Theory of a superfluid Fermi liquid

A. I. Akhiezer, V. V. Krasil'nikov, S. V. Peletminskiĭ, and A. A. Yatsenko

Khar'kov Physico-Technical Institute of the Ukrainian Academy of Sciences

(Submitted 13 August 1992)
Usp. Fiz. Nauk **36**, 1–32 (February 1993)

A Fermi-liquid approach to the theory of superfluidity is developed, based on a generalization of the combinatoric expression for the entropy to the case in which the state of the Fermi liquid is described not only by a normal but also by an anomalous distribution function. The energy of the superfluid Fermi liquid is given in the form of a functional of these distribution functions. In the weak-coupling approximation the Fermi-liquid approach is equivalent to the BCS theory. The interaction between the Fermi liquid and an electromagnetic field is gauge invariant. In addition to the singlet-pairing theory the triplet-pairing theory is also developed. A unified variational principle that gives rise to the theory of superfluidity (superconductivity) is used to obtain the London equations and the Ginzburg–Landau equation. Applications of this theory are made to the derivation of an equation for ideal two-component hydrodynamics.

I. INTRODUCTION

A system of particles can be in the liquid state when the particles interact strongly and the density is sufficiently high. Thus far no detailed general microscopic theory of the liquid state has been constructed. There is, however, an important exception to this assertion. We refer to a fermion gas at temperatures below the degeneracy temperature, e.g., an electron gas in a metal. The system of electrons in a metal cannot be regarded as an ideal gas, but must be viewed as a liquid, which, following Landau, we will refer to as a Fermi liquid. The Fermi liquid can be in a superfluid state. If it exhibits no superfluid properties, then the Fermi liquid is called normal. The theory of a normal Fermi liquid was developed by Landau in 1956.[1]

Note that a Fermi liquid is a quantum fluid. Another example of a quantum fluid is a Bose liquid, whose particles are bosons instead of fermions. A quantum fluid composed of bosons necessarily has superfluid properties. A Fermi liquid at low temperatures, however, can be either normal or superfluid.

The purpose of the present review is to present some topics of the theory of Fermi-liquid superfluidity. We begin with a brief review of the theory of a normal Fermi liquid, and then go on to present the theory of a superfluid Fermi liquid. Specifically, we use the concept of a Fermi liquid to derive the theory of singlet and triplet fermion pairing, and in particular we obtain the results of the *BCS* theory. We present the Fermi-liquid treatment of the London equation, the Ginzburg–Landau equation, and the equations of two-fluid hydrodynamics for a superfluid.

Our treatment is semiphenomenological: although we use concepts of the microscopic theory such as the normal and anomalous distribution function and the density matrix, we give the interaction between quasiparticles in terms of phenomenological amplitudes, which for the normal Fermi liquid were first introduced by Landau.[1]

In the present review we do not discuss questions associated with the microscopic basis of the theory of a Fermi liquid. A number of treatments of these topics have been given previously.[2–5]

The basic idea of the theory of a normal Fermi liquid is that as a result of the interaction each fermion is transformed into a quasiparticle, so that the Fermi gas becomes a gas of weakly interacting quasiparticles, which are also fermions. This means that the pair interaction between quasiparticles is weak, but the interaction between the quasiparticles and the self-consistent field of the other quasiparticles is substantial. As a result of the self-consistent interaction the total energy of the system is not equal to the sum of the individual quasiparticle energies. Since in the theory of a normal Fermi liquid it is assumed that there are no bound states, the overall number of quasiparticles is the same as the original number of fermions. The temperature of the Fermi liquid is assumed to be low. It is for this reason that the Fermi liquid can be regarded as an aggregate of weakly interacting quasiparticles, since the probability of a collision between fermions is proportional to the square of the temperature (we recall that the extent to which the discontinuity in the Fermi distribution is spread out is proportional to the temperature).

The individual state of a quasiparticle is characterized by the momentum $\mathbf{p}$ and the projection σ of the spin. At the basis of the quantitative theory of a normal Fermi liquid is the concept of the system energy $E=E(f)$ as a functional of the "distribution function" $f_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2}\equiv f_{12}$ (the one-particle density matrix) of the quasiparticles and the combinatoric expression for the entropy

$$S=-\mathrm{tr}[f\ln f+(1-f)\ln(1-f)]. \qquad (1.1)$$

In this equation tr represents the trace over one-particle states $1\equiv\mathbf{p}_1,\ \sigma_1,\ 2\equiv\mathbf{p}_2,\ \sigma_2$.

1063-7869/93/020035-30$10.00

The equilibrium one-particle quasiparticle density matrix must be found from the requirement that the entropy S be a maximum for specified values of the energy functionals $E(f)$, particle number

$$N=\operatorname{tr} f\equiv \sum_{\mathbf{p},\sigma} f_{\mathbf{p}\sigma,\mathbf{p}\sigma},$$

and momentum

$$\mathscr{P}_i=\operatorname{tr} f\hat{p}_i,$$

where $\hat{p}_i$ is the quasiparticle momentum operator. Introducing the Lagrangian multipliers Y corresponding to the quantities E, N, $\mathscr{P}_i$ we arrive at the problem of finding the absolute minimum of the functional

$$\Omega(f)=-S(f)+Y_0E(f)+Y_i\operatorname{tr} f\hat{p}_i+Y_4\operatorname{tr} f \tag{1.2}$$

(since the statistical equilibrium state is assumed to be spatially uniform, the quantities Ω, S, E, $\mathscr{P}_i$, and N are proportional to the volume V of the system). From this variational principle follows the self-consistency equation

$$f=[\exp(Y_0\hat{\varepsilon}(f)+Y_i\hat{p}_i+Y_4)+1]^{-1},\qquad \varepsilon_{12}(f)=\frac{\partial E(f)}{\partial f_{21}}, \tag{1.3}$$

a nonlinear equation determining the equilibrium one-particle Fermi-liquid density matrix; here

$$Y_0=T^{-1},\quad Y_i=-Y_0v_i,\quad Y_4=-Y_0\mu, \tag{1.4}$$

where T is the temperature, v_i is the average Fermi-liquid quasiparticle velocity, and μ is the chemical potential. The quantity $\hat{\varepsilon}(f)$, which is a functional of the one-particle density matrix, is the quasiparticle energy or Hamiltonian.

Since to leading order a gas of quasiparticles is ideal, the nonequilibrium statistical operator of the whole Fermi liquid can be written in the form[6]

$$\rho_0(f)=\exp\Big(\underline{\Omega}-\sum_{1,2}a_1^+A_{12}a_2\Big), \tag{1.5}$$

where a_1^+ and a_1 are the creation and annihilation operators for quasiparticles in the state $1\equiv p_1$, σ_1. The quantity $\underline{\Omega}$ is determined from the normalization condition $\operatorname{Sp}\rho_0(f)=1$. The matrix A_{12}, which characterizes the nonequilibrium state, is related to the one-particle density matrix f_{12} by

$$f_{12}=\operatorname{Sp}\rho_0(f)a_2^+a_1. \tag{1.6}$$

Using this relation we can easily show (see Ref. 6) that the entropy of the system, determined by

$$S=-\operatorname{Sp}\rho_0(f)\ln\rho_0(f),$$

is the same as the combinatoric entropy (1.1). We emphasize that in Eq. (1.1) the symbol tr stands for a trace in the space of one-particle states, while in Eq. (1.6) the symbol Sp stands for the trace in the second-quantization space of the entire system.

In the nonequilibrium case the one-particle density matrix f is time-dependent. This dependence is determined by the kinetic equation. Since $\hat{\varepsilon}(f)$ is the quasiparticle Hamiltonian, if we neglect collisions between quasiparticles this equation takes the form

$$i\frac{\partial f}{\partial t}=[\hat{\varepsilon}(f),f]. \tag{1.7}$$

Collisions between quasiparticles can be neglected under the condition $\omega\tau\gg 1$ (here $1/\omega$ is the characteristic time scale on which the one-particle density matrix varies and τ is the quasiparticle collision time). This equation is analogous to the Vlasov equation,[7] which is widely used in plasma theory.

In order that Eqs. (1.3) and (1.7) have a specific physical meaning, it is necessary to give the system energy $E(f)$ as a functional of the one-particle density matrix. This functional in the simplest case can be written in the form

$$E(f)=\sum_{1,2}\varepsilon_{12}f_{21}+\tfrac{1}{2}(f_0,F_1f_0)+\tfrac{1}{2}(f_i,F_2f_i), \tag{1.8}$$

where

$$(f_0,F_1f_0)=\frac{1}{V}\sum_{\mathbf{p},\mathbf{q};\mathbf{p}',\mathbf{q}'} f_{0\mathbf{pq}}(F_1)_{\mathbf{qp},\mathbf{p}'\mathbf{q}'}f_{0\mathbf{q}'\mathbf{p}'},$$

$$(f_i,F_2f_i)=\frac{1}{V}\sum_{\mathbf{p},\mathbf{q};\mathbf{p}',\mathbf{q}'} f_{i\mathbf{pq}}(F_2)_{\mathbf{qp},\mathbf{p}'\mathbf{q}'}f_{i,\mathbf{q}'\mathbf{p}'};$$

here $f_0=\operatorname{tr}_\sigma f$, $f_i=\operatorname{tr}_\sigma f\sigma_i$ and F_1 and F_2 are the amplitudes for the potential and exchange interactions of quasiparticles, which are called Landau amplitudes, σ_i is a Pauli matrix, and tr_σ is the trace over the quasiparticle spin variables. In writing down Eq. (1.8) we have assumed that the energy functional is invariant under spin rotations.

The Landau amplitudes can be introduced in a somewhat more general form, without specifying the form of the system energy. That is, these amplitudes can be determined from the formulas

$$\begin{aligned}\frac{\partial\varepsilon_{12}}{\partial f_{1'2'}}&=\frac{\partial^2E(f)}{\partial f_{1'2'}\partial f_{21}}\\&=F_{12,2'1'}=(F_1)_{\mathbf{p}_1\mathbf{p}_2,\mathbf{p}_2'\mathbf{p}_1'}\delta_{\sigma_1\sigma_2}\delta_{\sigma_2'\sigma_1'}\\&\qquad+(F_2)_{\mathbf{p}_1\mathbf{p}_2,\mathbf{p}_2'\mathbf{p}_1'}(\sigma^i_{\sigma_1\sigma_2}\sigma^i_{\sigma_2'\sigma_1'}),\end{aligned} \tag{1.9}$$

where after evaluating the derivatives we take the one-particle density matrix to be the equilibrium form. With this definition the Landau amplitudes, strictly speaking, are functions of the thermodynamic variables Y_0, Y_i, Y_4 (or T, v_i, μ). These amplitudes play a fundamental role both in studies of the thermodynamic properties of Fermi liquids and in investigating the high-frequency kinetic properties of a Fermi liquid[8] (zero sound[8,9]).

The principal physical objects to which the Fermi-liquid theory applies are liquid ^{3}He above the temperature of the transition to the superfluid state and an electron gas above the temperature of the transition to the superconducting state (if one exists). The theory of a normal Fermi liquid was originally derived by Landau in connection with liquid ^{3}He. The Fermi-liquid description of an electron gas in metals is due to V. P. Silin.[10]

A normal Fermi liquid can be unstable against perturbations associated with magnetic ordering. If this instability is present, then below some temperature the ordinary normal Fermi liquid undergoes a transition to a magnetically ordered normal Fermi liquid. The instability conditions for a normal Fermi liquid were studied by Pomeranchuk.[11] Magnetically ordered (ferromagnetic) normal Fermi liquids were studied by Stoner,[12], Abrikosov,[13] I. A. Akhiezer,[14] and others. The antiferromagnetic Fermi liquids were studied by Akhiezer and Chudnovskii.[15] (See also the review of Akhiezer[16] and the treatise of Kondrat'ev and Uzdin.[17])

A normal Fermi liquid can also be unstable against perturbations associated with the development of anomalous average values $\langle aa\rangle$. This means that below some temperature the superfluid (superconducting) state is the stable one. The present review is aimed at describing the theory of a superfluid Fermi liquid.

In 1986 high-temperature superconductors (HTSC) were discovered,[18] for which the transition temperature is above 30 K. At the present time there is no detailed theory of HTSC analogous to the BCS theory. Nevertheless, it is plausible to assume that the properties of HTSC are consistent with the concept of a Fermi liquid.[19] Hence the theory of a superfluid Fermi liquid may be applicable to HTSC also, but since this theory is semiphenomenological, it affords no insight into the microscopic mechanism responsible for HTSC. For this reason we deliberately do not concern ourselves in this review with the questions of HTSC. It is possible, however, that the superfluid model developed in Sec. 3.5 below has some connection with HTSC.

2. THEORY OF THE SUPERFLUID FERMI LIQUID

2.1. Entropy of a superfluid Fermi liquid

In the Introduction we outlined the theory of a normal Fermi liquid, described by the one-particle density matrix $f_{12}\equiv f_{\mathbf{p}_1\sigma_1,\mathbf{p}_2\sigma_2}$. Below some temperature the Fermi liquid can be in a superfluid state. This means that the Fermi liquid consists of two components, normal and superfluid, where the superfluid component flows without friction (viscosity). However, the one-particle density matrix f does not suffice to describe the superfluid state of a Fermi liquid; we need, in addition, to use the anomalous distribution functions g, g^+. As already noted, the normal "distribution function" is defined by the formula $f_{12}=\mathrm{Sp}\,\rho a_2^+a_1$, whereas the anomalous distribution functions g_{12}, g_{12}^+ are defined by $g_{12}=\mathrm{Sp}\,\rho a_2a_1$ and $g_{12}^+=\mathrm{Sp}\,\rho a_2^+a_1^+$ (here ρ is the nonequilibrium statistical operator of the Fermi liquid, and a_1^+ and a_1 are the creation and annihilation operators for fermions in the state $1\equiv\mathbf{p}_1,\sigma_1$). In a state of statistical equilibrium the anomalous distribution function vanishes above the superconducting transition temperature T_c.

Since Fermi-liquid quasiparticles interact weakly with one another, a nonequilibrium state of the quasiparticle (fermion) gas in the presence of anomalous average values $g\neq 0$ is described by the nonequilibrium statistical operator[6]

$$\rho_0(f,g,g^+)=\exp(\Omega-F),$$
$$F=\sum_{1,2}(a_1^+A_{12}a_2+\tfrac{1}{2}a_1^+B_{12}a_2^++\tfrac{1}{2}a_1B_{12}^+a_2), \tag{2.1}$$

where the quantities f, g, g^+ and A, B, B^+ are related by

$$\mathrm{Sp}\,\rho_0(f,g,g^+)a_2^+a_1=f_{12},$$
$$\mathrm{Sp}\,\rho_0(f,g,g^+)a_2a_1=g_{12}, \tag{2.2}$$
$$\mathrm{Sp}\,\rho_0(f,g,g^+)a_2^+a_1^+=g_{12}^+.$$

The quantity $\Omega(f,g,g^+)$ is found from the normalization condition $\mathrm{Sp}\,\rho_0(f,g,g^+)=1$. The nonequilibrium statistical operator (2.1) is characterized by the property that for averages of the form $\mathrm{Sp}\,\rho_0a_1^+\ldots a_n^+a_{n'},\ldots a_{1'}$, the Wick rules hold, in which the normal and anomalous distribution functions (2.2) serve as correlations, e.g.,

$$\mathrm{Sp}\,\rho_0a_1^+a_2^+a_3a_4=f_{32}f_{41}-f_{31}f_{42}+g_{21}^+g_{43}.$$

The absence of correlations on the average is a consequence of the fact that we are dealing with an ideal nonequilibrium gas of quasiparticles.

The entropy of a nonequilibrium Fermi liquid is determined by the general formula $S=-\mathrm{Sp}\,\rho_0\ln\rho_0$. We show that this expression can be represented in terms of the normal and anomalous distribution functions in the form

$$S=-\mathrm{Sp}\,\hat{f}\ln\hat{f}, \tag{2.3}$$

where

$$\hat{f}=\begin{pmatrix} f & g\\ g^+ & 1-\tilde{f}\end{pmatrix},\quad \tilde{f}_{12}=f_{21},\quad g_{12}^+=g_{21}^*$$

and Sp is taken in the extended space in which the matrices $\hat{f}$ act. We will call the matrix $\hat{f}$ the statistical operator of a nonequilibrium Fermi liquid. From the definition the quantities f and g which go into $\hat{f}$ satisfy the relations

$$f^+=f,\quad \tilde{g}=-g. \tag{2.4}$$

In the absence of anomalous average values of g the expression (2.3) for the entropy is identical to the entropy of a normal Fermi liquid [see (1.1)].

To prove Eq. (2.3) we represent the operator F in the form

$$F=[(\psi^+\hat{Q}\psi)/2]+[(\mathrm{tr}\,A)/2], \tag{2.5}$$

where

$$\psi_1\equiv\begin{pmatrix} a_1\\ a_1^+\end{pmatrix},\quad \psi_1^+\equiv(a_1^+,a_1),\quad \hat{Q}=\begin{pmatrix} A & B\\ B^+ & -\tilde{A}\end{pmatrix}.$$

We introduce the unitary Bogolyubov transformation for the creation and annihilation operators

$$\underline{U}^+ a_1 \underline{U} = u_{12} a_2 + v_{12} a_2^+ ,$$
$$\underline{U}^+ a_1^+ \underline{U} = u_{12}^* a_2^+ + v_{12}^* a_2 \tag{2.6}$$

(repeated indices imply summation). This transformation can be written in the form

$$\underline{U}^+ \psi \underline{U} = \underline{\hat{U}} \psi, \quad \underline{U}^+ \psi^+ \underline{U} = \psi^+ \underline{\hat{U}}^+, \tag{2.7}$$

where the matrix $\hat{U}$ is defined by

$$\hat{U} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}, \tag{2.8}$$

and the operators u and v satisfy the relations (the unitarity conditions for the matrix $\hat{U}$)

$$uu^+ + vv^+ = 1, \quad u\tilde{v} + v\tilde{u} = 0,$$
$$u^+ u + \tilde{v} v^* = 1, \quad u^+ v + \tilde{v} u^* = 0. \tag{2.9}$$

As is well known, the unitary transformation (2.8) can be taken so that a quadratic form in the operators a^+ and a

$$\underline{U}^+ F \underline{U} = \tfrac{1}{2} \psi^+ \hat{U}^+ \hat{Q} \hat{U} \psi + \tfrac{1}{2} \operatorname{tr} A$$

becomes quasidiagonal. This requirement is equivalent to the requirement that the matrix $\hat{U} + \hat{Q}\hat{U}$ be quasidiagonal:

$$\hat{U}^+ \hat{Q} \hat{U} = \hat{Q}_0, \quad \hat{Q}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & -\tilde{A}_0 \end{pmatrix}. \tag{2.10}$$

Taking (2.6) and (2.10) into account, we can represent the statistical operator ρ_0 in the form

$$\rho_0 = \underline{U} \rho_0^{(0)} \underline{U}^+,$$

where

$$\rho_0^{(0)} = \exp(\Omega - \tfrac{1}{2} \operatorname{tr} A - \tfrac{1}{2} \underline{Q}), \quad \underline{Q} = \psi^+ Q_0 \psi.$$

From (2.2) and (2.7) we therefore have

$$\hat{f} = \hat{U} \hat{f}_0 \hat{U}^+, \tag{2.11}$$

where

$$\hat{f}_0 = \begin{pmatrix} f_0 & 0 \\ 0 & 1 - \tilde{f}_0 \end{pmatrix},$$

$$(f_0)_{12} = \operatorname{Sp} \rho_0^{(0)} a_2^+ a_1, \quad (g_0)_{12} \equiv \operatorname{Sp} \rho_0^{(0)} a_2 a_1 = 0.$$

Since the entropy $S = -\operatorname{Sp} \rho_0 \ln \rho_0$ can be written in the form $S = -\operatorname{Sp} \rho_0^{(0)} \ln \rho_0^{(0)}$, we have (see Ref. 6)

$$S = -\operatorname{tr}[f_0 \ln f_0 + (1 - f_0) \ln(1 - f_0)] = -\operatorname{Sp} \hat{f}_0 \ln \hat{f}_0.$$

From (2.11) we therefore obtain Eq. (2.3), $S = -\operatorname{Sp} \hat{f} \ln \hat{f}$.

It can be shown that the general class of unitary transformations $\hat{U}$, which leave the structure of $\hat{f}$ invariant, i.e.,

$$\hat{f} \to \hat{f}' = \hat{U}^+ \hat{f} \hat{U}, \quad \hat{f}' = \begin{pmatrix} f' & g' \\ g'^+ & 1 - \tilde{f}' \end{pmatrix},$$
$$f'^+ = f', \quad \tilde{g}' = -g', \tag{2.12}$$

is defined by the following relations:

$$\hat{U} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}, \quad uu^+ + vv^+ = 1, \quad u^+ u + \tilde{v} v^* = 1,$$
$$u\tilde{v} + v\tilde{u} = 0, \quad u^+ v + \tilde{v} u^* = 0. \tag{2.13}$$

These relations show that the unitary transformations used here are identical with the $u - v$ transformations of Bogolyubov.[20] For infinitesimal unitary transformations they take the form $\hat{U} = 1 + i\varepsilon \hat{T} (\varepsilon \ll 1)$, where the generator $\hat{T}$ of these transformations, which is a Hermitian operator, has the structure

$$\hat{T} = \begin{pmatrix} t & \underline{t} \\ \underline{t}^+ & -\tilde{t} \end{pmatrix}, \quad t = t^+, \quad \tilde{\underline{t}} = -\underline{t}.$$

Since the physical quantities are generators of unitary transformations, to any physical variable A in the Fermi-liquid theory there corresponds an operator

$$\hat{A} = \begin{pmatrix} a & \underline{a} \\ \underline{a}^+ & -\tilde{a} \end{pmatrix}, \quad a = a^+, \quad \tilde{\underline{a}} = -\underline{a}.$$

It is easy to see that the operator $\hat{A}$ is transformed as follows under a unitary transformation $\hat{U}$:

$$\hat{A} \to \hat{A}' = \hat{U} \hat{A} \hat{U}^+, \quad \hat{A}' = \begin{pmatrix} a' & \underline{a}' \\ \underline{a}'^+ & -\tilde{a}' \end{pmatrix}, \tag{2.14}$$

where

$$a' = uau^+ + v\underline{a}^+ u^+ + u\underline{a} v^+ - \tilde{v} a v^+ \equiv a'^+,$$
$$\underline{a}' = ua\tilde{v} + v\underline{a}^+ \tilde{v} + u\underline{a}\tilde{v} - v\widetilde{au} \equiv -\tilde{\underline{a}}'.$$

In particular, for an ordinary unitary transformation we have

$$\hat{U} = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} (v = 0),$$

$$a' = uau^+, \quad \underline{a}' = u\underline{a}\tilde{u}.$$

Using the unitary transformation $\hat{U}$ we can always cause the quantity g' in Eq. (2.12) or the quantity a' in Eq. (2.14) to vanish.

To conclude this section we show that the statistical operator $\hat{f}$ is related to the entropy S by

$$\hat{f} = (e^{\hat{Q}} + 1)^{-1}, \tag{2.15}$$

where

$$\hat{Q} = \begin{pmatrix} A & B \\ B^+ & -\tilde{A} \end{pmatrix}, \quad A_{12} = \frac{\partial S}{\partial f_{21}}, \quad B_{12}^+ = 2 \frac{\partial S}{\partial g_{21}}. \tag{2.16}$$

This assertion follows from the fact that the statistical operator $\hat{f}$ can be reduced to quasidiagonal form by means of a unitary transformation U [see (2.11)]:

$$\hat{U}^+ \hat{f} \hat{U} = \begin{pmatrix} f_0 & 0 \\ 0 & 1 - \tilde{f}_0 \end{pmatrix} \equiv \hat{f}_0,$$
$$\hat{f}_0 = (e^{\hat{Q}_0} + 1)^{-1}, \quad \hat{Q}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & -\tilde{A}_0 \end{pmatrix}, \tag{2.17}$$

or [see (2.15)]

$$\hat{f}=(e^{\hat{Q}}+1)^{-1},\quad \hat{Q}=\begin{pmatrix} A & B \\ B^{+} & -\tilde{A} \end{pmatrix},$$

where

$$\hat{Q}=\hat{U}\hat{Q}_0\hat{U}^{+}.$$

From (2.17) it follows that the variation δS of the entropy takes the form

$$\delta S=-\mathrm{tr}\,\delta f_0 \ln\frac{f_0}{1-f_0}=\tfrac{1}{2}\,\mathrm{Sp}\,\delta f_0\hat{Q}_0=\tfrac{1}{2}\,\mathrm{Sp}\,U\delta\hat{f}_0U^{+}\hat{Q}.$$

But from (2.17) we have

$$\delta\hat{f}=\hat{U}\delta\hat{f}_0\hat{U}^{+}+\delta\hat{U}\hat{U}^{+}\hat{f}+\hat{f}\hat{U}\delta\hat{U}^{+}$$
$$=\hat{U}\delta\hat{f}_0\hat{U}^{+}+[\delta\hat{U}\cdot\hat{U}^{+},\hat{f}\,]$$

(we have taken into account the fact that $\hat{U}\delta\hat{U}^{+}=-\delta\hat{U}\hat{U}^{+}$). Since $[\hat{f},\hat{Q}]=0$, we have

$$\delta S=\mathrm{Sp}\,\delta\hat{f}\hat{Q}/2. \tag{2.17'}$$

Noting that

$$\delta\hat{f}=\begin{pmatrix} \delta f & \delta g \\ \delta g^{+} & -\delta\tilde{f} \end{pmatrix},$$

we arrive at Eqs. (2.15) and (2.16).

2.2. The energy functional and its symmetry properties

The theory of a normal Fermi liquid is based on expressing the energy as a functional of the distribution function f. In the case of a superfluid Fermi liquid we similarly assume that the system energy is a functional of both the normal and the anomalous distribution functions:

$$E=E(f,g,g^{+})\equiv E(\hat{f}\,).$$

Along with the total energy we introduce the energy density functional $\mathscr{E}(\mathbf{x};f,g,g^{+})\equiv\mathscr{E}(\mathbf{x};\hat{f}\,)$:

$$E(\hat{f}\,)=\int_V d^3x\,\mathscr{E}(\mathbf{x};\hat{f}\,).$$

Let us derive the symmetry properties of the functional $\mathscr{E}(\mathbf{x};\hat{f}\,)$ [and hence those of $E(\hat{f}\,)$] under phase and spin transformations, and also under spatial translations.

The average value of a physical quantity

$$\hat{a}\equiv\begin{pmatrix} a & 0 \\ 0 & -\tilde{a} \end{pmatrix}(\underline{a}=0)$$

associated with a fermion can be represented in the form

$$a(\hat{f}\,)\equiv\mathrm{tr}\,fa=\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}\hat{a} \tag{2.18}$$

[here we have neglected the term tr a, which does not depend on $\hat{f}$ and therefore does not contribute to the change in the quantity $a(\hat{f}\,)$]. Since 1, s_i, p_i are the operators of the particle number and the electron spin and momentum, the corresponding operators in the Fermi-liquid approach assume the form

$$\hat{\tau}_3=\int d^3x\hat{\rho}(\mathbf{x}),\quad \hat{s}_i=\int d^3x\hat{s}_i(\mathbf{x}),\quad \hat{p}_i=\int d^3x\hat{\pi}_i(\mathbf{x}),$$

where the operators of the particle number density $\hat{\rho}(\mathbf{x})$, spin density $\hat{s}_i(\mathbf{x})$, and momentum density $\hat{\pi}_i(\mathbf{x})$ are defined by

$$\hat{\rho}(\mathbf{x})=\begin{pmatrix} \rho(\mathbf{x}) & 0 \\ 0 & -\tilde{\rho}(\mathbf{x}) \end{pmatrix},\quad \hat{s}_i(\mathbf{x})=\begin{pmatrix} s_i(\mathbf{x}) & 0 \\ 0 & -\tilde{s}_i(\mathbf{x}) \end{pmatrix},$$

$$\hat{\pi}_i(\mathbf{x})=\begin{pmatrix} \pi_i(\mathbf{x}) & 0 \\ 0 & -\tilde{\pi}_i(\mathbf{x}) \end{pmatrix},$$

$$\rho(\mathbf{x})=\delta(\mathbf{x}-\underline{\mathbf{x}}),\quad s_i(\mathbf{x})=s_i\,\delta(\mathbf{x}-\underline{\mathbf{x}}),$$

$$\pi_i(\mathbf{x})=\tfrac{1}{2}\{p_i,\delta(\mathbf{x}-\underline{\mathbf{x}})\};$$

here we have written $s_i=\sigma_i/2$; $\mathbf{x}$ is the position operator and σ_i are the Pauli matrices.

The operator $\hat{\tau}_3$ is the generator of the phase unitary transformation

$$\hat{U}_\varphi=\exp(i\varphi\hat{\tau}_3).$$

The operator $\hat{s}_k$ is the generator of the spin rotations

$$\hat{U}_\omega=\exp(i\omega_k\hat{s}_k).$$

Finally, the operator $\hat{p}_k$ is the generator of the spatial translations

$$\hat{U}_\mathbf{y}=\exp(iy_k\hat{p}_k).$$

In accordance with (2.13) f and g transform under these operations according to

$$f\to f_\varphi=f,\quad g\to g_\varphi=e^{-2i\varphi}g,$$
$$f\to f_\omega=e^{-i\omega_k s_k}fe^{i\omega_k s_k},\quad g\to g_\omega=e^{-i\omega_k s_k}ge^{-i\omega_k\tilde{s}_k}, \tag{2.19}$$
$$f\to f_\mathbf{y}=e^{-iy_k p_k}fe^{iy_k p_k},\quad g\to g_\mathbf{y}=e^{-iy_k p_k}ge^{-iy_k\tilde{p}_k}.$$

We will assume that the functional $\mathscr{E}(\mathbf{x};\hat{f}\,)$ is always invariant under phase transformations,

$$\mathscr{E}(\mathbf{x};\hat{U}_\varphi^{+}\hat{f}\hat{U}_\varphi)=\mathscr{E}(\mathbf{x};\hat{f}\,). \tag{2.20}$$

If we disregard the relativistic interaction the functional $\mathscr{E}(\mathbf{x};\hat{f}\,)$ is invariant under spin rotations:

$$\mathscr{E}(\mathbf{x};\hat{U}_\omega^{+}\hat{f}\hat{U}_\omega)=\mathscr{E}(\mathbf{x};\hat{f}\,). \tag{2.21}$$

Finally, in the absence of nonuniform external fields the functional $\mathscr{E}(\mathbf{x};\hat{f}\,)$ is invariant under spatial translations:

$$\mathscr{E}(\mathbf{x}+\mathbf{y};\hat{U}_\mathbf{y}^{+}\hat{f}\hat{U}_\mathbf{y})=\mathscr{E}(\mathbf{x};\hat{f}\,). \tag{2.22}$$

The infinitesimal form of relations (2.20), (2.21), and (2.22) can easily be found. For this we note that the variation of the functional $\mathscr{E}(\mathbf{x};\hat{f}\,)$ with respect to the statistical operator $\hat{f}$ can be written in the form

$$\delta\mathscr{E}(\mathbf{x};\hat{f}\,)=\tfrac{1}{2}\,\mathrm{Sp}\,\delta\hat{f}\hat{\varepsilon}(\mathbf{x};\hat{f}\,), \tag{2.23}$$

where

$$\hat{\varepsilon}(\mathbf{x};\hat{f}\,)=\begin{pmatrix} \varepsilon(\mathbf{x};\hat{f}\,) & \Delta(\mathbf{x};\hat{f}\,) \\ \Delta(\mathbf{x};\hat{f}\,) & -\tilde{\varepsilon}(\mathbf{x};\hat{f}\,) \end{pmatrix},$$

$$\varepsilon_{12}(\mathbf{x};\hat{f}\,)=\frac{\partial\mathscr{E}(\mathbf{x};\hat{f}\,)}{\partial f_{21}},\quad \Delta_{12}(\mathbf{x};\hat{f}\,)=2\frac{\partial\mathscr{E}(\mathbf{x};\hat{f}\,)}{\partial g_{21}^{+}} \tag{2.24}$$

[the operator $\hat{\varepsilon}(\mathbf{x};\hat{f})$ is the quasiparticle energy density operator; see the next section]. By varying Eqs. (2.20), (2.21), and (2.22) with respect to φ, ω, $\mathbf{y}$ and using (2.23) we find in the case of phase transformations

$$i\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\mathbf{x};\hat{f}),\hat{\tau}_3]=0, \tag{2.25}$$

in the case of spin rotations

$$i\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\mathbf{x};\hat{f}),\hat{s}_i]=0 \quad . \tag{2.26}$$

and in the case of spatial translations

$$\frac{\partial\mathscr{E}(\mathbf{x};\hat{f})}{\partial x_k}=\frac{i}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\mathbf{x};\hat{f}),\hat{p}_k]. \tag{2.27}$$

These formulas will be used below to construct the flux densities of the corresponding physical quantities, which are necessary to study the hydrodynamic stage in the evolution of a superfluid.

By varying Eqs. (2.20)–(2.22) with respect to the statistical operator $\hat{f}$ and using Eq. (2.23) we find the relations

$$\hat{U}_\varphi\hat{\varepsilon}(\mathbf{x};\hat{U}_\varphi^+\hat{f}\hat{U}_\varphi)\hat{U}_\varphi^+=\hat{\varepsilon}(\mathbf{x};\hat{f}), \tag{2.28}$$

$$\hat{U}_\omega\hat{\varepsilon}(\mathbf{x};\hat{U}_\omega^+\hat{f}\hat{U}_\omega)\hat{U}_\omega^+=\hat{\varepsilon}(\mathbf{x};\hat{f}), \tag{2.29}$$

$$\hat{U}_\mathbf{y}\hat{\varepsilon}(\mathbf{x};\hat{U}_\mathbf{y}^+\hat{f}\hat{U}_\mathbf{y})\hat{U}_\mathbf{y}^+=\hat{\varepsilon}(\mathbf{x}-\mathbf{y};\hat{f}), \tag{2.30}$$

which express the symmetry properties of the quasiparticle energy density $\hat{\varepsilon}(\mathbf{x};\hat{f})$.

Finally, we consider the question of the invariance of the theory under Galilean transformations. Specifically, we will say that the energy density functional is invariant under a Galilean transformation if we have

$$\mathscr{E}(\mathbf{x};\hat{U}_\mathbf{v}\hat{f}\hat{U}_\mathbf{v}^+)=\mathscr{E}(\mathbf{x};\hat{f})-v_i\pi_i(\mathbf{x};\hat{f})+\frac{mv^2}{2}\rho(\mathbf{x};\hat{f}), \tag{2.31}$$

where the unitary operator $\hat{U}_v$ is defined by

$$\hat{U}_\mathbf{v}=e^{-imv_i\hat{x}_i}$$

$\pi_i(\mathbf{x};\hat{f})$ and $\rho(\mathbf{x};\hat{f})$ are the average values of the momentum density and particle number density, and m is the particle mass. By varying this expression with respect to v_i we find

$$\frac{i}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\mathbf{x};\hat{f}),\hat{x}_i]=\frac{1}{m}\pi_i(\mathbf{x};\hat{f})/m. \tag{2.31'}$$

And if we vary Eq. (2.31) with respect to $\hat{f}$ we find the transformation properties of the quantity $\hat{\varepsilon}(\mathbf{x};\hat{f})$ under Galilean transformations:

$$\hat{U}_\mathbf{v}^+\hat{\varepsilon}(\mathbf{x};\hat{U}_\mathbf{v}\hat{f}\hat{U}_\mathbf{v}^+)\hat{U}_\mathbf{v}=\hat{\varepsilon}(\mathbf{x};\hat{f})-v_i\hat{\pi}_i(\mathbf{x})+\frac{mv^2}{2}\hat{\rho}(\mathbf{x}).$$

2.3. Variational principle and the self-consistency equations

As already noted, the statistical equilibrium state of a Fermi liquid corresponds to the maximum value of the entropy for given values of the constants of motion, namely, the energy, momentum, and particle number (when the relativistic interaction is included the spin of the system is not a constant of motion). Just as in the theory of a normal Fermi liquid, instead of looking for the local maximum of the functional $S(\hat{f})$ we will look for the absolute minimum of the functional

$$\Omega(\hat{f})=-S(\hat{f})+Y_0E(\hat{f})+Y_i\,\mathrm{tr}\,fp_i+Y_4\,\mathrm{tr}\,f$$

[here Y_0, Y_i, and Y_4 are Lagrangian multipliers corresponding to the constants of motion $E(\hat{f})$, $\mathscr{P}_i(\hat{f})=\mathrm{tr}\,fp_i$, and $N(\hat{f})=\mathrm{tr}\,f$]. From Eqs. (2.17) and (2.23) we have

$$\delta\Omega(\hat{f})=\mathrm{Sp}\,\delta\hat{f}(-\hat{Q}+Y_0\hat{\varepsilon}(\hat{f})+Y_i\hat{p}_i+Y_4\hat{\tau}_3)/2, \tag{2.31''}$$

where the matrix $\hat{Q}$ is defined by Eq. (2.16) and

$$\hat{\varepsilon}(\hat{f})=\int_V d^3x\hat{\varepsilon}(\mathbf{x};\hat{f})=\begin{pmatrix}\varepsilon & \Delta\\ \Delta^+ & -\tilde{\varepsilon}\end{pmatrix},$$

$$\varepsilon=\int_V d^3x\varepsilon(\mathbf{x};\hat{f}),\quad \Delta=\int_V d^3x\Delta(\mathbf{x};\hat{f}).$$

From (2.16) and (2.31″) the requirement $\delta\Omega=0$ yields the nonlinear self-consistency equations[44]

$$\hat{f}=[\exp(Y_0\hat{\varepsilon}(\hat{f})+Y_i\hat{p}_i+Y_4\hat{\tau}_3)+1]^{-1}. \tag{2.32}$$

From this relation it follows that the quantity $\hat{\varepsilon}(\hat{f})$ must be interpreted as the quasiparticle energy. It is a functional of the normal and anomalous distribution functions.

Now let us derive the condition for uniform superfluidity of a Fermi liquid. For a normal Fermi liquid this condition takes the form $[f,p_i]=0$. In the case of systems where the phase invariance is destroyed ($[\hat{f},\hat{\tau}_3]\neq 0$), the condition for spatial uniformity must be formulated as

$$[\hat{f},\hat{p}_i-q_i\hat{\tau}_3]=0, \tag{2.33}$$

where q_i is the momentum of the condensed particles (the momentum of the superfluid component of the liquid). Note that in this case the average value $a(\mathbf{x})$ of a physical quantity $\hat{a}(\mathbf{x})$ depends in general on $\mathbf{x}$. However, if the operator $\hat{a}(\mathbf{x})$ is phase-invariant, i.e., $[\hat{a}(\mathbf{x}),\hat{\tau}_3]=0$, then from (2.33) it follows that the average value $\langle\hat{a}(\mathbf{x})\rangle$ does not depend on $\mathbf{x}$. It is specifically for this reason that we called the relation (2.33) the condition for spatial uniformity (but not the condition for translational invariance, which corresponds to the case $[\hat{f},\hat{p}_i]=0$).

Thus, in order to determine the spatially uniform states of a superfluid Fermi liquid we must find simultaneous solutions of Eqs. (2.32) and (2.33).

The fact that these equations hold simultaneously follows from Eqs. (2.30) and (2.24′), which show that if $[\hat{f},\hat{p}_i-q_i\hat{\tau}_3]=0$ holds then $[\hat{\varepsilon}(\hat{f}),\hat{p}_i-q_i\hat{\tau}_3]=0$.

We emphasize that in addition to the parameters $Y_0^{-1}=T$ (temperature), $-Y_i/Y_0=v_i$ (normal velocity), and $-Y_4/Y_0=\mu$ (chemical potential), which are characteristic of a normal Fermi liquid, the statistical equilibrium state of a superfluid Fermi liquid requires us to specify an additional thermodynamic parameter q_i, called the superfluid momentum. In the case of Galilean invariant systems, when we can introduce the fermion mass m, the quantity $q_i/m=u_i$ is the velocity of the superfluid component (see Sec. 2.2).

Equation (2.32) holds for both spatially uniform and nonuniform systems. It yields both the BCS theory with singlet and triplet pairing (the different superfluid phases of ^{3}He; see Sec. 3.4) and the London equation and the Ginzburg–Landau equation. In this sense the self-consistency equations in the form (2.32) are universal.

Note that by virtue of the phase-invariance property of the energy functional the self-consistency equations admit solutions with $g=0$, $\Delta=0$ (that is, $[\hat{f},\hat{\tau}_3]=0$). For these solutions all the results of the present section go over to the formulas of the normal Fermi-liquid theory. However, a phase transition can occur in the system, as a result of which solutions with $g\neq 0$, $\Delta\neq 0$ ($[\hat{f},\hat{\tau}_3]\neq 0$) occur, corresponding to states with spontaneously broken symmetry [the statistical equilibrium state has a lower symmetry than the functional $\mathscr{E}(\mathbf{x};\hat{f})$]. It is precisely these states which we will be treating in the present review.

2.4. Kinetic equation and differential conservation laws

We will now formulate the kinetic equations for the normal and anomalous distribution functions, or what is the same thing, for the statistical operator $\hat{f}$. Since we are developing a phenomenological approach to the theory of Fermi-liquid fluidity, by noting that $\hat{\varepsilon}(\hat{f})$ can be interpreted as the quasiparticle energy we will assume that the kinetic equation for $\hat{f}$ in the absence of collisions between quasiparticles assumes the form

$$i\frac{\partial\hat{f}}{\partial t}=[\hat{\varepsilon}(\hat{f}),\hat{f}]. \tag{2.34}$$

In component form this equation is equivalent to the following system of equations for the normal and anomalous distribution functions:

$$\begin{aligned}\frac{\partial f}{\partial t}&=i[f,\varepsilon]+i(g\Delta^+-\Delta g^+),\\ \frac{\partial g}{\partial t}&=-i(g\tilde{\varepsilon}+\varepsilon g)+i(f\Delta+\Delta\tilde{f})-i\Delta.\end{aligned} \tag{2.35}$$

These equations hold when the characteristic time scales on which the distribution functions vary are much less than the quasiparticle relaxation time τ. If we consider oscillations of the superfluid, then their frequency ω must satisfy $\omega\tau\gg 1$. Under these conditions we can neglect the quasiparticle collision integral.

Note that the equilibrium normal and anomalous distribution functions, which are solutions of the self-consistency equations (2.32), satisfy Eqs. (2.34) and (2.35). In this connection we use Eqs. (2.34) and (2.35) in Sec. 3.6 to derive the equations for ideal (in the absence of dissipation) two-fluid hydrodynamics (valid for $\omega\tau\ll 1$). The equations of motion for the phase and superfluid momentum are obtained from these equations.

Equations (2.35) can easily be obtained starting from the concept of weak interaction between quasiparticles in the Fermi liquid. As is well known, the kinetic stage of evolution[6] is described by the statistical operator $\rho(f,g,g^+)$, which is a functional of the distribution functions f,g,g^+. These distribution functions satisfy the kinetic equations

$$\begin{aligned}\dot{f}_{12}&=i\,\mathrm{Sp}\,\rho(f,g,g^+)[H,a_2^+a_1],\\ \dot{g}_{12}&=i\,\mathrm{Sp}\,\rho(f,g,g^+)[H,a_2a_1],\end{aligned} \tag{2.36}$$

where $H=H_0+V$ (here H_0 is the free Hamiltonian and V is the interaction Hamiltonian, whose structure can be completely arbitrary). We now take into account the fact that according to the definition we have

$$\mathrm{Sp}\,\rho(f,g,g^+)a_2^+a_1=f_{12},$$

$$\mathrm{Sp}\,\rho(f,g,g^+)a_2a_1=g_{12}.$$

Since the free Hamiltonian is quadratic in the operators a^+, a, then from (2.2) we have

$$i\,\mathrm{Sp}\,\rho(f,g,g^+)[H_0,a_2^+a_1]=i\,\mathrm{Sp}\,\rho_0(f,g,g^+)[H_0,a_2^+a_1],$$

$$i\,\mathrm{Sp}\,\rho(f,g,g^+)[H_0,a_2a_1]=i\,\mathrm{Sp}\,\rho_0(f,g,g^+)[H_0,a_2a_1].$$

Since the statistical operator $\rho(f,g,g^+)$ is identical to $\rho_0(f,g,g^+)$ if we disregard quasiparticle interactions, the kinetic equations (2.36) including the terms which are linear in the interaction can be written in the form

$$\dot{f}_{21}=i\,\mathrm{Sp}\,\rho_0(f,g,g^+)[H,a_1^+a_2],$$

$$\dot{g}_{21}=i\,\mathrm{Sp}\,\rho_0(f,g,g^+)[H,a_1a_2].$$

Note that when we take averages in the state $\rho_0(f,g,g^+)$ the Wick rules hold, according to which the normal and anomalous averages

$$a_1^+a_2=\mathrm{Sp}\,\rho_0a_1^+a_2=f_{21},\quad \hat{a}_1\hat{a}_2=\mathrm{Sp}\,\rho_0a_1a_2=g_{21}.$$

appear as contractions. It can easily be shown that relations of the form

$$\begin{aligned}&[a_1^+a_2,Aa_3^+Ba_4C]=(-1)^B(f_{41}\delta_{23}-f_{23}\delta_{41})ABC,\\ &[a_1^+a_2,Aa_3Ba_4C]=(-1)^Bg_{24}\delta_{13}ABC,\\ &[a_1a_2,Aa_3Ba_4^+C]=(-1)^B(-\delta_{24}\delta_{13}+\delta_{24}f_{13}\\ &\qquad+\delta_{13}f_{24})ABC,\end{aligned} \tag{2.37}$$

hold, where $Aa_3^+Ba_4C$ are normal ordering operators; $(-1)^B=1$ if the operator B contains an even number of a, a^+ operators, and $(-1)^B=-1$ if the operator B contains an odd number of these operators. Introducing the energy functional

$$E(f,g,g^+)=\mathrm{Sp}\,\rho_0(f,g,g^+)H$$

and the quantities

$$\varepsilon_{12}=\frac{\partial E}{\partial f_{21}},\quad \Delta_{12}=2\frac{\partial E}{\partial g_{21}^+},\quad \Delta_{12}^+=2\frac{\partial E}{\partial g_{21}}$$

and using relations of the form (2.37), we find

$$\mathrm{Sp}\,\rho_0(f,g,g^+)[a_1^+a_2,H]=([\varepsilon,f]-g\Delta^++\Delta g)_{21},$$

$$\mathrm{Sp}\,\rho_0(f,g,g^+)[a_1a_2,H]=(\varepsilon g+g\tilde{\varepsilon}+\Delta-\Delta\tilde{f}-f\Delta)_{21}.$$

Thus

$$i\dot{f}=[\varepsilon,f]-g\Delta^++\Delta g^+,$$

$$i\dot{g}=\varepsilon g+g\tilde{\varepsilon}-\Delta\tilde{f}-f\Delta+\Delta.$$

These equations coincide exactly with Eq. (2.35).

Starting from the kinetic equation (2.34) and using the symmetry properties of the energy functional $\mathscr{E}$, we can express the conservation laws and define the flux densities of the conserved physical quantities. (These results will be used below to derive the hydrodynamics of a superfluid.)

The time derivatives of the average value of the density of a physical quantity a, given by $a(\mathbf{x};\hat{f})=\mathrm{Sp}\,\hat{f}\hat{a}(\mathbf{x})/2$, can be represented in the form

$$\frac{\partial a(\mathbf{x};\hat{f})}{\partial t}=-\frac{\partial a_k(\mathbf{x};\hat{f})}{\partial x_k}+\frac{i}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\mathbf{x};\hat{f}),\hat{A}],\qquad(2.38)$$

where

$$\hat{a}_k(\mathbf{x})=i\int \mathrm{d}^3x'x'_k\int_0^1 \mathrm{d}\xi[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';\hat{f}),\hat{a}(\mathbf{x}+\xi\mathbf{x}')],\qquad(2.39)$$

$$\hat{A}=\int \mathrm{d}^3x\hat{a}(\mathbf{x}).$$

The proof of this assertion follows directly from the kinetic equation (2.34), according to which we have

$$\frac{\partial a(\mathbf{x};\hat{f})}{\partial t}=\frac{i}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\hat{f}),\hat{a}(\mathbf{x})],$$

and from the obvious relation

$$i[\hat{\varepsilon}(\hat{f}),\hat{a}(\mathbf{x})]=i[\hat{\varepsilon}(\mathbf{x};\hat{f}),\hat{A}]-\frac{\partial\hat{a}_k(\mathbf{x})}{\partial x_k}.\qquad(2.40)$$

Setting $\hat{a}(\mathbf{x})=\hat{\rho}(\mathbf{x})$ in Eq. (2.38) and using Eq. (2.25), which is a consequence of the phase invariance of $\mathscr{E}(\mathbf{x};\hat{f})$, we find the conservation law for the particle number density:

$$\frac{\partial\rho(\mathbf{x};\hat{f})}{\partial t}=-\frac{\partial j_k(\mathbf{x};\hat{f})}{\partial x_k},\qquad(2.41)$$

where the operator of the particle flux density is defined by

$$\hat{j}_k(\mathbf{x})=i\int \mathrm{d}^3x'x'_k\int_0^1 \mathrm{d}\xi[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';\hat{f}),\hat{\rho}(\mathbf{x}+\xi\mathbf{x}')].\qquad(2.42)$$

Setting $\hat{a}(\mathbf{x})=\hat{s}_i(\mathbf{x})$ in Eq. (2.38) and using (2.26), we find the law for the conservation of the spin density:

$$\frac{\partial s_i(\mathbf{x};\hat{f})}{\partial t}=-\frac{\partial j_{ik}(\mathbf{x};\hat{f})}{\partial x_k},\qquad(2.43)$$

where the operator of the spin flux density is defined by

$$\hat{j}_{ik}(\mathbf{x})=i\int \mathrm{d}^3x'x'_k\int_0^1 \mathrm{d}\xi[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';\hat{f}),\hat{s}_i(\mathbf{x}+\xi\mathbf{x}')].\qquad(2.44)$$

Finally, setting $\hat{a}(\mathbf{x})=\hat{\pi}_i(\mathbf{x})$ in Eq. (2.39) and using (2.27) we find the law for the conservation of the momentum density:

$$\frac{\partial\pi_i(\mathbf{x};f)}{\partial t}=-\frac{\partial t_{ik}(\mathbf{x};\hat{f})}{\partial x_k},\qquad(2.45)$$

where the average value of the stress tensor is defined by

$$t_{ik}(\mathbf{x};\hat{f})=-\mathscr{E}(\mathbf{x};\hat{f})\delta_{ik}+\frac{i}{2}\int \mathrm{d}^3x'x'_k\int_0^1 \mathrm{d}\xi\times\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';\hat{f}),\hat{\pi}_i(\mathbf{x}+\xi\mathbf{x}')].\qquad(2.46)$$

We next give the energy conservation law. From (2.23) and (2.34) we have

$$\frac{\partial\mathscr{E}(\mathbf{x};\hat{f})}{\partial t}=\frac{1}{2}\,\mathrm{Sp}\,\frac{\partial\hat{f}}{\partial t}\,\hat{\varepsilon}(\mathbf{x};\hat{f})=\frac{1}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(\hat{f}),\hat{\varepsilon}(\mathbf{x};\hat{f})].$$

Hence setting $\hat{a}(\mathbf{x})=\hat{\varepsilon}(\mathbf{x};\hat{f})$ in Eq. (2.40) we find

$$\frac{\partial\mathscr{E}(\mathbf{x};\hat{f})}{\partial t}=-\frac{\partial W_k(\mathbf{x};\hat{f})}{\partial x_k},\qquad(2.47)$$

where the operator of the energy flux density is defined by

$$\hat{W}_k(\mathbf{x})=\frac{1}{2}\int \mathrm{d}^3x'x'_k\int_0^1 \mathrm{d}\xi[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';\hat{f}),\hat{\varepsilon}(\mathbf{x}+\xi\mathbf{x}';\hat{f})].\qquad(2.48)$$

3. SINGLET AND TRIPLET FERMION PAIRING

3.1. Diagonalization of the operators of dynamic Fermi-liquid quantities

We return to the self-consistency equation (2.32) and show that from this relation we can obtain the basic results of the BCS theory. The problem consists essentially of block diagonalization of the matrix

$$\hat{\varepsilon}=\begin{pmatrix}\varepsilon & \Delta\\ \Delta^+ & -\tilde{\varepsilon}\end{pmatrix},$$

which determines the quasiparticle energy. To this end we note that an arbitrary physical quantity $\hat{A}$ having the structure

$$\hat{A}=\begin{pmatrix}a & \underline{a}\\ \underline{a}^+ & -\tilde{a}\end{pmatrix},$$

is transformed under the unitary transformations (2.13) according to Eqs. (2.14).

Block diagonalization means that we choose a unitary transformation so that the operator $\underline{a}'$ vanishes [see Eq. (2.14)]:

$$\hat{A}'=\hat{U}^+\hat{A}\hat{U}=\begin{pmatrix}a' & 0\\ 0 & -\tilde{a}'\end{pmatrix}.\qquad(3.1)$$

In other words, we require that

$$u^+av+u^+\underline{a}u^*+\tilde{v}\underline{a}^+v-\tilde{v}\tilde{a}u^*=0$$

or

$$aX+\underline{a}+\widetilde{X}\underline{a}^{+}X-X\widetilde{a}=0, \tag{3.2}$$

where $v=Xu^{*}$. From the unitary condition $[\hat{U}^{+}\hat{U}=1$; see Eq. (2.13)] it is easy to find

$$uu^{+}=(1-\widetilde{X}X^{+})^{-1}, \quad uu^{+}\widetilde{X}+X(uu^{+})^{*}=0. \tag{3.3}$$

Substituting the first relation in the second we find

$$(1-\widetilde{X}X^{+})(X+\widetilde{X})=0,$$

from which it follows that

$$\widetilde{X}=-X, \quad uu^{+}=(1+XX^{+})^{-1}\equiv K. \tag{3.4}$$

The equation determining X therefore takes the form

$$aX+\underline{a}+X\widetilde{a}-X\underline{a}^{+}X=0. \tag{3.5}$$

Now we find the operator a'. From (2.14) and (3.4) we have

$$a'=u^{+}Du,$$

$$D=a-\underline{a}X^{+}-X\widetilde{a}X^{+}-X\underline{a}^{+}=D^{+}. \tag{3.6}$$

Using Eq. (3.5) we can represent the quantity D in the form

$$D=(a-X\underline{a}^{+})(1+XX^{+}), \tag{3.7}$$

and consequently from (3.6)

$$a'=u^{+}(a-Xa^{+})u^{+^{-1}}. \tag{3.8}$$

Note that it follows from the Hermitian property of the matrix D and from Eq. (3.7) that

$$K(a-X\underline{a}^{+})=(a-\underline{a}X^{+})K. \tag{3.9}$$

We also have the relation

$$X\widetilde{K}=KX=-\widetilde{KX}. \tag{3.10}$$

Equation (3.4) determines the matrix u to within a transformation $u\to u\lambda$, where λ is some unitary matrix, $\lambda\lambda^{+}=1$. This arbitrariness in the determination of the matrix u can be used in order to carry out a diagonalization of the matrix a' in momentum and spin spaces.

We rewrite the self-consistency equation (2.32) in the form

$$\hat{f}=[\exp(Y_0\hat{\xi})+1]^{-1},$$

where

$$\hat{\xi}=\begin{pmatrix}\xi & \underline{\xi}\\ \underline{\xi}^{+} & -\xi\end{pmatrix}, \quad \xi=\varepsilon+\frac{Y_4}{Y_0}+\frac{Y_k}{Y_0}p_k, \quad \underline{\xi}=\Delta. \tag{3.11}$$

From what was said above it follows that the quantity ξ can be made quasidiagonal by means of a unitary transformation

$$\hat{U}^{+}\hat{\xi}\hat{U}=\begin{pmatrix}\xi' & 0\\ 0 & -\widetilde{\xi}'\end{pmatrix},$$

where

$$\xi'=u^{+}(\xi-X\underline{\xi}^{+})u^{+^{-1}}$$

and the matrix X satisfies the equation

$$\xi X+X\widetilde{\xi}+\underline{\xi}-X\underline{\xi}+X=0, \quad \widetilde{X}=-X. \tag{3.12}$$

Since

$$\hat{U}=\begin{pmatrix}u & v\\ v^{*} & u^{*}\end{pmatrix}, \quad \hat{f}=\hat{U}\hat{f}'\hat{U}^{+},$$

where

$$\hat{f}'=\begin{pmatrix}f' & 0\\ 0 & -f'\end{pmatrix}, \quad f'=(e^{Y_0\xi'}+1)^{-1}, \tag{3.13}$$

we have

$$f=uf'u^{+}+v(1-\widetilde{f}')v^{+}, \quad g=uf'\widetilde{v}+v(1-\widetilde{f}')\widetilde{u}.$$

Noting that from (3.13) and (3.8) we have

$$\begin{aligned} &f'=u^{+}nu^{+^{-1}},\\ &n=\{\exp[Y_0(\xi-X\underline{\xi}^{+})]+1\}^{-1}, \end{aligned} \tag{3.14}$$

it follows that

$$f=Kn+X(1-\widetilde{n})X^{+}K,$$

$$g=\widetilde{KnX}+K(1-n)X, \tag{3.15}$$

where [see Eq. (3.4)]

$$K=(1+XX^{+})^{-1}. \tag{3.16}$$

In deriving these formulas we have taken into account that $v=Xu^{*}$ and $X\widetilde{K}=KX$ [see (3.10)]. By virtue of the relation (3.9) the matrix f is Hermitian, while the matrix g is antisymmetric because $\widetilde{K}\widetilde{X}=-KX$.

Equations (3.15) are equivalent to the self-consistency equation (2.32). We see that when we go over to operators acting in spin and momentum space the self-consistency equation ceases to be physically transparent [specifically, Eq. (2.32) clearly exhibits the self-consistency principle]. However, Eqs. (3.15) are more convenient in specific applications.

Note that from (3.12) it follows that if X is the solution of Eq. (3.12), then $X' = -X^{+^{-1}}$ is also a solution of Eq. (3.12). Using Eqs. (3.9) and (3.10) we can transform Eqs. (3.15) to the symmetric form

$$f=Kn+K'n', \quad g=-KnX-K'n'X', \tag{3.17}$$

where

$$K=(1+XX^{+})^{-1}, \quad K'=(1+X'X'^{+})^{-1},$$

$$n=\{\exp[Y_0(\xi-X\underline{\xi}^{+})]+1\}^{-1},$$

$$n'=\{\exp[Y_0(\xi-X'\underline{\xi}^{+})]+1\}^{-1}.$$

From these relations it follows that

$$f|_{X\to X'}=f, \quad g|_{X\to X'}=g,$$

i.e., the matrices f and g do not depend on whether we use the matrix X or the matrix $X' = -X^{+^{-1}}$ in solving Eq. (3.12).

3.2. Energy density functional

In this section we go into details regarding the energy functional $E(\hat{f})$. If we are close to the phase transition point in the superfluid state, then the quantities g and g^+ are small and the functional $E(f,g,g^+)$ can be expanded in powers of g and g^+. In this expansion, by virtue of the phase symmetry, there will be no terms linear in g and g^+ or quadratic terms of the form gg, g^+g^+. Hence the functional $E(f,g,g^+)$ for small g has the following structure:

$$E(f,g,g^+)=E(f)+V^{-1}(g^+,Lg), \tag{3.18}$$

where $E(f)$ is the Fermi-liquid energy functional of the normal fluid and $L\equiv\{L_{12;34}\}$ is an operator that maps quantities of the form g_{34} into quantities of the same kind (this operator can in principle be the functional f). We now take into account the spin invariance. Note that under a unitary transformation u_ω the spin matrix s_i transforms according to

$$u_\omega^+ s_i u_\omega=a_{ik}(\omega)s_k, \tag{3.19}$$

where $a_{ik}(\omega)$ is an orthogonal matrix. From (2.19), the operator g transforms under the unitary transformation u_ω according to $g\rightarrow g_\omega=\widetilde{u}_\omega g u_\omega$. From (3.19) it can easily be seen that we have $\widetilde{u}_\omega\sigma_2 u_\omega=\sigma_2$ and $\widetilde{u}_\omega\sigma_2\sigma_i u_\omega=a_{ik}(\omega)\sigma_2\sigma_k$. Consequently, the energy $(g^+,Lg)/V$ which is invariant under spin rotations takes the form

$$(g^+,Lg)=(g_0^+,L_1 g)+(g_i^+,L_2 g_i), \tag{3.20}$$

where

$$g_0=\mathrm{Sp}_\sigma\,\sigma_2 g,\quad g_i=\mathrm{Sp}_\sigma\,\sigma_2\sigma_i g,$$

and is characterized by the two anomalous Fermi-liquid amplitudes L_1, L_2, which are operators acting only on the translational degrees of freedom

$$(g_0^+,L_1 g_0)=\sum_{\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4} g^*_{0\mathbf{p}_1\mathbf{p}_2}L_{1\mathbf{p}_1\mathbf{p}_2;\mathbf{p}_3\mathbf{p}_4}g_{0\mathbf{p}_3\mathbf{p}_4}.$$

If the energy functional $E(f)$ is characterized by the two Fermi-liquid Landau amplitudes F_1, F_2 [see (1.8)],

$$E(f)=\sum_1 \varepsilon_1 f_{11}+\tfrac{1}{2}(f_0,F_1 f_0)+\tfrac{1}{2}(f_i,F_2 f_i), \tag{3.21}$$

where

$$f_0=\mathrm{Sp}_\sigma f,\quad f_i=\mathrm{Sp}_\sigma f\sigma_i,$$

then we see that the superfluid Fermi liquid is characterized by the four interaction amplitudes F_i, L_i $(i=1,2)$. In Eq. (3.21) ε_1 is the quasiparticle energy neglecting Fermi-liquid interactions.

We can approach the determination of the Fermi-liquid energy E from the point of view of perturbation theory. If the microscopic Hamiltonian H has the form

$$H=\sum_1 \varepsilon_1\hat{a}_1^+\hat{a}_1+\sum_{1,2,3,4}\Phi(12;34)a_1^+a_2^+a_3a_4, \tag{3.22}$$

then to lowest order in the interaction we have $E=\mathrm{Sp}\,\rho_0 H$, where ρ_0 is the nonequilibrium statistical operator defined by Eq. (2.1). Consequently, taking (3.22) into account we have

$$E(\hat{f})=\sum_1\varepsilon_1 f_{11}+\sum_{1,2,3,4}\Phi(12;34)(f_{32}f_{41}-f_{31}f_{42}+g_{21}^+g_{43}). \tag{3.23}$$

In this equation $\Phi(12;34)$ is the spin-invariant interaction amplitude, which is nonvanishing only for $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4$ (we have taken into consideration the fact that the average value of the operator $a_1^+a_2$ is exactly determined even in the zeroth order approximation, i.e., by the statistical operator (2.1) (see Ref. 6). In this case the quantity $\Phi(12;34)$ determines both the usual Fermi-liquid amplitudes F_i and the Fermi-liquid amplitudes L_i which characterize the superfluid Fermi liquid.

3.3. Singlet pairing and the BCS theory

Equations (3.15) [or (3.17)] can serve as the basis for studying the different superfluid systems corresponding to the various physical phases. Each phase corresponds to a different solution of Eqs. (3.15) [or (3.17)]. A selection from these solutions must be made by taking into account the symmetry properties of the phase.

In this section we consider the case of singlet pairing, when the operator $\hat{f}$ commutes with the spin operator $\hat{s}_i$:

$$[\hat{f},\hat{s}_i]=0. \tag{3.24}$$

As follows from Eq. (2.33), the statistical operator is $\hat{f}_\mathbf{q}\equiv\hat{U}_\mathbf{q}^+\hat{f}\hat{U}_\mathbf{q}$, where $\hat{U}_\mathbf{q}=\exp(iq_i\hat{x}_i)$ is translationally invariant, i.e., it commutes with the momentum operator: $[\hat{f}_\mathbf{q},\hat{p}_i]=0$. From (2.32) it follows that this operator satisfies the self-consistency equation

$$\hat{f}_\mathbf{q}=\{\exp[Y_0\hat{\varepsilon}_\mathbf{q}(\hat{f}_\mathbf{q})+Y_i\hat{p}_i+(Y_4+Y_i q_i)\hat{\tau}_3]+1\}^{-1},$$

where

$$\varepsilon_\mathbf{q}(\hat{f}_\mathbf{q})=\hat{U}_\mathbf{q}^+\varepsilon(\hat{U}_\mathbf{q}\hat{f}_\mathbf{q}\hat{U}_\mathbf{q}^+)\hat{U}_\mathbf{q}.$$

In what follows we will assume $\mathbf{q}=0$. In order to find the operator $\hat{f}_\mathbf{q}$ and hence the operator $\hat{f}$ for $\mathbf{q}\neq0$, in the expressions found below it is necessary to make the substitution

$$\hat{f}\rightarrow\hat{f}_\mathbf{q},\quad \hat{\varepsilon}(\hat{f})\rightarrow\hat{\varepsilon}_\mathbf{q}(\hat{f}_\mathbf{q}),\quad Y_4\rightarrow Y_4+Y_i q_i.$$

From Eqs. (3.24) and (2.33) it follows that $[f,\sigma_i]=0$, $[f,p_i]=0$, $\sigma_i g+g\widetilde{\sigma}_i=0$, $p_i g+g\widetilde{p}_i=0$. Thus,

$$f_{12}=f_{\mathbf{p}_1}\delta_{\mathbf{p}_1\mathbf{p}_2}\delta_{\sigma_1,\sigma_2},\quad g_{12}=g_{\mathbf{p}_1}\delta_{\mathbf{p}_1,-\mathbf{p}_2}(\sigma_2)_{\sigma_1\sigma_2}, \tag{3.25}$$

where $f_\mathbf{p}=f_\mathbf{p}$, $g_\mathbf{p}=g_{-\mathbf{p}}$. From this and from (2.24) it follows that

$$\varepsilon_{12}=\varepsilon_{\mathbf{p}_1}\delta_{\mathbf{p}_1,\mathbf{p}_2}\delta_{\sigma_1,\sigma_2},\quad \Delta_{12}=\Delta_{\mathbf{p}_1}\delta_{\mathbf{p}_1,-\mathbf{p}_2}(\sigma_2)_{\sigma_1\sigma_2}.$$

Consequently, we have $X_{12}=X_{\mathbf{p}_1}\delta_{\mathbf{p}_1,-\mathbf{p}_2}(\sigma_2)_{\sigma_1\sigma_2}$, where from (3.12) $X_\mathbf{p}$ satisfies the equation

$$\xi_\mathbf{p}X_\mathbf{p}+\xi_{-\mathbf{p}}X_\mathbf{p}+\Delta_\mathbf{p}-\Delta_\mathbf{p}^*X_\mathbf{p}^2=0,$$

whence

$$X_{\mathbf{p}}=\frac{\xi_{\mathbf{p}}+\xi_{-\mathbf{p}}-2E_{\mathbf{p}}}{2\Delta^{*}_{\mathbf{p}}},$$

$$E_{\mathbf{p}}=\left[\left(\frac{\xi_{\mathbf{p}}+\xi_{-\mathbf{p}}}{2}\right)^{2}+\left|\Delta_{\mathbf{p}}\right|^{2}\right]^{1/2}. \tag{3.26}$$

From Eqs. (3.4) and (3.14) it follows that

$$n_{\mathbf{p}}=\left\{\exp\left[Y_0\left(\frac{\xi_{\mathbf{p}}-\xi_{-\mathbf{p}}}{2}+E_{\mathbf{p}}\right)\right]+1\right\}^{-1},$$

$$K_{\mathbf{p}}=\frac{1}{2E_{\mathbf{p}}}\left(E_{\mathbf{p}}+\frac{\xi_{\mathbf{p}}+\xi_{-\mathbf{p}}}{2}\right), \tag{3.27}$$

$$K_{\mathbf{p}}X^{2}_{\mathbf{p}}=\frac{1}{2E_{\mathbf{p}}}\left(E_{\mathbf{p}}-\frac{\xi_{\mathbf{p}}+\xi_{-\mathbf{p}}}{2}\right).$$

Thus,

$$f_{\mathbf{p}}=\frac{1}{2E_{\mathbf{p}}}\left[\left(E_{\mathbf{p}}+\frac{\xi_{\mathbf{p}}+\xi_{-\mathbf{p}}}{2}\right)n_{\mathbf{p}}+\left(E_{\mathbf{p}}-\frac{\xi_{\mathbf{p}}+\xi_{-\mathbf{p}}}{2}\right)(1-n_{-\mathbf{p}})\right],$$

$$g_{\mathbf{p}}=\frac{\Delta_{\mathbf{p}}}{2E_{\mathbf{p}}}\,(n_{\mathbf{p}}+n_{-\mathbf{p}}-1). \tag{3.28}$$

Noting that

$$\varepsilon_{\mathbf{p}}=\frac{1}{2}\frac{\partial E}{\partial f_{\mathbf{p}}},\quad \Delta_{\mathbf{p}}=\frac{\partial E}{\partial g^{*}_{\mathbf{p}}}, \tag{3.29}$$

we see that Eqs. (3.28) together with Eqs. (3.29) form a closed system of equations determining Δ and ε, provided the energy functional $E(\hat{f})$ is known.

As the energy functional we now choose the functional (3.18), (3.21). Then for the distributions (3.25) we have

$$E(f,g,g^{+})=\sum_{\mathbf{p}}\varepsilon_{\mathbf{p}}f_{\mathbf{p}}+\frac{2}{V}\sum_{\mathbf{p},\mathbf{p}_1}f_{\mathbf{p}_1}F_1(\mathbf{p}_1,\mathbf{p})f_{\mathbf{p}}$$
$$+\frac{2}{V}\sum_{\mathbf{p},\mathbf{p}_1}g^{*}_{\mathbf{p}_1}L_1(\mathbf{p}_1,\mathbf{p})g_{\mathbf{p}}$$

and consequently

$$\xi_{\mathbf{p}}=\varepsilon_{\mathbf{p}}+\frac{Y_4}{Y_0}+\frac{Y_i}{Y_0}p_i+\frac{2}{V}\sum_{\mathbf{p}_1}F_1(\mathbf{p},\mathbf{p}_1)f_{\mathbf{p}_1},$$

$$\Delta_{\mathbf{p}}=\frac{2}{V}\sum_{\mathbf{p}_1}L_1(\mathbf{p},\mathbf{p}_1)g_{\mathbf{p}_1}.$$

Thus, from (3.28) the equations determining $\xi_{\mathbf{p}}$, $\Delta_{\mathbf{p}}$ take the form

$$\xi_{\mathbf{p}}=\varepsilon_{\mathbf{p}}+\frac{Y_4}{Y_0}+\frac{Y_i}{Y_0}p_i+\frac{2}{V}\sum_{\mathbf{p}_1}F_1(\mathbf{p},\mathbf{p}_1)$$
$$\times\frac{1}{2E_{\mathbf{p}_1}}\left[\left(E_{\mathbf{p}_1}+\frac{\xi_{\mathbf{p}_1}+\xi_{-\mathbf{p}_1}}{2}\right)n_{\mathbf{p}_1}\right.$$
$$\left.+\left(E_{\mathbf{p}_1}-\frac{\xi_{\mathbf{p}_1}+\xi_{-\mathbf{p}_1}}{2}\right)(1-n_{-\mathbf{p}_1})\right],$$

$$\Delta_{\mathbf{p}}=\frac{1}{V}\sum_{\mathbf{p}_1}L_1(\mathbf{p},\mathbf{p}_1)\,\frac{\Delta_{\mathbf{p}_1}}{E_{\mathbf{p}_1}}\,(n_{\mathbf{p}_1}+n_{-\mathbf{p}_1}-1). \tag{3.30}$$

If we take the expression obtained from Eq. (3.23) as the Fermi-liquid amplitude L_1 and disregard the Fermi-liquid amplitude F_1, then we arrive at the basic equation of the BCS theory of superfluid systems.[21] In this theory the quantity $g_{\mathbf{p}}=\langle a_{-\mathbf{p}}a_{\mathbf{p}}\rangle$, which we have termed the anomalous distribution function, can be interpreted as the wave function of two paired electrons in the momentum representation. (This wave function is called the Cooper pair wave function.) According to Eq. (3.28) the function $g_{\mathbf{p}}$ vanishes for $|\mathbf{p}-\mathbf{p}_{\mathrm{F}}|\gg\Delta/v_{\mathrm{F}}$. We can therefore interpret the quantity $\xi_0=\hbar v_F/\pi\Delta$ as the size of a Cooper pair.

If in the self-consistency equation (3.30) (for $F_1=0$) we take $L_1(\mathbf{p},\mathbf{p}')$ to be a quantity which is nonvanishing in the narrow layer near the Fermi surface,

$$L_1(\mathbf{p},\mathbf{p}')=-L,\quad |\xi_{\mathbf{p}}|\leqslant\theta,\quad |\xi_{\mathbf{p}'}|\leqslant\theta,$$
$$=0,\quad |\xi_{\mathbf{p}}|>\theta\quad\text{or}\quad |\xi_{\mathbf{p}'}|>\theta,$$

where for the electron–phonon interaction mechanism θ is a quantity of the order of the Debye temperature, then, as is well known, for the transition temperature T_c determined from Eq. (3.3) in the limit $\Delta\to 0$

$$1=L\nu(\mu)\int_0^{\theta}\frac{\mathrm{d}\xi}{\xi}\,\mathrm{th}\,\frac{\xi}{2T},$$

we find the following well-known expression:

$$T_{\mathrm{c}}=\frac{1}{2}\gamma\theta\exp\left(-\frac{1}{L\nu(\mu)}\right),$$
$$\ln\gamma=-\int_{\theta}^{\infty}\mathrm{d}x\,\frac{\ln x}{\mathrm{ch}^2 x},\quad \gamma=2.268\ldots, \tag{3.31}$$

where

$$\nu(\mu)=2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\delta(\varepsilon_{\mathbf{p}}-\mu)$$

is the density of states at the Fermi surface. If in the self-consistency equation (3.30) along with the linear terms we also take into account the terms which are cubic in Δ and the fact that T is close to T_{c}, then we find the following expression for the quantity Δ, (T):

$$\Delta(T)=3.06T_{\mathrm{c}}\left(\frac{T_{\mathrm{c}}-T}{T_{\mathrm{c}}}\right)^{1/2}.$$

The quantity $\Delta(T)$ in the limit $T\ll T_{\mathrm{c}}$ is given by $\Delta(0)=4T_{\mathrm{c}}/\gamma$.

The self-consistency equation (3.30) for $\mu > 0$ has a nontrivial solution no matter how small the Fermi-liquid amplitude L is, provided that this amplitude is positive. It is of formal interest to explain the behavior that occurs when we solve the self-consistency equation (3.30) for $\mu < 0$.

For simplicity we will assume $L_1(\mathbf{p},\mathbf{q}) = -L$ if $|\mathbf{p}|$ and $|\mathbf{q}|$ are less than p_0 and $L(\mathbf{p},\mathbf{q}) = 0$ in the opposite case (here L is a positive constant). Then the self-consistency equation (3.30) assumes the form

$$1 = \frac{L}{2} \int_0^{\varepsilon_0} d\varepsilon \frac{\nu(\varepsilon)}{E} \operatorname{th} \frac{E}{2T}, \quad \varepsilon_0 = \varepsilon(p_0).$$

Hence we obtain the equation determining the critical temperature,

$$1 = \frac{L}{2} \int_0^{\varepsilon_0} d\varepsilon \frac{\nu(\varepsilon)}{\varepsilon + |\mu|} \operatorname{th} \frac{\varepsilon + |\mu|}{2T}. \tag{3.32}$$

Since

$$\operatorname{th} \frac{\varepsilon + |\mu|}{2T} < 1, \quad \frac{1}{\varepsilon + |\mu|} \leqslant \frac{1}{\varepsilon},$$

this equation which determines the critical temperature T_c, will not have a solution for any value $\mu < 0$ if $L < L_0$ holds, where

$$L_0^{-1} = \frac{1}{2} \int_0^{\varepsilon_0} d\varepsilon \frac{\nu(\varepsilon)}{\varepsilon}$$

[note that $\nu(\varepsilon) \to 0$ for $\varepsilon \to 0$]. Thus, in contrast to the case $\mu > 0$, for $\mu < 0$ superfluidity can arise only when the interaction is sufficiently strong. From Eq. (3.32) it follows that the solution corresponding to $T_c = 0$ is obtained with a chemical potential $\mu = \mu^*$ satisfying the condition

$$1 = \frac{L}{2} \int_0^{\varepsilon_0} d\varepsilon \frac{\nu(\varepsilon)}{\varepsilon + |\mu^*|}.$$

From this it follows that for $|\mu| > |\mu^*|$ Eq. (3.32) has no solution.

To conclude this section we note that the basic equations of the BCS theory derived from the modified perturbation theory (by canceling the most dangerous diagrams) were obtained by Bogolyubov.[22] He was also responsible for introducing the equilibrium anomalous distribution functions in the method of quasiaverages.[23] The basic equations of the BCS theory were derived in the temperature Green's function method by Gor'kov.[24] Eliashberg[25] and Nambu[26] derived the basic superconductivity equations using the Dyson field equations for the normal and anomalous Green's functions.

3.4. Triplet pairing and superfluid phases of ^{3}He

In the preceding section we studied the singlet pairing of fermions, where the total spin of the paired quasiparticles was equal to zero. Since each quasiparticle has a spin of 1/2, in principle it is possible to have another type of pairing, in which the total spin of the paired particles is equal to unity. This is called triplet pairing. Whereas the mechanism for singlet pairing of electrons is usually phonon exchange, the basis for triplet pairing, which occurs in ^{3}He, is the van der Waals interaction of ^{3}He atoms. Pitaevskiĭ[27] first drew attention to the possibility of this pairing.

Let us analyze the self-consistency equations (2.32) for the case of triplet pairing.[41] Here we will assume that the spatial-uniformity condition (2.33) holds with $q_i = 0$. In order to find the self-consistency equations with $q_i \neq 0$ it is necessary to make use of the same substitutions as in the case of singlet pairing. The anomalous distribution function g_{12} in the case of triplet pairing is a symmetric spinor of rank two and consequently can be expanded in a basis of symmetric unitary second-order matrices $\sigma_1\sigma_2$. Thus we have

$$g_{12} = g_{i\mathbf{p}_1} \delta_{\mathbf{p}_1, -\mathbf{p}_2} (\sigma_i \sigma_2)_{\sigma_1 \sigma_2},$$

where by virtue of the antisymmetry of the anomalous distribution function we have $g_{i,-\mathbf{p}} = -g_{i,\mathbf{p}}$. From this and from (2.24) it follows that

$$\Delta_{12} = \Delta_{i\mathbf{p}_1} \delta_{\mathbf{p}_1, -\mathbf{p}_2} (\sigma_i \sigma_2)_{\sigma_1 \sigma_2}, \quad \Delta_{i,-\mathbf{p}} = -\Delta_{i,\mathbf{p}}. \tag{3.33}$$

Just as in the case of singlet pairing, we use the general expressions (3.14) and (3.15). The solution of Eq. (3.12) for the matrix X is found in the form

$$X_{12} = X_{i\mathbf{p}_1} \delta_{\mathbf{p}_1, -\mathbf{p}_2} (\sigma_i \sigma_2)_{\sigma_1 \sigma_2}, \quad X_{i,-\mathbf{p}} = -X_{i,\mathbf{p}}.$$

As a result, Eq. (3.12) assumes the form

$$X_{i\mathbf{p}}(\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}) - 2X_{i\mathbf{p}}(X_{j\mathbf{p}}\Delta_{j\mathbf{p}}) + (X_{j\mathbf{p}}X_{j\mathbf{p}})\Delta^*_{i\mathbf{p}} + \Delta_{i\mathbf{p}} = 0. \tag{3.34}$$

We take $F_2 = 0$ [see (3.21)], so that $\xi_{12} = \xi_{\mathbf{p}_1} \delta_{\mathbf{p}_1, \mathbf{p}_2} \delta_{\sigma_1, \sigma_2}$. We introduce the notation

$$\alpha_{\mathbf{p}} \equiv X_{j\mathbf{p}} X_{j\mathbf{p}}, \quad \zeta_{\mathbf{p}} = X_{i\mathbf{p}} \Delta^*_{i\mathbf{p}} - \tfrac{1}{2} (\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}). \tag{3.35}$$

Then

$$X_{i\mathbf{p}} = \frac{1}{2\zeta_{\mathbf{p}}} (\Delta_{i\mathbf{p}} + \alpha_{\mathbf{p}} \Delta^*_{i\mathbf{p}}), \tag{3.36}$$

and the system of equations determining the quantities $\alpha_{\mathbf{p}}$ and $\zeta_{\mathbf{p}}$ from (3.35) takes the form

$$\begin{gathered} \alpha_{\mathbf{p}} = \frac{1}{2\zeta_{\mathbf{p}}} \left[\frac{\Delta_{i\mathbf{p}}\Delta_{i\mathbf{p}} + \alpha_{\mathbf{p}} \Delta_{i\mathbf{p}} \Delta^*_{i\mathbf{p}}}{2\zeta_{\mathbf{p}}} + \alpha_{\mathbf{p}} \left(\zeta_{\mathbf{p}} + \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2} \right) \right], \\ \zeta_{\mathbf{p}} = \frac{1}{2\zeta_{\mathbf{p}}} (\Delta_{i\mathbf{p}} \Delta^*_{i\mathbf{p}} + \alpha_{\mathbf{p}} \Delta^*_{i\mathbf{p}} \Delta^*_{i\mathbf{p}}) - \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2}. \end{gathered} \tag{3.37}$$

Solving (3.37) and taking (3.36) into account, we find

$$X_{i\mathbf{p}} = \frac{1}{2\zeta_{\mathbf{p}}} \left\{ \Delta_{i\mathbf{p}} + \frac{2\zeta_{\mathbf{p}}[\zeta_{\mathbf{p}} + (1/2)(\zeta_{\mathbf{p}} + \xi_{-\mathbf{p}})] - \Delta_{j\mathbf{p}} \Delta^*_{j\mathbf{p}}}{\Delta^*_{j\mathbf{p}} \Delta^*_{j\mathbf{p}}} \Delta^*_{i\mathbf{p}} \right\}, \tag{3.38}$$

where

$$(\zeta_{\mathbf p})^2_{1,2}=\tfrac12\,[\Delta_{i\mathbf p}\Delta^*_{i\mathbf p}+\tfrac14\,(\xi_{\mathbf p}+\xi_{-\mathbf p})^2+(G_{\mathbf p})^2_{1,2}], \tag{3.39}$$

$$(G_{\mathbf p})_{1,2}=\pm\{[\Delta_{i\mathbf p}\Delta^*_{i\mathbf p}+\tfrac14\,(\xi_{\mathbf p}+\xi_{-\mathbf p})^2]^2-|[\Delta_{\mathbf p},\Delta^*_{\mathbf p}]|^2\}^{1/2}.$$

Thus, using relations (3.17) and (3.38) for the anomalous distribution functions $g_{i\mathbf p}$ we have

$$\begin{aligned}g_{i\mathbf p}=\tfrac14\,[&(1-n^0_{\mathbf p}(\zeta_1)-n^0_{-\mathbf p}(\zeta_1))(\chi_{\mathbf p}(\zeta_1)\Delta_{i\mathbf p}+\varphi_{\mathbf p}(\zeta_1)\Delta^*_{i\mathbf p})\\&+(1-n^0_{\mathbf p}(\zeta_2)-n^0_{-\mathbf p}(\zeta_2))(\chi_{\mathbf p}(\zeta_2)\Delta_{i\mathbf p}+\varphi_{\mathbf p}(\zeta_2)\Delta^*_{i\mathbf p})];\end{aligned} \tag{3.40}$$

here

$$\chi_{\mathbf p}(\zeta)=\frac{2\zeta^2_{\mathbf p}-\Delta_{i\mathbf p}\Delta^*_{i\mathbf p}}{\zeta_{\mathbf p}G_{\mathbf p}},\quad \varphi_{\mathbf p}(\zeta)=\frac{\Delta_{i\mathbf p}\Delta_{i\mathbf p}}{\zeta_{\mathbf p}G_{\mathbf p}},$$

$$n^0_{\mathbf p}=\tfrac12\,(n^+_{\mathbf p}+n^-_{\mathbf p}),$$

$$n^\pm_{\mathbf p}=\left\{\exp\left[Y_0\left(-\zeta_{\mathbf p}+\frac{\xi_{\mathbf p}-\xi_{-\mathbf p}}{2}\pm\frac{|[\Delta_{\mathbf p},\Delta^*_{\mathbf p}]|}{2\zeta_{\mathbf p}}\right)\right]+1\right\}^{-1}.$$

The result (3.40) has been presented in a form in which it is clear that the functions g_α are invariant with respect to the choice of the root in (3.39), where for ζ_1 and ζ_2 we can take either the "plus" or the "minus" sign of the square root in (3.39). The structure of the normal distribution function f in the momentum and spin spaces is determined by the expression

$$f_{12}=(f_{0\mathbf p_1}\delta_{\sigma_1\sigma_2}+f_{i\mathbf p}(\sigma_i)_{\sigma_1\sigma_2})\delta_{\mathbf p_1\mathbf p_2}, \tag{3.41}$$

where, as follows from (3.17) and (3.35), the functions $f_{0\mathbf p}$ and $f_{i\mathbf p}$ take the form

$$\begin{aligned}f_{0\mathbf p}=\tfrac12\,[&1+n^0_{\mathbf p}(\zeta_{1,2})-n^0_{-\mathbf p}(\zeta_{1,2})+2(\zeta^2_{\mathbf p})_1\psi_{\mathbf p}(\zeta_1)(1-n^0_{\mathbf p}(\zeta_1)\\&-n^0_{-\mathbf p}(\zeta_1))+2(\zeta^2_{\mathbf p})_2\psi_{\mathbf p}(\zeta_2)(1-n^0_{\mathbf p}(\zeta_2)\\&-n^0_{-\mathbf p}(\zeta_2))],\end{aligned} \tag{3.42}$$

$$\begin{aligned}f_{\mathbf p}=-\tfrac12\,[&\psi_{\mathbf p}(\zeta_1)(1-n^0_{\mathbf p}(\zeta_1)-n^0_{-\mathbf p}(\zeta_1))+\psi_{\mathbf p}(\zeta_2)(1\\&-n^0_{\mathbf p}(\zeta_2)-n^0_{-\mathbf p}(\zeta_2))+|[\Delta_{\mathbf p},\Delta^*_{\mathbf p}]|^{-1}(n''_{\mathbf p}(\zeta_{1,2})\\&-n''_{-\mathbf p}(\zeta_{1,2}))]i[\Delta_{\mathbf p},\Delta^*_{\mathbf p}],\end{aligned}$$

$$\psi_{\mathbf p}(\zeta)=\frac{\xi_{\mathbf p}+\xi_{-\mathbf p}}{4\zeta_{\mathbf p}G_{\mathbf p}},\quad n''_{\mathbf p}=\tfrac12\,(n^+_{\mathbf p}-n^-_{\mathbf p}). \tag{3.43}$$

Where the quantity $\zeta_{1,2}$ appears in some places in (3.42) and (3.43) this means that these combinations are invariant themselves with respect to the choice of the root and as the argument we can take either the first root ζ_1 or the second root ζ_2.

From expression (3.20) and (3.21) for the energy functional it follows that

$$\xi_{\mathbf p}=\varepsilon_{\mathbf p}+\frac{Y_i}{Y_0}p_i+\frac{Y_4}{Y_0}+\frac1V\sum_{\mathbf p_1}F_1(\mathbf p,\mathbf p_1)f_{0\mathbf p_1},$$

$$\Delta_{i\mathbf p}=\frac2V\sum_{\mathbf p_1}L_2(\mathbf p,\mathbf p_1)g_{i\mathbf p_1}.$$

Hence from (3.40), (3.42), and (3.43) the self-consistency equations for the quantities $\xi_{\mathbf p}$ and $\Delta_{i\mathbf p}$ take the form

$$\begin{aligned}\xi_{\mathbf p}=\varepsilon_{\mathbf p}+\frac{Y_i}{Y_0}p_i+\frac{Y_4}{Y_0}\frac1V\sum_{\mathbf p_1}&F_1(\mathbf p,\mathbf p_1)[1+n^0_{\mathbf p_1}(\zeta_{1,2})\\&-n^0_{-\mathbf p_1}(\zeta_{1,2})+\psi_{\mathbf p_1}(\zeta_1)(1-n^0_{\mathbf p_1}(\zeta_1)-n^0_{-\mathbf p_1}(\zeta_1))\\&+\psi_{\mathbf p_1}(\zeta_2)(1-n^0_{\mathbf p_1}(\zeta_2)-n^0_{-\mathbf p_1}(\zeta_2))],\end{aligned} \tag{3.44}$$

$$\begin{aligned}\Delta_{i\mathbf p}=\frac1V\sum_{\mathbf p_1}&L_2(\mathbf p,\mathbf p_1)[(1-n^0_{\mathbf p_1}(\zeta_1)-n^0_{-\mathbf p_1}(\zeta_2))\\&\times(\chi_{\mathbf p_1}(\zeta_1)\Delta_{i\mathbf p_1}+\varphi_{\mathbf p_1}(\zeta_1)\Delta^*_{i\mathbf p_1})+(1-n^0_{\mathbf p_1}(\zeta_2)\\&-n^0_{-\mathbf p_1}(\zeta_2))(\chi_{\mathbf p_1}(\zeta_2)\Delta_{i\mathbf p_1}+\varphi_{\mathbf p_1}(\zeta_2)\Delta^*_{i\mathbf p_1})].\end{aligned} \tag{3.45}$$

These equations completely solve the general problem of the triplet pairing and are analogous to Eqs. (3.30) in the case of singlet pairing. Note that if the vector product $[\Delta_{\mathbf p},\Delta^*_{\mathbf p}]$ is nonzero, then in the superfluid there exists a spontaneous magnetization M_i occurring as a result of the transition to the superfluid phase, which is determined by the expression

$$\begin{aligned}M_i=-\beta\sum_{\mathbf p}&[\psi_{\mathbf p}(\zeta_1)(1-n^0_{\mathbf p}(\zeta_1)-n^0_{-\mathbf p}(\zeta_1))+\psi_{\mathbf p}(\zeta_2)\\&\times(1-n^0_{\mathbf p}(\zeta_2)-n^0_{-\mathbf p}(\zeta_2))]\cdot i[\Delta_{\mathbf p},\Delta^*_{\mathbf p}]_i,\end{aligned} \tag{3.46}$$

where $\beta=e\hbar/2mc$ is the Bohr magneton.

Let us consider the states of a superfluid Fermi liquid for which the product $\Delta\Delta^+$ is a multiple of the unit matrix in spin space (this is equivalent to the condition $[\Delta_{\mathbf p},\Delta^*_{\mathbf p}]=0$). Following Legget,[28] we will call these unitary states. It is clear that no spontaneous magnetization occurs in the transition to the superfluid phase in the case of unitary states.

The self-consistency equations (3.44) and (3.45) for unitary states take the form

$$\begin{aligned}\xi_{\mathbf p}=\varepsilon_{\mathbf p}+\frac{Y_i}{Y_0}p_i+\frac{Y_4}{Y_0}+\frac1V\sum_{\mathbf p_1}&F_1(\mathbf p,\mathbf p_1)\cdot\frac{1}{2E_{\mathbf p_1}}\left[\left(E_{\mathbf p_1}\right.\right.\\&\left.\left.+\frac{\xi_{\mathbf p_1}+\xi_{-\mathbf p_1}}{2}\right)n_{\mathbf p_1}+\left(E_{\mathbf p_1}-\frac{\xi_{\mathbf p_1}+\xi_{-\mathbf p_1}}{2}\right)(1-n_{-\mathbf p_1})\right],\end{aligned} \tag{3.47}$$

$$\Delta_{i\mathbf p}=\frac2V\sum_{\mathbf p_1}L_2(\mathbf p,\mathbf p_1)\,\frac{\Delta_{i\mathbf p_1}}{E_{\mathbf p_1}}\,(n_{\mathbf p_1}+n_{-\mathbf p_1}-1);$$

here

$$E_{\mathbf p}=[\tfrac14\,(\xi_{\mathbf p}+\xi_{-\mathbf p})^2+\Delta_{i\mathbf p}\Delta^*_{i\mathbf p}]^{1/2},$$

$$n_{\mathbf p}=\left\{\exp\left[Y_0\left(E_{\mathbf p}+\frac{\xi_{\mathbf p}+\xi_{-\mathbf p}}{2}\right)\right]+1\right\}^{-1}.$$

These equations agree in form with those [(3.30)] of the BCS theory. The only difference is that the functions $\Delta_{i\mathbf p}$ and $L_2(\mathbf p,\mathbf p_1)$ are odd in $\mathbf p$.

The unitary states include, in particular, those for which the projection of the spin on the quantization axis vanishes. For such states $\Delta_{i\mathbf{p}} = d_1\Delta_{\mathbf{p}}$ holds, where d_i is a real unit vector parallel to the spin quantization axis. The momentum dependence of the quantity $\Delta_{\mathbf{p}}$ must be found from an equation analogous to (3.47), where it is only necessary to replace $\Delta_{i\mathbf{p}}$ by $\Delta_{\mathbf{p}}$. For $Y_i=0$ the angular dependence of the quantity $\Delta_{\mathbf{p}}$ is determined solely by the structure of the angular dependence of the interaction amplitude $L_2(\mathbf{p},\mathbf{p}')$. If the system as a whole is rotationally invariant (as in the case of ^{3}He), then the interaction amplitude in the general case takes the form

$$L_2(\mathbf{p},\mathbf{p}_1)=L_2(\mathbf{p},\mathbf{p}_1;\eta\eta_1)$$

(here η is a unit vector in the direction $\mathbf{p}$) and can be expanded in Legendre polynomials:

$$L_2(\mathbf{p},\mathbf{p}_1;\eta\eta_1)=\sum_{l=0}^{\infty}(2l+1)V_l(\mathbf{p},\mathbf{p}_1)P_l(\cos\theta) \qquad (3.48)$$

(here θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{p}_1$). The expansion (3.48) gives rise to an expansion of the order parameter in the form

$$\Delta_{\mathbf{p}}=\sum_{l,m}\Delta_{lm}Y_{lm}$$

(in the case of triplet pairing L_2 and $\Delta_{\mathbf{p}}$ contain a superposition of spherical functions with odd l). If only one term appears in the expansion (3.48), corresponding to some value l_0 of the angular momentum, then the angular dependence of $\Delta_{\mathbf{p}}$ will also be described by a combination of spherical functions with $l=l_0$.

An example of unitary states is given by the so-called A and B phases of ^{3}He. The various superfluid phases correspond to different symmetry properties of the equilibrium distribution functions with respect to rotation in spin and orbit space. For example, in the case of electrons with singlet pairing the operator $\hat{f}$ must commute with the spin operator

$$[\hat{f},\hat{s}_i]=0.$$

Moreover, in the usual theory of superconductivity it is assumed that pairing occurs in the state with $l=0$, and so the relation

$$[\hat{f},\hat{\mathscr{L}}_k]=0,$$

must hold where

$$\hat{\mathscr{L}}_k=\hat{l}_k-i\varepsilon_{kli}\left(q_l\frac{\partial}{\partial q_i}+Y_l\frac{\partial}{\partial Y_i}\right); \qquad (3.48')$$

here the second term describes rotation in the space of the vector thermodynamic parameters Y_i, q_i;

$$\hat{l}_k=\begin{pmatrix} l_k & 0 \\ 0 & -\tilde{l}_k \end{pmatrix}$$

is the operator of the orbital angular momentum of a superfluid Fermi liquid.

In the case of the A phase of ^{3}He, pairing occurs in the state with $s=1$, $l=1$ (triplet pairing with $l=0$ is impossible by virtue of the Pauli principle). The operator $\hat{f}$ must satisfy the following symmetry conditions:

$$\begin{aligned} &[\hat{\mathbf{p}}-\mathbf{q}\hat{\tau}_3,\hat{f}]=0, \quad [d_i\hat{s}_i,\hat{f}]=0, \\ &[k_i\hat{\mathscr{L}}_i-\tfrac{1}{2}\hat{\tau}_3,\hat{f}]=0, \end{aligned} \qquad (3.49)$$

where d_1 is the unit vector which determines the quantization axis in spin space and k_1 is the unit vector which determines the quantization axis in momentum space. These relations show that the projection of the spin on the direction d_i vanishes, while the projection of the orbital angular momentum on the direction k_i is equal to unity.

If the projection of the spin on the axis d_i is equal to $m_s=\pm 1,0$ and the projection of the orbital angular momentum on the axis k_i is equal to $m_l=\pm 1,0$, then relations (3.49) must be replaced by

$$\begin{aligned} &[d_i\hat{s}_i-\tfrac{1}{2}m_s\hat{\tau}_3,\hat{f}]=0, \\ &[k_i\hat{l}_i-\tfrac{1}{2}m_l\hat{\tau}_3,\hat{f}]=0. \end{aligned} \qquad (3.50)$$

This interpretation follows from the fact that relations (3.50) imply that (we have set $\mathbf{Y}=\mathbf{q}=0$)

$$(\mathbf{s},\mathbf{d})\mathbf{g}=m_s\mathbf{g}, \quad (\mathbf{k},\mathbf{l})\mathbf{g}=m_l\,\mathbf{g}, \qquad (3.51)$$

where $\mathbf{s}=\mathbf{s}_1+\mathbf{s}_2$, $\mathbf{l}=\mathbf{l}_1+\mathbf{l}_2$. These relations show that if the anomalous distribution function g_{12} is interpreted as the wave function of a pair, then this wave function is the eigenfunction of the operators of the spin projection and the orbital angular momentum on the quantization axes d_i and k_i.

States with $m_s=\pm\mathbf{1}$ are obviously nonunitary, since in this case from Eqs. (3.51) we have $g_i\sim n_{1j}+in_{2j}$, where n_{1j} and n_{2j} are unit vectors which are mutually perpendicular and orthogonal to $\mathbf{d}$.

In the B phase pairing occurs for $s=1$, $l=1$, but the symmetry properties of the phase are determined by

$$[\hat{L}_i+\hat{s}_kR_{ki},\hat{f}]=0, \qquad (3.52)$$

where R is a rotation matrix ($R\tilde{R}=1$, $R=R^*$). In this case the projections of the spins and orbital angular momenta of a Cooper pair are indeterminate. The parameters d_i, k_i, R together with the parameters Y, q_i determine the state of statistical equilibrium of a superfluid. Whether the $l=0$, $s=0$ phase or the A or B phase in the case of ^{3}He ($l=1$, $s=1$) is actually realized is determined by the condition that the thermodynamic potential Ω be minimized; this in turn follows from the structure of the functional $E(f,g,g^+)$ [or the Fermi-liquid amplitudes; cf. Eq. (3.48)].

It is known that ^{3}He forms a liquid which does not freeze down to absolute zero at pressures less than 35 atm. The degeneracy temperature T_{F} is determined by

$$T_{\mathrm{F}}=\frac{1}{2m_{^3\mathrm{He}}}(3\pi^2\hbar^3n_{^3\mathrm{He}})^{2/3}$$

(here $m_{^3\mathrm{He}}$ is the mass of the atom and $n_{^3\mathrm{He}}$ is the ^{3}He atomic density) of order 1 K. The phase transition of liquid ^{3}He from the normal state to the superfluid state was

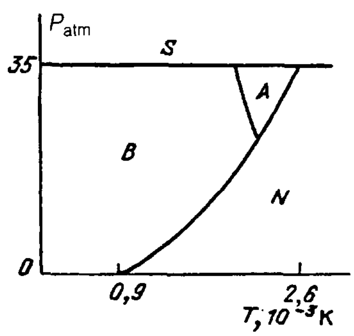

FIG. 1. Phase diagram of the phases of ^{3}He: S is the region of solid ^{3}He, N is the region of the normal liquid ^{3}He; A and B are the regions of superfluid ^{3}He-A and ^{3}He-B respectively.

observed in 1972 (Ref. 29). Superfluid ^{3}He can be found in the two phases A and B (when there is no magnetic field). The phase diagram of liquid ^{3}He is shown in Fig. 1. The phase transition from the normal state to the superfluid A phase at a pressure of 35 atm occurs for a temperature of $T_c=2.6\cdot 10^{-3}$ K, while the transition from the A to the B phase at the same pressure occurs at a temperature $T_{AB}=2.07\cdot 10^{-3}$ K (Ref. 30).

The A and B phases differ in their symmetry properties (see above). The phase transition from the normal state to the superfluid phases is a second-order transition. The phase transition between the A and B phases is accompanied by the production of latent heat and is a first-order transition.

The change in the specific heat as a function of temperature in the transition from the normal to the superfluid state (the second-order transition) is analogous to the behavior of the specific heat in ordinary superconductors.

3.5. Model of superfluidity with a nonquadratic energy functional

Thus far the energy function has been taken to be quadratic in the anomalous distribution function. In this section we include in the energy functional nonquadratic terms of the form $(gg^*)^2$ (terms which are cubic in g and terms of the form gg^*, g^4 cannot appear in the energy functional on account of phase invariance). Assuming that the fermion pairing is singlet and the superfluid momentum q_i vanishes, so that $g_{12} = g_{\mathbf{p}_1}\delta_{\mathbf{p}_1,-\mathbf{p}_2}(\sigma_2)_{\sigma_1\sigma_2}$ holds, we write the part of the energy $E_2(g)$ associated with the anomalous distribution function g in the form

$$E_2(g)=-\frac{1}{2V}\sum_{\mathbf{p},\mathbf{q}} I_{\mathbf{pq}}g_{\mathbf{p}}g^*_{\mathbf{q}}$$

$$-\frac{1}{4V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q},\mathbf{q}'}\tilde{I}(\mathbf{p},\mathbf{p}';\mathbf{q},\mathbf{q}')g_{\mathbf{p}}g_{\mathbf{p}'}g^*_{\mathbf{q}}g^*_{\mathbf{q}'}.$$

We will assume that the amplitude $\tilde{I}(\mathbf{p},\mathbf{p}';\mathbf{q},\mathbf{q}')$ has a sharp maximum at $\mathbf{p}'=\mathbf{p}$, $\mathbf{q}'=\mathbf{q}$, and consequently

$$E_2(g)=-\frac{1}{2V}\sum_{\mathbf{p},\mathbf{q}} I_{\mathbf{pq}}g_{\mathbf{p}}g^*_{\mathbf{q}}-\frac{1}{4V}\sum_{\mathbf{p},\mathbf{q}}\tilde{I}_{\mathbf{pq}}g^2_{\mathbf{p}}{g^*_{\mathbf{q}}}^2.$$

In addition, we assume that the amplitudes $I_{\mathbf{p}q}$ and $\tilde{I}_{\mathbf{p}q}$ can be factored:

$$I_{\mathbf{pq}}=I_{\mathbf{p}}I_{\mathbf{q}},\quad \tilde{I}_{\mathbf{pq}}=\tilde{I}_{\mathbf{p}}\tilde{I}_{\mathbf{q}}$$

(we recall that the amplitudes $I_{\mathbf{p}q}$ and $\tilde{I}_{\mathbf{p}q}$ are positive-definite, which corresponds to fermion attraction). Then the function $\Delta_{\mathbf{p}}\equiv 2\partial E/\partial g^*_{\mathbf{p}}$ is determined by

$$\Delta_{\mathbf{p}}=-\frac{1}{V}I_{\mathbf{p}}\sum_{\mathbf{q}} I_{\mathbf{q}}g_{\mathbf{q}}-\frac{1}{V}g^*_{\mathbf{p}}\tilde{I}_{\mathbf{p}}\sum_{\mathbf{q}}\tilde{I}_{\mathbf{q}}g^2_{\mathbf{q}},$$

and consequently from Eqs. (3.28) and (3.29) the self-consistency equation assumes the form

$$\Delta_{\mathbf{p}}=-AI_{\mathbf{p}}-B\tilde{I}_{\mathbf{p}}\Delta_{\mathbf{p}}\frac{2n_{\mathbf{p}}-1}{2E_{\mathbf{p}}},\tag{3.53}$$

where

$$A=\frac{1}{V}\sum_{\mathbf{q}} I_{\mathbf{q}}\frac{2n_{\mathbf{q}}-1}{2E_{\mathbf{q}}}\Delta_{\mathbf{q}},$$
$$B=\frac{1}{V}\sum_{\mathbf{q}}\tilde{I}_{\mathbf{q}}\frac{(2n_{\mathbf{q}}-1)^2}{4E^2_{\mathbf{q}}}\Delta^2_{\mathbf{q}}.\tag{3.54}$$

Substituting in (3.54) the solution of Eq. (3.53) for $\Delta_{\mathbf{p}}$ ($E_{\mathbf{p}}$ and $n_{\mathbf{p}}$ also depend on $\Delta_{\mathbf{p}}$), we find transcendental equations determining the constants A and B (independent of momentum), whose solution determines $\Delta_{\mathbf{p}}$ in accordance with (3.53).

It is easy to see that if $\tilde{I}_{\mathbf{p}}=0$ holds we return to the basic equations of the BCS theory. Let us consider a different case, in which for some reason the amplitude $I_{\mathbf{p}}$ is anomalously small. Then setting $I_{\mathbf{p}}=0$ in Eq. (3.53) we find from (3.54) that $A=0$ holds, and the quantity B is determined from the equations

$$\Delta_{\mathbf{p}}=-\tilde{I}_{\mathbf{p}}\frac{2n_{\mathbf{p}}-1}{2E_{\mathbf{p}}}B\Delta_{\mathbf{p}},$$
$$B=\frac{1}{V}\sum_{\mathbf{q}}\tilde{I}_{\mathbf{q}}\frac{(2n_{\mathbf{q}}-1)^2}{4E^2_{\mathbf{q}}}\Delta^2_{\mathbf{q}}.\tag{3.55}$$

Assume $\tilde{I}_{\mathbf{p}}=0$ outside the layer $|\xi|\geqslant\theta$ and $\tilde{I}_{\mathbf{p}}=\tilde{I}=$const inside it. Then from Eqs. (3.55) we have $\Delta_{\mathbf{p}}=0$ for $|\xi|>\theta$ and for $|\xi|<\theta$ we have the relation

$$1=-\tilde{I}\frac{2n_{\mathbf{p}}-1}{2E_{\mathbf{p}}}B,\quad |\xi|\leqslant\theta$$

with a nontrivial solution $\Delta_{\mathbf{p}}\neq 0$. Since

$$n_{\mathbf{p}}=\left(\exp\frac{E_{\mathbf{p}}}{T}+1\right)^{-1},$$

it follows from the last equation that inside the layer ($|\xi|\leqslant\theta$) the energy $E_{\mathbf{p}}$ does not depend on the momentum $\mathbf{p}$, and hence we find taking into account $E_{\mathbf{p}} = (\Delta^2_{\mathbf{p}} + \xi^2_{\mathbf{p}})^{1/2}$ the following dependence of the gap $\Delta_{\mathbf{p}}$ on the momentum p_i:

$$\Delta_{\mathbf{p}}=0,\quad |\xi|>\theta=(E^2_0-\xi^2_{\mathbf{p}})^{1/2},\quad |\xi|\leqslant\theta.\tag{3.56}$$

The constant E_0 is determined according to (3.55) and (3.56) from the equation

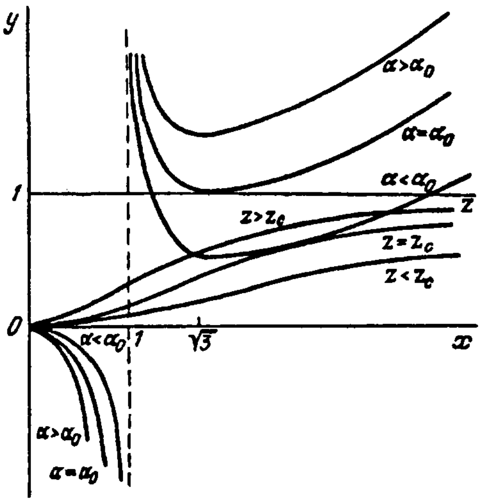

FIG. 2. Graphical solution of the equation $\alpha x^3/(x^2-1)=\tanh^3(xz)$ (here $z_c=\theta/2\sqrt{3}T_c$).

$$-\frac{1}{\tilde{I}}\frac{2E_0}{2n_0-1}=\frac{1}{V}\sum_{|\xi|<\theta}\frac{(2n_0-1)^2}{4E_0^2}(E_0^2-\xi_{\mathbf{p}}^2),$$

$$n_0=\left(\exp\frac{E_0}{T}+1\right)^{-1}.$$

Noting that (we assume $\mu\gg\theta$)

$$\frac{1}{V}\sum_{|\xi|<\theta}(E_0^2-\xi_{\mathbf{p}}^2)=\nu(\mu)\theta\left(E_0^2-\frac{\theta^2}{3}\right),$$

we find the relation

$$1=\tilde{I}\nu(\mu)\theta\frac{\mathrm{th}^3(E_0/2T)}{(2E_0)^3}\left(E_0^2-\frac{\theta^2}{3}\right),$$

or in dimensionless variables

$$\alpha\frac{x^3}{x^2-1}=\mathrm{th}^3(xz), \tag{3.57}$$

where

$$x=\sqrt{3}E_0/\theta,\quad z=\theta/2\sqrt{3}T,\quad \alpha=8(\tilde{I}\nu(\mu))^{-1}/\sqrt{3}.$$

We emphasize that the variable z is determined by the temperature and is independent of the interaction strength, while the variable α depends on the interaction strength but not on the temperature.

To analyze Eq. (3.57) we turn to Fig. 2, in which the curves $y=\alpha x^3/(x^2-1)$ trace the left side of Eq. (3.57), while the curves lying below the straight line $y=1$ trace the right side of the equation $y=\tanh^3(zx)$. For $T=0$ the quantity $\tanh^3(zx)$ degenerates into the straight line $y=1$. This straight line can intersect the curves $y=\alpha x^3/(x^2-1)$ only for $\alpha\leqslant\alpha_0=2/3\sqrt{3}$. For $\alpha>\alpha_0$ Eq. (3.57) has no solutions. This means that if the interaction $\tilde{I}$ is less than $\tilde{I}_0=12/\nu(\mu)$, then the self-consistency equation has no solutions $\Delta\neq0$. A solution $\Delta\neq0$ occurs only when the interaction is sufficiently strong, $\tilde{I}>\tilde{I}_0$. For $T=0$ and $\tilde{I}>\tilde{I}_0$ the quantity $x=\sqrt{3}E_0/\theta$ is found from the equation

$$\alpha x^3-x^2+1=0.$$

This equation has three real roots, one of which is negative; the second lies in the region $1<x<\sqrt{3}$, as can be seen from the figure; the third root satisfies $x>\sqrt{3}$. In order that the quantity $\Delta^2=E_0^2-\xi^2$ be positive for all $|\xi|\leqslant\theta$ it is necessary to choose a solution of Eq. (3.57) which is greater than $\sqrt{3}$.

Now assume $T>0$. Then Eq. (3.57) has two positive solutions, one of which is larger than $\sqrt{3}$. If the interaction is specified and $\alpha<\alpha_0$ holds, then part of the curve $y=\alpha x^3/(x^2-1)$ lies beneath the straight line $y=1$. At sufficiently high temperatures the curve $y=\tanh^3(zx)$ presses against the straight line $y=0$ and Eq. (3.57) has no solution. A solution occurs only at sufficiently low temperatures, when the curves $y=\tanh^3(zx)$ and $y=\alpha x^3/(x^2-1)$ are tangent. The temperature corresponding to the point of tangency also determines the phase transition temperature. We will not write down the equation which determines the point of tangency, but only remark that for $\alpha\lesssim\alpha_0$ the quantity z associated with the temperature T at the point of tangency will also be of order unity (since all the coefficients in the equation determining the point of tangency are of order unity). In this case we have $T_c\sim\theta$, $E_0\sim\theta$. In the strong-interaction limit ($\alpha\ll\alpha_0$) it can be shown that

$$T_c\approx0.03\tilde{I}^2\nu(\mu)\theta,\quad \tilde{I}^2\nu(\mu)\gg12.$$

We emphasize that for $\alpha\ll\alpha_0(\tilde{I}\gg\tilde{I}_0)$ the transition temperature T_c can be substantially greater than θ, while for $\tilde{I}\sim\tilde{I}_0$ the transition temperature is of order θ.

Note that at the transition point the gap $\Delta_{\mathbf{p}}$ appears discontinuously, in contrast to the usual BCS theory in which the quantity Δ vanishes at $T=T_c$. Thus, in this model the phase transition is of first order.

Now we analyze the case in which $I\neq0$ holds, but the interaction is a small quantity (see below). In this case the self-consistency equation (3.53) always has a solution corresponding to small values of $g_{\mathbf{p}}$, for which the term proportional to $\tilde{I}_{\mathbf{p}}$ plays no role no matter how large $\tilde{I}$ is. The transition temperature corresponding to this solution is determined by the usual formula $T_c=\theta\exp(-1/\nu_F I)$. For $\nu_F I\ll1$ (weak interaction) the transition temperature satisfies $T_c\ll\theta$. Consequently, for small values of I [such that $\exp(-1/\nu_F I)\ll\tilde{I}^2\nu_F$ holds] there are two solutions, one of which corresponds to the transition temperature (3.31), while the other to a higher temperature and is determined by the solution obtained in the present section. If the temperature of the normal phase is decreased, then for $T_c\sim\tilde{I}^2\nu_F\theta$ the phase transition we are considering occurs. Whether or not the phase transition with temperature T_c determined by Eq. (3.31) occurs as the temperature is further reduced depends on the relative magnitudes of the thermodynamic potential of the high-temperature phase and that of the BCS phase.

Here we do not go into the situation which arises when the interactions I and $\tilde{I}$ are comparable in the sense that $\exp(-1/\nu_F I)\sim\tilde{I}^2\nu_F$ holds.

Thus, in the model we are treating the transition temperature can be considerably greater than θ, and the phase transition is of first order. Note that in high-temperature superconductors the behavior of the order parameter Δ is very close to that of the order parameter in a first-order phase transition.

3.6. Thermodynamics and hydrodynamics of a Fermi liquid with singlet pairing

In this section we derive the second law of thermodynamics for reversible processes with superfluid Fermi liquids. We will assume for simplicity that pairing of quasiparticles in a Fermi liquid occurs in the singlet state.

In Sec. 2.3 we introduced the potential $\Omega(\hat{f})$. By minimizing it with respect to the function $\hat{f}$ we obtained the self-consistency equations (2.32). In the case of spatially uniform states we can introduce the density of the thermodynamic potential $\omega=\Omega/V$:

$$\omega(\hat{f})=-V^{-1}S(\hat{f})+Y_0\mathscr{E}(0;\hat{f})+Y_i\,\mathrm{tr}\,f\pi_i(0)$$
$$+Y_4\,\mathrm{tr}\,f\rho(0). \tag{3.58}$$

The equilibrium potential $\omega(Y,\mathbf{q})$ will result from substituting for $\hat{f}$ in $\omega(\hat{f})$ the equilibrium distribution function $\hat{f}_{\mathrm{eq}}$, which satisfies Eq. (2.32). As can be seen from Eq. (3.58), the potential $\omega(Y,\mathbf{q})$ depends on the thermodynamic parameters Y and $\mathbf{q}$ both explicitly and implicitly through the distribution function $\hat{f}_{\mathrm{eq}}$. In consequence of the variational principle, we can find the derivatives $\partial\omega/\partial Y_\alpha$ by differentiating with respect to Y_α, taking into account only the explicit dependence of ω on Y_α. As a result we find

$$\frac{\partial\omega}{\partial Y_0}=\mathscr{E},\quad \frac{\partial\omega}{\partial Y_i}=\pi_i,\quad \frac{\partial\omega}{\partial Y_4}=\rho, \tag{3.59}$$

where $\mathscr{E}$, π_i, and ρ are the equilibrium densities of energy, momentum, and particle number. Then we find the derivative $\partial\omega/\partial q_i$.

For this purpose we introduce the unitary operator

$$\hat{U}_{\mathbf{g}}=\exp(-ig_i\hat{x}_i),\quad \hat{x}_i=\begin{pmatrix}\underline{x}_i & 0\\ 0 & -\underline{x}_i\end{pmatrix} \tag{3.60}$$

(here x_i is the position operator), and taking (2.3) into account we rewrite Eq. (3.50) in the form

$$\omega(Y,\mathbf{q}+\mathbf{g})=V^{-1}\,\mathrm{Sp}\,\hat{f}_{\mathbf{g}}\ln\hat{f}_{\mathbf{g}}+Y_0\mathscr{E}(\hat{U}_{\mathbf{g}}^{+}\hat{f}_{\mathbf{g}}\hat{U}_{\mathbf{g}})+Y_i$$
$$\times\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}_{\mathbf{g}}\hat{U}_{\mathbf{g}}\hat{\pi}_i\hat{U}_{\mathbf{g}}^{+}+Y_4\cdot\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}_{\mathbf{g}}\hat{U}_{\mathbf{g}}\hat{\rho}\hat{U}_{\mathbf{g}}^{+}, \tag{3.61}$$

where

$$\hat{f}_{\mathbf{g}}=\hat{U}_{\mathbf{g}}\hat{f}(Y,\mathbf{q}+\mathbf{g})\hat{U}_{\mathbf{g}}^{+},\quad \hat{f}(Y,\mathbf{q})\equiv\hat{f}_{\mathrm{eq}}.$$

Since $[\hat{x}_i,\hat{p}_j]=i\hat{\tau}_3\delta_{ij}$ holds, we have

$$\hat{U}_{\mathbf{g}}\hat{p}_i\hat{U}_{\mathbf{g}}^{+}=\hat{p}_i+g_i\hat{\tau}_3.$$

Hence from (2.33) it follows that

$$[\hat{f}_{\mathbf{g}},\hat{p}_i-q_i\hat{\tau}_3]=0,$$

i.e., the operator $\hat{f}_{\mathbf{g}}$ belongs to the class of operators (2.33). Consequently, using Eqs. (2.23) and (3.58), according to the variation principle [cf. Eq. (2.3)] we have from (3.61)

$$\frac{\partial\omega}{\partial q_i}=Y_0\cdot\frac{i}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(0;\hat{f}),\hat{x}_i]+Y_k\cdot\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}[\hat{\pi}_k,\hat{x}_i].$$

Noting further that

$$[\hat{\pi}_k(\mathbf{x}),\hat{\rho}(\mathbf{x}')]=-i\hat{\rho}(\mathbf{x})\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{x}')$$

and that according to (2.42)

$$j_i(\mathbf{x};\hat{f})\equiv\mathrm{Sp}\,\hat{f}_{\mathrm{eq}}\hat{j}_i(0)/2=i\,\mathrm{Sp}\,\hat{f}[\hat{\varepsilon}(0;\hat{f}),\tilde{x}_i]/2, \tag{3.62}$$

we finally obtain

$$\frac{\partial\omega}{\partial q_i}=Y_0\left(j_i+\frac{Y_i}{Y_0}\rho\right). \tag{3.63}$$

Thus, the second law of thermodynamics for reversible processes in superfluid Fermi liquids can be derived in the form

$$\mathrm{d}\omega=\mathscr{E}\,\mathrm{d}Y_0+\pi_i\mathrm{d}Y_i+\rho\,\mathrm{d}Y_4+Y_0\left(j_i+\frac{Y_i}{Y_0}\rho\right)\mathrm{d}q_i. \tag{3.64}$$

Recall that we have written $Y_0=T^{-1}$ (here T is temperature), $Y_i=-Y_0v_i$ (where v_i is the normal velocity), and $Y_4=-\mu Y_0$ (where μ is the chemical potential). For Galilean invariant systems according to (3.62) and (2.31′) the momentum density π_i coincides with the mass flux density mj_i, $\pi_i=m\cdot j_i$.

Noting that the energy density is given by $s=-\omega+\mathscr{E}Y_0+\pi_iY_i+\rho Y_4$, we can rewrite Eq. (3.64) in the form

$$\mathrm{d}\mathscr{E}=\frac{1}{Y_0}\mathrm{d}s-\frac{Y_i}{Y_0}\mathrm{d}\pi_i-\frac{Y_4}{Y_0}\mathrm{d}\rho+\left(j_i+\frac{Y_i}{Y_0}\rho\right)\mathrm{d}q_i. \tag{3.65}$$

The thermodynamic relations (3.63) and (3.64) which we have obtained can be used to construct the ideal hydrodynamics of a superfluid Fermi liquid with singlet pairing. For this purpose, in consequence of the principle that a state of statistical equilibrium is localized, it is necessary to find the flux densities of particle number, momentum, and energy in thermodynamic equilibrium. The flux density of the particle number is determined by the formula (3.63) which we have just derived. Next we will obtain an expression for the stress tensor t_{ik} and the energy flux density W_k in terms of the thermodynamic potential ω. We begin with the stress tensor t_{ik}.

Let a_{ik} be the coefficients of arbitrary affine transformations $x_i\rightarrow x_i'=a_{ik}x_k$. Then it is easy to see that the operators $a_{ik}^{-1}\hat{x}_k$ and $\tilde{a}_{jk}\hat{p}_k$ satisfy the same permutation relations as the operators $\hat{x}_i$, $\hat{p}_k$. From this it follows, as is well known, that

$$\hat{U}_a\hat{x}_i\hat{U}_a^{+}=a_{ik}^{-1}\hat{x}_k,\quad \hat{U}_a\hat{p}_i\hat{U}_a^{+}=\tilde{a}_{ik}\hat{p}_k, \tag{3.66}$$

where $\hat{U}_a$ is some unitary operator. In the limit of infinitesimal transformations $a_{ik}=\delta_{ik}+\xi_{ik}$ we have $\hat{U}_a=1-i\xi_{ik}\hat{\Gamma}_{ki}$ where the generator $\hat{\Gamma}_{ik}$ of the unitary transformation (3.66) is determined by

$$\hat{\Gamma}_{ik}=\int \mathrm{d}^3x x_i\hat{\pi}_k(x)=\frac{1}{2}\begin{pmatrix}\{\underline{x}_i,p_k\} & 0\\ 0 & -\widetilde{\{\underline{x}_i,p_k\}}\end{pmatrix}.$$

From Eqs. (3.66) it follows that

$$\begin{aligned}&\hat{U}_a\hat{\rho}(\mathbf{x})\hat{U}_a^+=(\det a)\hat{\rho}(a\mathbf{x}),\\ &\hat{U}_a\hat{\pi}_i(\mathbf{x})\hat{U}_a^+=(\det a)\widetilde{a}_{ik}\hat{\pi}_k(a\mathbf{x}).\end{aligned} \tag{3.67}$$

The equilibrium statistical operator $\hat{f}$ is determined by the thermodynamic parameters Y, q. From relations (2.33) it is clear that

$$[\hat{U}_a\hat{f}(Y,\widetilde{a}q)\hat{U}_a^+,\hat{p}_i-q_i\hat{\tau}_3]=0. \tag{3.68}$$

Let $|\mathbf{x},\lambda\rangle$ $(\lambda=s,\tau,\tau=\pm 1)$ be the eigenvector of the operators $\hat{\mathbf{x}}$, $\hat{s}_3$, $\hat{\tau}_3$:

$$\hat{\mathbf{x}}|\mathbf{x},\lambda\rangle=\mathbf{x}|\mathbf{x},\lambda\rangle,\quad \hat{s}_3|\mathbf{x},\lambda\rangle=s|\mathbf{x},\lambda\rangle,$$

$$\hat{\tau}_3|\mathbf{x},\lambda\rangle=\tau|\mathbf{x},\lambda\rangle.$$

Then from (3.66) we have $\hat{U}_a^+|\mathbf{x},\lambda\rangle=\zeta|a^{-1}\mathbf{x},\lambda\rangle$. The normalization condition $\langle\mathbf{x},\lambda|-x'\lambda\rangle=\delta(\mathbf{x}-\mathbf{x}')$ leads to a normalization factor ζ given by $\zeta=(\det a)^{-1/2}$. Hence

$$\hat{U}_a|\mathbf{x},\lambda\rangle=(\det a)^{1/2}|a\mathbf{x},\lambda\rangle,$$

$$\hat{U}_a|0,\lambda\rangle=(\det a)^{1/2}|0,\lambda\rangle.$$

Calculating the trace in the $(\mathbf{x},\lambda)$ representation, we can represent the entropy per unit volume in the form

$$s(\hat{f})=-\frac{1}{V}\operatorname{Sp}\hat{f}\ln\hat{f}=-\frac{1}{V}\sum_\lambda\int_V \mathrm{d}^3x\langle\mathbf{x},\lambda|\hat{f}\ln\hat{f}|\mathbf{x},\lambda\rangle.$$

Since for spatially uniform states (2.33) the matrix element $\langle\mathbf{x},\lambda|\hat{f}\ln\hat{f}|\mathbf{x},\lambda\rangle$ is independent of $\mathbf{x}$, we have

$$s(\hat{f})=-\sum_\lambda\langle 0\lambda|\hat{f}\ln\hat{f}|0\lambda\rangle.$$

Alternatively, using (3.69) we have

$$s(\hat{f})=-(\det a)\sum_\lambda\langle 0\lambda|\hat{U}_a\hat{f}\hat{U}_a^+\ln\hat{U}_a\hat{f}\hat{U}_a^+|0\lambda\rangle.$$

Thus,

$$s(\hat{f})=(\det a)s(\hat{U}_a\hat{f}\hat{U}_a^+).$$

Consequently from (3.58)

$$\begin{aligned}\omega(Y,\widetilde{a}q)=&\frac{\det a}{V}\operatorname{Sp}\hat{f}_a\ln\hat{f}_a+Y_0\mathscr{E}(\hat{U}_a^+\hat{f}_aU_a)\\ &+Y_i\tfrac{1}{2}\operatorname{Sp}\hat{f}_a\hat{U}_a\hat{\pi}_i\hat{U}_a^++Y_4\tfrac{1}{2}\operatorname{Sp}\hat{f}_a\hat{U}_a\hat{\rho}\hat{U}_a^+,\\ \hat{f}_a=&\hat{U}_a\hat{f}(Y,\widetilde{a}q)\hat{U}_a^+\end{aligned} \tag{3.69}$$

or from (3.67)

$$\begin{aligned}\frac{\omega(Y,\widetilde{a}q)}{\det a}=&\frac{1}{V}\operatorname{Sp}\hat{f}_a\ln\hat{f}_a+\frac{Y_0}{\det a}\mathscr{E}(\hat{U}_a^+\hat{f}_a\hat{U}_a)\\ &+Y_i\widetilde{a}_{ik}\tfrac{1}{2}\operatorname{Sp}\hat{f}_a\hat{\pi}_k(0)+Y_4\tfrac{1}{2}\operatorname{Sp}\hat{f}_a\hat{\rho}(0).\end{aligned} \tag{3.70}$$

The right-hand side of this relation depends on a both explicitly and in terms of $\hat{f}_a$. Variation of the right-hand side of (3.70) with respect to $\hat{f}_a$ (followed by taking the limit $a_{ik}\rightarrow\delta_{ik}$) leads to a null result according to the variational principle [see Eq. (2.3)]. Consequently, setting $a_{ik}=\delta_{ik}+\xi_{ik}$ in (3.67) and expanding in powers of ξ_{ik} we find

$$\begin{aligned}&\omega\delta_{ik}-q_k\frac{\partial\omega}{\partial q_i}+Y_i\pi_k(\hat{f})-Y_0\mathscr{E}\delta_{ik}+Y_0\\ &\times\frac{i}{2}\operatorname{Sp}\hat{f}[\hat{\varepsilon}(0,\hat{f}),\hat{\Gamma}_{ik}]=0.\end{aligned}$$

Note that according to (2.46), for states $\hat{f}$ satisfying the spatial uniformity condition the average value of the stress tensor $t_{ik}(\hat{f})$ is equal to

$$t_{ik}(\hat{f})=-\mathscr{E}\delta_{ik}+i\operatorname{Sp}\hat{f}[\hat{\varepsilon}(0;\hat{f}),\hat{\Gamma}_{ik}]/2.$$

Consequently, taking (3.59) into account we have

$$t_{ik}(\hat{f})=\frac{q_k}{Y_0}\frac{\partial\omega}{\partial q_i}-\frac{\partial}{\partial Y_k}\frac{\omega Y_i}{Y_0}. \tag{3.71}$$

In order to find the energy flux density $\bar{W}_k$ we note that for spatially uniform states [see (2.33)] the following lemma holds: if we have $[\hat{f},p_i-q_i\hat{\tau}_3]=0$ then

$$\bar{W}_k\equiv\frac{i}{2}\int \mathrm{d}^3x x_k\operatorname{Sp}\hat{f}[\hat{Q}(\mathbf{x}),\hat{Q}(0)]=0, \tag{3.72}$$

where

$$\hat{f}=(e^{\hat{Q}}+1)^{-1},\quad \hat{Q}=\int \mathrm{d}^3x\hat{Q}(\mathbf{x}).$$

To prove this we introduce the general eigenfunctions of the commuting operators $\hat{p}_i-q_i\hat{\tau}_3$ and $\hat{f}$:

$$(\hat{p}_i-q_i\hat{\tau}_3)\chi_{\mathbf{p}'}=p_i'\chi_{\mathbf{p}'},\quad \hat{f}\chi_{\mathbf{p}'}=f_{\mathbf{p}'}\chi_{\mathbf{p}'}.$$

The trace which appears in Eq. (3.72) is taken in the system of eigenfunctions $\chi_{\mathbf{p}}$:

$$\begin{aligned}&\operatorname{Sp}\hat{f}[\hat{Q}(\mathbf{x}),\hat{Q}(0)]\\ &=\frac{V^2}{(2\pi)^6}\int \mathrm{d}^3\mathbf{p}_1\mathrm{d}^3p_2\operatorname{Sp}'f_{\mathbf{p}_2}[\hat{Q}_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{x})\\ &\quad\times\hat{Q}_{\mathbf{p}_2\mathbf{p}_1}(0)-\hat{Q}_{\mathbf{p}_1\mathbf{p}_2}(0)\hat{Q}_{\mathbf{p}_2\mathbf{p}_1}(\mathbf{x})],\end{aligned}$$

$$\hat{Q}_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{x})=e^{-i(\mathbf{p}_1-\mathbf{p}_2)\mathbf{x}}\hat{Q}_{\mathbf{p}_1\mathbf{p}_2}(0);$$

here $\hat{Q}_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{x})=\langle\ldots\mathbf{p}_1|\hat{Q}(\mathbf{x})|\mathbf{p}_2\ldots\rangle$ is the operator in spin space and in the space in which the matrix $\hat{\tau}_3$ acts (in this space we use the trace Sp′). Consequently we have

$$\bar{W}_k \equiv \frac{i}{2}\frac{V^2}{(2\pi)^3}\int \mathrm{d}^3p\mathrm{d}^3p_1\,\mathrm{Sp}'\hat{f}_{\mathbf{p}}\hat{Q}_{\mathbf{pp}_1}(0)\hat{Q}_{\mathbf{p}_1\mathbf{p}}(0)$$

$$\times\left(i\frac{\partial}{\partial p_k}\delta(\mathbf{p}-\mathbf{p}_1)-i\frac{\partial}{\partial p_{ik}}\delta(\mathbf{p}-\mathbf{p}_1)\right)$$

$$=-\frac{V}{(2\pi)^3}\int \mathrm{d}^3p\,\mathrm{Sp}'\hat{f}_{\mathbf{p}}\left(\left.\frac{\partial\hat{Q}_{\mathbf{pp}'}(0)}{\partial p_k}\right|_{\mathbf{p}'=\mathbf{p}}\right.$$

$$\left.\times\hat{Q}_{\mathbf{p}}+\hat{Q}_{\mathbf{p}}\left.\frac{\partial\hat{Q}_{\mathbf{p}_1\mathbf{p}}}{\partial p_k}\right|_{\mathbf{p}_1=\mathbf{p}}\right),$$

where $\hat{Q}_{\mathbf{p}}=V\hat{Q}_{\mathbf{pp}}(0)$. It is easy to see that

$$\int \mathrm{d}^3x\hat{Q}_{\mathbf{pp}'}(x)=\delta_{\mathbf{pp}'}\hat{Q}_{\mathbf{p}},$$

and consequently we have $\hat{f}_{\mathbf{p}}=(e^{\hat{Q}_{\mathbf{p}}}+1)^{-1}$ [compare Eq. (2.15)]. Then, noting that $[\hat{f}_{\mathbf{p}},\hat{Q}_{\mathbf{p}}]=0$ we have

$$\bar{W}_k=-\frac{V}{(2\pi)^3}\int \mathrm{d}^3p\,\mathrm{Sp}'\hat{f}_{\mathbf{p}}\hat{Q}_{\mathbf{p}}\left(\left.\frac{\partial\hat{Q}_{\mathbf{pp}_1}(0)}{\partial p_k}\right|_{\mathbf{p}_1=\mathbf{p}}\right.$$

$$\left.+\left.\frac{\partial\hat{Q}_{\mathbf{p}_1\mathbf{p}}(0)}{\partial p_k}\right|_{\mathbf{p}_1=\mathbf{p}}\right)$$

$$=-\frac{1}{(2\pi)^3}\int \mathrm{d}^3p\,\mathrm{Sp}'\hat{f}_{\mathbf{p}}\frac{\partial}{\partial p_k}\hat{Q}^2_{\mathbf{p}}$$

$$=-\frac{1}{(2\pi)^3}\int \mathrm{d}^3p\frac{\partial}{\partial p_k}\int_{-\infty}^{\hat{Q}^2_p}\mathrm{d}z(e^{\sqrt{z}}+1)^{-1},$$

or

$$-(2\pi)^3\bar{W}_k=\int \mathrm{d}s_k\,\mathrm{Sp}'\int_{-\infty}^{\hat{Q}^2_p}\mathrm{d}z(e^{\sqrt{z}}+1)^{-1}|_{p\to\infty}=0$$

(here $\mathrm{d}s_k$ is an element of the surface of integration). Thus, we have $\bar{W}_k=0$, which was to be shown.

Using the definitions of the operators $\hat{\pi}_i(\mathbf{x})$, $\hat{\rho}(\mathbf{x})$ it is easy to find the following commutation relations:

$$[\hat{\rho}(\mathbf{x}),\hat{\rho}(\mathbf{x}')]=0,$$

$$[\hat{\pi}_i(\mathbf{x}),\hat{\rho}(\mathbf{x}')]=-i\hat{\rho}(\mathbf{x})\frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{x}'),$$

$$[\hat{\pi}_i(\mathbf{x}),\hat{\pi}_k(\mathbf{x}')]=-i\hat{\pi}_k(\mathbf{x})\frac{\partial\delta(\mathbf{x}-\mathbf{x}')}{\partial x_i}$$

$$-i\hat{\pi}_i(\mathbf{x}')\frac{\partial\delta(\mathbf{x}-\mathbf{x}')}{\partial x_k}. \qquad (3.73)$$

We now employ the lemma (3.72) for the state of statistical equilibrium, in which

$$\hat{Q}(\mathbf{x})=Y_0\hat{\varepsilon}(\mathbf{x};\hat{f})+Y_i\hat{\pi}_i(\mathbf{x})+Y_4\hat{\rho}(\mathbf{x}).$$

Using expressions (2.42), (2.46), and (2.48) for the operators $\hat{j}_i$, $\hat{t}_{ik}$, and $\hat{W}_k$ and taking into account Eqs. (3.73) we find

$$W_k(\hat{f})=-\frac{Y_4+Y_iq_i}{Y_0}\frac{\partial\omega}{\partial q_k}-\frac{\partial}{\partial Y_0}\frac{\omega Y_k}{Y_0}. \qquad (3.74)$$

If we use $\zeta_{\alpha k}(\alpha=0,\ 1,\ 2,\ 3,\ 4)$ to represent the average values of the flux densities of the operators $[\zeta_{0k}=W_k(\hat{f}),\zeta_{ik}=t_{ik}(\hat{f}),\zeta_{4k}=j_k(\hat{f})]$, then we can rewrite Eqs. (3.63), (3.71), and (3.74) in the compact form (cf. Ref. 31):

$$\zeta_{\alpha k}=\frac{\partial\omega}{\partial q_k}\frac{\partial}{\partial Y_k}\frac{Y_4+Y_iq_i}{Y_0}-\frac{\partial}{\partial Y_\alpha}\frac{\omega Y_k}{Y_0},\quad \alpha=0,1,2,3,4.$$

From the principle of localization of a state of statistical equilibrium the hydrodynamic equations of an ideal superfluid Fermi liquid follow:

$$\frac{\partial\zeta_\alpha}{\partial t}=-\frac{\partial\zeta_{\alpha k}}{\partial x_k}, \qquad (3.75)$$

where $\zeta_\alpha=\partial\omega/\partial Y_\alpha$ [see Eq. (3.59)], $\zeta_0=\mathscr{E}(\hat{f})$, $\zeta_k=\pi_k(\hat{f})$, $\zeta_4=\rho(\hat{f})$. In these equations ζ_α and $\zeta_{\alpha k}$ depend on $\mathbf{x}$ and t through the intermediacy of the spatially and temporally slowly varying functions $Y_\alpha=Y_\alpha(x,t)$, $q_i=q_i(x,t)$. In order to find a closed system of hydrodynamic equations it is still necessary to find an equation for the superfluid momentum $q_i(x,t)$. In order to derive this equation we note that the equilibrium distribution function does not commute with the quasiparticle Hamiltonian $\hat{\varepsilon}(\hat{f})$. In fact, from (2.33) and (2.32) we have

$$[\hat{\varepsilon},\hat{f}]=-\left[\frac{Y_i}{Y_0}\hat{p}_i+\frac{Y_4}{Y_0}\hat{\tau}_3,\hat{f}\right]=-\frac{Y_4+Y_iq_i}{Y_0}[\hat{\tau}_3,\hat{f}]. \qquad (3.76)$$

Consequently, the operator $\hat{f}$ describing the state of statistical equilibrium must depend on time. It is easy to see that this dependence must reduce to some phase transformation

$$\hat{f}_t=e^{i\varphi(t)\hat{\tau}_3}\hat{f}e^{-i\varphi(t)\hat{\tau}_3}. \qquad (3.77)$$

In fact, using Eq. (3.76) we see that the operator $\hat{f}_t$ satisfies Eq. (2.34):

$$i\frac{\partial\hat{f}_t}{\partial t}=[\hat{\varepsilon}(\hat{f}_t),\hat{f}_t], \qquad (3.78)$$

if the phase $\varphi(t)$ satisfies the equation

$$\dot{\varphi}(t)=\frac{Y_4+Y_iq_i}{Y_0}. \qquad (3.79)$$

Thus, the general structure of the equilibrium statistical operator is determined by Eqs. (2.32), (3.77), and (3.79).

We now give the general definition of a phase depending on $\mathbf{x}$, t. Consider the operator g in the x representation, $g(\mathbf{x},\mathbf{x}')$. Then the phase $\varphi(\mathbf{x})$ in the case of singlet pairing is determined by

$$\mathrm{tr}_\sigma\,\sigma_2g(\mathbf{x},\mathbf{x})=g(\mathbf{x})e^{2i\varphi(\mathbf{x})},$$

where the trace is taken in spin space [$g(\mathbf{x})$ is a real function]. It is easy to see that this definition is equivalent to the following:

$$\varphi(\mathbf{x})=\operatorname{Im}\ln \operatorname{Sp}\hat{f}\hat{\psi}(\mathbf{x}),\qquad \hat{\psi}(\mathbf{x})=\begin{pmatrix}0 & 0\\ s_2(\mathbf{x}) & 0\end{pmatrix}. \tag{3.80}$$

Using Eq. (3.79) it is easy to show that for the equilibrium statistical state (3.77) we have

$$\varphi_t(\mathbf{x})\equiv\tfrac{1}{2}\operatorname{Im}\ln\operatorname{Sp}\hat{f}_t\hat{\psi}(\mathbf{x})=\frac{Y_4+Y_iq_i}{Y_0}\,t+q_ix_i+\varphi_0(0), \tag{3.81}$$

from which we obtain

$$\dot{\varphi}_t(\mathbf{x})=\frac{Y_4+Y_iq_i}{Y_0},\quad \frac{\partial\varphi_t(\mathbf{x})}{\partial x_i}=q_i. \tag{3.82}$$

If we neglect the gradients of the parameters of the reduced description Y_α, q_i, these equations also hold for weakly nonuniform states. Consequently, from (3.82) we have

$$\dot{q}_i=\frac{\partial}{\partial x_i}\frac{Y_4+Y_iq_i}{Y_0}.$$

This equation together with Eqs. (3.75) constitutes a closed system of equations for an ideal superfluid Fermi liquid (cf. Ref. 31).

These relations apply to the general case in which the energy functional has no Galilean invariance property, so that it is impossible to introduce the concept of particle mass, and hence of superfluid velocity. This situation arises, e.g., for electrons in a metal with an arbitrary dispersion law.

We now present expressions for the flux densities in a form corresponding to two-fluid hydrodynamics. The thermodynamic potential ω is a function of Y_0, Y^2, Y_4, q^2, and Y_iq_i. We introduce the quantities $\sigma_n\sigma_s$, m*, which are functions of these thermodynamic variables:

$$\sigma_n\equiv-2Y_0\frac{\partial\omega}{\partial Y^2},\quad \sigma_s\equiv\frac{2}{Y_0}\frac{\partial\omega}{\partial q^2}m^{*2},\qquad \frac{\sigma_n}{m^*}=\rho-\frac{\partial\omega}{\partial Y_iq_i}. \tag{3.83}$$

Then if we use (3.83) the fluxes j_k, t_{ik}, and W_k acquire the form

$$j_k=\frac{\sigma_n}{m^*}v_{nk}+\frac{\sigma_s}{m^{*2}}q_k,$$

$$t_{ik}=-\frac{\omega}{Y_0}\delta_{ik}+\sigma_nv_{ni}v_{nk}+\sigma_s\frac{q_iq_k}{m^{*2}}, \tag{3.84}$$

$$W_k=v_{nk}\left[-\frac{\omega}{Y_0}+\varepsilon+\left(\rho-\frac{\sigma_n}{m^*}\right)q_0\right]-\frac{\sigma_sq_k}{m^{*2}}q_0,$$

where $q_0=(Y_4+Y_iq_i)/Y_0$. From this it is clear that σ_n acts as the "mass" density of the normal component and σ_s as the "mass" density of the superfluid component. If the quantity m^* is interpreted as an effective "particle mass," then q_i/m^* must be interpreted as the superfluid velocity. Note that the total density $\sigma=m^*\rho=m^*\partial\omega/\partial Y_4$, generally speaking, differs from the sum of the normal density σ_n and superfluid density σ_s: $\sigma\neq\sigma_n+\sigma_s$.

Assume that the energy functional is invariant under Galilean transformations. Then from (2.31) we have

$$\mathscr{E}(\mathbf{x};e^{-imv_i\hat{x}_i}\hat{f}e^{imv_i\hat{x}_i})=\mathscr{E}(\mathbf{x};\hat{f})-v_i\pi_i(\mathbf{x};\hat{f})+\frac{mv^2}{2}\rho(\mathbf{x};\hat{f}), \tag{3.85}$$

and consequently, using the definition of $\varepsilon(\hat{f})$ in (2.24) we have

$$e^{imv_i\hat{x}_i}\hat{\varepsilon}(\hat{f})e^{-imv_i\hat{x}_i}=\hat{\varepsilon}(e^{imv_i\hat{x}_i}\hat{f}e^{-imv_i\hat{x}_i})-v_i\hat{p}_i+(1/2)mv^2\hat{\tau}_3.$$

If we note that

$$e^{imv_i\hat{x}_i}\hat{\tau}_3e^{-imv_i\hat{x}_i}=\hat{\tau}_3,\qquad e^{imv_i\hat{x}_i}\hat{p}_je^{-imv_i\hat{x}_i}=\hat{p}_j-mv_j\hat{\tau}_3, \tag{3.86}$$

the self-consistency equation (2.32) and the spatial uniformity condition (2.33) can be rewritten in the form

$$e^{imv_i\hat{x}_i}\hat{f}e^{-imv_i\hat{x}_i}=\left[\exp\left\{Y_0\left[\hat{\varepsilon}(e^{imv_i\hat{x}_i}\hat{f}e^{-imv_i\hat{x}_i})-v_i\hat{p}_i+\hat{\tau}_3\times\frac{mv^2}{2}\right]+Y_i(\hat{p}_i-mv_i\hat{\tau}_3)+Y_4\hat{\tau}_3\right\}+1\right]^{-1},$$

$$[e^{imv_i\hat{x}_i}\hat{f}e^{-imv_i\hat{x}_i},\quad p_i-mv_i\hat{\tau}_3-q_i\hat{\tau}_3]=0.$$

When we compare these formulas with (2.32) and (2.33) we find

$$e^{imv_i\hat{x}_i}\hat{f}(Y_0,Y_i,Y_4,q_i)e^{-imv_i\hat{x}_i}=\hat{f}\left(Y_0,Y_i-Y_0v_i,Y_4-mv_iY_i+\frac{mv^2}{2}Y_0,\quad q_i+mv_i\right),$$

where $\hat{f}(Y_0,Y_i,Y_4,q_i)$ is the equilibrium statistical operator determined by the thermodynamic parameters Y, $\mathbf{q}$. Using this relation, and also Eqs. (3.85) and (3.86) and the definition of the thermodynamic potential $\omega(Y,\mathbf{q})$ given in (3.58) we can easily see that in the case of a Galilean-invariant Fermi liquid we have

$$\omega(Y_0,Y_i,Y_4,q_i)=\omega\left(Y_0,Y_i+Y_0\frac{q_i}{m},\ Y_4+Y_iq_i+\frac{q^2}{2m}Y_0,0\right),$$

from which we obtain

$$\omega(Y,\mathbf{q})=\omega(Y',0)\equiv\omega(Y'), \tag{3.87}$$

where

$$Y'_0=Y_0,\quad Y'_k=Y_k+Y_0v_{sk},\qquad Y'_4=Y_4+Y_kmv_{sk}+Y_0\cdot\frac{mv_s^2}{2}, \tag{3.88}$$

and the parameter $v_{si}=q_i/m$ acts as the superfluid velocity. Thus, taking rotational invariance into account we find that the thermodynamic potential ω of Galilean invariant systems is a function of three independent variables Y_0', Y'^2, and Y_4'. Equation (3.87) shows that for Galilean-invariant systems, Eq. (3.83) implies that we have

$$m^*=m, \quad \sigma=\sigma_s+\sigma_n, \quad mj_k=\pi_k, \tag{3.89}$$

and Eqs. (3.84) assume the form

$$j_k=\frac{\sigma_n}{m}v_{nk}+\frac{\sigma_s}{m}v_{sk},$$

$$t_{ik}=-\frac{\omega}{Y_0}\delta_{ik}+\sigma_s v_{si}v_{sk}+\sigma_n v_{ni}v_{nk},$$

$$W_k=v_{nk}\left(-\frac{\omega}{Y_0}+\varepsilon+\frac{\sigma_s}{m}\frac{Y_4+Y_i v_{si}m}{Y_0}\right)-v_{sk}\frac{\sigma_s}{m}\frac{Y_4+Y_i m v_{si}}{Y_0}. \tag{3.90}$$

Using Eqs. (3.75), (3.86), and (3.90) we can easily convince ourselves that the ideal hydrodynamic equations we have obtained with singlet pairing in the case of Galilean invariant systems go over to the two-fluid Landau hydrodynamic equations.^{32}

4. INTERACTION BETWEEN A CHARGED FERMI LIQUID AND AN ELECTROMAGNETIC FIELD

4.1. Self-consistency equations and kinetic equations when an electromagnetic field is present

Thus far we have not included the electromagnetic interaction associated with the possible presence of an electric charge in the quasiparticles. If the quasiparticles have an electric charge, then there is an additional electromagnetic interaction between the quasiparticles and also an interaction between the quasiparticles and the external electromagnetic field. The effects associated with the interaction between the Fermi-liquid electron superfluid and the electromagnetic field play an important role in studies of superfluid phenomena.

It is well known that if the particle Hamiltonian is given in the absence of an electromagnetic field, then electromagnetic interactions can be included in a universal fashion. Specifically, the particle momentum p_i is replaced by $p_i+(e/c)A_i(\mathbf{x})$ [here $A_i(\mathbf{x})$ is the vector potential of the electromagnetic field] and the interaction with the scalar potential $\varphi(\mathbf{x})\equiv A_0(\mathbf{x})$ is included in the usual fashion, i.e., a term $e\int d^3x A_0(\mathbf{x})\rho(\mathbf{x})$ is added to the Hamiltonian [here $\rho(\mathbf{x})$ is the particle number density].

In the Fermi-liquid theory the situation is complicated by the fact that it is impossible in the general case to distinguish terms corresponding to the particle kinetic energy in the energy functional $E(\hat{f})$.

We will assume that in the presence of an electromagnetic field the energy density $\mathscr{E}(\mathbf{x};\hat{f})$ of the Fermi liquid is replaced by $\mathscr{E}(\mathbf{x};A;\hat{f})$:

$$\mathscr{E}(\mathbf{x};A;\hat{f})=\mathscr{E}(\mathbf{x};\hat{U}^+(\mathbf{x},A)\hat{f}\hat{U}(\mathbf{x};A))+eA_0(\mathbf{x},t)\times\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}\hat{\rho}(\mathbf{x}), \tag{4.1}$$

where

$$\hat{U}(\mathbf{x};A)=\exp\left(i\frac{e}{c}\int d^3x'\Phi(\mathbf{x},\mathbf{x}')\hat{\rho}(\mathbf{x}')\right),$$
$$\Phi(\mathbf{x},\mathbf{x}')=\int_{\mathbf{x}}^{\mathbf{x}'}dy_k A_k(\mathbf{y},t) \tag{4.2}$$

and the integration in $\Phi(\mathbf{x},\mathbf{x}')$ is performed over an arbitrary contour C connecting the points $\mathbf{x}$ and $\mathbf{x}'$. If we take a straight-line segment between the points $\mathbf{x}$ and $\mathbf{x}'$ as the contour C, then

$$\Phi(\mathbf{x},\mathbf{x}')=(\mathbf{x}'-\mathbf{x})_k\int_0^1 d\xi A_k(\mathbf{x}+\xi(\mathbf{x}'-\mathbf{x}),t). \tag{4.3}$$

This definition of the functional $\mathscr{E}(\mathbf{x};A;\hat{f})$ is related to the fact that for gauge transformations of the electromagnetic field and the statistical operator $\hat{f}$

$$A\to A_\chi\equiv A+\partial\chi=\left(A_k+\frac{\partial\chi}{\partial x_k},A_0+\frac{\partial\chi}{c\partial t}\right),$$
$$\hat{f}\to\hat{f}_\chi=\hat{U}_\chi\hat{f}\hat{U}_\chi^+,$$
$$\hat{U}_\chi=\exp\left(i\frac{e}{c}\int d^3x\chi(\mathbf{x},t)\hat{\rho}(\mathbf{x})\right) \tag{4.4}$$

[here $\chi(\mathbf{x},t)$ is an arbitrary function of $\mathbf{x}$ and t], the energy density (4.1) is transformed according to the usual formula

$$\mathscr{E}(\mathbf{x},A_\chi;\hat{f}_\chi)=\mathscr{E}(\mathbf{x};A;\hat{f})+e\frac{\partial\chi(\mathbf{x},t)}{c\partial t}\cdot\frac{1}{2}\,\mathrm{Sp}\,\hat{f}\hat{\rho}(\mathbf{x}). \tag{4.5}$$

This formula follows directly from Eq. (4.1) if we note that

$$\hat{U}(\mathbf{x};A_\chi)=\hat{U}(\mathbf{x};A)\hat{U}_\chi\exp\left(-\frac{ie}{c}\chi(\mathbf{x},t)\hat{\tau}_3\right).$$

For the momentum density $\pi_i(\mathbf{x})=\{\hat{p}_i,\hat{\tau}_3\hat{\rho}(\mathbf{x})\}/2$ [see Eq. (2.18′)] we have

$$\hat{U}(\mathbf{x};A)\hat{\pi}_i(\mathbf{x})\hat{U}^+(\mathbf{x};A)=\tfrac{1}{2}\left\{\hat{p}_i+\frac{e}{c}A_i(\mathbf{x})\hat{\tau}_3,\tau_3\hat{\rho}(\mathbf{x})\right\}$$
$$=\hat{\pi}_i(\mathbf{x})-\frac{e}{c}A_i(\mathbf{x})\hat{\rho}(\mathbf{x}).$$

Similarly, if the kinetic energy density is determined by

$$\mathscr{E}(\mathbf{x};\hat{f})=\mathrm{Sp}\,\hat{f}\hat{p}_i\hat{\rho}(\mathbf{x})\hat{p}_i/4m,$$

then

$$\hat{U}(\mathbf{x};A)\hat{p}_i\hat{\rho}(\mathbf{x})\hat{p}_i\hat{U}^+(\mathbf{x};A)$$
$$=\left(\hat{p}_i-\frac{e}{c}A_i(\mathbf{x})\tau_3\right)\hat{\rho}(\mathbf{x})\left(\hat{p}_i-\frac{e}{c}A_i(\mathbf{x})\right),$$

and consequently

$$\mathscr{E}(\mathbf{x};A,\hat{f})=\frac{1}{4m}\,\mathrm{Sp}\,\hat{f}\left(\hat{p}_i-\frac{e}{c}A_i(\mathbf{x})\hat{\tau}_3\right)\times\hat{\rho}(\mathbf{x})\left(\hat{p}_i-\frac{e}{c}A_i(\mathbf{x})\hat{\tau}_3\right).$$

Since the variation of the functional $\mathscr{E}(\mathbf{x};\hat{f}\,)$ with respect to the statistical operator $\hat{f}$ can be written in the form [see (2.23)]

$$\delta_f\mathscr{E}(\mathbf{x};\hat{f}\,)=\mathrm{Sp}\,\delta\hat{f}\hat{\varepsilon}(\mathbf{x};\hat{f}\,)/2,$$

where $\hat{\varepsilon}(\mathbf{x};\hat{f}\,)$ is the quasiparticle energy density, it is easy to see that the following relation holds:

$$\delta_f\mathscr{E}(\mathbf{x};\hat{U}^+\hat{f}\hat{U})=\mathrm{Sp}\,\hat{U}\hat{\varepsilon}(\mathbf{x};\hat{U}^+\hat{f}\hat{U})\hat{U}^+\delta\hat{f}/2.$$

Hence by varying the energy density functional (4.1) with respect to the statistical operator $\hat{f}$ we arrive at the following expression for the quasiparticle energy density in the presence of an electromagnetic field:

$$\hat{\varepsilon}(\mathbf{x};A;\hat{f}\,)=\hat{U}(\mathbf{x};A)\hat{\varepsilon}(\mathbf{x};\hat{U}^+(\mathbf{x},A)\hat{f}U(\mathbf{x},A))\hat{U}^+(\mathbf{x},A)$$
$$+eA_0(\mathbf{x},t)\hat{\rho}(\mathbf{x}). \qquad (4.6)$$

Consequently, the quasiparticle energy operator is equal to

$$\hat{\varepsilon}(A;\hat{f}\,)=\int \mathrm{d}^3x\hat{\varepsilon}(\mathbf{x};A;\hat{f}\,).$$

Then from this equation and (4.5) we find the gauge transformation law for the quasiparticle energy density:

$$\hat{\varepsilon}(A_\chi,\hat{f}_\chi)=\hat{U}_\chi\hat{\varepsilon}(A,\hat{f}\,)\hat{U}_\chi^+$$
$$+e\int \mathrm{d}^3x(\partial\chi(\mathbf{x},t)/c\partial t)\hat{\rho}(\mathbf{x}), \qquad (4.7)$$

and hence

$$\hat{\varepsilon}(\mathbf{x},A_\chi;\hat{f}_\chi)=\hat{U}_\chi\hat{\varepsilon}(\mathbf{x};A,\hat{f}\,)\hat{U}_\chi^++e(\partial\chi(\mathbf{x},t)/c\partial t)\hat{\rho}(\mathbf{x}). \qquad (4.8)$$

The kinetic equation (2.34) for the statistical operator $\hat{f}$ when an electromagnetic field is present takes the form

$$i\frac{\partial\hat{f}}{\partial t}=[\hat{\varepsilon}(A,\hat{f}\,),\hat{f}\,]. \qquad (4.9)$$

It has the gauge invariance property. Specifically, the operator $\hat{f}_\chi\equiv\hat{U}_\chi\hat{f}\hat{U}_\chi^+$ satisfies the equation

$$i\frac{\partial\hat{f}_\chi}{\partial t}=\left[\hat{U}_\chi\hat{\varepsilon}(A,\hat{f}\,)\hat{U}_\chi^++i\frac{\partial\hat{U}_\chi}{\partial t}\hat{U}_\chi^+,\hat{f}_\chi\right]$$
$$=\left[\hat{U}_\chi\hat{\varepsilon}(A,\hat{f}\,)\hat{U}_\chi^++e\int \mathrm{d}^3x\frac{\partial\chi(\mathbf{x},t)}{c\partial t}\hat{\rho}(\mathbf{x}),\hat{f}_\chi\right].$$

From this by virtue of (4.7) we have

$$i\frac{\partial\hat{f}_\chi}{\partial t}=[\hat{\varepsilon}(A_\chi,\hat{f}_\chi),\hat{f}_\chi],$$

which proves the gauge invariance of Eq. (4.9).

Let us find the variation of the total energy of the Fermi liquid

$$E(A,\hat{f}\,)=\int \mathrm{d}^3x\mathscr{E}(\mathbf{x};A,\hat{f}\,)$$

with respect to the vector potential A. From (2.23) we have

$$\delta_A\mathscr{E}(\mathbf{x};\hat{U}^+\hat{f}\hat{U})=\mathrm{Sp}\,\hat{\varepsilon}(\mathbf{x};\hat{U}^+\hat{f}\hat{U})\delta_A(\hat{U}^+\hat{f}\hat{U})/2,$$

$$\hat{U}\equiv\hat{U}(\mathbf{x},A).$$

Noting that $\delta_A(\hat{U}^+\hat{f}\hat{U})=\hat{U}^+[\hat{f},\delta\hat{U}\cdot\hat{U}^+]\hat{U}$ holds and using expression (4.6) for $\hat{\varepsilon}(\mathbf{x};A,\hat{f}\,)$ we find

$$\delta_AE(A,\hat{f}\,)$$
$$=\int \mathrm{d}^3x\,\mathrm{Sp}\,\hat{f}[\delta_A\hat{U}(\mathbf{x},A)\hat{U}^+(\mathbf{x},A),\hat{\varepsilon}(\mathbf{x};A,\hat{f}\,)]/2. \qquad (4.10)$$

Choosing the integration contour C in the form of a straight-line segment [cf. Eq. (4.3)] in the transformation $\hat{U}(\mathbf{x},A)$ [see Eq. (4.2)], we find

$$\delta_AE(A,\hat{f}\,)=-(e/c)\int \mathrm{d}^3x\delta A_k(\mathbf{x},t)j_k(\mathbf{x},t), \qquad (4.11)$$

where

$$j_k(\mathbf{x},t)=\frac{i}{2}\int \mathrm{d}^3x'x'_k\int_0^1 \mathrm{d}\xi\,\mathrm{Sp}\,\hat{f}$$
$$\times[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';A,\hat{f}\,),\hat{\rho}(\mathbf{x}+\xi\mathbf{x}')] \qquad (4.12)$$

[cf. the definition of the particle number flux density (2.44)]. The variation of the energy with respect to the scalar potential A_0 is found directly from expression (4.1):

$$\delta_{A_0}E(A,\hat{f}\,)=e\int \mathrm{d}^3x\delta A_0(\mathbf{x},t)\cdot\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}\hat{\rho}(\mathbf{x}). \qquad (4.13)$$

In order to find the total energy functional when a magnetic field is present ($A_0=0$) it is necessary to add the energy of the self-consistent magnetic field $b_i(\mathbf{x})$ to the functional (4.1). Thus, the total energy functional has the form

$$E_t(\hat{f},A)=E(\hat{f},A)+\int \mathrm{d}^3x\frac{b(\mathbf{x})^2}{8\pi},$$
$$\mathbf{b}(\mathbf{x})=\mathrm{curl}\,\mathbf{A}. \qquad (4.14)$$

Just as in Sec. 3, the self-consistency equations for f and $\mathbf{A}$ are found by determining the extremum (minimum) of the functional

$$\Omega(\hat{f},\mathbf{A})=-S(\hat{f}\,)+Y_0E_t(\mathbf{A},\hat{f}\,)+Y_4\cdot\tfrac{1}{2}\,\mathrm{Sp}\,\hat{f}\hat{\tau}_3.$$

Variation with respect to $\mathbf{A}$ yields the magnetostatic equations

$$\mathrm{curl}\,\mathbf{b}=\frac{4\pi}{c}\mathbf{j},$$

where j_i is defined by Eq. (4.12). Here we have taken into account the fact that

$$\delta\int \mathrm{d}^3x\frac{b^2}{8\pi}=\int \mathrm{d}^3x\delta\mathbf{A}\cdot\mathrm{curl}\,\mathbf{b}/4\pi.$$

Variation with respect to $\hat{f}$ leads to the self-consistency equation for the normal and anomalous distribution functions, in which, however, in contrast to Eq. (2.32) the quasiparticle energy operator $\hat{\varepsilon}$ is determined by the total functional (4.14). Thus, the full system of self-consistency equations for $\hat{f}$ and $\mathbf{b}(\mathbf{x})$ takes the form

$$\hat{f}=[\exp(Y_0\hat{\varepsilon}(\mathbf{A},\hat{f})+Y_4\hat{\tau}_3)+1]^{-1},$$

$$\operatorname{curl}\mathbf{b}=\frac{4\pi}{c}\mathbf{j},$$

$$\hat{\varepsilon}(\mathbf{A},\hat{f})=\begin{pmatrix}\varepsilon(\mathbf{A},\hat{f}) & \Delta(\mathbf{A},\hat{f})\\ \Delta^{+}(\mathbf{A},\hat{f}) & -\tilde{\varepsilon}(\mathbf{A},\hat{f})\end{pmatrix},$$

$$j_k(\mathbf{x})=\frac{ie}{c}\int d^3x'x'_k\int_0^1 d\xi\,\mathrm{Sp}\,\hat{f}$$
$$\times[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';\mathbf{A},\hat{f}),\hat{\rho}(\mathbf{x}+\xi\mathbf{x}')],\quad(4.15)$$

$$\varepsilon_{12}(\mathbf{A},\hat{f})=\frac{\partial E(\mathbf{A},\hat{f})}{\partial f_{21}},$$

$$\Delta_{12}(\mathbf{A},\hat{f})=2\frac{\partial E(\mathbf{A},\hat{f})}{\partial g_{21}^{+}}.\qquad(4.16)$$

These equations are the ones which describe superconductivity (superfluidity) phenomena in the presence of a magnetic field. From these equations [see also (4.4)] follows the gauge invariance property of the statistical operator $\hat{f}\equiv\hat{f}(\mathbf{A})$ and the current $j_i\equiv j_i(\mathbf{A},\hat{f})$,

$$U_\chi\hat{f}(\mathbf{A})U_\chi^{+}=f(\mathbf{A}_\chi),\quad j_i(\mathbf{A},f)=j_i(\mathbf{A}_\chi,f_\chi).$$

The first of these equations leads to the relation

$$\frac{ie}{c}[\hat{f}(\mathbf{A}),\hat{\rho}(\mathbf{x})]=\frac{\partial}{\partial x_k}\frac{\delta\hat{f}(\mathbf{A})}{\delta\mathbf{A}_k(\mathbf{x})}.$$

4.2. The London equation

In this section we simplify the self-consistency equation (4.15) and show that in this case, when the magnetic field is a slowly varying function of position, the expression for the current is given by

$$j_i(\mathbf{x})=-\frac{e^2}{m^*c}\rho_s A_i(\mathbf{x}),\qquad(4.17)$$

where ρ_s and m^* are determined by Eqs. (3.83). For this purpose we note that when the gradients of the vector potential $A_i(\mathbf{x})$ are neglected in the calculation of the current $j_i(\mathbf{x})$ [see (4.16)] from (4.2) it follows that we can represent the unitary transformation $U(\mathbf{x};\mathbf{A})$ in the form

$$U(\mathbf{x};\mathbf{A})=U_0(\mathbf{x};\mathbf{A})+\ldots,$$
$$U_0(\mathbf{x};\mathbf{A})=\exp\left(\frac{ie}{c}\mathbf{A}\hat{\mathbf{x}}\right)\exp\left(-\frac{ie}{c}\mathbf{A}\mathbf{x}\hat{\tau}_3\right)\qquad(4.18)$$

[the vector potential $A_i(x)$ refers to the same spatial point as does the current $j_i(\mathbf{x})$]. We will assume in solving the self-consistency equation (4.15) that the source of the nonuniformity of the statistical operator $\hat{f}(\mathbf{A})$ is the nonuniformity of the vector potential $A_i(\mathbf{x})$. Consequently, again recalling that the current density $j_i(\mathbf{x})$ is calculated at the point x, by virtue of the localization principle we could write the expansion

$$\hat{f}(\mathbf{A})\equiv\hat{\underline{f}}(\mathbf{A})=\hat{\underline{f}}_0(\mathbf{A}(\mathbf{x}))+\hat{\underline{f}}_1\left(\mathbf{A}(\mathbf{x}),\frac{\partial\mathbf{A}(\mathbf{x})}{\partial x_k}\right)+\ldots,\qquad(4.19)$$

where

$$\underline{f}_0(\mathbf{A}(\mathbf{x}))\equiv\underline{f}(\mathbf{A})|_{\mathbf{A}=\mathbf{A}(\mathbf{x})},$$

with (see Ref. 6)

$$[\hat{p}_i,\underline{f}_0]=0,\quad[\hat{p}_i,[\hat{p}_j,\underline{f}_1]]=0.\qquad(4.20)$$

However, taking into account the possibility of carrying out the gauge transformations

$$A_i\to A_i+\frac{\partial\chi}{\partial x_i},$$

we see from (4.4) that the phase of the statistical operator $\hat{f}$ is indeterminate. Consequently, the solution of Eq. (4.15) will be written in the form

$$\hat{f}(\mathbf{A})=e^{i\hat{\varphi}}\hat{\underline{f}}(\mathbf{A})e^{-i\hat{\varphi}},$$

where the expansion $\underline{f}(\mathbf{A})$ is determined by Eq. (4.19) and

$$\hat{\varphi}=\int d^3x\varphi(\mathbf{x})\hat{\rho}(\mathbf{x}).$$

The expansion of the phase $\hat{\varphi}$, analogous to the expansion (4.19), takes the form

$$\hat{\varphi}=\hat{\varphi}_0+\hat{\varphi}_1+\ldots,$$

where

$$\hat{\varphi}_0=\left(\varphi(\mathbf{x})-x_k\frac{\partial\varphi(\mathbf{x})}{\partial x_k}\right)\hat{\tau}_3+\frac{\partial\varphi(\mathbf{x})}{\partial x_k}\hat{x}_k,\qquad(4.21)$$

and $\hat{\varphi}_1\sim\partial^2\varphi(\mathbf{x})/\partial x_i\partial x_k$. This expansion induces the expansion

$$\hat{f}(\mathbf{A})=\hat{f}_0(\mathbf{A})+\hat{f}_1(\mathbf{A})+\ldots,$$

where

$$\hat{f}_0(\mathbf{A})=e^{i\hat{\varphi}_0(\mathbf{x})}\hat{\underline{f}}_0(\mathbf{A})e^{-i\hat{\varphi}_0(\mathbf{x})}.$$

From this equation and Eqs. (4.20) and (4.21) it follows that

$$\left[\hat{p}_i-\hat{\tau}_3\frac{\partial\varphi(\mathbf{x})}{\partial x_i},\hat{f}_0(\mathbf{A})\right]=0.$$

Introducing the statistical operator

$$\underset{\sim}{f}_0=\exp\left(-\frac{ie}{c}\mathbf{A}\hat{\mathbf{x}}\right)\hat{f}_0\exp\left(i\frac{e}{c}\mathbf{A}\hat{\mathbf{x}}\right)\qquad(4.22)$$

and noting that from (4.6) and (4.18) we have

$$\varepsilon(\mathbf{A};\hat{f})\approx e^{ie\mathbf{A}\hat{\mathbf{x}}/c}\varepsilon(e^{-ie\mathbf{A}\hat{\mathbf{x}}/c}\hat{f}_0e^{-ie\mathbf{A}\hat{\mathbf{x}}/c})e^{-ie\mathbf{A}\hat{\mathbf{x}}/c}\qquad(4.23)$$

[we have taken into account the phase invariance property of $\varepsilon(\hat{f})$; cf. Eq. (2.28)], we find the self-consistency equation for $\underset{\sim}{f}_0$:

$$\hat{\underline{f}}_0=[\exp(Y_0\hat{\varepsilon}(\underset{\sim}{\hat{f}}_0)+Y_4\hat{\tau}_3)+1]^{-1},\qquad(4.24)$$

and also a supplementary condition expressing the "spatial uniformity" of the leading approximation $\underset{\sim}{f}_0$:

$$\left[\hat{p}_i-\hat{\tau}_3\left(\frac{\partial\varphi(\mathbf{x})}{\partial x_i}-\frac{e}{c}A_i(\mathbf{x})\right),\underset{\sim}{f}_0\right]=0. \tag{4.25}$$

Comparing Eqs. (4.24) and (4.25) with Eqs. (2.32) and (2.33) we see that the statistical operator $\hat{\underset{\sim}{f}}_0$ can be expressed in terms of the equilibrium statistical operator $\hat{f}(\mathbf{q},Y)$ (in the absence of a magnetic field) when there is a superfluid momentum q_i:

$$\hat{\underset{\sim}{f}}_0=\hat{f}(\mathbf{q},Y),\quad Y_i=0,\quad q_i=\frac{\partial\varphi(\mathbf{x})}{\partial x_i}-\frac{e}{c}A_i(\mathbf{x}),$$

or, using Eq. (4.22),

$$\begin{gathered}\hat{f}_0(\mathbf{A})=e^{ie\mathbf{A}\hat{\mathbf{x}}/c}\hat{f}(\mathbf{q},Y)e^{-ie\mathbf{A}\hat{\mathbf{x}}/c},\\ Y_i=0,\quad q_i=\frac{\partial\varphi(\mathbf{x})}{\partial x_i}-\frac{e}{c}A_i(\mathbf{x}).\end{gathered} \tag{4.26}$$

The electric current in the leading approximation [neglecting the gradients of $A_i(x)$] according to (4.16) and (4.23) is determined by

$$J_1(x)=\frac{ie}{2}\int \mathrm{d}^3x'x_i'\int_0^1 \mathrm{d}\xi\,\mathrm{Sp}\,\hat{f}_0[e^{ie\mathbf{A}\hat{\mathbf{x}}/c}$$
$$\times\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x};\underset{\sim}{f}_0)e^{-ie\mathbf{A}\hat{\mathbf{x}}/c},\hat{\rho}(\mathbf{x}+\xi\mathbf{x}')],$$

or, using Eq. (4.25),

$$J_i(\mathbf{x})=\frac{ie}{2}\int \mathrm{d}^3x'x_i'\int_0^1 \mathrm{d}\xi\,\mathrm{Sp}\,f(\mathbf{q},Y)$$
$$\times[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}';f(\mathbf{q},Y)),\hat{\rho}(\mathbf{x}+\xi\mathbf{x}')],$$
$$Y_i=0,\quad q_i=\frac{\partial\varphi}{\partial x_i}-\frac{e}{c}A_i.$$

From (2.42) this can be rewritten in the form

$$J_i(\mathbf{x})=ej_i(\mathbf{q},Y),\quad Y_i=0,\quad q_i=\frac{\partial\varphi}{\partial x_i}-\frac{e}{c}A_i(\mathbf{x}), \tag{4.27}$$

where $j_i(\mathbf{q},Y)$ is the particle current density in the statistical equilibrium state in the absence of a magnetic field [see Eq. (3.63)]. In Sec. 3 we saw that the current density $j_i(\mathbf{q},Y)$ in the absence of an electromagnetic field in the limit of a small superfluid velocity q_i takes the form

$$j_i(\mathbf{q},Y)=\beta_{ik}q_k, \tag{4.28}$$

where

$$\beta_{ik}=\left.\frac{\partial^2\omega'}{\partial q_i\partial q_k}\right|_{\mathbf{q}=0,Y=0},\quad \omega'=\frac{\omega}{Y_0}.$$

Hence in the absence of a vector potential the electric current is given by

$$J_i(\mathbf{x})=e\beta_{ik}\left(\frac{\partial\varphi(\mathbf{x})}{\partial x_k}-\frac{e}{c}A_k(\mathbf{x})\right). \tag{4.29}$$

Since under gauge transformations the phase $\varphi(\mathbf{x})$ and the vector potential $A_i(\mathbf{x})$ transform according to

$$\varphi(\mathbf{x})\to\varphi(\mathbf{x})+\frac{e}{c}\chi(\mathbf{x}),\quad A_i(\mathbf{x})\to A_i(\mathbf{x})+\frac{\partial\chi(\mathbf{x})}{\partial x_k},$$

the current (4.19) is gauge-invariant. Noting that according to Eqs. (3.83) we have

$$\left.\frac{\partial^2\omega'}{\partial q_i\partial q_k}\right|_{\mathbf{q}=0,Y=0}=\delta_{ik}\frac{\rho_s}{m^*},$$

we thus arrive at Eq. (4.17) in the gauge $\varphi(\mathbf{x})=0$.

We emphasize that m^* is a thermodynamic quantity depending on temperature and the chemical potential; only in the case of Galilean invariance does m^* reduce to the ordinary electron mass m, i.e., $m^*=m$.

The magnetostatic equation (4.16) together with the definition (4.17) of the current can be written in the form

$$\mathbf{b}+\lambda_{\mathrm{L}}^2\,\mathrm{curl\,curl}\,\mathbf{b}=0, \tag{4.30}$$

where

$$\lambda_{\mathrm{L}}^2=m^*c^2/4\pi\rho_se^2.$$

This equation is called the London equation.[33] It is usually employed in determining the magnetic-field distribution in a superconductor.

Let us consider the simplest case, in which the superconductor fills the half-space $z>0$, and the magnetic field outside the superconductor is steady and directed parallel to the z axis. Then the field b_i inside the superconductor also points in the z direction and by virtue of the condition div $\mathbf{b}=0$ is independent of $\mathbf{z}$; hence from Eq. (4.30) it follows that inside the superconductor we have $b_i=0$ and $j_i=0$. Now assume that an external field b_i points in the x direction. Then inside the superconductor the field is also parallel to the x axis and is a function only of z. From Eq. (4.30) it follows that

$$\mathrm{d}^2\mathrm{b}/\mathrm{d}z^2=\lambda_{\mathrm{L}}^2\mathrm{b},$$

and consequently,

$$b_x(z)=b_x(0)e^{-z/\lambda_{\mathrm{L}}}.$$

The current density in this case is in the y direction, and from (4.17) is given by

$$j_y(z)=-(4\pi/c)\lambda_{\mathrm{L}}e^{-z/\lambda_{\mathrm{L}}}.$$

Thus, the quantity λ_{L} in Eq. (4.30) can be interpreted as the depth to which the magnetic field penetrates into the superconductor. The quantity λ_{L} is small (it is usually of the order of 10^{-6} cm at $T=0$ K), and hence the magnetic field does not penetrate into a massive superconductor (the Meissner effect).[34,35]

The London equations apply when the quantity λ_{L} is large compared to all other microscopic parameters with the dimensions of length. In the BCS theory this parameter is the characteristic size ξ_0 of a Cooper pair, $\xi_0=\hbar v_{\mathrm{F}}/\pi\Delta(0)$. Superconductors for which $\lambda_{\mathrm{L}}\gg\xi_0$ holds are called London superconductors or superconductors of the second kind.

In the opposite limiting case $\lambda_{\mathrm{L}}\ll\xi_0$, the London equations are invalid and must be modified. Specifically, in the London theory the relation between the current and the potential **A** is local and linear. In the case $\lambda_L < \xi_0$ the field changes considerably over distances of order ξ_0, as a result of which the relation between the current and the potential must be nonlocal. In the London gauge div $\mathbf{A}=0$, $A_n|_s=0$ the phenomenological relation between the current and the potential, proposed by Pippard, takes the form

$$j_i(\mathbf{x})=C\int \frac{\mathbf{A}(x')\mathbf{R}}{R^4} R_i e^{-R/\xi_0} \mathrm{d}^3x', \qquad (4.31)$$

where we have written $\mathbf{R}=\mathbf{x}-\mathbf{x}'$ and the coefficient C is found by combining Eq. (4.31) and Eq. (4.17) for a slowly varying field $\mathbf{A}(\mathbf{x})$:

$$C=-\frac{3\rho_s e^2}{4\pi m^* c^2}\xi_0^{-1}.$$

Superconductors for which $\lambda_L \ll \xi_0$ holds are called superconductors of the first kind or Pippard superconductors. When we treat the problem of the penetration of a magnetic field into a superconductor of the first kind we obtain the following expression for the depth of penetration λ:

$$\lambda^3=0.62\lambda_L^2\xi_0.$$

4.3. The Ginzburg–Landau equation

In this section we study the solution of the self-consistency equations (2.32) in the presence of a nonuniform steady magnetic field. Here we will assume that the temperature of the Fermi liquid is close to critical, and hence the anomalous distribution function g is small.

In the region of small g the energy density functional can be written in the form

$$\mathscr{E}(\mathbf{x};\hat{f})=\mathscr{E}_0(\mathbf{x};f)+\mathscr{E}_1(\mathbf{x};f)+\mathscr{E}_2(\mathbf{x};g),$$

where $\mathscr{E}_2(\mathbf{x};g)\sim g^+g$ [see Eq. (3.20)], $\mathscr{E}_1(\mathbf{x};f)$ is the functional of the interaction energy density of the normal state, and

$$\mathscr{E}_0(\mathbf{x};f)=\int \mathrm{d}^3x_1\mathrm{d}^3x_2 f_0(\mathbf{x}_1,\mathbf{x}_2)\varepsilon(\mathbf{x};\mathbf{x}_2,\mathbf{x}_1),$$

$$f_0(\mathbf{x}_1,\mathbf{x}_2)=\mathrm{Sp}_\sigma f(\mathbf{x}_1,\mathbf{x}_2)$$

(we employ the coordinate representation). The quantity $\varepsilon_0(\mathbf{x};\mathbf{x}_2,\mathbf{x}_1)\equiv\delta\mathscr{E}_0(\mathbf{x};f)/\delta f_0(\mathbf{x}_1,\mathbf{x}_2)$ is obviously the matrix element of the energy density operator $\varepsilon_0(\mathbf{x})$ of a free quasiparticle in the coordinate representation,

$$\varepsilon(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2)=\langle\mathbf{x}_1|\varepsilon_0(\mathbf{x})|\mathbf{x}_2\rangle.$$

We assume that the interaction energy density functional $\mathscr{E}_{\mathrm{int}}(\mathbf{x};\hat{f})\equiv\mathscr{E}_1(\mathbf{x};f)+\mathscr{E}_2(\mathbf{x};g)$ is invariant under general phase transformations:

$$\mathscr{E}_{\mathrm{int}}(\mathbf{x};e^{i\hat{\varphi}}\hat{f}e^{-i\hat{\varphi}})=\mathscr{E}_{\mathrm{int}}(\mathbf{x};\hat{f}), \quad \hat{\varphi}=\int \mathrm{d}^3x\varphi(\mathbf{x})\hat{\rho}(\mathbf{x}).$$

In this case the energy density of the Fermi liquid in the presence of a magnetic field is given by [see Eq. (4.1)]

$$\mathscr{E}(\mathbf{x};\mathbf{A},\hat{f})=\mathscr{E}_0(\mathbf{x};\mathbf{A},f)+\mathscr{E}_1(\mathbf{x};f)+\mathscr{E}_2(\mathbf{x};g),$$

where

$$\mathscr{E}_0(\mathbf{x};\mathbf{A},f)=\mathscr{E}_0(\mathbf{x};u^+(\mathbf{x};\mathbf{A})fu(\mathbf{x};\mathbf{A})),$$

$$u(\mathbf{x};\mathbf{A})=\exp\left(\frac{ie}{c}\Phi(x,\underline{\mathbf{x}})\right)$$

(here $\underline{\mathbf{x}}$ is the position operator).

Since the anomalous distribution functions transform under phase transformations according to

$$g(\mathbf{x}_1,\mathbf{x}_2)\rightarrow g'(\mathbf{x}_1,\mathbf{x}_2)=g(\mathbf{x}_1,\mathbf{x}_2)e^{i(\varphi(\mathbf{x}_1)+\varphi(\mathbf{x}_2))},$$

the phase-invariant functional $\mathscr{E}_2(\mathbf{x};g)$, which is quadratic in $g(\sim gg^+)$ can be written in the form [cf. Eq. (3.20)]

$$\mathscr{E}_2(\mathbf{x};g)=\int \mathrm{d}^3x_1\mathrm{d}^3x_2L_1(\mathbf{x}_1-\mathbf{x}_2)\delta\left(\mathbf{x}-\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right)g_0(\mathbf{x}_1,\mathbf{x}_2)$$
$$\times g_0(\mathbf{x}_1,\mathbf{x}_2)^*+\int \mathrm{d}^3x_1\mathrm{d}^3x_2L_2(\mathbf{x}_1-\mathbf{x}_2)$$
$$\times\delta\left(\mathbf{x}-\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right)g_i(\mathbf{x}_1,\mathbf{x}_2)g_i(\mathbf{x}_1,\mathbf{x}_2)^*, \qquad (4.32)$$

where

$$g_0(\mathbf{x}_1,\mathbf{x}_2)=\mathrm{Sp}_\sigma\,\sigma_2 g(\mathbf{x}_1,\mathbf{x}_2),$$

$$g_i(\mathbf{x}_1,\mathbf{x}_2)=\mathrm{Sp}_\sigma\,\sigma_2\sigma g(\mathbf{x}_1,\mathbf{x}_2).$$

The functional $\mathscr{E}_0(\mathbf{x};\mathbf{A},f)$ can obviously be written in the form

$$\mathscr{E}_0(\mathbf{x};\mathbf{A},f)=\int \mathrm{d}^3x_1\mathrm{d}^3x_2 f_0(\mathbf{x}_1,\mathbf{x}_2)$$
$$\times\langle x_2|u(\mathbf{x},\mathbf{A})\varepsilon_0(\mathbf{x};\mathbf{p},\underline{\mathbf{x}})u^+(\mathbf{x},\mathbf{A})|\mathbf{x}_1\rangle.$$

Thus, the quasiparticle free energy density operator in the presence of a magnetic field is given by

$$\varepsilon_0(\mathbf{x};\mathbf{A})=u(\mathbf{x};\mathbf{A})\varepsilon_0(\mathbf{x};\mathbf{p},\underline{\mathbf{x}})u^+(\mathbf{x};\mathbf{A})$$

[cf. Eq. (4.6)], while the quasiparticle energy operator in a magnetic field is given by

$$\varepsilon_0(\mathbf{A})=\int \mathrm{d}^3xu(\mathbf{x};\mathbf{A})\varepsilon_0(\mathbf{x};\mathbf{p},\underline{\mathbf{x}})u^+(\mathbf{x};\mathbf{A}). \qquad (4.33)$$

[We write $\varepsilon_0(\mathbf{x};\mathbf{p},\underline{\mathbf{x}})$ instead of $\varepsilon_0(\mathbf{x})$ in order to emphasize that the operator $\varepsilon_0(\mathbf{x})$ is constructed from the operators $\mathbf{p}$, $\underline{\mathbf{x}}$.] Note that

$$u(\mathbf{x};\mathbf{A})\underline{\mathbf{x}}u^+(\mathbf{x};\mathbf{A})=\underline{\mathbf{x}},$$

$$u(\mathbf{x};\mathbf{A})\mathbf{p}u^+(\mathbf{x};\mathbf{A})=\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x})-e\mathbf{Z}(\mathbf{x},\underline{\mathbf{x}}),$$

where

$$Z_k(\mathbf{x}_1,\mathbf{x}_2)=\int_0^1 \mathrm{d}\lambda\frac{\partial y_l(\lambda;\mathbf{x}_1,\mathbf{x}_2)}{\partial\lambda}\frac{\partial y_s(\lambda;\mathbf{x}_1,\mathbf{x}_2)}{\partial x_{2k}}\varepsilon_{slp}H_p(\mathbf{y}) \qquad (4.34)$$

and $\mathbf{y}=\mathbf{y}(\lambda;\mathbf{x}_1,\mathbf{x}_2)$ is a parametric equation for the integration contour C connecting the points $\mathbf{x}_1$ and $\mathbf{x}_2$ ($y(0;\mathbf{x}_1,\mathbf{x}_2)=\mathbf{x}_1$, $\mathbf{y}(1;\mathbf{x}_1,\mathbf{x}_2)=\mathbf{x}_2$); $\mathbf{H}=$curl $\mathbf{A}$ is the intensity of the magnetic field.

If we take for the quasiparticle energy density operator the operator

$$\varepsilon_0(\mathbf{x})=p_i\delta(\mathbf{x}-\underline{\mathbf{x}})p_i/2m, \tag{4.35}$$

then, noting that we have $\mathbf{Z}(\mathbf{x}_1,\mathbf{x}_1)=0$, we find

$$\varepsilon_0(\mathbf{x},\mathbf{A})=\frac{1}{2m}\left(p_i-\frac{e}{c}A_i(\underline{\mathbf{x}})\right)\delta(\mathbf{x}-\underline{\mathbf{x}})\left(p_i-\frac{e}{c}A_i(\underline{\mathbf{x}})\right),$$

$$\varepsilon_0(\mathbf{A})=\frac{1}{2m}\left(p_i-\frac{e}{c}A_i(\underline{\mathbf{x}})\right)\left(p_i-\frac{e}{c}A_i(\underline{\mathbf{x}})\right),$$

and consequently, the operators $\varepsilon_0(\mathbf{x}, \mathbf{A})$ and $\varepsilon_0(\mathbf{A})$ do not depend on the choice of the integration contour C in Eq. (4.35). Note, however, that if we were to choose the expression

$$\varepsilon_0(\mathbf{x})=\frac{1}{2}\left\{\frac{p^2}{2m},\delta(\mathbf{x}-\underline{\mathbf{x}})\right\},$$

as the energy density operator $\varepsilon_0(\mathbf{x})$ rather than Eq. (4.35), then $\varepsilon(\mathbf{x};\mathbf{A})$ and $\varepsilon(\mathbf{A})$ would still depend on the contour C.

Let us find an approximate expression for the energy $E_0(\mathbf{A},f)$ of the system of free quasiparticles in the presence of a magnetic field:

$$E_0(\mathbf{A},f)=\int \mathrm{d}^3x_1\mathrm{d}^3x_2\mathrm{d}^3x f_0(\mathbf{x}_1,\mathbf{x}_2)\langle\mathbf{x}_2|\varepsilon_0(\mathbf{x};\mathbf{A})|\mathbf{x}_1\rangle,$$

In this expression the electron distribution function $f_0(\mathbf{x}_1,\mathbf{x}_2)$ is substantially different from zero when $\mathbf{x}_1$ is close to $\mathbf{x}_2$ ($|\mathbf{x}_1-\mathbf{x}_2|\lesssim a$, where a is the lattice constant), since the electrons of the Fermi liquid are close to the Fermi surface ($a\sim p_{\mathrm{F}}{}^{-1}$, where p_{F} is the Fermi momentum). Hence in the matrix element

$$\left\langle\mathbf{x}_2\middle|\varepsilon_0\left(\mathbf{x};\mathbf{p}-\frac{e}{c}\mathbf{A}(\underline{\mathbf{x}})-\frac{e}{c}\mathbf{Z}(\mathbf{x},\underline{\mathbf{x}}),\underline{\mathbf{x}}\right)\middle|\mathbf{x}_1\right\rangle$$

the quantity $\mathbf{Z}(\mathbf{x},\underline{\mathbf{x}})$, which is proportional to the gradients of the vector potential $\mathbf{A}(\mathbf{x})$ [cf. Eq. (4.16)], can be disregarded and hence the energy functional $E_0(\mathbf{A},f)$ can be written in the form

$$E_0(\mathbf{A},f)\approx \mathrm{Sp}\, f\varepsilon_0\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(\underline{\mathbf{x}})\right), \tag{4.36}$$

where the momentum $\mathbf{p}$ has been replaced by $\mathbf{p}-(e/c)\mathbf{A}(\mathbf{x})$ in the quasiparticle energy.

Let us consider the solutions of the self-consistency equation (2.32) in the spatially nonuniform case for temperatures T close to the critical temperature T_{c}. In order to find the quantities f and g from Eq. (2.32) we use the general expressions (3.15). Here the quantity $\underline{\xi}=\Delta$ in the equation for X [see (3.12)] can be taken to be small. Applying perturbation theory in the small parameter Δ, we find the following expansion for the operator X in the momentum representation:

$$X_{\mathbf{pp}'}=X^1_{\mathbf{pp}'}+X^3_{\mathbf{pp}'}+\dots,$$

where

$$X^1_{\mathbf{pp}'}=-\frac{\Delta_{\mathbf{pp}'}}{\xi+\xi'},$$

$$X_{\mathbf{pp}'}=\frac{1}{\xi+\xi'}\sum_{\mathbf{p}'',\mathbf{p}'''}\frac{\Delta_{\mathbf{pp}''}\Delta^*_{\mathbf{p}'''\mathbf{p}''}\Delta_{\mathbf{p}'''\mathbf{p}'}}{(\xi+\xi'')(\xi'''+\xi')}; \tag{4.37}$$

here we have introduced the notation $\xi=\xi_{\mathbf{p}}$, $\xi'=\xi_{\mathbf{p}'}$, $\xi''=\xi_{\mathbf{p}''}$, $\xi'''=\xi_{\mathbf{p}'''}$. The expansion in powers of Δ of the operator n which enters Eqs. (3.14) takes the form

$$n_{\mathbf{pp}'}=n^0_{\mathbf{pp}'}+n^2_{\mathbf{pp}'}+\dots,$$

where

$$\begin{gathered}n^0_{\mathbf{pp}'}=n_{\mathbf{p}}\delta_{\mathbf{pp}'},\quad n_{\mathbf{p}}=(e^{Y_0\xi}+1)^{-1},\\ n^2_{\mathbf{pp}'}=\frac{n-n'}{\xi-\xi'}\sum_{\mathbf{p}''}\frac{\Delta_{\mathbf{pp}''}\Delta^*_{\mathbf{p}'\mathbf{p}''}}{\xi+\xi''},\quad n_{\mathbf{p}}\equiv n,\quad n_{\mathbf{p}'}\equiv n'.\end{gathered} \tag{4.38}$$

From these relations and Eq. (3.15) we derive an expansion in powers of Δ for the normal distribution function

$$f_{\mathbf{pp}'}=f^0_{\mathbf{pp}'}+f^2_{\mathbf{pp}'}+\dots,$$

where

$$\begin{gathered}f^0_{\mathbf{pp}'}=n\delta_{\mathbf{pp}'},\\ f^2_{\mathbf{pp}'}=\sum_{\mathbf{p}''}\left[\frac{1-n''}{(\xi+\xi'')(\xi'+\xi'')}-\frac{n}{(\xi'-\xi)(\xi+\xi'')}\right.\\ \left.-\frac{n'}{(\xi-\xi')(\xi'+\xi'')}\right]\Delta_{\mathbf{pp}''}\Delta^*_{\mathbf{p}'\mathbf{p}''},\end{gathered} \tag{4.39}$$

and an expansion in powers of Δ for the anomalous distribution function g

$$\begin{gathered}g_{\mathbf{pp}'}=g^1_{\mathbf{pp}'}+g^3_{\mathbf{pp}'}+\dots,\\ g^1_{\mathbf{pp}'}=-\frac{1-n-n'}{\xi+\xi'}\Delta_{\mathbf{pp}'},\\ g^3_{\mathbf{pp}'}=\frac{1}{4\xi^2}\left[\frac{1-2n}{\xi}-Y_02n(1-n)\right](\Delta\Delta^+\Delta)_{\mathbf{pp}'}.\end{gathered} \tag{4.40}$$

The quantities $f_{\mathbf{pp}'}$, $g_{\mathbf{pp}'}$ in the Wigner representation are given by

$$\begin{gathered}f(\mathbf{x},\mathscr{P})=\frac{V}{(2\pi)^3}\int \mathrm{d}^3k e^{i\mathbf{kx}}f_{\mathbf{pp}'},\\ g(\mathbf{x},\mathbf{k})=\frac{V}{(2\pi)^3}\int \mathrm{d}^3\mathscr{P} e^{i\mathscr{P}\mathbf{x}}g_{\mathbf{pp}'},\\ \mathbf{p}=\frac{\mathbf{k}}{2}+\mathscr{P},\quad \mathbf{p}'=\mathscr{P}-\frac{\mathbf{k}}{2}.\end{gathered} \tag{4.41}$$

From (4.19), in this representation we can use Eqs. (4.39) and (4.40), assuming in the latter that the quantity ξ depends on x through the vector potential of the magnetic field [here we have neglected the variation of $\mathbf{A}(\mathbf{x})$ over distances of order p_{F}^{-1}]:

$$\xi_{\mathbf{p}}=\varepsilon_0\left(\mathbf{p}+\frac{e}{c}\mathbf{A}(\mathbf{x})\right)+Y_0Y_4^{-1}+\varepsilon_1(\mathbf{p}),$$

where $\varepsilon_1(\mathbf{p})=\partial E_1(f)/\partial f_{\mathbf{p}}$. In the case of singlet pairing we have $g_i=0$, and hence

$$\Delta_0(\mathbf{x},\mathbf{x}')=\frac{\partial E}{\partial g_0^*(\mathbf{x}',\mathbf{x})}=2L_1(\mathbf{x}-\mathbf{x}')g(\mathbf{x}',\mathbf{x}),$$

$$\Delta_i(\mathbf{x},\mathbf{x}')=\frac{\partial E}{\partial g_i^*(\mathbf{x}',\mathbf{x})}=0.$$

We will assume in what follows that $L_1(\mathbf{x}-\mathbf{x}')=L_1\delta(\mathbf{x}-\mathbf{x}')$. Then

$$\Delta_0(\mathbf{x}-\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}')\Delta(\mathbf{x}), \qquad \Delta(\mathbf{x})=2L_1g(\mathbf{x}), \tag{4.42}$$

where

$$g(\mathbf{x})\equiv g(\mathbf{x},\mathbf{x})=\frac{1}{V}\sum_{\mathscr{P}}g(\mathbf{x},\mathscr{P})=\frac{1}{V}\sum_{\mathbf{p},\mathbf{p}'}g_{\mathbf{pp}'}g_{\mathbf{pp}'}e^{i\mathbf{x}(\mathbf{p}+\mathbf{p}')}.$$

Substituting expression (4.40) for $g^1_{\mathbf{pp}'}$ in (4.42) and noting that

$$\Delta_{\mathbf{pp}'}=\frac{1}{V}\int d^3x\Delta(\mathbf{x})e^{-i\mathbf{x}(\mathbf{p}+\mathbf{p}')}, \tag{4.43}$$

we find the linearized self-consistency equation in the presence of a magnetic field

$$\Delta(\mathbf{x})=-\frac{4L_1}{V}\sum_{\mathbf{p},\mathbf{p}'}\int d^3x'\Delta(\mathbf{x}')e^{i(\mathbf{x}-\mathbf{x}')(\mathbf{p}+\mathbf{p}')}\frac{1-n-n'}{\xi+\xi'}. \tag{4.44}$$

Hence expanding in the small gradients of the order parameter $\Delta(\mathbf{x})$ and the vector potential $\mathbf{A}(\mathbf{x})$ we find

$$(G-1)\Delta(\mathbf{x})+G_{ij}\left(\frac{\partial}{\partial x_i}-i\frac{2e}{c}A_i(\mathbf{x})\right)\times\left(\frac{\partial}{\partial x_{ij}}-i\frac{2e}{c}A_j(\mathbf{x})\right)\Delta(\mathbf{x})=0, \tag{4.45}$$

where

$$G=-\frac{2L_1}{(2\pi)^3}\int d^3k\,\frac{1-2n(\varepsilon)}{\varepsilon-\mu}, \quad \mu=-Y_4Y_0^{-1},$$

$$G_{ij}=\frac{2L_1}{(2\pi)^3}\int d^3k\,\frac{\partial^2}{\partial\mathscr{P}_i\partial\mathscr{P}_j}\frac{1-n(\varepsilon)-n(\varepsilon')}{\varepsilon+\varepsilon'-2\mu}\bigg|_{\mathscr{P}=0},$$

$$\varepsilon=\varepsilon_0(\mathbf{p})+\varepsilon_1(\mathbf{p}), \quad \varepsilon'=\varepsilon_0(\mathbf{p}')+\varepsilon_1(\mathbf{p}'),$$

$$\mathbf{p}=\tfrac{1}{2}\mathscr{P}+\mathbf{k}, \quad \mathbf{p}'=\tfrac{1}{2}\mathscr{P}-\mathbf{k}.$$

Substituting $g^1_{\mathbf{pp}'}$ in Eq. (4.42) and also expression (4.40) for $g^3_{\mathbf{pp}'}$, and using Eq. (4.45), we include in the self-consistency equation a term which is nonlinear in Δ. Thus, we obtain a nonlinear self-consistency equation for the gap $\Delta(\mathbf{x})$ in the presence of a magnetic field (we have disregarded the effect of the electric field on the nonlinear term):

$$(G-1)\Delta(\mathbf{x})+G_{ij}\left(\frac{\partial}{\partial x_i}-i\frac{2e}{c}A_i(\mathbf{x})\right)\left(\frac{\partial}{\partial x_j}-i\frac{2e}{c}A_j(\mathbf{x})\right)\Delta(\mathbf{x})+B|\Delta(\mathbf{x})|^2\Delta(\mathbf{x})=0, \tag{4.46}$$

where

$$B=\frac{L_1}{(2\pi)^3}\int d^3k\,\frac{1}{(\varepsilon-\mu)^2}\left[\frac{1-2n(\varepsilon)}{\varepsilon-\mu}-Y_02n(\varepsilon)(1-n(\varepsilon))\right].$$

Since the chemical potential $\mu=-Y_4Y_0^{-1}$ is considerably greater than T_c, at temperatures T close to the critical value the coefficients G, G_{ij}, and B are given by

$$G=1-2L_1\nu(\varepsilon_F)\frac{T_c-T}{T_c},$$

$$G_{ij}=-\delta_{ij}\frac{7\zeta(3)}{24\pi^2}\frac{L_1v_F^2\nu(\varepsilon_F)}{T_c^2},$$

$$B=\frac{7\zeta(3)}{4\pi^2}\frac{L_1\nu(\varepsilon_F)}{T_c^2},$$

where ε_F and v_F are the Fermi energy and the quasiparticle velocity, $\nu(\varepsilon_F)$ is the density of states at the Fermi surface, and $\zeta(x)$ is the Riemann zeta function. Equation (4.38) assumes the well-known form (see Ref. 36)

$$\left[\frac{1}{4}\left(\frac{\partial}{\partial x_i}-i\frac{2e}{c}A_i(\mathbf{x})\right)^2+\frac{2\pi^2T_c}{7\zeta(3)v_F^2}\times\left(\frac{T_c-T}{T}-\frac{7\zeta(3)}{8\pi^2T_c^2}|\Delta(\mathbf{x})|^2\right)\right]\Delta(\mathbf{x})=0. \tag{4.47}$$

Let us now calculate the current density $\mathbf{j}$ in the presence of a magnetic field. We determine the current density $\mathbf{j}(\mathbf{x})$ from the formula

$$j_i(\mathbf{x})=\frac{2}{(2\pi)^3}\int d^3\mathscr{P}f(\mathbf{x},\mathscr{P})\frac{\partial\varepsilon}{\partial P_i}=\frac{2V}{(2\pi)^6}\int d^3\mathscr{P}\frac{\partial\varepsilon}{\partial P_i}\int d^3ke^{i\mathbf{kx}}f_{\mathbf{pp}'},$$

$$\mathbf{p}=\frac{\mathscr{P}}{2}+\mathbf{k}, \quad \mathbf{p}'=\frac{\mathscr{P}}{2}-\mathbf{k}. \tag{4.48}$$

Using Eq. (4.30) for the matrix element $f^2_{\mathbf{pp}'}$ in the limit $\mathbf{A}(\mathbf{x})=0$, we find an expansion for the Wigner distribution function $f(\mathbf{x},\mathscr{P})$ in the gradients of the order parameter:

$$f(\mathbf{x},\mathscr{P})=f^{(0)}(\mathbf{x},\mathscr{P})+f^{(2)}(\mathbf{x},\mathscr{P})+\dots,$$

$$f^{(0)}(\mathbf{x},\mathscr{P})=-Y_0^2|\Delta(\mathbf{x})|^2Q(\mathscr{P},\mathscr{P};-\mathscr{P}),$$

$$f^{(2)}(\mathbf{x},\mathscr{P})=\frac{i}{2}Y_0^2\left(\Delta^*(\mathbf{x})\frac{\partial\Delta(\mathbf{x})}{\partial x_i}-\Delta(\mathbf{x})\frac{\partial\Delta^*(\mathbf{x})}{\partial x_i}\right)$$

$$\times\frac{\partial}{\partial p_i}Q(\mathscr{P},\mathscr{P};\mathbf{p})\,|_{\mathbf{p}=-\mathscr{P}},$$

where

$$Q(\mathbf{p},\mathbf{p}',\mathbf{p}'')=-\frac{1-n''}{(\xi'+\xi'')(\xi+\xi'')}$$

$$+\frac{1}{\xi-\xi'}\left(\frac{n'}{\xi'+\xi''}-\frac{n}{\xi+\xi''}\right).$$

The current density in the absence of a vector potential $\mathbf{A}(\mathbf{x})$ assumes the form

$$j_i^0(\mathbf{x})=D_{ik}i\left(\Delta^*(\mathbf{x})\frac{\partial\Delta(\mathbf{x})}{\partial x_k}-\Delta(\mathbf{x})\frac{\partial\Delta^*(\mathbf{x})}{\partial x_k}\right),$$

where

$$D_{ik}=Y_0^3\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial\varepsilon}{\partial p_i}\frac{\partial\varepsilon}{\partial p_k}\frac{Q(\xi,\xi;\xi)}{\xi}. \tag{4.49}$$

Now we take into account the effect of the vector potential $\mathbf{A}(\mathbf{x})$ in first-order perturbation theory. Since

$$\partial\varepsilon\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(x)\right)\Big/\partial p_i=\partial\varepsilon(\mathbf{p})/\partial p_i$$

$$-(e/c)A_k(\mathbf{x})\partial^2\varepsilon/\partial p_k\partial p_i,$$

the correction to the current density $j^A(\mathbf{x})$ due to the vector potential A can be written as follows:

$$j_i^A(\mathbf{x})=-2eA_k\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial^2\varepsilon}{\partial p_k\partial p_i}f_A^{(0)}(\mathbf{p},\mathbf{x})\,|_{\mathbf{A}=0}$$

$$+2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial\varepsilon}{\partial p_i}(f_A^{(0)}(\mathbf{p},\mathbf{x})-f_A^{(0)}(\mathbf{p},\mathbf{x})\,|_{\mathbf{A}=0}),$$

where

$$f_A^{(0)}(\mathbf{p},\mathbf{x})=-Y_0^2|\Delta(\mathbf{x})|^2Q\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x}),\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x});\right.$$

$$\left.-\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x})\right).$$

Expanding the quantity Q in powers of $\mathbf{A}(\mathbf{x})$, we find

$$j_i^A(\mathbf{x})=2|\Delta(\mathbf{x})|^2\cdot 2eD_{ik}A_k(\mathbf{x}),$$

where D_{ik} is given by (4.49). Hence the current density $j(\mathbf{x})=j^{(0)}(\mathbf{x})+j^A(\mathbf{x})$ in the presence of a magnetic field is equal to

$$j_i(\mathbf{x})=D_{ik}i\left[\Delta^*(\mathbf{x})\left(\frac{\partial}{\partial x_k}-i\frac{2e}{c}A_k(\mathbf{x})\right)\Delta(\mathbf{x})-\Delta(\mathbf{x})\right.$$

$$\left.\times\left(\frac{\partial}{\partial x_k}+i\frac{2e}{c}A_k(\mathbf{x})\right)\Delta^*(\mathbf{x})\right]. \tag{4.50}$$

The coefficient D_{ik} in the limit $\mu\gg T_c$, $T\approx T_c$ is given by

$$D_{ik}=-\delta_{ik}\frac{7\zeta(3)}{24\pi^2}\frac{\nu(\varepsilon_F)v_F^2}{T_c^2}.$$

Now that we have the expression (4.50) for the current, we can write the equation for the magnetic field

$$\operatorname{curl}\mathbf{b}=\frac{4\pi}{c}\mathbf{j},\quad \mathbf{b}=\operatorname{curl}\mathbf{A}. \tag{4.51}$$

Equation (4.50) for the current contains the unknown function $\Delta(\mathbf{x})$, which is determined by Eq. (4.46). The magnetostatic equations (4.51) together with Eq. (4.46) constitute a closed system of equations determining $\Delta(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ [or $\mathbf{b}(\mathbf{x})$]. These equations are called the Ginzburg–Landau equations.[37] In particular, they can be used to solve the problem of how a constant magnetic field penetrates into superconductors. These equations were obtained by Gor'kov[36] in the microscopic treatment based on the temperature Green's function.

In the weak-field limit to lowest order in the field the order parameter Δ can be regarded as independent of position. In this case the current given by Eq. (4.50) takes the form

$$j_i=-\frac{4e^2}{mc}|\Delta|^2A_i,$$

and consequently the equations determining the magnetic field are the same as the London equations (4.30),

$$\mathbf{b}+\lambda^2(T)\operatorname{curl}\operatorname{curl}\mathbf{b}=0,$$

where

$$\lambda^{-2}(T)=\frac{16\pi e^2}{mc^2}\Delta. \tag{4.52}$$

Noting that $\Delta(T)\sim(T_c-T)^{1/2}$ holds, we have for the depth of penetration

$$\lambda(T)=\frac{1}{\sqrt{2}}\left(1-\frac{T}{T_c}\right)^{-1/2}\lambda_L(0), \tag{4.53}$$

where $\lambda_L^{-2}(0)=4\pi ne^2/mc^2$. Observe that Eqs. (4.52) and (4.53) are equivalent to Eq. (4.30) if we set $m^*=m$ in the latter and evaluate ρ_s by starting from Eqs. (3.83) with T close to T_c.

Thus, the Ginzburg–Landau equation leads to a local relation between the current and the vector potential. However, this equation can be used to describe superconductors of both the first and second kinds. This is because it contains two parameters with the units of length: the Cooper pairing length $\xi(T)$, which is contained in the terms that are not associated with the vector potential of the magnetic field, and the London penetration depth $\lambda(T)$, which is contained in the terms associated with the vector potential.

A nonlocal relation between the current and the vector potential $\mathbf{A}(\mathbf{x})$ can arise when we improve the approximations which lead to the Ginzburg–Landau equation; specifically, we take into account the following terms in the expansion in the order parameter $\Delta(\mathbf{x})$ and in the gradients of the order parameter.

To conclude this section we discuss the topic of the structure of the Ginzburg-Landau equation for neutral particles (such as ^{3}He atoms) in the case of triplet pairing.[42,43] It can be shown that the transition temperature T_c in the case of triplet pairing is given by the equation

$$\Delta_j(\mathbf{r},\mathbf{p}) = -\frac{1}{(2\pi)^3}\int \mathbf{d}^3q L_2(|p-q|)\Delta_j(\mathbf{r},\mathbf{q})\,\frac{1-2n_q}{2\xi_q}.$$

The order parameter Δ_j is a vector in spin space. Close to the transition temperature this order parameter is small. We expand the self-consistency equation (2.32) in this quantity to terms of order Δ^3, and also in gradients of this parameter. We further assume that the amplitude $L_2(\mathbf{x}_1-\mathbf{x}_2)$ has a sharp maximum at $\mathbf{x}_1=\mathbf{x}_2$. Then the Fourier component $L_2(\mathbf{k})$ of this amplitude can be expanded in powers of $\mathbf{k}$:

$$L_2(\mathbf{k}) = L_2(0) + k^2L + \ldots\,. \tag{4.54}$$

Using this assumption, we can look for a solution of the self-consistency equation in the form

$$\Delta_j(\mathbf{r},\mathbf{p}) = p_F^{-1} p_l A_{jl}(\mathbf{r}), \tag{4.55}$$

where the quantities $A_{jl}(\mathbf{r})$ determining the nonuniform order parameter $\Delta_j(\mathbf{r},\mathbf{p})$ satisfy the equation

$$\frac{T_c-T}{T_c}A_{ji} + \frac{7\zeta(3)v_F^2}{80\pi^2T_c^2}\left(\frac{\partial^2 A_{ji}}{\partial x_l^2} + 2\frac{\partial^2 A_{jl}}{\partial x_i \partial x_l}\right)$$
$$+\frac{7\zeta(3)}{40\pi^2T_c^2}\,[A^*_{ji}\,\mathrm{Sp}(A\tilde{A}) - 2A_{ji}\,\mathrm{Sp}(AA^+)$$
$$-2A_{js}\tilde{A}_{sr}A^*_{ri} - 2A_{js}A^+_{sr}A_{ri} + 2A^*_{js}\tilde{A}_{sr}A_{ri}] = 0. \tag{4.56}$$

This equation is a generalization of the Ginzburg–Landau equation to the case of triplet pairing, in which on account of the approximation (4.54) pairing takes place in the state with orbital angular momentum $l=1$. We refrain from writing the expression here for the current density, assuming that the system is a neutral Fermi liquid (^{3}He).

5. CONCLUSION

In this review we have treated topics related to the semi-phenomenological description of a superfluid Fermi liquid. This description is based on the ideas put forward by Landau in constructing the theory of a normal Fermi liquid. In the weak-interaction approximation the theory we have developed goes over to the BCS theory of superconductivity. In this review we have considered triplet pairing in superfluid systems in detail. In particular, we have derived the self-consistency equations for the nonunitary phases, and have also considered the symmetry properties of the superfluid A and B phases of ^{3}He. We have studied questions involving gauge-invariant switching-on of the electromagnetic interaction in the theory of superfluid Fermi liquids.

Many problems in the kinetics of superfluid liquids lie outside the scope of our review; in particular, these include the problem of the propagation of high-frequency oscillations, the determination of the dissipative kinetic coefficients, and the treatment of impurities. In this connection we note that the development of the hydrodynamics of the superfluid phases of ^{3}He, which we do not enter into here, is the subject of reviews by Mineev[30] and Volovik;[38] the theory of high-frequency oscillations in the superfluid phases of ^{3}He has been treated, e.g., by Brusov and Popov.[39] We have not treated the effect of the electric field on superconductors. These questions are explored by Svidzinskii.[40] We have also not touched on the important applications of the theory of superfluidity in nuclear physics and astrophysics.

We emphasize, however, that in this review we have developed a mathematical apparatus by means of which it is possible in principle to investigate both problems associated with the static properties of superfluids and with various kinetic problems in terms of a semi-phenomenological approach.

NOTATION

S, total entropy of the Fermi liquid
s, entropy density of the Fermi liquid
$\hat{f}$, statistical operator of a quasiparticle
f, g, normal and anomalous distribution functions, respectively
f_{eq}, equilibrium quasiparticle statistical operator
n, diagonalized quasiparticle distribution function
$E(\hat{f})$, total energy functional of the Fermi liquid
$\mathscr{E}(x,\hat{f})$, total energy density functional of the Fermi liquid
N, total particle number of the Fermi liquid
P_i, total momentum of the Fermi liquid
$\hat{\varepsilon}$, quasiparticle energy operator
ε, normal part of the quasiparticle energy operator
Δ, anomalous part of the quasiparticle energy operator (energy gap)
$\hat{\mathbf{p}}_i$, momentum operator of a quasiparticle
$\hat{s}_i$, spin operator of a quasiparticle
$\hat{\tau}_3$, quasiparticle number operator
$a(x,\hat{f})$, average value of the density of a physical quantity
$\hat{s}_i(x)$, spin density operator
$\hat{\pi}_i$, momentum density operator
$\hat{\rho}(x)$, particle density operator
Ω, total thermodynamic potential of the system
ω, density of the thermodynamic potential
x, spatial coordinate
t, time
V, system volume
F_i, normal Fermi-liquid amplitudes
L_i, anomalous Fermi-liquid amplitudes
a^+, a, quasiparticle creation and annihilation operators
σ_i, Pauli matrices
$\hat{j}_i(x)$, particle flux density operator
$\hat{j}_{ik}(x)$, spin flux density operator
$W_i(x)$, energy flux density operator
$t_{ik}(x;\hat{f})$, average value of the stress tensor
$\underline{U}$, Bogolyubov unitary transformation in "one-particle" space

u, v, Bogolyubov transformation coefficients
U_ϕ, unitary phase transformation
U_ω, unitary transformation for rotation in spin space
U_y, unitary transformation for spatial translations
Γ_{ki}, generator for arbitrary affine transformations
H, microscopic Hamiltonian
$\Phi(1,2;3,4)$, microscopic fermion interaction amplitude
q_i, superfluid momentum
$\phi(x)$, phase of the anomalous distribution function
$Y_0=T^{-1}$ (T is absolute temperature)
$Y_i=-Y_0v_i$ (v_l is the normal velocity)
$Y=-Y_0\mu$ (μ is the chemical potential)
T_c, transition temperature
p_F, ε_F, Fermi momentum and energy respectively
$\mathbf{v}_F$, Fermi velocity
ν_F, density of states at the Fermi surface
M_i, spontaneous magnetization
β, Bohr magneton
l, m, orbital angular momentum quantum numbers
σ_n, σ_s, mass densities of the normal and superfluid components
m^*, effective quasiparticle mass of a superfluid Fermi liquid
$\mathbf{v}_n$, $\mathbf{v}_s$, velocities of the normal and superfluid components of a Fermi liquid
A_0, scalar potential of the self-consistent electromagnetic field
A_i, vector potential of the self-consistent electromagnetic field
χ, gauge function
$U(x,A)$, unitary operator for switching on the electromagnetic interaction
b_i, self-consistent magnetic field
λ_L, London depth of penetration into a superconductor by a magnetic field
ξ_0, coherence length (size of a Cooper pair)
$\zeta(x)$, Riemann zeta function

[1] L. D. Landau, Zh. Éksp. Teor. Fiz. **30**, 1058 (1956) [Sov. Phys. JETP **3**, 920 (1957)].

[2] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics*, Prentice-Hall, Englewood Cliffs, N.J., 1963 [Russ. original, Fizmatgiz, M., 1962].

[3] L. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, Benjamin, N.Y., 1962 [Russ. transl., Mir, M., 1964].

[4] A. S. Kondrat'ev and A. G. Kuchma, *The Electron Fluid of Normal Metals* [in Russian], Leningrad State University, L., 1980.

[5] E. M. Lifshitz and L. P. Pitaevskiĭ, *Statistical Physics (Part II)*, Pergamon, Oxford, 1980. [Russ. original, Nauka, M., 1978].

[6] A. I. Akhiezer and S. V. Peletminskiĭ, *Methods of Statistical Physics*, Pergamon, N.Y., 1981. [Russ. original, Nauka, M., 1977].

[7] A. A. Vlasov, Zh. Éksp. Teor. Fiz. **8**, 291 (1939).

[8] L. D. Landau, Zh. Éksp. Teor. Fiz. **32**, 59 (1957) [Sov. Phys. JETP **5**, 161 (1957)].

[9] V. P. Silin, Zh. Éksp. Teor. Fiz. **23**, 641 (1952).

[10] V. P. Silin, Zh. Éksp. Teor. Fiz. **33**, 495 (1957) [Sov. Phys. JETP **6**, 387 (1958)].

[11] I. Ya. Pomeranchuk, Zh. Éksp. Teor. Fiz. **35**, 524 (1958) [Sov. Phys. JETP **8**, 361 (1959)].

[12] E. C. Stoner, Proc. R. Soc. London, A **165**, No. 992, 372 (1938).

[13] A. A. Abrikosov and I. E. Dzyaloshinskii, Zh. Éksp. Teor. Fiz. **35**, 771 (1958) [Sov. Phys. JETP **8**, 535 (1959)].

[14] I. A. Akhiezer, V. G. Syshchenko, and E. M. Chudnovskii, Pis'ma Zh. Éksp. Teor. Fiz. **20**, No. 11, 739 (1974) [JETP Lett. **20**, 342 (1974)].

[15] I. A. Akhiezer and E. M. Chudnovskii, Fiz. Tverd. Tela **17**, 1907 (1975) [Sov. Phys. Solid State **17**, 1252 (1975)].

[16] I. A. Akhiezer, Ukr. Fiz. Zh. **25**, 177 (1980).

[17] A. A. Kondrat'ev and V. M. Uzdin, *Electron Fluid of Magnetically Ordered Metals* [in Russian], Leningrad State University, L., 1988.

[18] J. G. Bednorz and K. A. Müller, Z. Phys. B **64**, 189 (1986).

[19] L. D. Gor'kov and N. B. Kopnin, Usp. Fiz. Nauk **156**, 117 (1988) [Sov. Phys. Usp. **31**, 850 (1988)].

[20] N. N. Bogoliubov, *Lectures on Quantum Statistics*, Gordon & Breach, N. Y., 1967 [Ukr. original, Kiev, 1949].

[21] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).

[22] N. N. Bogolyubov, Zh. Éksp. Teor. Fiz. **34**, 58, 73 (1958) [Sov. Phys. JETP **7**, 41 (1958); **7**, 51 (1958)].

[23] N. N. Bogolyubov, "Quasiaverages in statistical mechanics problems," (in Russian) Preprint No. R-1451, Joint Institute for Nuclear Research, Dubna, 1963; *Collected Works* [in Russian], Vol. 3, Naukova Dumka, Kiev, 1971.

[24] L. D. Gor'kov, Zh. Éksp. Teor. Fiz. **34**, 735 (1958) [Sov. Phys. JETP **7**, 505 (1958)].

[25] G. M. Éliashberg, Zh. Éksp. Teor. Fiz. **39**, 735 (1958) [Sov. Phys. JETP **12**, 1000 (1961)].

[26] Y. Nambu, Phys. Rev. **117**, 648 (1960).

[27] L. P. Pitaevskiĭ, Zh. Éksp. Teor. Fiz. **37**, 1794 (1959) [Sov. Phys. JETP **10**, 1267 (1960)].

[28] A. I. Legget, Rev. Mod. Phys. **47**, 331 (1975); R. Balian and N. R. Werthamer, Phys. Rev. **131**, 1553 (1963).

[29] D. D. Osheroff, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. **28**, 885 (1972).

[30] V. P. Mineev, Usp. Fiz. Nauk **139**, 303 (1983) [Sov. Phys. Usp. **26**, 160 (1983)].

[31] N. N. Bogolyubov, Jr. *et al.*, Fiz. Élem. Chast. Atom. Yad. **16**, 875 (1985) [Sov. J. Part. Nucl. **16**, 389 (1985)].

[32] L. D. Landau, Zh. Éksp. Teor. Fiz. **11**, 592 (1941).

[33] H. London and F. London, Proc. R. Soc. A **149**, 71 (1935).

[34] W. Meissner and R. Ochsenfeld, Naturwissenschaften **33**, No. 4, 787 (1933).

[35] Yu. N. Ryabinin and L. V. Shubnikov, Zh. Éksp. Teor. Fiz. **5**, 130 (1935).

[36] L. P. Gor'kov, Zh. Éksp. Teor. Fiz. **36**, 1918 (1959) [Sov. Phys. JETP **9**, 1364 (1959)].

[37] V. L. Ginsburg and L. D. Landau, Zh. Éksp. Teor. Fiz. **20**, 1064 (1950).

[38] G. E. Volovik, Usp. Fiz. Nauk **143**, 73 (1984) [Sov. Phys. Usp. **27**, 363 (1984)].

[39] P. N. Brusov and V. N. Popov, *Superfluidity and Collective Properties of Quantum Fluids* [in Russian], Nauka, M., 1988.

[40] A. V. Svidzinskii, *Spatially Nonuniform Problems in the Theory of Superconductivity* [in Russian], Nauka, M., (1982).

[41] A. A. Isaev and S. V. Peletminskiĭ, Ukr. Fiz. Zh. **37**, No. 6, 952 (1992).

[42] V. Ambegackar, P. G. De Gennes, and D. Rather, Phys. Rev. A **9**, No. 6, 2676 (1974).

[43] A. N. Tarasov, Preprint No. 92-q2, Kharkov Physico-Technical Institute, 1992.

[44] V. V. Krasil'nikov, S. V. Peletminskiĭ, A. A. Yatsenko, and A. A. Rozhkov, Fiz. Élem. Chast. Atom. Yad. **19**, 1440 (1988) [Sov. J. Part. Nucl. **19**, 674 (1988)].

Translated by D. L. Book