Axially symmetric steady-state flows in the vicinity of a Kerr black hole and the nature of the activity of galactic nuclei

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A generalization of the force-free equation describing steady-state axially symmetric flows in the vicinity of a Kerr black hole is carried out to the case of ideal magnetohydrodynamics. It is shown how this equation changes from an elliptic to a hyperbolic type in the vicinity of the horizon. The limits of applicability of the approach under discussion are examined for different astrophysical sources. It is shown that a high photon density strongly restricts the limits of applicability of the MHD approach.

1. INTRODUCTION

Axially symmetric steady-state magnetohydrodynamic flows in the vicinity of a central compact body have already been studied for a long time in connection with many astrophysical sources. Spherically symmetrical accretion onto ordinary stars[1,2] and black holes,[2] axially symmetric stellar (solar) wind,[3–6] jets from young stellar objects,[7] outflows from the axially symmetric magnetosphere of a rotating neutron star[8–12]–they are all flows of the type under consideration. It can not be excluded that such magnetohydrodynamic flows also play an important role in the galactic sources which are regarded as candidates for black holes.[13,14] It is the equations describing axially symmetric steady-state flows that are the basis of the progress in our understanding of more complicated systems, for example radio pulsars, which have no axial symmetry.[15,16]

MHD-models are now actively developed in connection with the theory of the magnetospheres of rotating supermassive black holes ($\mathscr{M} \sim 10^8$–$10^9\, \mathscr{M}_\odot$), which are believed to be a 'central engine' in active galactic nuclei and quasars.[17–22] In fact, it is an accretion of material onto such compact objects that provides insight into the nature of their extremely high energy production 10^{46}–10^{48} erg/s and the stability of observed jets from it. The energetics of these jets may reach 10^{43}–10^{45} erg/s.[17] Since further we are basically interested specifically in black hole magnetospheres, we shall everywhere give all expressions in the framework of general relativity. Selfgravitation of matter and fields, i.e. their influence on the space-time metric of a black hole, will be neglected, which corresponds to real astrophysical conditions.[17]

Let us stress that the necessity of taking into account general relativity effects is not so obvious for many compact sources. For instance, there exist some indications that the jets in young stellar objects are connected not with the central rotating star but with the accretion disk around it.[18] If jet formation in galactic nuclei and quasars has the same nature as in young stellar objects, than one can not exclude that the black hole plays only a passive role in the jet formation process, and general relativistic effects in this case may not be fundamental for a description of flows in the region of jet formation.

At the same time gravitational effects make, apparently, a noticeable contribution in determination of physical conditions in compact objects. First of all this is indicated by hard spectra and annihilation lines observations in galactic X-ray sources, which are believed to be solar mass black holes.[14] Such characteristics are never observed in X-ray sources which are firmly established to show accretion not onto a black hole but onto a neutron star. Another indication comes from superluminal motion of some details in quasars[17] which may be due to relativistic electron-positron plasma flow ejected along with the weakly relativistic jet.[23] All this testifies in favor of the existence of an additional mechanism for particle creation and acceleration, for which general relativistic effects may be of principal importance. So it is undoubtedly interesting to consider the magnetosphere structure under the most general conditions, i.e. in the presence of a rotating black hole.

In this paper, the basic equation describing a steady axisymmetric flow in the vicinity of a Kerr black hole is given. An analysis of this equation is carried out. Particularly, it is shown how the type of the equation changes at a fast MHD-point from elliptic to hyperbolic in the proximity of the event horizon. The validity limits of the MHD treatment of the phenomena occurring in active galactic nuclei and quasars is discussed. It is shown that the photon density in the vicinity of the black hole must be sufficiently small for the MHD treatment to be applicable.

2. BASIC EQUATIONS

Let us consider a steady-state axially symmetric MHD plasma flow in the gravitational field of a Kerr black hole. In Boyer-Lindquist coordinates the Kerr metric is[24]

$$ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt) \qquad (1)$$

where

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$$\alpha=\frac{\rho}{\Sigma}\sqrt{\Delta},\ \beta^r=\beta^\theta=0,$$

$$\beta^\varphi=-\omega=-\frac{2a\mathscr{M}r}{\Sigma^2},$$

$$g_{rr}=\frac{\rho^2}{\Delta},\ g_{\theta\theta}=\rho^2,\ g_{\varphi\varphi}=\tilde{\omega}^2, \tag{2}$$

and

$$\Delta=r^2+a^2-2\mathscr{M}r,$$

$$\rho^2=r^2+a^2\cos^2\theta,$$

$$\Sigma^2=(r^2+a^2)^2-a^2\Delta\sin^2\theta, \tag{3}$$

$$\tilde{\omega}=\frac{\Sigma}{\rho}\sin\theta$$

Here, as usual, $\mathscr{M}$ and a are the black hole mass and the angular momentum per unit mass respectively, i.e. $a=J/\mathscr{M}$. Units where $c=G=1$ are used throughout the paper except for some estimates where this is pointed out explicitly.

Due to axial symmetry and steady-state conditions there exist two Killing vectors $\mathbf{k}=\partial/\partial t$ and $\mathbf{m}=\partial/\partial\varphi$ (see Ref. 24), so we have two conservation laws: conservation of energy E and conservation of the z-component of angular momentum L_z

$$P^\alpha=k^\beta T^\alpha_\beta,\ J^\alpha=-m^\beta T^\alpha_\beta, \tag{4}$$

where T^α_β is the total energy-momentum tensor of matter and field. This fact remarkably permits us to divorce the problem of the structure of the poloidal magnetic field and electric currents from the problem of particle acceleration and of the toroidal magnetic field structure. This possibility is connected with the general existence of five integrals of motion describing a steady-state axially symmetric flow in the one-fluid ideal magnetohydrodynamic approximation. Given this, the solution of the latter problem in a prescribed poloidal field is expressed by simple algebraic relations.

2.1. A flow in a given poloidal magnetic field

In this section we list the main algebraic relations which permit us to determine all the characteristics of a plasma flow in the magnetosphere when the poloidal magnetic field is given. At first we are going to show how the five integrals of motion appear in the formalism under consideration.

In what follow we shall everywhere carry out our computations using the so-called '3+1'-formalism. This means that all the quantities are expressed as 3-dimensional vectors measured by so-called zero angular momentum observers.[24] These observers are moving with angular velocity ω (2) around a rotating black hole. Roman subscripts and superscripts without carets denote vector and tensor components referred to the coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\varphi$ in 'absolute' 3-space, while those with carets denote the same components referred to the orthonormal basis

$$\mathbf{e}_{\hat{r}}=\frac{\sqrt{\Delta}}{\rho}\frac{\partial}{\partial r},\ \mathbf{e}_{\hat{\theta}}=\frac{1}{\rho}\frac{\partial}{\partial\theta},\ \mathbf{e}_{\hat{\varphi}}=\frac{1}{\tilde{\omega}}\frac{\partial}{\partial\varphi}.$$

Everywhere the symbol ∇ means covariant differentiation in 'absolute' 3-space having the metric g_{ik} (2). For details see chapters 2 and 3 in the book by Thorne *et al.*[24]

Thus in the steady-state axially symmetric case the poloidal magnetic field may be written in the form

$$\mathbf{B}_P=\frac{\nabla\Psi\times\mathbf{e}_{\hat{\varphi}}}{2\pi\tilde{\omega}}, \tag{5}$$

so the Maxwell equation $\nabla\cdot\mathbf{B}=0$ holds identically. One can easily see that $\mathbf{B}\nabla\Psi=0$. Therefore, the magnetic surfaces are given by the expression $\Psi(r,\theta)=\text{const}$. The proportionality coefficient in Eq. (5) is chosen such that the quantity Ψ is equal to the magnetic flux inside the magnetic tube $\Psi=\text{const}$. It is convenient for us to write down the toroidal magnetic field in the form

$$B_{\hat{\varphi}}=-\frac{2I}{\alpha\tilde{\omega}}. \tag{6}$$

Here $I(r,\theta)$ is the total electric current flowing inside the region $\Psi<\Psi(r,\theta)$ and the caret denotes, as usual, the physical component of the vector (for more explanation see the book by Thorne *et al.*[24]).

In line with other authors we assume that the magnetosphere contains enough plasma to screen the longitudinal electric field component and the corresponding condition $\mathbf{E}\cdot\mathbf{B}=0$ holds. Therefore, the electric field $\mathbf{E}$ must be parallel to $\nabla\Psi$. The electric field is determined directly from the Maxwell equation[24,25]

$$\nabla\times(\alpha\mathbf{E})=\mathscr{L}_\beta\mathbf{B}, \tag{7}$$

where

$$\mathscr{L}_\beta\mathbf{B}\equiv(\beta\nabla)\mathbf{B}-(\mathbf{B}\nabla)\beta$$

is a Lee derivative. From the expression (7) one can obtain that $\mathbf{B}\nabla\Omega^F=0$. As a result we arrive at the following expression for the electric field

$$\mathbf{E}=-\frac{\Omega^F-\omega}{2\pi\alpha}\nabla\Psi. \tag{8}$$

Then the angular velocity Ω^F must have a constant value on the entire magnetic surface $\Psi=\text{const}$ as is evident from the condition $\mathbf{B}\nabla\Omega^F=0$.

$$\Omega^F=\Omega^F(\Psi). \tag{9}$$

The next equation we must involve expresses continuity of the particles flow

$$\nabla\cdot(\alpha n\mathbf{u})=0, \tag{10}$$

where n is the particle concentration in the reference frame comoving with the hydrodynamic flow, $\mathbf{u}=\mathbf{v}/\sqrt{1-v^2}$ is the space component of 4-velocity of the flow. The relation (10) together with the Maxwell equation $\nabla\cdot\mathbf{B}=0$ and the frozen-in condition enables the poloidal component of 4-velocity of the flow $\mathbf{u}_P$ to be expressed as

$$\mathbf{u}_P=\frac{\eta}{n\alpha}\mathbf{B}_P. \tag{11}$$

Here we introduced a new quantity η which, due to the relation $\nabla\cdot(\eta\mathbf{B}_P)=0$, must also be constant on the entire magnetic surface $\Psi=\mathrm{const}$

$$\eta=\eta(\Psi). \tag{12}$$

Thus the quantity $\eta(\Psi)$ has the meaning of an integral of motion similar to the angular velocity Ω^F. In the MHD treatment, the following frozen-in condition is an exact statement

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0. \tag{13}$$

Using this equation one can obtain the expression for $\mathbf{u}$

$$\mathbf{u}=\frac{\eta}{\alpha n}\mathbf{B}+\gamma(\Omega^F-\omega)\frac{\widetilde{\omega}}{\alpha}\mathbf{e}_{\hat{\varphi}} \tag{14}$$

where $\gamma=(1-v^2)^{-1/2}$ is the Lorentz factor of the plasma flow.

To find two more integrals of motion corresponding to relations (4) we have to write down the energy-momentum conservation law $T^{\mu}_{\nu;\mu}=0$. In the '3+1'-formalism and under the condition $\partial/\partial t=0$ it takes the form[24]

$$-\frac{1}{\alpha}(\boldsymbol{\beta}\nabla)\varepsilon=-\frac{1}{\alpha^2}\nabla\cdot(\alpha^2\mathbf{S})+H_{ik}T^{ik} \tag{15}$$

$$\nabla_k T_i^k+\frac{1}{\alpha}S_\varphi\frac{\partial\omega}{\partial x^i}+(\varepsilon\delta_i^k+T_i^k)\frac{1}{\alpha}\frac{\partial\alpha}{\partial x^k}=0. \tag{16}$$

Here $g=-(1/\alpha)\nabla\alpha$ is the gravitational acceleration, $H_{ik}=(1/\alpha)\nabla\beta_k$ is the gravimagnetic tensor field; the gravitational redshift α and the bias function β are defined by the relations (2). The energy density ε, the energy flow $\mathbf{S}$ and the stress tensor T_{ik} are exactly identical to those in a flat space-time and are expressed as

$$\varepsilon^{\mathrm{em}}=\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2)$$

$$\mathbf{S}^{\mathrm{em}}=\frac{1}{4\pi}(\mathbf{E}\times\mathbf{B}) \tag{17}$$

$$T_{ik}^{\mathrm{em}}=\frac{1}{4\pi}\left[-E_iE_k-B_iB_k+\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)g_{ik}\right],$$

and

$$\varepsilon^{\mathrm{matter}}=(\rho_m+Pv^2)\gamma^2$$

$$\mathbf{S}^{\mathrm{matter}}=(\rho_m+P)\gamma^2\mathbf{v} \tag{18}$$

$$T_{ik}^{\mathrm{matter}}=(\rho_m+P)\gamma^2v_iv_k+Pg_{ik}.$$

Substitution of the expressions (17), (18) for components of the energy-momentum tensor into the energy equation (15) and into $i=\varphi$-component of the momentum equation (16) gives us, after using the definitions (5), (6), (8), (14) and carrying out some algebra, the following relations

$$\mathbf{B}_P\nabla\left[\frac{\Omega^F I}{2\pi}+\alpha\mu\gamma\eta\right]-\frac{\omega}{2\pi}\mathbf{B}_P\nabla I+(\mathbf{B}_P\nabla\omega)\mu\eta u_\varphi=0,$$

$$\mathbf{B}_P\nabla\left[\frac{I}{2\pi}+\eta\mu u_\varphi\right]=0, \tag{19}$$

where the quantity μ is the specific enthalpy per particle

$$\mu=\frac{\rho_{\mathrm{m}}+P}{n}. \tag{20}$$

It immediately follows from the relations (19) that the quantities E and L must be conserved along a magnetic field line

$$E=E(\Psi)=\frac{\Omega^F I}{2\pi}+\mu\eta(\alpha\gamma+\omega u_\varphi), \tag{21}$$

$$L=L(\Psi)=\frac{I}{2\pi}+\mu\eta u_\varphi. \tag{22}$$

The possibility of defining $E(\Psi)$ and $L(\Psi)$ is connected with the existence of conserved flows of energy and z-component of the angular momentum (4).

One must add the equation of state to the hydrodynamic equations. To specify the equation of state, it is convenient to use the pressure P and the entropy per particle s as thermodynamical variables. The corresponding thermodynamical potential in this case would be just the specific enthalpy μ (20). The first law of thermodynamics implies[26]

$$\mathrm{d}\mu=\frac{1}{n}\mathrm{d}P+T\mathrm{d}s, \tag{23}$$

so that

$$n=n(P,s),\quad T=T(P,s).$$

The relations above allow one to express μ, T and P as functions of n and s. For instance, if the equation of state is polytropic, i.e. $P=k_0n^\Gamma$, then we have[2]

$$\mu=m+\frac{\Gamma}{\Gamma-1}k_0(s)n^{\Gamma-1},$$

where m is the rest mass of a particle. In the special case of cold plasma this yields simply $\mu=m$. It is worth noting that one must use for m the averaged mass of all the particles constituting the plasma. In particular, for an e^-p plasma $m\simeq m_p/2$, for an e^+e^- plasma $m=m_e$. n is the total concentration of all the particles in the plasma.

We also make an additional assumption that the matter flow is isentropic

$$\mathbf{u}\nabla s=0.$$

In the axially symmetric case this condition and (10) yield

$$s=s(\Psi),$$

so the entropy per particle $s(\Psi)$ is, in fact, the fifth integral of motion.

The next step is to show that for the known poloidal field $\mathbf{B}_P$ and for the given five integrals of motion $\Omega^F(\Psi)$, $\eta(\Psi)$, $s(\Psi)$, $E(\Psi)$ and $L(\Psi)$ one can reconstruct the toroidal magnetic field B_φ, the matter density n and the velocity $\mathbf{v}$. For this purpose we use the conservation laws (21), (22) which together with the φ-component of equation (14) allow expressing the electric current I, the Lorentz factor γ and the physical toroidal velocity $u_{\hat\varphi}=u_\varphi/\tilde\omega$ as follows

$$\frac{I}{2\pi}=\frac{\alpha^2L-(\Omega^F-\omega)\tilde\omega^2(E-\omega L)}{\alpha^2-(\Omega^F-\omega)^2\tilde\omega^2-M^2} \quad (24)$$

$$\gamma=\frac{1}{\alpha\mu\eta}\frac{\alpha^2(E-\Omega^FL)-M^2(E-\omega L)}{\alpha^2-(\Omega^F-\omega)^2\tilde\omega^2-M^2} \quad (25)$$

$$u_{\hat\varphi}=\frac{1}{\tilde\omega\mu\eta}\frac{(E-\Omega^FL)(\Omega^F-\omega)\tilde\omega^2-LM^2}{\alpha^2-(\Omega^F-\omega)^2\tilde\omega^2-M^2}. \quad (26)$$

From now on we denote

$$M^2=\frac{4\pi\eta^2\mu}{n}. \quad (27)$$

Introducing the Alfvén velocity $u_A = B_P/\sqrt{4\pi n\mu}$ we can write M^2 in the form

$$M^2=\alpha^2\frac{u_P^2}{u_A^2}.$$

Thus M^2 is (up to the factor α^2) the Mach number calculated for the poloidal velocity u_P with respect to the Alfvén velocity u_A. It is the value of M^2 which is convenient to be used subsequently because it remains finite at the horizon provided the field and matter fluxes are regular there. According to equation (23) $\mu=\mu(n,s)$ and hence Eq. (27) allows the particle number density n (and, hence, the specific enthalpy μ) to be expressed as a function of η, s and M^2. It means that besides five integrals of motion $\Omega^F(\Psi)$, $\eta(\Psi)$, $s(\Psi)$, $E(\Psi)$ and $L(\Psi)$ only one additional quantity (the Mach number M) enters the expressions for I, γ and $u_{\hat\varphi}$.

To determine the Mach number M one should use the obvious relation

$$\gamma^2-\mathbf{u}^2=1. \quad (28)$$

Substitution of expressions for γ and $u_{\hat\varphi}$ (Eqs. (25)–(26)), and Eq. (11) for $\mathbf{u}_P$ into this relation gives

$$\frac{K}{\tilde\omega^2A^2}=\frac{1}{64\pi^4}\frac{M^4(\nabla\Psi)^2}{\tilde\omega^2}+\alpha^2\eta^2\mu^2, \quad (29)$$

where

$$A=\alpha^2-(\Omega^F-\omega)^2\tilde\omega^2-M^2 \quad (30)$$

and

$$K=\alpha^2\tilde\omega^2(E-\Omega^FL)^2[\alpha^2-(\Omega^F-\omega)^2\tilde\omega^2-2M^2] +M^4[\tilde\omega^2(E-\omega L)^2-\alpha^2L^2]. \quad (31)$$

Equations (24–26) and (29) are the desired algebraic relations, which for the known poloidal field $\mathbf{B}_P$ (5) (and, hence, for the known stream function Ψ) and for the given integrals of motion allow us to find all components of the plasma four-velocity u^i and toroidal magnetic field B_φ (6).

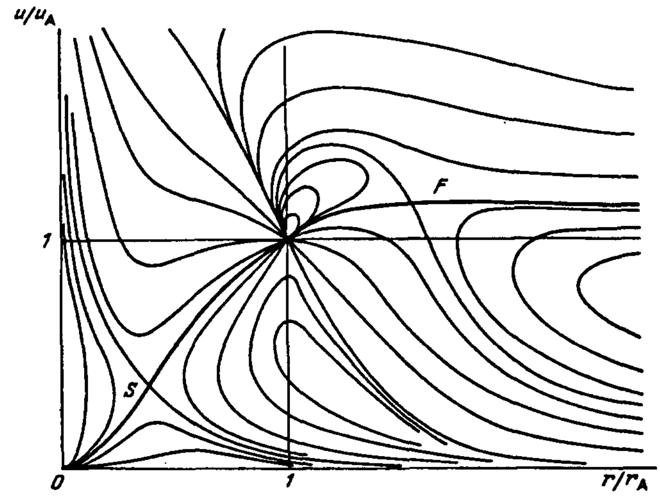

FIG. 1. The phase diagram representing the dependence of the poloidal flow velocity u on the distance r to the pointlike Newtonian gravitating center. The MHD flow is nonrelativistic and is at the equatorial plane of the monopole poloidal magnetic field $\Psi=\Psi_0(1-\cos\theta)$. The diagram is obtained as a result of solving the algebraic equation (29) for different values of E. The quantities u_A, r_A are the values of u and r at the Alfvén point; F is the fast magnetosonic point and S is the slow one. The values of the other four integrals of motion are chosen so that the solution passing through all three critical points exists. This solution is subsonic when $r\to 0$ and supersonic when $r\to\infty$. It is shown by the heavy line (Weber and Davis[4]).

It is equations (24–26, 29) that have been analyzed in the papers on solar wind,[4–6] on accretion onto neutron stars and black holes,[1,2] on plasma ejection from pulsars[10–12] and in the most general case of the Kerr metric in Refs. 19–22.

2.2. Critical points

The MHD-flow, described by the algebraic relations obtained above, is characterized by the following singular points. These points are

1. *The Alfvén point A* defined from the condition of setting to zero the denominator A (30) in the relations (24–26)

$$A=0. \quad (32)$$

As pointed out in Ref. 4 for a monopole magnetic field, the Alfvén point is a higher order singular point in coordinates u^r-r than those of ordinary "X"- or "O"-type (see Fig. 1). At the same time all tracks with positive squares of energy E pass through this point irrespectively of the values of the integrals of motion.[22]

2. *The fast magnetosonic point F* is defined as a singularity in the expression for the gradient of M (and, hence, as a singularity in the gradient of four-velocity $\mathbf{u}$). Indeed, Eqs. (29–31) may be expressed in the form

$$(\nabla\Psi)^2=F(M^2,\ E,\ L,\ \eta,\ \Omega^F,\mu), \quad (33)$$

where

$$F=\frac{64\pi^4}{M^4}\frac{K}{A^2}-\frac{64\pi^4}{M^4}\alpha^2\tilde\omega^2\eta^2\mu^2. \quad (34)$$

Taking the gradient of both sides of Eq. (33) one can obtain the following expression for ∇M^2

$$\nabla_a M^2 = -\frac{A}{(\nabla\Psi)^2 D}\nabla^b\Psi\nabla_a\nabla_b\Psi + \frac{A}{2}\frac{\nabla'_a F}{(\nabla\Psi)^2 D}. \quad (35)$$

Here and again later the subscripts a, b run through the values r, θ only and the gradient ∇' acts on all the variables except M^2. We denote by D in (35) the quantity

$$D = -\frac{A}{2F}\frac{\partial F}{\partial M^2}$$

which may be written as

$$D = \frac{A}{M^2} + \frac{\alpha^2}{M^2}\frac{B_{\hat\varphi}^2}{B_P^2} - \frac{1}{u_P^2}\frac{A}{M^2}\frac{a_s^2}{1-a_s^2}, \quad (36)$$

where $a_s^2 = (1/\mu)(\partial P/\partial n)_s$ is the square of the speed of sound. In deriving Eq. (36) we used the first law of thermodynamics (23), which implies (see, for instance, Ref. 2)

$$d\mu = \frac{a_s^2}{1-a_s^2}\mu\left[2\frac{d\eta}{\eta} - \frac{dM^2}{M^2}\right] + \frac{1}{1-a_s^2}\left[\frac{1}{n}\left(\frac{\partial P}{\partial s}\right)_n + T\right]ds. \quad (37)$$

It is the condition $D=0$ that determines the position of the fast magnetosonic point. In contrast to the Alfvén point, in a monopole magnetic field a fast magnetosonic one is of the saddle point-type.[22] Due to this fact regular solutions without any stagnation points and infinite derivatives exist only when an appropriate relation between the integrals of motion holds and the numerator in (35) vanishes at the fast magnetosonic point (see Fig. 2). The condition $D=0$ determines the position of the slow magnetosonic point in the same way.

Finally, a characteristic point is

3. *The light cylinder* R_L defined as a surface where the electric field **E** is equal in magnitude to the poloidal component of the magnetic field $\mathbf{B}_P$.

One more property inherent exclusively in the black hole magnetosphere should be noted. It lies in the fact that general relativistic effects lead to the appearance of the second family of singularities near the black hole horizon side by side with the usual outer singularities—the "outer" light cylinder R_L, the Alfvén point A lying inside it and, probably, the "outer" fast magnetosonic point F (Ref. 25). According to Takahashi *et al.*,[22] in the case of a monopole poloidal magnetic field the F-point is always between the horizon and the inner Alfvén point A (see Fig. 2).

We should point out that the outer Alfvén point (as has already been mentioned, all tracks pass through this point) lies in the upper half of the phase plane $u^r > 0$ corresponding to the outflowing plasma, while the inner Alfvén point always lies in the lower one $u^r < 0$ corresponding to the accreting plasma. This extremely important fact is in contradiction with the assumption that the function η is constant along a given field line Ψ=const because different signs of longitudinal velocity u^r must correspond to different signs of η according to Eq. (11). Consequently, a plasma flow cannot be continuous everywhere in the black

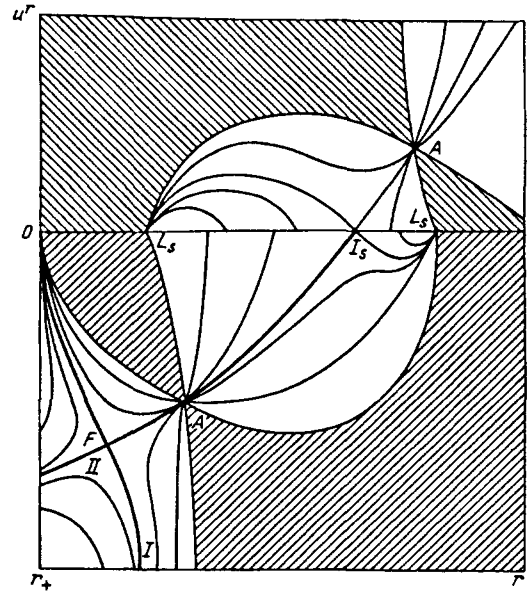

FIG. 2. The solutions of the algebraic equation (29) for different values of E at the equatorial plane $\theta = \pi/2$ of a Kerr black hole with $a = 0.8$.// for cold matter ($s=0$). The magnetic field is monopole, $0 < \Omega^F < \Omega^H$. Outer and inner Alfvén points are denoted by A, F is an inner fast magnetosonic point, I_s is the point of matter injection with zero speed as a result of plasma generation, L_s is the light cylinder. In the hatched region there is no solution, i.e. equation (29) *cannot* be satisfied for any values of E. The separation lines of the saddlepoint-type singularity F are shown by heavy lines. The quantities E, L, η have a discontinuity when $u_r = 0$ caused by the plasma creation at a point I_s (Takahashi *et al.*[22]).

hole magnetosphere. It must have a discontinuity on the field lines penetrating the horizon. So, one has to allow the existence of a plasma source between two Alfvén points determining different values of η in the outer and inner parts of the black hole magnetosphere.

2.3. The equation for a poloidal magnetic field

Thus, the system of algebraic relations (24–26, 29) describing plasma motion in a given poloidal magnetic field allows us to advance substantially in our understanding of the main features of steady-state axially symmetric flows. But on the basis only of these algebraic relations one cannot determine the values of the integrals of motion Ω^F, η, s, E and L themselves, which is of principal importance for obtaining one of the main characteristics of the system, namely, the energy loss by the central object. Moreover, in the overwhelming majority of studies plasma flow was considered only for a monopole magnetic field (Refs. 10, 12, 22) which is, in general, not selfconsistent (i.e. does not satisfy the poloidal field equation). Obviously, it is impossible to understand the nature of the jets observed in active galactic nuclei and quasars remaining in the framework of the monopole magnetic field geometry. Finally, it is clear that in the case of an equation in partial derivatives, the behavior of the solutions in proximity of the singular points must be specially investigated.

The situation with the structure of the poloidal magnetic field and longitudinal currents is much less defined.

At the same time, only by considering this problem can one hope to investigate, for instance, the possible jet collimation to the rotational axis. The greatest progress in this problem has been achieved only in the force-free approximation, when the magnetic field energy density greatly exceeds the plasma energy density and it is possible to disregard particle masses and to write down $T=T^{\rm em}$. This approximation is interesting, first of all, for the description of a neutron star magnetosphere and an appropriate equation for the structure of the poloidal magnetic field was already obtained long ago.[8,9] This equation is a generalization of the Grad–Shafranov equation derived for toroidal plasma configurations as far back as the 1950s. In the general case of the Kerr metric it has the form[25]

$$\frac{1}{\alpha}\nabla\left\{\frac{\alpha}{\tilde{\omega}^2}\left[1-\frac{(\Omega^F-\omega)^2\tilde{\omega}^2}{\alpha^2}\right]\nabla\Psi\right\}+\frac{\Omega^F-\omega}{\alpha^2}(\nabla\Psi)^2\frac{d\Omega^F}{d\Psi}+\frac{16\pi^2}{\alpha^2\tilde{\omega}^2}I\frac{dI}{d\Psi}=0. \qquad (38)$$

Equation (38) is an elliptic second-order equation for the stream function $\Psi(r,\theta)$ which contains as sources two unknown functions—the angular velocity $\Omega^F(\Psi)$ and the current $I(\Psi)$. We note that in the force-free approximation, the current $I(\Psi)$ becomes an integral of motion as well, as it is seen from the definition (22).

However, it is clear that in the force-free approximation the information about the influence of particle masses on the magnetic field structure is lost. But this influence must undoubtedly be taken into account, for instance, in the outer regions of a neutron star magnetosphere.[10,11] As we shall see later, Eq. (38) is deliberately violated near the black hole event horizon as well. So it is of indubitable interest to investigate a more general equation for a poloidal field including effects related to nonzero particle masses. Similar to Eq. (38), such an equation must contain only the stream function Ψ, the integrals of motion $\Omega^F(\Psi)$, $\eta(\Psi)$, $L(\Psi)$, $E(\Psi)$ depending on Ψ and, in general, the entropy per unit particle $s(\Psi)$.

Ardavan[27] was the first to obtain in 1979 a similar equation for an axially symmetric steady-state magnetosphere disregarding gravitational effects. This equation was subsequently discussed in Refs. 28–30. The equation for a poloidal magnetic field was numerically investigated for a nonrelativistic plasma in Refs. 18, 31–34 and for a relativistic plasma in the presence of a gravitational field—in Refs. 35–40. Finally, recently in Ref. 41 this equation was obtained in the most general case of the Kerr metric but for cold matter, i.e. when $s(\Psi)\equiv 0$.

So, let us write down the poloidal component of the momentum conservation equation (16)

$$\nabla_k T^k_a+\frac{1}{\alpha}S_\varphi\nabla_a\omega+(\varepsilon\delta^k_a+T^k_a)\frac{1}{\alpha}\nabla_k\alpha=0,$$

where the indices a and b run through the r and θ only. Substituting the energy density ε, the energy flow $\mathbf{S}$ and the stress tensor T from Eqs. (17), (18) into this equation, we obtain

$$Z=Z_{\rm em}+Z_{\rm matter}=0,$$

$$Z_{\rm em}=\frac{1}{16\pi^3\alpha}\left\{\nabla_k\left\{\frac{\alpha}{\tilde{\omega}^2}\left[1-\frac{(\Omega^F-\omega)^2\tilde{\omega}^2}{\alpha^2}\right]\nabla^k\Psi\right\}\nabla_a\Psi+\frac{\Omega^F-\omega}{\alpha}(\nabla\Psi)^2\frac{d\Omega^F}{d\Psi}\nabla_a\Psi+\frac{16\pi^2}{\alpha\tilde{\omega}^2}I\nabla_a I\right\}, \qquad (39)$$

$$Z_{\rm matter}=nu^b\nabla_b(\mu u_a)+\nabla_a P-\mu n(u_{\hat\varphi})^2\frac{1}{\tilde{\omega}}\nabla_a\tilde{\omega}+\frac{1}{\alpha}\mu n\gamma(\tilde{\omega}u_{\hat\varphi})\nabla_a\omega+\mu n\gamma^2\frac{1}{\alpha}\nabla_a\alpha. \qquad (40)$$

Here the first term (Eq. (39)) corresponds to the electromagnetic contribution (17) and the second one (Eq. (40)) corresponds to the hydrodynamic contribution (18).

We see that in the force-free approximation, where $I=I(\Psi)$ and, hence, $\nabla_a I=(dI/d\Psi)\nabla_a\Psi$, the first term (Eq. (39)) is proportional to the l.h.s. of Eq. (38). At the same time, contrary to Eq. (38), the equation $Z_{\rm em}=0$ is of a vector type and cannot be reduced to one second-order equation for the stream function Ψ in the general case. However, we shall show that the equation $Z=0$ can be reduced to a scalar one.

Indeed, using Eq. (14) which relates the four-velocity $\mathbf{u}$ to the magnetic field $\mathbf{B}$, and substituting in the last term of Eq. (40) the quantity $1+u^bu_b+(u_{\hat\varphi})^2$ for γ^2 one can obtain after some simple but awkward calculations the following vector equation

$$\frac{1}{\alpha}\nabla_k\left\{\frac{1}{\alpha\tilde{\omega}^2}[\alpha^2-(\Omega^F-\omega)^2\tilde{\omega}^2-M^2]\nabla^k\Psi\right\}\nabla_a\Psi+\frac{1}{\alpha^2}\frac{1}{2M^2}\times\nabla_a\left[\frac{M^4(\nabla\Psi)^2}{\tilde{\omega}^2}\right]+\frac{\Omega^F-\omega}{\alpha^2}\frac{d\Omega^F}{d\Psi}(\nabla\Psi)^2\nabla_a\Psi+\frac{16\pi^2}{\alpha^2\tilde{\omega}^2}I\nabla_a I$$
$$+16\pi^3\nabla_a P+\frac{16\pi^3}{\alpha^2}\mu n(\gamma\alpha)(\tilde{\omega}u_{\hat\varphi})\nabla_a\omega-16\pi^3\mu n(u_{\hat\varphi})^2\times\frac{1}{\tilde{\omega}}\nabla_a\tilde{\omega}+16\pi^3\mu n(1+u^2_{\hat\varphi})\frac{1}{\alpha}\nabla_a\alpha=0. \qquad (41)$$

Here $M^2=4\pi\mu\eta^2/n$ as before.

Then we substitute into Eq. (41) the quantity $M^4(\nabla\Psi)^2/\tilde{\omega}^2$ its value determined from the relation (29), differentiate and reduce similar terms. As a result, it turns out that the coefficients at the gradients $\nabla_a M^2$, $\nabla_a\alpha$, $\nabla_a\tilde{\omega}$ and $\nabla_a\omega$ are identically equal to zero. For transformation of the pressure gradient $\nabla_a P$ we must use the first law of thermodynamics in the form (23) which gives us

$$dP=nd\mu-nTds. \qquad (42)$$

Here we take into account that the flow is isentropic, so the entropy s must be constant along a magnetic field line

$$s=s(\Psi) \qquad (43)$$

(for the polytropic equation of state $P=k_0n^\Gamma$ the fifth integral of motion corresponding to $s(\Psi)$ will be the quantity $k_0=k_0(\Psi)$). As a result, the vector $\nabla_a P-n\nabla_a\mu$ must be orthonormal to the magnetic field line and, hence, parallel to $\nabla_a\Psi$. Finally, it is obvious that the gradients of the integrals of motion $\Omega^F(\Psi)$, $\eta(\Psi)$, $E(\Psi)$ and $L(\Psi)$ are proportional to $\nabla_a\Psi$ as well.

Finally, the equation for the poloidal field (39), (40) was reduced to one scalar second-order equation multiplied by the $\nabla_a\Psi$ similar to that in the force-free approximation. It can be expressed as

$$\frac{1}{\alpha}\nabla_k\left[\frac{1}{\tilde{\omega}\alpha}A\nabla^k\Psi\right]+\frac{\Omega^F-\omega}{\alpha^2}(\nabla\Psi)^2\frac{d\Omega^F}{d\Psi}+\frac{64\pi^4}{\alpha^2\tilde{\omega}^2}\frac{1}{2M^2}\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right)-16\pi^3\mu n\frac{1}{\eta}\frac{d\eta}{d\Psi}-16\pi^3 nT\frac{ds}{d\Psi}=0, \quad (44)$$

where

$$G=\alpha^2\tilde{\omega}^2(E-\Omega^F L)^2+\alpha^2M^2L^2-M^2\tilde{\omega}^2(E-\omega L)^2. \quad (45)$$

The fact that the terms in Eqs. (39), (40), which are not parallel to $\nabla_a\Psi$ are cancelled may be understood as follows. Let us write down the energy-momentum conservation in the orthonormal basis $\mathbf{e}_{\tilde{0}}$, $\mathbf{e}_{\tilde{\varphi}}$, $\mathbf{e}_{\tilde{\lambda}}$, $\mathbf{e}_{\tilde{\Psi}}$, where $\mathbf{e}_{\tilde{0}}=\mathbf{e}_{\hat{0}}$, $\mathbf{e}_{\tilde{\varphi}}=\mathbf{e}_{\hat{\varphi}}$, $\mathbf{e}_{\tilde{\lambda}}=\mathbf{B}_p/|\mathbf{B}_p|$, $\mathbf{e}_{\tilde{\Psi}}=\nabla\Psi/|\nabla\Psi|$. We have $T^{\tilde{\mu}\tilde{\nu}}_{;\tilde{\nu}}=0$, where $T^{\mu\nu}$ is a sum of the parts given by the expressions (17) and (18). The $\tilde{0}$ and $\tilde{\varphi}$ components of this equation can be integrated according to (19) and lead to the relations (21), (22) with two arbitrary functions $E(\Psi)$, $L(\Psi)$ arising due to the integration. Then one can show that, provided the frozen-in condition $u^{\tilde{\mu}}F_{\tilde{\mu}\tilde{\nu}}=0$ is fulfilled (this condition is identical to Eq. (13)), the energy-momentum conservation law projection onto the plasma 4-velocity $u_{\tilde{\mu}}T^{\tilde{\mu}\tilde{\nu}}_{;\tilde{\nu}}=0$ leads to the flow being adiabatic (43). Adiabaticity was explicitly used when deriving Eq. (44). Thus, only one of the four components $T^{\tilde{\mu}\tilde{\nu}}_{;\tilde{\nu}}=0$ is not identically satisfied by the relations (21), (22), (29) and (43). It is this component that we obtain in the form (44).

Thus, the poloidal component of the momentum equation (16) is in fact the equilibrium equation for the magnetic surfaces $\Psi=$const.

Finally, expressing in Eq. (44) the terms $\nabla_a M^2$ according to Eq. (35) we obtain our main equation

$$A\left[\frac{1}{\alpha}\nabla_k\left(\frac{1}{\alpha\tilde{\omega}^2}\nabla^k\Psi\right)+\frac{1}{\alpha^2\tilde{\omega}^2(\nabla\Psi)^2}\frac{\nabla^a\Psi\nabla^b\Psi\nabla_a\nabla_b\Psi}{D}\right]+\frac{1}{\alpha^2\tilde{\omega}^2}\nabla'_kA\nabla^k\Psi-\frac{A}{\alpha^2\tilde{\omega}^2(\nabla\Psi)^2}\frac{1}{2D}\nabla'_kF\nabla^k\Psi+\frac{\Omega^F-\omega}{\alpha^2}\frac{d\Omega^F}{d\Psi}(\nabla\Psi)^2+\frac{64\pi^4}{\alpha^2\tilde{\omega}^2}\frac{1}{2M^2}\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right)-16\pi^3\mu n\frac{1}{\eta}\frac{d\eta}{d\Psi}-16\pi^3nT\frac{ds}{d\Psi}=0, \quad (46)$$

where the gradient ∇'_a denotes the action of ∇_a under the condition that M is fixed and the derivative $\partial/\partial\Psi$ is acting only on the integrals of motion while the other variables are considered as constants. Let us stress that in equation (46) the pressure P, the temperature T and the specific enthalpy μ are to be expressed via an equation of state in terms of the entropy $s(\Psi)$ and the square of the Mach number M^2. In turn, the quantity M^2 is to be considered as a function of $(\nabla\Psi)^2$ and the integrals of motion

$$M^2=M^2[(\nabla\Psi)^2,\ E(\Psi),\ L(\Psi),\ \eta(\Psi),\ \Omega^F(\Psi),\ s(\Psi)]. \quad (47)$$

The latter relation is an unexplicit form of Eq. (29). The stream equation (46) with the definitions (30), (34), (36), (45) is the desired equation for the poloidal field which contains only the magnetic flux Ψ and the five integrals of motion $\Omega^F(\Psi)$, $\eta(\Psi)$, $s(\Psi)$, $E(\Psi)$ and $L(\Psi)$ depending on it.

3. DISCUSSION

3.1. Basic properties of the equation for a poloidal magnetic field

Let us discuss the basic properties of equation (46)—a nonlinear second-order differential equation. It is a mixed type equation. One can easily check that in the region where $D>0$ the equation is elliptic and where $D<0$, it is hyperbolic.

First of all, consider the region near the black hole event horizon. If we suppose that the square of the Mach number M^2 (27) does not approach 0 at the horizon $r=r_+$ then the quantity $D(r_+)$ (36) may be rewritten as follows

$$D(r_+)=-1+\frac{\alpha^2}{M^2B_P^2}(B_{\hat{\varphi}}^2-E^2) \quad (48)$$

Here the second term in the r.h.s. is finite when $\alpha\to 0$ by virtue of the definitions (6), (8). Equation (29) implies that if $M^2(r_+)\neq 0$ the following relation must be fulfilled at the horizon

$$\frac{(E-\Omega^H L)^2}{[(\Omega^F-\Omega^H)^2\tilde{\omega}^2-M^2]^2}=\frac{1}{64\pi^4}\frac{(d\Psi/d\theta)^2}{\rho^2\tilde{\omega}^2}, \quad (49)$$

where $\Omega^H=\omega(r_+)$ is the angular velocity of the black hole rotation. Then, taking into account Eqs. (6) and (24) we obtain the equality

$$B_{\hat{\varphi}}(r_+)=E_{\hat{\theta}}(r_+). \quad (50)$$

This result is in full agreement with the basic proposition of the "membrane paradigm" that the zero angular momentum observer situated close to the horizon must detect only the φ-component of the magnetic field and the θ-component of the electric field, both diverging like $1/\alpha$ with the Pounting vector directed toward the black hole.[24] Hence, the following condition must be fulfiled at the event horizon

$$D(r_+)=-1 \quad (51)$$

so for the case $M^2(r_+)\not\equiv 0$ equation (46) near the black hole horizon is hyperbolic. In Fig. 2 the hyperbolicity regions are lying to the left of the separation line I. In particular, equation (46) is hyperbolic for the inner part of separation line II passing through the fast magnetosonic point.

Thus, we come to an important conclusion that a region where equation (46) is hyperbolic exists between the black hole horizon and the fast magnetosonic point for the integrals of motion corresponding to separation line *II*. So, if a plasma source is situated in the ellipticity region then equation (46) is of mixed type for an accreting plasma. As for relation (49), in this case it does not mean the boundary condition at the event horizon but it may be understood as a relation for determining the value of $M^2(r_+)$.

On the other hand, for the integrals of motion corresponding to tracks on the phase plane $u_{\hat{p}}-r$ deflecting to the separation line *I* so that $u_{\hat{p}}\to 0$, $M^2\to 0$ as $r\to r_+$ on these tracks, one can formally see that the quantity D remains positive there and equation (46) is elliptic. However, it is clear that in this case the particle number density must go to infinity on approaching to the horizon. So the approximations used for the derivation of equation (46) will be violated near a black hole event horizon. The result is that plasma emission and other processes not allowed for by Eq. (46) become significant in this region. This makes senseless the discussion of "boundary conditions" at the black hole horizon.

Let us mention some other important properties of the equation for a poloidal magnetic field. First of all it is necessary to stress that the solution of this equation can be obtained only when the five integrals of motion, $\Omega^F(\Psi)$, $\eta(\Psi)$, $s(\Psi)$, $E(\Psi)$ and $L(\Psi)$, are known. These integrals of motion must be determined first of all by a particular particle creation mechanism and must be given, in fact, as boundary conditions.[15,16] As we have already seen, on the magnetic field lines penetrating into the black hole such a source must ensure both an outflow to infinity and an accretion onto the horizon. At the same time, because the poloidal field equation (46) is highly nonlinear, it may have physically admissible solutions only if the integrals of motion themselves are appropriately chosen. This problem is well known in connection with the construction of a self-consistent model for a neutron star magnetosphere and arises even for the force-free equation (38) in a flat space-time.[16]

Finally, it is interesting to compare the question itself about posing the problem for a black hole magnetosphere with an analogous one arising in the investigation of a radio pulsar magnetosphere. In a neutron star the magnetosphere plasma is believed to be created in the vacuum gap near the star surface[15,16] and, probably, in the outer gap.[42] Such a plasma source can determine, in general, four parameters—the longitudinal velocity component $v_{\parallel}$, the density, the electric current and the entropy, i.e. four integrals of motion $E(\Psi)$, $L(\Psi)$, $\eta(\Psi)$ and $s(\Psi)$. But for a neutron star magnetosphere, besides these four integrals of motion a fifth one, $\Omega^F(\Psi)$, is also defined. Indeed, the electrical potential determining the value of Ω^F (and held constant on the magnetic surface $\Psi=$const, as, is done for Ω^F is uniquely defined by the rotational angular velocity of the neutron star and by the voltage drop across the gap (see Refs. 15, 16 for details). The situation is substantially different in the case of a black hole magnetosphere. The reason is that the black hole event horizon does not have a causal relationship with the plasma generation region.[20,21] As a result, the fifth integral of motion Ω^F, which is responsible for the energy flow in the magnetosphere according to relation (21), remains what we think of as a free parameter of the problem (see, however, Refs. 19, 25).

3.2. Particular cases of the equation for a poloidal magnetic field

a) Force-free approximation $\mu\to 0$

One can immediately see from the relation (39) that equation (46) goes over to the force-free Eq. (38) in the limit $\mu\to 0$, i.e. when the magnetic field energy density is much higher than the energy density of particles. This is a second-order elliptical equation. In application to active galactic nuclei and quasars it has been investigated in Refs. 43–48.

b) Hydrodynamic limit $M^2\to\infty$

We now consider a hydrodynamic limit when the plasma energy density is much higher than that of the magnetic field. In this case it is natural to introduce a new potential $\Phi(\Psi)$ according to the following relation

$$\eta(\Psi)=\frac{d\Phi}{d\Psi}. \qquad (52)$$

Pure hydrodynamics corresponds to the limit $\Psi\to 0$, $\eta\to\infty$ where, however, the product $\eta\Psi$ remains finite. By the definitions (5) and (11), the expression for plasma flux density is

$$\alpha n\mathbf{u}_P=\frac{1}{2\pi\tilde{\omega}}(\nabla\Phi\times\mathbf{e}_{\hat{\varphi}}). \qquad (53)$$

The surfaces of $\Phi=$const are the plasma flow surfaces.

It is easy to check that from the mathematical point of view the replacement (52) corresponds to the condition $\eta=1$. As a result, in this approximation there will be only three integrals of motion

$$E(\Phi)=\mu(\alpha\gamma+\tilde{\omega}\omega u_{\hat{\varphi}}), \qquad (54)$$

$$L(\Phi)=\mu\tilde{\omega}u_{\hat{\varphi}} \qquad (55)$$

and $s(\Phi)$. Then, the algebraic relation (29) will be rewritten in the form

$$(E-\omega L)^2=\alpha^2\mu^2+\frac{\alpha^2}{\tilde{\omega}^2}L^2+\frac{\hat{M}^4}{64\pi^4\tilde{\omega}^2}|\nabla\Phi|^2, \qquad (56)$$

where the square of the "Mach number" $\hat{M}^2$ is

$$\hat{M}^2=\frac{4\pi\mu}{n}. \qquad (57)$$

For the equation for the stream function Φ one can easily obtain the expression

$$-\hat{M}^2\left[\frac{1}{\alpha}\nabla_k\left(\frac{1}{\alpha\tilde{\omega}^2}\nabla^k\Phi\right)+\frac{1}{\alpha^2\tilde{\omega}^2|\nabla\Phi|^2}\frac{\nabla^a\Phi\nabla^b\Phi\nabla_a\nabla_b\Phi}{D}\right]$$
$$+\frac{\hat{M}^2\nabla'_k\hat{F}\nabla^k\Phi}{2\alpha^2\tilde{\omega}^2(\nabla\Phi)^2D}+\frac{32\pi^4}{\alpha^2\tilde{\omega}^2\hat{M}^2}\frac{\partial}{\partial\Phi}[\tilde{\omega}^2(E-\omega L)^2$$
$$-\alpha^2L^2]-16\pi^3nT\frac{ds}{d\Phi}=0, \qquad (58)$$

where

$$D=-1+\frac{1}{u_P^2}\frac{a_s^2}{1-a_s^2}, \tag{59}$$

$$\hat{F}=\frac{64\pi^4}{M^4}[\tilde{\omega}^2(E-\omega L)^2-\alpha^2L^2-\tilde{\omega}^2\alpha^2\mu^2]. \tag{60}$$

Equation (58) contains only one singular point, i.e., the sound point determined by the condition $D=0$. Just as should be expected, at a sound point the poloidal four-velocity is equal to the sound four-velocity[2]

$$u_P^2=\frac{a_s^2}{1-a_s^2}.$$

Equation (58) describes an axially symmetric steady-state hydrodynamic flow in the vicinity of a Kerr black hole. In the special case of Schwarzschild space-time when $L(\Phi)\equiv 0$ and for spherically symmetrical boundary conditions it has a simple solution corresponding to a spherically symmetrical accretion. This solution of Eq. (58) is the following monopole flow field

$$\Phi=\Phi_0(1-\cos\theta). \tag{61}$$

As for the condition of passing through a sound point (which in this case is a saddle point), it determines in fact the accretion rate $\dot{\mathcal{M}}=2m\Phi_0$

$$\dot{\mathcal{M}}=4\pi m\mathcal{M}^2\frac{n_s}{u_s}\left(1+\frac{1}{4u_s^2}\right) \tag{62}$$

where n_s and u_s are quantities evaluated at the sound point. In the case of the polytropic equation of state, $P=k_0(\Phi)n^{\Gamma}$, one can explicitly express n_s and u_s and, hence, Φ_0 via two nonzero integrals of motion, $E(\Phi)$ and $k_0(\Phi)$,

$$u_s^2=\frac{5\Gamma-6-2(\Gamma-1)\left(\frac{m}{E}\right)^2+\sqrt{(3\Gamma-2)^2-12(\Gamma-1)\left(\frac{m}{E}\right)^2}}{2\left[8-4\Gamma+(\Gamma-1)\left(\frac{m}{E}\right)^2\right]},$$

$$n_s=\frac{\Gamma-1}{\Gamma k_0}\left(\frac{mu_s^2}{\Gamma-1+(\Gamma-2)u_s^2}\right)^{1/\Gamma-1}$$

Of course, the relation (62) coincides with the Bondi condition for spherically symmetrical accretion.[1,2]

3.3. Validity area of the equation for a poloidal magnetic field

In conclusion we shall make some remarks on the validity area of the equations discussed above. First, it is clear that following the approach under consideration, one cannot take into account particle interaction with radiation which may undoubtedly be important for the case of active galactic nuclei and quasars.[17] The radiation will produce a force acting on each particle. So, it is clear that Eq. (46) can be applied to objects with a low luminosity L only (and, correspondingly, with a small photon number density U). The natural limit for luminosity in the case of an electron-proton plasma is the Eddington luminosity $L_{\rm Ed}\simeq 10^{38}\mathcal{M}/\mathcal{M}_{\odot}$erg/s. But for an e^+e^- plasma the restriction $L<L_{\rm Ed}$ will be invalid because the force produced by the photons will be the same for e^- and e^+ and, as a result, there will be no charge separation and no associated polarizational electric field.[49]

At the same time, regardless of plasma composition, particles have losses due to inverse Compton scattering in the photon field.[50] For an e^+e^- plasma the condition for these losses $\sigma_T U\gamma^2\varepsilon_{\rm ph}$ being much smaller than the energy gain rate $d\gamma/dr\sim\delta\gamma/r_+$, $\delta\ll 1$ may be written in the form

$$L<\frac{4\pi R_L^2 cm_e c^2}{\gamma R_+\sigma_T}\simeq 1.4\cdot 10^{44}\frac{1}{\gamma}\left(\frac{R_L}{3R_+}\right)^2\left(\frac{\mathcal{M}}{10^8\mathcal{M}_{\odot}}\right)\text{erg/s}, \tag{63}$$

where $L=\varepsilon_{\rm ph}Uc4\pi R_L^2$ is the luminosity of the photon gas with the photon energy $\varepsilon_{\rm ph}$ and the number density U occupying a region with a characteristic dimension R_L. The restriction (63) is probably the most crucial for a possible application of the approach considered in the present paper. If the inequality (63) is violated then particle retardation (or acceleration) in a photon field will play a significant role in the plasma dynamics and, as a result, the energy $E(\Psi)$ (22) and the angular momentum $L(\Psi)$ (23) will no longer be integrals of motion. It should be noticed that pure kinetic effects must be important under such conditions.[51]

We stress that in the case of a sufficiently rare plasma $M^2\ll 1$, when the particle energy density is much smaller than that of the electromagnetic field, a violation of the algebraic relations (24)–(31) does not lead to a noticeable distortion of the force-free equation (38). At the same time the interaction of particles with the radiation field would be essential irrespectively of the value of M^2 in the vicinity of the singular points where the value of M^2 is very close to the difference $\alpha^2-(\Omega^F-\omega)^2\tilde{\omega}^2$. Any small nonzero pressure and temperature must also exert an influence upon the plasma motion in the vicinity of singular points.

Then, the pattern discussed here may be distorted because of the plasma selfradiation leading to a change of the entropy s along a magnetic line of fore. However, one can notice that the term $nTds/d\Psi$ in equation (46) is significant only for a nonrelativistic plasma. Indeed, it is easy to verify that the pressure gradient $\nabla_a P$ in Eq. (41) is of the order of the terms connected to 1 in the relation $\gamma^2-u^2=1$, i.e. is of the order of the term $\alpha^2\eta^2\mu^2$ in Eq. (29) and of the last term in Eq. (41). In the ultrarelativistic limit these terms may be neglected, so in the region where $\gamma\gg 1$, in particular in the vicinity of the black hole event horizon, a change of the entropy s along a field line cannot lead to a strong violation of the approximation considered in the present paper.

There is evidence (particularly, observations made in the COMPTON γ-ray observatory[52]) that many active galactic nuclei and quasars are sources of hard γ radiation which may be generated in the process of effective heating occurring in the inner regions of accreting matter.[53] A sufficiently large number density of γ quanta with energy exceeding $m_e c^2$ may lead to direct e^+e^--pair creation due to the reaction $\gamma+\gamma\to e^+e^-$ as pointed out by Kardashev *et al.*[54] In this case, when $M^2\gg 1$, the accretion picture differs significantly from the model discussed in the present paper because of the particle number N changing along a magnetic field line.

It is clear that the secondary particle creation will be sufficiently effective when an optical depth with respect to the pair creation process $\tau\sim\sigma_T U_\gamma R$ is greater than 1. This gives us the limiting value for the γ-ray luminosity L_γ of an object below which our ideal MHD approach is valid

$$L_\gamma<\frac{\varepsilon_\gamma Rc}{\sigma_T}\simeq 10^{43}\left(\frac{\mathcal{M}}{10^8\mathcal{M}_\odot}\right)\mathrm{erg/s}. \qquad (64)$$

For active galactic nuclei and quasars with $R\sim 10^{14}$ cm it will be $L_\gamma\leqslant 10^{43}$ erg/s, for solar mass black holes $L_\gamma\leqslant 10^{35}$ erg/s. We see that the restriction (64) is also strong enough.

The next possible limitation of applicability of equation (46) under real astrophysical conditions originates from the fact that turbulence and various plasma instabilities are not included in our model. Therefore we do not claim to describe a turbulent α-disk,[53,55,56] magnetic field generation and turbulent diffusion of the magnetic field.[57,59] Particularly, the boundary conditions imposed on the accretion disk must be like those for the case of a force-free field. A specific form of restrictions is determined by the plasma composition, boundary conditions on the accretion disk, geometry of the disk, photon number density in the vicinity of the black hole and depends upon the physical conditions in the source. A discussion of these conditions is beyond the scope of the present paper.

Possible limitations of the validity area of equation (46) may be due to a violation of the frozen-in condition (13). First, it is clear that equation (46) cannot be used for a plasma generation region where one can expect the existence of a strong longitudinal electric field (see, for example, the work by Beskin *et al.*[48]). As we have already mentioned, in this region it is necessary to impose boundary conditions determined by an appropriate particle creation mechanism.

Furthermore, a natural limitation on the validity of equation (46) would arise if an electric field E were equal to the magnetic one. Such a violation of the MHD approach will occur if the longitudinal current I and, consequently, the toroidal magnetic field $B_{\hat\phi}$ are not strong enough for the total magnetic field $B=\sqrt{B_P^2+B_{\hat\phi}^2}$ to be greater than the electric field $E\sim\Omega^F\tilde\omega B_{\hat P}$ (Ref. 16). This case has been investigated in detail for the outer light surface in a neutron star magnetosphere. As a result, in a transition layer near the light surface $E=B$ the frozen-in condition (13) is violated and the particles begin to cross equipotential magnetic surfaces gaining a large amount of energy there. So the curves in Fig. 2 corresponding to small current values I have an infinite derivative du^r/dr at some point. From the physical point of view the fact that $E\to B$ and $du^r/dr\to\infty$ means that in a transition layer it is necessary to take particle masses into account, i.e. to use the following equation of motion

$$\frac{d\mathbf{p}}{dt}=\mathbf{E}+\mathbf{v}\times\mathbf{B}. \qquad (65)$$

Generally speaking, in a transition layer a region of multistream flow arises. As pointed out by Beskin *et al.* in their book[16] in the case of the outer light surface closing of the electric current circulating in the magnetosphere also occurs in this layer. Beyond the region of particle acceleration and current closing a magnetohydrodynamic wave may be generated where the energy of the particles constitutes a significant fraction of the total wave energy. Here in order to describe accretion onto a black hole one should also use quite different equations.

4. CONCLUDING REMARKS

The equation for a poloidal magnetic field (46) now obtained in the general Kerr space-time metric case provides the possibility to describe a wide range of MHD flows in the vicinity of a rotating black hole. Thus one obtains a tool for a consistent study of the processes occurring in compact sources. In particular, construction of selfconsistent magnetosphere models for such objects now becomes possible.

Evidently, there are many important phenomena beyond the scope of the present paper which can play an important role in a real accretion picture. In particular, we do not discuss at all the problem of discontinuities, whose presence in the magnetospheres of compact object cannot be excluded. And moreover it is known that in some cases only an introduction of discontinuities permits obtaining a noncontradictory picture of a material flow.[26] An analysis of discontinuous flows in a neutron star magnetosphere in connection with equation (46) was discussed in Refs. 30, 39, 40.

The significance of kinetic effects was not discussed in this paper either. We did not touch upon the problem of the outer magnetosphere structure (outer singular points, jet formation), which can be investigated by using a simplified equation (46) without general relativistic effects.[18,31–34] Separate investigations of physical conditions in plasma generation regions and of the regular magnetic field generation mechanism are also needed (see Refs. 57–59 on the latter topic). It is clear that the answers to all these questions may be obtained only when the problem is posed specifically and correctly. We hope that the use of Eq. (46) will allow us to understand the main characteristics and specific features of matter accretion onto black holes which is believed to occur in active galactic nuclei, quasars and, probably, in the vicinity of the galactic solar mass black holes (of course, if the accretion there is really magnetohydrodynamic in nature!).

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