad \text{for all } n \in Z^+. \qquad (4.137)$$

Looking from the point of view of integrable hierarchies, one immediately recognises statement (4.137) as an example of the reduction condition Eqn (4.40). It corresponds to the so-called p-reduction of the KP hierarchy, of which KdV ($p = 2$) and Boussinesq ($p = 3$) are the most celebrated examples. See [30, 40] for all details and references, the only thing to mention here is that the slightly weaker version of the constraint (4.137),

$$\frac{\partial Z_V}{\partial T_{pn}} = a_n = \text{const}, \qquad (4.138)$$

where the a_n do not depend on any time variables, can be expressed simply in the Miwa parametrisation: it is just the statement that the φ-functions in

$$Z_V = \frac{\det_{\gamma\delta}\,\varphi_\gamma(\lambda_\delta)}{\Delta(\lambda)}$$

satisfy the p-reduction condition

$$\lambda^p \varphi_\gamma(\lambda) = \sum_{\delta=1}^{\gamma+p} \hat{\mathcal{V}}_{\gamma\delta}\varphi_\delta(\lambda). \qquad (4.139)$$

This is a restrictive relation, because the φs are infinite series in $1/\lambda$, while on the r.h.s. of Eqn (4.139) there is only a finite number of terms. In the GKM it is satisfied for monomial potential just as a corollary of the Gross–Newman equation, or, more exactly, of the WI for the integral

$$\varphi_\gamma(\lambda) \sim \int x^{\gamma-1}\exp[-V(x) + V'(\lambda)x]\mathrm{d}x.$$

Indeed, the integral does not change under the shift $\delta x =$ const, and this implies

$$\int x^{\gamma-1}\left[V'(x) - V'(\lambda) - \frac{\gamma-1}{x}\right]\exp[-V(x) + V'(\lambda)x]\,\mathrm{d}x = 0,$$

that is,

$$\sum_{k=1}^{p+1} k\upsilon_k[\varphi_{\gamma+k-1}(\lambda) - \lambda^{k-1}\varphi_\gamma(\lambda)] - (\gamma-1)\varphi_{\gamma-1} = 0. \qquad (4.140)$$

If only $\upsilon_{p+1} \neq 0$, this leads to an identity of the required form of Eqn (4.139). This description of reduction can be modified to allow for nonmonomial potentials, making use of the concept of 'equivalent hierarchies', see [40, 100]: in this framework the reduction condition is

$$V'(\lambda)\,\varphi_\gamma(\lambda) = \sum_\delta \mathcal{V}_{\gamma\delta}\varphi_\delta, \qquad (4.141)$$

† In the Miwa parametrisation, $\hat{T}_k = (1/k)\mathrm{tr}[V_p'(\Lambda)]^{-k/p}$. Throughout these notes I have used different time-variables $T_k = (1/k)\mathrm{tr}\,\Lambda^{-k}$, which are independent of the potential V; instead the V-dependence of Z_V — which we did not really study — was rather nontrivial. If expressed in terms of $\hat{T}$, the partition function $\hat{Z}_V\{\hat{T} + r\} = Z_V\{T\}$ becomes almost independent of V: it changes — abruptly — *only* when the degree, p, of the potential changes. This second type of description is, of course, in better accord with the symmetries of the particular model, which are different in different 'vacua' (for different p). Therefore, the variables $\hat{T} + r$, rather than T, arise naturally in the WIs as we saw in Sections 2.5 and 2.6. Ts and $\hat{T}$s are suited to different purposes: the Ts are nice where the *universality* aspects of the GKM are concerned, while the $\hat{T}$s arise when specific features of particular models (orbits, vacua) are considered.

‡ This property was technically implicit in Kontsevich's original work [22] for $p + 1 = 3$, where it was related to certain combinatorial identities. A tricky proof, relying upon properties of τ-functions, was given in [30] for any p. An example of a straightforward proof, again for $p + 1 = 3$, (just in terms of Kontsevich matrix integrals) can be found in [99].

but classes of essentially different reductions are labelled by the degree of the potential only.

As already discussed in the previous subsection, linear constraints like Eqn (4.139) are not restrictive enough to fix the shape of the τ-function (the point G in the universal module space) unambiguously; the string equation should also be imposed. If expressed in terms of φs, the string equation is just the property (4.106):

$$\varphi_{\gamma+1} = \mathcal{A}\varphi_\gamma \ , \tag{4.142}$$

where the Kac–Schwarz operator is given by

$$\mathcal{A} = \frac{1}{V''(\lambda)}\frac{\partial}{\partial\lambda} - \frac{1}{2}\frac{V'''(\lambda)}{[V''(\lambda)]^2} + \lambda \ , \tag{4.143}$$

and has an obvious generalisation of the form

$$\mathcal{A}_{p,q} = \frac{\partial}{\partial V'_p(\lambda)} - \frac{1}{2}\frac{V'''_p(\lambda)}{[V''_p(\lambda)]^2} + Q'_q(\lambda) \ , \tag{4.144}$$

where $Q_q(\lambda)$ is a polynomial of degree $q+1$ and Eqn (4.142) is replaced by

$$\varphi_{\gamma+q} = \mathcal{A}_{p,q}\varphi_\gamma \ . \tag{4.145}$$

This generalisation is naturally related to the string equation in (p,q)-models; see [41] and references therein. The generic (p,q) LGG model can be described by a system of constraints,

$$(\lambda^p - \mathcal{V}_p)\{\varphi\} = 0 \ ,$$

$$\mathcal{A}_{p,q}\{\varphi\} = 0 \ , \tag{4.146}$$

where the operators $\mathcal{V}_p$ and $\mathcal{A}_{p,q}$ are not uniquely fixed by the choice of p and q, and there is also the freedom to change variables $\lambda \to \mathrm{f}(\lambda)$ and to make a triangular transformation of the basis $\varphi_\gamma \to \varphi_\gamma + \sum_{\delta<\gamma} C_{\gamma\delta}\varphi_\delta$. Altogether, the set of Eqns (4.146), modulo these allowed transformations, is finite: $(p-1)(q-1)$-dimensional, which is the dimension of the module space of LGG models with given p and q. The Kontsevich integral can now be used to establish duality transformation from the (p,q)- to the (q,p)-model [41]:

$$Z_{V,Q}(\Lambda) = C^{-1}_{V,Q}(\Lambda) \times \int_{n\times n} \mathrm{d}X \exp[-\mathrm{tr}\, S_{V,Q}(X,\Lambda) + \mathrm{tr}\, V'(\Lambda)Q'(X)]Z_{Q,V}(X) \ . \tag{4.147}$$

Here,

$$S_{V,Q}(x,\lambda) = \int^x V'(y)\,Q''(y)\,\mathrm{d}y = \int^x V'(y)\,\mathrm{d}Q'(y) \ . \tag{4.148}$$

As usual, $C_{V,Q}(\Lambda)$ is the quasiclassical approximation to the integral, and

$$Z_{V,Q}(\Lambda) \equiv \frac{\det_{\gamma\delta}\varphi_\gamma(\lambda_\delta)}{\Delta(\lambda)} \ ,$$

where φ are solutions to Eqns (4.146) with $\mathcal{V}_p$ and $\mathcal{A}_{p,q}$ defined by Eqns (4.141) and (4.144), respectively.†

† Also, the expression for the t_k-variables is now modified:

$$r_k = \frac{p}{k(p-k)}\mathrm{Res}[V'_p(\mu)]^{1-k/p}\mathrm{d}Q'_p(\mu) \ .$$

For monomial V_p and Q_q,

$$r_k = -\frac{p}{p+q}\delta_{k,p+q} \ .$$

This relation does *not* provide any formula for $Z_{V_p,Q_q}(\Lambda)$ unless $q=1$. The case of $q=1$ is distinguished because Z_{V_p,Q_1} is trivial. Indeed, the 1-reduction constraint, $\lambda\varphi_\gamma = \varphi_{\gamma+1} + \sum_{\delta\leqslant\gamma}\mathcal{V}_{\gamma\delta}\varphi_\delta$, implies that $\det_{\gamma\delta}\varphi_\gamma(\lambda_\delta) = \Delta(\lambda)\prod_\delta\varphi_1(\lambda_\delta)$, and hence $Z_{Q_1,V_p} = \exp\sum_k a_k T_k$, which is essentially the same as $Z_{Q_1,V_p} = 1$,‡ and Eqn (4.147) is just our old formula (4.133) for the $(p,1)$ version of GKM. [In fact, $Q_1(X) \sim X^2$, and Z_{V_p,Q_1} is nothing but the gaussian Kontsevich model. It is trivial for the 'zero-time' condition $N=0$, as is assumed here.] The matrix model realisation of Z_{V_p,Q_q} for $q \neq 1$ is as yet unknown.

This is not the only important further generalisation of GKM Eqn (4.133). Another one is implied by the formula for $\mathcal{F}_V$ in terms of the eigenvalues from Section 3.3,

$$\mathcal{F}_V \sim \prod_{\gamma=1}^n \int \mathrm{d}x_\gamma \exp[-V(x_\gamma)]\Delta^2(x)\,I(x,l) \ . \tag{4.149}$$

As was already mentioned in Section 3.3, the Itzykson–Zuber integral,

$$I(x,l) \sim \int [\mathrm{D}U]\exp(\mathrm{tr}\,UXU^\dagger L) \sim \frac{\det_{\gamma\delta}\exp(x_\gamma l_\delta)}{\Delta(x)\Delta(l)} \ , \tag{4.150}$$

is, in fact, a coadjoint orbit integral and has a group theoretical interpretation: under certain conditions it becomes a character $\chi_R(\mathrm{g}) = \mathrm{Tr}_R\, g$ of the group $\mathrm{GL}(n)$. Here, $g \equiv \exp(L)$ is considered as a group element, and the representation R is labelled by integer-valued parameters $m_1,\ldots,m_n$ — essentially the lengths of the rows in the Young diagram. The exact statement is

$$I(m,l)\frac{\Delta(l)}{\Delta(g)} = \frac{\det_{\gamma\delta} g_\gamma^{m_\delta}}{\Delta(m)\Delta(g)} = \frac{\chi_R(g)}{d_R} \ , \tag{4.151}$$

i.e. in order to get a character one should integrate over matrices, X, with integer-valued eigenvalues.§ The dimension, $d_R R(I)$, of the representation can also be expressed in terms of m-variables: $d_R = \Delta(m)$. As regards the traces, $\mathrm{tr}\,X^k = \sum_\gamma x_\gamma^k \to \sum_\gamma m_\gamma^k$, which appear in the action of the GKM, they are very similar to the k-th Casimir eigenvalue $C_k(R)$ (though not exactly the same). Thus, we see that the integral in Eqn (4.149) is in fact very similar to

$$\mathcal{F}_V^{\mathrm{qu}}\{g,\bar{g}\} \equiv \sum_R \chi_R(\bar{g})\,\chi_R(g)\exp\left[-\sum_{k=0}^\infty v_k C_k(R)\right], \tag{4.152}$$

‡ Since $\varphi_1(\lambda) = 1 + \sum_{k>0} b_k\lambda^{-k}$, $\ln\varphi_1(\lambda) = \sum_{k>0}(a_k/k)\lambda^{-k}$, and the sum $\sum_\delta \ln\varphi_1(\lambda_\delta) = \sum_{k>0}(a_k/k)\sum_\delta\lambda_\delta^{-k} = \sum_{k>0} a_k T_k$. Addition of any *linear* combination of time-variables to $\ln\tau$ does not essentially change the τ-function. For example, the ordinary integrable equations (like KdV or KP) are usually written in terms of variables like $u = (\partial^2/\partial T_1^2)\ln\tau$, which are *second* derivatives of $\ln\tau$.

§ The ratio

$$\frac{\Delta(l)}{\Delta(g)} = \prod_{\gamma>\delta}\frac{l_\gamma - l_\delta}{\exp(l_\gamma) - \exp(l_\delta)}$$

is the usual correction factor, which is the price for the possibility of reducing the quantum-mechanical problem of motion on the orbit to a *single* matrix integral. The full problem of matrix quantum mechanics can and should be considered as a multimatrix (in fact, infinite-matrix) generalisation of GKM Eqn (4.133), which incorporates all the (p,q) LGG models.

evaluated at the point $\bar{g} = I$. The only real difference is that instead of the integral we have a sum over discrete values of m [sum over all the representations, or a *model* of GL(n)]. This 'discretised' (quantum?) GKM is more general than the continuum one, which can be obtained by various limiting procedures. It is now obvious that the theory of the discretised GKM largely overlaps that of 2d Yang–Mills theory. The simplest ingredient of this theory is the classical result, [101] that GL(N) characters are in fact (singular) Toda-lattice and KP τ-functions. Moreover, the entire sum on the r.h.s. of Eqn (4.152), if considered as a function of $T_k = (1/k)\operatorname{tr} g^k$ and $\bar{T}_k = [(1/k)\operatorname{tr}\bar{g}^k]$ is in fact a Toda-lattice τ-function. There are also features parallel to Eqn (4.134). See [102] for a little more details about the discretised GKM (see also the recent paper [81]). This is one more very important direction for the further investigation of GKM.

4.10 The 1-matrix model versus the Toda-chain hierarchy

At the end of this section I use an explicit example of a discrete 1-matrix model [26] to illustrate how a more familiar Lax description of integrable hierarchies arises from determinant formulas. This example will also be useful in Section 5.3 below, when one of the ways to take the double-scaling continuum limit of the 1-matrix model will be discussed. The Lax representation appears usually after some coordinate system is chosen in the Grassmannian. In the example which is now being considered this system is introduced by the use of orthogonal polynomials.

We already know from Section 3.6 that the partition function of the 1-matrix model (which is a *one*-component model) is given by

$$Z_N = \mathrm{Det}_{0<i,j\leqslant N}\langle h^i|h^j\rangle = \prod_{i=0}^{N-1}\exp(\phi_i) = Z_1\prod_{i=1}^{N-1}R_i^{N-i}\,, \tag{4.153}$$

where the last two representations are in terms of the norms of orthogonal polynomials:

$$\langle Q_n|Q_m\rangle = \exp(\phi_n)\,\delta_{nm}\,, \tag{4.154}$$

and the parameter of the 3-term relation

$$hQ_n(h) = Q_{n+1}(h) + c_nQ_n(h) + R_nQ_{n-1}(h)\,,$$

$$Z_1 = \exp(\phi_0) = \langle 1|1\rangle, \qquad R_n = \exp(\phi_n - \phi_{n-1})\,.$$

Of course, all the information is contained in the determinant formula together with the rule which defines the time-dependence of $\mathcal{H}^{\mathrm{f}}_{ij} = \langle h^i|h^j\rangle = \hat{\mathcal{H}}^{\mathrm{f}}_{i+j}$:

$$\frac{\partial\mathcal{H}^{\mathrm{f}}_{ij}}{\partial t_k} = \mathcal{H}^{\mathrm{f}}_{i+k,j} = \mathcal{H}^{\mathrm{f}}_{i,j+k}\,, \quad\text{or}\quad \frac{\partial\hat{\mathcal{H}}^{\mathrm{f}}_{i}}{\partial t_k} = \hat{\mathcal{H}}^{\mathrm{f}}_{i+k}\,. \tag{4.155}$$

(The possibility of expressing everything in terms of $\mathcal{H}^{\mathrm{f}}_i$ with a single matrix index i is a feature of the Toda-chain reduction of the generic Toda-lattice hierarchy.)

However, in order to reveal the standard Lax representation I need to go into somewhat more involved considerations. Namely, I consider representation of two operators on the basis of orthogonal polynomials. First,

$$h^kQ_n(h) = \sum_{m=0}^{n+k}\frac{\langle n|h^k|m\rangle}{\langle m|m\rangle}Q_m(h) = \sum_{m=0}^{n+k}\gamma^{(k)}_{nm}Q_m(h) \tag{4.156}$$

(here a simplified notation is introduced for

$$\langle n|\mathrm{f}(h)|m\rangle \equiv \langle Q_n|\mathrm{f}(h)|Q_m\rangle \quad\text{and}\quad \gamma^{(k)}_{nm} \equiv \frac{\langle n|h^k|m\rangle}{\langle m|m\rangle}\,.$$

Second,

$$\frac{\partial Q_n(h)}{\partial t_k} = -\sum_{m=0}^{n-1}\frac{\langle n|h^k|m\rangle}{\langle m|m\rangle}Q_m(h) = -\sum_{m=0}^{n-1}\gamma^{(k)}_{nm}Q_m(h)\,,$$

$$\frac{\partial\phi_n}{\partial t_k} = \frac{\langle n|h^k|n\rangle}{\langle n|n\rangle} = \gamma^{(k)}_{nn}\,. \tag{4.157}$$

[These last relations arise from differentiation of the orthogonality condition (4.154):

$$\exp(\phi_n)\frac{\partial\phi_n}{\partial t_k}\delta_{nm} = \frac{\partial\langle Q_n|Q_m\rangle}{\partial t_k}$$
$$= \left\langle\frac{\partial Q_n}{\partial t_k}\middle|Q_n\right\rangle + \left\langle Q_n\middle|\frac{\partial Q_m}{\partial t_k}\right\rangle + \langle Q_n|h^k|Q_m\rangle\,,$$

by looking at the cases of $m < n$ and $m = n$, respectively.]

From these relations one immediately derives the Lax-like formula

$$\frac{\partial\gamma^{(k)}_{nm}}{\partial t_q} = -\sum_{l=m-k}^{n-1}\gamma^{(q)}_{nl}\gamma^{(k)}_{lm} + \sum_{l=m+1}^{n+k}\gamma^{(k)}_{nl}\gamma^{(q)}_{lm}\,, \tag{4.158}$$

or, in matrix form,

$$\frac{\partial\gamma^{(k)}}{\partial t_q} = [R\gamma^{(q)},\gamma^{(k)}]\,, \tag{4.159}$$

where

$$R\gamma^{(k)}_{mn} \equiv \begin{cases} -\gamma^{(k)}_{mn} & \text{if } m > n \\ \gamma^{(k)}_{mn} & \text{if } m < n \end{cases}\,. \tag{4.160}$$

[I remind the reader that usually the R-matrix acts on a function

$$\mathrm{f}(h) = \sum_{n=-\infty}^{+\infty} f_n h^n$$

according to the rule

$$R\,\mathrm{f}(h) = \sum_{n\geqslant l} f_n h^n - \sum_{n<l} f_n h^n$$

with some 'level' l.] These $\gamma^{(k)}$ are not symmetric matrices, but one can also rewrite all the formulas above in terms of symmetric ones:

$$\mathcal{L}^{(k)}_{mn} \equiv \exp\left[\tfrac{1}{2}(\phi_n - \phi_m)\right]\gamma^{(k)}_{mn} = \frac{\langle m|h^k|n\rangle}{\sqrt{\langle m|m\rangle\langle n|n\rangle}}\,. \tag{4.161}$$

From Eqns (4.158) one can easily deduce Toda-equations for ϕ_n:

$$\frac{\partial^2\phi_n}{\partial t_k\partial t_l} = \frac{\partial}{\partial t_k}\frac{\langle n|h^l|n\rangle}{\langle n|n\rangle}$$
$$= \left(\sum_{m>n} - \sum_{m<n}\right)\frac{\langle n|h^k|m\rangle\langle m|h^l|n\rangle}{\langle m|m\rangle\langle n|n\rangle}\,, \tag{4.162}$$

where the r.h.s. can be expressed in terms of R_m, where $R_m = \exp(\phi_m - \phi_{m-1})$. In particular,

$$\frac{\partial^2\phi_n}{\partial t_1\partial t_1} = R_{n+1} - R_n$$
$$= \exp(\phi_{n+1} - \phi_n) - \exp(\phi_n - \phi_{n-1})\,. \tag{4.163}$$

Let me also mention that in this formalism the WIs (Virasoro constraints) follow essentially from the relation

$$\left(\frac{\partial}{\partial h}\right)^{\dagger} = -\frac{\partial}{\partial h} - \sum_{k>0} k t_k h^{k-1} , \tag{4.164}$$

where the Hermitean conjugation is with respect to the scalar product $\langle | \rangle$. For example, this relation implies, that

$$\left\langle Q_n \middle| \frac{\partial Q_n}{\partial h} \right\rangle = -\left\langle \frac{\partial Q_n}{\partial h} \middle| Q_n \right\rangle - \sum_{k>0} k t_k \langle Q_n | h^{k-1} | Q_n \rangle . \tag{4.165}$$

Now note that $\partial Q_n/\partial h$ is a polynomial of degree $n-1$, thus $\langle Q_n | \partial Q_n/\partial h \rangle = 0$. In fact,

$$\frac{\partial Q_n}{\partial n} = -\sum_{k>0} k t_k \left(\sum_{m=0}^{n-1} \gamma_{nm}^{(k-1)} Q_m \right) = -\sum_{k>0} k t_k \frac{\partial Q_n}{\partial t_{k-1}} .$$

Also, recall that

$$\langle Q_n | h^{k-1} | Q_n \rangle = \langle Q_n | Q_n \rangle \frac{\partial \phi_n}{\partial t_{k-1}} ,$$

to obtain:

$$\sum_{k>0} k t_k \frac{\partial \phi_n}{\partial t_{k-1}} = 0 \tag{4.166}$$

for any n. This should be supplemented by the relation $\partial \phi_n/\partial t_0 = \phi_n$. In order to get the lowest Virasoro constraint (string equation), $L_{-1} Z_N = 0$ or $L_{-1} \ln Z_N = 0$, it is enough just to sum over n from 0 to $N-1$.

For more details about the 1-matrix model, Toda-chain hierarchy, and application of the formalism of orthogonal polynomials in this context, see [26].

5. Continuum limits of discrete matrix models

5.1 What is the continuum limit?

The continuum limit of matrix models is, of course, the crucial issue for their phsycial application whenever these models are interpreted as discrete (lattice) approximations to continuum theory. The very first thing to be kept in mind is that it is *not* the only possible view on matrix models. Another approach considers them as describing *topological* (and thus also in a certain sense 'discrete') properties of the theory. Such models, when appearing in the field of, say, quantum gravity (which after all is a sort of pure topological theory) do *not* require any continuum limit to be taken: their discrete nature (occurrence of *integer*-valued matrix indices) reflects *not* the discrete approximation to the spacetime (which does not really exist in quantum gravity), but rather the essential discreteness of the underlying structures: the topology of the module spaces of geometries. Examples of matrix models which allow for this kind of interpretation — in terms of the topology of module spaces of bundles over Riemann surfaces — are provided by Kontsevich models, and this is why they usually do not require any continuum limit and why they were called 'continuous matrix models' in the introduction. The models which are usually interpreted in a more traditional way — as lattice theories — are represented by our 'discrete' models, the 1-matrix, conventional, and 'conformal' multimatrix models being included in this class. More sophisticated examples are provided by 'c = 1' theories, the Kazakov–Migdal model, and, say, Wilson's quantum chromodynamics (QCD) (and infinitely many other lattice theories). It is not surprising that the continuum limits of some discrete models provide theories of the Kontsevich type: this happens whenever continuum theory is supposed to have a kind of topological nature. This is usually the case for quantum gravity (which, as I said, is conceptually a topological theory in the 'module space of geometries', the notion of which is already made more or less explicit in the 2d case), but in principle this can also be true for many other theories, including the exhaustive quantum theory of Yang–Mills (YM) fields (again there is already considerable progress in this direction, as far as the 2d YM model is concerned). There should not be confusion about the presence of gauge particles in dimensions greater than 2 (for YM) and 3 (for gravity): there is no reason to prevent generic topological theory from possessing a continuum spectrum of excitations, through an explicit analogue of the Kontsevich-like description of such situations has not yet been found (as I have mentioned many times, it should probably rely upon noneigenvalue models).

I shall not discuss the nontrivial history of invention and understanding of all these notions (the crucial steps being the discovery of the 'multiscaling continuum limits [19, 20], which preserve the integrable structure of the discrete models in the continuum case; and the hypothesis of the equivalence of quantum and topological 2d gravities [9] and its proof [23, 24], provided by discovery of Kontsevich models [22] as a peculiar and powerful tool for description of the topology of the module spaces). Instead, following the main theme of these notes, I shall concentrate on the intrinsic relation between (multi-scaling) continuum limits and integrability: the notion of continuum limits is, in fact, built into the theory of integrable hierarchies and the underlying representation theory of Kac–Moody algebras.

In the case of the eigenvalue models the central issue here is the interrelation between Toda-lattice and Kadomtsev–Petviasvili (KP) hierarchies, even its more narrow aspect: elimination of the zero-time N, present in the Toda-lattice case. In terms of representation theory (or conformal field theory, which is essentially the same) the zero-time (which labels the filling level of the Dirac sea in the fermionic picture) is associated with the zero-modes of a *scalar* field and its elimination is just the change of boundary conditions, which eliminates zero-modes. The simplest example of this 'twisting' procedure is just the transformation from periodic to antiperiodic scalars — it still preserves the possibility of a a fermionic description (where it looks like a switch from the Ramond to the Neveu–Schwarz sector), and thus does not take us out of the field of conventional integrable hierarchies. In representation theory one can interpret the same operation simply as a switch from the homogeneous to the principal representation (associated with the Toda-lattice and KP hierarchies respectively).

This remarkably simple description is, of course, far from obvious if one investigates the continuum limit in a naive way, without taking integrable structure into account explicitly, but just sending the number of degrees of freedom in the discrete theory (i.e. the matrix size N) to infinity (together with the inverse lattice spacing, if any). See the classical review [18] for a discussion of the naive continuum limits in lattice gauge theories, i.e. the conditions for obtaining the second-order phase transitions, which allows for a continuum-like scaling behaviour in the vicinity of the critical point, with critical exponents defining all the continuum physics, from the quantum dimension of the spacetime to the spectrum of particles. The problem with

naive continuum limits is that they can easily destroy the integrable structure of the theory (the underlying hidden symmetries), unless special precaution is taken: the critical point (which is in fact a low-codimensional hypersurface in the infinite-dimensional space of parameters) should be approached from certain directions, so that the Ward identities (WIs) are not explicitly broken.

As soon as one considers WIs one is already into the field of integrable systems and the issue can be discussed inside this field. The above-mentioned switch from periodic to antiperiodic fields is, of course, apparent if the discrete and continuous Virasoro constraints [represented by formulas (1.2) and (1.3)] are compared, but this is *a posteriori* information, because so far I have interpreted 'continuous Virasoro constraints' as the WIs for the $V = X^3$ Kontsevich model, and it still remains to be explained why the Kontsevich model is indeed what arises after the continuum limit is taken. The simplest approach to *this* problem is to make use of the identity between the discrete 1-matrix model and the gaussian Kontsevich model [56], established in Section 3.8. Then the X^3 model arises in the large-N limit, just when the matrix integral is evaluated by the steepest-descent method [36]. I shall present this simple calculation in Section 5.4, but before that, I take a somewhat more direct (and complicated) approach in order to reveal at least some of ideas underlying the entire theory of continuum limits.

5.2 From the Toda-chain to the Korteveg de Vries equation

I begin with the simplest existing example: the continuum limit, in which the lowest equation of the 'Volterra hierarchy',

$$\frac{\partial R_n}{\partial t} = -R_n(R_{n+1} - R_{n-1}) \ , \tag{5.1}$$

turns into the lowest Korteveg de Vries (KdV) equation:

$$\frac{\partial r}{\partial T_3} = -\tfrac{1}{3} r''' - 2rr' \ . \tag{5.2}$$

The Volterra hierarchy is a reduction of the Toda-chain hierarchy, with $R_n = \exp(\phi_n - \phi_{n-1})$, arising when all the odd-times $t_{2k+1} = 0$ and all ϕ_n are supposed to be independent of them. More precisely,

$$\left.\frac{\partial \phi_n}{\partial t_{2_{k+1}}}\right|_{t_{\rm odd}=0} = 0 \ .$$

Therefore, this hierarchy is clearly related to the discrete 1-matrix model. I shall turn to the study of the 1-matrix model in the next subsection, but here I address the transformation from Eqns (5.1) to (5.2) [26, 103].

The basic idea of taking the continuum limit is to change the discrete 'zero-time' n for the *continuum* variable x (to be substituted by T_1 of the continuous hierarchy). In other words, the idea is to consider a subset of functions R_n, which satisfy the Volterra equation and depend on n very smoothly, so that they can actually be substituted by a smooth function $R(x)$. This is a very natural thing to do, of course, when one is interested in the large-n limit of the equation. Namely, one replaces Eqn (5.1) by

$$\frac{\partial R(x)}{\partial t} = -R(x)[R(x+\varepsilon) - R(x-\varepsilon)] \ , \tag{5.3}$$

and takes the limit $\varepsilon \to 0$, which, after rescaling $x \to \varepsilon x$, gives rise to the 'Bateman equation' (or 'Hopf equation'),

$$\frac{\partial R(x)}{\partial t} = -R(x)R'(x) \ . \tag{5.4}$$

This is a very interesting equation (see [104] for a description of the amusing aspects of the related theory, which is in fact intimately related to the theory of jets). However, it is much simpler than the KdV equation (for example, it is completely integrable in the most trivial sense of the word: the *entire set* of solutions satisfying *any* boundary conditions can be written down immediately, see [104]). The KdV equation can be considered as a sort of 'quantisation' of Eqn (5.4) (unfortunately this very interesting subject has not yet attracted enough attention and has not been studied well enough).

Remarkably, the Bateman equation is not the *only* possible limit of the Volterra equation: a fine-tuning procedure ['double-scaling (d.s) limit'] exists, which can provide a less trivial — KdV — equation [103]. Indeed, suppose that in the continuum limit R_n tends to a constant R_0, and the function $r(x)$ arises only as a scaling approximation to this constant: $R(x) = R_0[1 + \varepsilon^s r(x)]$. Then, the leading term on the r.h.s. of Eqn (5.4) is $\varepsilon RR'(x) = -2\varepsilon^s r(x)[1 + \mathrm{O}(\varepsilon^2, \varepsilon^s)]$, and instead of Eqn (5.4) we would get

$$\frac{\partial r}{\partial t} = -2\varepsilon R_0 r'(x)[1 + \mathrm{O}(\varepsilon^2, \varepsilon^s)] \ . \tag{5.5}$$

This equation is even simpler than Eqn (5.4) — it is just linear, but in fact it is too simple to preserve its form: by a simple change of variables,†

$$\tilde{x} = x - 2\varepsilon R_0 t \ , \tag{5.6}$$

$$\tilde{t} = \varepsilon^3 R_0 t \ , \tag{5.7}$$

it can be transformed into

$$\frac{\partial r}{\partial \tilde{t}} = \varepsilon^{-2}\mathrm{O}(\varepsilon^2, \varepsilon^s) \ ,$$

and terms on the r.h.s. also deserve to be taken into account. Then we get

$$\frac{\partial r(x)}{\partial t} = -2\varepsilon R_0[1 + \varepsilon^s r(x)][r'(x) + \tfrac{1}{6}\varepsilon^2 r'''(x) + \mathrm{O}(\varepsilon^4)]$$

$$= -2\varepsilon R_0[r'(x) + \tfrac{1}{6}\varepsilon^2 r'''(x) + \varepsilon^s rr'(x) + \varepsilon^2\mathrm{O}(\varepsilon^2, \varepsilon^s)],$$

and, after the change of variables (5.7),

$$\frac{\partial r(\tilde{x})}{\partial \tilde{t}} = -\tfrac{1}{3} r'''(\tilde{x}) - 2\varepsilon^{s-2} rr'(\tilde{x}) + \mathrm{O}(\varepsilon^2, \varepsilon^s) \ .$$

It is now clear that the choice $s = 2$ is to be preferred (a critical point), and at this point we get

$$\frac{\partial r}{\partial T_3} = -\frac{1}{3}\frac{\partial^3 r}{\partial T_1^3} - 2r\frac{\partial r}{\partial T_1} \ , \tag{5.8}$$

where new notation, T_1 and T_3, is introduced for $\tilde{x}$ and $\tilde{t}$, respectively. This is already the KdV Eqn (5.2), so we reach the following conclusion.

While the naive continuum limit of the Volterra equation is just a simple Bateman equation, the scaling limit can be fine tuned so that the KdV equation arises instead. The crucial ingredient of this adjustment is the change of time-variables, $\{t\} \to \{T\}$, which involves a singular parameter ε. The procedure can be easily generalised to the entire Volterra

† This change of variables is implied by the relation:

$$\frac{\partial}{\partial t} + 2\varepsilon^s R_0 \frac{\partial}{\partial x} = \left(\frac{\partial \tilde{t}}{\partial t} + 2\varepsilon^s R_0 \frac{\partial \tilde{t}}{\partial x}\right)\frac{\partial}{\partial \tilde{t}} + \left(\frac{\partial \tilde{x}}{\partial t} + 2\varepsilon^s R_0 \frac{\partial \tilde{x}}{\partial x}\right)\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial \tilde{t}} \ .$$

hierarchy, and fine tuning allows one to get the entire KdV hierarchy in the limit of $\varepsilon \to 0$. Uusally, transformation to the 'Kazakov variables' $\{T\}$ (they are a little different from those originally introduced by Kazakov [19]) from $\{t\}$ is some linear *triangular* transformation.

An important detail is that this procedure requires restriction to only *even* time-variables t_{2m}, $m \geqslant 0$. (If odd times are also involved, a *pair* of KdV hierarchies arises in the continuum limit — this is not a 'minimal' case.) Thus 'irreducible' realisation of the continuum limit requires a *reduction* of the original hierarchy. This can also be seen from the fact that the lowest KdV equation arises from the lowest Volterra equation, which is related to the *second* equation of the Toda-chain hierarchy.

Unfortunately, this simple piece of theory (continuum limits in terms of hierarchies) has never been worked out in full detail (for the entire Toda-lattice hierarchy, its multi-component generalisations and their reductions). As already mentioned, this theory will involve the general relation between homogeneous and principal representations of the (level $k = 1$) Kac–Moody algebras.

5.3 Double-scaling limit of the 1-matrix model

Now I proceed to a discussion of a slightly different approach to continuum limits, which is directly suited to the needs of matrix models. The naive idea [20, 29] is to forget about integrability and just look at the WIs (Virasoro constraints in the 1-matrix case) and take a continuum limit of these identities. This approach makes close contact with the standard technique of 'loop equations' (Makeenko–Migdal equations [105]) in the theory of matrix models, of which Virasoro and W-constraints are just particular examples.†

However, careful analysis of the continuum limit of discrete Virasoro constraints [28] makes it clear that the pro-cedure is far less simple than one might have thought (usually, derivations are not very careful and details are 'swept under the carpet'). The crucial problem is that what is needed is a peculiar (double scaling) limit rather than a naive limit, and, as mentioned in the previous subsection, this also requires a certain reduc-tion (elimination of the odd-times t_{2m+1}). If parity symmetry (with respect to the change of $H \to -H$ in the original matrix integral) is taken into account, one can easily throw away *first* derivatives with respect to the odd-times t_{2m+1}, just because

$$\left.\frac{\partial Z_N}{\partial t_{2m+1}}\right|_{t_{2k+1}=0} = 0 ,$$

but this is no longer true as far as the *second* derivatives

$$\left.\frac{\partial^2 Z_N}{\partial t_{2m+1}\partial t_{2l-1}}\right|_{t_{2k+1}=0} ,$$

are concerned, which appear in (the 'quantum portion' of) the Virasoro constraints (1.2). It is a highly nontrivial feature of loop equations (having its origin in their integrable structure!) that in the continuum limit these terms can in fact be carefully eliminated. The thing is that the second derivatives of $\ln Z_N$ appear to be *local* objects, in the sense that they depend only on $Z_{\widetilde{N}}$ with the difference $|\widetilde{N} - N| \leqslant m + l$, which does *not* blow up as $N \to \infty$ in continuum limit. Moreover, the differences

$$\frac{\partial^2 \ln Z_N}{\partial t_{2m+1}\partial t_{2l-1}} - \frac{\partial^2 \ln Z_N}{\partial t_{2m}\partial t_{2l}}$$

almost tend to zero, leaving some simple (though vitally important) correction to the arising continuous loop equations. This locality property allows one to get rid of these dangerous odd-time derivatives, substituting them by second derivatives with respect to the even-times. Since such a substitution is possible only for *logarithms* of Z_N, continuous constraints appear imposed on the *square root* of the original partition function [or on the $(1/p)$-th power in the case of the $(p-1)$-component conformal models]. Another aspect of this trick to deal with the odd-time derivatives is that it makes the entire derivation dependent on the fact that the theory is integrable — this is what guarantees the above-mentioned locality. Since the way to reveal integrability, by looking at the loop equations themselves, is not yet very well understood, the whole calculation becomes not quite self-contained (but of course, if everything is known about integrable structure this is not a real drawback, just a limitation of the particular approach starting from the loop equations). In particular, this is the only loophole which is still not filled in the description of the continuum limit of conformal (multi-component) matrix models, which in all other respects goes through exactly in parallel with the 1-component (1-matrix) case.‡

I shall now describe briefly the steps of this calculation for the 1-matrix model, referring for all the details to [28 and 45]. The previous discussion already contains motivations for the main steps, so I do not need to go into detailed explana-tions. Manipulations below, involving Kazakov variables can look a little artificial, but I repeat that they can be interpreted as a switch from the Toda-type to the KP-type hierarchies, which, was already seen in the previous subsection, is naturally associated with the double-scaling continuum limit.

I start from the discrete Virasoro constraints (1.2), rewritten in terms of a generating functional ('stress tensor' on the spectral plane):

$$L_-(z)Z_N = 0 , \tag{5.9}$$

where

$$L_-(z) = \sum_{n \geqslant -1}^{\infty} L_n z^{-n-2} = \tfrac{1}{2}[J^2(z)]_- , \tag{5.10}$$

and

† One of the puzzles in the theory of noneigenvalue models is to identify the group theoretical meaning of generic loop equations: they are usually introduced as equations of motion rather than as WIs (see the discussion at the beginning of Section 2), and thus their implications are a more obscure and technical means to deal with them and are much more restricted. When a group-theoretical description has been found, it will very soon reveal the (generalised) integrable structure of noneigenvalue models and it will be a big step forward in the whole theory.

‡ It transforms discrete W-constraints into continuum W-constraints, which, in their turn, arise from the generalised Kontsevich model (GKM) with the appropriate potential [30, 55]. Unfortunately, since the GKM interpretation of *discrete* multicomponent models (like the one existing in the 1-matrix case, see Section 3.8) is as yet unknown, the direct way to take their continuum limit — like the one to be described in the next subsection for the 1-matrix case — is also as yet unavailable. For more details about conformal matrix models, their integrable structure, and continuum limits, see [39].

$$J(z) = \partial\phi(z) = \sum_{n=-\infty}^{\infty} J_n z^{-n-1} ,$$

$$\phi(z) = \frac{1}{\sqrt{2}} \sum_{k \geqslant 0} t_k z^k - \sqrt{2} \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k} ,$$

$$J_{-k} = \sqrt{2} \frac{\partial}{\partial t_k}, \quad J_k = \frac{1}{\sqrt{2}} k t_k, \quad k \geqslant 0 ,$$

$$\frac{\partial}{\partial t_0} Z_N = N Z_N . \tag{5.11}$$

Next, one needs to reduce the original partition function:

$$Z_N\{t\} \to Z_N^{\text{red}}\{t_{\text{even}}\} \equiv Z_N\{t_{\text{odd}} = 0, t_{\text{even}}\} . \tag{5.12}$$

All odd Virasoro generators, L_{2n+1}, act trivially on Z_N^{red}, since

$$\left.\frac{\partial Z_N}{\partial t_{2k+1}}\right|_{t_{\text{odd}}=0} = 0$$

and it is necessary to consider only L_{2n}. Introduce also†

$$\phi^{\text{red}}(z) \equiv \frac{1}{\sqrt{2}} \sum_{k \geqslant 0} t_{2k} z^{2k} - \sqrt{2} \sum_{k>0} \frac{z^{-2k}}{k} \frac{\partial}{\partial t_{2k}} ,$$

$$L^{\text{red}}(z) = \tfrac{1}{2} [\partial\phi^{\text{red}}(z)]^2 ,$$

$$L_{2n}^{\text{red}} \equiv \sum_{k>0} k t_{2k} \frac{\partial}{\partial t_{2k+2n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_{2k} \partial t_{2n-2k}} . \tag{5.13}$$

Now there are two issues to be discussed separately. The first is the change from t_{2k} to Kazakov variables, T_{2m+1}. The second is the difference between the constraints imposed on Z^{red} and Z.

The simplest way to *describe* Kazakov variables is to introduce one more — antiperiodic — scalar field:

$$\Phi(u) = \frac{1}{\sqrt{2}} \sum_{k \geqslant 0} T_{2k+1} u^{k+\frac{1}{2}} - \sqrt{2} \sum_{k \geqslant 0} \frac{u^{-k-\frac{1}{2}}}{k+\frac{1}{2}} \frac{\partial}{\partial \widetilde{T}_{2k+1}} . \tag{5.14}$$

Here $\widetilde{T}$ and T are related by the transformation

$$T_{2k+1} = \widetilde{T}_{2k+1} + \varepsilon^2 \frac{k}{k+\frac{1}{2}} \widetilde{T}_{2k-1} + 2\varepsilon N \delta_{k,0} . \tag{5.15}$$

Now impose the relation

$$\partial\phi^{\text{red}}(z) = \frac{1}{\varepsilon^2} U^{-1} \partial\Phi(u) U ,$$

$$z^2 = 1 + \varepsilon^2 u , \tag{5.16}$$

and in the continuum limit ε is assumed to vanish. This is a relation which maps homogeneous into principal representations, but its invariant meaning (especially from the point of view of CFT) does not seem to be well-enough understood. Anyhow, these relations establish a relation between t_{even} and T. Namely, comparing the coefficients in front of the positive powers of u on both sides of this equation, one obtains

$$T_{2k+1} = \tfrac{1}{2} \varepsilon^{2k+1} \sum_{m \geqslant k}^{\infty} \frac{g_m \Gamma(m+\frac{1}{2})}{(m-k)!\,(k+\frac{3}{2})}, \qquad k \geqslant 0 ;$$

$$g_m = m t_{2m}, \qquad m \geqslant 1; \qquad g_0 = 2N . \tag{5.17}$$

The inverse transformation looks as follows

$$g_m = 2 \sum_{k \geqslant m} (-1)^{k-m} \frac{T_{2k+1} \Gamma(k+\frac{3}{2})}{\varepsilon^{2k+1} (k-m)!\, \Gamma(m+\frac{1}{2})} . \tag{5.18}$$

Now,

$$\frac{\partial}{\partial t_{2k}} = \frac{1}{2} \sum_{m=0}^{k-1} \frac{\Gamma(k+\frac{1}{2}) \varepsilon^{2k+1}}{(k-m-1)!\, \Gamma(m+\frac{3}{2})} \frac{\partial}{\partial \widetilde{T}_{2m+1}} , \tag{5.19}$$

and, using the formula when comparing the negative powers of u, one finds that

$$U = \exp\left(\sum_{m,n} A_{mn} \widetilde{T}_{2m+1} \widetilde{T}_{2n+1} \right) ,$$

$$A_{mn} = 2 \frac{(-1)^{m+n}}{\varepsilon^{2(m+n+1)}} \frac{\Gamma(m+\frac{3}{2}) \Gamma(n+\frac{3}{2})}{m!\, n!\, (m+n+1)(m+n+2)} . \tag{5.20}$$

The square of relation (5.16) is

$$\left(\partial\phi^{\text{red}}\right)^2(z) = \frac{1}{\varepsilon^4} U^{-1} (\partial\Phi)^2(u) U , \tag{5.21}$$

or

$$\sum_{p \geqslant 0} L_{2p}^{\text{red}} z^{-2p-2} = \frac{1}{\varepsilon^4} U^{-1} \left(\sum_{n \geqslant -1} \widetilde{\mathcal{L}}_{2n} u^{-n-2} \right) U . \tag{5.22}$$

This equality implies that

$$U^{-1} \widetilde{\mathcal{L}}_{2n} U = \varepsilon^4 \sum_{p \geqslant 0} L_{2p}^{\text{red}} \oint_{\infty} \frac{u^{n+1} \mathrm{d}u}{z^{2p+2}}$$

$$= \varepsilon^{-2n} \sum_{p=0}^{n+1} (-1)^{n+1-p} C_{n+1}^{p} L_{2p}^{\text{red}} , \tag{5.23}$$

since

$$\varepsilon^4 \oint_{\infty} \frac{u^{n+1} \mathrm{d}u}{z^{2p+2}} = \oint_{\infty} \frac{u^{n+1} \mathrm{d}u}{(1+\varepsilon^2 u)^{p+1}} = \frac{1}{\varepsilon^{2n}} \frac{\Gamma(-p)}{(n+1-p)!\, \Gamma(-n-1)}$$

$$= \frac{(-1)^{n+p+1}}{\varepsilon^{2n}} \frac{(n+1)!}{p!\, (n+1-p)!} = \frac{(-1)^{n+1-p}}{\varepsilon^{2n}} C_{n+1}^{p} .$$

Explicit expressions for the generators $\widetilde{\mathcal{L}}_{2n}$ [which are harmonics of the stress tensor $\frac{1}{2}(\partial\Phi)^2$ (u) of the antiperiodic field $\Phi(u)$], are

$$\widetilde{\mathcal{L}}_{-2} = \sum_{k \geqslant 1} (k+\tfrac{1}{2}) T_{2k+1} \frac{\partial}{\partial \widetilde{T}_{2(k-1)+1}} + \frac{T_1^2}{4} ,$$

$$\widetilde{\mathcal{L}}_0 = \sum_{k \geqslant 0} (k+\tfrac{1}{2}) T_{2k+1} \frac{\partial}{\partial \widetilde{T}_{2k+1}} ,$$

$$\widetilde{\mathcal{L}}_{2n} = \sum_{k \geqslant 0} (k+\tfrac{1}{2}) T_{2k+1} \frac{\partial}{\partial \widetilde{T}_{2(k+n)+1}}$$

$$+ \frac{1}{4} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial \widetilde{T}_{2k+1} \partial \widetilde{T}_{2(n-k-1)+1}} - \frac{(-1)^n}{16\varepsilon^{2n}} , \quad n > 0 . \tag{5.24}$$

† Note that $\phi^{\text{red}}(z) \neq \phi(z)|_{t_{\text{odd}}=0}$ and similarly $L_{2n}^{\text{red}}(z) \neq L_{2n}|_{t_{\text{odd}}=0}$, some factors of 2 in Eqns (5.13) being responsible for this discrepancy. In fact, L^{red} are related to generators of the Virasoro constraints in the *complex*-matrix model [28],

$$Z_N^{\text{C}} = \int \mathrm{d}M \exp\left[\sum_{k \geqslant 0} t_{2k} \operatorname{Tr}(MM^{\dagger})^k \right] ,$$

and, in the continumm limit, $Z_N^{\text{C}} \sim \sqrt{Z_{2N}^{\text{red}}}$.

So far, all that has been done is to change the variables, and all the relations were exact for any ε; no limits were taken.

Operators (5.24) are very similar to $\mathcal{L}_{2n}$, arising in the 'continuous Virasoro constraints' (1.3), imposed on the partition function of the X^3-Kontsevich model. There are, however, two discrepancies.

First, $\partial/\partial\widetilde{T}$ instead of $\partial/\partial T$ appears in (5.24). One can argue that this difference is not really essential, since $\widetilde{T}_{2k+1}$ and T_{2k+1} differ by terms which are proportional to ε^2 and thus vanish in the continuum limit $\varepsilon \to 0$. [Note, however, that this reasoning can be applied only for each particular constraint $\widetilde{L}_{2n}Z = 0$ $(n \geqslant -1)$ not to the entire generating functional, where different terms are summed, multiplied by different powers of ε.]

The second discrepancy is a little more serious: it is the occurrence of an extra term $(-1)^{n+1}/16\varepsilon^{2n}$ for all $n \geqslant 0$ [this difference is present for $n = 0$ as well, because $\mathcal{L}_0$ contains the coefficient $\frac{1}{16}$, which is lacking in (5.24)]. This extra term cannot be eliminated just by taking the continuum limit; moreover, it blows up instead of vanishing when $\varepsilon \to 0$. Remarkably enough, this term disappears when considering actual Virasoro constraints, not just a formal choice of time variables. It cancels completely with the other potential source of problems for the derivation of continuum WIs. I proceed now to this, the most sophisticated matter in this whole subsection.

The point is that, as mentioned before, the reduction of the discrete Virasoro constraint $L_{2n}Z_N = 0$ contains some nonvanishing terms with odd-time derivatives:

$$\left(\sum_{k>0} 2kt_{2k}\frac{\partial}{\partial t_{2k+2n}} + 2\sum_{k=0}^{n}\frac{\partial^2}{\partial t_{2k}\partial t_{2n-2k}}\right)Z_N^{\text{red}} = \left(\sum_{k=0}^{n}\frac{\partial^2}{\partial t_{2k}\partial t_{2n-2k}} - \sum_{k=0}^{n-1}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2n-2k-1}}\right)Z_N^{\text{red}} . \tag{5.25}$$

An extra term with second even-time derivatives has been added to both sides of the identity in order to get on the r.h.s. a combination which has a chance to vanish in the continuum limit. [This formula still needs to be corrected, see Eqn (5.29) below.]

In order to find a rigorous reason for eliminating the terms on the r.h.s. I need to address the explicit formulas from Section 4.10 (no simpler way is known so far). The crucial formula needed is

$$\frac{\partial^2\phi_n}{\partial t_k\partial t_l} = \frac{\partial}{\partial t_k}\frac{\langle n|h^l|n\rangle}{\langle n|n\rangle} = \left(\sum_{m>n} - \sum_{m<n}\right)\frac{\langle n|h^k|m\rangle\langle m|h^l|n\rangle}{\langle m|m\rangle\langle n|n\rangle}, \tag{5.26}$$

and the most important feature of it is its R-matrix structure (the fact that a *difference* occurs on the r.h.s.). This structure implies an almost complete cancellation of terms when one sums over n in order to get $\ln Z_N = \sum_0^{N-1}\phi_n$, leaving only a finite sum of the length *independent* of N:

$$\frac{\partial^2 \ln Z_N}{\partial t_k\partial t_l} = \sum_{0<j<\min(k,l)}\left(\sum_{n=N-j}^{N-1}\frac{\langle n|h^k|n+j\rangle\langle n+j|h^l|n\rangle}{\langle n|n\rangle\langle n+j|n+j\rangle}\right) . \tag{5.27}$$

The finite sum on the r.h.s. can be expressed in terms of $R_n = \exp(\phi_n - \phi_{n-1})$, and contains exactly the quantities to satisfy the equations of the Volterra hierarchy and tending to a *constant* (denoted by R_0 in the previous section) in the continuum limit. The locality property — the finiteness of the sum on the r.h.s. of Eqn (5.26) — implies that the r.h.s. tends to a constant value as $N \to \infty$. This constant does not completely cancel in the difference

$$\left(\sum_{k=0}^{n}\frac{\partial^2}{\partial t_{2k}\partial t_{2n-2k}} - \sum_{k=0}^{n-1}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2n-2k-1}}\right)\ln Z_N^{\text{red}} , \tag{5.28}$$

and the remaining contributions appear to be exactly those necessary to cancel the dangerous term $(-1)^{n+1}/16\varepsilon^{2n}$ which appeared in the difference between $\widetilde{\mathcal{L}}_n$ and $\mathcal{L}_n$. See [28] for more details on these cancellations, and the only thing to discuss in the rest of this subsection is the difference between the r.h.s. of (5.25) and (5.28). In the second expression the second derivatives are taken of $\ln Z$, while they are of z itself in the first one. Of course,

$$\frac{\partial^2 \ln Z_N^{\text{red}}}{\partial t_{2k+1}\partial t_{2n-2k-1}} = \frac{1}{Z_N^{\text{red}}}\frac{\partial^2 Z_N^{\text{red}}}{\partial t_{2k+1}\partial t_{2n-2k-1}}\bigg|_{t_{\text{odd}}=0} ,$$

but this is *not* true for even derivatives. So identity (5.25) still needs to be transformed a little more in order to contain exactly Eqn (5.26) on its r.h.s. If this is achieved, the l.h.s. acquires an additional contribution and turns into

$$\sum_{k>0} 2kt_{2k}\frac{\partial Z_N^{\text{red}}}{\partial t_{2k+2n}} + \sum_{k=0}^{n}\left(2\frac{\partial^2 Z_N^{\text{red}}}{\partial t_{2k}\partial t_{2n-2k}} - \frac{1}{Z_N^{\text{red}}}\frac{\partial Z_N^{\text{red}}}{\partial_{2k}}\frac{\partial Z_N^{\text{red}}}{\partial t_{2n-2k}}\right) = 4\sqrt{Z_N^{\text{red}}}L_{2n}^{\text{red}}\sqrt{Z_N^{\text{red}}} . \tag{5.29}$$

As a result of all this reasoning it is possible to conclude that the double-scaling continuum limit of the reduced 1-matrix can be described by the following relation:

$$\lim_{\text{d.s.}\,\varepsilon\to 0,\, N\to\infty}\sqrt{Z_N^{\text{red}}\{t_{\text{even}}\}} = U^{-1}Z_{V=\frac{1}{3}X^3}\{T\} , \tag{5.30}$$

where the factor U is defined in Eqn (5.20), the relation between the t and T variables is given by Eqn (5.17), and $Z_{V=\frac{1}{3}X^3}\{T\}$ is the X^3-Kontsevich model. The motivation for this conclusion is that both sides of the equation satisfy the same continuous Virasoro constraints Eqn (1.3).

This whole derivation can be straightforwardly generalised to the case of the multiscaling limit in conformal matrix models and the analogous relation contains roots of the pth degree, see Ref. 39 for a detailed discussion.

5.4 From the gaussian to the X^3-Kontsevich model

I shall now abandon these complicated matters and give a simple illustration of how things can work, if expressed in adequate terms. Namely, as an alternative to the sophisticated procedure involving an explicit switch to Kazakov variables and the study of the limits of the WIs (loop equations), I shall use just the equivalence of the discrete 1-matrix model and gaussian Kontsevich model, proved in Section 3.8, in order to take the continuum limit just of this simplest Kontsevich model. This procedure, suggested in [36] appears to be a kind of standard evaluation of the integral in the large-N limit by the steepest-descent method. It is important here that the GKM is not sensitive to the size n of the matrix in the Kontsevich integral; therefore, this limit, when expressed in terms of the GKM, has nothing to do with the infinitely large matrices.

The relation to be proved below is

$$\lim_{\text{d.s.}\,N\to\infty} \mathcal{F}_{\{\hat{V}\}} = \mathcal{F}^2_{\{V\}} \,, \tag{5.31}$$

where $\hat{V}(X) = \frac{1}{2}X^2 - N\ln X$ and $V(X) = \frac{1}{3}X^3$.

Very naively, what happens as $N\to\infty$ is that in the Kontsevich integral,

$$\int \mathrm{d}X \exp \mathrm{tr}\left(-\tfrac{1}{2}X^2 + N\ln X + \Lambda X\right) \,, \tag{5.32}$$

a stationary point arises at $X = X_0$, such that

$$X_0 = \frac{N}{X_0} + \Lambda \,. \tag{5.33}$$

Expansion of this action in powers of $\widetilde{X} = \gamma^{-1}(X - X_0)$ comes entirely from the logarithmic expression

$$\begin{aligned} S - S_0 &= \frac{\gamma^2}{2}\widetilde{X}^2 - N\left[\ln\left(1 + \frac{\gamma\widetilde{X}}{X_0}\right) - \frac{\widetilde{X}}{X_0}\right] \\ &= \frac{\gamma^2}{2}\left(1 + \frac{N}{X_0^2}\right)\widetilde{X}^2 + \sum_{k\geqslant 3}\frac{N}{k}\left(-\gamma\frac{\widetilde{X}}{X_0}\right)^k . \end{aligned} \tag{5.34}$$

In the continuum limit, γ should be adjusted in such a way that the quadratic term is finite, i.e. $\gamma \sim (1 + N/X_0^2)^{-1/2}$. Now, if Λ remains finite as $N\to\infty$, $X_0 \sim \sqrt{N}$, $\gamma \sim 1$ and all the terms with $k \geqslant 3$ in the sum are damped as $\gamma^k N X_0^{-k} \sim N^{1-k/2}$. This is the naive continuum limit. However, it is clear, that one can usually ask Λ to behave more adequately — blow up together with the growth of N — and fine tune the way in which it tends to infinity so that in the end the first term with $k = 3$ also survives. For this purpose Λ, and thus X_0, should scale in such a way, that both quantities $\gamma^2(1 + N/X_0^2)$ and $N\gamma^3/X_0^3$ remain finite. This requirement in the case of the latter expression means that $\gamma \approx X_0 N^{-1/3}$ and then

$$\gamma^2\left(1 + \frac{N}{X_0^2}\right) \approx \frac{N + X_0^2}{N^{2/3}} \,.$$

This is never finite, unless $N + X_0^2 \to 0$ as $N\to\infty$. This in turn implies that $X_0 \approx \mathrm{i}\sqrt{N}$ and $\Lambda \to 2X_0 \approx 2\mathrm{i}\sqrt{N}$ should be pure imaginary. One can also check that the terms with $k > 3$ in the sum in Eqn (5.34) all tend to zero in this specific limit. Thus, we are left with a model which has only cubic and quadratic terms in the action. By a simple shift of variables, the quadratic term can be changed to a linear one and we get a description of the theory in the vicinity of the stationary point in terms of an X^3-Kontsevich model.

In practice things are a little more complicated because reduction to even-times should also be taken into account. However, this does not add too many new problems. We need that only even times, $t_{2k} = (1/2k)\,\mathrm{tr}\,(1/\Lambda^{2k})$, remain nonvanishing, while all the odd times,

$$t_{2k+1} = \frac{1}{2k+1}\mathrm{tr}\frac{1}{\Lambda^{2k+1}} = 0 \,.$$

This obviously implies that the matrix Λ should be of block form:

$$\Lambda = \begin{bmatrix} \mathcal{M} & 0 \\ 0 & -\mathcal{M} \end{bmatrix} , \tag{5.35}$$

and, therefore, the matrix integration variable is also naturally decomposed into block form:

$$X = \begin{bmatrix} \mathcal{X} & \mathcal{Z} \\ \bar{\mathcal{Z}} & \mathcal{Y} \end{bmatrix} . \tag{5.36}$$

Then,

$$\begin{aligned} \mathcal{F}_{\{\hat{V}=\frac{1}{2}X^2 - N\ln X\}} &= \int \mathrm{d}\mathcal{X}\,\mathrm{d}\mathcal{Y}\,\mathrm{d}^2\mathcal{Z} \,, \\ \det\left(\mathcal{X}\mathcal{Y} - \bar{\mathcal{Z}}\frac{1}{\mathcal{Y}}\mathcal{Z}\mathcal{Y}\right)^N &\exp[-\mathrm{tr}\{|\mathcal{Z}|^2 \\ &+ \tfrac{1}{2}\mathcal{X}^2 + \tfrac{1}{2}\mathcal{Y}^2 - \mathcal{M}\mathcal{X} + \mathcal{M}\mathcal{Y}\}] \,. \end{aligned} \tag{5.37}$$

To take the limit $N\to\infty$, one should assume a certain scaling behaviour for $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. Moreover, previous naive consideration gave us some feeling of what the *fine-tuned* scaling behaviour can look like. So I take

$$\begin{aligned} &\mathcal{X} = \gamma(\mathrm{i}\beta I + x) \,, \quad \mathcal{Y} = \gamma(-\mathrm{i}\beta I + y) \,, \quad \mathcal{Z} = \gamma\zeta \,, \\ &\mathcal{M} = \gamma^{-1}(\mathrm{i}\alpha I + m) \,, \end{aligned} \tag{5.38}$$

with some large real parameters α, β, and γ. If expressed through these variables, the action becomes:

$$\begin{aligned} &\mathrm{tr}\Big[|\mathcal{Z}|^2 + \tfrac{1}{2}\mathcal{X}^2 + \tfrac{1}{2}\mathcal{Y}^2 - \mathcal{M}\mathcal{X} + \mathcal{M}\mathcal{Y} \\ &\quad - N\ln\left(\mathcal{X}\mathcal{Y} - \bar{\mathcal{Z}}\frac{1}{\mathcal{Y}}\mathcal{Z}\mathcal{Y}\right)\Big] = \gamma^2\,\mathrm{tr}\,\big[\tfrac{1}{2}(\mathrm{i}\beta I + x)^2 \\ &\quad + \tfrac{1}{2}\mathrm{tr}(\mathrm{i}\beta I - y)^2 + |z|^2\big] - \mathrm{tr}(\mathrm{i}\alpha I + m)(2\mathrm{i}\beta I + x - y) \\ &\quad - N\,\mathrm{tr}\ln(\beta^2\gamma^2)\left\{1 - \mathrm{i}\frac{x-y}{\beta} + \frac{xy}{\beta^2} - \frac{|\zeta|^2}{\beta^2}\left[1 + \mathrm{O}\left(\frac{1}{\beta}\right)\right]\right\} \\ &\quad = (2\alpha\beta - \beta^2\gamma^2 - 2N\ln\beta\gamma)\,\mathrm{tr}\,I - 2\mathrm{i}\beta\,\mathrm{tr}\,m && \text{(A)} \\ &\quad + \mathrm{i}\left(\beta\gamma^2 - \alpha + \frac{N}{\beta}\right)(\mathrm{tr}\,x - \mathrm{tr}\,y) + \frac{1}{2}\left(\gamma^2 - \frac{N}{\beta^2}\right)(\mathrm{tr}\,x^2 + \mathrm{tr}\,y^2) && \text{(B)} \\ &\quad + \left(\gamma^2 + \frac{N}{\beta^2}\right)\mathrm{tr}|\zeta|^2 && \text{(C)} \\ &\quad - \mathrm{tr}\,mx + \mathrm{tr}\,my + \frac{\mathrm{i}N}{3\beta^3}\mathrm{tr}(x^3 - y^3) && \text{(D)} \\ &\quad + \mathrm{O}\left(\frac{N}{\beta^4}\right) + \mathrm{O}\left(|\zeta|^2\frac{N}{\beta^3}\right) . && \text{(E)} \end{aligned} \tag{5.39}$$

We want to adjust the scaling behaviour of α, β, and γ in such a way that only the terms in line (D) survive. This goal is achieved in several steps.

Line (A) describes the normalisation of the functional integral, and does not contain x and y. Thus, it is not of interest to us at the moment.

Two terms in line (B) are eliminated by adjustment of α and γ:

$$\gamma^2 = \frac{N}{\beta^2} \,, \qquad \alpha = \frac{2N}{\beta} \,. \tag{5.40}$$

As we shall see soon, $\gamma^2 = N/\beta^2$ is large in the limit of $N\to\infty$. Thus, the term (C) implies that the fluctuations of the ζ-field are severely suppressed, and this is what makes the terms of the second type in the line (E) negligible. More generally, this is the reason for the integral $Z_{\{\hat{V}\}}$ to split into a product of two independent integrals leading to the square of

the partition function in the limit $N \to \infty$ [this splitting is evident as, if $\mathcal{Z}$ can be neglected, the only mixing term

$$\ln \det \begin{bmatrix} \mathcal{X} & \mathcal{Z} \\ \mathcal{Z} & \mathcal{Y} \end{bmatrix}$$

turns into $\ln \mathcal{X}\mathcal{Y} = \ln \mathcal{X} + \ln \mathcal{Y}$].

Thus, we remain with a single free parameter β, which can be adjusted so that

$$\frac{\beta^3}{N} \to \text{const} \quad \text{as} \quad N \to \infty \tag{5.41}$$

(i.e. $\beta \sim N^{1/3}$, $\gamma^2 \sim N^{1/3}$, $\alpha \sim N^{2/3}$), making the terms in the last line (E) vanishing and the third term in line (D) finite.

This proves the statement Eqn (5.31) in a rather straightforward way. Unfortunately no generalisation of this procedure for other discrete models has so far been found, the main problem being identification of GKM-type realisation of other (for example, conformal) discrete matrix models.

6. Conclusion

I have come to the end of my brief review of the facts that are currently known about the relation between matrix models and integrable hierarchies. There are still several topics which are discussed in the literature but are not presented in these notes.

First, I did not discuss the relation between matrix models and theories of topological (Landau–Ginzburg) activity (LGG). This field has been developing rapidly in recent months and will soon be ready for inclusion in reviews of this kind. This list of things which are sufficiently clarified includes the realisation of the Ward identities in the form of 'recursion relations' for topological gravity [9]. Also, the relation between quasiclassical hierarchies, arising in the spherical approximation to topological theories [96], to the integrable structure of the generalised Kontsevich model is more or less understood [40]. Of special importance is the chapter on this theory, which provides a matrix-model description of module spaces associated with Riemann surfaces [22, 106]. What sill deserves better understanding is the acionatic construction of topological gravity, similar to the remarkably simple construction of topological LG models (before they are coupled to 2d gravity) in terms of the Grothendieck residues and chiral rings [107] : see [108] for a very nice presentation of the latter case, and [17] for the first big steps towards a similar construction in the former case. Also, the relation to the theory of nonconformal LG models [109], deserves clarification. A piece which is essentially lacking so far is the clear description of minimal (p, q)-models coupled to 2d gravity in the case of $p \neq 1$. In this situation the generalised Kontsevich model is known to describe nothing more than duality transformation between (p, q) and (q, p) models [41], rather than the models themselves. This subject is also connected with the theory of the Kac–Schwarz operator [110]. The work in this direction is extremely important for the understanding of the unification of various string models and of the essential symmetries of future string field theory (in particular, generic BRST and Batalin–Vilkovisky symmetries are very close analogues of the complete sets of the Ward identities, as described in the general framework in the beginning of Section 2). All these things would constitute a natural next section to these notes, but I chose to wait a little longer until further clarification is achieved in this fragment of the theory.

Second, I did not touch at all the physical interpretations of matrix models, which include quantum gravity. Yang–Mills theory, and many other possible applications. This should be a subject of a very different review, for which the whole content of these notes is just a list of techniques involved in the study of physical phenomena.

Third, the biggest *terra incognita* in this branch of science, which remained beyond the scope of these notes, is the theory of noneigenvalue matrix models, which are related to physical theories in spacetime dimensions $d \geqslant 2$. It is indeed a *terra incognita*, at least from the point of view of the semirigorous analysis which I am reviewing. The recent breakthrough in this field is due to the appearance of the Kazakov–Migdal model [25] (see also the latest review [111] and references therein), which for the first time creates the possibility to treat a wide class of noneigenvalue models by the extra methods of localisation theory (other names for this field, which in fact is developing into the generic theory of integra-bility, are the Duistermaat–Heckman theorem or Fourier analysis on group manifolds). Work in this direction is, however, only at the early stages and this is why I decided not to present the first nonsystematised results in these notes. A part of it which is very close to being satisfactorily understood is the 'boundary model' of the $c = 1$ string ('$d = 2$ dilaton gravity')—a very important one from the point of view of general string theory. For the present state of knowledge about this model see [112], and its relation to integrability theory is partly revealed in [41, 113].

In the domain which *was* actually reviewed, the weakest points are the theory of continuum limits and that of the multicomponent hierarchies. These theories, when developed, can also help to move in the most important direction, mention many times above: towards the creation of a more general theory of integrability. The next natural step, when approached form this side, should be generalisation of the conventional integrable hierarchies, which would life the restriction to level $k = 1$ simply-laced Kac–Moody algebras and unitary representations. The emerging theory will, of course, have much to do with both localisation theory and noneigenvalue matrix models, and when it is created we shall find ourselves at a new level of understanding, which will be one step closer to the goal of constructing the entire building of string theory (mathematical physics) and will probably provide us with an unexpected new means for investigating the features of the real physical world around us.

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