Angular momentum in classical electrodynamics

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Certain paradoxes of classical electrodynamics that relate to the field angular momentum are discussed. Explicit expressions are obtained for the first terms of the expansion of the angular momentum flux radiated by a set of nonrelativistic charges. The consequences of ambiguities in the definition of the angular momentum tensor in classical electrodynamics are examined.

1. INTRODUCTION

Recent publications[1,2] devoted to the angular momentum of the classical electromagnetic field have rightly noted that inadequate attention has been paid in the scientific literature to this question. The result has been that many simple matters have been either totally ignored or accounts of them have suffered from inaccuracies. The latter seem to be due to inadequate understanding of the origin of paradoxes associated with the definition of the angular momentum of a field in a volume V, namely,

$$\mathbf{M}_f=\frac{1}{4\pi c}\int_V d^3r[\mathbf{r}\times[\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)]]. \qquad (1.1)$$

The simplest of these paradoxes may be formulated as follows. It is readily verified that the integral in (1.1) vanishes when it is evaluated over a spherical volume V for a circularly-polarized plane wave (the vector $\mathbf{r}$ is measured from the center of the sphere). At the same time, it is clear that charged particles that are initially at rest within this volume will revolve under the influence of the circularly polarized wave and the field angular momentum within the volume V will be transferred to them. This paradox is examined in detail in Ref. 2 in a somewhat different formulation (the angular momentum flux of a circularly polarized plane wave in the direction of the Poynting vector is zero).

In this note, we present some useful results on angular momenta in classical electrodynamics. Some of them may be looked upon as extensions of statements made in Refs. 1 and 2.

2. MULTIPOLE EXPANSION OF THE ANGULAR MOMENTUM FLUX

The intensity of electromagnetic radiation emitted by a nonrelativistic classical source can be written in the form of the following multipole expansion (a series in powers of the parameter $v/c\ll 1$):[3]

$$I=\frac{2}{3c^3}\ddot{\mathbf{d}}^2+\frac{2}{3c^3}\ddot{\mathbf{m}}^2+\frac{1}{180c^5}\dddot{Q}_{ij}\dddot{Q}_{ij}+\dots; \qquad (2.1)$$

where $\mathbf{d}$,$\mathbf{m}$, and Q_{ij} are the usual electric dipole, magnetic dipole, and quadrupole moments of the radiating system. There is an analogous expansion for the flux of the ith component of radiated angular momentum $F_i^{(M)}$ (Ref. 4) whose first (dipole) term is[3,5]

$$F_i^{(M)}=\frac{2}{3c^3}[\dot{\mathbf{d}}\times\ddot{\mathbf{d}}]_i+\dots. \qquad (2.2)$$

As far as we have been able to establish, there are no explicit expressions in the literature for the higher-order terms of this expansion. We shall derive these expressions in this Section and will show that the representation of $F_i^{(M)}$ by the sum of the contributions of individual multipoles is not entirely correct.

The conservation of angular momentum can be derived in classical electrodynamics by applying the identity transformation to the Maxwell equations and to the equations of motion of N charged particles in a volume V (see, for example, Ref. 6):

$$\frac{d}{dt}(M_{pi}+M_{fi})=-F_i^{(M)}, \qquad (2.3)$$

where $\mathbf{M}_p$ is the mechanical angular momentum of the N particles

$$\mathbf{M}_p=\sum_{a=1}^{N}[\mathbf{r}_a(t)\times\mathbf{p}_a(t)], \qquad (2.4)$$

M_f is the field angular momentum in the volume V as given by (1.1), and $F_i^{(M)}$ is the ith component of the angular momentum flux vector over the surface bounding V,

$$F_i^{(M)}=e_{ijk}\oint dS_l r_j\sigma_{kl}, \qquad (2.5)$$

expressed in terms of the Maxwell stress tensor[3] σ_{kl}. If V is a spherical volume of radius $r\gg\lambda$ (λ is the characteristic wavelength of the radiation), then (see Appendix)

$$F_i^{(M)}=\frac{1}{4\pi}\oint r^2 d\Omega[\mathbf{r}\times[\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)]]_i. \qquad (2.6)$$

The transformation to the multipole expansion for the flux $F_i^{(M)}$ relies on the substitution in (2.6) of the expan-

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sions for $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ in terms of the electric $(\alpha=e)$ and magnetic $(\alpha=m)$ multipole potentials $\mathbf{B}_{LM}(\mathbf{r},\alpha)$ for diverging waves.[6] However, this expansion is found to consist of not only contributions of the individual multipoles $F_i^{(\alpha L)}$, but also terms due to interference between the (eL) and $(mL\pm 1)$ sources. In particular, the complete expression for the angular momentum flux $F_i^{(M)}$ radiated by a nonrelativistic source has the following form to within terms of order $(v/c)^2$:

$$F_i^{(M)}=\frac{2}{3c^3}[\dot{\mathbf{d}}\times\ddot{\mathbf{d}}]_i+\frac{2}{3c^3}[\dot{\mathbf{m}}\times\ddot{\mathbf{m}}]_i+\frac{1}{90c^5}e_{ijk}\ddot{Q}_{jl}\dddot{Q}_{kl}$$

$$-\frac{1}{15c^4}\frac{\mathrm{d}}{\mathrm{d}t}(\ddot{Q}_{ij}\dot{m}_j)+\dots . \tag{2.7}$$

The interference term arises for the following reason. The conservation law given by (2.3) for 'ordinary' angular momenta of the particles $(\mathbf{M}_\mathrm{p})$ and of the field $(\mathbf{M}_\mathrm{f})$ is transformed identically (without using any gauge conditions) into the law of conservation of canonical angular momenta of the particles $(\mathbf{J}_\mathrm{p})$ and the field $(\mathbf{J}_\mathrm{f})$ (see Ref. 2 and also Sec. 4 below):

$$\frac{\mathrm{d}}{\mathrm{d}t}(J_{\mathrm{p}i}+J_{\mathrm{f}i})=-F_i^{(J)}, \tag{2.8}$$

$$\mathbf{J}_\mathrm{p}=\sum_{a=1}^{N}[\mathbf{r}_a(t)\times\mathbf{P}_a(t)],$$

$$\mathbf{P}_a(t)=\mathbf{p}_a(t)+\frac{e_a}{c}\mathbf{A}(\mathbf{r}_a(t),t), \tag{2.9}$$

$$\mathbf{J}_\mathrm{f}=\frac{1}{4\pi c}\int_V \mathrm{d}^3r(iE_j(\mathbf{r},t)(\hat{\mathbf{J}})_{jk}A_k(\mathbf{r},t)), \tag{2.10}$$

where $\hat{\mathbf{J}}$ is the total angular momentum operator of a spin 1 particle

$$(\hat{J}_i)_{jk}=\hat{L}_i\delta_{jk}-ie_{ijk},\quad \hat{\mathbf{L}}=-i\left[\mathbf{r}\times\frac{\partial}{\partial\mathbf{r}}\right]. \tag{2.11}$$

In the general case, the ith component of the flux of the canonical angular momentum $F_i^{(J)}$ is related to $F_i^{(M)}$ by

$$F_i^{(M)}=F_i^{(J)}+\frac{1}{4\pi c}\frac{\mathbf{d}}{\mathrm{d}t}\oint \mathrm{d}S_lE_l(\mathbf{r},t)[\mathbf{r}\times\mathbf{A}(\mathbf{r},t)]_i . \tag{2.12}$$

The flux $F_i^{(J)}$ out of the spherical surface of radius $r\gg\lambda$ is (see Appendix)

$$F_i^{(J)}=\frac{1}{4\pi}\oint r^2\mathrm{d}\Omega(iE_j(\mathbf{r},t)(\hat{J}_i)_{jk}A_k(\mathbf{r},t)). \tag{2.13}$$

The multipole potentials $\mathbf{B}_{LM}(\mathbf{r},\alpha)$ are the eigenfunctions of the operator $\hat{J}_z$, or $\hat{J}_0$ if we transform to the spherical coordinate frame. This must be taken into account when the expansions of $\mathbf{E}$ and $\mathbf{A}$ in terms of $\mathbf{B}_{LN}(\mathbf{r},\alpha)$ are substituted in (2.13) (it has been assumed in this problem that the potentials φ and $\mathbf{A}$ that describe radiation satisfy the standard Lorentz gauge condition). If we now use the Wigner–Eckert theorem and the orthogonality of the multipole potentials on a sphere, we can readily represent the flux $F_i^{(J)}$ by the sum of the contributions of the individual multipoles $F_i^{(\alpha L)}$ $(\alpha=e,m;L\geqslant 1)$. The surface integral on the right hand side of (2.12), on the other hand, does not vanish because the longitudinal components of $\mathbf{E}(\mathbf{r},t)$ decrease as $\sim 1/r^2$ (the radius r of the spherical surface for which the flux is calculated is as large as desired but finite) and yields precisely the above interference terms.

These terms are total differentials with respect to time and vanish on averaging. They were therefore obviously missed in Ref. 4 in which the initial equation for the angular momentum flux $F_i^{(M)}$ was averaged over time. Expressions for the flux of angular momentum $\mathbf{M}_\mathrm{f}(1.1)$ radiated by a source of given multipolarity α,L,M were obtained in Refs. 7--9. The interference terms could not, of course, appear in such calculations.

3. TRANSFER OF THE ANGULAR MOMENTUM OF A CIRCULARLY-POLARIZED PLANE WAVE AND RADIATION FROM A DIPOLE ROTATOR

The paradox associated with the transfer of the angular momentum of a circularly-polarized plane wave was formulated in the Introduction. It was shown with reasonable generality in Ref. 2 that, for a plane wave, the angular momentum flux across a plane perpendicular to the direction of propagation was zero and that this was not in conflict with the possibility that the wave could transfer its angular momentum to the charged particles interacting with it. In actual fact, and in accordance with the conservation law given by (2.3), a change in the angular momentum $\mathbf{M}_\mathrm{p}$ of the particles cannot be directly related to the absolute magnitude of the angular momentum $\mathbf{M}_\mathrm{f}$ of the field in the volume V or to the angular momentum flux of the plane wave crossing the surface bounding the volume V. Charged particles interacting with the plane wave radiate, so that the total field on the boundary of the volume V is a superposition of these radiated fields and the incident plane wave. It is shown in Ref. 2 that the change in $M_{\mathrm{p}i}$ is precisely cancelled by the terms in the flux $F_i^{(M)}$ due to interference between the radiated and incident fields. It is well-known that energy conservation is assured in a similar way during the interaction of an electromagnetic wave and a set of charges. The same phenomenon of interference between incident and scattered waves arises in quantum theory where we have the conservation of probability (see, for example, the derivation of the optical theorem in the Ref. 10).

A simple problem illustrating these ideas is examined in our previous paper.[11] Consider a circularly-polarized plane wave propagating along the z axis. Its electric field $\mathbf{E}_0(t)$ forces a free nonrelativistic particle of charge e into a circular orbit of radius r_0. The energy (intensity I) radiated by this particle per unit time and the z component of the angular momentum (the flux $F_z^{(M)}$) are given by (2.1) and (2.2) (ω_0 is the angular frequency):

$$I=\frac{2e^2\omega_0^4r_0^2}{3c^3},\quad F_z^{(M)}=\frac{2e^2\omega_0^3r_0^2}{3c^3}. \tag{3.1}$$

These fluxes are exactly compensated by terms due to interference between the incident plane wave and the field radiated by the particle (since the problem is time independent, the energy and the angular momentum of the particle and the field are constant within the spherical volume V). We note that this is a natural result from the standpoint of the correspondence principle. In quantum theory, a circularly polarized wave constitutes a flux of photons polarized parallel or antiparallel to their momentum. The absorption of n photons of total energy $n\hbar\omega_0=2e^2\omega_0^4r_0^2/3c^3$ should be accompanied by the transfer of angular momentum $n\hbar=2e^2\omega_0^3r_0^2/3c^3$ to the scattering target.

Thus, the fact that the vector $\mathbf{M}_f$ given by (1.1) is equal to zero within the volume V is not in conflict with the ability of a circularly polarized plane wave to transport angular momentum and transfer it to charged particles within the same volume V. The energy and the angular momentum transfers, δE_f and ΔM_{fz} calculated in classical theory are then related by

$$\Delta M_{fz}=\frac{\Delta E_f}{\omega_0}, \tag{3.2}$$

which is also valid in quantum theory. The source of the apparent paradoxes here is the implied simple-minded treatment of the vector $\mathbf{M}_f$ as an angular momentum 'contained' within the volume V. According to (2.3), the physical significance must, however, be assigned only to *changes* in the vector $\mathbf{M}_f$ within the volume V and not to the absolute values of these vectors, and also to the integral fluxes of angular momentum across the surfaces bounding V. In the same way, only the integral flux of the Poynting vector across a closed surface and not the Poynting vector itself can be physically meaningful.[12]

We shall now show that an apparent paradox similar to that examined above arises in the case of radiation from a dipole rotator.[1] In accordance with (1.1) and (2.6), the ith component of the flux of radiated angular momentum $F_i^{(M)}$ is equal to the ith projection of the angular momentum dM_{fi} of a spherical layer of thickness dr divided by the time dr/c taken by the wave front to cross this layer:

$$F_i^{(M)}=\oint\frac{dM_{fi}}{dr/c}. \tag{3.3}$$

In its turn, the field angular momentum within the volume $V=r^2 dr d\Omega$ is given by the following expression in the dipole approximation:

$$dM_{fi}=dr/c\,d\Omega\,\frac{1}{2\pi c^3}(\mathbf{n}\dot{\mathbf{d}})[\mathbf{n}\times\ddot{\mathbf{d}}]_i. \tag{3.4}$$

Substituting (3.4) in (3.3) and integrating with respect to Ω we obtain formula for $F_i^{(M)}$ given by (2.2). The formula given by (3.4) is interesting in that it is obtained only if we take into account the terms in the electric field $\mathbf{E}(\mathbf{r},t)$ that fall as $\sim 1/r^2$. This point is usually the subject of discussion.[3,5]

Another feature of (3.4) was also noted in Ref. 1. Suppose that the radiator is a nonrelativistic charged particle revolving at the end of a spring on a circle around the z-axis. The polarization of the radiation emitted by this particle is a function of the polar angle ϑ between the z-axis and the direction $\mathbf{M}$ of an element of the wavefront within the solid angle $d\Omega$. In general, this is an elliptic polarization that becomes linear when $\vartheta=\pi/2$ and circular when $\vartheta=0$ or π. It is clear from physical considerations that the angular momentum is transported by the circularly polarized waves propagating along the z-axis. Moreover, as was shown in Ref. 1, the differential angular momentum flux is a maximum for $\vartheta=\pi/2$ and 0 for $\vartheta=0$ or π. In point of fact, in accordance in (3.4), the average angular momentum $\langle dM_{fz}\rangle$ [i.e., the differential flux, see (3.3)] averaged over time is proportional to $\sin^3\vartheta d\vartheta$. This paradox can be avoided if, in accordance with the foregoing, we consider that it is only the integral angular momentum flux that is physically meaningful.

4. CANONICAL ANGULAR MOMENTUM OF PARTICLES AND FIELD

In Sec. 2, we gave a formal transformation from the conservation law (2.3) for the 'usual' angular momenta $\mathbf{M}_p$ and $\mathbf{M}_f$ to the conservation law (2.8) for the vectors $\mathbf{J}_p$ and $\mathbf{J}_f$. On the other hand, this transformation relies on the well-known ambiguity (see, for example, Refs. 13 and 14) of the energy-momentum tensor and the field angular momentum tensor in classical electrodynamics. In accordance with the Noether theorem, the invariance of the field Lagrangian L_f under 4-translations and 4-rotations leads to

$$\frac{\partial}{\partial x^\mu}T_f^{\mu\nu}=0,\quad \frac{\partial}{\partial x^\eta}M_f^{\eta\rho\sigma}=0, \tag{4.1}$$

where $T_f^{\nu\upsilon}$ is the canonical energy-momentum tensor and $M_f^{\eta\alpha\sigma}$ is the angular momentum tensor ($\sigma_f^{\eta\rho\sigma}$ is the spin tensor)

$$M_f^{\eta\rho\sigma}=T_f^{\eta\rho}x^\sigma-T_f^{\eta\sigma}x^\rho+\sigma_f^{\eta\rho\sigma}. \tag{4.2}$$

In general, the canonical energy momentum tensor $T_f^{\mu\upsilon}$ is screw-symmetric. For example, this is so with the usual choice of the Lagrangian for the free-electromagnetic field: $L_f=-(1/16\pi c)F_{\mu\nu}F^{\mu\nu}$.

Classical electrodynamics usually employs the symmetric energy-momentum tensor $T_{f(\text{sym})}^{\mu\nu}$ and the angular momentum tensor $M_{f(\text{sym})}^{\eta\rho\sigma}$ expressed in terms of $T_{f(\text{sym})}^{\nu\upsilon}$ (see, for example, Ref. 3)

$$M_{f(\text{sym})}^{\eta\rho\sigma}=T_{f(\text{sym})}^{\eta\rho}x^\sigma-T_{f(\text{sym})}^{\eta\sigma}x^\rho. \tag{4.3}$$

The tensors $T_{f(\text{sym})}^{\mu\nu}$ and $M_{f(\text{sym})}^{\eta\rho\sigma}$ differ from $T_f^{\mu\nu}$ and $M_f^{\eta\rho\sigma}$ by total 4-divergences such that (the field is considered free)

$$\frac{\partial}{\partial x^\mu}T_{f(\text{sym})}^{\mu\nu}=0,\quad \frac{\partial}{\partial x^\eta}M_{f(\text{sym})}^{\eta\rho\sigma}=0. \tag{4.4}$$

In accordance with this ambiguity, there are two ways in which we can introduce the angular momentum (more precisely, the pseudovector):

$$J_{fi}=-\frac{1}{2}e_{ijk}\oint_V d^3r M_f^{0jk},$$

$$M_{fi}=-\frac{1}{2}e_{ijk}\oint_V d^3rM^{0jk}_{f(sym)}. \tag{4.5}$$

In 'three-dimensional' notation, the vector $\mathbf{J}_f$ is given by (2.10) and $\mathbf{M}_f$ by (1.1). Since the second term under the integral sign in (2.10) is expressed in terms of the spin tensor $\sigma_f^{\eta\rho\sigma}$, the vector

$$\mathbf{S}_f=(1/4\pi c)\int_V d^3r[\mathbf{E}\times\mathbf{A}]$$

is sometimes referred to as the spin angular momentum of the field.[15,16]

When charged particles are present, the transition from the 'usual' field angular momentum $\mathbf{M}_f$ to the canonical angular momentum $\mathbf{J}_f$ is accompanied, as we have seen in Sec. 2, by the transition from the 'usual' angular momentum $\mathbf{M}_p$ of the particles to the canonical angular momentum $\mathbf{J}_p$ expressed in terms of the canonical momenta $\mathbf{P}_a$ of the particles in the external field. Although this entire argument can be applied to the conservation law (2.8) for the canonical angular momentum, as it was in the case of (2.3), and, accordingly, only changes in $\mathbf{J}_f$ and in integral fluxes $F_i^{(J)}$ can be given physical meaning, we cannot ignore the following well-known fact (see, for example, Ref. 16). When we evaluate the canonical angular momentum of a circularly-polarized plane wave, described in the Coulomb gauge by the vector potential

$$\mathbf{A}(\mathbf{r},t)=\mathrm{Re}A_0(\mathbf{e}_x\pm i\mathbf{e}_y)\exp[-i(\omega t-\mathbf{kr}+\alpha)],$$

we obtain

$$J_{fz}=S_{fz}=\pm\frac{1}{4\pi c}A_0^2kV. \tag{4.6}$$

where V is a spherical volume with center at the origin. Since the energy of a plane wave within V is

$$E_f=\frac{1}{8\pi}\int_V d^3r(\mathbf{E}^2+\mathbf{H}^2)=\frac{1}{4\pi}A_0^2k^2V, \tag{4.7}$$

the relationship between J_{f_z} and E_f is

$$J_{fz}=\pm\frac{E_f}{\omega}, \tag{4.8}$$

so that J_f is effectively the classical pressure original of the angular momentum of the quantized electromagnetic field. This is meaningful for the following reason. The classical analog of the orbital angular momentum operator

$$\hat{\mathbf{L}}_p=-i\hbar\sum_a[\mathbf{r}_a\times\partial/\partial\mathbf{r}_a]$$

is the canonical angular momentum $\mathbf{J}_p$ since it is the canonical momenta $\mathbf{P}_a$ that are replaced with the operators (see, for example, Ref. 17) $-i\hbar(\partial/\partial\mathbf{r}_a)$. This is why $\mathbf{J}_f$, which appears together with $\mathbf{J}_p$, is naturally associated with the angular momentum of the quantized electromagnetic field.

To conclude this Section, we reproduce the formula that relates the vectors $\mathbf{M}_f$ and the $\mathbf{J}_f$ for the free classical electromagnetic field:

$$\mathbf{M}_f=\mathbf{J}_f-\frac{1}{4\pi c}\oint dS_lE_l(\mathbf{r},t)[\mathbf{r}\times\mathbf{A}(\mathbf{r},t)]. \tag{4.9}$$

This relation was first obtained in Ref. 18 where it was also applied to the analysis of the angular momentum $\mathbf{M}_f$ of a 'packet' of a circularly-polarized field localized in space. If we suppose that the surface integral in (4.9) must vanish because of localization, we would expect that for a 'packet' propagating along the z axis, the projection of the angular momentum M_{fz} and the energy E_f should be related by (4.8). This idea is developed in detail in Ref. 2.

5. ANGULAR MOMENTUM OF A CHARGED PARTICLE IN A MAGNETIC FIELD

The difference between the 'usual' and the canonical particle angular momenta of particles and fields becomes explicit when an external magnetic field is present ($\mathrm{A}\neq0$). It is not fortuitous, therefore, that it is precisely in a magnetic field that we immediately encounter angular momentum paradoxes (when the formulas are not properly understood, of course). Here is one of them (other examples can be found in Refs. 19 and 20). A nonrelativistic particle of charge e and mass m travels on a circle of radius r_0, centered on the origin of coordinates in a uniform magnetic field H_0 which points in the direction of the negative z-axis, so that the angular momentum of the particle points along the z-axis (the angular frequency $\omega_0=eH_0/mc$ is independent of the radius). In the dipole approximation, the energy lost by the particle (in a magnetic field, this is only the kinetic energy) and the z-component of the angular momentum would appear to be given by (3.1):

$$\frac{d}{dt}\left(\frac{m\omega_0^2r_0^2}{2}\right)\overset{?}{=}-\frac{2e^2\omega_0^4r_0^2}{3c^3}, \tag{5.1}$$

$$\frac{d}{dt}(m\omega_0r_0^2)\overset{?}{=}-\frac{2e^2\omega_0^3r_0^2}{3c^3}. \tag{5.2}$$

However, these two equations for $r_0(t)$ are clearly incompatible. Such complications do not arise for a particle rotating, for example, on a spring in the absence of a magnetic field: its total energy is greater by a factor of two than its kinetic energy (due to the potential energy), so that equations (5.1) and (5.2) have the same solution $r_0(t)$.

A paradox arises because the law of conservation of angular momentum as given by (2.3) is incorrectly used. On the left-hand side of the equation we must have the angular momentum of the field that arises when we cross the Coulomb field of the charged particle $\mathbf{E}(\mathbf{r},t)=e(\mathbf{r}-\mathbf{r}_0(t))/|\mathbf{r}-\mathbf{r}_0(t)|^3$ with the constant magnetic field $\mathbf{H}=-H_0\mathbf{e}_z$

$$\frac{d}{dt}\left(m\omega_0r_0^2+\frac{1}{4\pi c}\int_V d^3r[\mathbf{r}\times[\mathbf{E}(\mathbf{r},t)\times\mathbf{H}]]_z\right)=-\frac{2e^2\omega_0^3r_0^2}{3c^3}. \tag{5.3}$$

This is precisely the case in which the field angular momentum $\mathbf{M}_f(1.1)$ within the volume V does not depend on time and the physically meaningful quantity is the derivative and not the absolute magnitude of the vector $\mathbf{M}_f$.

It is readily seen that no paradox will arise if we examine the compatibility of the law of conservation of energy (5.1) and the law of conservation of the canonical angular momentum (2.8). Substituting the explicit expression for the vector potential of the uniform magnetic field in (2.9) and (2.10) (ρ is the distance from the z-axis and $\mathbf{e}_\psi$ is the azimuthal unit vector),

$$\mathbf{A}(\mathbf{r})=\tfrac{1}{2}[\mathbf{H}_0\times\mathbf{r}]=-\tfrac{1}{2}H_0\rho\mathbf{e}_\psi=-\frac{c}{2e}m\omega_0\rho\mathbf{e}_\psi, \quad (5.4)$$

we obtain

$$J_{pz}=\tfrac{1}{2}m\omega_0r_0^2, \quad J_{fz}=0. \quad (5.5)$$

The particle canonical angular momentum J_{pz} is now none other but the generalized momentum that is the conjugate of the azimuthal angle ψ. If we neglect radiation, this generalized momentum is conserved, because the charged-particle Lagrangian

$$L=\tfrac{1}{2}m\mathbf{v}^2+\frac{e}{c}\mathbf{v}\mathbf{A}(\rho,z) \quad (5.6)$$

is independent of the azimuthal angle ψ. Accordingly ($\dot{\psi}=\omega_0$)

$$J_{pz}=\frac{\partial}{\partial\dot{\psi}}L=m\omega_0r_0^2+\frac{e}{c}r_0A_\psi. \quad (5.7)$$

The conservation of J_{pz} is used in the analysis of the motion of charged particles in axially symmetric time-independent magnetic fields.[21]

We shall now show that (5.3) is in fact compatible with (5.1). Suppose that a uniform magnetic field exists only in limited portions of spaces, e.g., inside a long solenoid ($R \ll L$ where R,L are the radius and length of the solenoid) or inside a rotating sphere that is uniformally charged on its surface. In any case, the static magnetic field can be described throughout space by potentials satisfying the following gauge conditions:

$$\varphi(\mathbf{r})=0, \quad \mathrm{div}\mathbf{A}(\mathbf{r})=0, \quad \mathbf{A}(\mathbf{r})\to 0 \quad \text{for } r\to\infty. \quad (5.8)$$

The vector potential $\mathbf{A}(\mathbf{r})$ then falls as $1/r^2$ at infinity.[3] With this gauge, the field angular momentum $\mathbf{M}_f$ due to crossed Coulomb field of the charged particle and the external constant magnetic field of arbitrary reconfiguration is given by

$$\mathbf{M}_f=\frac{1}{4\pi c}\int_{V\to\infty}d^3r\left[\mathbf{r}\times\left[\frac{e(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}\times\mathrm{rot}\ \mathbf{A}(\mathbf{r})\right]\right]$$
$$=\frac{e}{c}[\mathbf{r}_0\times\mathbf{A}(\mathbf{r}_0)]. \quad (5.9)$$

Comparing this result with (2.4) and (2.9), we see that, in the gauge defined by (5.8), the choice between the 'usual' ($\mathbf{M}_p$) and the canonical ($\mathbf{J}_p$) angular momentum of the relativistic particle is equivalent to the choice to between two possible ways of taking into account the angular momentum due to interference between the Coulomb field of the particle and the external magnetic field. In the one case, this is the angular momentum of the field whereas in the other it is part of the canonical angular momentum of the particle. It follows that, on the left-hand side of (5.3) the time differentiation symbol is followed by J_{pz} which in turn is given by (5.5) in the region of space in which the field is uniform; as we have already established the variation of J_{pz} is compatible with the variation in the particle energy.

Similarly, the crossed particle field and external static magnetic field generate additional momentum that has the following form in the gauge (5.8):

$$\mathbf{P}_f=\frac{1}{4\pi c}\int_{V\to\infty}d^3r\left[\frac{e(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}\times\mathrm{rot}\ \mathbf{A}(\mathbf{r})\right]$$
$$=\frac{e}{c}\mathbf{A}(\mathbf{r}_0). \quad (5.10)$$

It is precisely this quantity that is the difference between the 'usual' ($\mathbf{p}$) and the canonical ($\mathbf{P}$) momenta of the particle in the field [see (2.9)]. In contrast to (5.9), this result can be found in some textbooks (see, for example, Refs. 22 and 23). The formula given by (5.9) appears to have been given for the first time in a paper on the Aharonov–Bohm effect.[24]

To complete the picture, we must mention that the paradox formulated at the beginning of this Section can also be resolved in a completely different way. The total time derivative of the mechanical angular momentum of the particle is equal to the total moment of forces acting on the particle. Consequently, equation (5.2) can be corrected by adding to its right hand side the moment of the Lorentz force

$$\frac{d}{dt}M_{pz}=-\frac{2e^2\omega_0^3r_0^2}{3c^3}+\frac{e}{c}[\mathbf{r}(t)\times[\dot{\mathbf{r}}(t)\times\mathbf{H}]]_z \quad (5.11)$$

and then integrating the first term on the right-hand side in its role as the moment of the force of radiative friction. The moment of the Lorentz force is not zero because a radiating particle travels on a spiral and not a circle. Since $\mathbf{r}(t)\mathbf{H}=0$ and $\mathbf{H}=-H_0\mathbf{e}_z$, we now transform (5.11) to the form

$$\frac{d}{dt}\left(M_{pz}-\frac{m\omega_0r_0^2}{2}\right)=-\frac{2e^2\omega_0^3r_0^2}{3c^3}, \quad (5.12)$$

which is compatible with conservation of energy (5.1).

This method of resolving the paradox is actually closely related to what we have done before. To see this, consider the similar problem that arises in electrostatics. The change in the kinetic energy $E_p=m\mathbf{v}^2/2$ of a radiating nonrelativistic particle moving in the static electric field $\mathbf{E}(\mathbf{r})=-\mathrm{grad}\varphi(\mathbf{r})$ in a time dt is equal to the total work done by forces acting on the particle during the time dt. Hence,

$$\frac{d}{dt}E_p=-e\dot{r}(t)\text{grad}\varphi(\mathbf{r})-I,. \quad (5.13)$$

where the radiation intensity I is interpreted as the work done by the radiative friction force per unit time. However, (5.13) can also be written in the form

$$\frac{d}{dt}(E_p+e\varphi(\mathbf{r}(t)))=-I, \quad (5.14)$$

which describes the change in the total energy of the particle as it radiates. It is well-known[3] that the quantity $e\varphi(\mathbf{r}(t))$ in the usual gauge employed in electrostatics:

$$\varphi(\mathbf{r})\to 0 \text{ for } r\to\infty, \quad \mathbf{A}(\mathbf{r})=0. \quad (5.15)$$

It can be interpreted as the potential energy of the particle and is equal to the part of the field energy E_p which is due to interference between the Coulomb field of the particle and the external electric field $\mathbf{E}(\mathbf{r})$. We thus see that there is a clear analogy between the transition from (5.13) to (5.14) and from (5.11) to (5.12). Consequently, the additional term that arises under the time differentiation sign on the right-hand side of (5.12) can be interpreted as the 'potential' angular momentum of the charged particle in the time-independent uniform magnetic field, which is numerically equal, as we have seen, to the field angular momentum $\mathbf{M}_f$ (5.9) due to interference between the Coulomb field of the particle and the external magnetic field.

6. CONCLUSION

We can now summarize our results in the form of the following two propositions.

First, the angular momentum arises in the theory as a quantity that satisfies a particular conservation law. This law relates the total derivative of the angular momentum in the volume V and the integral angular-momentum flux across the surface bounding V. To avoid paradoxes, we must take as physically meaningful only *changes* in the field angular momentum in the volume V and not its absolute magnitude, and only the integral flux of the angular momentum across the surface V and not the differential of this flux.

Second, the theory contains both the conservation of the 'usual' angular momentum (2.3) and the canonical angular momentum (2.8). This duality gives rise to an ambiguity in the choice of the energy-momentum tensor and the angular momentum tensor of the classical electromagnetic field. None of the terms in the conservation law (2.8) are formally gauge invariant, so that they are not directly physically meaningful. The point is that in problems solved under particular gauges, the canonical angular momenta may acquire explicit physical significance (see Secs. 2, 4, and 5). It is also appropriate to recall here that $e\varphi(\mathbf{r})$ is also formally gauge noninvariant, but this does not prevent us from widely using the expressions for the potential energy of a particle in an electric field by fixing the gauge condition (5.15). In a constant magnetic field (5.8), the canonical angular momentum (2.9) of a nonrelativistic charge can be interpreted as the total angular momentum of the system in precisely the same way as the generalized momentum of the charge is equal to the total electromagnetic momentum.

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APPENDIX

Substituting the explicit expressions for the stress tensor σ_{kl} (Ref. 3) in (2.5) and (2.12), we obtain the following expressions for the fluxes $F_i^{(M)}$, $F_i^{(J)}$ of the ith components of the 'usual' and the canonical angular momenta across a spherical surface of radius r ($\mathbf{n}=\mathbf{r}/r$)

$$F_i^{(M)}=\frac{1}{4\pi}\oint r^2 d\Omega([\mathbf{H}\times\mathbf{r}]_i(\mathbf{nH})-[\mathbf{r}\times\mathbf{E}]_i(\mathbf{nE})), \quad (A1)$$

$$F_i^{(J)}=\frac{1}{4\pi}\oint r^2 d\Omega(i(\mathbf{nE})\hat{L}_i\varphi+i[\mathbf{H}\times\mathbf{n}]_k\hat{L}_iA_k + [[\mathbf{H}\times\mathbf{n}]\times\mathbf{A}]_i). \quad (A2)$$

In the integral expressions we must, of course, retain only those terms that do not decrease in the wave zone (for $r\to\infty$). Since $\mathbf{n}\cdot\mathbf{H}$ and $\mathbf{n}\cdot\mathbf{E}$ can be nonzero only because of components that decrease as $\sim 1/r^2$, it is sufficient to take into account only the components of $\mathbf{H}$ and $\mathbf{E}$ that falls as $\sim 1/r$ in $\mathbf{H}\times\mathbf{r}$ and $\mathbf{r}\times\mathbf{E}$. However, for these components,

$$\mathbf{H}(\mathbf{r},t)\approx[\mathbf{n}\times\mathbf{E}(\mathbf{r},t)], \qquad \mathbf{E}(\mathbf{r},t)\approx[\mathbf{H}(\mathbf{r},t)\times\mathbf{n}]. \quad (A3)$$

Hence, we have

$$F_i^{(M)}=\frac{1}{4\pi}\oint r^2 d\Omega[\mathbf{r}\times[\mathbf{E}\times\mathbf{H}]]_i. \quad (A4)$$

instead of (A1).

Similarly, since the operator $\hat{L}_i$ in (2.11) operates only on the angular variables, and only components that decrease as $\sim 1/r$ in (2.11) operates only on the angular variables, and only components that decrease as $\sim 1/r$ need to be taken into account in the quantities $\varphi,\mathbf{A},\mathbf{E},\mathbf{H}$ that appear in the integral relation (A2). However, we then have

$$F_i^{(J)}=\frac{1}{4\pi}\oint r^2 d\Omega(iE_k\hat{L}_iA_k+[\mathbf{E}\times\mathbf{A}]_i). \quad (A5)$$

These can be given clear significance if we multiply and divide (A4) and (A5) by dr/c: the fluxes $F_i^{(M)}$ and $F_i^{(J)}$ are thus found to be equal to the projections dM_{fi}, dJ_{fi} of the angular momentum (1.1) and (2.10) of the field in a spherical layer of thickness dr, divided by the time dr/c taken by the wave front to cross this layer.

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