Nonuniversality of the classical concept of the tangential discontinuity

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It is pointed out that the classical rules for matching at a tangential discontinuity, viz., continuity of the pressure and displacement, are nonuniversal. It is shown that in open systems both the pressure and the displacement can be discontinuous. The presence of a jump in the displacement renders meaningless the textbook use of the hydrodynamic concept of a perturbed surface of tangential discontinuity: the perturbed tangential discontinuity is not a single surface but a small spatial region bounded by nonparallel surfaces. It turns out that matching rules which are independent of the structure of the jump can be obtained only for a narrow (one-parameter) class of tangential discontinuities. In general the matching rules are different for different structures of the jumps in the main parameters of the medium. The examples cited in this paper are based on the real tangential discontinuity observed in the gaseous disk of the Milky Way galaxy.

1. IN WHAT WAY IS THE CLASSICAL CONCEPT OF THE TANGENTIAL DISCONTINUITY NONUNIVERSAL?

Traditionally two types of hydrodynamic discontinuities are distinguished: tangential and shock-wave.[1] The tangential discontinuity is characterized by the absence of transport of material through the surface of discontinuity. By using the expression for the momentum flux density tensor $\Pi_{ik}=P\delta_{ik}+\rho v_i v_k$, writing the Euler equation in the form

$$\frac{\partial}{\partial t}(\rho v_i)=-\frac{\partial}{\partial x_k}\Pi_{ik},$$

and employing the definition of the tangential discontinuity $v_i=0$, where v_i is the velocity component normal to the surface of discontinuity, one finds that the pressure P on the discontinuity must be a continuous function:[1]

$$[P]=0. \tag{1}$$

According to the classical concept,[1] a discontinuity can occur on some single surface. With this definition of the discontinuity, the displacement must also be continuous:

$$[\xi]=0. \tag{2}$$

It should be noted that relations (1) and (2) were obtained without taking into account any external forces acting on the medium or any external sources of heat, i.e., they were derived for a closed system.

In the pioneering studies[2,3] of the stability of the tangential discontinuity, conditions (1) and (2) were used as the matching conditions for the perturbed quantities. For an open system, however, it is incorrect to use them this way, since the perturbed forces near the perturbed surface cannot be described in the framework of a linearized system of equations, as is demonstrated by Fig. 1.

Indeed, from the equilibrium condition we have near the unperturbed boundary (let this be $x=0$)

$$\rho_0\frac{d\Psi_0}{dx}=-\frac{dP_0}{dx}. \tag{3}$$

It is seen in Fig. 1 that in the absence of perturbation of the surface of discontinuity, the functions Ψ_0 and P_0 have a kink at the point $x=0$, i.e., their first derivatives are discontinuous there. On the perturbed boundary $x=\xi$ the functions Ψ_0 and P_0 behave differently. The form of the function Ψ_0 is not affected in any way by the existence of the perturbation itself, since this function is determined by the external source. Therefore, the kink in Ψ_0 occurs as before at the point $x=0$. At the same time, a kink in the pressure P_0 arises on the perturbed boundary, at the point $x=\xi$. Consequently, at an arbitrary point $0\leqslant x\leqslant\xi$ there is an excess force $\rho_0 d\Psi_0/dx+dP_0/dx$ of zero order in smallness, which can be compensated only by a perturbed pressure dP_1/dx. Thus, near the perturbed boundary we unavoidably go beyond the framework of the linear approximation. This contradiction means that for the linearized equations, matching rules on a discontinuity surface of zero thickness cannot be used, i.e., one cannot assume that $|\xi|\gg|L|$, where L is the width of the discontinuity.

Our further analysis is based on the canonical method of deriving the matching rules, which consists in the following (see, e.g., Refs. 4–9). The tangential discontinuity is treated as a small region $(-\varepsilon,\varepsilon)$ in which the gradients of three parameters are nonzero, $\nabla v_0,\nabla\rho_0,\nabla c_s\neq 0$, while outside this region these gradients are equal to zero (the latter assumption is not at all obligatory, and we adopt it exclusively for consistency with Refs. 2 and 3). Then, integrating the linearized dynamical equations over the layer $(-\varepsilon,\varepsilon)$ and neglecting terms which are small for $\varepsilon\to 0$, we obtain the matching rules. Here it is easily shown that for the comparatively simple conditions adopted in Refs. 2 and 3 we arrive at the matching rules (1) and (2).

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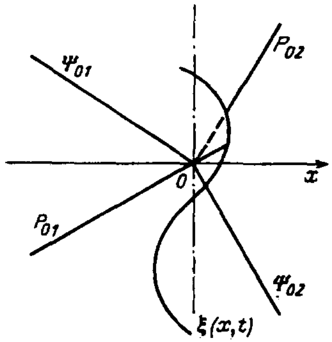

FIG. 1. Classical representation of the tangential discontinuity as a surface of zero thickness:[1-3] $|\xi| \gg L$, where ξ is the displacement and L is the characteristic width of the discontinuity. The unperturbed surface, the plane $x=0$, is shown by the broken line, the perturbed surface by the solid curve $\xi=\xi(x,t)$. In the absence of perturbation the potential Ψ_0 and pressure P_0 have a bend at the plane $x=0$ and for all x satisfy the equilibrium condition (3). When the discontinuity is perturbed, the form of the gravitational potential $\Psi_0(x)$, which is determined by the distribution of external masses, does not change, while the function $P_0(x)$ varies in the regions between the perturbed and unperturbed discontinuity surfaces ($0 \ll x \ll \xi$ for $\xi > 0$ and $\xi \ll x \ll 0$ for $\xi < 0$) as shown in the figure: the dashed line segment in the absence of the perturbation goes over to the solid line segment in the presence of the perturbation. Thus in these regions there arises a force which is unbalanced in the zero order approximation (in the amplitude of the perturbation) and causes perturbed forces (pressure forces, centrifugal forces, etc.) of the same order of magnitude. As a result, one goes beyond the framework of the linear approximation. In a closed system or in the case of a smooth external potential Ψ_0 such a force unbalanced in zero order does not arise.

In actual situations, however, one is more often dealing with open systems, for which the hydrodynamic medium in question is found in the field of external forces. For example, for the overwhelming majority of astrophysical objects these forces are gravitational and/or magnetic. In addition, practically all astrophysical objects are rotating.

It is natural to ask whether matching rules (1) and (2) can be used in these cases; moreover, there was no mention in Refs. 1–3 of their domain of applicability. This question, as far as we know, is raised here for the first time, even though previously, in the solution of the problem of stability of the tangential discontinuity in a gaseous galactic disk, matching rules different from (1) and (2) were obtained. For example, the matching rules at a discontinuity of the angular velocity of rotation[8] presuppose a discontinuity of the (total) pressure, and in Ref. 9, where the discontinuity of the unperturbed pressure was also taken into account, it was found that the displacement as well as the total pressure are discontinuous. We note that in the latter case not only are matching rules (1) and (2) incorrect, but so is the classical concept of the perturbed tangential discontinuity in the form of a single surface.

Our goal in this paper is to obtain matching rules at the tangential discontinuity for a rather wide class of open systems and to elucidate the reasons why, and the conditions under which, they are different from the classical matching rules.

In Sec. 2, using as an example the gaseous disk of the Milky Way galaxy with a jump in the unperturbed pressure, we derive matching rules different from the classical ones:[1-3] the perturbed pressure and the displacement turn out to be discontinuous. The reasons for this are discussed in Sec. 3. In Sec. 4 we introduce the concept of a two-parameter tangential discontinuity, characterized by jumps in two independent parameters, viz., the angular velocity of rotation and the unperturbed pressure. In this case the matching rules are no longer universal but depend on the specific structure of the discontinuity. We conclude Sec. 3 by considering the general case of a two-parameter discontinuity. In Sec. 5 we state our conclusions.

2. MATCHING RULES DIFFERING FROM THE CLASSICAL AT A ONE-PARAMETER CONTACT DISCONTINUITY

To avoid an artificial formulation of the problem, we refer to tangential discontinuities actually observed in nature. As an example, we consider the gaseous disk of the Milky Way, in which abrupt changes are observed[10] in the angular velocity of rotation $\Omega_0(r)$, sound velocity $c_{0s}(r)$, and surface density of the gas $\sigma_0(r)$. The equilibrium condition for such a disk along the coordinate r has the form

$$\Omega_0^2 r=\sigma_0^{-1}P_0'+\Psi_0'+\Psi_{0*}'. \tag{4}$$

The primes here and below denote differentiation with respect to r: $(...)'\equiv d(...)/dr$. All quantities refer to the gaseous disk of the Milky Way except for the last term, which describes the acceleration of an element of gas in the external field of the stars. Since the mass of the latter is much greater than the mass of the gaseous disk, the term $d\Psi_0/dr$ can be dropped.

From the equilibrium condition (4) with the numerical values of the parameters substituted in, it follows that the gas pressure can vary appreciably over a distance L_P which is much smaller than the characteristic scales of the problem along r, viz., the radius R and the wavelength λ_r of the perturbations that can be excited:[9]

$$\Lambda_P \ll 1 < \Lambda_w, \quad \Lambda_P \equiv L_P/R, \quad \Lambda_w \equiv \lambda_r/R. \tag{5}$$

Indeed, the ratio of the pressure force to the centrifugal force can be estimated as

$$\zeta \equiv P_0'/\sigma_0\Omega_0^2 R \approx c_{s0}^2(L_P R\Omega_0^2)^{-1}$$

$$\approx (\mathrm{Ma}^2\Lambda_P)^{-1} \approx \Lambda_h^2\Lambda_P^{-1}, \tag{6}$$

where $\mathrm{Ma}\equiv\Omega_0 R/c_{s0}$ is the Mach number, $\Lambda_h\equiv h/R\approx \mathrm{Ma}^{-1}$; the latter relation is obtained with allowance for the fact that the characteristic thickness of the disk can be estimated[7] in order of magnitude as $h\approx c_{s0}/\Omega_0$. In the Milky Way a jump in the gas pressure by 2.5 orders of magnitude (the surface density σ_0 changes from 300 $M\cdot \mathrm{pc}^{-2}$ to 5 $M\cdot\mathrm{pc}^{-2}$, the sound velocity c_{s0} from 80 to 9 km/s) is observed at a distance $R=0.35$ kpc from the center, which corresponds to the edge of the so-called "nuclear" disk. Although the "smearing" of the edge of the disk has not yet been established from observations, it is natural to assume that it is of the order of the thickness of the disk, i.e., $\Lambda_P\approx\Lambda_h\ll 1$. It can be seen from Eq. (6) that in this case $\zeta\approx\Lambda_h\ll 1$, i.e., the term $\sigma_0^{-1}P_0'$ in the equation

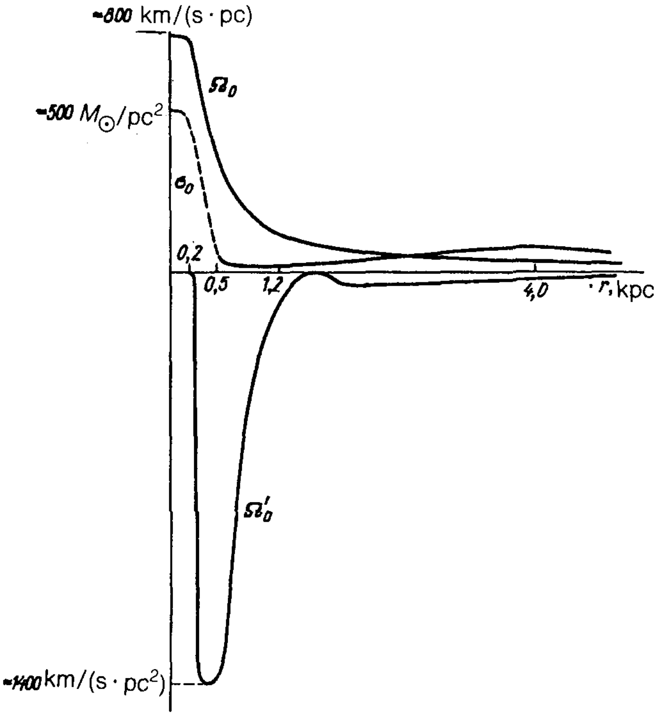

FIG. 2. Profiles of the angular velocity of rotation Ω_0, surface density σ_0, and derivative of the angular velocity Ω_0' of the gaseous component of the Milky Way (within the observational accuracy). The narrow jumps of the surface density and derivative of the angular velocity correspond to the edge of the molecular gas ring (0.2–0.5 kpc). The region in which the angular velocity changes appreciably is relatively large (0.2–1.2 kpc).

of equilibrium (4) can be neglected in spite of the sharp change (in effect, a "jump") in the pressure at distances $h \ll R$.

We note, however, that the equilibrium equation (4) does not prohibit L_P from being even smaller than h. In fact, if $\zeta \approx 1$ then $L_P \approx h\Lambda_h \ll h$, and it becomes even smaller if $\zeta \gg 1$. In the latter case the pressure gradient is counterbalanced by the external gravitational field of the stars. Thus the approximation of a "jump" in pressure ($\Lambda_P \ll 1$) holds for any value of the parameter $\zeta \gg \Lambda_h$.

The edge of the nuclear disk in the Milky Way also coincides with a narrow region of a jump in the derivative Ω_0' of the angular velocity of rotation. In addition, this place in the galactic disk marks the beginning of a region of sharp change in the angular velocity of rotation itself. Although in general the ratio of the widths of the jumps in pressure and angular velocity can be arbitrary, in the Milky Way the latter is substantially greater than the former[10] (see Fig. 2). Therefore, as a first step we consider the relatively simple case in which the angular velocity can be assumed continuous, while its derivative and the pressure are discontinuous:

$$\begin{aligned} P_0(r) &= P_{01} = \text{const} \quad \text{for } r \leqslant R - (L_P/2), \\ &= P_{02} = \text{const} \quad \text{for } r \geqslant R + (L_P/2), \\ \Omega_0'(r) &= \Omega_{01}' = \text{const} \quad \text{for } r \leqslant R - (L_\delta/2), \\ &= \Omega_{02}' = \text{const} \quad \text{for } r \geqslant R + (L_\delta/2), \end{aligned} \tag{7}$$

$$\Lambda_P \approx \Lambda_\delta \ll \Lambda_\Omega. \tag{8}$$

The symmetry of the jumps of the functions $P_0(r)$ and $\Omega_0'(r)$ with respect to the point $r=R$ that is assumed in formulas (7) and in Fig. 2 is extremely conjectural, since the small number of observation points do not permit any conclusions as to the behavior of these functions in the jump region. We shall see below that the forms of the functions $P_0(r)$ and $\Omega_0'(r)$ can differ from (7) without altering the main conclusion of this paper.

As was shown in Ref. 9, the system of linearized hydrodynamic equations can be reduced to the two equations

$$\begin{aligned} \frac{d}{dr}(r\sigma_0\tilde{\xi}) = -r\sigma_0[2m\Omega_0 r^{-1}\hat{\omega}^{-1}\tilde{\xi} \\ + (1 - m^2 c_{g0}^2 r^{-2}\hat{\omega}^{-2})\tilde{\eta}], \end{aligned} \tag{9}$$

$$\frac{d}{dr}(c_{g0}^2\tilde{\eta}) = 2m\Omega_0 r^{-1}\hat{\omega}^{-1}c_{g0}^2\tilde{\eta} + (\hat{\omega}^2 - \varkappa^2)\tilde{\xi}. \tag{10}$$

Equation (9) is the equation of continuity, and (10) is the equation to which the equations of motion can be reduced. In (9) and (10) we have used the notation[9]

$$\begin{aligned} &\eta \equiv \frac{\sigma}{\sigma_0}, \quad \hat{\omega} = \omega - m\Omega_0, \quad \varkappa^2 = 2\Omega_0(2\Omega_0 + r\Omega_0'), \\ &v_r \equiv \frac{d\xi}{dt} = \left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \varphi}\right)\xi = -i\hat{\omega}\xi. \end{aligned} \tag{11}$$

All the stationary quantities are denoted by the subscript 0, and the remaining perturbed quantities are functions of the form $A(r,\varphi,t) = \tilde{A}(r)\exp[i(m\varphi - \omega t)]$; c_{g0} is the sound velocity in a self-gravitating medium and is always less than c_{s0}. Inside the jump the force of self-gravitation is relatively small, and we can therefore assume that $c_{g0} \approx c_{s0}$. The value of c_{g0} outside the jump, which is required for the matching conditions, has been calculated in Ref. 9: $c_{g0}^2 \equiv c_{s0}^2 - 2\pi G\sigma_0(|k|R_g)^{-1}$, $R_g \approx [1 + (|k|h/2)]^{-1}$, $c_{s0}^2 \equiv dP_0/d\sigma_0$, where k is the radial wave number, which comes in here through the solution of the linearized Poisson equation (see Ref. 9), and G is the gravitational constant.

(The self-gravitation of the disk is taken into account here exclusively for reasons of generality: the results obtained below will not be altered in the absence of self-gravitation.)

The initial system of linearized dynamical equations used in the derivation of (9) and (10) were closed by the equation of state of an isentropic medium ($S=$const, $\tilde{S}=0$)

$$\tilde{P} = c_{s0}^2\tilde{\sigma}. \tag{12}$$

This is a particular case (for $S_0' = 0$) of the more general equation of state of adiabatic perturbations ($dS/dt=0$):

$$\tilde{P} = c_{s0}^2\tilde{\sigma} - \left(\frac{\partial P}{\partial S}\right)_\sigma S_0'\tilde{\xi} = c_{s0}^2\tilde{\sigma} + (c_{s0}^2\sigma_0' - P_0')\tilde{\xi}; \tag{13}$$

here we have used the thermodynamic relation $\tilde{P} = (\partial P/\partial\sigma)_S\tilde{\sigma} + (\partial P/\partial S)_\sigma\tilde{S}$ and the definition of the displacement in the form (11).

Integrating Eqs. (9) and (10) over a radial layer $(R-L_P, R+L_P)$ and using (8), we obtain the following "matching" conditions:[9]

$$[\sigma_0\tilde{\xi}]=0, \quad [c_{g0}^2\tilde{\eta}]=0. \tag{14}$$

In deriving the first of these matching rules we have used the condition $\mathrm{Ma}^2\Lambda_P^2 \ll 1$.

We see that the displacement $\tilde{\xi}$, like the pressure $\tilde{P}$, is not a continuous function. In the absence of self-gravitation the second matching condition gives

$$[\tilde{P}\sigma_0^{-1}]=0. \tag{15}$$

3. REASONS FOR THE DISCONTINUITY OF THE PERTURBED PRESSURE AND DISPLACEMENT

The reason for the discontinuity of the perturbed pressure in the presence of the external gravitational field of the stars, which causes a sharp change in the unperturbed pressure, is illustrated in Fig. 3. Figure 3a shows the jumps in the unperturbed pressure P_0 and unperturbed density σ_0 resulting from the unperturbed gravitational force shown in Fig. 3b. The perturbed gravitational force (Fig. 3c), whose overall profile repeats that of the unperturbed force, is lower in amplitude by a factor of $\tilde{\sigma}/\sigma_0$. The corresponding profiles of the jumps in the perturbed pressure and density are shown in Fig. 3d. Thus the jump in the perturbed pressure stems from the need to balance the forces: $-\nabla\tilde{P}\approx\tilde{\sigma}\nabla\Phi_0$, where $\nabla\Phi_0$ must be sharp in order to counterbalance the sharp gradient of the unperturbed pressure. In the absence of such a balance of forces there would arise arbitrarily large accelerations of the medium.

The cause of the discontinuity of the displacement $\tilde{\xi}$ is explained in Fig. 3e, which shows the interior of the jumps in the unperturbed pressure and density on the (r,φ) plane. The wavy lines are the perturbed surfaces through which matter does not move (matter moves only along these lines). Since the density falls off with increasing radius, the volume of matter that is displaced by each succeeding layer increases from left to right, i.e., the displacement $\tilde{\xi}$ increases. Thus the product $\tilde{\xi}\sigma_0$ turns out to be constant, according to the first of matching rules (14): a jump in σ_0 causes a jump in $\tilde{\xi}$.

In this case the tangential discontinuity no longer reduces to a perturbed surface but is a small spatial region bounded by nonparallel surfaces.

4. DEPENDENCE OF THE MATCHING RULES ON THE DETAILS OF THE STRUCTURE OF THE TWO-PARAMETER TANGENTIAL DISCONTINUITY

4.1. Example of two-parameter discontinuity

As our next step, let us consider the new aspects arising in the situation when the widths of the jumps in density and angular velocity of rotation are comparable.

This case can be regarded as a particular example of a two-parameter tangential discontinuity: the behavior of the characteristics of the medium in the region of the discontinuity is set by the behavior of two independent parameters, in this case the density and angular velocity of rota-

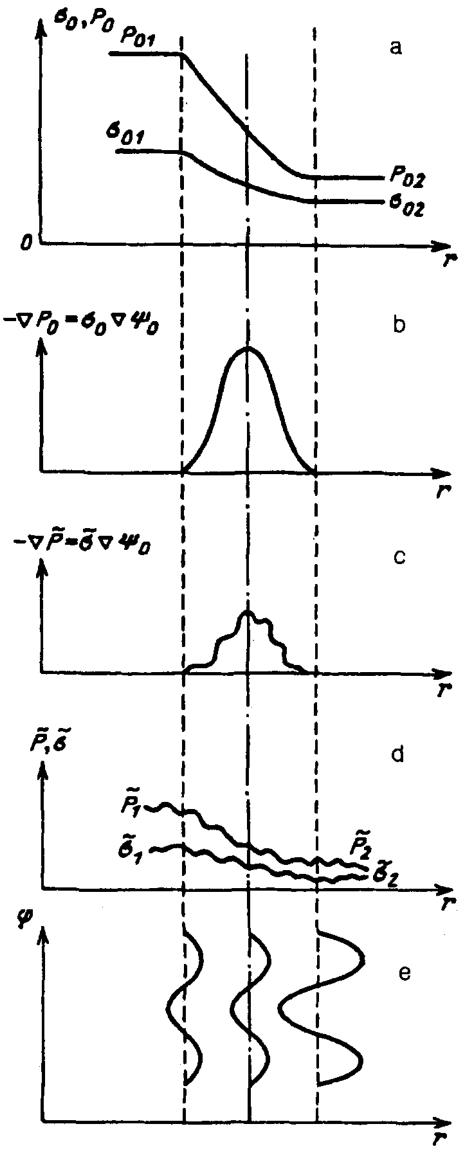

FIG. 3. Illustration of the cause of the jumps of the perturbed pressure $\tilde{P}$ and the displacement $\tilde{\xi}$. a: Jumps of the unperturbed pressure P_0 and unperturbed density σ_0. b: Profile of the unperturbed gravitational force causing the jumps in P_0 and σ_0 shown in part a of the figure. c: Profile of the perturbed gravitational force which in its overall shape resembles the unperturbed profile but is smaller by a factor of $\tilde{\sigma}/\sigma_0$. d: Profiles of the perturbed pressure and density constructed by integrating the function $\nabla\tilde{P}$ shown in part c to obtain $\tilde{P}$ and then using the equation of state $\tilde{\sigma}=\tilde{P}/c_{s0}^2$ to find $\tilde{\sigma}$. e: A representation of the tangential discontinuity that is opposite from the "classical" representation,[1-3] i.e., with $\tilde{\xi}\ll L$ (cf. caption to Fig. 1). In this case the linear approximation for describing the perturbed quantities is correct. The illustration shows the "jump" of the displacement $\xi(r,t)$ on different sides of the "discontinuity". The unperturbed boundaries of the "discontinuity" region are shown by the broken lines and the perturbed boundaries by the solid curves. Outside the "discontinuity" region the stationary parameters of the medium can be assumed homogeneous (in accordance with Refs. 2 and 3). No transport of matter occurs through the perturbed surfaces. Since the density falls off with increasing radius, the matter displaced by each succeeding layer increases from left to right, i.e., the displacement $\tilde{\xi}$ increases. It follows from Eqs. (14) that $\tilde{\xi}_1\sigma_{01}=\tilde{\xi}_2\sigma_{02}$. The case considered in the figures is for $\sigma_{01}>\sigma_{02}$; consequently, $\tilde{\xi}_1<\tilde{\xi}_2$.

tion. Such a situation becomes possible owing to the appearance of a "free" parameter (the external force) in the equation of equilibrium. In general the number of independent parameters necessary for the description of a discontinuity is one greater than the number of external influences acting on the system. For example, if in addition to external forces one includes external sources of heat, the discontinuity can turn out to be a three-parameter one. According to such a classification, the discontinuities in closed system are one-parameter.

The matching rules for the case considered in this Section are analogous to the previous ones. The matching rule for the displacement is unchanged, while Eq. (10) must include a term $(\Omega_0^2)'\xi$ in the discontinuity region. When there is no pressure jump this term can be taken into account without difficulty, and it leads to a jump in the perturbed pressure:[8]

$$[\tilde{P}] = -[\Omega_0^2]\sigma_0\xi R. \tag{16}$$

In the case of a two-parameter discontinuity (with a jump in P_0) the result of integrating Eq. (10) over the layer $(-\varepsilon,\varepsilon)$ is not so obvious. In fact, when the discontinuity $\bar{\xi}\equiv\xi_0 r\xi$ is taken into account according to (14), we obtain,

$$[c_{g0}^2\tilde{\eta}] = -\bar{\xi}\lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{\varepsilon}\sigma_0^{-1}(\Omega_0^2)'\mathrm{d}r\right]. \tag{17}$$

The value of the limit on the right-hand side depends on the detailed structure of the jumps and cannot be evaluated in general form. Let us demonstrate this for several examples.

The simplest case is when the narrow jumps of Ω_0 and σ_0 do not coincide but are separated by such a small distance that this fact is unobservable. In this situation the limit can be easily evaluated, and one obtains the matching condition

$$[c_{g0}^2\tilde{\eta}] = -\bar{\xi}\sigma_{01}^{-1}[\Omega_0^2] \tag{18}$$

if the density jump is farther from the center than the jump in angular velocity, and

$$[c_{g0}^2\tilde{\eta}] = -\bar{\xi}\sigma_{02}^{-1}[\Omega_0^2] \tag{19}$$

in the opposite case.

As an intermediate case between these two we can consider the case when the jumps of Ω_0^2 and σ_0^{-1} coincide in the sense that

$$\Omega_0^2 = (\Omega_0^2)_+ + f(r-R)(\Omega_0^2)_-, \tag{20}$$

$$\sigma_0^{-1} = (\sigma_0^{-1})_+ + g(r-R)(\sigma_0^{-1})_-, \tag{21}$$

where

$$\begin{gathered}(\Omega_0^2)_+ \equiv \tfrac{1}{2}(\Omega_{01}^2+\Omega_{02}^2), \quad (\Omega_0^2)_- \equiv \tfrac{1}{2}(\Omega_{01}^2-\Omega_{02}^2), \\ (\sigma_0^{-1})_+ \equiv \tfrac{1}{2}(\sigma_{01}^{-1}+\sigma_{02}^{-1}), \quad (\sigma_0^{-1})_- \equiv \tfrac{1}{2}(\sigma_{01}^{-1}-\sigma_{02}^{-1}),\end{gathered} \tag{22}$$

and f and g are narrowly localized odd functions of their arguments that take the values $+1$ and -1 outside the jump region. In this case the integral in (17) reduces to the sum of two integrals, one over an even integrand and the other over an odd. Only the first integral contributes to the limit in (17), and we obtain the matching condition

$$[c_{g0}^2\tilde{\eta}] = -\bar{\xi}[\Omega_0^2](\sigma_0^{-1})_+. \tag{23}$$

From the examples given it is clear that for the tangential discontinuity under discussion there is no matching condition *per se* in the classical sense, i.e., a condition independent of the details of the structure, relating quantities on different sides of the discontinuity. In the general case a condition similar to a matching condition can be obtained by using the theorem of the mean:

$$\begin{aligned}[c_0^2\tilde{\eta}] &= -\bar{\xi}\int\sigma_0^{-1}(\Omega_0^2)'\mathrm{d}r = -\bar{\xi}\int\sigma_0^{-1}\mathrm{d}\Omega_0^2 \\ &= -\bar{\xi}[\Omega_0^2]\sigma_0^{-1}(\bar{r}),\end{aligned} \tag{24}$$

where $R-\varepsilon\leqslant\bar{r}\leqslant R+\varepsilon$ is a certain intermediate point and, accordingly, $\sigma_0(\bar{r})\in[\sigma_{01},\sigma_{02}]$ is a certain intermediate value of the surface density, which can be evaluated exactly only if the structure of the jumps in σ_0 and Ω_0 is known.

This example shows that in general neither the perturbed nor the initial unperturbed tangential discontinuity can be reduced to a single surface.

4.2. General case

We shall conclude by showing that the effects considered above are not specific to tangential discontinuities with a sharp pressure differential (a "jump"). To do this, let us consider the behavior of tangential discontinuities in a nonuniform rotating medium with a continuous pressure profile. This system is convenient because it permits one to track at once the role of noninertial effects, the effect of inhomogeneity of the density of the medium, and the influence of external forces, which lead, for example, to kinks in the pressure at the jumps in the density of the medium.

Let F_0 be the external force per unit mass acting in the radial direction, and suppose that the following equilibrium condition holds:

$$P_0' = \rho_0\Omega_0^2 r + \rho_0 F_0. \tag{25}$$

The equations, analogous to (9) and (10), for the perturbations in this case become[10]

$$\rho_0 c_0^2 r^{-1}((r\tilde{\xi})' + 2m\Omega_0\hat{\omega}^{-1}\tilde{\xi}) - P_0'\tilde{\xi} = (\bar{k}^2 c_0^2\hat{\omega}^{-2}-1)\tilde{P}, \tag{26}$$

$$\begin{aligned}&\tilde{P}' - 2m\Omega_0 r^{-1}\hat{\omega}^{-1}\tilde{P} - P_0'(\rho_0 c_0^2)^{-1}\tilde{P} \\ &= \rho_0(\hat{\omega}^2-\varkappa_0^2)\tilde{\xi} + \beta T_0 C_P^{-1}S_0'P_0'\tilde{\xi};\end{aligned} \tag{27}$$

here ρ_0, T_0, β, and C_P are the volume density, temperature, temperature coefficient of volume expansion, and heat capacity at constant pressure, respectively, $\bar{k}^2 = k_z^2 + (m^2/r^2)$, and the perturbed quantities are all denoted by a tilde.

From the first equation we see the intimate connection between the jump in displacement and the jump in the unperturbed pressure. If the latter jump is absent, the displacement is continuous:

$$[\tilde{\xi}] = 0. \tag{28}$$

Integration of the second equation over the layer yields the classical matching conditions only for a narrow class of discontinuities. An indication of this can be seen from the fact that the "singular" (at the discontinuity) term $\rho_0(\Omega_0^2)' r\tilde{\xi}$ contained in $\rho_0 \varkappa_0^2 \tilde{\xi}$ gives a total derivative only in the case of a continuous density profile; otherwise the result of its integration depends on the details of the structure of the jumps in ρ_0 and Ω_0.

For a more accurate analysis let us transform the last two terms in Eq. (27), assuming a continuous pressure profile (the profile of P_0' can be discontinuous):

$$\rho_0 \varkappa_0^2 \tilde{\xi} - \beta T_0 C_P^{-1} S_0' P_0' \tilde{\xi} \approx \rho_0(\Omega_0^2)' r\tilde{\xi} + \rho_0^{-1} \rho_0' P_0' \tilde{\xi} \approx (\rho_0 \Omega_0^2 r\tilde{\xi})' + \rho_0' F_0 \tilde{\xi}. \qquad (29)$$

It is clear that the result of integration of the expression obtained reduces to the classical condition $[\tilde{P} + P_0'\tilde{\xi}] = 0$ only if the mass force F_0 is constant, as is the case, for example, in the traditional problem of gravitational waves (or the Rayleigh–Taylor instability): $F_0 = g = \text{const}$. Otherwise, everything that was said in Sec. 4.1 applies.

Thus it follows from what we have said that matching rules that do not depend on the detailed structure of the jumps can be obtained only for one-parameter tangential discontinuities. For two-parameter or higher discontinuities one needs to know the detailed structure of the jumps of the individual parameters in order to obtain the matching rules.

CONCLUSIONS

I. In open systems the classical concept of the tangential discontinuity, which is based on analysis of closed systems,[1–3] is modified in the following fundamental ways:

1) The perturbed tangential discontinuity is not a single surface[1–3] but a small spatial region bounded by nonparallel surfaces. (We are referring to the boundaries of the region outside of which the parameters of the medium can be considered homogeneous, in accordance with Refs. 2 and 3.)

2) The conventional approximation[1–3] of assuming a discontinuity of zero thickness, $|\xi| \gg L$ (ξ is the displacement and L is the width of the discontinuity), is incompatible with the linear approximation in the stability analysis of the tangential discontinuity.

3) Universal matching rules can be obtained when the tangential discontinuity is one-parameter. In the case of a two-parameter tangential discontinuity the matching rules depend on the details of the specific structure of the jumps of the individual parameters.

II. A study of various particular cases of the tangential discontinuity in the presence of an external field (in the presence of a jump in potential) has led to the following results that are qualitatively different from the analogous results in closed systems.[1–3]

4) In an isentropic medium the perturbation of the pressure $\tilde{P}$ and the displacement $\tilde{\xi}$ are not continuous functions but rather exhibit jumps in accordance with the jump in the unperturbed density σ_0: $[\sigma_0\tilde{\xi}] = 0$, $[\tilde{P}]/\sigma_0 = 0$. This is an example of a one-parameter jump, and the matching rules obtained do not depend on the detailed structure of the discontinuity.

5) In the case of a two-parameter discontinuity with jumps in the density and angular velocity of rotation, the following matching rules are obtained: $[\sigma_0\tilde{\xi}] = 0$, $[c_{g0}^2\tilde{\sigma}/\sigma_0] = -R\tilde{\xi}_i[\Omega_0^2]$, $i = 1, 2$. Here $i = 1$ when the jump in the density is closer to the center than the jump in the angular velocity, and $i = 2$ in the opposite case. When the two jumps are both localized in the same region, the matching condition takes the form $[c_{g0}^2\tilde{\sigma}/\sigma_0] = -R\tilde{\xi}[\Omega_0^2](\sigma_{01}^{-1} + \sigma_{02}^{-1})/2$.

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