Coherent population trapping in quantum systems

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Coherent population trapping (CPT) is a technique in ultrahigh-resolution spectroscopy, developed in recent years. It relies on the nonlinear coherent interaction between an atomic system and incident electromagnetic radiation and is followed by an analysis of the fine structure of the response to this interaction, which contains information about the spectral characteristics of the quantum system. The basic physics of CPT are reviewed, and the effects of level relaxation and structure, and of external factors, on CPT dynamics and characteristics are examined. Experimental results confirming the validity of theoretical ideas on CPT are presented. Possible applications of CPT to frequency stabilization, laser cooling of atoms, and so on are discussed.

1. INTRODUCTION

It is clear from the history of the subject that three-level systems occupy a special place in the development of laser spectroscopy and quantum electronics. Suffice it to recall that early studies of such systems led to the discovery of Raman scattering in liquids and gases, and to the implementation of the idea of optical pumping of atoms, which was of fundamental importance to the development of the first radiofrequency generators (masers). Finally, we must not forget the now well-established phenomena such as level crossing, the Hanle effect, and quantum beats.

Nonlinear spectroscopy employing three-level systems is usually concerned with the absorption of a weak (probe) field when another (adjacent) transition is produced by an intense electromagnetic field that saturates it. The absorption of the probe field then displays features that are due to the nonlinear interaction between the atomic system and the saturating electromagnetic field, and this can be exploited in determinations of the spectroscopic parameters of the particular transition in the three-level system. Such studies have provided the foundations for modern ultrahigh-resolution laser spectroscopy, and the most important of them are discussed in well-known monographs.[1–5]

Interest in three-level systems grew in the late 70s because of the previously unknown properties of such systems in the field of two electromagnetic waves. The efforts of many researches finally revealed a new phenomenon that occurs during the excitation of three-level systems (and, generally, multilevel systems), namely, coherent population trapping (CPT).

It was found that multilevel systems cannot always be excited to higher levels because special superposition states that do not interact with optical fields are present in the system. These states play a decisive part only when certain conditions apply to the frequency detunings and the light-wave intensities. When these conditions are met, the system is in the CPT state and has practically no interaction with the field. We emphasize that this behavior is encountered in systems in which interference between several excitation channels can occur.

A simple system of this kind is the three-level Λ-system of energy levels (we shall refer to it simply as the Λ-system or Λ-atom) interacting with two light waves of frequency ω_m $(m=1,2)$ (Fig. 1a).

Under the same resonance conditions for adjacent transitions $|1\rangle-|3\rangle, |2\rangle-|3\rangle$, i.e., for equal light-wave frequency detunings $\Omega_m=\omega_m-\omega_{3m}$ $(m=1,2)$ from the frequencies of the corresponding transitions

$$\Omega=\Omega_1-\Omega_2=-(\omega_{31}-\omega_{32})+(\omega_1-\omega_2)=0, \qquad (1.1)$$

the probability of finding an atom in the upper state $|3\rangle$ is close to zero. In other words, when condition (1.1) is met, the Λ-atom (Fig. 1a) is not excited to the upper state $|3\rangle$, so that it cannot absorb or emit resonant photons. It is precisely for this reason that a valley or a 'black line' is observed in the fluorescence spectrum of a Λ-system.

This property of the three-level system is called coherent population trapping (CPT) in which the population of the Λ-system is distributed among the lower levels, and it is in this sense that the word 'trapping' is to be interpreted.

We thus encounter a situation that is relatively unfamiliar for problems of this kind because the resonance radiation acts on each of the transitions in the three-level system, but the system as a whole does not go over to the upper state. On the contrary, when the well-studied two-level system is excited by saturating resonance radiation, the probability of finding the system in the upper state is a maximum and close to 1/2.

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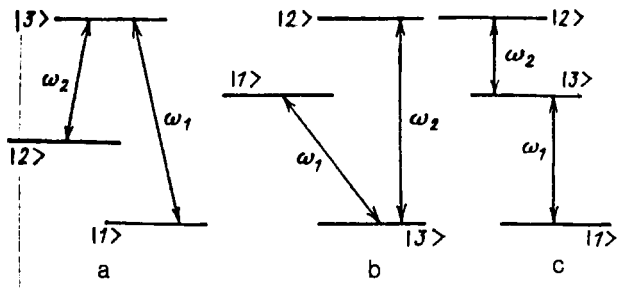

FIG. 1. Types of three-level system interacting with light.

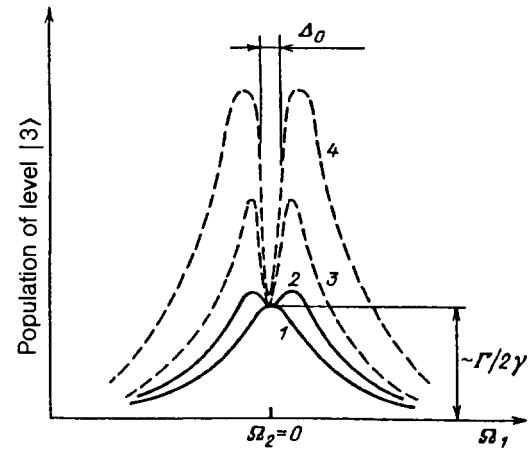

FIG. 2. Schematic dependence of the population of the upper level in a Λ-system when one of the light waves of frequency ω_2 is in precise resonance ($\Omega_2=0$) and the other frequency ω_1 is scanned: *1*—no CPT valley; (1.2) not satisfied; low intensity, *2–4*—no CPT; (1.2) satisfied. The relative valley depth increases with increasing light-wave intensity (*3,4*).

After the early theoretical papers[7] that actually predicted the effect itself, and the pioneering experiments[8–10] that confirmed its existence,[1)] the efforts of researchers were directed mostly toward the elucidation[11–14] of the dynamics of processes occurring in three-level systems obeying condition (1.1) and to the explanation of the fact that the upper level was not populated. It became clear that the absence of particles in the upper state was evidence for a transition of the system to a new state that is reached after a certain time τ has elapsed since the interaction was turned on. The order of magnitude of τ is determined by the radiative lifetime γ^{-1} of the atom in the excited state $|3\rangle$. At the end of this time, the system is found entirely in the new state and radiative processes cease in the system.

The CPT phenomenon occurs only for a certain light intensity, e.g., in the Λ-system, the necessary intensity is

$$I \gg I_c = I_n \frac{\Gamma}{\gamma}, \tag{1.2}$$

where I_n is the saturating intensity for the particular optical transition and Γ is the rate of transverse relaxation between the lower levels. If we suppose that $\Gamma \ll \gamma$ we have $I_c \ll I_n$ (for alkali-metal atoms $I_n \approx 0.1$ W cm^{-2}).

In real experiments, transverse relaxation between the $|1\rangle$ and $|2\rangle$ states can be produced by a number of factors, e.g., finite width of the laser spectrum, transit broadening, collisions between atoms, and so on.

When (1.2) is not met, CPT is practically absent even when (1.1) is satisfied by the frequency detunings. The excitation of a Λ-atom does not then display the above properties.

Figure 2 shows a typical graph of the population of the upper level $|3\rangle$ in a Λ-system for a fixed frequency of one of the light fields as a function of the frequency of the field acting on the other transition. The characteristic valley of width Δ_0 is the CPT region in which condition (1.1) is reasonably well satisfied. This width is given by

$$\Delta_0 \approx \Gamma + (g^2/\gamma), \tag{1.3}$$

where $g = dE_0/\hbar$ is the Rabi frequency, E_0 is the amplitude of the light field that is the same in both transitions of the Λ-atom, and d is the matrix element of the dipole interaction (the same for both transitions).

It is interesting to note that, according to (1.3), the width Δ_0 can be made much smaller than the natural line width γ of the optical transitions: $\Delta_0 \ll \gamma$, provided[2)] $g, \Gamma \ll \gamma$. However, it must then be remembered that the light intensity must satisfy (1.2) if the effect is to appear. The relative valley depth decreases with decreasing intensity (curves *4*, *3*, *2*); it vanishes almost entirely for $g^2 \approx \Gamma\gamma$ (curve *1*; see Fig. 2), showing that CPT has also vanished,

Even if (1.2) is satisfied and CPT is possible in the system, its experimental detection may be impeded by inadequate frequency stabilization of the fields. It is shown in Refs. 11 and 13 that, when the spectrum width is of the order of the optical excitation rate W

$$\beta_i \approx W \approx \frac{g^2}{\gamma} \quad (i=1,2),$$

the CPT valley disappears. It follows that we must have $\beta_i \ll W$ for CPT to be observable.

The excitation of a three-level system by mutually correlated fields constitutes a special case (the so-called cross correlation).[13] It is interesting that CPT is then independent of the width of the spectrum and the valley is always present. This means that, for cross-correlated fields, CPT can be observed even for $\beta_i \gg W$.

In practice, the important spectroscopic property of CPT is the presence in the fluorescence spectrum of a valley with small width Δ_0 (1.3). It is therefore natural that the ultrahigh-resolution spectroscopic techniques that have been suggested should frequently rely on this particular CPT property.[15–19] This also applies to the competing two-photon ionization spectroscopy[20–22] and to the development of frequency standards.[23–27]

Applications of strong nonlinearity during CPT offer interesting possibilities. This applies in particular to work on optical bistability[28–30] and to laser cooling of atoms.[31–37] We note that, in the latter case, it is possible to achieve temperatures of the order of 10^{-6}, which is well below the

Doppler limit found for the two-level atom.[38] Moreover, special cooling techniques can be used with CPT to produce temperatures several orders of magnitude lower than the recoil temperature $T_R \approx 10^{-6}$ K (Refs. 36 and 37). A separate group consists of studies of the propagation of radiation (both pulsed and continuous wave) in a resonant three-level optically dense medium.[40–44] It is shown in Refs. 41–42 that, for two-frequency continuous-wave radiation satisfying (1.1), the medium transmits this radiation because the atoms do not scatter resonant photons. Transmission also occurs in the case of pulsed excitation of a three-level medium, but only for a particular pulse repetition frequency that is a multiple of the frequency difference ω_{21} between the lower levels of the Λ-system.[40]

There is considerable interest in the studies (Refs. 76–83) in the development of noninversion lasers based on CPT, and in the destruction, followed by reinstatement, of the state of coherent trapping in the Λ-system closed by a third resonant field.[72–74]

We now emphasize two further points. First, the CPT phenomenon is a fundamental property of a quantum system[48–57] in which there can be interference between different excitation channels. The specific nature of these systems is therefore quite irrelevant. They can be quantum transitions in atoms or molecules, color or impurity centers in semiconductors, and γ-transitions in excited nuclei.

The second point involves the study of multilevel excitation systems. Since the three-level system is the simplest multilevel system in which CPT is possible, and a multilevel system can often been divided into three-level subsystems, it follows that if we know the nature of the excitation of the simpler system, we can qualitatively analyze the excitation of very complicated systems. Moreover, we can then also determine the excitation properties of multilevel systems with CPT.[54,55] We note that recent years have seen the publication of researches into CPT in cases where the levels forming the Λ-system belong to the continuous spectrum.[60–66]

The plan of our presentation is as follows. We begin with the excitation of three-level systems (Fig. 1) in the absence of spontaneous relaxation, and show that the system dynamics changes radically depending on the sign ratio of the initial phases of the lower-lying states.[14] Next, we examine the dynamics of excitation of a three-level system during spontaneous decay to thermostat levels. This is based on the formalism of state vectors. By directly solving the Schrödinger equation, we obtain the level population probability. The remainder of the analysis is based on the solution of the exact equations for the density matrix of the three-level Λ-system. We derive the conditions for CPT and examine the properties of CPT during the motion of atoms. We then go on to examine CPT in a cascade system (subsequently referred to as the Ξ-system) and the conditions that must be met for it to arise. Next, we consider the manifestations of CPT in complex quantum systems, i.e., multilevel and continuous-spectrum systems. We conclude with an analysis of experimental situations in which CPT is found to occur, and with possible applications of the effect in atomic physics and spectroscopy.

2. THREE-LEVEL SYSTEMS IN THE ABSENCE OF SPONTANEOUS RELAXATION

We begin our study of CPT with three-level systems that interact with two light fields of frequency ω_1 and ω_2. If the field amplitudes are equal, we can write

$$E = E_0(e^{i\omega_1 t} + e^{i\omega_2 t}) + \text{c.c.} \tag{2.1}$$

We shall also assume the absence of spontaneous relaxation and describe the system by a wave function of the form

$$\Psi(\mathbf{r},t) = a_1(t)\Psi_1(\mathbf{r}) + a_2(t)\Psi_2(\mathbf{r})e^{i\omega_{21}t} + a_3(t)\Psi_3(\mathbf{r})e^{i\omega_{31}t}, \tag{2.2}$$

where energy is measured from the energy of the state $|1\rangle$. The functions $\Psi_m(r)$ and $a_m(t)$ $(m=1,2,3)$ are the eigenfunctions of the stationary states of the system and the time-dependent probability amplitudes, respectively.

We emphasise that the absence of spontaneous relaxation leads to the indistinguishability of the three-level systems shown in Fig. 1. Hence the equations for the probability amplitudes and their solution are equally valid for all three-level systems. However, to be specific, we shall confine our attention to the Λ-system of atomic levels. Three-level systems with relaxation that interact with the field (2.1) are discussed in Sec. 3.

Let us now consider the excitation of a Λ-system to the upper state $|3\rangle$; in other words, let us evaluate the probability $|a_3|^2$ of finding the system in the state $|3\rangle$ subject to (1.1). The probability $|a_3|^2$ can then depend on the type of initial conditions, and the initial conditions can be chosen so that the probability of populating the state $|3\rangle$ is zero for at all times. This means it is possible to select special states of this type of system from which the atom cannot be excited even by resonant fields. Such states will henceforth be referred to as coherent states of the system.

We note that the existence of coherent states when (1.1) is satisfied is not a trivial fact because it might have been expected that resonant fields should ensure the population of the upper state, just as in the case of the two-level atom. However, analysis shows that the probability of populating the state $|3\rangle$ is very dependent on the phase relations between the wave functions of the lower levels in the initial state.

We now turn to the evaluation of the probability of finding the Λ-system in the state $|3\rangle$. To do this, we write the time-dependent probability amplitudes obtained after substituting (2.2) in the time-dependent Schrödinger equation:

$$\dot{a}_3 = iga_1 \exp(-i\Omega_1 t) + iga_2 \exp(-i\Omega_2 t),$$

$$\dot{a}_2 = ig^* a_3 \exp(i\Omega_2 t),$$

$$\dot{a}_1 = ig^* a_3 \exp(i\Omega_1 t), \tag{2.3}$$

where $g = dE_0/\hbar$ is the Rabi frequency and $\Omega_m = \omega_m - \omega_{3m}$ $(m=1,2)$ are the frequency detunings of the light fields. We now substitute $r = a_1 - a_2$, $s = a_1 + a_2$ and consider the case of equal detunings $\Omega_m = \Omega$ $(m=1,2)$, i.e., the case in

which CPT is observed [see condition (1.1.)]. The set of equations given by (2.3) then assumes the form

$$\dot{a}_3 = igs\exp(-i\Omega t),$$

$$\dot{s} = 2ig^* a_3 \exp(i\Omega t), \quad \dot{r} = 0. \tag{2.4}$$

It is clear from (2.4) that new superposition states r, s appear when the detunings are equal; one of them, r, is optically unrelated to the state $|3\rangle$, and is exclusively determined by the initial conditions for the probability amplitudes $a_{1,2}(0)$ (Ref. 9).

The fact that there are new superposition states when (1.1) is satisfied does not by itself ensure that the system cannot be excited. Indeed, everything depends on the type of the initial conditions, i.e., the distribution of the total population at the initial time. For example, we shall encounter situations in which a Λ-system will not be excited unless certain initial conditions are satisfied. For other initial conditions, everything proceeds as usual despite the existence of new superposition states.

The subsequent solution of (2.4) presents no real difficulty and we finally obtain

$$a_3 = A\exp(i\alpha_1 t) + B\exp(i\alpha_2 t),$$

$$s = g^{-1}\exp(i\Omega t)[\alpha_1 A\exp(i\alpha_1 t) + \alpha_2 B\exp(i\alpha_2 t)], \tag{2.5}$$

$$r = C,$$

where the constants A, B, C are determined from the initial conditions and

$$\alpha_{1,2} = -\Omega/2 \pm \Delta/2$$

are the roots of the corresponding characteristic equation $\Delta^2 = \Omega^2 + 8g^2$ and the amplitudes are given by

$$a_1 = (s+r)/2, \quad a_2 = (s-r)/2.$$

The solution of (2.5) enables us to examine the population of the upper state $|3\rangle$ as a function of the type of the initial conditions.

One of the lower levels initially populated. Let us suppose that at the initial time $t=0$, the probability amplitudes are

$$a_1 = \pm 1, \quad a_2 = a_3 = 0 \quad \text{for } t=0, \tag{2.6}$$

which corresponds to the initial population

$$|a_1|^2 = 1, \quad |a_2|^2 = |a_3|^2 = 0 \quad \text{for } t=0.$$

According to (2.5) the constants of integration then become

$$A = \pm g/\Delta, \quad B = \mp g/\Delta, \quad C = \pm 1,$$

and, correspondingly, the probability of population of $|3\rangle$ is given by

$$|a_3|^2 = 2g^2(1-\cos\Delta\cdot t)/\Delta^2. \tag{2.7}$$

It is clear from (2.7) that, although (1.1) is satisfied, the population of the upper level exhibits Rabi oscillations in the same way that this happens in the two-level atom (Fig. 3, curve *1*).

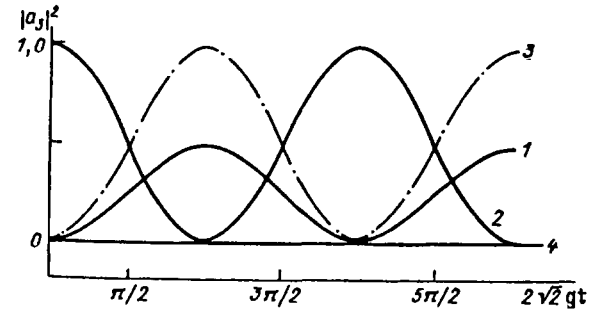

FIG. 3. Population of the upper level in a Λ- system without relaxation: *1*—(2.7), *2*—(2.9), *3*—(2.10), *4*—(2.11).

Upper level initially populated. The initial probability amplitudes are

$$a_1 = a_2 = 0, \quad a_3 = \pm 1 \quad \text{for } t=0, \tag{2.8}$$

and the initial populations are

$$|a_1|^2 = |a_2|^2 = 0, \quad |a_3|^2 = 1 \quad \text{for } t=0.$$

The constants of integration then follow from (2.5):

$$A = B = \pm(1-\alpha_1\Delta^{-1}), \quad C = 0,$$

and the probability of finding the system in the state $|3\rangle$ is

$$|a_3|^2 = [2\Omega^2 + 8g^2(1+\cos\Delta\cdot t)]/2\Delta^2. \tag{2.9}$$

It is clear that $|a_3|^2$ is a periodic function of time for (2.8) (Fig. 3, curve *2*).

Both lower levels initially populated. Let us now consider the population in which the lower levels are initially equally populated:

$$|a_1|^2 = |a_2|^2 = 1/2, \quad |a_3|^2 = 0 \quad \text{for } t=0.$$

The probability amplitudes can now have four types of initial conditions (for $t=0$) (see Table I). Substituting the conditions from the table in (2.5), we can readily show that the probability $|a_3|^2$ depends significantly on the mutual signs of the initial probability amplitudes. Thus, if for the conditions in the first and fourth rows of the table we have

$$A = -B = \pm\sqrt{2}g/\Delta, \quad C = 0$$

and

$$|a_3|^2 = 4g^2(1-\cos\Delta\cdot t)/\Delta^2, \tag{2.10}$$

whereas for the conditions from the second and third rows

TABLE I.

a_1	a_2	a_3
$1/\sqrt{2}$	$1/\sqrt{2}$	0
$1/\sqrt{2}$	$-1/\sqrt{2}$	0
$-1/\sqrt{2}$	$1/\sqrt{2}$	0
$-1/\sqrt{2}$	$-1/\sqrt{2}$	0

$$A=B=0, \quad C=\sqrt{2},$$

and

$$|a_3|^2=0 \tag{2.11}$$

for any time t, i.e., the atom remains in the initial state and does not 'feel' the presence of the resonant field (2.1) (Fig. 3, curve *4*).

It follows that, when the frequency detunings (1.1) are equal, excitation is absent from a three-level system only if the initial probability amplitudes (at $t=0$) in the lower levels have opposite signs (see the Table). This suggests that the system (Λ-system) has a series of states in which it ceases to interact with an external field. We shall called them coherent states.

We note that nothing new is introduced by the nonzero population of the state $|3\rangle$ for the corresponding initial probability amplitudes in the lower levels (which have opposite signs). The smaller fraction of atomic population that was initially in level $|3\rangle$ is then found to oscillate whereas the greater part of that population is trapped in the coherent state. This is a decisive factor in attempts to produce noninversion lasers based on CPT.[76–83]

The signs of the initial probability amplitudes in the lower states are determined by the symmetry of the system under the interchange of the phase factors of the wave functions. The significant point is that this symmetry manifests itself (i.e., the Λ-system is not excited to the upper state) only if (1.1) is satisfied. If (1.1) is not met, the system can always be excited, even from these coherent states.

The fact that three-level systems have special coherent states was also noted in Ref. 14 which reported the collapse and restoration of the dynamics of three-level systems. The calculations were performed for the nonrelaxing V-system (Fig. 1b) interacting with the quantized electromagnetic field with the Lorentz distribution over the photon number n:

$$F_n=[(n-\bar{n})^2+\Gamma_0]^{-1}\left\{\sum_n [(n-\bar{n})^2+\Gamma_0^2]^{-1}\right\}^{-1},$$

where $\bar{n}$ is the average number of photons and Γ_0 is the half-width of the distribution. The analysis is based on the solution of the time-dependent Schrödinger equation for the probability amplitudes. It is found that there are initial probablity amplitudes in the upper state of a V-system in which the probability of finding the system in a lower state is not only close to zero, but the evolution of the system in time is qualitatively similar to the evolution of the Λ-system. The fundamental difference between the results reported in Ref. 14 and the corresponding results for the Λ-system is that, in the case of the quantized field, these special initial conditions depend additionally on the average number of photons $\bar{n}$.

3. PHENOMENOLOGICAL ALLOWANCE FOR SPONTANEOUS RELAXATION IN THREE-LEVEL SYSTEMS

3.1. The levels of Λ- and Ξ-systems

We shall now examine the temporal evolution of the populations of Λ- and Ξ-systems with allowance for spontaneous relaxation. We shall use the formalism of probability amplitudes in which the decay of the state $|3\rangle$ to thermostat states is phenomenologically taken into account at the rate γ_0, whereas the states $|1\rangle$ and $|2\rangle$ are assumed to be nondecaying.[11,12] Of course, this approach suffers from certain disadvantages, including the fact that input terms cannot be taken into account. We reproduce the solutions and their analysis because this approach is relatively simple and clear, and reveals the characteristic features of CPT. A comprehensive approach that includes spontaneous relaxation and other factors can be based on the more laborious density matrix method which enables us to obtain the solution of many problems that are important in practice.

Our approach will give us the populations of the states $|m\rangle$ $(m=1,2,3)$ as functions of the time of observation.

The set of equations for the probability amplitude, given by (2.3) and augmented by spontaneous decay, can be written in the form[1,3,4] $(\Omega_1=\Omega_2=0)$

$$\begin{aligned} \dot{a}_3&=ig(a_1+a_2)-\gamma_0 a_3,\\ \dot{a}_2&=ig^* a_3,\\ \dot{a}_1&=ig^* a_3. \end{aligned} \tag{3.1}$$

We shall now write down the solutions of (3.1) for the following initial conditions:

$$a_1=C_1, \quad a_2=C_2, \quad a_3=0 \quad \text{for } t=0. \tag{3.2}$$

The population probabilities in states $|m\rangle$ are given by (for $g\ll\gamma_0$)

$$\begin{aligned} |a_3|^2&=|C_1+C_2|^2\frac{g^2}{\gamma_0^2}(e^{-2g^2t/\gamma_0}-e^{-\gamma_0 t})^2,\\ |a_{1,2}|^2&=\frac{1}{4}\,|\pm(C_1-C_2)+(C_1+C_2)e^{-2g^2t/\gamma_0}|^2, \end{aligned} \tag{3.3}$$

whereas for $g\gg\gamma_0$

$$\begin{aligned} |a_3|^2&=\frac{1}{2}\,|C_1+C_2|^2e^{-\gamma_0 t}(1-\cos^2 x),\\ |a_{1,2}|^2&=\frac{1}{4}\,|\pm(C_1-C_2)+(C_1+C_2)e^{-\gamma_0 t/2}\cos x|^2, \end{aligned} \tag{3.4}$$

where $x=\sqrt{2}gt$.

The solutions given by (3.3) and (3.4) provide a complete description of the temporal behavior of populations in Λ- and Ξ-systems for initial conditions such as (3.2). It is clear from (3.3) and (3.4) that, with initial conditions for $a_{1,2}$ of the form $C_1=-C_2$, the intermediate level $|3\rangle$ is not populated whatever the dependence on the Rabi frequency and decay rate γ_0. The initial populations remain unal-

tered: $|a_1|^2=|a_2|^2, |a_3|^2=0$, which confirms the existence of a state of the system in which it does not interact with resonant fields.

For initial values such that $C_1 \neq -C_2$, the temporal evolution of populations depends on the Rabi frequency: for high Rabi frequencies $g \gg \gamma_0$, there are periodic damped oscillations with time constant γ_0^{-1}; for $g \ll \gamma_0$, the damping is aperiodic with time constant $t \approx \gamma_0/g^2$.

We note two further points. First, the character of the dynamics is virtually unaltered in the case of detuning from exact resonance. All that is required is that the two-photon resonance condition is satisfied; for the Λ-system this takes the form

$$\Omega_1-\Omega_2=0, \tag{3.5}$$

and for the Ξ-system

$$\Omega_1+\Omega_2=0. \tag{3.6}$$

Second—and this is important for the understanding of CPT—some of the atomic population is always trapped in extreme levels when (3.5) and (3.6) are satisfied in the case of the three-level system. The population thus remains in the system in spite of decays occurring outside the system, which is significantly different from the analogous analysis of the two-level atom[1,3] for which the total population decreases with time ($N \to 0$ as $t \to \infty$), which is due to the loss of population by the system by excited-state decay.

3.2. *V*-scheme of levels

Let us now consider the V-scheme of levels shown in Fig. 1b. The equations for the probability amplitudes now take the form (using the same notation as before)

$$\begin{aligned} &\dot{a}_3=igs, \\ &\dot{s}+2\gamma s=2ig^* a_3, \\ &\dot{r}+2\gamma r=0, \end{aligned} \tag{3.7}$$

where 2γ is the rate of decays from levels $|1\rangle$ and $|2\rangle$ to thermostat states and the light-wave frequency detunings are zero, i.e., $\Omega_m=0$ ($m=1,2$). The solution of (3.7) can be written as

$$\begin{aligned} &a_3=ig\exp(-\gamma t)[A\alpha_1^{-1}\exp(\Delta_1 t)-B\alpha_2^{-1}\exp(-\Delta_1 t)], \\ &s=\exp(-\gamma t)[A\exp(\Delta_1 t)+B\exp(-\Delta_1 t)], \\ &r=C\exp(-2\gamma t), \end{aligned} \tag{3.8}$$

where

$$\alpha_{1,2}=-\gamma\pm\Delta_1, \quad \Delta_1^2=\gamma^2-g^2.$$

It is clear from (3.8) that, in this case, the entire behavior of the system is determined by decay from levels $|1\rangle$ and $|2\rangle$. Correspondingly, the system is excited to these states for any initial conditions. The necessary condition for the absence of coherent trapping is the decay of the superposition state r at the rate 2γ, which does not occur in the case of the Λ- and Ξ-systems.

The excitation of this kind of *V*-system is completely analogous to the excitation of the two-level system whose upper state decays.[1,3] It is therefore not surprising that, as in the two-level system, the normalization condition is not met and the total population in the system tends to zero, i.e., $|a_1|^2+|a_2|^2+|a_3|^2=N\to 0$ as $t\to\infty$.

4. COHERENT POPULATION TRAPPING IN THE *L*-SYSTEM

4.1. Establishment of coherent trapping in the Λ-system

We now continue our study of coherent trapping in Λ-systems, using the density matrix formalism in which relaxation processes can be most fully taken into account.

It will be helpful for the ensuing discussion to start by specifying the field with which the Λ-system interacts in the form of two plane light waves with frequencies $\omega_{1,2}$, wave vectors $\mathbf{k}_{1,2}$, and polarization unit vectors $\mathbf{e}_{1,2}$[3)]

$$\mathbf{E}=E_1\mathbf{e}_1\exp(i\omega_1 t-i\mathbf{k}_1\mathbf{r})+E_2\mathbf{e}_2\exp(i\omega_2 t-i\mathbf{k}_2\mathbf{r})+\text{c.c.} \tag{4.1}$$

The equations for the elements of the density matrix[1-5]—the Λ-atom—are as follows:

$$\begin{aligned} i\frac{d\rho_{11}}{dt}&=-g_1\rho_{31}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})+\text{c.c.}+2i\gamma_1\rho_{33}, \\ i\frac{d\rho_{22}}{dt}&=-g_2\rho_{32}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})+\text{c.c.}+2i\gamma_2\rho_{33}, \\ i\frac{d\rho_{33}}{dt}&=g_1\rho_{31}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})+g_2\rho_{32}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r}) \\ &\quad -\text{c.c.}-2i(\gamma_1+\gamma_2)\rho_{33}, \\ i\frac{d\rho_{13}}{dt}&=-g_1(\rho_{33}-\rho_{11})\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r}) \\ &\quad +g_2\rho_{12}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})-i\gamma\rho_{13}, \\ i\frac{d\rho_{23}}{dt}&=-g_2(\rho_{33}-\rho_{22})\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r}) \\ &\quad +g_1\exp(i\Omega_1 t)-i\mathbf{k}_1\mathbf{r})\rho_{21}-i\gamma\rho_{23}, \\ i\frac{d\rho_{12}}{dt}&=-g_1\rho_{32}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r}) \\ &\quad +g_2\rho_{13}\exp(-i\Omega_2 t+i\mathbf{k}_2\mathbf{r})-i\Gamma\rho_{12}; \end{aligned} \tag{4.2}$$

where g_m are the Rabi frequencies, $\Omega_m=\omega_m-\omega_{3m}$ ($m=1,2$) are the frequency detunings, $2\gamma_1$, $2\gamma_2$ are the partial decay rates from level $|3\rangle$ to levels $|1\rangle$ and $|2\rangle$, $\gamma=\gamma_1+\gamma_2$, and Γ is the rate of relaxation of low-frequency coherence ρ_{12}.

Next, we introduce the following substitutions for the off-diagonal elements in (4.2):

$$\begin{aligned} &\rho_{m3}=\rho_{m3}\exp(i\Omega_n t-i\mathbf{k}_m\mathbf{r}) \quad (m=1,2), \\ &\rho_{12}=\rho_{12}\exp(i(\Omega_1-\Omega_2)t-i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}), \end{aligned} \tag{4.3}$$

and, assuming that the Λ-atom is at rest, we put $\mathbf{v}=0$ on the left-hand side of (4.2). This gives a set of equations

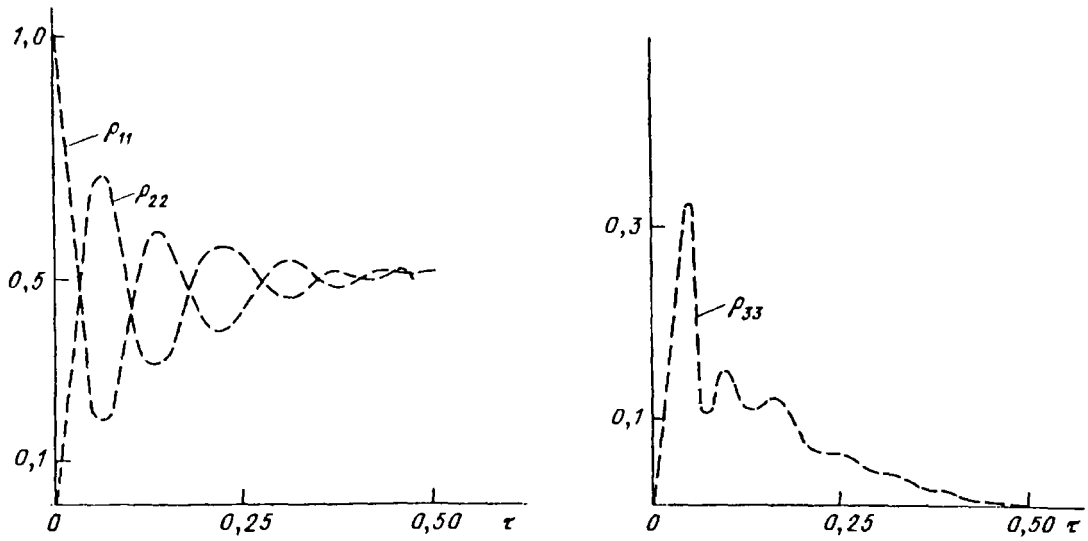

FIG. 4. Evolution of populations $\rho_{mm}(t)$ ($m=1,2,3$) in a Λ-system for (1.1) and $g=2\gamma$, $\gamma=2\cdot 10^7\,\mathrm{s}^{-1}$, $\Gamma=10^3\,\mathrm{s}^{-1}$, $\tau=\gamma t$ and initial conditions $\rho_{11}(0)=1$, $\rho_{22}(0)=\rho_{33}(0)=\rho_{ik}(0)=0$ ($i\neq k=1,2,3$).

containing only time derivatives, which can be solved for zero detunings $\Omega_m=0$ and, correspondingly, we can find the time dependence of the population of level $|3\rangle$, i.e., the result obtained above by the formalism of state vectors.

For equal partial rates $\gamma=\gamma_i$ ($i=1,2$) and equal Rabi frequencies, we find[45] from (4.2) that, when the initial conditions are $\rho_{11}(0)=\rho_{22}(0)=1/2$, $\rho_{33}(0)=0$, $\rho_{ik}(0)=0$ ($i\neq k=1,2,3$),

$$\rho_{33}=\frac{g^2}{\gamma^2}\left[1+\exp(-\tau)-2\exp\left(-\frac{3}{2}\tau\right)\right], \tag{4.4}$$

where $g^2\ll\Gamma\gamma$, whereas for $g^2\gg\Gamma\gamma$

$$\rho_{33}=\frac{\Gamma}{2\gamma}[1-\exp(-\tau)]+\frac{2g}{\gamma}\exp(-\tau)\left[\sqrt{2}\left(\frac{9}{4}+\frac{2g^2}{\gamma^2}\right)\right]^{-1}$$
$$\times\left[\sqrt{2}\frac{g}{\gamma}+\exp\left(-\frac{\tau}{4}\right)\left(\frac{1}{2}\sin x-\sqrt{2}\frac{g}{\gamma}\cos x\right)\right], \tag{4.5}$$

where $\tau=\sqrt{2}\gamma t$ and $x=\sqrt{2}gt$.

It is clear from (4.4)–(4.5) that the population of the third level in the Λ-system depends significantly on both the light-wave intensity and the relaxation rates Γ, γ.[4)] The character of the populations of the upper level depends on the ratio of g^2 and $\Gamma\gamma$. Let us compare (4.4), (4.5) with the solutions (3.3), (3.4) obtained in the formalism of state vectors. This will enable us to estimate the degree of rigor of this approach. For example, for high Rabi frequencies, the population of the third level in (3.4) tends, in its stationary state, to zero and not to the constant value $\Gamma/2\gamma$. This is a consequence of the fact that it is impossible to take transverse relaxation into account in the formalism of amplitudes of states.

We emphasize that allowance for the relaxation of low-frequency coherence ρ_{12} in (4.2) is fundamental because, on the one hand, Γ determines the magnitude of the CPT effect (the population of level $|3\rangle$) and, on the other hand, it indicates the light-wave intensities for which the effect is possible.

Figure 4 shows the populations in the Λ-system as functions of time of observation. It is interesting to note that, in the case of the exact solution of (4.2), obtained from the equations for the density matrix, the evolution depends significantly on the initial conditions. If the initial populations of levels $|1\rangle$ and $|2\rangle$ are equal to 1/2 and the initial coherence is $\rho_{12}=-1/2$, the system is already in the coherent trapping state and, in general, is not excited by a field (cf. the data in Table I). In other words, particular initial population densities facilitate the transition to CPT. This shows once again that definite coherent states, from which the system undergoes weak excitation, exist in the Λ-system. These coherent states have fully determined amplitudes and phases (see Sec. 2). At the same time, when spontaneous relaxation occurs in the system, a transition from the upper state to the lower levels takes the system to a statistical mixture of pure states in the lower levels. There is then a finitie probability that the system will reach a coherent state, after which (as shown in Secs. 2 and 3), no excitation takes place in the system. If, as a result of spontaneous decay, the Λ-system does not reach a coherent state, then (3.3) and (3.4) show that it will, as before, be excited to the upper state so long as spontaneous decay does not take it to the state (2.11). It is precisely because of these processes which 'pull out' the atoms from the interaction with the field in each new decay that oscillatory behavior such (3.4) and (3.5) takes place.

Figure 5 shows the behavior of the Λ-system population when the conditions for coherent trapping are not met.

We note that, for low light-wave intensities, coherent trapping is absent altogether. It is also clear from (4.4) that the time-independent population of level $|3\rangle$ is proportional in this case to g^2/γ^2 The conditions for the onset of CPT will be discussed in greater detail below.

4.2. Conditions for the onset of coherent population trapping

We noted above that the CPT phenomenon is a fundamentally nonlinear effect that occurs only for particular light-wave intensities. This is already clear from (4.4) and (4.5) as we pass to the limit $t\to\infty$, i.e., when time-

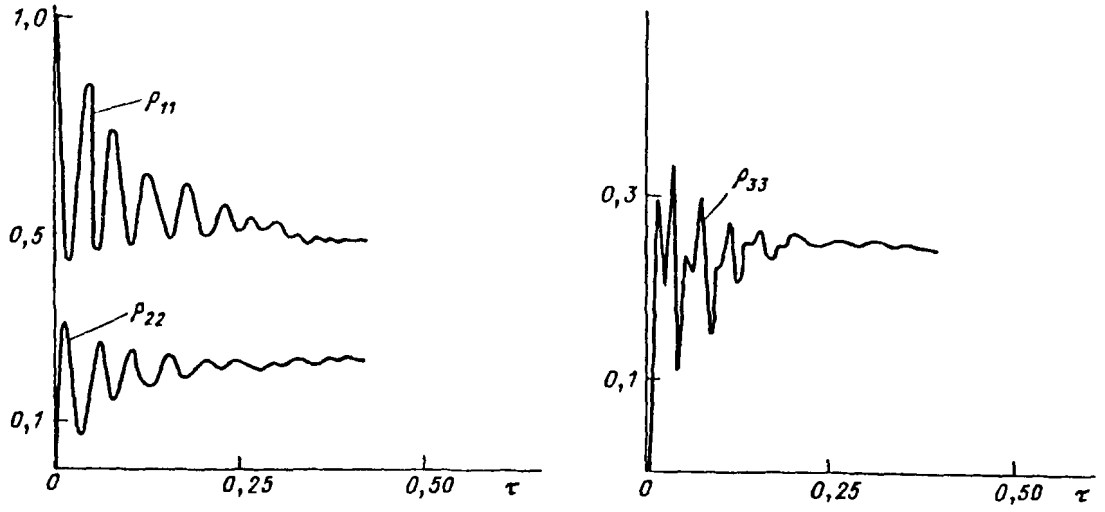

FIG. 5. Same as Fig. 4 with (1.1) not satisfied; $\Omega_1=5\gamma$, $\Omega_2=0$, $g=2.5\gamma$.

independent population is determined. However, at this point, we shall proceed in a different way and will find the intensity for the onset of CPT directly from the expression for the time-independent population of level $|3\rangle$.

Substituting (4.3) in (4.2), and assuming that $\Omega_1=\Omega_2\equiv\Omega$ and that the atom is at rest ($\mathbf{v}=0$) for $t\gg\gamma^{-1},\Gamma^{-1}$, we obtain a set of algebraic equations, the solution of which yields

$$\rho_{33}=\frac{\Gamma}{\gamma}\frac{g^2}{2g^2+\Gamma\gamma+(\Gamma\Omega^2/\gamma)}, \tag{4.6}$$

where we have put $\Gamma\ll\gamma$ which is valid for all cases that are of importance in practice.

It is clear from (4.6) that the population in the upper state depends significantly on the relationship between the quantities g,Ω,γ,Γ:

$$\begin{aligned}\rho_{33}&=\frac{g^2}{\gamma^2+\Omega^2}\quad\text{for } g^2\ll\gamma\Gamma\left(1+\frac{\Omega^2}{\gamma^2}\right),\\ &=\frac{\Gamma}{2\gamma}\quad\text{for } g^2\gg\gamma\Gamma\left(1+\frac{\Omega^2}{\gamma^2}\right).\end{aligned} \tag{4.7}$$

In the second case, i.e., when $g^2\gg\gamma\Gamma(1+\Omega^2\gamma^{-2})$, the population does not depend on the light-wave intensity and is determined entirely by the relaxation constants of the system. This is a fundamental point because it shows that nonlinear phenomena (similar to saturation in the two-level atom) occur in the system. Next, since it is assumed that $\gamma\gg\Gamma$ and that $\rho_{33}\ll1$, it follows that, in atoms with strong optical lines, $\gamma=10$ MHz and Γ is practically never less than 1000 GHz, we find that $\rho_{33}\approx10^{-4}$. On the contrary, in the first case in (4.7), we find that the population of the upper level is a linear function of the field intensity, which may be looked upon as the linear approximation in the light-wave intensity.

As in the case of equal detunings, we now obtain the time-independent population of level $|3\rangle$ when one wave is in resonance with a transition in the Λ-atom and the other is scanned:

$$\rho_{33}=g^2\gamma aL^{-1}, \tag{4.8}$$

$$\begin{aligned}L=a[\gamma_1(\gamma^2+\Omega_1^2)+\gamma_2\gamma^2+3\gamma g^2]-2g^2\Omega_1^2\gamma_1\gamma\\ +g^2\gamma^2(4g^2-\Omega_1^2+2\Gamma\gamma),\end{aligned} \tag{4.9}$$

where $a=\gamma\Omega_1^2+\Gamma(2g^2+\Gamma\gamma)$ and the Rabi frequencies of the two light waves are both equal to g. The dependence of the population ρ_{33} on the detuning Ω_1 is shown if Fig. 2 for different light-wave intensities. Curve *1* corresponds to the first condition in (4.7) and curves *2–4* to the second. It is clear that a sharp reduction in ρ_{33} in the case of exact resonance ($\Omega_m=0$) occurs only for $g^2\gg\Gamma\gamma$. Let us now estimate the intensity for which coherent trapping can be observed. From the second row in (4.7) we have $\Omega_m=0$

$$g^2/\gamma^2=I/I_n\gg\Gamma/\gamma\approx10^{-4}, \tag{4.10}$$

and, considering that the saturation intensity is $I_n\approx0.1$ W/cm^2, we find that the required intensity is $I\approx10^{-5}$ W/cm^2.

We emphasize once again that the correct physical picture is obtained only for a nonzero transverse relaxation rate Γ. On the other hand, when $\Gamma=0$, then (1.1) shows that, formally CPT should occur for any light wave intensity which, strictly speaking, is not true.

4.3. Coherent population trapping in partially coherent light waves

When we considered coherent trapping phenomena, we implicitly assumed that the spectral width of the exciting fields was much smaller than the natural level width in the upper state. However, tunable dye lasers are usually employed in CPT observations, and their spectral widths are comparable with, or even greater than, the natural width of the upper level of the Λ-system. It is therefore important to examine the onset of coherent trapping in this situation, i.e., to investigate trapping for partially coherent light fields.[11–13]

We note that, the properties of CPT in fields of finite spectral width are of major practical importance because they enable us to identify the conditions that these widths have to satisfy to ensure that the CPT effect is observed.

To solve this problem, let us suppose that the Λ-system interacts with a light-wave field whose phases are random functions:

$$E=E_0\cos(\omega_1 t-\mathbf{k}_1\mathbf{r}+\varphi_1(t))+E_0\cos(\omega_2 t-\mathbf{k}_2\mathbf{r}+\varphi_2(t)), \tag{4.11}$$

where E_0 is a determined amplitude and $\varphi_i(t)$ $(i=1,2)$ are random phases whose initial values $\varphi_i^0=\varphi_i(0)$ $(i=1,2)$ are uniformly distributed on the interval $(0,2\pi)$. We shall also consider that the time derivatives $\dot{\varphi}_i(t)$ are delta-correlated random processes[4] with zero average values

$$\langle\dot{\varphi}_i(t)\rangle=0, \quad i=1,2,$$

correlation functions

$$\langle\dot{\varphi}_i(t)\dot{\varphi}_i(t')\rangle=2\beta_i\delta(t-t'), \tag{4.12}$$

$$\langle\dot{\varphi}_1(t)\dot{\varphi}_2(t')\rangle=2\nu\delta(t-t'), \tag{4.13}$$

and

$$|\nu|\leqslant(\beta_1\beta_2)^{1/2},$$

where (4.13) describes the possible case of mutual correlation (cross-correlation) of light waves (4.11). Under these conditions the intensity spectrum $I(\omega_i')$ of the random fields (4.11) is a Lorentzian with the half-width β_i $(i=1,2)$

$$i(\omega_i')=E_0^2\cdot 2\beta_i[(\omega_i'-\omega_i)^2+\beta_i^2]^{-1}, \tag{4.14}$$

Under these conditions, we can write down the microscopic equations for the elements of the density matrix of the form (4.2), remembering that the expression for the light waves (4.1) must be replaced with (4.11). The next step is to replace the off-diagonal elements of the density matrix by analogy with (4.3).

We note that the equations obtained in this way are stochastic because they contain the derivatives $\dot{\varphi}_i(t)$ of the random phases. It follows that, henceforth, we shall have to confine our attention to the average characteristics evaluated over the ensemble of phases (including the average population of level $|3\rangle$, evaluated over the ensemble). To transform from the equations for the random phases to equations for their averages, we must average the resulting set over the ensemble of phases.[4] If we suppose that the correlation functions are specified by (4.12) and (4.13), we thus obtain the following replacements for the relaxation constants in the equations for the average off-diagonal elements (cf. Ref. 11):

$$\begin{aligned}&\rho_{13}:\gamma_1\to\gamma+\beta_1,\\&\rho_{23}:\gamma_2\to\gamma+\beta_2,\end{aligned} \tag{4.15}$$

and

$$\rho_{12}:\Gamma\to\Gamma+\beta_1+\beta_2-2\nu. \tag{4.16}$$

The last expression determines the low-frequency coherence relaxation rate Γ due to frequency fluctuations. Three important facts follow from it. First, if the relaxation rate in the absence of frequency fluctuations is zero ($\Gamma=0$), a 'noisy' field such as (4.11) will always broaden the

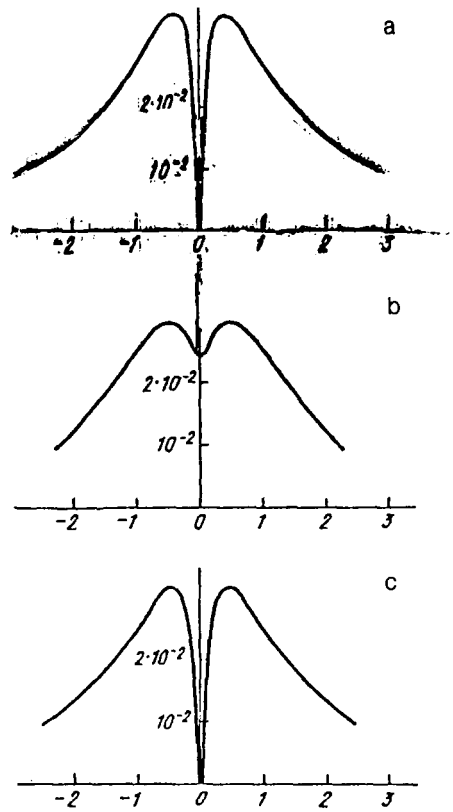

FIG. 6. Population of the upper level of a Λ-system as a function of Ω_2 for $\Omega_1=0$; $g_1=g_2=0.4$, a—$\beta_1=\beta_2=\nu=0$. b—$\beta_1=\beta_2=0.1$, $\nu=0$. c—$\beta_1=\beta_2=\nu=0.1$. (arb. units; taken from Ref. 11).

1–2 transition. It is precisely this that we had in mind when we considered the light wave intensities that are necessary for CPT to occur (cf. Sec. 4.2)

Second, the depth of the CPT valley as a function of the size of the fluctuations and the cross-correlation of the light wave is also an important consideration. For complete correlation

$$\beta_1=\beta_2=\nu \tag{4.17}$$

between the waves in (4.11) we then again obtain coherent trapping[11] for the corresponding intensity conditions (4.7). If, on the other hand, we take $\beta_1\neq\beta_2$ (and, correspondingly, $\nu<(\beta_1+\beta_2)/2$), then for substantial fluctuations

$$\beta_1+\beta_2-2\nu\gg 2g_i^2/(\gamma+\beta_i) \quad (i=1,2) \tag{4.18}$$

there is no coherent trapping,[11] which is also consistent with (4.7).

Figure 6 shows the population of level $|3\rangle$ in the Λ-system for different light-wave correlations.[11] It is clear that, even for small widths β_i and completely uncorrelated fields (Fig. 6b), the valley depth is considerably reduced. In other words, for completely uncorrelated fields, CPT is very dependent on the spectral width β_i of the exciting waves.

At the same time, the presence of correlation between the light waves makes it much easier to observe the

coherent-trapping resonance. CPT is then observed even when the spectral width β_i is considerable. This is why cross-correlated light waves are used in experiments. In practice, this is achieved with the help of an opto-acoustic modulator.

The case of two correlated fields with significantly different spectral widths $\beta_1 \neq \beta_2$ requires particular attention. Coherent trapping can be destroyed or restored (as was shown above), depending on the type of coupling between the fields (positive or negative correlation, partial or total). The results are then different for different types of the three-level systems.[13]

The case of fluctuating exciting-wave amplitudes can be considered in a similar way. However, amplitude fluctuations have a much smaller influence on the existence of CPT when the intensity conditions (4.10) are satisfied.

4.4. Effect of the motion of atoms on the onset of coherent population trapping

When the CPT effect was examined above, it was assumed that the atoms interacting with the light field (4.1) were at rest. We shall now take into account the translational motion of the atoms. In the frame in which the atoms are at rest, the condition for CPT such as (4.1) is replaced by the condition for equal Doppler-shifted frequency detunings

$$\omega_1 - \mathbf{k}_1\mathbf{v} = \Omega_2 - \mathbf{k}_2\mathbf{v}. \tag{4.19}$$

It then follows from (4.19) that the condition for the onset of CPT depends significantly on the mutual directions of the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ of the light waves and the velocity $\mathbf{v}$ of the atoms. Let us now consider (4.19) for some special configurations that are important in practice.

Parallel light beams. Suppose that the wave vectors are parallel. We then find from (4.19) that

$$\Omega_1 - \Omega_2 - (k_1 - k_2)v = \Omega_1 - \Omega_2 - qv = 0, \tag{4.20}$$

where k_m $(m=1,2)$ are the moduli of the wave vectors and v is the component of the velocity of an atom in the direction of propagation $(q = k_1 - k_2)$.

If we consider a Λ-system in which the two wave vectors are approximately equal, then (4.20) becomes identical with (1.1). If, on the other hand, the wave vector difference q is large, then CPT will occur only for particles with a particular projection of their velocity (for fixed detuning Ω_m), namely,

$$v = (\Omega_1 - \Omega_2)/q. \tag{4.21}$$

We note that direct observation of CPT in a Λ-system was reported for (4.21) in Ref. 70. The results of these experiments will be discussed in Sec. 10.2 which is devoted to applications of CPT in ultrahigh-resolution spectrocopy.

Counterpropagating light waves. We shall now consider (4.19) in the case of counterpropagating waves assuming that $\mathbf{k}_1 = -\mathbf{k}_2$, so that (4.19) gives

$$\Omega_1 - \Omega_2 - (k_1 + k_2)v = 0. \tag{4.22}$$

This means that both here and in (4.21), the upper level is depleted only for a particular particle velocity.

If we now consider the case of two waves oriented arbitrarily in space, we can write (4.19) in a Cartesian frame in the form

$$\Omega_1 - \Omega_2 + \sum_{m=1}^{3} (k_2^m - k_1^m) v_m = 0, \tag{4.23}$$

where $k_{1,2}^m$, v_m are the projections of the wave vectors and the velocity vector of the atom along the coordinate axes. We can also use (4.23) to determine the velocity vector of the Λ-atom for which the atom will not be excited to the upper state for a given light-field configuration.

We note that all the above conditions can be obtained naturally from (4.2) by direct evaluation of the time-independent population of the upper state.

5. COHERENT POPULATION TRAPPING IN OPTICALLY DENSE MEDIA

It is well-known[3] that light is attenuated by absorption in an optically dense medium. This is described by the linear absorption coefficient α in the Bouguer–Lambert law. The reciprocal $l = \alpha^{-1}$ is a measure of the thickness of the medium in which the light intensity is reduced by a factor of e. Moreover, we also have to take into account the effect of interatomic collisions on CPT. We have already seen that the typical time constant for the establishment of CPT is the lifetime of the atom in the upper state $\tau \approx 10^{-7}$ s. Let us therefore estimate the time interval between successive collisions, using the expression

$$\tau_s \approx (n\bar{v}\sigma_s)^{-1}, \tag{5.1}$$

where $\bar{v}$ is the average velocity, σ_s is the gas-kinetic cross section, and n is the concentration of the atoms. It is clear from (5.1) that for typical values ($\bar{v} \approx 10^4$ cm/s, $\sigma_s \approx 10^{-16}$ cm^2 and $n \approx 10^{15}$ cm^{-3}), we have $\tau_s \approx 10^{-3}$s $\gg \tau$ and CPT can always be established in the interval between two collisions. On the other hand, frequent interatomic collisions produce an increase in the transverse relaxation rate between the lower states.[5)] This must be taken into account when the light-wave intensity (4.10) that is necessary for CPT to occur is chosen.

It is qualitatively clear that when two-frequency radiation passes through a medium with CPT, the absorption of light should be lower. Since, as was shown above, an atom cannot scatter resonant photons when the detuning condition (1.1) is satisfied, the medium transmits radiation within a narrow coherent-trapping resonance. It is interesting to note that if the coherent trapping condition (1.1) is not met, ordinary absorption of light by the resonant medium is observed. The free transmission by the medium is then lost even when (1.1) is satisfied, and the radiation intensity is reduced to the extent that the CPT condition (4.10) which limits the light-wave intensity is not satisfied. CPT can thus lead to important consequences for the propagation of laser radiation in optically dense media.[40–44]

Several cases have now been investigated, including continuous two-frequency laser radiation interacting with an optically dense medium, the case where the radiation is a periodic sequence of ultrashort pulses of width that cov-

ers both of the lower levels of the Λ-system, and the propagation of continuous non-monochromatic laser radiation with spectrum covering both of the lower levels.

The change in the intensity of radiation propagating in a medium must be taken into account when the transmission of light by an optically dense medium is analyzed. The equations for the elements of the atomic density matrix, from which the microscopic characteristics of the medium are obtained, and which are given by (4.2), must therefore be augmented by the equations for the fields in the medium. We shall follow Refs. 40–42 and write down the truncated wave equations for the field amplitude in the interior of the medium in the form (we assume that all the rays at entry to the medium are parallel)

$$\frac{\partial E_m}{\partial z}+\frac{1}{c}\frac{\partial E_m}{\partial t}=i2\pi d^2 n\omega_m E_n\sigma_{3m}/c\hbar\gamma \quad (m=1,2), \tag{5.2}$$

where d_{m3} are the transition dipole elements for the $|m\rangle$–$|3\rangle$ transitions, n is the concentration of the atoms, $E_n=\hbar\gamma/d$ is the field amplitude that saturates optical transitions in the Λ-system, and

$$\sigma_{3m}=\rho_{3m}\exp[i(\omega_m t-k_m z)] \quad (m=1,2). \tag{5.3}$$

Next, substituting (4.3) in (4.2), and solving the equations for the time-independent case, we obtain the off-diagonal elements of the density matrix.

It will be convenient to introduce the dimensionless optical length l and field intensity $J_m(l)$ $(\omega\equiv\omega_1\approx\omega_2)$:

$$l=\frac{2\pi n\omega d^2}{c\hbar\gamma}z,$$

$$J_m(l)=\frac{|E_m(l)|^2}{|E_m(0)|^2}=\frac{I_m(l)}{I_m(0)},$$

where $I_m(0)$ is the radiation intensity of the nth field entering the medium. For exact resonance between the waves and the transitions $\Omega_m=0$, $m=1,2$ and $E_1(0)=E_2(0)\equiv E(0)$, we obtain from (5.2) the equation describing the propagation of laser radiation during CPT:

$$\frac{dJ}{dl}=-J(l)[1-2g^2(\Gamma\gamma+2g^2J(l))^{-1}], \quad J(0)=1. \tag{5.4}$$

Solution of this nonlinear differential equation (5.4) gives

$$\frac{2g^2}{\gamma\Gamma}(1-J(l))-\ln J(l)=l, \tag{5.5}$$

from which it is clear that the law of propagation inside the medium depends on the initial laser intensity at entry to the medium $[J(0)=J(l=0)]$ and the optical length l. As in (4.10), the characteristic parameter is the coherent intensity I_c

$$I_c=\frac{\Gamma}{\gamma}I_n, \quad I_n=\frac{\hbar^2\gamma^2 c}{8\pi d^2}.$$

When the initial intensity is $I(0)\gg I_c$ $(g^2\gg\Gamma\gamma)$, it follows from (5.5) that the law of propagation in the medium is linear, i.e.,

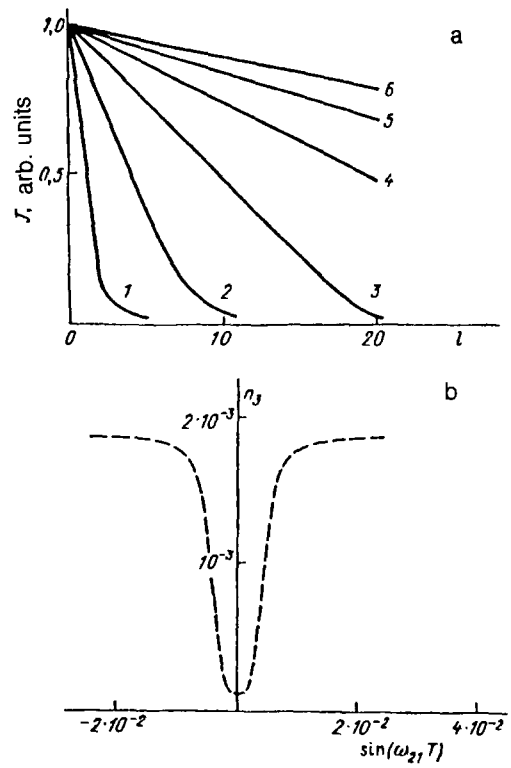

FIG. 7. a—Propagation of laser radiation under CPT condition (1.1) for $\gamma=1.9\cdot10^7$ s^{-1}, $\Gamma=10^2$ s^{-1}. $1-g=10^4$ s^{-1}, $2-g=10^5$ s^{-1}, $3-g=2\cdot10^{-5}$ s^{-1}, $4-g=3\cdot10^5$ s^{-1}, $5-g=4\cdot10^5$ s^{-1}, $6-g=5\cdot10^5$ s^{-1} (taken from Ref. 42). b—Resonant population n_3 of the upper level of a Λ-system as a function of sin $(\omega_{21}T)$ which characterizes the detuning $\Omega=2\pi/T$ of the pulse repetition frequency from the resonant value $\Omega_0=\omega_{21}m$ $(m=0,\pm1,...)$ (from Ref. 40).

$$J(l)=1-\frac{\Gamma\gamma}{2g^2}l. \tag{5.6}$$

However, when the intensity $I(l)$ decreases with increasing l until $I(l)<I_c$, the radiation intensity falls exponentially for any l (Fig. 7a).

As already noted, the physical meaning of the condition $I\gg I_c$ is that it ensures coherent trapping. Optical communication with level $|3\rangle$ then breaks off and the medium becomes a weak absorber of light.

This weak absorption produces a transparency window under CPT conditions, which appears when the frequency of one of the laser fields is scanned. Actually, the solution given by (5.6) is valid only in the region of CPT resonance whose width in our case is given by[42]

$$\Delta\approx\Gamma+J(l)\frac{g^2}{\gamma} \tag{5.7}$$

and depends on the optical length l. Outside the frequency interval corresponding to the CPT resonance we have ordinary exponential absorption of light by the resonant medium. It follows that, when the frequency of one of the lasers is scanned, the medium acquires a "transparency window" but only in the region of the CPT resonance (5.7). We note that this can be exploited in an optical modulator.[42,45] An interesting consequence of the nonlin-

ear interaction between resonant radiation and the medium is that the width of the transparency window (5.7) decreases with increasing optical thickness l.

We now turn to coherent transmission in a three-level medium of a periodic sequence of ultrashort pulses[40] of length τ_p and repetition frequency T^{-1}. We shall suppose that τ_p^{-1} and $T^{-1} \gg \omega_{21}$, where ω_{21} is the separation between levels $|1\rangle$ and $|2\rangle$ in the Λ-system, and that each pulse interacts with two optical transitions. The equation describing the propagation of these pulses in the medium are analogous to (5.2), and the equation for the density matrix is given in Ref. 40. Proceeding as before, we obtain a set of equations describing both the propagation of the light pulses in the medium and the variation in the parameters of the medium. Next, by considering the settling down of the solution describing periodic oscillations in the medium with frequency equal to the pulse repetition frequency, we find[40] that, when $\Gamma^{-1} \gg T$, and the low-frequency coherence does not succeed in relaxing between successive pulses, the population n_3 of level $|3\rangle$ has a resonant dependence on the pulse repetition frequency (Fig. 7b). Resonances thus occur when

$$\omega_{21}=2\pi m/T \quad (m=1,2,\ldots), \tag{5.8}$$

or

$$\sin \omega_{21}T=0.$$

Naturally, complete trapping of populations by a pulse train can occur only for $\Gamma \ll \gamma$ because otherwise coherence is lost and there is no population trapping by the lower levels. When $I \gg I_c$ pulse propagation in the medium is analogous to (5.6)

Radiation transfer within spectral lines in an optically dense medium with CPT was examined in Ref. 43 for a nonmonochromatic laser field. It was shown that if the radiation spectrum covers the ω_{21} frequency separation between levels $|1\rangle$ and $|2\rangle$, and this separation is less than the optical excitation rate $W=g^2/\gamma$, the atoms become trapped by these levels and the medium transmits. The integrated intensity of the laser radiation then decreases linearly in the medium and is independent of the shape of the incident and absorbing profiles.

We note, finally, that transmission by a medium with CPT was observed experimentally in Ref. 44

6. POPULATION TRAPPING IN A CASCADE SYSTEM

6.1. Approach to the steady state in a Ξ-system

We shall now consider CPT in a cascade system of levels (Fig. 1c). We shall suppose that the field interacting with the system is given, as it was in the case of the Λ-system, in the form of two plane light waves (4.1)

The equations for the elements of the atomic density matrix take the following form in the case of the Ξ system[4]

$$\begin{aligned}
i\frac{d\rho_{11}}{dt}&=-g_1\rho_{31}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})+\text{c.c.}+2i\gamma_1\rho_{33},\\
i\frac{d\rho_{22}}{dt}&=g_2\rho_{23}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})+\text{c.c.}-2i\gamma_2\rho_{22},\\
i\frac{d\rho_{33}}{dt}&=g_1\rho_{31}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})-g_2\rho_{23}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})\\
&\quad-\text{c.c.}-2i\gamma_1\rho_{33}+2i\gamma_2\rho_{22},\\
i\frac{d\rho_{13}}{dt}&=-g_1(\rho_{33}-\rho_{11})\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})\\
&\quad+g_2\rho_{12}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})-i\gamma_1\rho_{13},\\
i\frac{d\rho_{32}}{dt}&=g_2(\rho_{22}-\rho_{33})\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})\\
&\quad+g_1\rho_{12}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})-i\gamma\rho_{32},\\
i\frac{d\rho_{12}}{dt}&=-g_1\rho_{32}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r})\\
&\quad-g_2\rho_{13}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r})-i\gamma_2\rho_{12},
\end{aligned} \tag{6.1}$$

where g_m $(m=1,2)$ are the Rabi frequencies, $\Omega_m=\omega_m-\omega_{3m}$ are the frequency detunings, $2\gamma_1$ is the rate of decay from level $|3\rangle$ to level $|1\rangle$, $2\gamma_2$ is the rate of decay from level $|2\rangle$ to level $|3\rangle$, and $\gamma=\gamma_1+\gamma_2$. Next, we introduce the following replacement of the off-diagonal elements by analogy with (4.3):

$$\begin{aligned}
\rho_{13}&=\rho_{13}\exp(i\Omega_1 t-i\mathbf{k}_1\mathbf{r}),\\
\rho_{32}&=\rho_{32}\exp(i\Omega_2 t-i\mathbf{k}_2\mathbf{r}),\\
\rho_{12}&=\rho_{12}\exp[i(\Omega_1+\Omega_2)t-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}].
\end{aligned} \tag{6.2}$$

If we now consider a resting Ξ-systems ($\mathbf{v}=0$), we find from (6.1) and (6.2) a set of equations that contains only the time derivatives. We must now try to solve this system for zero detunings $\Omega_m=0$ $(m=1,2)$, equal Rabi frequencies $g_m=g$ $(m=1,2)$, and $\gamma_2 \ll \gamma_1$. We shall see later that CPT is then possible in the cascade system. The population of the intermediate level $|3\rangle$ in the case of initial conditions $\rho_{11}(0)=1$, $\rho_{22}(0)=0$, $\rho_{33}(0)=0$, $\rho_{ik}(0)=0$ $(i\neq k=1,2,3)$ has the following form[45] when $g^2 \gg \gamma_1\gamma_2$:

$$\rho_{33}=\frac{\gamma_2}{\gamma_1}+\exp(-\tau)\times\left[\frac{1}{4}(1-\cos x)-\frac{\gamma_2}{\gamma_1}\left(1+2^{-3/2}\frac{\gamma_1}{g}\sin x\right)\right], \tag{6.3}$$

and for $g^2 \ll \gamma_1\gamma_2$

$$\rho_{33}=\frac{g^2}{\gamma_1^2}\left[1+\exp\left(-\frac{\tau}{2}\right)-2\exp\left(-\frac{\tau}{4}\right)\right]. \tag{6.4}$$

In the last two expressions, the dimensionless time $\tau=2\gamma_1 t$ is expressed in units of γ_1^{-1} and $x=\sqrt{2}gt$.

It is clear that the evolution in time of the population of the intermediate level of the cascade system is close to the behavior of $\rho_{33}(t)$ for the Λ-system [cf. (6.3) and (6.4) with (4.5) and (4.4), respectively]. However, it is the ratio of the Rabi frequencies and rates of spontaneous relaxation in the system that is decisive here. For sufficiently high Rabi frequencies, ρ_{33} oscillates and asymptotically ap-

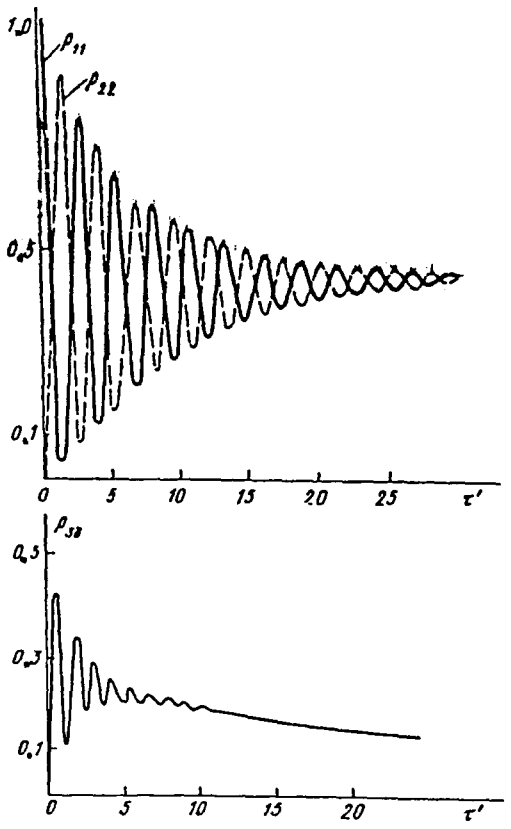

FIG. 8. Evolution of populations in the Ξ-system under CPT conditions for $\gamma_2=0.1\gamma_1$, $g=10\gamma_1, \gamma_1=10^7$ s^{-1} and initial conditions $\rho_{11}(0)=1, \rho_{22}(0)=\rho_{33}(0)=\rho_{ik}(0)=0 (i\neq k=1,2,3,)$ for $\Omega_m=0$; $\tau'=64\gamma_1 t$.

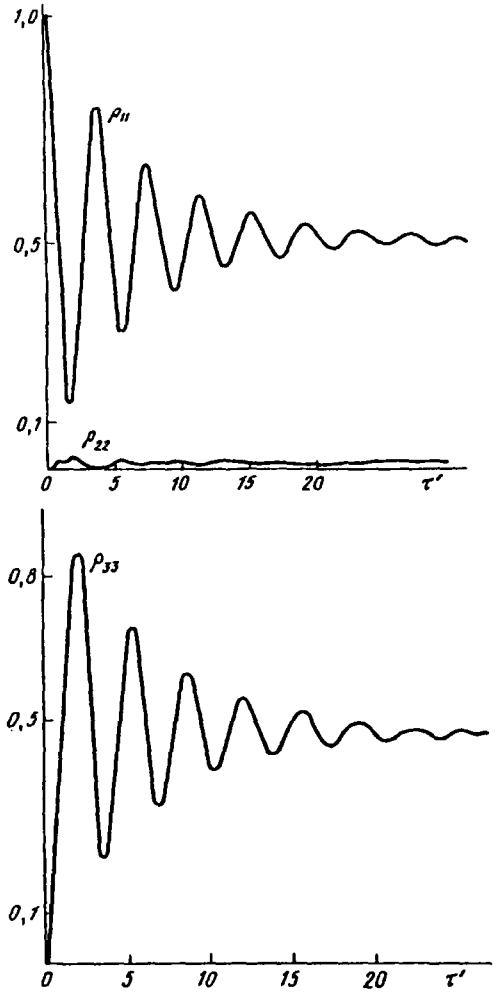

FIG. 9. Same as Fig. 8, but with $\Omega_1=0, \Omega_2=30\gamma_1, g=5\gamma_1$.

proaches the small quantity γ_2/γ_1 [we recall that (6.3) and (6.4) were obtained on the assumption that $\gamma_2 \ll \gamma_1$] which is independent of the light-wave intensity and is an internal characteristic of the system. In the case of low Rabi frequencies, the population of level $|3\rangle$ does not oscillate and, in the stationary state, is proportional to the intensity of the fields interacting with the atom $\rho_{33}=g^2/\gamma_1^2$. The time functions $\rho_{mm}(\tau)$ $(m=1,2,3)$ for $g_{1,2}^2 \gg \gamma_1\gamma_2$ are shown in Fig. 8. The detunings Ω_m $(m=1,2)$ from resonance have practically no effect on the time dependence of populations in the systems if $\Omega_1+\Omega_2=0$. On the other hand, when $\Omega_1+\Omega_2\neq 0$, then despite the fact that the condition $g^2 \gg \gamma_1\gamma_2$ is satisfied, CPT is not observed and the intermediate-level population reaches values comparable with ρ_{11} whilst the upper level is only slightly populated (Fig. 9). A similar evolution of populations in a cascade system can be observed when the condition imposed on the decay rates $(\gamma_2 \ll \gamma_1)$ is not satisfied.[46] However, in order to understand why this happens, it is more convenient to consider the time-independent solution of (6.1).

6.2. Spectroscopy of a Ξ-system

To investigate the conditions for CPT in a cascade system, consider the time-independent solution of (6.1). Assuming that the atom is at rest ($\mathbf{v}=0$), we find from (6.1) for $t \gg \gamma_m^{-1}$ $(m=1,2)$ that, at exact resonance $(\Omega_m=0,\ m=1,2)$, the expressions for the populations are

$$\rho_{33}=g_1^2\gamma_2[\gamma_2(\gamma_1^2+2g_1^2)+g_2^2\gamma_1]^{-1},$$

$$\rho_{22}=g_2g_1^2(g_1\gamma_1+g_2\gamma_2)\{(\gamma_2\gamma+g_1^2+g_2^2)[\gamma_2(\gamma_1^2+2g_1^2)+g_2^2\gamma_1]\}^{-1},$$

$$\rho_{11}=1-\rho_{22}-\rho_{33}. \tag{6.5}$$

It is clear from this that, when $\gamma_2 \ll \gamma_1$ and $g_1^2=g_2^2 \gg \gamma_1\gamma_2$, the populations are $\rho_{33}=\gamma_2/\gamma_1$, $\rho_{11}=\rho_{22}\approx 1/2$ and population trapping is observed for the upper and lower levels. It is readily verified that, for field intensities $g_1^2, g_2^2 \ll \gamma_1\gamma_2$ and arbitrary γ_1, γ_2, we have $\rho_{33}=g_1^2/\gamma_1^2$ and the intermediate-level population is proportional to the intensity of the optical radiation. We thus find that, for CPT to occur in the Ξ-system, we must have not only $g_m^2 \gg \gamma_1\gamma_2$, but also a definite relation between the decay rates, $\gamma_2 \ll \gamma_1$. For example, when $\gamma_1 \approx \gamma_2$, $g_1^2=g_2^2 \gg \gamma_1\gamma_2$, the level populations are given by the simple expression $\rho_{mm}=1/3$ $(m=1,2,3)$, and $\rho_{12}=0$ (Ref. 46). It is physically clear that, as in the case of the Λ-system, a coherent superposition of states 1 and 2 should be formed during CPT and should not interact with

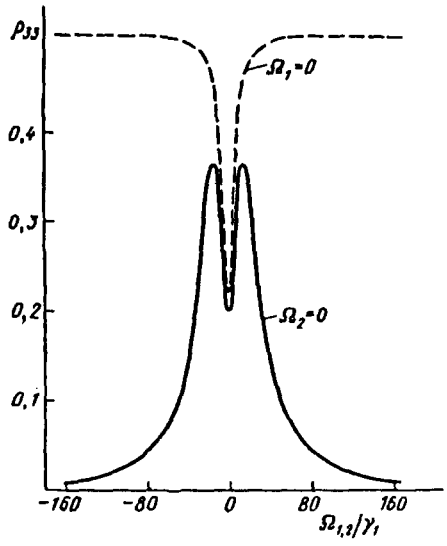

FIG. 10. The population ρ_{33} when one frequency is fixed and the other scanned: $\gamma_1=10^7\ s^{-1}$, $\gamma_2=0.1\gamma_1$, $g_1=2\gamma_1$, $g_2=\gamma_1$.

resonant fields. The decay of the states in the superposition should therefore be relatively slight which in the case of the cascade system leads to the condition $\gamma_2 \ll \gamma_1$.

By analogy with (6.5), we can obtain expressions for the time-independent populations when one detuning is zero and the other is scanned. Figure 10 shows ρ_{33} as a function one of the detunings. It is clear that the functions ρ_{33} $(\Omega_1, \Omega_2=0)$ and ρ_{33} $(\Omega_2, \Omega_1=0)$ are different in character. If the field applied to the $|3\rangle-|2\rangle$ transition is in exact resonance $(\Omega_2=0)$, and the field applied to the $|1\rangle-|3\rangle$ transition is scanned, then level $|3\rangle$ remains practically unpopulated when the detuning Ω_1 is large. When $\Omega_1=0$ and Ω_2 is scanned, we find that for large Ω_2 we obtain a saturated (for $g_1^2 \gg \gamma_1\gamma_2$) two-level system $|1\rangle-|3\rangle$ that is weakly coupled to the upper level $|2\rangle$. In both cases we then have a typical well-defined coherent-trapping resonance at $\Omega_m=0$ $(m=1,2)$. We note that CPT in a cascade level system has been observed experimentally.[69] A fast $^{20}Ne^*$ atomic beam was illuminated by two colinear laser beams from the same source. One of the beams was parallel to the atomic beam and the other antiparallel. This meant that a change in the frequency of the laser was accompanied by a change in the total detuning $\Omega_1+\Omega_2$ from the two-photon resonance. On the other hand, the single-photon detunings Ω_1 (of the $|1\rangle-|3\rangle$ transition) and Ω_2 (for the $|3\rangle-|2\rangle$ transition) could be varied by varying the beam velocity by exploiting the change in the Doppler shift of the resonance frequencies. The following $^{20}Ne^*$ levels were used in the cascade scheme: $3s[3/2]_2$, $3p'[3/2]_2$, and $4d'[5/2]_3$ were taken to be the lower 1, intermediate 3, and upper 2 levels, respectively (Fig. 11a). The fluorescence from the intermediate level was recorded. We note that the rates of spontaneous relaxation from levels $|3\rangle$ and $|2\rangle$ were $\gamma_1 \equiv \gamma_{3\to1} \approx 21$ MHz and $\gamma_2 \equiv \gamma_{2\to3} \approx 1$ MHz, respectively, i.e., the condition $\gamma_2 \ll \gamma_1$ for CPT in a cascade scheme was satisfied. When the Rabi frequency g_1 is small, the spectrum of the signal as a function of the laser frequency consists of the Doppler profile with a sharp peak superimposed upon it, which is a 'reflection' of the two-photon absorption peak in the population of the upper level $|2\rangle$, and is due to the $|2\rangle-|3\rangle$ spontaneous decay. The peak becomes accompanied by a valley as the Rabi frequency is increased. The valley on the intermediate-level population as a function of the detuning $\Omega_1+\Omega_2$ is therefore observed for $\Omega_1+\Omega_2=0$, $\gamma_2 \ll \gamma_1$, $g_{1,2}^2 \gg \gamma_1\gamma_2$, which indicates the presence of CPT in this experimental situation. Figure 11b shows the fluoresence intensity from level $|3\rangle$ as a function of the laser detuning for different velocities of the atomic beam (governed by the accerlating voltage applied to the primary ion beam).[69]

We note, finally, that the excitation of the Ξ-system by correlated fields is discussed in Ref. 13. The main properties of the excitation process are found to be the same as for the Λ-system (Sec. 4.3) with the only difference that the transverse relaxation rate Γ is replaced with γ_2 in the case of the Ξ-system and negative cross correlation of the fields

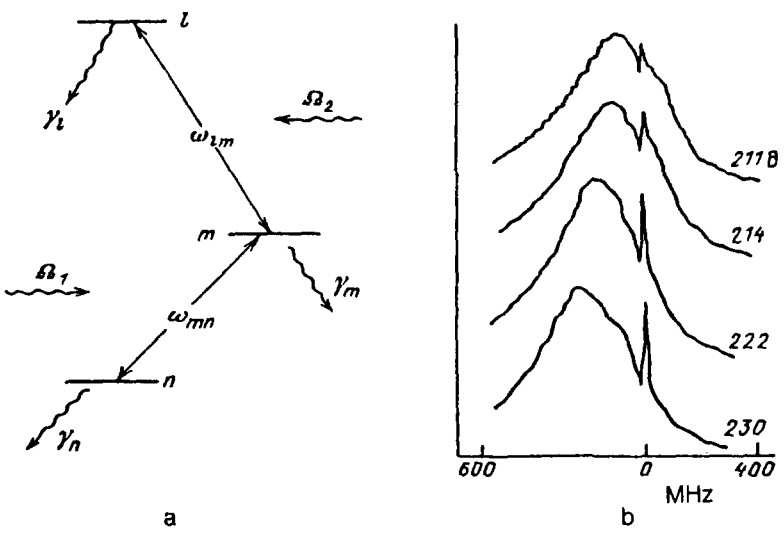

FIG. 11. a—Fragment of the energy-level diagram of $^{20}Ne^*$. Levels (n,l,m) correspond to the $(3s[3/2]_2$, $3p'[3/2]_2$, $4d'[5/2]_3)$ levels of neon; b—population of the intermediate level $|3\rangle$ as a function of the laser detuning $g_1=7$ MHz, $g_2=2$ (from Ref. 69).

is necessary for the observation of CPT in this case.

7. COHERENT TRAPPING IN SYSTEMS WITH DEGENERACY

So far, we have confined our attention to different types of the three-level systems (Fig. 1). However, more complicated multilevel systems can also display properties associated with the phenomenon of coherent population trapping. Such systems include, for example, the two-level systems that are degenerate in magnetic sublevels and interact with polarized radiation. Here again there are states from which a system will not be excited.

Following Ref. 47, we consider a two-level system that is degenerate in magnetic sublevels $E_n(J_n)\to E_m(J_m)$, $((E_n<E_m)$ where J_n, J_m are the total angular momenta of levels with energy $E_{n,m})$ with a complete set of orthonormal wave functions

$$\left\{\exp\left(-i\frac{E_n}{\hbar}t\right)\Psi_k^n(|k|\leqslant J_n),\right.$$

$$\left.\exp\left(-i\frac{E_m}{\hbar}t\right)\Psi_j^m(|j|\leqslant J_m)\right\}.$$

We shall consider that this two-level system interacts with the field given by

$$\mathbf{E}=E_0\mathbf{e}\exp(-i\omega t)+\text{c.c.}, \tag{7.1}$$

where $\mathbf{e}$ is the elliptic polarization vector and the quantization axis (the z axis) is orthogonal to $\mathbf{e}$. The field (7.1) can be written in the form

$$\mathbf{E}=E_0(q_+\mathbf{e}_++q_-\mathbf{e}_-)\exp(-i\omega t)+\text{c.c.}, \tag{7.2}$$

and in this case the dipole interaction operator is

$$\hat{V}=E_0(q_+\hat{d}_-+q_-\hat{d}_+)\exp(-i\omega t)+\text{h.c.}, \tag{7.3}$$

where

$$q_\pm=\exp(\mp i\varphi)\sin\left(\varepsilon+\frac{\pi}{4}\right)\cos\left(\varepsilon+\frac{\pi}{4}\right),$$

ε is the ellipticity of the light beam $(-\pi/4\leqslant\varepsilon\leqslant\pi/4)$, and φ is the angle between the semimajor axis of the ellipse and the x axis.

As before (Sec. 2) we start with the time-dependent Schrödinger equation and seek its solution in the form

$$\Psi=\exp\left(-i\frac{E_m}{\hbar}t\right)\sum_{|j|<J_m}a_j(t)\Psi_j^m$$

$$+\exp\left(-i\frac{E_n}{\hbar}t\right)\sum_{|k|<J_m}b_k(t)\Psi_k^n. \tag{7.4}$$

Substition of (7.4) in the time-dependent Schrödinger equation gives a set of differential equations of the form of (2.4).

The main interest is in solutions of this system that are not perturbed by the light field, i.e., solutions that satisfy the condition (see also Sec. 8)

$$\hat{V}\Psi=0. \tag{7.5}$$

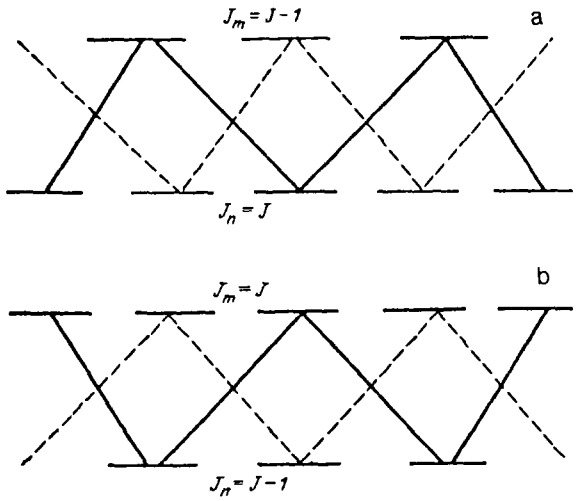

FIG. 12. a—Λ-chain for the $J\to J-1$ transition; b—V-chain for the $J-1\to J$ transition.

Nontrivial solutions of (7.5) maybe referred to as stationary coherent states or, simply, coherent states (see Sec. 2).

Subsequent analysis, performed in Ref. 47, examined different cases of total angular momenta $J_{n,m}$.

For example, Fig. 12 shows two special cases: first transition $J\to J-1$ and second transition $J-1\to J$. It is clear that for the $J\to J-1$ transition, the two-level degenerate system splits into Λ-chains, and for the $J\to 1\to J$ transition it splits into V-chains of levels.

The same method is used in both cases to obtain the coherent states. It involves finding a nontrivial solution of (7.5). Coherent states were obtained in Ref. 47 for both Λ- and V-chains. Naturally, coherent states are constructed from the initial amplitudes of the lower levels in the case of Λ-chains and the upper levels in the case of V-chains, which corresponds to our simple approach to Λ- and V-systems (Sec. 2). As for a single Λ-system, the two-level degenerate system is found to have states that are wholly determined by initial amplitudes and are not excited by the light field (7.1).

Further generalizations to the case of a two-level system with relaxation from the upper level, introduced in Ref. 47, are analogous to those discussed in Sec. 3 for three-level systems. The analogy is indeed so complete that coherent trapping states were found in Ref. 47 for both the Λ-chain and the Λ-system, but no such states were found for the V-chain (as for the V-scheme of Sec. 3.2).

8. CONSERVATION LAWS AND COHERENT POPULATION TRAPPING IN *N*-LEVEL QUANTUM SYSTEMS

So far, we have been largely concerned with CPT in three-level systems. However, it is found that coherent population trapping can occur in a wider class of models of interaction between atomic systems and coherent fields. The effect is observed under certain conditions in systems with an arbitrary number $N>2$ of levels participating in the interaction. The conditions that must be satisfied for

CPT to occur determine the symmetry properties of the system under investigation and, in this sense, CPT is a consequence of the symmetries of atom-plus-field system. We know, however, that symmetries give rise to specific integrals of motion (e.g., via the Noether theorem). It follows that CPT can be associated with a particular integral of motion, and this presents us with a clear mathematical criterion for the onset or otherwise of CPT. This "symmetry approach" to coherent processes in complex quantum systems is promising, but, so far, it has been developed in sufficient detail and depth for only one (though very extensive) branch of coherent optics, namely, the interaction of a set of discrete levels with short laser pulses for which slow relaxation processes in the system can be neglected. It is precisely such systems that demonstrate particularly clearly the fundamental importance of the symmetry properties of the atom-plus-field system in coherent phenomena, including CPT.

These problems are solved in Refs. 48–57. The first publications on the dynamics of the interaction of an N-level atom with pulses of radiation were Refs. 48 and 49 in which this was tackled in the language of rotations [in the SU(N) group of transformations] of the state vector in Hilbert space.

The evolution of a quantum system is described by the Liouville equation

$$i\hbar\frac{\partial\hat{\rho}}{\partial t}=[\hat{H},\hat{\rho}]. \tag{8.1}$$

The two operators in (8.1) can be expressed in terms of N^2-1 generators $\hat{S}_j$ of SU(N):

$$\hat{\rho}(t)=N^{-1}\hat{I}+\frac{1}{2}\sum_{j=1}^{N^2-1}S_{j(t)}\hat{S}_j, \tag{8.2}$$

$$\hat{H}(t)=N^{-1}\left(\sum_{k=1}^{N}E_k\right)\hat{I}+\frac{1}{2}\hbar\sum_{j=1}^{N^2-1}\Gamma_j(t)\hat{S}_j, \tag{8.3}$$

where I is the unity operator and E_k is the energy of the kth level. The operator $\hat{S}_j$ is constructed as follows. First, we introduce the operators $\hat{u}_{jk}=(\hat{P}_{jk}+\hat{P}_{kj})$, $\hat{v}_{jk}=-i(\hat{P}_{jk}-\hat{P}_{kj})$ and $\hat{w}_l=-[2/l(l+1)]^{1/2}$ $(\hat{P}_{11}+\ldots+\hat{P}_{ll}-l\hat{P}_{l+1,l+1})$ where $1\leqslant j<k\leqslant N$ and $1\leqslant l\leqslant N-1$ and the "projection operators" are $\hat{P}_{mn}=|m\rangle\langle n|$, in which $|m\rangle$ are the eigenfunctions of the unperturbed Hamiltonian. We shall use $\hat{S}_j$ to denote the elements of the ordered set $\hat{u}_{jk}$, $\hat{v}_{jk}$,$\hat{w}_l$. It is then readily verified that

$$[\hat{S}_j,\hat{S}_k]=2i\sum_{l=1}^{N^2-1}f_{jkl}\hat{S}_l, \tag{8.4}$$

$$\mathrm{Tr}\{\hat{S}_j\hat{S}_k\}=2\delta_{jk}, \tag{8.5}$$

where δ_{ik} is the Kronecker symbol and f_{ikl} are completely antisymmetric structures of the SU(N) algebra. Because of the completeness property of (8.5), the coefficients in (8.2) and (8.3) can be written in the form

$$S_j(t)=\mathrm{Tr}\{\hat{\rho}(t)\hat{S}_j\}, \tag{8.6}$$

$$\hbar\Gamma_j(t)=\mathrm{Tr}\{\hat{H}(t)\hat{S}_j\}. \tag{8.7}$$

Substituting (8.2) and (8.3) in (8.1), we obtain

$$\frac{\mathrm{d}S_j(t)}{\mathrm{d}t}=\sum_{k=1}^{N^2-1}\sum_{l=1}^{N^2-1}f_{ikl}\Gamma_kS_l, \tag{8.8}$$

which describes the evolution of the "coherent vector" $S(t)=(S_1,S_2,\ldots,S_{N^2-1})$ in N^2-1-dimensional space. The complete antisymmetry f_{ikl} ensures that the length S is conserved, so that the evolution of the coherent vector constitutes its rotation.

Some of the advantages of this vector description are demonstrated in Ref. 49. In particular it is shown that the conservation of the length of the vector, $|\mathbf{S}|^2$, alone means that total inversion is impossible in the quantum system excited by a sequence of laser pulses. Complete inversion, for example, in a three-level system, implies a situation in which the entire population is confined to a single level (level $|3\rangle$ in Fig. 1a). It follows that part of the population should be trapped by the lower levels (levels $|1\rangle$ and $|2\rangle$ in Fig. 1a) independently of the laser-pulse characteristics.

We now note one further fact. We know (see, for example, Refs. 48, 49, and 58) that, for the density matrix of the N-level quantum system satisfying the Liouville equation (8.1), the quantities

$$C_n=\mathrm{Tr}\{\hat{\rho}^n(t)\},\quad n=1,2,\ldots,N \tag{8.9}$$

are constants of motion [which is readily verified by direct substitution of (8.9) in (8.1)]. It is not difficult to show that $|\mathbf{S}|^2$ and $C_2=\mathrm{Tr}\hat{\rho}^2$ can be expressed in terms of one another. Hence, for a 3-level system[48]

$$C_2=\tfrac{1}{3}+\tfrac{1}{2}|\mathbf{S}|^2. \tag{8.10}$$

In this sense, we can say that population trapping in quantum systems is a consequence of the conservation of $\mathrm{Tr}\,\hat{\rho}^2$.

Subsequent work[51–55] has demonstrated the power of the formalism based on the description of the dynamic evolution of quantum systems in terms of the state vector $\mathbf{S}$ constructed with the help of the generators of the SU(N) algebra. By using the properties of the coefficients Γ_j, determined by the structure of the Hamiltonian (8.3), we can find the integrals of motion of the systems[51–55] for a variety of physically interesting cases; we can also obtain analytic descriptions of the evolution of the density matrix[53,55,57] and deduce qualitatively predictions about the motion of these quantum systems.[51–57] From the standpoint of symmetry properties, the models used for the systems discussed in quantum optics are often found to be analogous to those used in nuclear physics and in quark physics.[51,54]

In our view, this offers considerable possibilities for the study of the dynamics of complex quantum systems.

Next, let us consider the results obtained on coherent population trapping, and return to the possibility of expressing the operators $\hat{\rho}(t)$ and $\hat{H}(t)$ in terms of the $\hat{S}_j$ in (8.2) and (8.3). In principle, the $N\times N$ matrices that represent the density matrix $\hat{\rho}(t)$ and the Hamiltonian $\hat{H}(t)$ of any system can be expanded over the N^2 generators of the $U(N)$ algebra. However, the choice of the basis operators is very crucial if we try to exploit the dynamic symmetry of the system. Following Refs. 51 and 52, we shall assume that the system is dynamically symmetric if the Hamiltonian $\hat{H}(t)$ is expressed in terms of a particular subsystem of N^2 basis operators. The linear space resting on the vector **S** can then be factorized into a direct product of smaller independent subspaces. The same system can lead to different realizations of the $U(N)$ symmetry under different physical conditions. Thus, for a three-level system in two-photon resonance, the Hamiltonian $\hat{H}(t)$ is expressed in terms of only the generators of the subgroup $SU(2)\times U(1)$ of $SU(3)$ (Gell–Mann symmetry), and for equal and time independent detunings and Rabi frequencies, in terms of the generators of the subgroup $O(3)$ of $SU(3)$ (Elliott symmetry).

Multiphoton resonance is the necessary condition for coherent population trapping. We shall therefore be interested in the consequences of Gell–Mann type symmetries in the system. The concept of Gell–Mann symmetry of an N-level system will be defined later. For the moment, lets us consider the well known three-level system without relaxation (Sec. 2) for which both the CPT conditions and the necessary criteria are well-known. Thus, it is shown in Ref. 9 that the three-level system in two-photon resonance has a particular state characterized by the linear combination

$$r(t)=a_1(t)-\frac{g_2}{(g_1^2+g_2^2)^{1/2}}-a_2(t)\,\frac{g_1}{(g_1^2+g_2^2)^{1/2}}=\text{const}, \tag{8.11}$$

where $a_1(t)$ and $a_2(t)$ are the amplitudes of the "initial" and "final" states $|1\rangle$ and $|2\rangle$, g_1 and g_2 are the Rabi frequencies of fields applied to the $|1\rangle-|3\rangle$ and $|2\rangle-|3\rangle$ transitions, respectively, and $|3\rangle$ is the intermediate state. This state is decoupled from the remainder of the system, i.e., it does not interact with applied fields, so that $r(t)$ remains constant in time. In particular, if the state of the atom is prepared in advance so that $r(0)=1$, then it will not interact at all with resonant fields and its population will remain distributed over levels $|1\rangle$ and $|2\rangle$ in accordance with $r=1$. This means that the integral of motion $r(t)=\text{const}$ can serve under these conditions as the criterion for the formation of the CPT state in the three-level system.

The generators $\hat{S}_j$ of the $SU(3)$ algebra over which the density matrix and the Hamiltonian $\hat{H}(t)$ in (8.2) and (8.3) are resolved, take the form of the well known Gell–Mann matrices[51] in the case of the three-level system:

$$\hat{S}_1=\begin{vmatrix}0&1&0\\1&0&0\\1&0&0\end{vmatrix},\quad \hat{S}_2=\begin{vmatrix}0&-i&0\\i&0&0\\0&0&0\end{vmatrix},$$
$$\hat{S}_3=\begin{vmatrix}1&0&0\\0&-1&0\\0&0&0\end{vmatrix},\quad \hat{S}_4=\begin{vmatrix}0&0&-i\\0&0&0\\i&0&0\end{vmatrix},$$
$$\hat{S}_5=\begin{vmatrix}0&0&0\\0&0&1\\0&1&0\end{vmatrix},\quad \hat{S}_6=\begin{vmatrix}0&0&0\\0&0&-i\\0&i&0\end{vmatrix},$$
$$\hat{S}_7=\begin{vmatrix}0&0&1\\0&0&0\\1&0&0\end{vmatrix},\quad \hat{S}_8=\frac{1}{\sqrt{3}}\begin{vmatrix}1&0&0\\0&1&0\\0&0&-2\end{vmatrix}. \tag{8.12}$$

They form three combinations: $\hat{S}_1,\hat{S}_2,\hat{S}_3$ belong to group A of isospin components, $\hat{S}_4,\hat{S}_5,\hat{S}_6,\hat{S}_7$ belong to group B of operators that mix states of different strangeness, and $\hat{S}_8$ belong to group C which, for $SU(3)$, consists of a single strangeness operator. The commutation relations for the operators in these groups can be written symbollically in the form

$$[\hat{A},\hat{A}]=\hat{A},\quad [\hat{A},\hat{B}]=\hat{B},\quad [\hat{A},\hat{C}]=0,$$
$$[\hat{B},\hat{B}]=\hat{A}+\hat{C},\quad [\hat{B},\hat{C}]=\hat{B},\quad [\hat{C},\hat{C}]=0, \tag{8.13}$$

where, for example, $[\hat{A},\hat{A}]=\hat{A}$ shows that the commutator of two different members of group A is equal to a member of group A (possibly multiplied by a constant).

However, it is readily verified that, when the generators (8.12) are chosen as the basis operators, the Hamiltonian $\hat{H}$ contains the operators from all three groups A,B,C under all conditions, and the set of equations of motion given by (8.8) for the coherent vector **S** will not exhibit any special symmetry properties, e.g., it will not factorize into smaller independent subsystems.

If on the other hand, we take the basis operators

$$\hat{\Lambda}_j=\hat{U}+\hat{S}_j\hat{U}, \tag{8.14}$$

obtained from $\hat{S}_j$ by a unitary transformation with the matrix

$$U=U^+=\frac{1}{\varepsilon}\begin{vmatrix}g_1&0&g_2\\0&\varepsilon&0\\g_2&0&-g_1\end{vmatrix}, \tag{8.15}$$

where $\varepsilon=(g_1^2+g_2^2)^{1/2}$, the Hamiltonian $\hat{H}(t)$ will involve only the operators from the groups $A=\{\hat{\Lambda}_1,\hat{\Lambda}_2,\hat{\Lambda}_3\}$ and $C=\{\hat{\Lambda}_8\}$ in two-photon resonance:

$$\hat{H}=\frac{1}{2}\hbar\left(-2\varepsilon\hat{\Lambda}_1+\Delta\cdot\hat{\Lambda}_3-\frac{1}{\sqrt{3}}\Delta\cdot\hat{\Lambda}_8\right), \tag{8.16}$$

where Δ is the single-photon detuning.

The dynamic space of the Liouville equation (8.1) for the observables $\Lambda_j(t)=\mathrm{Tr}\{\hat{\rho}(t)\hat{\Lambda}_j\}$ can be resolved into three independent subspaces, so that

$$\frac{d}{dt}\begin{vmatrix}\Lambda_1(t)\\ \Lambda_2(t)\\ \vdots\\ \Lambda_8(t)\end{vmatrix}=\begin{vmatrix}H^{\rm I}_{3\times3} & O_{3\times4} & O_{3\times1}\\ O_{4\times3} & H^{\rm II}_{4\times4} & O_{4\times1}\\ O_{1\times3} & O_{1\times4} & O\end{vmatrix}\times\begin{vmatrix}\Lambda_1(t)\\ \Lambda_2(t)\\ \vdots\\ \Lambda_8(t)\end{vmatrix}, \tag{8.17}$$

where $O_{m\times n}$ are $m\times n$ zero matrices and $H^{\rm I}_{3\times3}$ and $H^{\rm II}_{4\times4}$ are antisymmetric 3×4 and 4×4 matrices respectively; their matrix elements (j,k) are equal to $(2i\hbar)^{-1}\mathrm{Tr}\{\hat H(t)[\hat\Lambda_j,\hat\Lambda_k]\}$. The conservation laws follow immediately from (8.17):

$$\Lambda_1^2(t)+\Lambda_2^2(t)+\Lambda_3^2(t)=\text{const}, \tag{8.18a}$$

$$\Lambda_4^2(t)+\Lambda_5^2(t)+\Lambda_6^2(t)+\Lambda_7^2(t)=\text{const}, \tag{8.18b}$$

$$\Lambda_8(t)=\text{const}. \tag{8.18c}$$

In particular, the combination of population and coherence

$$\Lambda_8(t)=\varepsilon^{-2}[(g_1^2-2g_2^2)\rho_{11}(t)+\varepsilon^2\rho_{22}(t)+(g_2^2-2g_1^2)\rho_{33}(t)+3g_1g_2u_{12}(t)] \tag{8.19}$$

remains constant in time. If we now subtract $\rho_{11}+\rho_{22}+\rho_{33}=1$ from (8.19), we obtain the square of the modulus of the complex constant of motion (8.11) that characterizes the establishment of CPT in the three-level system.

The symmetry approach to the Liouville equation thus enables us to confirm the conclusion that the three-level system has a special 'closed' state in the case of two-photon resonance. On the other hand, systems in which the number of levels is $N>3$ have no known integrals of motion such as (8.11) that are responsible for the formation of superposition states that are not coupled by an interaction to the remainder of the system. The above dynamic symmetry formalism has been used to establish such integrals for a number of important cases.[52,54,56] One of them is the multilevel system with Gell–Mann symmetry.[54] It is analyzed by analogy with three-level systems. Instead of the Hamiltonian (for two-photon resonance)

$$\hat H=-\hbar\begin{vmatrix}0 & g_1(t) & 0\\ g_1(t) & \Delta(t) & g_2(t)\\ 0 & g_2(t) & 0\end{vmatrix}, \tag{8.20}$$

expressed in terms of the generators $\hat\Lambda_j$ of the group SU(3) with the help of (8.16), we can introduce the transformed Hamiltonian

$$\hat H'=\hat U\hat H\hat U^+, \tag{8.21}$$

which, clearly, can be written in the form

$$\hat H'=\frac{1}{2}\hbar\left(-2\varepsilon\hat S_1+\Delta\cdot\hat S_3-\frac{1}{\sqrt3}\Delta\cdot\hat S_8\right). \tag{8.22}$$

By analogy with the eight Gell–Mann matrices given by (8.12), the SU(N) generators can be taken to be the following $N\times N$ matrices. The "isospin" matrices $\hat S_1,\hat S_2,\hat S_3$ in group A with the Pauli matrices

$$\hat\sigma_x=\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix},\quad \hat\sigma_y=\begin{vmatrix}0 & -i\\ i & 0\end{vmatrix},\quad \hat\sigma_z=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}$$

in the top left hand corner and zeros elsewhere:

$$\hat S_i=\begin{vmatrix}\hat\sigma_i & O_{2\times(N-2)}\\ O_{(N-2)\times2} & O_{(N-2)\times(N-2)}\end{vmatrix}\quad (i=1,2,3); \tag{8.23a}$$

the $N-2$ diagonal matrics in group C, denoted by $\hat S_{n^2-1}(n=3,4,...,N)$ and given by

$$\hat S_{n^2-1}=\frac{1}{\left[\dfrac{n(n-1)}{2}\right]^{1/2}}\times\begin{vmatrix}\hat I_{n-1} & O_{(n-1)\times1} & O_{(n-1)\times(N-n)}\\ O_{1\times(n-1)} & -(n-1) & O_{1\times(N-n)}\\ O_{(N-n)\times(n-1)} & O_{(N-n)\times1} & O_{(N-n)\times(N-n)}\end{vmatrix}, \tag{8.23b}$$

where $\hat I_m$ is the $m\times m$ unit matrix; and the remaining N^2-N-2 generators in group B can be readily obtained. Thus, the important point here is that the N^2-1 generators from SU(N), introduced in this way, satisfy the commutation relations given by (8.13). In principle, this enables us to find for systems with particular properties of their Hamiltonian a unitary transformation $\hat U$ such that the transformed Hamiltonian can be expressed in terms of only the generators from A and C, i.e.,

$$\hat H'(t)=\hat U\hat H\hat U^+=\sum_{j=1}^{3}a_j(t)\hat S_j+\sum_{j=3}^{N}c_{j^2-1}(t)\hat S_{j^2-1}. \tag{8.24}$$

Such systems are called systems with Gell–Mann symmetry in Ref. 54. The explict form of the Hamiltonian $\hat H$ and of the corresponding transformations $\hat U$ is given in Ref. 54 together with the conditions that must be satisfied by the interaction parameters in these operators for Gell–Mann symmetry. Briefly, the interaction of N-level systems and the field due to laser pulses should be as follows: (1) the only $|i\rangle-|j\rangle$ transitions that are excited are those with odd $|i-j|$ (in accordance with the selection rule for electromagnetic transitions), (2) the Rabi frequencies $g_{ij}(t)$ of all the fields must have the same time dependence, and (3) the multiphoton resonance must be of a special type, namely, it must be a two-photon resonance with equal single-photon detunings. We note that the most important consequence of Gell–Mann symmetry in $\hat H(t)$ is that the (N^2-1)-dimensional dynamic space resolves into three groups of independent subspaces, namely, (a) a three-dimensional subspace (A group), (b) a (N^2-N-2)-dimensional subspace (B group), (c) $N-2$ one-dimensional subspaces (c). This, in turn, leads to the following set of constants of motion

$$\sum_{j=1}^{3}[\Lambda_j(t)]^2=\text{const}, \tag{8.25a}$$

$$\Lambda_{n^2-1}(t)=\text{const}\quad (n=3,4,...,N), \tag{8.25b}$$

$$\mathrm{Tr}\hat{\rho}^2(t)-\sum_{j=1}^{3}[\Lambda_j(t)]^2-\sum_{n=3}^{N}[\Lambda_{n^2-1}(t)]^2=\mathrm{const}, \tag{8.25c}$$

where

$$\Lambda_j(t)=\mathrm{Tr}\{\hat{\rho}(t)\hat{\Lambda}_j\},$$

and

$$\Lambda_j=\hat{U}+\hat{S}_j\hat{U}.$$

The constants in (8.25b) determine the time-independent linear combinations of populations and coherences in multilevel systems that characterize special superposition states that are not coupled by interaction to the remainder of the system.

The following are two examples of the constant (8.25b) for a four-level system with Gell–Mann symmetry:

$$\Lambda_8(t)=\frac{1}{\sqrt{3}}\{(g_{12}^2+g_{23}^2)^{-1}[(g_{12}^2-2g_{23}^2)\rho_{11}+(g_{23}^2-2g_{12}^2)\rho_{33}+3g_{12}g_{23}u_{13}]+(g_{23}^2+g_{34}^2)^{-1}(g_{23}2\rho_{22}+g_{34}^2\rho_{44}+g_{23}g_{34}u_{24})\},$$

$$\Lambda_{15}(t)=\frac{1}{\sqrt{6}}\{\rho_{11}+\rho_{33}+(g_{23}^2+g_{34}^2)^{-1}[(g_{23}^2-3g_{34}^2)\rho_{22}+(g_{34}^2-3g_{23}^2)\rho_{44}+4g_{23}g_{34}u_{24}]\},$$

where g_{ij} is the Rabi frequency of the field that couples the $|i\rangle-|j\rangle$ transition.

The existence of states that do not interact with applied fields in multiphoton resonances can be clearly demonstrated for atoms in a pure state, i.e., in the Schrödinger formalism.[47,56,59,72] Moreover, it is possible to find CPT-type integrals of motion such as (8.11) and (8.25b) for a somewhat wider class of conditions than in the case of systems with Gell–Mann symmetry. The conditions that must be satisfied by the Rabi frequencies g_{ij} are practically the same:

$$\begin{aligned}&\text{(a) } g_{ij}=0, \quad |i-j| \text{ even},\\ &\text{(b) } g_{ij}(t)=g_{ij}f(t) \quad |i-j| \text{ odd and } i \text{ odd}\\ &\qquad\quad =g_{ij}f^*(t) \quad |i-j| \text{ odd and } i \text{ even},\end{aligned} \tag{8.26}$$

and at the same time the multiphoton resonance can be achieved in different ways.

CPT studies in the formalism of state amplitudes are based on the obvious but not very fruitful idea[47,56] that CPT must be due to states with wave function

$$\Psi_s=\sum_{i=1}^{N}k_i\psi_i,$$

where ψ_i are the eigenfunctions of the unperturbed Hamiltonian for which, in the resonant approximation,

$$\hat{V}\Psi_s=0; \tag{8.27}$$

where $\hat{V}$ is the operator representing the dipole interaction between the atom and the field. If we know Ψ_s we can readily obtain the constant of motion responsible for CPT by multiplying the Schrödinger equation for the wave function of the atom

$$\Psi=\sum_{i=1}^{N}a_i\psi_i$$

in the resonance approximation from the left by the row vector Ψ_s^+. The result is that, since $\Psi_s^+\hat{V}=0$ and $\Psi_s(t)=\mathrm{const}$, we have

$$\frac{\partial}{\partial t}(\Psi_s^+\Psi)=0, \tag{8.28}$$

or

$$\sum_{i=1}^{N}k_i^*a_i=\mathrm{const}. \tag{8.29}$$

It is often possible to find the explicit form of the wave functions Ψ_s satisfying (8.27). This was done in Ref. 47 for the two-level atom with levels that were degenerate in the angular momentum, and in Ref. 56 for the N-level atom in multiphoton resonance. The following two cases were examined:

1) A special type of multiphoton resonance, i.e. two-photon resonance; odd level number N. According to (8.26) the determinant of the matrix $\hat{V}$ is then zero, so that the eigenvalue of $\hat{V}$ is is also zero. Hence Ψ_s is the eigenvector of $\hat{V}$ that corresponds to its zero eigenvalue. Again, by (8.26), the wave function Ψ_s contains only the odd i component that can be expressed in terms of, for example, k_1:

$$k_{2j+1}=-(D_{2j+1}/D)k_1 \quad (j=1,2,\ldots,(N-1)/2), \tag{8.30}$$

where $D=\det A, D_{2j+1}=\det A_{2j+1}$, A is the Rabi frequency g_{ij}

$$A=\begin{vmatrix} g_{23} & g_{25} & \cdots & g_{2N}\\ g_{43} & g_{45} & \cdots & g_{4N}\\ \vdots & \vdots & & \vdots\\ g_{N-1,3} & g_{N-1,5} & \cdots & g_{N-1,N}\end{vmatrix},$$

and A_{2j+1} is obtained from A by replacing the $(2j+1)$th column with the column

$$a_1=\begin{vmatrix} g_{21}\\ g_{41}\\ \vdots\\ g_{N-1,1}\end{vmatrix}.$$

The noninteracting coherent superposition state is formed from odd levels. The constant (8.29) can then be written in the form

$$a_1(t)-\sum_{j=1}^{(N-1)/2}\frac{D_{2j+1}^*}{D^*}a_{2j+1}(t)=const, \tag{8.31}$$

which, as can be readily verified, is identical for $N=3$ with (8.11) apart from a multiplying factor.

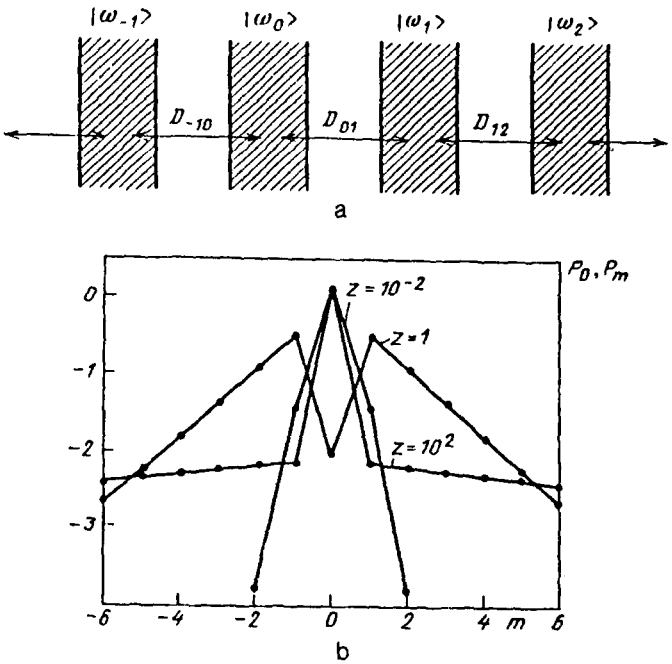

FIG. 13. a—Schematic of coupled continua; b—total population of different continua for three values of z (m represents the mth continuum).

2) An r-photon resonance occurs (r odd and $2<r<N$) between any two levels in the atom under consideration. If

$$\frac{g_{21}}{g_{2,r+1}}=\frac{g_{41}}{g_{4,r+1}}=\ldots=\frac{g_{N-1,1}}{g_{N-1,r+1}}=\varkappa \tag{8.32}$$

then there exists a vector Ψ_s satisfying (8.27) whose only two nonzero components are related by

$$k_{r+1}=-\varkappa k_1, \tag{8.33}$$

i.e., the noninteracting coherent superposition state arises from levels $|1\rangle$ and $|r+1\rangle$. The corresponding constant of motion is

$$g^*_{2,r+1}a_1(t)-g_{21}a^*(t)=\text{const}. \tag{8.34}$$

It follows that multilevel quantum systems pumped by coherent fields in multiphoton resonance have special superposition states that are not coupled to the rest of the system. The population occupying these states does not interact with the external pump fields, i.e., we have coherent population trapping. The parameters of the resulting superposition states can be determined by determining the kernel of the atom-field interaction operator (8.27). The CPT produced in this way is characterized by conservation laws of a particular type [such as (8.25b) and (8.29)] that are allowed by the equations of motion of the pumped quantum system. These equations of motion can serve as criteria for the onset of CPT, and the conditions for them are then the necessary conditions or CPT.

9. COHERENT POPULATION TRAPPING IN THE CONTINUUM

The CPT phenomenon was earlier related to the presence of discrete levels in the system. Under certain conditions, a superposition of these levels can be constructed and the population is trapped by it. The question now is: can CPT occur in the absence of discrete states?

It is shown in Refs. 20, 60, and 61 that population trapping is also possible in the continuum, e.g., when an autoinizing resonance is excited by intense laser radiation.

We shall now consider a quantum system in which in place of individual states there are bands of continuous states. We shall essentially follow Refs. 62–66 and consider an infinite set of separate continuous structureless bands. We shall suppose that the entire population is initially concentrated in a single continuum $|\omega_0\rangle$, the continua are coupled by the coherent filed through the operator $\hat{D}$, and each continuum is coupled only to its nearest neighbors (Fig. 13a).

In the Schrödinger picture, the wave function of the system is

$$\Psi(t)=\sum_m \int d\omega_m C_{\omega_m}|\omega_m\rangle. \tag{9.1}$$

The equations of motion are

$$\dot{C}_{\omega_0}=-iE_{\omega_0}C_{\omega_0}-i\int d\omega_1 D_{0,1}C_{\omega_1}-i\int d\omega_{-1}D_{0,-1}C_{\omega_{-1}}, \tag{9.2}$$

$$\dot{C}_{\omega_m}=-iE_{\omega_m}C_{\omega_m}-i\int d\omega_{m+1}D_{m,m+1}C_{\omega_{m+1}}$$

$$-i\int d_{m-1}D_{m,m-1}C_{\omega_{m-1}},$$

where E_ω is the continuum energy and $D_{m,m\pm1}=\langle\omega_{m\pm1}|\hat{D}|\omega_m\rangle$. We shall assume that the initial probability amplitude distribution in the continuum $|\omega_0\rangle$ is given by

$$C_{\omega_0}=(\gamma_0/\pi)^{1/2}(\omega_0+i\gamma_0)^{-1}, \tag{9.3}$$

where γ_0 is the width of the distribution of states in the continuum. The equations of motion (9.2) can be solved by means of the Laplace transformation. Simple analytic results are obtained when all the Ds are equal, i.e., $D_{m,m\pm 1}\equiv D$, and we consider the steady-state limit $t\to\infty$. The continuum populations are then given by

$$P_0=|(1-2F)/(1+2F)|^2,$$
$$P_m=\left|\frac{2}{1+2F}\left(\frac{F}{\pi D}\right)^{|m|}\right|^2, \tag{9.4}$$

where

$$F=-\tfrac{1}{2}[1-(1+4\pi^2D^2)^{1/2}].$$

It follows from (9.4) that the population does not become uniform even for $D\to\infty$. We now put $z=\pi^2D^2$ which is an effective parameter that can be interpreted as the coupling strength. In the limit of strong coupling, ($z\gg 1$) $P_0=1$, $P_m=0$, $m=\pm 1,\pm 2,\ldots$. Here we find coherent population trapping in the initial continuum $|\omega_0\rangle$. However, when $z\sim 1$ ($z=3/4$) we find that $P_0=0$ and, as noted in Ref. 63, we have population 'antitrapping.' Figure 13b shows the continuum populations as functions of the coupling strength parameter.

Coherent trapping in discrete states occurs only when there are two transition channels to a single excited state and compensation of these transitions takes place. The picture in continuum-continuum transitions is very similar: there are two channels of departure from a given continuum and, when there is strong coupling, they may compensate one another. The result is that the continuum that was occupied by the atoms at $t=0$ will be occupied. Estimates reported in Ref. 62 suggest that CPT is possible in the continuum-continuum system for laser intensities in excess of 10^{14} W cm^{-2} (as indicated by autoionization experiments).

However, this differs fundamentally from CPT in discrete systems. The excited-state population in discrete quantum states is close to zero independently of the coupling strength, whereas in continuum-continuum systems this coupling is very significant and coherent 'antitrapping' in which the atoms leave the initial continuum is possible.

The results reported in Ref. 62 were generalized to arbitrary time in Ref. 63. It was shown that

$$|C_0(t)|^2=J_0^2\left(4D\sin\frac{E_0t}{2\hbar}\right), \tag{9.5}$$

where E_0 is the initial-state energy that oscillates like a Bessel function. The results obtained in Ref. 62 are generalized in Ref. 64 to the case where the continua are not structureless, i.e., the transition matrix elements between the continua are functions of energy. The results presented in Ref. 64 become identical with those in Ref. 62 in the time-independent limit. It is also shown in Ref. 64 that the system of coupled continua need not be infinite and that CPT is possible for a finite number of coupled continua.

Two and three coupled continua are discussed in Ref. 65. Three types of continuum are examined: (a) wide, whose "width" is given by (9.3) and is much greater than the Rabi frequency that characterizes the coupling between the continua, (b) narrow, whose "width" is smaller than the Rabi frequency, and (c) a strictly-limited continuum that differs from the narrow continuum only by the way the width is defined. The "width" is defined here as the size of the region of containing energy levels. The strictly-limited continuum is a system with sharp boundaries (similar to the Π-shaped distribution). In the limit of strong coupling, pure CPT is possible only for a "wide" continuum.

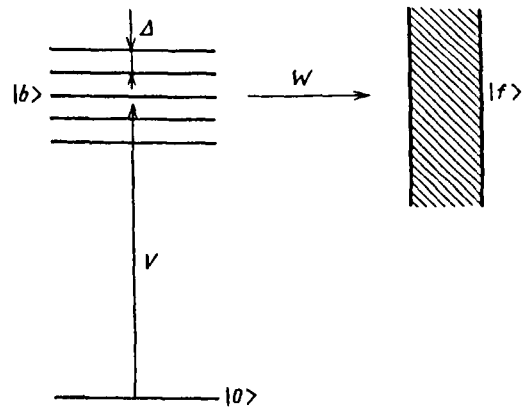

FIG. 14. Mixed scheme: $|0\rangle$—ground state, $|b\rangle$—quasicontinuum, $|f\rangle$—pure continuum; V and W—coupling parameters.

Finally, in Ref. 66, the mixed scheme consists of a discrete ground state 0 coupled to the quasicontinuum $|b\rangle$ (coupling parameter V) which, in turn, is coupled to the pure continuum $|f\rangle$ (coupling parameter W; Fig. 14). The quasicontinuum $|b\rangle$ is a set of closely spaced energy levels with level separation Δ. In the time-independent limit, the population trapped in the ground state is

$$P_{|0\rangle}=(1+\gamma)^{-2},$$

the population of the quasicontinuum is

$$P_{|b\rangle}=\gamma(1+\gamma)^{-2},$$

and, finally, the population of the continuum is

$$P_{|f\rangle}=(1+\gamma)^{-1},$$

where $\gamma=\pi^2V^2/\Delta^2$.

The condition for CPT in this case replaces the usual conditions for two-photon resonance and takes the form

$$\Delta_{|b\rangle}=\left(b+\frac{1}{2}\right)\Delta,$$

where $\Delta_{|b\rangle}$ is the detuning between the ground state and $|b\rangle$ is the quasicontinuum level. This condition means that the ground state is in resonance with a certain dressed state lying between $|b\rangle$ and $|b+1\rangle$. It is interesting to note that all these populations are independent of the coupling parameter W between the quasicontinuum and the continuum. The important fact is that, when $\gamma>1$, the quasicontinuum population becomes greater than the ground-state population, and we have population inversion (we recall,

however, that the quasicontinuum decay is not taken into account here).

Trapping is thus seen to occur in this system as well. The continuum population can lie between 0 and 1, depending on the parameter γ. It depends on V and not on W.

We note in conclusion that the possibility of CPT in systems with a continuous spectrum extends very substantially the classes of media in which it can be exploited: they now include condensed media and solids, and not only gases.

10. BASIC EXPERIMENTS AND POSSIBLE APPLICATIONS OF COHERENT POPULATION TRAPPING

We shall now review the more important experiments that marked the beginning of systematic studies of coherent population trapping, and will show how we can exploit the remarkable properties of CPT resonance (such as the anomalously small width) in ultrahigh-resolution spectroscopy, frequency stabilization, laser cooling of atoms, optical bistability, and other important branches of physics.

10.1. Absorption of light by atoms in a magnetic field

Studies of the absorption of light by atoms in different magnetic-field configurations have a long history[6] and continue to attract the attention of researchers.[15–18,67,68] The main effect here is that light induces coherence between Zeeman sublevels, which leads to a number of unusual phenomena. The effect of Zeeman coherence on the absorption of light by an atom was originally noted[6] as far back as 1961 in the course of an investigation of the pumping of alkali-metal atoms in a magnetic field. Magnetic resonance in these atoms was detected in Ref. 6 by means of the amplitude modulation of circularly polarized light. There was no light absorption when the precesion frequency of the atoms in the magnetic field was equal to the light modulation frequency. On the other hand, strong absorption was observed when when these frequencies were different. There are two explanations of this. The quasiclassical explanation is given in Ref. 8 and a physical interpretation of coherent trapping can be found in Ref. 10. The authors of Ref. 6 anticipated much of subsequent work on induced coherence between the low-lying levels of alkali-metal atoms by short light pulses. It is irrelevant how such pulses are produced: they can be generated by amplitude modulation of wideband radiation (e.g., from a sodium lamp) or they can take the form of ultrashort laser pulses.[15–18] The only significant point is that the radiation must cover the Zeeman sublevels of the ground state of the alkali-metal atom.

The so-called "black lines" were found in Ref. 8 in the fluorescence of sodium atoms pumped by continuous mulimode dye-laser radiation in a magnetic field. The origin and properties of these lines were established in Ref. 10. Their physical origin is the same as before, i.e., excitation of coherence between different Zeeman components and the assumption by the atom of a superposition in which it is weakly excited by the field. We note that this interpretation enabled the authors of Ref. 10 to explain not only their own experiments, but also a number of others on the excitation of Zeeman resonances by modulated light (see, for example, the review of the literature given in Ref. 10).

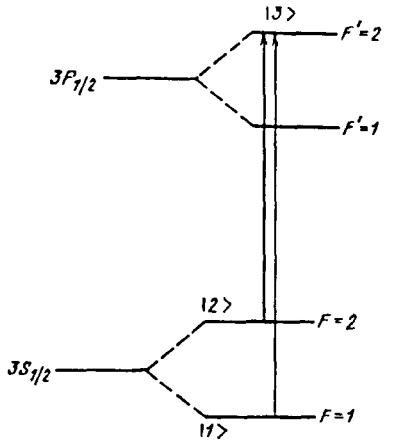

FIG. 15. Energy-level diagram and transitions in the sodium atom (D_1 line).

10.2. Ultrahigh-resolution spectroscopy

We now turn to ultrahigh-resolution spectroscopy based on resonant coherent trapping. This resonance occurs when a three-level atom is illuminated by two wideband light waves. We showed earlier that the width of this resonance can be made much narrower than the radiation width γ (1.3). It is therefore not surprising that there have been suggestions[9,68–70] that these ultranarrow resonances could be exploited in atomic and molecular spectroscopy. For example, the dependence of upper-level fluorescence on the frequency of the field acting on one of the transitions in a Λ-system was investigated in Ref. 9 whilst a fixed-frequency field was acting on another transition in the system. A beam of Na^{23} atoms was used in the experiment reported in Ref. 9. The light propagating at right angles to the atomic beams was used to excite the transitions $3S_{1/2}(F=1)-3P_{1/2}(F'=2)$, $3S_{1/2}(F=2)-3P_{1/2}(F'=2)$ (Fig. 15). The frequency stabilization of the light beam was better than ± 1 MHz. These measurements and a numerical calculation are shown in Fig. 16. There is a well-defined valley corresponding to coherent trapping.

The fluorescence from the upper level of the Λ-system, due to the $4s4p^3P_1-4s5s^3S_1$(612.2 nm), $4s4p^3P_2-4s5s^3S_1$(616.3 nm) transitions in metastable calcium ($^{40}Ca^*$), is also investigated in Ref. 70. The calcium atomic beam interacts with two parallel linearly-polarized light waves. The 612.2-nm transition occurs in a saturating field of 100 mW cm^{-2} and the $\lambda=616.3$-nm transition in a probe field of less than 1 mW cm^{-2}. The frequency stability is better than ± 1 MHz. Figure 17 shows a typical experimental spectrum. The saturating laser is run at a fixed detuning from precise resonance, and the frequency of the probe field is scanned. It is clear that, in the vicinity of the detuning, the width of the observed CPT resonance

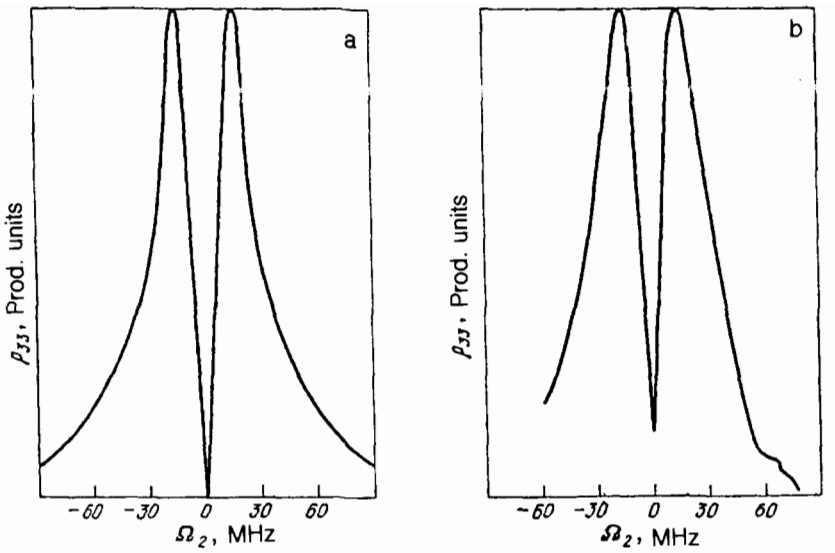

FIG. 16. 'Black lines' in fluorescence from the upper level. Results of calculations (a) and of experiment (b); $\Omega_1=0$, Ω_2-scanned (from Ref. 9).

is $\Delta_0 \approx 4$ MHz. This can be reduced to 1.5 MHz by reducing the intensity [in complete accord with (1.3)]. At the same time, the radiation width of the intermediate level amounts to $\gamma \approx \tau^1$ ($\tau = 10.7$ ns), so that $\Delta_0 \ll \gamma$.

A further property of the Λ-system is that the lower levels are well separated, and calculations (such as those performed in Ref. 70) must allow for the difference between the wave vectors of the pump waves.

We note that a similar dependence is also obtained for the intermediate-level fluorescence (Fig. 11) from the Ξ-system.[69] A metastable neon beam ($^{20}Ne^*$) is used in this experiment. Typically, a dispersive dependence of the CPT resonance is observed[69,70] for detunings from precise resonance, due to the difference between the Rabi frequencies for adjacent transitions.

At the same time, a coherent trapping resonance is observed[18] in a cell filled with, for example, the molecular gas I_2. The lower states of the Λ-system are in this case the hyperfine components of vibrational sublevels of the ground state $V''=0$, $J''=15$; $V''=11$, $J''=15$ and the upper state is the hyperfine component of the first excited state with $V'=43$, $J'=16$. The natural width of the excited state is 100 kHz. The system is pumped by an argon laser producing 514.5-nm radiation; a 582.8-nm probe beam is

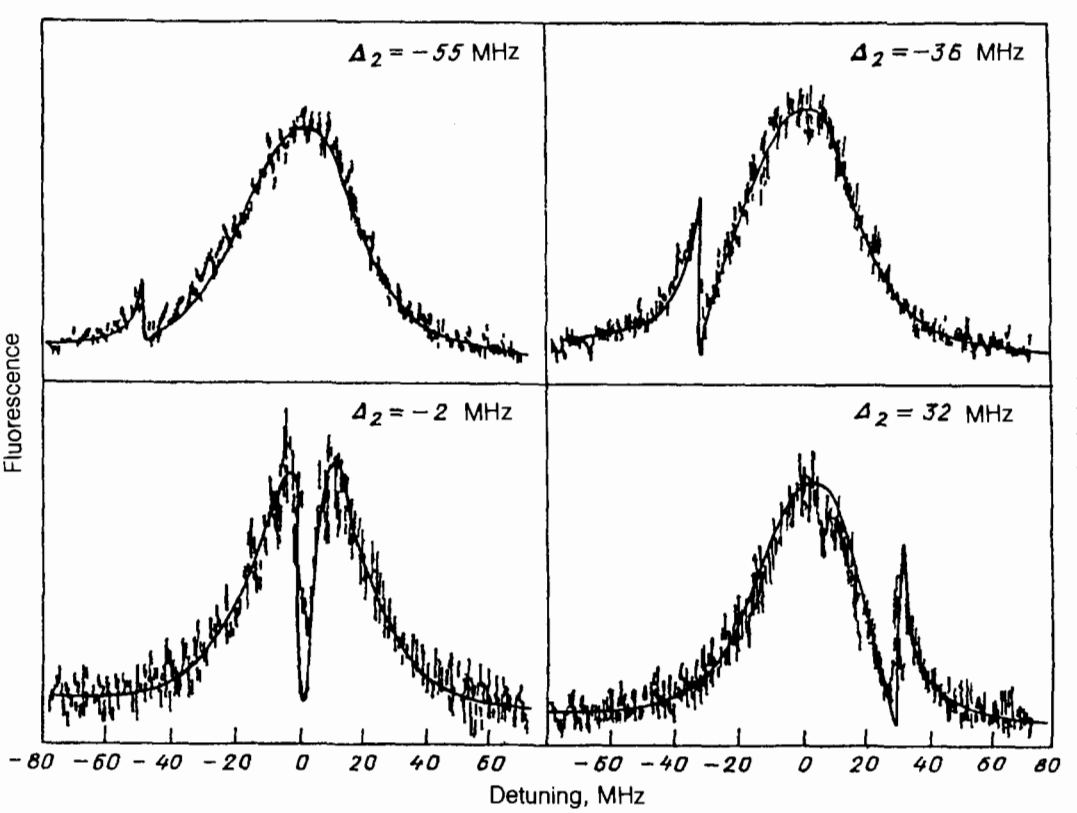

FIG. 17. Fluorescence signal from the upper level as a function of the detuning Ω_2 (from Ref. 70).

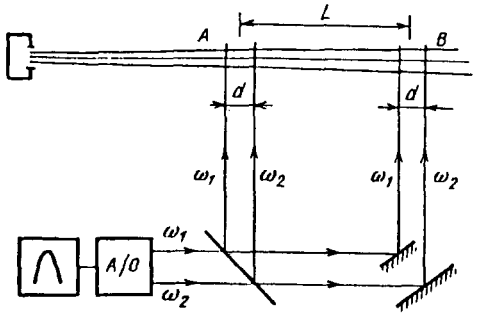

FIG. 18. Principle of the experimental arrangement (from Ref. 27).

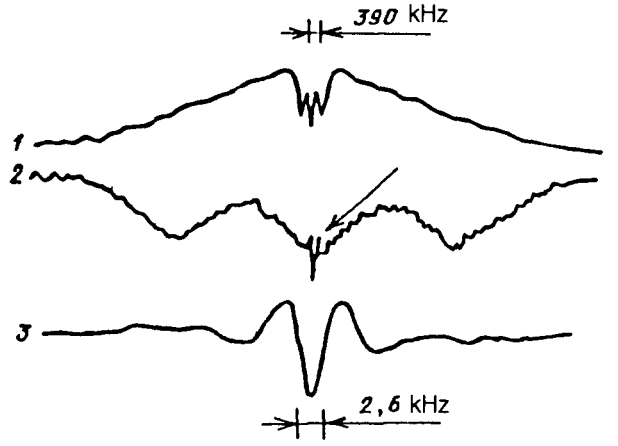

FIG. 19. Experimental results[27] on fluorescence from coherent trapping resonances (*1*). Trace *1* shows the coherent trapping resonances and the Ramsey line (arrow) for $L=15$ cm (*2*) and trace *3* shows the Ramsey line.

applied to the other transition in the Λ-system. Measurements are made of the transmission of the latter beam. The width of the resulting resonance is $\Delta_0 \approx 58$ kHz, which is less than the natural linewidth of the excited state. The width Δ_0 is largely due to transit effects.

10.3. Frequency stabilization

The narrow coherent trapping resonance whose width is given by (1.3) provides a promising basis for beam devices for frequency stabilization.[23–27] An examination of the possibility of this is reported in Ref. 27. As noted above, the resonance width Δ_0 is not determined by the lifetime γ^{-1} of the upper level in the system, and can be made much smaller than γ by adjusting the intensity. This means that, when atomic beams are used in frequency stabilization devices, the main contribution to the broadening of the CPT resonance is due to transit broadening $\Delta\omega_t \approx \bar{v}/d$ where d is the diameter of the laser beam and $\bar{v}$ is the mean velocity of atoms in the beam. The traditional method of separated fields (Ramsey's method) is used in Ref. 27 to increase the duration of the interaction between the atoms and the light beam and to reduce further the width of the coherent trapping resonance.

The experiment employs a beam of sodium atoms excited from the hyperfine states $3S_{1/2}(F=1)-3P_{1/2}(F'=2)$, $3S_{1/2}(F=2)-3P_{1/2}(F'=2)$ (Fig. 15). Dye-laser radiation of frequency ω_1 acts on the $|1\rangle-|3\rangle$ transition. A light field of frequency ω_2 that resonates with the $|2\rangle-|3\rangle$ transition is obtained from ω_1 with the help of an opto-acoustic modulator controlled by a microwave generator quartz-stabilized near 1772 MHz (the hyperfine splitting frequency of the ground state of the sodium atom). This method of producing two-frequency radiation gives good correlation between the light fields which, in turn, gives rise to the CPT resonance and to a satisfactory contrast for the observation of the effect.[11,13] The radiation leaving the modulator is intercepted by a splitter and finally interacts with the atomic beam in the two regions marked A and B in Fig. 18. The separation L between these regions can be varied between 15 and 30 cm, and the size of the light beams in the interaction regions is $d \approx 2.5$ mm. A weak magnetic field is present along the entire length of the interaction region.

Two types of experiment were performed. The first was concerned with fluorescence from the upper state $|3\rangle$ in region B in the absence of interaction in region A. The second examined both the CPT valley and the Ramsey line structure (intercation in both A and B). The narrow Ramsey fluorescence structures were then used to tune the frequency of the opto-acoustic modulator and thus stabilize the frequency relative to the hyperfine splitting of the ground state of the sodium atom.

Figure 19 shows the fluorescence from region B, obtained by scanning the modulator frequency. The power carried by each of the two beams of frequency $\omega_{1,2}$ is 16 mW. The wide structure corresponds to the $|3\rangle-|2\rangle$ transition (D_1 line) in the sodium atom, whose width is 10 MHz. The three narrow valleys are the coherent trapping resonances. They appear because a magnetic field of 300 mG is applied to the entire length of the region of interaction with the beam. The central valley in Fig. 19, *1* corresponds to the transition with $m=0$, $\Delta m=0$, which is insensitive to the magnetic field. The polarizations of the beams with frequencies $\omega_{1,2}$ are linear and mutually perpendicular. Each of the three valleys in Fig. 19, *2* has a width of about 390 kHz, which is consistent with the time of flight across the laser beam.

The narrow central feature (arrow) is the Ramsey line obtained for a separation of $L=15$ cm between regions A and B.

An expanded scan is shown in Fig. 19, *3*. The central line width is 2.6 kHz which is consistent with the time of flight between A and B ($L=15$ cm).

We note that, to obtain a symmetric Ramsey line, we have to ensure that A and B are optically equidistant from the splitter (Fig. 18).

The above paper also reports the development of a frequency standard using a sodium atomic beam. Figure 20 shows the measured relative frequency stability as a function of the averaging time interval τ. For $\tau=1000$ s, the frequency stability is about 1.5×10^{-11}. It is clear that the

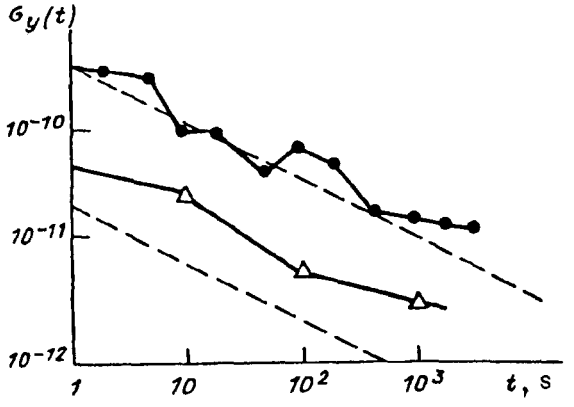

FIG. 20. Measured relative frequency stability as a function of averaging interval. Upper dashed line—calculated for shot noise; lower dashed line—proposed stability when sodium is replaced by cesium; triangles—stability of commercial clocks.[27]

preliminary results for sodium are comparable with commercial cesium clocks up to averaging times $\tau = 1000$ s.

Much of the above paper was concerned with the sources of uncertainty in determinations of frequency for standards of this type. The most important of them are listed below:

(1) relative departure of $\omega_{1,2}$ from a common direction

(2) optical phase shifts due to changes in the polarization of the laser beams

(3) phase shifts due to differences between laser path lengths

(4) deviation of the laser frequency ω_1 from precise resonance with the $|1\rangle - |3\rangle$ transition.

10.4. Optical bistability

The possible use of a three-level system with CPT to generate a bistable response was reported in Ref. 28. A strong nonlinearity was produced in this case by a sharp change in the absorption coefficient, subject to condition (1.1) under which CPT takes place. It is noted in Ref. 28 that there are two possible manifestations of bistability, namely, the output intensity may be a function of intensity, and the output intensity may be a function of laser detuning (we shall refer to them as type one and type two bistabilities, respectively). Consider a Λ-system placed in a resonator and illuminated by a single-mode laser beam of frequency ω and light field strength E. The laser is tuned to the resonant mode $\omega = (\omega_{31} + \omega_{32})/2 \approx \omega_c$. The upper-level population of is then given by

$$\rho_{33} = \frac{2Y^2\delta^2}{\delta^4 + 4\delta^2 + 2Y^2\delta^2 + 4Y^4}, \tag{10.1}$$

where $\delta = \omega_{21}/2\gamma$ is the detuning parameter and $Y = |dE|/\sqrt{2}\hbar\gamma$. It is clear from (10.1) that ρ_{33} and, consequently, absorption by the medium, are nonlinear functions of the detuning δ and of the intensity Y of the input signal. This is responsible for both types of bistability in this kind of system.

If we now put $X = |dE_c|/\sqrt{2}\hbar\gamma$, where E_c is the field strength in the resonator that characterizes the output signal intensity, we can readily show that

$$Y = X\left(1 + \frac{4C \cdot 2\delta^2}{\delta^4 + 4\delta^2 + 2X^2\delta^2 + 4X^4}\right), \tag{10.2}$$

where $C = N|d|^2/4\gamma k_0$, N is the concentration of the atoms, and k_0 is the resonator damping rate. Figure 21 shows the functions $X = X(Y)$ and $X = X(\delta)$. Both types of instability can be seen for certain parameter values. The dashed line shows, for comparison, the result for the two-level system.

These ideas were advanced further by the suggestion[29] for a laser with a three-level absorber whose atoms have a Λ-system of levels. It is shown in Ref. 29 that this laser displays type one bistability. Lasers with a three-level absorber have an important advantage[71] as compared with the laser incorporating a saturable absorber: it is possible to use an absorber for which the saturation intensity is greater than the saturation intensity for the amplifying medium. This is so because the physical effect responsible for bistability is not optical saturation but the intensity dependence of the absorption coefficient during CPT.

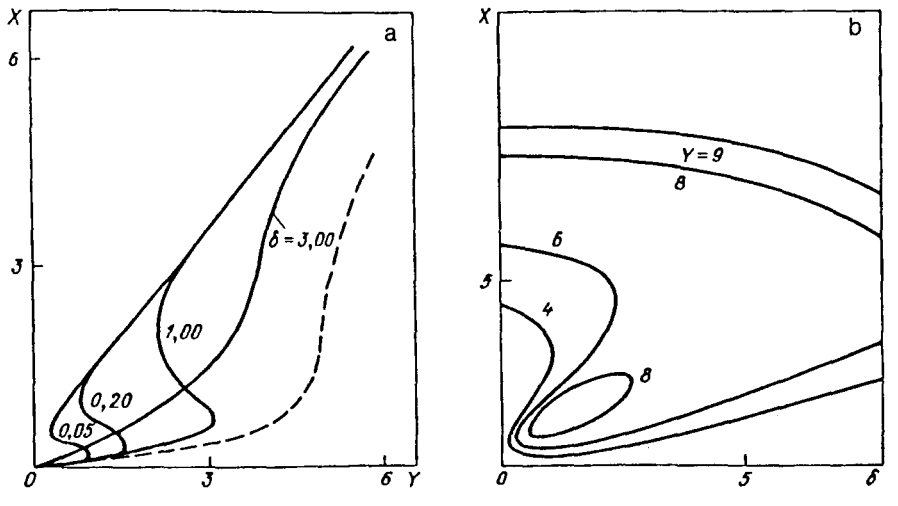

FIG. 21. a—Output field X as a function of the input field Y for different values of δ ($C=4$); dashed curve—two-level absorbing scheme; b—output field X as a function of detuning δ for different values of Y ($C=10$).

Both types of bistability have been observed experimentally[30] in a medium of three-level atoms with CPT. Sodium atoms were placed in a constant magnetic field and exposed to laser radiation of frequency ω. The sublevels of the ground state $^2S_{1/2}$ were thus split by the Zeeman effect in the magnetic field. In addition, a variable magnetic field was also applied to the system and was used to vary the detuning of the laser beam from precise resonance. CPT occurred at resonance, and the susceptibility depended on the input signal intensity.[28–29] This was exploited to produce the bistable response of the system. Both types of instability were thus investigated.

10.5. Laser cooling of neutral atoms

Laser cooling of atoms has been phenomenally successful in recent years.[31,34,39] When atoms are cooled in this way, so that they become "optical molasses",[39] the resulting temperatures are significantly lower than the Doppler limit $T_D=\hbar\gamma/2k_B\approx 10^{-4}K$ (k_B is Boltzmann's constant). When atoms are accumulated in coherent superposition states,[34] temperatures less than $T_R=R/k_B\approx 10^{-6}$ K (which corresponds to the atom recoil energy $R=(\hbar k)^2/2M$) can be reached, where k is the wave number and M the mass of the atom.

CPT provides the basis for one of the mechanisms whereby such deeep cooling of atoms can be achieved.[32,33] We shall illustrate this by considering the example of a beam of Λ-atoms interacting with the field of two counterpropagating waves.

We shall use the quasiclassical theory[38] to describe the translational motion of the atoms. This description is convenient because it enables us to introduce such concepts as light presure, momentum diffusion tensor, dynamic friction, and so on, i.e., to describe the motion of an atom as a classical Brownian particle. The limit of validity of this quasiclassical approach is the temperature T_R (Ref. 38), since it is for atomic velocities $v\approx v_R=\hbar k/M$ (v_R is is the recoil velocity) that we have to take directly into account the relative recoil shift of photon emission and absorption lines.

When the frequency detunings are equal, $\Omega_m=\Omega$ ($m=1,2$), and if we meet the conditions for the light-wave intensity (4.10), we find from (4.2) and (4.3) that we can determine the upper-level population density ρ_{33} for counterpropagating waves:[33]

$$\rho_{33}=g^2aL^{-1}, \tag{10.3}$$

$$a=4(kv_z)^2+(2g^2\Gamma/\gamma),$$

$$L=4(kv_z)^4+8\Omega b(kv_z)^3+4(\Omega^2+\gamma^2+g^2)(kv_z)^2$$
$$-4g^2\Omega b(kv_z)+2g^2(\Omega^2\Gamma/\gamma+2g^2);$$

$$b=(\gamma_1-\gamma_2)/\gamma,\quad \gamma=\gamma_1+\gamma_2.$$

The expression given by (10.3) defines comletely both the light pressure F_z acting on the atom[38]

$$F_z=2\hbar k\gamma b\rho_{33}, \tag{10.4}$$

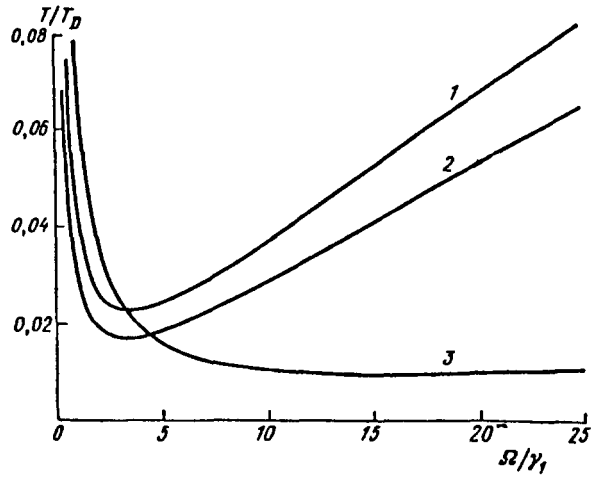

FIG. 22. The temperature T as a function of detuning Ω. $1-g=0.1\gamma_1$, $\Gamma=10^{-3}\gamma_1$, $\gamma_2=0.2\gamma_1$; $2-\gamma_2=0.1\gamma_1$; $3-g_1=0.1\gamma_1$, $g_2=0.3\gamma_1$, $\gamma_2=0.2\gamma_1$, $\Gamma=10^{-3}\gamma_1$.

and the momentum diffusion tensor whose component in the direction of propagation is

$$D_{zz}=2\hbar^2k^2\gamma\rho_{33}. \tag{10.5}$$

Next, let us determine the temperature of the atomic beam in the region of zero velocity, using Einstein's formula[38]

$$T=-D_{zz}\left(\nabla_{v_z}F_z\big|_{v_z=0}\right)^{-1}=2T_D\frac{2g^2+(\Omega^2\Gamma/\gamma)}{b^2\gamma|\Omega|}, \tag{10.6}$$

which is related to the width of the velocity distribution

$$\Delta v_z=(2k_BT/M)^{1/2}.$$

Figure 22 shows the temperature (10.6) of cold atoms as a function of the detuning $|\Omega|$. It is clear that there is a wide range of values of detuning $|\Omega|$ in which the temperature T can be less than the Doppler limit. The minimum temperature

$$T_{\min}=T_D\frac{8g}{b^2\gamma}\left(\frac{\Gamma}{2\gamma}\right)^{1/2} \tag{10.7}$$

is reached for

$$\Omega_{\rm opt}=(2g^2\gamma/\Gamma)^{1/2},$$

and, choosing as an example the cooling of sodium atoms by the $3S_{1/2}-3P_{1/2}$ transition, we have $T_{\min}=5\times 10^{-6}$ K for $\Omega_{\rm opt}\approx 8\gamma$ for $g\approx 0.1\gamma$, $b\approx 0.7$, $\Gamma\approx 10^{-3}\gamma$ ($\gamma\approx 10$ MHz).

Figure 23 shows the evolution of the velocity distribution of atoms experiencing the force F_z. It is clear that a narrow peak with the effective temperature given by (10.6) appears on the distribution.

The possibility of the cooling of atoms to temperatures below T_D is thus physically related to the sharp reduction in the effect of momentum diffusion in the coherent trapping resonance with a high dynamic friction coefficient.[33]

The cooling of atoms below the recoil temperature T_R, reported in Ref. 34, can be explained[35] in terms of CPT in

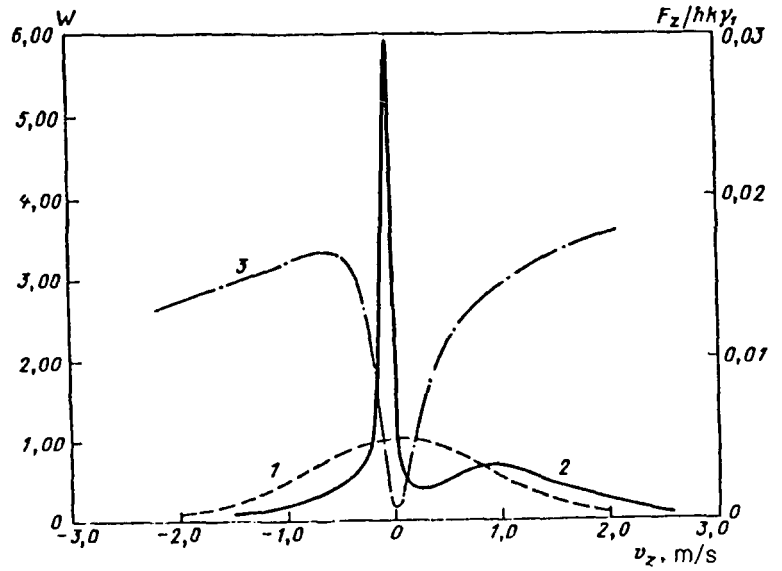

FIG. 23. Deformation of transverse velocity distribution of atoms in a beam in the presence of a force (10.4) for one-dimensional collimation. Time of interaction with radiation $t=2\times10^{-5}$ s. *1*—initial transverse distribution, *2*—distribution after interaction with radiation, *3*—resonant light pressure on atoms for $g=0.3\gamma_1$, $\gamma_2=0.2\gamma_1$, $\Gamma=10^{-3}\gamma_1$, $\Omega=2\gamma_1$.

velocity space. This directly takes into account the influence of the recoil effect on the atomic emission and absorption lines, so that the quasiclassical approach is invalid and we have to use the complete quantum desciption of the translational motion of the atom.

It is shown in Ref. 35 that, since the atoms in different quantum states in the Λ-system can have different velocities, the system can have specific coherent states with momenta $\pm\hbar k$ that do not interact with radiation. If there is spontaneous relaxation, the atoms accumulate in these states, and this is responsible for the two-peak structure of the velocity distribution (Fig. 24). The width Δv_z of an individual peak is then less than the recoil velocity v_R $\Delta v_z < v_R$, and the temperature T of the atoms is less than the recoil temperature ($T < T_R$). An effective temperature of 2 μK was achieved in Ref. 34 for helium atoms (the recoil temperature was $T_R = 4\ \mu K$).

CPT in velocity space was proposed in Refs. 36 and 37 as a basis for several two-dimensional and three-dimensional cooling schemes exploiting different atomic transitions. It was shown that temperatures lower by several orders of magnitude than the recoil temperatures could be produced in this way.

10.6. Effect of relative phases of pump fields on CPT

When we discussed coherent trapping in Λ-systems we ignored the effect of the phases of the pump fields. It is, however, well-known[72–74] that these phases do have a significant effect in a system closed by a third resonant field (Fig. 25), especially when the three-frequency resonance condition

$$\omega_r = \omega_1 - \omega_2 = \omega_{21}$$

is satisfied. The population dynamics is investigated in Ref. 72 for this type of closed system with $\tau \ll \gamma^{-1}$. It has been shown[79] that the field phases have a significant effect on CPT itself.

The physical phase of the third resonant field influences the coherence between the lower levels that is induced by the other two fields. Hence, depending on the phase of the third field, we observe the destruction or conservation of this coherence. This, in turn, leads to the destruction or restoration of the coherent trapping state.

We shall demonstrate this in the special case of a Λ-system (Fig. 25) interacting with light fields with frequencies ω_m ($m=1,2$), closed by an r.f. field of frequency ω_r applied between the lower levels. We shall assume that $|m\rangle - |3\rangle$ ($m=1,2$) are electric dipole transitions whereas $|1\rangle - |2\rangle$ are magnetic dipole transitions. We shall also assume that the three-photon resonance condition is met.

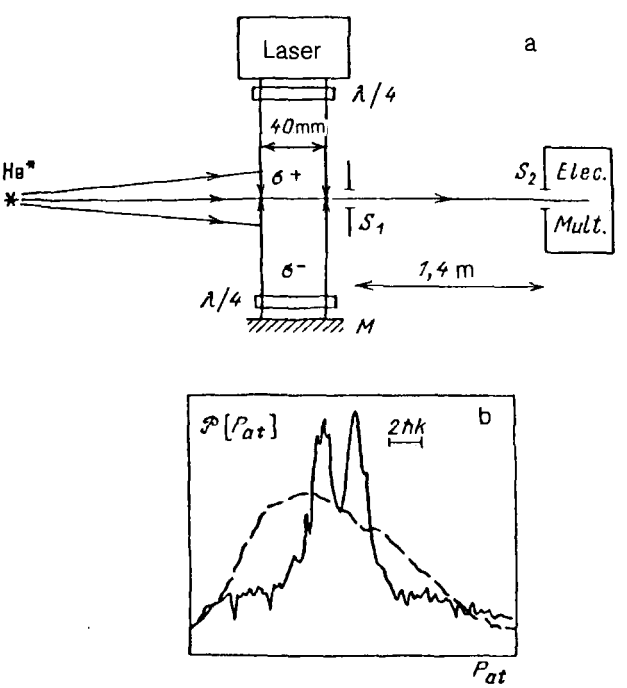

FIG. 24. Observation[34] of the transverse collimation of a beam of $^4He^*$ metastable atoms: a—principle of the experiment, b—measured velocity distributions; dashed curve shows the distribution without interaction with the light field.

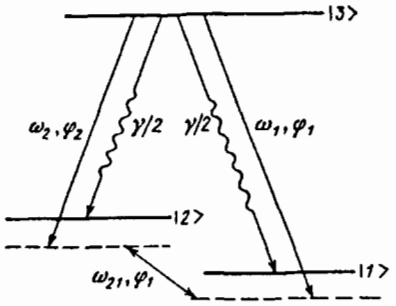

FIG. 25. Λ-scheme interacting with two light fields (frequencies $\omega_{1,2}$) and closed by a radio field (frequency ω_{21}).

The Hamiltonian of the atom-field interaction will be written in the form $\hat{H}=\hat{V}+\hat{U}$ where

$$\hat{V}=\hbar^{-1}\sum_{m=1,2}\hat{d}_m E_0 \exp[i(\chi_m-\vartheta_\mu-\omega_m t)],$$
$$\hat{U}=\hbar^{-1}\hat{\mu}H_0\exp[i(\chi_r-\vartheta_r-\omega_r t)], \tag{10.8}$$

where $\hat{d}_m, \hat{\mu}$ $(m=1,2)$ are the dipole moment operators (electric and magnetic, respectively), E_0, H_0 are the field amplitudes, ϑ_p $(p=1,2,3,r)$ are the field phases, and χp $(p=1,2,3,r)$ are the dipole moment phases.

Next, we can write down the equations for the elements of the density matrix ρ_{ij}, using (10.8) and the solution for this system in the steady state $(\tau \gg \gamma^{-1})$. The results are

$$\rho_{33}=(4g^2u^2\sin^2\Phi)M_0^{-1},$$
$$\rho=\rho_{22}-\rho_{11}=(2g^2u\gamma\sin\Phi)M_0^{-1}, \tag{10.9}$$

where

$$M_0=u^2\gamma^2+4(u^2-g^2)^2+12g^2u^2\sin^2\Phi,$$

and for coherence between the lower levels

$$\rho^{12}=\frac{1}{2}\left\{a_0\rho\exp\left[i\left(\varphi_r-\frac{\pi}{2}\right)\right]+(1-3\rho_{33})\right.$$
$$\left.\times\exp[-i(\varphi_2-\sigma_1+\pi)]\right\}, \tag{10.10}$$

where

$$a_0=u(4u^2+\gamma^2-4g^2)/2g\gamma.$$

In (10.9) and (10.10), the relative phases are given by $\varphi_p=\vartheta_p-\chi_p(p=1,2,r)$, $\Phi=\varphi_r+\varphi_2-\varphi_1$, and $g=dE_0/\hbar$, $u=\mu H_0/\hbar$ are the corresponding rabi frequencies. Moreover, it is assumed that the detunings Ω_m satisfy the condision for exact resonance, i.e., $\Omega_m=\omega_m-\omega_{3m}=0(m=1,2)$.

Equations (10.9) and (10.10) determine the effect of the resonant r.f. field on the existence of CPT in the Λ-system. It is clear that when the r.f. field is absent ($u=0$), the upper level is empty and CPT exists. We emphasize that there is then no dependence on the optical-field

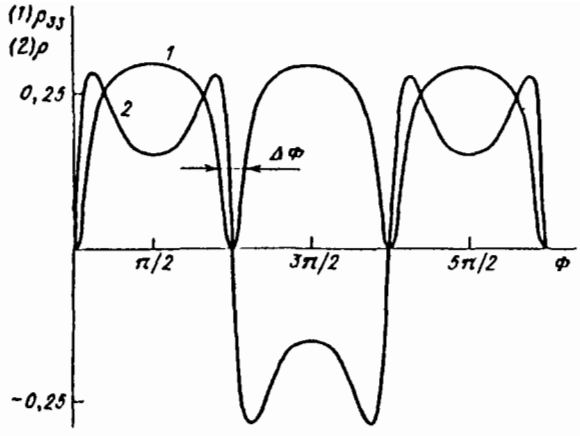

FIG. 26. The population ρ_{33} (*1*) and the population difference $\rho=\rho_{22}-\rho_{11}$ (*2*) as functions of the phase Φ for $g=u=10^7$ s^{-1} and $\gamma=10^7$ s^{-1}.

phases, since a change in the relative phases leads only to a change in the resultant phase of the coherence ρ_{12} without a change in its modulus (10.10), which ensures that there is no special direction on the complex plane of ρ_{12}.

If we now consider the case $u\neq 0$, we see from (10.9) and (10.10) that the dependence on the relative phases is the dominant factor. Thus, when

$$\Phi=\pi n \quad (n=0,\pm 1,\pm 2,...)$$

coherent trapping occurs as before, but when

$$\Phi=\pi(2n+1)/2 \quad (n=0,\pm 1,\pm 2,...)$$

coherent trapping ceases and the level $|3\rangle$ is populated despite the fact that CPT ondition is satisfied. In the limit of high Rabi frequencies $g\approx u\gg\gamma$, all the levels are equally populated: $\rho_{mm}\approx 1/3$ $(m=1,2,3)$. Figure 26 shows the level population (10.9) as a function of the phase Φ. It is clear that the phase can be used to destroy or restore the population trapping state.

Figure 27 shows the population of the upper level of

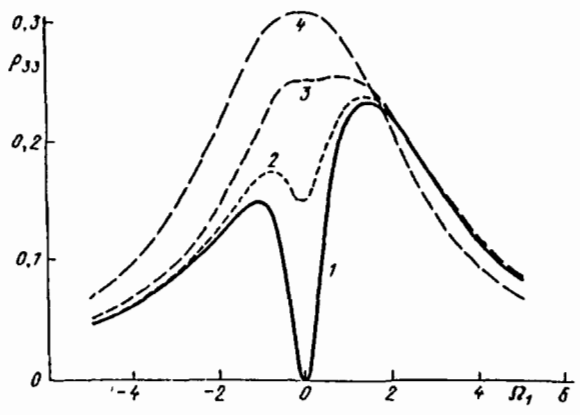

FIG. 27. Third-level population for $\Omega_2=0$. Ω_1 scanned, $\gamma=g=u=10^7$ s^{-1}. *1*—$\Phi=0$, *2*—$\Phi=\pi/12$, *3*—$\Phi=\pi/6$, *4*—$\Phi=\pi/2$.

the Λ-system as a function of one of the detunings of the optical field when the other is held constant and (10.8) is satisfied. There are well-defined coherent trapping valleys that occur for some of the phases and vanish altogether for the others.

This sensitivity of atomic systems to the phase Φ during CPT can be exploited in quantum electronics. For example, atomic interferometers for measuring the phase and amplitude of radiation fields and the corresponding transition dipole moments are proposed in Ref. 72, whereas in Ref. 45 the above sensitivity is used in optical modulators. The most interesting is the possibility of controlling light amplitude modulation by modulating the phase of a resonant r.f. field.

The effect of the relative phase of the fields on the upper-level population in the Λ-system was first oberved experimentally in Ref. 74.

10.7. Lasers without inversion

Finally, we must consider a further promising application of coherent population trapping, namely, noninversion lasers, i.e., a new branch of quantum electronics[76–83] in which light beams are generated and amplified without producing the population inversion that is essential in classical lasers.[3]

The inversionless laser relies on the following principle. If the Λ-system is in the coherent trapping state, most of the population resides in the lower levels and, according to (4.9), the $|3\rangle$ state is occupied by the smaller fraction $\rho_{33}\approx\Gamma/\gamma$. When the population of the upper level is raised (if only slightly) in some way, e.g., by electron impact, then because the atoms occupying the lower levels are in a particular coherent state that does not interact with the light field, we have a kind of 'inversion' between the weakly populated level $|3\rangle$ which can nevertheless interact with the highly populated lower levels $(\rho_{11}+\rho_{22}\approx 1-(\Gamma/\gamma))$ that do not interact with the field. Next, light that resonates with optical transitions is thus amplified because of the low population of the level $|3\rangle$ of the Λ-system. As noted above, the highly populated lower levels do not participate in this interaction at all.

We note that the very phrase 'laser without inversion' emphasizes that there is no real inversion and that the amplification process is entirely due to the weakly populated upper level.

The amplification of light pulses in a Λ-system during the excitation of coherence between the lower levels by resonant $\pi/2$ microwave pulses is discussed in Ref. 76. The amplification of optical fields in Λ- and V-systems is also investigated in Refs. 78 and 79 for times much shorter than the relaxation times. The phase of the resonant r.f. field applied between close levels was found to affect the condition for the amplification of the optical signal. The effect of the phase of exciting fields on generation conditions was also demonstrated in Refs. 80–82 for the case of the so-called double Λ-system.

We emphasize that the use of the 'noninversion' principle of light amplification may be very useful for transitions for which inversion is difficult to achieve.

11. CONCLUSIONS

We have reviewed coherent population trapping—a new nonlinear phenomenon. We have tried to give a history of the discovery and to outline its physical significance and possible applications. Of course, each new nonlinear effect results not only in a more complete understanding of nonlinear phenomena generally, but also stimulates its practical applications. This is, in our view, the present situation with coherent population trapping. The many publications reviewed above are mostly concerned with basic concepts and the physical significance of the phenomenon. Practical applications of coherent population trapping, on the other hand, have begun to appear only recently. There are still relatively few such applications. They include frequency stabilization and ultrahigh-resolution spectroscopy, lasers without inversion, and ultradeep cooling of atoms.

There is, however, no doubt that important new applications of CPT will emerge in the future. This will involve, in the first instance, ultranarrow resonances and the fact that the continuing advances in laser technology will provide the necessary light sources for such applications. For example, CPT was used in Ref. 84 as a basis for a suggested spatial superlocalization of atoms on a scale much shorter than the wavelength of light in the caustics of nonuniformly polarized wavefronts.

On the other hand, coherent population trapping reveals itself in an increasing number of experimental situations such as, for example, the behavior of Ba^+ ions in magnetic traps[85] or in light-induced drift.[86] CPT is likely to be detected in condensed media and also (once coherent γ-sources become available) in nuclear spectroscopy, as well.

We have not reviewed publications in which CPT is discussed in terms of the quantum description of radiation. We had neither the facilities nor the intention to review the entire enormous volume of published material. Moreover, we do not claim to provide a comprehensive coverage of the theory of coherent population trapping and its applications in physics. Some of the topics that were not touched upon above may be found, for example, in the review given in Ref. 87.

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[1)]Here we must recall the undeservingly forgotten paper by Bell and Bloom[6] on optical pumping in a transverse magnetic field and the absorption of light by alkali-metal atoms. This paper essentially laid the foundations for the phenomenon of coherent trapping and provided an explanation of the first experiments on the excitation of Zeeman coherence by optical pumping.[8]

[2)]We note that, Letokhov and Chebotaev[1] in their 1975 monograph (page 169) discussed the possibilities of spectroscopy within the radiative width, and in this context considered nonlinear resonances whose widths were not determined by the width of the common level in a three-level Λ-system.

[3)]*It is important to note that the segregation of a three-level interaction* system from the infinite number of energy levels of a real system relies on the definition of radiation properties such as polarization, intensity, and monochromaticity. Examples of three-level systems will be considered in Sec. 10 in connection with experimental studies of CPT.

[4)]We assume that there is no relaxation of the populations (longitudinal relaxation) of levels $|1\rangle$ and $|2\rangle$.

[5)]However, it was shown in Ref. 75 that for Λ-atoms in a buffer gas that interact with sufficiently intense two-frequency laser radiation threre is a change in atomic collision dynamics (opto-collisional nonlinearity effects). In particular, during CPT, the rate of collisional relaxation of the coherence ρ_{12} decreases with increasing field intensity and, in the limit of very strong fields, the transition to the collisionless limit takes place.

[6)]The phrase *coherent population trapping* originally appeared in studies of three-level systems. However, it is now used for multilevel systems and systems with a continuous spectrum.

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