Particle kinetics in highly turbulent plasmas (renormalization and self-consistent field methods)

A. M. Bykov and I. N. Toptygin

A. F. Ioffe Physicotechnical Institute, Russian Academy of Sciences, St. Petersburg State Technical University

(Submitted 25 June 1993; resubmitted after revision 31 August 1993)
Usp. Fiz. Nauk. **160**, 19–56 (November 1993)

This review presents methods available for calculating transport coefficients for impurity particles in plasmas with strong long-wave MHD-type velocity and magnetic-field fluctuations, and random ensembles of strong shock fronts. The renormalization of the coefficients of the mean-field equation of turbulent dynamo theory is also considered. Particular attention is devoted to the renormalization method developed by the authors in which the renormalized transport coefficients are calculated from a nonlinear transcendental equation (or a set of such equations) and are expressed in the form of explicit functions of pair correlation tensors describing turbulence. Numerical calculations are reproduced for different turbulence spectra. Spatial transport in a magnetic field and particle acceleration by strong turbulence are investigated. The theory can be used in a wide range of practical problems in plasma physics, atmospheric physics, ocean physics, astrophysics, cosmic-ray physics, and so on.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

The diffusion of a test particle (passive impurity) in a turbulent medium has been under investigation by statistical methods for many decades (see, for example, Ref. 1), but a complete solution is still eluding us. This is so because the problem is a complex one even in its simplest formulation, and the situation is complicated still further by the increasing range of applications whereby new phenomena, closely related to one another, continue naturally to emerge. This includes, above all, the interaction of charged impurities with turbulent plasmas, and also the diffusion, conduction, and generation of magnetic fields in turbulent conducting media. Hence an adequate description of systems encountered in nature and in laboratory experiments necessarily involves a further complication of the original formulation of the problem as well as a considerable number of special cases.

The transport of an impurity of low concentration $n(\mathbf{r},t)$ that does not affect the dynamics of the host medium in the form of an unionized fluid or gas is described by the equation[2]

$$\frac{\partial n}{\partial t}+\nabla(n\mathbf{u})=\nabla(\varkappa\nabla n). \tag{1.1}$$

In a medium with hydrodynamic-type turbulence, the Euler velocity $\mathbf{u}(\mathbf{r},t)$ is a random function of the coordinates and of time. It exhibits stochastic fluctuations over scales $l\leqslant L$ and times $\tau\leqslant\tau_c\approx L/u$ where L is the principal (maximum) turbulence scale. In a compressible medium, the 'molecular' diffusion coefficient $\varkappa(\mathbf{r},t)$ is a random variable because of fluctuations in the density of the medium. Equation (1.1) is valid provided the turbulent pulsation scale l is large in comparison in the transport mean free path Λ of the impurity particles.

To describe the propagation of an impurity to a distance $R\gg L$ (for example, of the order of the linear dimensions of the systems of a whole), we have to average (1.1) over regions with linear dimensions exceeding L or over an ensemble of turbulent pulsations. These two methods of averaging are equivalent if the correlations in the turbulent medium decay rapidly enough.[3] The averaging procedure should establish the form of the equations for the average concentration $\langle n(\mathbf{r},t)\rangle$ and the coefficients of this equation as functions of the average parameters of the turbulent velocity field. Taylor[1] was the first to show that, for times much greater than the turbulent-velocity correlation time τ_c, the transport of an impurity is asymptotically a diffusion with an effective diffusion coefficient which, for isotropic and homogeneous turbulence, is given by an integral of the correlation function of Lagrange velocities

$$\chi=\frac{1}{3}\int_0^\infty\langle\mathbf{v}(\mathbf{a},t)\mathbf{v}(\mathbf{a},t+\tau)\rangle d\tau; \tag{1.2}$$

where $\mathbf{v}(\mathbf{a},t)$ differs from $\mathbf{u}(\mathbf{r},t)$ in being the Lagrange velocity of an element of the medium located at $\mathbf{a}$ at the initial time, and molecular diffusion is neglected. The relation between the Lagrange and Euler correlation functions has not been rigorously established, and only approximate expressions are available for the turbulent diffusion coefficient χ in terms of the observed Euler turbulence characteristics.[3–5]

When the passive impurity takes the form of charged particles in a plasma with a magnetic field and MHD-type turbulence, the evolution of the impurity differs from the above simple case in at least the following three respects:

1. In addition to transport in space, the particles may become accelerated, so that the transport analysis should describe not only the propagation of the particles in space, but also their energy distribution. The evolution of the spectra of nonthermal particles in astrophysical plasmas is

particularly important in the context of the origin of cosmic rays,[6] but the phenomenon is also significant in the laser plasma[7] and in tokamaks.[8]

2. Transport in space in the presence of a strong magnetic field is sharply anisotropic and takes place mainly along the field. At the same time, anomalous transport is observed across the magnetic field which, as a rule, is faster than predicted 'classical' theories. This applies to the propagation of heat and of impurity ions in thermonuclear installations[9-12] as well as to the diffusion of relativistic particles (cosmic rays) in the Galaxy.[13,14]

3. In cosmic media, the sources of turbulence are often shock waves, and the turbulence itself can assume ultrasonic character, so that it contains an ensemble of shock fronts with a certain distribution over the shock amplitude or values of the Mach number.[16] It was explained quite a long time ago[17-20] that strong shock fronts in a turbulent medium are efficient accelerators of charged particles. This means that studies of the transport of charged particles in turbulent plasmas containing shock fronts is a topical modern problem in high-energy astrophysics. Many problems that involve the above properties of impurity charged particles in plasmas with large-scale MHD-type turbulence can be solved on the basis of the transport equation[21]

$$\frac{\partial N}{\partial t}=\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial N}{\partial r_\beta}-u_\alpha\frac{\partial N}{\partial r_\alpha}+\frac{p}{3}\frac{\partial N}{\partial p}\frac{\partial u_\alpha}{\partial r_\alpha}; \tag{1.3}$$

where $N(\mathbf{r},p,t)$ is the isotropic part of the particle distribution function normalized by the condition

$$\int_0^\infty N(\mathbf{r},p,t)p^2\mathrm{d}p=n(\mathbf{r},t),$$

where $\varkappa_{\alpha\beta}(\mathbf{r},t)$ is the local tensor describing the diffusion of particles due to small-scale electromagnetic fields and Coulomb collisions, $\mathbf{u}(\mathbf{r},t)$ is the Euler turbulent-velocity field whose inhomogeneity scales exceed the local transport mean free path $\Lambda_\parallel$ of the particles which determines the values of the tensor $\varkappa_{\alpha\beta}$. Equation (1.3) describes particle diffusion and convective transport, as well as changes in particle energy due to changes in the density of the medium ($\partial u_\alpha/\partial r_\alpha \neq 0$). In other words, we may say that the effects of changes in energy can be described by explicitly introducing the electric field vector $\mathbf{E}=-\mathbf{u}\times\mathbf{B}/c$ where $\mathbf{B}$ is the magnetic field. Incompressible motion ($\operatorname{div}\mathbf{u}=0$) of the turbulent plasma also leads to particle acceleration, but the contribution of this is smaller than that due to the adiabatic effect represented in (1.3) by the factor $(\Lambda_\parallel/L)^2$ (Ref. 22) where L is, as above, the principal (energy bearing) turbulence scale. If we integrate (1.3) over the entire momentum space, assuming that the diffusion tensor is independent of particle momentum (or if we replace it by its average over the spectrum), we obtain (1.1).

We note that (1.2) describes particles that have already been injected into the acceleration regime. Their velocities are $v\gg u$, and their energy losses in collisions with particles of the host plasma can be neglected. On the other hand, the majority of impurity particles with thermal energies is not involved in the acceleration process because of losses, and the transport of these particles is described by (1.1) (with an anisotropic diffusion coefficient).

The averaging of (1.3) with arbitrary turbulence parameters, especially those that are typical for astrophysical objects, encounters considerable difficulties because the turbulence is strong and the quasilinear theory[23-25] that successfully describes weakly turbulent states ceases to be valid for the problems outlined above.

The theory for the turbulent dynamo[26-28] is another important modern aspect of the problem of passive-impurity diffusion. Such studies became particularly topical after Steenbeck *et al.*,[26] discovered the mechanism responsible for the generation of a magnetic field by gyrotropic turbulence (the α-effect). The theory of this effect is usually based on an examination of the induction equation

$$\frac{\partial \mathbf{H}}{\partial t}=\operatorname{curl}[\mathbf{u}\times\mathbf{H}]+\nu_m\Delta\mathbf{H}, \tag{1.4}$$

where $\nu_m=c^2/4\pi\sigma$ is the magnetic viscosity (magnetic-field diffusion coefficient). In the linear formulation, the turbulent velocity field $\mathbf{u}(\mathbf{r},t)$ and its static characteristics may be assumed to be given, so that the problem reduces to the derivation from (1.4) of the average equations for the large-scale magnetic field and its highest moment, followed by an examination of special cases. Evaluation of the coefficients of the average equations is subject to difficulties similar to those mentioned in connection with (1.1).

The aim of this paper is to provide a review of the methods available for averaging the above equations and for the evaluation of the transport coefficients without imposing restrictions on the amplitudes of the turbulent velocity and the magnetic-field pulsations, or on the correlation lengths and times, i.e., we shall be dealing with strong turbulence. In this respect, the material presented here differs from most published papers in which these limitations are employed in one form or another. Most of the results presented in this review refer to the transport description of the behavior of ensembles of particles in three-dimensional systems with statistically homogeneous MHD-type fluctuations that can be described in terms of low-rank correlation functions. It is also assumed that field correlations decay sufficiently rapidly. An important exception to this restriction is encountered in systems with ensembles of shocks (shock waves) in which intermittency effects are significant and require more detailed statistical information. Such systems are examined in detail in Sec. 7.

In all the above systems, physically different processes can be described by a unifying formalism, namely, the renormalization method. On the other hand, descriptions of systems with special realizations, which rely on percolation and statistical topography techniques, are described in detail in a number of other reviews,[11,29,54] (see, in particular, the recent detailed review by Isichenko[29]) and are mentioned here only in passing.

The structure and content of this review are as follows. In Sec. 2 we examine the simplest methods for averaging equations (1.3)–(1.4) that do not require renormalization. They are valid for sufficiently small turbulent pulsation amplitudes and for sufficiently rapid (in the limit, infinite) destruction of correlations between turbulence variables. These conditions are not satisfied in many real systems, so that nonperturbative approaches to these problems have to be developed.

In Sec. 3 we briefly describe, the most productive approaches used by different authors to the evaluation of the transport coefficients for impurity particles in systems with strong MHD-type turbulence. We have tried to cover all published papers in which the transport coefficients have been calculated in some particular approximation and the corresponding transport equations were constructed. General treatments are mentioned only in passing.

We emphasize that particular attention is devoted in this review to impurity particle kinetics, so that we cannot pretend that we have provided a presentation of the renormalization method for the investigation of strong turbulence dynamics, in which there have been considerable recent advances and which has been the subject of several recent reviews.[50,51] This topic is briefly examined in Sec. 3, but only to establish the relation between strong turbulence theories and theories of turbulent transport.

Most of this review (Secs. 4–8) is devoted to the presentation and application of the method developed by the authors in recent years for the description of spatial transport and acceleration of charged particles in plasmas with strong MHD-type turbulence. We hope that this will be of particular interest because this approach provides us with a unified method for solving a number of difficult problems whose solution is essential for the understanding of the physics of nonequilibrium processes in tenuous plasmas (especially in astrophysical studies) and which, as far as we know, have not been solved before.

In Sec. 4 we consider the fundamentals of the method and examine calculations of transport coefficients in systems with strong turbulence without shock fronts. We go on to analyze different special cases, including the process of strong acceleration of particles within the correlation length, which can be described by equations other than the Fokker-Planck equations.

In Sec. 5 we apply our method to the derivation of the equations for the large-scale magnetic field in the theory of the turbulent dynamo. We find that it is possible to calculate the renormalized turbulent viscosity and the coefficient of magnetic-field generation (α effect) without resorting to perturbation theory or the δ-correlated turbulence model.

Section 6 is devoted to an analysis of the transport of particles across a large-scale magnetic field when there are stochastic magnetic-field fluctuations and velocities of arbitrary amplitude in the medium. It is precisely such conditions that are typical for the Galaxy. The transverse diffusion coefficient had not been previously considered for arbitrary fluctuations amplitudes.

Section 7 examines turbulent systems in which random ensembles of shock waves with arbitrary Mach numbers are excited. They can be looked upon as systems with strong intermittency. Studies of particle transport in such systems are important because they are commonly encountered in astrophysics. Our Galaxy as a whole and its individual active regions in particular are plasma systems with strong turbulence and ensembles of random shocks.

Finally, in Sec. 8 we give a brief resumé of the method and discuss the advantages and disadvantages of our theory as well as possible future developments and more rigorous derivations.

2. IMPURITY TRANSPORT IN A MEDIUM WITH LONG-WAVE TURBULENCE

In this Section, we examine the conditions under which the averaging of the initial equation can be accomplished by simple methods.

2.1. Perturbation theory

We begin with the incompressible motion ($\operatorname{div}\mathbf{u}=0$) with zero average velocity: $\langle\mathbf{u}\rangle=0$. To average (1.3) over the ensemble of turbulent motions, we take the distribution function in the form

$$N(\mathbf{r},p,t)=F(\mathbf{r},p,t)+\delta F(\mathbf{r},p,t),$$
$$F=\langle N\rangle,\quad |\delta F|\ll F. \tag{2.1}$$

The inequality in (2.1) is the condition for the validity of perturbation theory. In this case, the acceleration effect is absent and the local diffusion tensor $\varkappa_{\alpha\beta}$ can be regarded as given and nonfluctuating. Substitution of (2.1) in (1.3) yields the following set of two equations:

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial F}{\partial r_\beta}-\left\langle u_\alpha\frac{\partial\delta F}{\partial r_\alpha}\right\rangle, \tag{2.2}$$

$$\frac{\partial\delta F}{\partial t}-\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial\delta F}{\partial r_\beta}=-u_\alpha\frac{\partial F}{\partial r_\alpha}. \tag{2.3}$$

The latter equation has been linearized in the fluctuating quantities $u_\alpha,\delta F$.

If we look upon (2.3) as the inhomogeneous diffusion equation, we can write its solution in terms of the Green function $G(\mathbf{r},\mathbf{r}',t,t')$ which contains the local diffusion tensor $\varkappa_{\alpha\beta}$:

$$\delta F(\mathbf{r},p,t)=\int G(\mathbf{r},\mathbf{r}',t,t')u_\alpha(\mathbf{r}',t')\,\frac{\partial F(\mathbf{r}',p,t')}{\partial r'_\alpha}\,\mathrm{d}^3r'\mathrm{d}t'. \tag{2.4}$$

Substitution of (2.4) and (2.2) gives the average transport equation that takes into account turbulent diffusion in the incompressible medium:

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial F}{\partial r_\beta}+\frac{\partial}{\partial r_\alpha}\int G(\mathbf{r},\mathbf{r}',t,t')$$
$$\times\langle u_\alpha(\mathbf{r},t)u_\beta(\mathbf{r}',t')\rangle\,\frac{\partial}{\partial r'_\alpha}F(\mathbf{r}',p,t')\mathrm{d}^3r'\mathrm{d}t'. \tag{2.5}$$

If the medium is stationary and homogeneous, the Green's function and the correlation tensor of the Euler velocities depend only on the differences between the arguments, and equation (2.5) reduces to

$$\frac{\partial F}{\partial t}=\varkappa_{\alpha\beta}\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}+\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\int G(\mathbf{r}-\mathbf{r}',t-t')$$

$$\times\langle u_\alpha(\mathbf{r},t)u_\beta(\mathbf{r}',t')\rangle F(\mathbf{r}',p,t')\mathrm{d}^3r'\mathrm{d}t'. \quad (2.6)$$

Averaging over the ensemble of turbulent motion is thus seen to lead to an integrodifferential equation with a nonlocal interaction within the correlation length L and correlation time τ_c, which are determined by the properties of the correlation velocity tensor $\langle u_\alpha u'_\beta\rangle$. If we are interested in the behavior of the distribution function over times and distances that are significantly greater than the turbulence correlation time and length, then (2.6) can be simplified. According to the ergodic theorem for homogeneous stationary random processes,[3] averaging over the ensemble is then equivalent to averaging over time or over space. The average distribution function F then varies slowly within the range of integration, and equation (2.6) assumes the differential form

$$\frac{\partial F}{\partial t}=\chi_{\alpha\beta}\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta} \quad (2.7)$$

$$\chi_{\alpha\beta}=\varkappa_{\alpha\beta}+\int \mathrm{d}^3\rho\int_0^\infty \mathrm{d}\tau G(\vec{\rho},\tau)K_{\alpha\beta}(\vec{\rho},\tau), \quad (2.8)$$

with the effective diffusion coefficient given by (2.8) in which $K_{\alpha\beta}(\vec{\rho},\tau)$ represents the turbulent velocity correlation tensor.

If we estimate the second correction to the distribution function and compare it with the first correction in (2.4), we find that the expansion parameter depends on the nature of the turbulence. For strong turbulence in an incompressible medium (vortices), the expansion parameter is the Péclet number $\beta=uL/\varkappa$ (where $u\equiv\langle u^2\rangle^{1/2}$ and local diffusion is isotropic, $\varkappa_{\alpha,\beta}=\varkappa\delta_{\alpha\beta}$. When a sufficiently strong magnetic field is present, so that $c_A\gg\varkappa/L$ where $c_A=B(4\pi\rho)^{-1/2}$ is the Alfven velocity, the expansion parameter is u/c_A (weak Alfven waves). The equation (2.7) and (2.8) are therefore valid with one of the following two inequalities is satisfied:

$$uL\ll\varkappa, \quad u\ll c_A. \quad (2.9)$$

The correction in (2.8) to the local diffusion tensor is quadratic in one of the above small parameters.

For compressible motion of a turbulent medium ($\operatorname{div}\mathbf{u}\neq 0$), the averaging of (1.3) is a more complicated task because we then have to take into account the acceleration term containing $\partial M/\partial p$ and also the fluctuations in the local diffusion tensor $\varkappa_{\alpha\beta}$. We shall assume that these fluctuations are due to changes in the particle number density $n(\mathbf{r},t)$ in the turbulent medium. In perturbation theory

$$\varkappa_{\alpha\beta}=\overline{\varkappa_{\alpha\beta}}+\overline{\delta\varkappa_{\alpha\beta}}\frac{\delta n}{n_0}, \quad \overline{\delta\varkappa_{\alpha\beta}}=\overline{\frac{\partial\varkappa_{\alpha\beta}}{\partial n}}n\bigg|_0. \quad (2.10)$$

If we adopt the above procedure, we obtain the following average equations for a statistically homogeneous medium:

$$\frac{\partial F}{\partial t}=\chi_{\alpha\beta}\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D(p)+D^*_{\alpha\beta}(p)\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\right)$$

$$\times\frac{\partial F}{\partial p}+\frac{2}{3}p\frac{\partial}{\partial p}\chi^{(3)}_{\alpha\beta}\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}, \quad (2.11)$$

where

$$\chi_{\alpha\beta}=\overline{\varkappa_{\alpha\beta}}+\chi^{(1)}_{\alpha\beta}+\chi^{(2)}_{\alpha\beta}+\chi^{(3)}_{\alpha\beta},$$

$$\chi^{(1)}_{\alpha\beta}=-\int \mathrm{d}^3\rho\int_0^\infty \mathrm{d}\tau\rho_\alpha\langle u_\beta u'_\gamma\rangle\frac{\partial G}{\partial\rho_\gamma}, \quad (2.12)$$

$$\chi^{(2)}_{\alpha\beta}=\overline{\delta\chi_{\alpha\nu}}\,\overline{\delta\chi_{\beta\mu}}\int \mathrm{d}^3\rho\int_0^\infty \mathrm{d}\tau\frac{\partial^2 G(\rho,\tau)}{\partial\rho_\nu\partial\rho_\mu}\frac{\langle\delta n\delta n'\rangle}{n_0^2}, \quad (2.13)$$

$$\chi^{(3)}_{\alpha\beta}=-\overline{\delta\chi_{\alpha\nu}}\int \mathrm{d}^3\rho\int_0^\infty \mathrm{d}\tau\frac{\partial G}{\partial\rho_\nu}\rho_\beta\left\langle\operatorname{div}\mathbf{u}\frac{\partial n'}{n_0}\right\rangle, \quad (2.14)$$

$$D(p)=-\frac{p^2}{9}\int \mathrm{d}^3\rho\int_0^\infty \mathrm{d}\tau\frac{\partial^2 G(\rho,\tau)}{\partial\rho_\nu\partial\rho_\mu}\langle u_\mu u'_\nu\rangle, \quad (2.15)$$

$$D^*_{\alpha\beta}(p)=-\frac{p^2}{18}\int \mathrm{d}^3\rho\int_0^\infty \mathrm{d}\tau\frac{\partial^2\langle u_\mu u'_\nu\rangle}{\partial\rho_\nu\partial\rho_\mu}G(\rho,\tau)\rho_\alpha\rho_\beta. \quad (2.15a)$$

in which all the correlation functions depend on $\vec{\rho}=\mathbf{r}-\mathbf{r}'$ and $\tau=t-t'$. The quantities ν and δn are related by the continuity equation

$$\frac{\delta n(\mathbf{r},t)}{n_0}=-\int_{-\infty}^{t}\operatorname{div}\mathbf{u}(\mathbf{r},\tau)\mathrm{d}\tau. \quad (2.16)$$

A clearer picture of the transport coefficients emerges if written in terms of the Fourier transforms of the correlation functions. Let

$$\langle u_\alpha(\mathbf{r},t)u_\beta(\mathbf{r}',t')\rangle=\int K_{\alpha\beta}(\mathbf{k},\omega)e^{i(\mathbf{k}\vec{\rho}-\omega\tau)}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}, \quad (2.17)$$

where for homogeneous and isotropic turbulence

$$K_{\alpha\beta}(\mathbf{k},\omega)=T(k,\omega)(\delta_{\alpha\beta}-k_\alpha k_\beta k^{-2})$$

$$+S(k,\omega)k_\alpha k_\beta k^{-2}. \quad (2.18)$$

We now specify explicitly the dependence on frequency by introducing the correlation time for the harmonics $\tau_c(k)=\Gamma_k^{-1}$:

$$S(k,\omega)=S(k)\frac{k}{2}\left[\frac{1}{(\omega-\omega_0)^2+(_k^2/4)}\right.$$

$$\left.+\frac{1}{(\omega+\omega_0)^2+(_k^2/4)}\right].$$

We have chosen the Lorentz or the dispersion form of the frequency dependence of the spectral function because it is universal and simple.

Transforming to the spectral functions in (2.12)–(2.15) and integrating with respect to frequency, we obtain an expression for the transport coefficients in the form of single integrals of the spectral functions $T(k)$ and $S(k)$. The diffusion coefficient in momentum space is given by

$$D(p)=\frac{p^2}{9}\int d^3k k^2 S(k)\,\frac{\bar{\varkappa}k^2+(\Gamma_k/2)}{\omega_0^2(k)+[\varkappa k^2+(\Gamma_k/2)]^2}. \tag{2.19}$$

The expression given by (2.19) shows that there is a significant dependence on both the correlation time (or the broadening of the resonances Γ_k) and on the spatial diffusion coefficient $\bar{\varkappa}$. As $\Gamma_k\to 0$ and $\omega_0(k)=v_{\rm ph}k$ (weak acoustic or magnetoacoustic turbulence), we obtain from (2.19) the result reported in Ref. 30: for weak turbulence, we have a superposition of monochromatic waves with different phases, and there is no acceleration effect in the limit as $\bar{\varkappa}\to 0$. The conditions for the validity of (2.19) are then

$$\langle u_{\rm pot}^2\rangle=4\pi\int_0^\infty dk k^2 S(k)\ll v_{\rm ph}^2. \tag{2.20}$$

If the broadening of the resonances Γ_k is finite, the particles are accelerated even in a system with very strong particle scattering ($\bar{\varkappa}\to 0$):

$$D(p)=\frac{p^2}{9}\int d^3k k^2 S(k)\,\frac{\Gamma_k/2}{\omega_0^2(k)+(\Gamma_k^2/4)}. \tag{2.21}$$

The conclusions about the role of diffusion and finite correlation time of turbulence are quite general and unrelated to particular forms of the function $S(k,\omega)$. In particular, analogous results are obtained for a Gaussian dependence of the spectrum on frequency.[31]

The presence of the acceleration effect in the limit of strong scattering ($\bar{\varkappa}\to 0$) and finite correlation time $\tau_c(k)=\Gamma_k^{1}$ is due to the presence in this case of the Fermi stochastic velocity field that is necessary for acceleration. As $\tau_c\to\infty$, the velocity field is found to consist of stationary modes into which the particles are 'frozen', so that there is no acceleration. The turbulent increments $\chi_{\alpha\beta}^{(1)}$, $\chi_{\alpha\beta}^{(2)}$, and $\chi_{\alpha\beta}^{(3)}$ in the spatial diffusion are also small in this case and are of the order of $u^2/v_{\rm ph}^2$. The vortical and potential components of the velocity provide comparable contributions to $\chi_{\alpha\beta}^{(1)}$.

We note that the above criteria for the validity of perturbation theory may cease to be valid in the presence of strong large-scale turbulence. The effective phase velocity $v_{\rm ph}\approx u$ and the Péclet number β are then found to be large for impurities in the atmosphere and in the ocean, and also for the nuclei of heavy elements and moderate-energy cosmic rays in the Galaxy. The validity of perturbation theory for the equation of induction (1.4) in the case of an incompressible medium is restricted by the condition that the magnetic Reynolds number must be small ($\mathrm{Re_m}=uL/\nu_{\rm m}\ll 1$), and this condition is also violated for most astrophysical objects.

2.2. Short correlation times

In order to emerge from the confines of perturbation theory, it is usual[32–34] to consider a hypothetic turbulence in which turbulent velocities have zero correlation time. In particular, this corresponds to the correlation tensor

$$K_{\alpha\beta}(\vec{\rho},\tau)=K_{\alpha\beta}(\vec{\rho})\cdot 2\tau_0\delta(\tau). \tag{2.22}$$

An analogous assumption can be made in relation to other correlation tensors. We note that δ-correlated turbulence (2.22) is a simplified model that does not correspond to any particular real case and is interesting only to the extent that averaging of equations such as (1.3) and (1.4) can be performed exactly. In reality, the correlation time is finite and of the order given by $\tau_c\approx L/u$ for strong turbulence and $\tau_c\approx L/v_{\rm ph}$ for weak turbulence.

Let us now average (1.4) by a method that is different from that used in Refs. 33 and 34. In the case of short correlation time (in the limit, zero correlation time), the turbulent velocities for $\tau<t$ and $\tau>t$ are uncorrelated. Any variable that depends upon them can then be averaged independently within these two time intervals. Let

$$\mathbf{B}(\mathbf{r},t)=\langle\mathbf{H}(\mathbf{r},t)\rangle \tag{2.23}$$

and take (1.4) in the form

$$\frac{\partial H_\alpha}{\partial\tau}-\nu_{\rm m}\Delta H_\alpha=A_{\gamma\sigma}^{\alpha\mu}\frac{\partial}{\partial r_\alpha}u_\gamma H_\sigma + B_\alpha\delta(\tau-t),\quad \tau\geqslant t, \tag{2.24}$$

where $A_{\gamma\sigma}^{\alpha\mu}=e_{\alpha\mu\nu}e_{\nu\gamma\sigma}$ in which the δ-term is effectively the initial condition for the unaveraged field $\mathbf{H}(\mathbf{r},t)$. The integral form of (2.24) is

$$H_\alpha(\mathbf{r},\tau)=A_{\gamma\sigma}^{\alpha\mu}\int d\tau'\int d^3r' G(\mathbf{r}-\mathbf{r}',\tau-\tau')\times\frac{\partial}{\partial r'_\mu}u'_\gamma H'_\sigma+B_\alpha^{(0)}(\mathbf{r},\tau), \tag{2.25}$$

where

$$B_\alpha^{(0)}(\mathbf{r},\tau)=\int d^3r' G(\mathbf{r}-\mathbf{r}',\tau-t)\,B_\alpha(\mathbf{r}',t), \tag{2.26}$$

and G is the Green's function of the diffusion operator

$$\frac{\partial}{\partial t}-\nu_{\rm m}\Delta,$$

satisfying the condition

$$G(\mathbf{r},\tau)\xrightarrow[\tau\to 0]{}\delta(\mathbf{r}).$$

We can now use (2.25) and the iteration method to calculate H_α at time $t+\Delta t$ to within terms that are linear in Δt. Since $H_\alpha(\mathbf{r},t+\Delta t)$ depends explicitly on the turbulent velocities, averaging over them can be performed directly. If we then assume that Δt is small, we obtain the differential equation for the mean field $\mathbf{B}(\mathbf{r},t)$.

To implement this program, we must first perform two iterations of (1.25). The quantity

$$B_\alpha^{(0)}(\mathbf{r},\tau)=B_\alpha(\mathbf{r},t)+(\tau-t)\nu_{\rm m}\Delta B_\alpha(\mathbf{r},t)$$

can then be looked upon as the zero-order iteration in the random velocity. The second iteration takes the form

$$H_\alpha^{(2)}(\mathbf{r},t+\Delta t)$$

$$=A^{\alpha\mu}_{\gamma\sigma}A^{\sigma\nu}_{\varkappa\varepsilon}\int_t^{t+\Delta t}d\tau_1\int d^3r_1G(\mathbf{r}-\mathbf{r}_1,\tau-\tau_1)$$

$$\times\frac{\partial}{\partial r_{1\mu}}u_\gamma(\mathbf{r}_1,t_1)\int_t^{\tau_1}d\tau_2\int d^3r_2G(\mathbf{r}_1-\mathbf{r}_2,\tau_1-\tau_2)$$

$$\times\frac{\partial}{\partial r_{2\mu}}u_\gamma(\mathbf{r}_2,t_2)B_\varepsilon^{(0)}(\mathbf{r}_2,\tau_2)+A^{\alpha\mu}_{\gamma\sigma}\int_t^{t+\Delta t}d\tau_1$$

$$\times\int d^3r_1G(\mathbf{r}-\mathbf{r}_1,\tau-\tau_1)$$

$$\times\frac{\partial}{\partial r_{1\mu}}u_\gamma(\mathbf{r}_1,t_1)B_\alpha^{(0)}(\mathbf{r}_1,\tau_1)+B_\alpha^{(0)}(\mathbf{r},t+\Delta t). \quad (2.27)$$

When we take the average of the last equation, we shall consider as before that the turbulence is uniform and isotropic, $\langle\mathbf{u}\rangle=0$, but is not invariant under reflection of the spatial axes (gyrotropy). It is only then that we have instability[28] with respect to a growing magnetic field. In contrast to (2.18), the velocity correlation tensor should now contain a further term with a pseudoscalar that represents gyrotropy, so that in the coordinate representation

$$K_{\alpha\beta}(\vec{\rho})=A(\rho)\delta_{\alpha\beta}+R(\rho)\rho_\alpha\rho_\beta+C(\rho)e_{\alpha\beta\sigma}\rho_\sigma \quad (2.28)$$

where A and R are related to T and S in (2.18) and (2.18′) by

$$T(k)=\tilde{A}(k)-\frac{d\tilde{R}(k)}{kdk},$$

$$S(k)=T(k)-\frac{d^2\tilde{R}(k)}{dk^2}, \quad (2.29)$$

in which $\tilde{A}(k),\tilde{R}(k)$ are the Fourier transforms of the functions $A(\rho),R(\rho)$.

If we average (2.27) with the help of (2.22) and (2.28), and assume that the field $\mathbf{B},(\mathbf{r},t)$ is smooth within the correlation length, we obtain the following equation:[35]

$$\frac{\partial\mathbf{B}}{\partial t}=\alpha\,\mathrm{curl}\,\mathbf{B}+(\nu_m+\nu_{turb})\Delta\mathbf{B}, \quad (2.30)$$

where

$$\nu_{turb}=A(0)\tau_c=\tfrac{1}{3}\langle u^2\rangle\tau_c,$$

$$\alpha=-2C(0)\tau_c=\tfrac{1}{3}\langle\mathbf{u}(\mathbf{r})\mathrm{curl}\,\mathbf{u}(\mathbf{r})\rangle\tau_c. \quad (2.30')$$

In the model of sharply-correlated turbulence, equations (2.30)–(2.30′) are valid for arbitrary Re_m, including $\mathrm{Re}_m\gg 1$.

Similarly, equation (1.3) can be readily averaged in this model of turbulence. The final result is

$$\frac{\partial F}{\partial t}=\chi_{\alpha\beta}\frac{\partial^2F}{\partial r_\alpha\partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D(p)\frac{\partial F}{\partial p}$$

$$+\frac{2}{3}p\frac{\partial}{\partial p}\chi'_{\alpha\beta}\frac{\partial^2F}{\partial r_\alpha\partial r_\beta}+\chi^{\alpha\beta}_{\mu\nu}\frac{\partial^4F}{\partial r_\alpha\partial r_\beta\partial r_\mu\partial r_\nu}; \quad (2.31)$$

where

$$\chi_{\alpha\beta}=\langle\bar{\chi}_{\alpha\beta}\rangle+\langle u_\alpha u'_\beta\rangle\tau_c-\left\langle\frac{\partial\delta\varkappa_{\alpha\nu}}{\partial r_\nu}\frac{\partial\delta\varkappa_{\mu\beta}}{\partial r_\mu}\right\rangle\tau_c$$

$$+\left\langle\delta\varkappa_{\alpha\beta}(\nabla\mathbf{u})\tau_c+\left\langle\delta\varkappa_{\nu\beta}\frac{\partial u_\alpha}{\partial r_\nu}-\delta\varkappa_{\beta\nu}\frac{\partial u_\alpha}{\partial r_\nu}\right\rangle\tau_c,\right.$$

$$\chi'_{\alpha\beta}=\langle\delta\varkappa_{\alpha\beta}(\nabla\mathbf{u})\rangle\tau_c,\quad \chi^{\alpha\beta}_{\mu\nu}=\langle\delta\varkappa_{\alpha\beta}\delta\varkappa_{\mu\nu}\rangle\tau_c,$$

$$D(p)=\frac{p^2}{9}\langle(\nabla\mathbf{u})^2\rangle\tau_c, \quad (2.32)$$

and $\delta\varkappa_{\alpha\beta}(\mathbf{r},t)$ is the fluctuating part of the diffusion coefficient. All the averages are evaluated at the same point in space.

Let us now consider the range of validity of (2.31). The δ-correlation of turbulence means that the actual correlation time L/u must be shorter than all other times and, in particular, shorter than the time $L^2/\bar{\varkappa}$ taken by a particle to cross the correlation region. This leads to the condition

$$uL/\bar{\varkappa}\gg 1. \quad (2.33)$$

Since $\varkappa<0$ and $\delta\varkappa\ll\bar{\varkappa}$, it follows that, in the zero order in the parameter of (2.33), the average equation assumes the form

$$\frac{\partial F}{\partial t}=\chi_{\alpha\beta}\frac{\partial^2F}{\partial r_\alpha\partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D(p)\frac{\partial F}{\partial p}, \quad (2.34)$$

and $\chi_{\alpha\beta}$ is determined exclusively by the contribution due to the turbulent velocity. This case was examined in our previous paper.[36]

If, on the other hand, we omit the acceleration terms from (2.31), we obtain the following fourth-order differential equation for isotropic diffusion:

$$\frac{\partial F}{\partial t}=\chi\Delta F+\eta\Delta^2F. \quad (2.35)$$

For scales $\Delta r\gg l=(\eta/\chi)^{1/2}$, the last term in (2.35) is small and solutions of the above equation have the usual diffusion form, but the character of the solution changes[37] for scales $\Delta r>1$. It is, however, important to remember that, by virtue of the criterion given by (2.33), the last term in (2.35) can only be a small correction and the length l should be small in comparison with the correlation length L. The question of possible intermittency in the distribution of a diffusing impurity must then be solved by starting with the transport equation and not the diffusion equation given by (2.35). This case is discussed in Sec. 4.

3. RENORMALIZATION METHODS FOR TRANSPORT COEFFICIENTS IN HYDRODYNAMICS AND IN PLASMA PHYSICS

The examples cited in the last Section demonstrate the limited range of approximate methods and the need for a more rigorous theory in which limitations such as (2.9) or (2.33) can be relaxed and it is possible to take correctly into account the finite correlation time between the harmonics, which has a significant effect on the rate of acceleration of charged particles. The required theory must take into account with sufficient precision the interaction between the diffusing impurity and the moving medium over correlation times and lengths prevailing in the turbulence. In this sense, the problem of averaging (1.3) and (1.4) with $uL \gg \varkappa$, $uL \gg \nu_m$, $u \gg v_{ph}$ is a typical problem in the theory of strong interactions for a stochastic process. It is not surprising that the many attempts to solve this problem and the problems of turbulence dynamics have exploited the methods of the theory of strong interactions developed in field theory, nuclear physics, and solid-state physics.

3.1. Descriptions of turbulent transport in hydrodynamics

In 1961, Wyld[38] and Kraichnan[39] formulated a perturbation theory for the solution of the Navier–Stokes equations in the dynamics of incompressible fluids under the influence of external forces. This approach can be used to obtain a formal solution in the form of a series which, as a rule, can readily be used in special cases.

At present, the most widely used method transplanted from the theory of strong interactions and quantum field theory into turbulence theory is the renormalization group method.[48–54] Yakhot and Orszag[49] and Dannevik *et al.*[52] have carried out a renormalization-group analysis of the Navier-Stokes equation for an incompressible fluid in the inertial interval of the turbulent spectrum. The effect of the source of turbulent energy arising in the fluid from the side of energy-containing scales was modeled by random forces with δ-correlation in time. The authors of Ref. 49 and 52, who did not use any adjustable parameters, calculated the spectral densities of the turbulent energy $E(k)$ and the passive scalar impurity $E_n(k)$, including the relevant numerical constants:

$$E(k)=1.617\varepsilon^{-2/3}k^{-5/3},$$

$$E_n(k)=1.146\bar{R}\bar{\varepsilon}^{1/3}k^{-5/3}, \tag{3.1}$$

where $\bar{\varepsilon}$ is the rate of dissipation of the turbulent energy per unit volume and the rate of dissipation of fluctuations in the scalar impurity is given by

$$\bar{R}=\frac{\partial}{\partial t}\frac{1}{2}\int n^2(r,t)\mathrm{d}^3r. \tag{3.2}$$

They also calculated the universal ratio of the renormalized (i.e., including the turbulent motion) kinematic viscosity ν to the turbulent diffusion coefficient χ for high Reynolds numbers:

$$\nu/\chi=0.7179. \tag{3.3}$$

Particular attention was concentrated in all these papers on incompressible fluids and, correspondingly, the Kolmogorov model of turbulence. It is therefore difficult to transfer this method directly to plasma systems, and to include the magnetic field and particle acceleration, although attempts have been made[56] to use the renormalization-group approach in the analysis of anisotropic diffusion in large-scale turbulent shear flow. It is appropriate at this point to note that the renormalization method has very extensive possible applications in the solution of the problem of anomalous transport of angular momentum during disk accretion of matter on gravitating centers, which is an important topic in theoretical astrophysics.[57] Anomalous viscosity in the accretion disk may be due to fluctuations associated with the universal instability established in Ref. 58.

A simplified form of the renormalization method as applied to the diffusion of a scalar impurity was discussed earlier by Howells[55] (see also Moffat[5]). We shall consider their approach in greater detail, since it allows a generalization to the transport and acceleration of charged particles in systems with strong fluctuations in magnetic and electric fields (see Sec. 4). The method consists of the following. Suppose that we have a statistical turbulent-velocity field u in an incompressible fluid. Let us resolve this field into a large number of components u_r, $r=1,2,...s$. We assign to the rth component the portion of the field with Fourier harmonics whose wave vectors k fall into a spherical shell of thickness $\Delta_r=k_r-k_{r+1}$, $k_r<k<r_{r+1}$, where we are assuming an isotropic distribution of wave vectors. A similar expansion is made for the concentration of this scalar impurity:

$$n=\langle n\rangle+\sum_{r=1}^{s} n_r, \tag{3.4}$$

where $\langle n\rangle$ describes the concentration averaged over the largest turbulence scale L and n_r are small-scale components for which $\langle n_r\rangle=0$. The averaging of the static transport equation

$$\mathbf{u}\nabla n=\varkappa\Delta n \tag{3.5}$$

is performed successively over spherical layers in k space, beginning with the smallest scales (largest k). Perturbation theory can then be used in each spherical layer because the expansion parameter, which does not exceed $u_r,l_r/\varkappa$, is small because u_r is small. When the averaging procedure is applied to the rth layer, we take into account the motion of smaller scales, but totally ignore transport by larger-scale motion, which is probably the main disadvantage of the method.

When we consider transport under the influence of the velocity field u_s, we replace u with u_s in (3.5) and employ the perturbative procedure (see Sec. 2.1). This leads to the following two equations:

$$\mathbf{u}_s\nabla n_s'=\varkappa\Delta n_s,\quad \langle\mathbf{u}_s\nabla\mathbf{n}_s\rangle=\varkappa\Delta n_s'; \tag{3.6}$$

where $n_s' = \langle n\rangle + \Sigma_{r=1}^{s-1}n_r$ is the 'large-scale' (as compared with n_s) concentration. Next, we use the first of the above equations to express n_s in terms of n_s' and evaluate the average

$$\langle u_s n_s\rangle=-\chi_s\Delta n_s', \tag{3.7}$$

where χ_s is the contribution of the field u_s to the turbulent diffusion coefficient:

$$\chi_s=\frac{1}{3\varkappa}\int_{(\Delta_s)}k^{-2}\langle u^2\rangle_k\mathrm{d}k. \tag{3.8}$$

The quantity $\langle u^2\rangle_k$ represents the turbulent spectral density normalized by the condition $\int_0^\infty\langle u^2\rangle_k dk=\langle u^2\rangle$.

The total diffusion coefficient that includes molecular diffusion and turbulent pulsations is then equal to the sum $\varkappa+\chi_s$. The unaveraged part of the impurity concentration $\langle n\rangle + n_1 + ...n_s - 1 = n_s'$ evolves with this effective diffusion coefficient in the turbulent field $u_1+u_2+...u_{s-1}$.

Similarly, if we take the average of the field u_{s-1} in the next spherical layer, we obtain the following addition to the effective diffusion coefficient:

$$\chi_{s-1}=\frac{1}{3(\varkappa+\chi_s)}\int_{(\Delta_{s-1})} k^{-2}\langle u_k\rangle \mathrm{d}k. \tag{3.9}$$

Repeating this procedure and assuming that the thickness of each spherical layer is small ($\Delta_r \to dk$), we obtain the differential relation

$$\mathrm{d}\chi(k)=\frac{\langle u^2\rangle_k \mathrm{d}k}{3[\varkappa+\chi(k)]k^2}, \tag{3.10}$$

where $\chi(k)$ is the effective turbulent diffusion coefficient that includes a contribution due to all harmonics with wave vectors greater than k. We note once again that it is calculated without taking into account large-scale turbulence with wave vectors smaller than k. Integrating (3.10) subject to the boundary condition $\chi(\infty)=0$, we obtain

$$[\varkappa+\chi(k)]^2=\tfrac{2}{3}\int_k^\infty k'^{-2}\langle u^2\rangle_{k'}\mathrm{d}k'+\varkappa^2. \tag{3.11}$$

The diffusion coefficient χ_0 that takes into account the entire turbulent velocity field can be obtained by going to the limit as $k\to 0$. When the Péclet number is large on the main turbulence scale, so that $uL/\varkappa \gg 1$, molecular diffusion is relatively unimportant and (3.11) gives

$$\chi_0=\left(\tfrac{2}{3}\int_0^\infty k^{-2}\langle u^2\rangle_k \mathrm{d}k\right)^{1/2}. \tag{3.12}$$

We then have the order-of-magnitude relation $\chi_0 \approx uL$, which is a reasonably quantitative estimate for turbulent diffusion.

A model of the transport of a scalar impurity by a stationary random turbulent-velocity field is investigated in Refs. 34 and 59. The velocity field can contain vortical and potential components. It is important to remember that these are essentially model calculations because an inhomogeneous but stationary velocity field (at least in three-dimensional space) cannot be realized in nature. The renormalization group method is used by these authors to investigate the asymptotic time dependence of the square of the particle displacement $\overline{R^2(t)}$ from the initial position. Depending on the relationship between the dimension d of space and the behavior of the velocity correlation function over large distances (we assuming a power law of the form r^{-a}), we have either normal diffusion ($\overline{R^2(t)} \sim t$) or different anomalous regimes in which $\overline{R^2(t)} \sim t^{2\nu}$ with $\nu \neq 1$. In some cases, a logarithmic time dependence $\overline{R^2(t)}$ is found to ensue. The equations for the average distribution function were not constructed in these papers and the transport coefficients were not calculated. Obukhov[60] has discussed slow nondiffusion regimes of particle propagation (localization) in highly inhomogeneous two-dimensional media. He showed that the phenomenon of localization was due to the presence of random 'traps', i.e., a distribution of particle transit probabilities for which the particle moves on almost closed paths.

Phythian and Curtis[4] have solved the problem of averaging equations such as (1.1) with the view to evaluating the effective diffusion coefficient, taking into account turbulent transport. The local diffusion coefficient $\varkappa$ was assumed to be a given regular quantity, the motion was taken to be incompressible, and the probability distribution for the velocity field was Gaussian, i.e., the higher-order even correlators were expressed in terms of the pair correlator, whereas odd correlators were set equal to zero. The method employed in these calculations was analogous to the self-consistent field method in atomic physics: a certain unknown diffusion coefficient was introduced into the original equation in order to represent local diffusion and part of the turbulent diffusion, and this was followed by a determination of the perturbation-theory correction, obtained on the basis of the above assumptions about the properties of the turbulence. The self-consistent conditions chosen by these authors led to the following equation for the required diffusion coefficient χ:

$$\chi-\varkappa=F(\chi), \tag{3.13}$$

where the right-hand side was calculated in the form of an expansion in the parameter $\langle u^2\rangle$. Drummond *et al.*[61] have performed a numerical simulation of diffusion in a turbulent medium by the Kraichnan method[62] in order to verify the accuracy of the self-consistent theory of Phythian and Curtis. They achieved good agreement with the results reported in Ref. 4 (to within less than a few percent) for reflection-invariant turbulence. On the other hand, for gyrotropic turbulence, numerical simulation gave a value of χ that was greater by tens of percent than the theoretical values. The reason for this discrepancy is still unclear.

The processes involved in the mixing of passive scalar and vector impurities in hydrodynamic flows are exceptionally important for a very wide range of technological and scientific problems. They have attracted a large number of papers and monographs.

We shall not attempt to review all these publications and will confine our attention to concepts and methods that have contributed in some way to the development and application of renormalization and self-consistent field methods in quantitative descriptions of particle transport in stochastic media. Nevertheless, to bring the topic to a sharper focus, we note that there are two possible types of stochastic mixing of particles in fluids and gases. The first can occur even in the laminar flow of a fluid if we neglect molecular diffusion effects: it is due to the appearance of stochastic particle trajectories in determined flows. Modern methods for the description of such systems are based on the theory of dynamic systems and allow a transport description.[40] This is particularly clear when one tries to describe mixing in two-dimensional flows of an incompressible fluid. The equations of motion of a fluid particle then assume the Hamiltonian form

$$\frac{dx}{dt}=-\frac{\partial\phi}{\partial y}, \quad \frac{dy}{dt}=\frac{\partial\phi}{\partial x}, \tag{3.14}$$

where $\phi(x,y,t)$ is the velocity potential. The structure of the phase space of the set of equations given by (3.14) with a one-dimensional 'Hamiltonian' can be investigated in detail, and the stochastic particle trajectories mix the medium in the determined laminar flow. Similarly, stochastic particle trajectories lead to mixing in plasma systems with complex magnetic field configurations (cf., for example, Refs. 11 and 41).

The interaction of particles with stochastic fields is another possible cause of turbulent transport. In this review we shall be mostly concerned with systems of particles in stochastic fields with strong long-wave fluctuations, described by low-order statistical parameters (see, however, Sec. 7).

The transport description of mixing in a medium with a passive-impurity gradient has recently attracted particular attention.[42,43] The intermittency of the passive-impurity distribution plays an important role in this problem. Gollub *et al.*[42] have shown experimentally that, when the Reynolds numbers are high enough, the distribution of the temperature fluctuations in a medium with a forced mean temperature gradient is exponential, i.e., of the form $\exp(|\Delta T|/\zeta)$, rather than Gaussian. The quantity ζ is then of the order of the product of the mean temperature gradient and the correlation length for fluctuations in flow velocity. This result can lead to significant corrections in the theory of stellar evolution because the radiative power of the medium often depends on local temperature. A wide (non-Gaussian) fluctuation probability distribution produces appreciable corrections to the radiation balance. Another example of intermittency, e.g., the distribution of charged particles in a system with ultrasonic and ultra-Alfven turbulence is discussed in detail in Sec. 7.

3.2. Renormalization of transport coefficients in plasma physics

Renormalization methods have been widely used in plasma kinetics ever since the paper by Dupree[63] who examined nonlinear broadening of resonances in wave-particle type interactions in turbulent plasmas. The renormalized Liouville equation for test particles had previously been obtained by Kubo.[46] Calculations of renormalized transport coefficients are discussed in detail Kadanoff and Martin[47] and by Martin *et al.*,[47a] using the functional method and the equation for the mean Green's function for the response of a system to a small perturbation.

Galeev and Zelena[10] examine the diffusion of electrons across a magnetic field in plasma with broken-up magnetic surfaces and infrequent collisions. Starting with the drift kinetic equation and the BGK type collision integral, and taking into account the nonlinear broadening of the wave-particle resonant interaction during effective transverse diffusion, the authors of Ref. 10 obtain the following equation for the transverse diffusion coefficient:

$$\chi_\perp=\chi_\parallel \sum_k \frac{B_{\perp k}^2}{B_0^2}\frac{\nu_{\rm eff}}{k_\parallel^2\chi_\parallel+\nu_{\rm eff}}, \tag{3.15}$$

where the effective frequency of scattering of electrons by fluctuations, $\nu_{\rm eff}$, is proportional to the required coefficient $\chi_\perp$. The solution of (3.15) for $\chi_\perp$ led to the following result:

$$\chi_\perp\approx\chi_\parallel\left(\sum_k \frac{B_{\perp k}^2}{B_0^2}\right)^2\frac{\lambda_e^2}{L_x^2}, \tag{3.16}$$

where λ_e is the Coulomb range of electrons and L_x depends on the longitudinal (relative to the unperturbed magnetic field) wave vector of the perturbations as a function of the transverse coordinate.[10] An important feature of (3.16) is that $\chi_\perp$ is proportional to the fourth power of the amplitude of the magnetic field fluctuations (see Sec. 6).

Kadomtsev and Pogutse[9] use renormalization to calculate the diffusion coefficient for the magnetic lines of force of a stochastic magnetic field. The imposition of a small transverse random component $\mathbf{B}'$ on a uniform magnetic field B_0 ensures that the line of force of the resultant magnetic field undergoes random deviations from the direction of $\mathbf{B}_0$. For $\mathbf{b}=\mathbf{B}'/B_0$, the deviation produced along a path length z along $\mathbf{B}_0$ is given by the integral

$$\mathbf{r}_\perp=\int_0^z \mathbf{b}(z,\mathbf{r}_\perp)dz. \tag{3.17}$$

A simple estimate can be obtained for this interval, but only when $b\ll 1$ for which we can put $r_\perp\approx 0$ in the integrand. Squaring and averaging, we obtain

$$\langle r_\perp^2\rangle=\int_0^z\int_0^z\langle\mathbf{b}(z',0)\mathbf{b}(z'',0)\rangle dz'dz''. \tag{3.18}$$

When $z\gg L_\parallel$, where the latter is the longitudinal correlation length of the random field, we obtain[9]

$$\langle r_\perp^2\rangle=4\chi_F z, \tag{3.19}$$

where

$$\chi_F=\frac{1}{4}\int_{-\infty}^{\infty}\langle\mathbf{b}(z,0)\mathbf{b}(0,0)\rangle dz=\frac{1}{4}\frac{\langle B'^2\rangle}{B_0^2}L_\parallel \tag{3.20}$$

is the diffusion coefficient for the magnetic lines of force. The last equation is the definition of the correlation length $L_\parallel$.

The expressions given by (3.18) and (3.19) were obtained by perturbation theory. They are valid when $b(z,\mathbf{r}_\perp)$ is a slowly-varying function of $\mathbf{r}_\perp$ along the integration path, which gives the condition $r_\perp\ll L_\perp$ or, according to (3.19) or (3.20),

$$(B'/B_0)L_\parallel/L_\perp\ll 1. \tag{3.21}$$

If the transverse correlation length $L_\perp$ is much shorter than the longitudinal length, the condition given by (3.21) may not be satisfied even for small relative amplitudes of the random field $B'\ll B_0$. The diffusion coefficient is often calculated in the following way for the case of 'strong' turbulence. The analysis involves the particle-number density N of certain 'tagged' random lines of force. By definition, the line number densities remain constant as we move along these lines, so that

$$\frac{\partial N}{\partial z}+\mathbf{b}\nabla N=0. \tag{3.22}$$

Next, equation (3.22) is averaged by the usual procedure and, assuming that $N=N_0+N'$, $N_0=\langle N\rangle$, $\langle N'\rangle=0$, we obtain the following two equations:

$$\frac{\partial N_0}{\partial z}=-\operatorname{div}(\mathbf{b}N'), \tag{3.23}$$

$$\frac{\partial N'}{\partial z}+\mathbf{b}\nabla N'-\langle \mathbf{b}\nabla N'\rangle=-\mathbf{b}\nabla N_0. \tag{3.24}$$

We now seek a diffusion-type equation for N_0 and, to achieve this, we put

$$-\operatorname{div}(\mathbf{b}N')=\chi_F\Delta_\perp N_0; \tag{3.25}$$

where χ_F is the required diffusion coefficient which can be calculated by finding the relation between N' and N_0 from (3.24). Equation (3.24) is then simplified by replacing it with the inhomogeneous diffusion equation

$$\frac{\partial N'}{\partial z}-\chi_F\Delta_\perp N'=-\mathbf{b}\nabla N_0. \tag{3.26}$$

The main error introduced by this replacement can be traced to the use of the same diffusion coefficient χ_F for both the rapidly-fluctuating quantity M' and the smoothed function N_0. It seems that the error introduced in this way cannot be estimated on the basis of the above considerations. However, if we adopt this approximation, then (3.25) and (3.26) readily yield a self-consistent non-linear equation for the diffusion coefficient for the magnetic lines of force:[9]

$$\chi_F=\frac{1}{2}\int b_k^2(ik_z+\chi_F k_\perp^2)^{-1}\frac{d^3k}{(2\pi)^3} \tag{3.27}$$

where the factor 1/2 is due to averaging over the azimuthal angle $k_\perp$. When $|k_0|\gg\chi_F k_\perp^2$, which corresponds to the condition for the validity of perturbation theory (3.21), we obtain the quasilinear result (3.20). In the opposite limiting case, $|k_z|\ll\chi_F k_\perp^2$, we have the renormalized diffusion coefficient

$$\chi_F=\left(\frac{1}{2}\int b_k^2 k_\perp^{-2}\frac{d^3k}{(2\pi)^3}\right)^{1/2} \tag{3.28}$$

which should be compared with (3.12). The order-of-magnitude result $\chi_F\approx B' L_\perp/B_0$ is proportional to the first power of the amplitude of the random field and not to its square.

The electronic coefficient of heat transfer across the mean magnetic field is also evaluated in Ref. 9. We shall examine the corresponding results in Chapter 6 where we shall be concerned with the similar problem of transverse diffusion of particles.

Renormalization methods have also found applications in the problem of charged-particle transfer in systems with inhomogeneous plasma turbulence (see Horton's review[64]).

4. PARTICLE TRANSPORT IN STRONGLY TURBULENT PLASMA AND RENORMALIZATION OF TRANSPORT COEFFICIENTS

In Sec. 2, we used perturbation theory to derive the particle transport equation (2.11) averaged over the ensemble of turbulent pulsations. It is readily seen that, under certain conditions, this equation retains its form [with suitably modified transport coefficients in (2.12)–(2.15)] even in the case of long-wave fluctuations with $uL>\varkappa$ for which perturbation theory is not valid. Actually, to ensure that the required equation for the average distribution function takes the form of the Fokker-Planck equation, we have to assume that F is sufficiently smooth. This condition would be satisfied if averaging were performed over spatial regions whose linear dimensions are appreciably greater than the principal turbulent scale L. To obtain the differential form of the acceleration term in the average transport equation, we have to ensure that the change in the particle momentum Δp within the correlation length L is small, i.e., $\Delta p\ll p$.

When the above conditions are satisfied, the average transport equation retains the form given by (2.11) even for strong turbulence, and the problem reduces to the determination of the diffusion coefficients χ and D which are now no longer described by (2.12) and (2.15). These coefficients will be calculated below in a number of stages, using the method developed in one of our previous papers,[31] thus gradually complicating the problem. We begin with the case of an incompressible medium, div $\mathbf{u}=0$, from which the adiabatic acceleration effect is absent and the small-scale diffusion coefficient $\varkappa$ can be looked upon as a nonfluctuating, constant quantity.

4.1. Turbulent transport of an impurity in an incompressible medium

We shall seek the transport equation, averaged over large-scale fluctuations, in its diffusion form

$$\frac{\partial F}{\partial t}=\chi_{\alpha\beta}\frac{\partial^2 F}{\partial r_\alpha\partial r_\beta} \tag{4.1}$$

where $\chi_{\alpha\beta}$ is the unknown diffusion tensor. In addition to (4.1), we shall also consider the equation for the distribution function in which averaging has been carried out over all the harmonics of the velocity field with the exception of those lying in a narrow wave-number interval Δk:

$$\frac{\partial\tilde F}{\partial t}=(\chi_{\alpha\beta}-\Delta\chi_{\alpha\beta})\frac{\partial^2\tilde F}{\partial r_\alpha\partial r_\beta}-\delta u_\alpha\frac{\partial\tilde F}{\partial r_\alpha}; \tag{4.2}$$

where

$$\delta\mathbf{u}(\mathbf{r},t)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{\Delta k}\frac{d^3k}{(2\pi)^3}u_{\mathbf{k}\omega}\times\exp[i(\mathbf{kr}-\omega t)] \tag{4.3}$$

is the unaveraged part of the velocity, the wave number integral is evaluated within a spherical layer of thickness Δk near the arbitrarily chosen k, $\tilde{F}$ is the part of the average distribution function which must be further averaged over the random velocity $\delta\mathbf{u}$, and $\chi_{\alpha\beta}-\Delta\chi_{\alpha\beta}$ is the diffusion coefficient due to the turbulent velocity field after $\delta\mathbf{u}$ has been subtracted. Averaging of (4.2) over the ensemble $\delta\mathbf{u}$ should cancel out the increment $\Delta\chi_{\alpha\beta}$ and should produce equation (4.1) with the resultant diffusion tensor $\chi_{\alpha\beta}$.

We note that the partially averaged distribution function $\tilde{F}$, which is an auxiliary function, depends on the radius k of the chosen spherical layer. However, the renormalized transport coefficients that are physically observable are expressed in terms of integrals over the entire wave-number space and, naturally, do not depend on the arbitrary choice of the spherical wave. The fully averaged observed distribution function F does not depend upon it either.

Equation (4.2) can be averaged in accordance with perturbation theory, using the fact that $\delta\mathbf{u}$ is small. Since $\Delta k \ll k$ is chosen arbitrarily, this approach does not limit the precision of the final results. However, it was assumed that the Fourier harmonics of the velocity field originating from the interval Δk were not correlated with harmonics outside this interval. We shall assume throughout that the turbulence is homogeneous and stationary, and will use the following averaging rule for the Fourier harmonics:

$$\langle \mathbf{u}_{\mathbf{k}\omega}\mathbf{u}_{\mathbf{k}'\omega'}\rangle=(2\pi)^4\langle u^2_{\mathbf{k}\omega}\rangle\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}'). \tag{4.4}$$

This assumption is in agreement with model ideas on Kolmogorov turbulence and with the fact that it is possible to describe it by specifying the spectral energy density.

If we assume that

$$\tilde{F}=F+\delta F, \langle\delta F\rangle=0, \tag{4.5}$$

where the angle brackets represent averaging over the ensemble $\delta\mathbf{u}$, and if we take the average of (4.2), we obtain

$$\frac{\partial F}{\partial t}=(\chi_{\alpha\beta}-\Delta\chi_{\alpha\beta})\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}-\left\langle \delta u_\alpha \frac{\partial \delta F}{\partial r_\alpha}\right\rangle. \tag{4.6}$$

The correction δF to the distribution function can be calculated from the equation

$$\frac{\partial \delta F}{\partial t}-\chi_{\alpha\beta}\frac{\partial^2 \delta F}{\partial r_\alpha \partial r_\beta}=-\delta u_\alpha \frac{\partial F}{\partial r_\alpha}, \tag{4.7}$$

in which we have discarded terms that are quadratic in δu_α, including $\Delta\chi_{\alpha\beta}$. The solution of the last equation will be expressed in terms of the Green's function G of the operator on the left-hand side:

$$\frac{\partial G}{\partial t}-\chi_{\alpha\beta}\frac{\partial^2 G}{\partial r_\alpha \partial r_\beta}=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{4.8}$$

If we now substitute the solution of (4.7) in (4.6), we obtain the following integrodifferential equation:

$$\frac{\partial F}{\partial t}=(\chi_{\alpha\beta}-\Delta\chi_{\alpha\beta})\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}+\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\int G(\mathbf{r}-\mathbf{r}',t-t')\cdot\langle\delta u_\alpha(\mathbf{r},t)\delta u_\beta(\mathbf{r}',t')\rangle F(\mathbf{r}',t')\mathrm{d}^3r'\mathrm{d}t'. \tag{4.9}$$

To reduce this to the form of (4.1), we have to assume that the distribution function F is sufficiently smooth within the correlation length L and within the correlation time L/u. This enables us to replace $F(\mathbf{r}',t')$ in (4.9) with $F(\mathbf{r},t)$, and to determine the increase in the diffusion tensor due to the velocity field $\delta\mathbf{u}$:

$$\Delta\chi_{\alpha\beta}=\int G(\vec{\rho},\tau)\langle\delta u_\alpha(\mathbf{r},t)\delta u_\beta(\mathbf{r}',t')\rangle\mathrm{d}^3\rho\mathrm{d}r, \tag{4.10}$$

where $\vec{\rho}=\mathbf{r}-\mathbf{r}'$ and $\tau=t-t'$. This formula differs from the analogous expression given by (2.12) in three important respects. In contrast to (2.4), it contains the correlator for the small part of the velocity field $\delta\mathbf{u}$, but the Green's function now includes the total ($\chi_{\alpha\beta}$) and not the small-scale ($\varkappa_{\alpha\beta}$) diffusion tensor that represents the total large-scale velocity field.

Transforming to the Fourier representation, and taking the Green's function in accordance with (4.8) in the form

$$G_{k\omega}=\frac{1}{-i\omega+k_\mu k_\nu \chi_{\mu\nu}}, \tag{4.11}$$

we obtain

$$\Delta\chi_{\alpha\beta}=k^2\Delta k\int\frac{\mathrm{d}\Omega_k\mathrm{d}\omega}{(2\pi)^4}\frac{\langle u_\alpha u'_\beta\rangle_{k\omega}}{i\omega+k_\mu k_\nu\chi_{\mu\nu}}, \tag{4.12}$$

where the integral is evaluated with respect to the angles of $\mathbf{k}$ and the frequency. By integrating this expression over all k-space with $\chi_{\alpha\beta}=\varkappa_{\alpha\beta}$ for $u=0$, we are able to write down the self-consistent set of equations for the components of the renormalized diffusion tensor $\chi_{\alpha\beta}$:

$$\chi_{\alpha\beta}=\varkappa_{\alpha\beta}+\int\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}\frac{\langle u_\alpha u'_\beta\rangle_{k\omega}}{i\omega+k_\mu k_\nu\chi_{\mu\nu}}. \tag{4.13}$$

Since the unknown quantities $\chi_{\nu\mu}$ appear under the integral sign as parameters, the set of equations given by (4.13) is algebraic rather than integral. It becomes simpler in the case of isotropic turbulence. Substituting the velocity correlation tensor in the form of (2.18) in (4.13) [with $S(k,\omega)=0$], and taking into account the parity of the spectral function $T(k,\omega)$ with respect to ω, and also the isotropy of the tensors $\chi_{\alpha\beta}=\chi\delta_{\alpha\beta}$ and $\varkappa_{\alpha\beta}=\varkappa\delta_{\alpha\beta}$, we obtain the following single transcendental equation for χ:

$$\chi=\varkappa+\frac{2\chi}{3}\int\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}\frac{k^2T(k,\omega)}{\omega^2+k^4\chi^2}. \tag{4.14}$$

The analytic solution of this equation can readily be obtained for 'frozen' turbulence, i.e., on the assumption that $T(k,\omega)\sim\delta(\omega)$:

$$\chi=\frac{1}{2}\varkappa+\left[\frac{\varkappa^2}{4}+\frac{2}{3}\int\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}k^{-2}T(k,\omega)\right]^{1/2}. \tag{4.15}$$

The solution that satisfies the natural condition $\chi \geqslant 0$ always exists because $T(k,\omega) \geqslant 0$. Nonphysical (negative or complex) values of χ indicate that the method is invalid because of the absence of diffusive particle propagation (anomalous diffusion). As $\varkappa \to 0$, the solution of (4.15) is found to be different from the Moffat result given by (3.10) by the factor $2^{-1/2}$, since

$$\langle u^2 \rangle_k dk = 2dk \int \frac{k^2 d\Omega_k d\omega}{(2\pi)^4} T(k,\omega).$$

Numerical solutions of (4.14) have to be obtained for more realistic turbulence spectra. This is readily done and the results of such calculations will be presented in Sec. 4.2. Here, we confine our attention to the comparison of the calculated renormalized diffusion coefficient with the data of a numerical experiment. This type of comparison is very important for our purposes because the above method of calculating the renormalized transport coefficients is approximate. The main error in the method is probably due to the fact that we consider the diffusive propagation of particles for all turbulence scales $l \leqslant L$, whereas this approach is valid for distances greater than $R \geqslant L$. Moreover, the renormalized diffusion tensor is expressed exclusively in terms of the pair turbulence correlation tensor $\langle u_\alpha u'_\beta \rangle$. We must therefore expect considerable error when the higher-order correlators contain significant information about the structure of the turbulence (strong intermittency).

Drummond *et al.*[61] describe numerical simulations of spatial particle transport by an incompressible single-scale hydrodynamic flow with a Gaussian distribution of realizations of velocity amplitudes. The spectrum of the velocity field realized in the numerical experiment[61] is

$$\frac{T(k,\omega)}{(2\pi)^4} = \frac{u_0^2}{4(2\pi)^{3/2} k_0^2 \omega_0} \delta(k-k_0) \exp\left(-\frac{\omega^2}{2\omega_0^2}\right). \tag{4.16}$$

This is a single-scale spectrum, so that the above formal approach is invalid for the evaluation of transport coefficients. It is readily verified, however, that the entire argument presented above can be reformulated for a small segment $\Delta\omega$ of the frequency spectrum, in which case we again arrive at (4.14) as the expression for the renormalized diffusion coefficient. The numerical calculation can therefore be performed for the spectrum (4.16) with the help of (4.14). The results obtained for a wide range of values of ω_0 and k_0 are in good agreement with the numerical experiments,[61] the maximum deviation being less than 10% and amounting to 3–6% for most points. This confirms that the approximations are reasonable and that the above method can be used to describe transport and acceleration of particles by large-scale turbulence. The inadequacy of the above method, due to the application of the diffusion approximation to small scales, can be reduced by using the above self-consistent approach. To do this, we shall seek the average transport equation, valid for any distances $R \gg \Lambda$ where Λ is the local (small-scale) transport range, in the integral form

$$\frac{\partial F}{\partial t} = \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \int \chi_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t') F(\mathbf{r}', t') d^3r' dt'. \tag{4.17}$$

The kernel $\chi_{\alpha\beta}(\vec{\rho},\tau)$ must correctly describe the propagation of particles in a single large-scale correlation region, averaged over the ensemble of turbulence realizations. If the distributions change little over such distances and times, we obtain the usual diffusion equation (4.1) with the diffusion coefficient given by

$$\chi_{\alpha\beta} = \int d^3\rho \int_0^\infty d\tau \chi_{\alpha\beta}(\vec{\rho},\tau). \tag{4.18}$$

Let us now calculate the kernel $\chi_{\alpha\beta}(\rho,\tau)$ in accordance with the above self-consistent scheme. We start with the partially averaged transport equation analogous to (4.2):

$$\frac{\partial \tilde{F}}{\partial t} = \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \int [\chi_{\alpha\beta}(\vec{\rho},\tau) - \Delta\chi_{\alpha\beta}(\vec{\rho},\tau)] \tilde{F}(\mathbf{r}', t') \times d^3r' dt' - \delta u_\alpha \frac{\partial \tilde{F}}{\partial r_\alpha}, \tag{4.19}$$

where $\delta\mathbf{u}$ is defined above as a small part of the velocity field corresponding to a spherical layer of thickness Δk in k-space. Next, we use perturbation theory and average the last equation. This gives the contribution to the kernel due to the field $\delta\mathbf{u}$:

$$\Delta\chi_{\alpha\beta}(\vec{\rho},\tau) = G(\vec{\rho},\tau) \langle \delta u_\alpha(\mathbf{r},t) \delta u_\beta(\mathbf{r}',t') \rangle. \tag{4.20}$$

The required kernel can therefore be written in the form of

$$\chi_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t') = \varkappa_{\alpha\beta} \delta(\mathbf{r}-\mathbf{r}') \delta(t-t') + G(\mathbf{r}-\mathbf{r}', t-t') \times \langle \delta u_\alpha(\mathbf{r},t) \delta u_\beta(\mathbf{r}',t') \rangle, \tag{4.21}$$

where G is the Green's function satisfying the equation

$$\frac{\partial G}{\partial t} = \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \int \chi_{\alpha\beta}(\mathbf{r}-\mathbf{r}_1', t-t_1) G(\mathbf{r}_1-\mathbf{r}', t_1-t') d^3r_1 dt_1 = \delta(\mathbf{r}-\mathbf{r}') \delta(t-t'). \tag{4.22}$$

Only its Fourier transform can be found without difficulty:

$$G_{\mathbf{k}\omega} = [-i\omega + k_\mu k_\nu \chi_{\mu\nu}(\mathbf{k},\omega)]^{-1}, \tag{4.23}$$

where now $\chi_{\nu\mu}(\mathbf{k},\omega)$ is the Fourier transform of the required kernel.

Transforming to the Fourier representation in (4.21), and using (4.23), we obtain the following nonlinear integral equation for $\chi_{\mu\nu}(\mathbf{k},\omega)$:

$$\chi_{\mu\nu}(\mathbf{k},\omega) = \varkappa_{\mu\nu} + \int \frac{d^3k' d\omega'}{(2\pi)^4} \frac{\langle u_\alpha u'_\beta \rangle_{\mathbf{k}-\mathbf{k}', \omega-\omega'}}{-i\omega' + k'_\mu k'_\nu \chi_{\mu\nu}(\mathbf{k}',\omega')}. \tag{4.24}$$

This equation generalizes (4.13) and can serve for the description of particle transport to any distances exceeding the local transport range, including those that are smaller than the large-scale correlation length L. To transform from (4.24) to (4.13), we must use two approximations, namely, (a) we must put $k \ll k'$, $\omega \ll \omega'$ and neglect the dependence on $\mathbf{k}$ and ω on both right- and left-hand sides of the equation and (b) we must replace the function $\chi_{\mu\nu}(\mathbf{k},\omega)$ under the integral sign with the constant tensor $\chi_{\nu\mu}(0,0)$ which, according to (4.18), is the diffusion tensor. The mathematical structure is thus significantly more complicated when we use a more accurate description of transport over short distances.

4.2. Particle transport in a compressible medium and weak acceleration within the correlation length

We shall now apply the averaging procedure to the resultant equation given by (1.3), including the acceleration term. In this particular approximation, the acceleration effect arises only in a compressible medium (div $\mathbf{u} \neq 0$). However, for the sake of simplicity, we shall ignore fluctuations in the small-scale diffusion tensor due to density fluctuations, which is permissible for strong turbulent transport, $uL \gg \varkappa$, when the small-scale diffusion coefficient $\varkappa$ may be looked upon as a 'seed' quantity whose precise value is not important. The role of density fluctuations in the medium will be examined below in Sec. 4.5.

As noted above, the average transport equation has the differential form

$$\frac{\partial F}{\partial t}=\chi_{\alpha\beta}\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2\left[D(p)+D^*_{\alpha\beta}(p)\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\right]\frac{\partial F}{\partial p}, \tag{4.25}$$

if acceleration within the correlation length is small and the distribution function is averaged over spatial regions with linear dimensions exceeding L. The contribution to the diffusion coefficients due to a small variation in the velocity field $\delta\mathbf{u}$ can be calculated from the equation

$$\frac{\partial F}{\partial t}=(\chi_{\alpha\beta}-\Delta\chi_{\alpha\beta})\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2\left[(D-\Delta D)+(D^*_{\alpha\beta}-\Delta D^*_{\alpha\beta})\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\right]\frac{\partial F}{\partial p}-\left\langle \delta u_\alpha \frac{\partial \delta F}{\partial r_\alpha}\right\rangle+\frac{p}{3}\frac{\partial}{\partial p}\left\langle \delta F\frac{\partial \delta u_\alpha}{\partial r_\alpha}\right\rangle, \tag{4.26}$$

in which the correction $\delta\varkappa$ is found by perturbation theory from the equation

$$\frac{\partial \delta F}{\partial t}-\chi_{\alpha\beta}\frac{\partial^2 \delta F}{\partial r_\alpha \partial r_\beta}-\frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D(p)+D^*_{\alpha\beta}(p)\frac{\partial^2}{\partial r_\alpha \partial r_\beta}\right)\frac{\partial \delta F}{\partial p}=-\delta u_\alpha\frac{\partial F}{\partial r_\alpha}+\frac{p}{3}\frac{\partial F}{\partial p}\frac{\partial \delta u_\alpha}{\partial r_\alpha}. \tag{4.27}$$

Since the solution of this equation need only be constructed for distances not exceeding the correlation length L within which the acceleration of particles is assumed to be small, the acceleration effect need not be taken into account in the corresponding Green's function, and we can use (4.8) and its solution (4.11).

As in Sec. 4.1, we obtain

$$\chi_{\alpha\beta}=\varkappa_{\alpha\beta}-\int d^3\rho\int_0^\infty d\tau\langle u_\beta(\mathbf{r},t)u_\gamma(\mathbf{r}',t')\rangle\rho_\alpha\frac{\partial G}{\partial\rho_\gamma}, \tag{4.28}$$

$$D=\frac{p^2}{9}\int d^3\rho\int_0^\infty d\tau\langle \operatorname{div}\mathbf{u}(\mathbf{r},t)\operatorname{div}\mathbf{u}(\mathbf{r}',t')\rangle G(\rho,\tau), \tag{4.29}$$

$$D^*_{\alpha\beta}=\frac{p^2}{18}\int d^3\rho\int_0^\infty d\tau\langle \operatorname{div}\mathbf{u}(\mathbf{r},t)\operatorname{div}\mathbf{u}(\mathbf{r}',t')\rangle \times G(\rho,\tau)\rho_\alpha\rho_\beta. \tag{4.29'}$$

Equation (4.28) determines the spatial diffusion tensor $\chi_{\alpha\beta}$ as a solution of a set of transcendental equations. It appears not only on the left-hand side of this equation, but also on the right-hand side (via the Green's function G). Once we have found $\chi_{\alpha\beta}$, we can calculate the diffusion coefficient D in momentum space with the help of (4.29).

For isotropic turbulence we can use the correlation tensor (2.18) and assume that $\varkappa_{\alpha\beta}=\varkappa\delta_{\alpha\beta}$, $\chi_{\alpha\beta}=\chi\delta_{\alpha\beta}$, $D_{\alpha\beta}=D^*\delta_{\alpha\beta}$. We thus obtain the simpler equations

$$\chi=\varkappa+\frac{1}{3}\int\frac{d^3k\,d\omega}{(2\pi)^4}\left[\frac{2T+S}{i\omega+k^2\chi}-\frac{2k^2\chi S}{(i\omega+k^2\chi)^2}\right], \tag{4.30}$$

$$D(p)=\frac{p^2}{9}\chi\int\frac{d^3k\,d\omega}{(2\pi)^4}\frac{k^4S(k,\omega)}{\omega^2+k^4\chi^2}, \tag{4.31}$$

$$D^*=\frac{p^2}{9}\chi\int\frac{d^3k\,d\omega}{(2\pi)^4}k^2S(k,\omega)\times\left[\frac{1}{(i\omega+k^2\chi)^2}-\frac{4k^2\chi}{3(i\omega+k^2\chi)^3}\right]. \tag{4.32}$$

The term in (4.25) that contains the coefficient $\Delta^*_{\alpha\beta}$ describes the coupling between acceleration effects and spatial diffusion. It can be of the same order of magnitude as the first, diffusion, term on the right-hand side of (2.45). In particular, if the momentum dependence of the distribution function in a certain part of the momentum space is of the power form $F(p)\sim p^{-\gamma}$, the presence of the term containing the cross derivatives will produce a change in the spatial diffusion coefficient χ which then becomes $\chi_{\text{eff}}=\chi+\gamma(\gamma-3)D^*p^{-2}$.

4.3. Transport equation with strong particle acceleration within the correlation length

Charged-particle transport in random electromagnetic fields in different physical systems can be examined in detail within the framework of the Fokker–Planck approximation (see, for example, Refs. 65 and 66). The condition for the validity of this approximation usually reduces to the requirement that the change in momentum and (or) the energy of a particle should be small within the correlation length L or the correlation time τ_c of the random field. The use of the Fokker–Planck approximation is therefore restricted either to the consideration of particles with high enough energies or to systems with short-correlated fields. On the other hand, the solution of many problems such as, for example, the evolution of cosmic-ray spectra at low energies ($E<1$ GeV) in the Galaxy, forces us to leave the framework of the Fokker–Planck approximation. In this section and in Sec. 4.4, we shall consider the evolution of charged-particle spectra due to substantial changes in the particle energy within the correlation length in a system with strong MHD-type long-wave turbulence (see Ref. 67).

To analyze systems with a large change in momentum within the correlation scale, we take the transport equation averaged over the ensemble of fluctuations in the integral form

$$\frac{\partial F}{\partial t}=\int_0^\infty d\eta'\chi_{\alpha\beta}(\eta-\eta')\,\frac{\partial^2 F(\mathbf{r},\eta',t)}{\partial r_\alpha\partial r_\beta}+\left(\frac{\partial^2}{\partial\eta^2}+3\,\frac{\partial}{\partial\eta}\right)\times\int_{-\infty}^{\infty}D(\eta-\eta')F(\mathbf{r},\eta',t)d\eta', \qquad (4.33)$$

in which momentum has been replaced with the new variable $\eta=\ln(p/p_0)$, so that

$$\frac{\partial^2}{\partial\eta^2}+3\,\frac{\partial}{\partial\eta}=\frac{1}{p^2}\frac{\partial}{\partial p}p^4\frac{\partial}{\partial p}. \qquad (4.34)$$

Having taken the average over all the field harmonics with the exception of the narrow interval Δk, we find from the transport equation (4.1) that

$$\frac{\partial\tilde F}{\partial t}=\int_{-\infty}^\infty d\eta'\chi'_{\alpha\beta}\frac{\partial^2\tilde F(\mathbf{r},\eta',t)}{\partial r_\alpha\partial r_\beta}+\left(\frac{\partial}{\partial\eta^2}+3\,\frac{\partial}{\partial\eta}\right)\times\int_{-\infty}^{\infty}D'(\eta-\eta')\tilde F d\eta'-\delta u_\alpha\frac{\partial}{\partial r_\alpha}\tilde F+\frac{1}{3}\frac{\partial}{\partial\eta}\tilde F\frac{\partial\delta u_\alpha}{\partial r_\alpha}. \qquad (4.35)$$

It is convenient to perform the Fourier transformation with respect to the momentum variable η. Denoting the Fourier variable by s and the Fourier transforms of $\tilde F$, $\chi'_{\alpha\beta}$, and D' by $\tilde F_s$, $\tilde\chi'_{\alpha\beta}(s)$, and $D'(s)$, we obtain

$$\frac{\partial\tilde F_s}{\partial t}=\tilde\chi'_{\alpha\beta}(s)\,\frac{\partial^2\tilde F_s}{\partial r_\alpha\partial r_\beta}-(s^2+3is)\,\tilde D'(s)\tilde F_s-\delta u_\alpha\frac{\partial}{\partial r_\alpha}\tilde F_s-\frac{is}{3}\tilde F_s\frac{\partial}{\partial r_\alpha}\delta u_\alpha. \qquad (4.36)$$

The problem has thus been reduced to averaging an equation of the form of (4.18) and can be solved by the method used in (4.3). The fluctuating extra term δF_s is given by

$$\delta\tilde F_s(\mathbf{r},t)=-\int d^3r\int_{-\infty}^t dt'G(\mathbf{r}-\mathbf{r}',t-t',s)\times\left\{\delta u_\alpha(\mathbf{r}',t')\,\frac{\partial}{\partial r'_\alpha}F_s(\mathbf{r}',t')+\frac{is}{3}F_s(\mathbf{r}',t')\,\frac{\partial}{\partial r'_\alpha}\delta u_\alpha(r',t')\right\}, \qquad (4.37)$$

where the Green's function is given by

$$G(\rho,\tau,s)=(4\pi\tilde\chi(s)\tau)^{-3/2}\exp[-(\rho^2/4\tilde\chi\tau)-(s^2+3is)\tilde D\tau]. \qquad (4.38)$$

After averaging (4.35) with the help of (4.38), we obtain

$$\frac{\partial\tilde F_s}{\partial t}=(\tilde\chi'_{\alpha\beta}(s)+\delta\tilde\chi_{\alpha\beta}(s))\,\frac{\partial^2\tilde F_s}{\partial r_\alpha\partial r_\beta}-(s^2+3is)(\tilde D'+\delta\tilde D)\tilde F_s, \qquad (4.39)$$

i.e., the Fourier transform of (4.33) in which

$$\delta\tilde\chi_{\alpha\beta}=\int d^3\rho\int_0^\infty d\tau G(\rho,\tau,s)\Big[\delta K_{\alpha\beta}(\rho,\tau)+\rho_\alpha\frac{\partial}{\partial\rho_\gamma}\delta K_{\gamma\beta}(\rho,\tau)+\frac{1}{18}(s^2+3is)\,\rho_\alpha\rho\,\frac{\partial^2}{\beta\partial\rho_\alpha\partial\rho_\delta}\delta K_{\gamma\delta}\Big], \qquad (4.40)$$

$$\delta\tilde D=-\frac{1}{9}\int d^3\rho\int_0^\infty d\tau G(\rho,\tau,s)\,\frac{\partial^2\delta K_{\alpha\beta}}{\partial\rho_\alpha\partial\rho_\beta}. \qquad (4.41)$$

The above relations are analogs of (4.24). They enable us to write down the transcendental equations for the Fourier transforms of the kernels of the integral operators $\tilde\chi(s),\tilde D(s)$:

$$\tilde\chi(s)=\varkappa+\frac{1}{3}\int\frac{d^3kd\omega}{(2\pi)^4}\left[\frac{2T(k,\omega)+S(k,\omega)}{\tilde\chi(s)k^2+\lambda\tilde D(s)+i\omega}-\frac{2S(k,\omega)\tilde\chi k^2(1+\lambda/6)}{(\tilde\chi k^2+i\omega+\lambda\tilde D)^2}+\frac{4\lambda\tilde\chi^2Sk^4}{9(\tilde\chi k^2+i\omega+\tilde\lambda D)^3}\right], \qquad (4.42)$$

$$\tilde D(s)=\frac{1}{9}\int\frac{d^3kd\omega}{(2\pi)^4}\frac{k^2S(k,\omega)}{\tilde\chi(s)k^2+i\omega+\lambda\tilde D(s)}, \qquad (4.43)$$

$$\lambda=s^2+3is.$$

In contrast to (4.26) and (4.27), these equations contain the variable s as an independent parameter. The solutions of these equations are functions of this parameter. Moreover, both of the required functions $\tilde\chi(s)$ and $\tilde D(s)$ appear in the integrand.

It was assumed in the derivation of (4.33), (4.42), and (4.43) that the seed diffusion coefficient $\varkappa$ was independent of the particle momentum.

4.4. The spectra of particles interacting with strong long-wave turbulence

Consider a plasma system with a developed, statistically homogeneous, isotropic MHD-type turbulence. The electric field induced by the turbulent motions of a perfectly conducting medium with a frozen-in magnetic field produce a statistical change in the energy of suprathermal charged particles. Magnetic-field fluctuations with scales smaller than or of the order of the gyrotropic radius of a particle lead (in the absence of Coulomb collisions) to the effective isotropization of the particle momentum and determine their transport range Λ. As before, we consider long-wave fluctuations with scales $L \gg \Lambda$. The distribution function $G(\mathbf{r},\eta,t)$ of the suprathermal particles, averaged over the ensemble of turbulent fluctuations of all scales, satisfies equation (4.3) obtained in Sec. 4.3:

$$\frac{\partial G}{\partial t}-\int_{-\infty}^{\infty}\mathrm{d}\eta'\chi_{\alpha\beta}(\eta-\eta')\frac{\partial^2 G(r,\eta',t)}{\partial r_\alpha \partial r_\beta} -\left(\frac{\partial^2}{\partial \eta^2}+3\frac{\partial}{\partial \eta}\right)\int_{-\infty}^{\infty}D(\eta-\eta')G(r,\eta',t)\mathrm{d}\eta' =Q\delta(\eta), \tag{4.44}$$

where the particle momentum p has been replaced with the variable $\eta=\ln(p/p_0)$ and G is the rate of generation of low-energy particles with momentum p_0 by a stationary source. Equation (4.44) is conveniently solved in the Fourier representation in η, whose transform will be denoted by s. The Fourier transforms of the kernels of the integral equation given by (4.44) can be expressed in terms of the turbulence correlation functions.

Consider the very general case of a compressible system with a wide velocity fluctuation spectrum[67]

$$T(k)=S(k)=\frac{\langle u^2\rangle}{4}C(\nu)\frac{k_0^{\nu-1}}{(k^2+k_0^2)^{\nu/2+1}}, \tag{4.45}$$

$$k \leqslant k_{\max}$$

and a Lorentz frequency dependence characterized by the dispersion relation $\omega_0=uk$ and 'resonance width'

$$\gamma/2=uk_0(k/k_0)^{(3-\nu)/2},$$

where

$$u\equiv\langle u^2\rangle^{1/2},$$

and

$$C(\nu)\approx\frac{4\Gamma(\nu/2+1)}{3\pi^{3/2}\Gamma(\nu/2-1/2)}$$

is the normalization constant. This leads to the following set of transcendental equations for the Fourier transforms of the kernels:

$$D_1(s)=\varepsilon+C(\nu)\int_0^b\frac{x^2\mathrm{d}x}{(x^2+1)^{(\nu/2)+1}}\left\{\frac{\psi}{\psi^2+\omega_0^2} -\frac{2}{3}D_1x^2\left(\frac{\lambda}{6}+1\right)\frac{\psi^2-\omega_0^2}{(\psi^2+\omega_0^2)^2} +\frac{4}{9}\lambda D_1^2x^4\frac{\psi^3-3\psi\omega_0^2}{(\psi^2+\omega_0^2)^3}\right\}, \tag{4.46}$$

$$D_2(s)=\frac{C(\nu)}{9}\int_0^b\frac{x^4\mathrm{d}x}{(x^2+1)^{\nu/2+1}}\frac{\psi}{\psi^2+\omega_0^2}, \tag{4.47}$$

which were obtained from (4.42) and (4.44) with allowance for the finite change in the particle energy within the field correlation length. The dimensionless variables in (4.46) and (4.47) are defined by $\chi_{\alpha\beta}(s)\equiv uk_0^{-1}D_1(s)\delta_{\alpha\beta}$, $D(s)\equiv uk_0D_2(s)$, $b=k_{\max}/k_0$, $x=k/k_0$, $\varepsilon=\varkappa k_0/u$ where $\varkappa$ is the 'seed' diffusion coefficient ($\varkappa\sim\Lambda$) due to the scattering of particles by small-scale magnetic-field fluctuations and

$$\psi(x,\lambda)=D_1(s)x^2+\lambda D_2(s)+\frac{\gamma(x)}{2}, \tag{4.48}$$

$$\lambda\equiv s^2+3is.$$

The transport equation given by (4.44) with kernels determined from (4.46) and (4.47) describes the energy spectra due to stochastic changes in the energy of particles (initially low-energy particles) produced by long-wave *electric-field fluctuations. The derivation of* (4.44) *rests on* the assumption that the velocities of the suprathermal particles are higher than the turbulent velocities u of the medium. The parameter of ε can then be arbitrary.

When $\varepsilon>1$, i.e., particle transport due to scattering by small-scale magnetic-field fluctuations predominates, the solutions of (4.46) and (4.47) are given by $D_1(s)\approx\varepsilon$ and $D_2(s)\approx(9\varepsilon)^{-1}$.

This gives

$$\chi_{\alpha\beta}(\eta-\eta')\approx\varkappa\delta(\eta-\eta')\delta_{\alpha\beta}, \tag{4.49}$$

$$D(\eta-\eta')\approx u^2(9\varkappa)^{-1}\delta(\eta-\eta'). \tag{4.50}$$

Equation (4.44) is thus reduced to the Fokker–Planck form.

When $\varepsilon \ll 1$, particle transport is determined by long-wave fluctuations in the velocity of the medium. The renormalization of the kernels of (4.44) is then significant. The dependence of D_1 and D_2 on s is not trivial. When $s \leqslant 3$, the transforms of the kernels are slowly-varying functions of s: $D_1(s)\approx D_1(0)$ and $D_2(s)\approx D_2(0)$ Next, the character of these functions depends significantly on the size of the fluctuation spectrum, characterized by the parameter $b>1$. When $\varepsilon b^{(\nu+1)/2}>1$ and $s \leqslant a(\nu)$ where

$$a(\nu)=\left[\frac{9(3-\nu)}{C(\nu)}\right]^{1/2}\sim 10,$$

the dependence of the kernels on s is weak. When $s \gg a(\nu)$, we have the following asymptotic expressions for the Fourier transforms of the kernels:

$$D_1(s)\to\varepsilon/2, \tag{4.51}$$

$$D_2(s)\to b^{(3-\nu)/2}(a(\nu)\lambda^{1/2})^{-1}. \tag{4.52}$$

For a wide fluctuation spectrum, $\varepsilon b^{(\nu+1)/2} \gg 1$, the asymptotic functions (4.51) and (4.52) are valid for $s \gg a(\nu)\varepsilon b^{(\nu+1)/2}$. The behavior of the functions $D_1(s)$ and $D_2(s)$ for $s>3$, and before the asymptotic behavior is reached, depends on the specific form of the resonant width $\gamma(k)$ and the parameters ε and b. In particular, the behavior of $D_2(s)$ in this region can be nonmonotonic. Typically, the limiting value in (4.51) is smaller than the seed diffu-

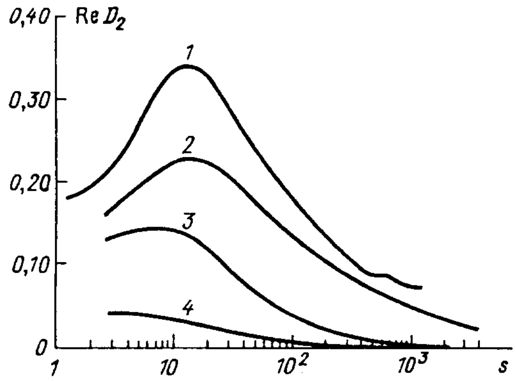

FIG. 1. Fourier transforms of the real part of the kernel D_2 for $\varepsilon=0.01$, $b=10,10,10^3,10^4$.

sion coefficient ε. This property is due to the compressibility of the system and is general in character (see also Ref. 54).

The equations for the Fourier transforms of the kernels, given by (4.46) and (4.47), can be solved numerically. Figures 1–4 show the results of a numerical calculation of $D_1(s)$ and $D_2(s)$ in order to illustrate their dependence on the size b of the velocity fluctuation spectrum and the seed diffusion coefficient ε. The kernel transforms are complex (whereas the kernels themselves are, of course, real). For the fluctuation spectrum (4.45) that was considered, the imaginary parts of the kernels are small in comparison with the real parts. There is particular interest in the function $\mathrm{Re}D_2(s)$ that determines the asymptotic behavior of the particle energy spectrum. We shall therefore examine this function in some detail for different values of ε and b. The results of the numerical calculation are in agreement with the approximate analytic results presented above.

We note that although the derivation of (4.44), (4.46), and (4.47) was based on the assumption that, as noted in Sec. 4.3, ε does not depend on the particle momentum, this is a significant limitation when (4.33), (4.44), and (4.43) are employed. Actually, for very general types of fluctuation spectra (4.45), the effects associated with the significant difference between the transport equation (4.33) and the corresponding Fokker-Planck equation are well-defined for small values of ε (dimensionless $\varkappa$). However, in the limit of small ε, the asymptotic behavior of $D_2(s)$ (4.52) does not, as has been shown, depend on ε. The subsequent results are therefore valid even in the case of a momentum-dependent seed diffusion coefficient.

The analytic expressions given by (4.51) and (4.52) enable us to investigate the charged-particle spectra for momenta approaching the injection momentum. For the sake of simplicity, but without significant loss of generality, we can replace the spatial diffusion operator in (4.44) with its eigenvalue. The Fourier transform of the stationary particle distribution function (here, this is the Green's function) then becomes

$$G(s)=\frac{Q(uk_0)^{-1}}{\xi D_1(s)+\lambda D_2(s)}, \qquad (4.53)$$

where $\xi\approx(k_0R)^{-2}$ in which R is the size of the region occupied by the turbulent fluctuations. From (4.51), (4.52), and (4.53) we then obtain the following asymptotic particle spectrum for momenta close to the injection momentum p_0:

$$G(p,p_0)\propto-Q(uk_0)^{-1}b^{(\nu-3)/2}a(\nu)\ln|\ln(p/p_0)|,$$
$$p\to p_0. \qquad (4.54)$$

This spectrum has an integrable logarithmic singularity near p_0, which is related to the possible finite change in the particle energy within the correlation scale. For momenta $p\gg p_0$, the particle spectrum (4.53) is determined by the behavior of $D_1(s)$ and $D_2(s)$ for $s\to0$:

$$G(p,p_0)\propto Q(uk_0)^{-1}(p/p_0)^{\mu}, \qquad (4.55)$$

where

$$\mu=-1.5-[2.25+\xi D_1(0)D_2^{-1}(0)]^{0.5}. \qquad (4.56)$$

The expressions given by (4.54) and (4.56) are also found to describe the Fokker-Planck spectrum for all $p\gg p_0$. In contrast to (4.54), as $p\to p_0$, the distribution function (4.55) has a finite limit and a weak dependence on the width b of the fluctuation spectrum.

We note that the above result can be used to calculate the photon spectra formed as a result of the Thomson scattering in an optically dense turbulent medium.

4.5. Effect of fluctuations in the density of a medium on particle diffusion

In the preceding Sections, we examined in detail the acceleration and diffusion of particles in a compressible medium ($\mathrm{div}\,\mathbf{u}\neq0$). We did not, however, take into account the change in the density of the medium, which produces fluctuations in the effective scattering frequency and the small-scale ('molecular') particle diffusion coefficient. This effect may be significant, but only if the small-scale diffusion itself plays an appreciable part against a background of turbulent transport, i.e., for $uL\leqslant\varkappa$. The limiting case of this situation is particle diffusion over chaotically distributed 'clouds' in a medium, whose motion can be neglected. We shall now consider global diffusion over distances exceeding the characteristic scale of density fluctuations. Particle acceleration will be neglected for the sake of simplicity.

It is convenient to start not with the transport equation, but with the kinetic equation for the distribution function $f(\mathbf{r},\mathbf{p}t)$ in the relaxation-time approximation:

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial\mathbf{r}}=-\nu\left[f-1+\frac{3\mathbf{uv}}{v^2}\bar{f}\right], \qquad (4.57)$$

where $\nu(\mathbf{r},t,p)$ is the effective particle scattering frequency that fluctuates in both space and time as a result of fluctuations in the density of the medium

$$\nu=\nu_0(p)+\tilde{\nu}(\mathbf{r},t,p),\quad\langle\tilde{\nu}\rangle=0. \qquad (4.58)$$

The statistical properties of the scattering frequency $\tilde{\nu}$ are assumed known. The right-hand side of (4.57) describes the scattering of particles in a moving medium for $u \ll v$ and the bar represents averaging over the angles of the vector $\mathbf{p}$; the derivatives with respect to momentum, i.e., particle accelerations are discarded (see Sec. 8 in Ref. 25 for further details).

The equation for the distribution function $F(r,p,t) = \langle f(r,p,t) \rangle$, averaged over the ensemble of velocity and density fluctuations in the medium will be taken in the form

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{r}} - \chi_{\alpha\beta}(\mathbf{p})\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta} = -\nu_e(F - \bar{F}), \tag{4.59}$$

where $\nu_e(p)$ is the effective scattering frequency and $\chi_{\alpha\beta}(\mathbf{p})$ is the diffusion tensor describing turbulent transport. Both these quantities must be calculated the global diffusion tensor $D_{\alpha\beta}$ is expressed in terms of them. Its relation to ν_e and $\chi_{\alpha\beta}(\mathbf{p})$ can be determined by transforming in (4.59) to the diffusion approximation

$$F = \frac{1}{4\pi}\left(N + \frac{4\mathbf{nJ}}{v}\right), \quad \mathbf{n} = \frac{\mathbf{v}}{v}, \tag{4.60}$$

where N is the isotropic part of the distribution function and J is the differential (with respect to the momenta) particle flux density. Substituting the distribution function (4.60) in (4.59) and eliminating the vector J in the usual way, we obtain the diffusion equation (4.1) with the diffusion tensor given by

$$D_{\alpha\beta} = \frac{v^2}{3\nu_e}\delta_{\alpha\beta} + \overline{\chi_{\alpha\beta}(\mathbf{p})}. \tag{4.61}$$

Henceforth, we shall put $v^2/3\nu_e = \varkappa$. In accordance with the general method presented in the last few sections, we shall consider (4.59) together with the partially averaged equation with the still unaveraged harmonics from the narrow wave-number integral Δk:

$$\frac{\partial \tilde{F}}{\partial t} + \mathbf{v}\frac{\partial \tilde{F}}{\partial \mathbf{r}} - \chi'_{\alpha\beta}(p)\frac{\partial^2 \tilde{F}}{\partial r_\alpha \partial r_\beta} = -(\nu'_e + \delta\tilde{\nu})\left(\tilde{F} - \bar{\tilde{F}} - \frac{3\delta\mathbf{uv}}{v^2}\bar{\tilde{F}}\right). \tag{4.62}$$

The primes in this equation indicate the transport coefficients associated with the spectrum of turbulent quantities after subtraction of Δk. Applying the perturbation procedure, we obtain from (4.62) the set of equations for the mean distribution function F

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{r}} - \chi'_{\alpha\beta}(p)\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta} = -\nu'_e(F - \bar{F}) - \langle \delta\tilde{\nu}(\delta F - \delta\bar{F})\rangle + \nu'_e\frac{3\mathbf{v}}{v^2}\langle \delta\mathbf{u}\delta F\rangle, \tag{4.63}$$

and the oscillatory increment δF:

$$\frac{\partial \delta F}{\partial t} + \mathbf{v}\frac{\partial \delta F}{\partial \mathbf{r}} - \chi'_{\alpha\beta}(\mathbf{p})\frac{\partial^2 \delta F}{\partial r_\alpha \partial r_\beta} + \tilde{\nu}_e(\delta F - \delta\bar{F}) = Q(\mathbf{r},\mathbf{p},t), \tag{4.64}$$

where

$$Q = -\delta\tilde{\nu}(F - \bar{F}) + \nu'_e\frac{3\mathbf{v}}{v^2}\delta\mathbf{u}F. \tag{4.65}$$

The solutions of (4.64) are conveniently expressed in terms of the Green's function $g(\mathbf{r}-\mathbf{r}', t-t', \mathbf{p},\mathbf{p}')$ of the operator on the left-hand side. In the Fourier representation, we have the following equation for the Green's function:

$$g_{k\omega}(\mathbf{p},\mathbf{p}') = \frac{\nu_e \bar{g}_{k\omega} + \delta(\mathbf{p}-\mathbf{p}')}{\nu_e - i\omega + i\mathbf{kv} + k_\mu k_\nu \chi_{\mu\nu}(\mathbf{p})}. \tag{4.66}$$

This simple equation is obtained only in the diffusion approximation, i.e., in the limit where $\omega/\nu_e \ll 1$, $|\mathbf{k}\cdot\mathbf{v}|/\nu_e \ll 1$. In this limit, and for the isotropic case in which $\chi_{\alpha\beta} = \chi\delta_{\alpha\beta}$, we find that

$$\begin{aligned} g_{k\omega}(\mathbf{p},\mathbf{p}') = {} & \nu_e^2\delta(p-p')(4\pi^2)^{-1}(-i\omega + k^2 D)^{-1} \\ & \times(\nu_e - i\omega + i\mathbf{kv} + k^2\chi)^{-1} \\ & \times(\nu_e - i\omega + i\mathbf{kv}' + k^2\chi)^{-1} \\ & + \delta(\mathbf{p}-\mathbf{p}')(\nu_e - i\omega + i\mathbf{kv} + k^2\chi)^{-1}. \end{aligned} \tag{4.67}$$

This equation must then be expanded in terms of the above small ratios. Using (4.67), we evaluate the term on the right-hand side of (4.63) and determine the contributions to the effective collision frequency ν_e and the turbulent diffusion coefficient $s\overline{\chi}\mathbf{p}$ due to the small wave-number range Δk.

This procedure gives the following set of transcendental equations for the transport coefficients $\nu_e, \bar{\chi}$:

$$\nu_e = \nu_0 - \frac{\langle \tilde{\nu}^2\rangle}{\nu_e} + \frac{\varkappa}{3\nu_e}\int \frac{k^2\langle \tilde{\nu}\tilde{\nu}'\rangle_{k\omega}}{i\omega + k^2 D}\frac{d^3k d\omega}{(2\pi)^4} + \frac{1}{3}\int \frac{ik_\beta\langle u_\beta \tilde{\nu}'\rangle_{k\omega}}{i\omega + k^2 D}\frac{d^3k d\omega}{(2\pi)^4}, \tag{4.68}$$

$$\begin{aligned} \bar{\chi}(p) = {} & \frac{1}{3}\int \frac{d^3k d\omega}{(2\pi)^4}\left[\frac{\langle u_\alpha u'_\beta\rangle_{k\omega}}{i\omega + k^2 D} - \frac{2Dk_\alpha k_\beta\langle u_\alpha u'_\beta\rangle_{k\omega}}{(i\omega + k^2 D)^2}\right] \\ & - \frac{1}{3}\int \frac{d^3k d\omega}{(2\pi)^4}\frac{ik_\beta(3i\omega + k^2 D)}{(i\omega + k^2 D)^2}\frac{\langle \tilde{\nu}u'_\beta\rangle_{k\omega}}{\nu_e}; \end{aligned} \tag{4.69}$$

where $D = \varkappa + \bar{\chi}$, $\varkappa = v^2/3\nu_e$. Fluctuations in the scattering frequency $\tilde{\nu}$ are proportional to fluctuations in the density of the medium, so that the correlator $\langle \tilde{\nu}\tilde{\nu}'\rangle$ and $\langle \tilde{\nu}u'_\beta\rangle$ can be expressed, at least in principle, in terms of the velocity correlator by using the equations of motion.

A simple analytic solution of (4.68)–(4.69) is possible only for static inhomogeneites when $u=0$ and $\tilde{\nu}$ is independent of time, i.e., $\langle \tilde{\nu}\tilde{\nu}'\rangle_{k\omega} \sim \delta(\omega)$. We then have $\bar{\chi} = 0$ and the effective scattering frequency satisfies the simple relations

$$\nu_e = \nu_0 - \frac{2}{3}\frac{\langle\tilde{\nu}^2\rangle}{\nu_e}, \tag{4.70}$$

where ν_0 is the scattering frequency averaged over the ensemble. The physical (positive) value of the root

$$\nu_e = \frac{1}{2}\nu_0 - \left(\frac{\nu_0^2}{4} - \frac{2}{3}\langle\tilde{\nu}^2\rangle\right)^{1/2} \tag{4.71}$$

is realized for $\langle\tilde{\nu}^2\rangle \leqslant (3/8)\nu_0^2$. In this range, the effective scattering frequency is less than the mean value ν_0 and decreases with increasing fluctuations. For sufficiently large fluctuations $\langle\tilde{\nu}^2\rangle > (3/8)\nu_0^2$, the roots become complex and physically meaningless. This is probably due to the failure of the above method of calculation which operates only with pair correlators of random variables for $|\tilde{\nu}| \geqslant \nu_0$ when there is strong intermittency in the structure of the scattering medium. Moreover, the diffusion approximation for the Green's function g used above will also fail under these conditions.

For strong intermittency, $|\tilde{\nu}| \gg \nu_0$, we can readily obtain an approximate estimate for the effective collision frequency. Suppose that the homogeneous medium in which the scattering frequency for the particular particles is ν_1 contains clouds in which the collision frequency is $\nu_1 + \nu_2 \gg \nu_1$. The mean separation between the clouds is L and the linear dimensions of the clouds are l. Suppose that the mean collision frequency is

$$\nu_0 = \nu_1 + \nu_2 (l/L)^3 = \text{const} \tag{4.72}$$

and let us vary the ratio l/L (assuming that the collision frequency is proportional to the density of the medium and that ν_2 depends on l). If the clouds fill the space almost completely ($l \approx L$), we have the case of homogeneous scattering with $\nu_e = \nu_0 = \nu_1^+ \nu_2$. In the opposite limiting case of dense and rare clouds of small size ($l \ll L$), the particles propagate mostly in the tenuous phase with effective collision frequency $\nu_e \gtrsim \nu_1 \ll \nu_0$. In these two limiting cases, the mean square collision frequency fluctuation is $\langle\tilde{\nu}^2\rangle \approx 0$ and $\langle\tilde{\nu}^2\rangle \approx (\nu_0 - \nu_1)^2 (l/L)^3 \gg \nu_0^2$, respectively, where in the latter limit we have a sharp asymmetry between frequency deviations frequency in the direction of increasing and decreasing values, respectively. In accordance with (4.71), the collision frequency decreases with increasing inhomogeneity of the medium. In particular when dense clouds are placed in a vacuum, the effective collision frequency decreases as $\nu_e \gtrsim \upsilon l^2/L^3 \to 0$ with increasing concentration of material in the cloud ($l \to 0$). This is independent of the mean density of the medium and mean collision frequency. The significant point is that the properties of the medium are characterized in this method by two scales that cannot be introduced by specifying the pair correlation tensor alone.

5. RENORMALIZATION OF THE EQUATION FOR THE MEAN MAGNETIC FIELD IN THE TURBULENT-DYNAMO THEORY

The method used in the last Section to renormalize the transport coefficients can also be used to describe the transport of a vector impurity by a turbulent flow. Consider the averaged equation of magnetic induction in turbulent-dynamo theory:

$$\frac{\partial \mathbf{b}}{\partial t} = \operatorname{curl}[\mathbf{u} \times \mathbf{b}] + \eta^{\mathrm{m}}_{\alpha\beta} \frac{\partial^2 \mathbf{b}}{\partial r_\alpha \partial r_\beta}; \tag{5.1}$$

where $\mathbf{u}(\mathbf{r},t)$ is the turbulent velocity field which, as before, we will be assumed to be specified by its statistical characteristics, and $\eta^{\mathrm{m}}_{\alpha\beta}$ is the local magnetic viscosity tensor. Since turbulent magnetic viscosity is usually much greater than $\eta^{\mathrm{m}}_{\alpha\beta}$, we shall look upon the latter as a seed quantity whose precise value is unimportant and taken in the form $\eta^{\mathrm{m}}_{\alpha\beta} = \eta_{\mathrm{m}} \delta_{\alpha\beta}$ where $\eta_{\mathrm{m}} = c^2/4\pi\sigma$ is determined by the electrical conductivity σ of the plasma. Possible anisotropy and gyrotropy of turbulence will be taken into account (we shall assume, as before, that the turbulence is homogeneous) and we shall take the Fourier transform of the velocity correlation tensor in the form

$$\tilde{K}_{\alpha\beta}(\mathbf{k},\omega) = P_{\alpha\beta}(\mathbf{k},\omega) + iC_{\alpha\beta}(\mathbf{k},\omega), \tag{5.2}$$

where

$$P_{\alpha\beta}(\mathbf{k},\omega) = P_{\beta\alpha}(\mathbf{k},\omega) = P_{\alpha\beta}(-\mathbf{k},\omega) \tag{5.3}$$

is a symmetric tensor that is invariant under the replacement of $\mathbf{k}$ with $-\mathbf{k}$ whereas

$$C_{\alpha\beta}(\mathbf{k},\omega) = C_{\beta\alpha}(-\mathbf{k},\omega) = -C_{\alpha\beta}(-\mathbf{k},\omega) \tag{5.4}$$

is a skew-symmetric reflection-noninvariant tensor describing the gyrotropy of the turbulence.

To investigate the possible generation of a magnetic field with scale exceeding the principal turbulent-velocity scale L, we have to average equation (5.1). We shall do this by the method developed in the last Section. We start by seeking the equation for the average large-scale field $\mathbf{B} = \langle \mathbf{b} \rangle$ in the form

$$\frac{\partial B_\alpha}{\partial t} = A_{\alpha\mu\nu} \frac{\partial B_\nu}{\partial r_\mu} + \eta^{\mathrm{tot}}_{\mu\nu} \frac{\partial^2 B_\alpha}{\partial r_\mu \partial r_\nu}; \tag{5.5}$$

where $A_{\alpha\mu\nu}$ and $\eta^{\mathrm{tot}}_{\mu\nu}$ are constant tensorial coefficients for homogeneous turbulence and repeated Greek indices indicate summation. If there are no special directions in space other than the direction of $\mathbf{k}$, then for an incompressible medium we have

$$P_{\alpha\beta}(\mathbf{k},\omega) = T(k,\omega)(\delta_{\alpha\beta} - k_\alpha k_\beta k^{-2}), \tag{5.6}$$

$$C_{\alpha\beta}(\mathbf{k},\omega) = C(k,\omega) e_{\alpha\beta\gamma} k_\gamma, \tag{5.7}$$

where $e_{\alpha\beta\gamma}$ is a skew-symmetric unit tensor and (5.5) assumes the well-known form[27,28]

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\alpha \mathbf{B}) + \eta^{\mathrm{tot}} \Delta \mathbf{B}. \tag{5.8}$$

The coefficient α that leads to the generation of a large-scale magnetic field is nonzero only for gyrotropic turbulence $[C(k,\omega) \neq 0]$.

To calculate the coefficients in the more general equation (5.5), we extract from the velocity field a small part $\delta\mathbf{u}$ which, as before, contains harmonics within a narrow wave-number interval Δk. If we now average (5.5) over all harmonics except those in this narrow interval, and denote the magnetic field obtained in this way by $\widetilde{\mathbf{B}}$, we obtain the following equation:

$$\frac{\partial \widetilde{B}_\alpha}{\partial t}=A'_{\alpha\mu\nu}\frac{\partial \widetilde{B}_\nu}{\partial r_\mu}+\eta'^{\,\rm tot}_{\mu\nu}\frac{\partial^2 \widetilde{B}_\alpha}{\partial r_\mu \partial r_\nu}+\mathrm{curl}_\alpha[\delta\mathbf{u}\times\widetilde{\mathbf{B}}], \quad (5.9)$$

in which $A'_{\alpha\mu\nu}$ and $\eta'^{\,\rm tot}_{\mu\nu}$ are not very different (because Δk is small) from the exact coefficients $A_{\alpha\mu\nu}$ and $\eta^{\rm tot}_{\mu\nu}$.

We shall now use perturbation theory to average the last equation over the realizations of $\delta\mathbf{u}$. Taking

$$\widetilde{\mathbf{B}}=\mathbf{B}+\delta\widetilde{\mathbf{B}}, \quad \langle\delta\widetilde{\mathbf{B}}\rangle=0, \quad (5.10)$$

we find from (5.9) that

$$\frac{\partial B_\alpha}{\partial t}=A'_{\alpha\mu\nu}\frac{\partial B_\nu}{\partial r_\mu}+\eta'^{\,\rm tot}_{\mu\nu}\frac{\partial^2 B_\alpha}{\partial r_\mu \partial r_\nu}+\mathrm{curl}_\alpha\langle[\delta\mathbf{u}\times\delta\widetilde{\mathbf{B}}]\rangle, \quad (5.11)$$

$$\frac{\partial \delta\widetilde{B}_\alpha}{\partial t}=A'_{\alpha\mu\nu}\frac{\partial \delta\widetilde{B}_\nu}{\partial r_\mu}+\eta'^{\,\rm tot}_{\mu\nu}\frac{\partial^2 \delta\widetilde{B}_\alpha}{\partial r_\mu \partial r_\nu}+\mathrm{curl}_\alpha[\delta\mathbf{u}\times\mathbf{B}]. \quad (5.12)$$

We note that the sharp difference between the turbulent pulsation scales ($l\ll L$) and the regular field $\mathbf{B}(R\gg L)$ is possible only if the gyrotropic part $C_{\alpha\beta}$ of the correlation tensor (5.2) is small in comparison with its nongyrotropic part $P_{\alpha\beta}$. This is so because magnetic-field generation occurs for scales[35] exceeding

$$L_{\rm crit}\approx 2\pi\eta^{\rm tot}/\alpha\approx 2\pi L\langle u^2\rangle^{1/2}/\alpha, \quad (5.13)$$

and the condition $L_{\rm crit}\gg L$ is satisfied for $\alpha\ll\langle u^2\rangle^{1/2}$.

Since we are assuming that the gyrotropic term in (5.12) is small, we can neglect the first term on the right-hand side and write the solution in the form

$$\delta\widetilde{B}_\alpha(\mathbf{r},t)=B_\beta(\mathbf{r},t)\int G_{\rm m}(\mathbf{r}-\mathbf{r}',t-t')\times\frac{\partial \delta u_\alpha(\mathbf{r}',t')}{\partial r'_\beta}\,\mathrm{d}^3r'\mathrm{d}t' -\frac{\partial B_\alpha(\mathbf{r},t)}{\partial r_\beta}\int G_{\rm m}(\mathbf{r}-\mathbf{r}',t-t')\times\delta u_\alpha(\mathbf{r}',t')\mathrm{d}^3r'\mathrm{d}t'; \quad (5.14)$$

where the Green's function $G_{\rm m}$ represents the turbulent transport

$$\frac{\partial G_{\rm m}}{\partial t}-\eta^{\rm tot}_{\mu\nu}\frac{\partial^2 G_{\rm m}}{\partial r_\mu\partial r_\nu}=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (5.15)$$

and we have used the incompressibility condition $\mathrm{div}\,\delta\mathbf{u}=0$.

By substituting (5.14) into the last term in (5.11), we can find the contributions $\Delta A_{\alpha\mu\nu}$ and $\Delta\eta^{\rm tot}_{\mu\nu}$ to the transport coefficients due to harmonics in the wave-number interval Δk:

$$\Delta A_{\alpha\mu\nu}=-2\int_{(\Delta k)} G_{\rm m}(\mathbf{r},\tau)\frac{\partial}{\partial r_\nu}\delta C_{\alpha\mu}(\mathbf{r},\tau)\mathrm{d}^3r\mathrm{d}\tau, \quad (5.16)$$

$$\Delta\eta^{\rm tot}_{\mu\nu}=\int_{(\Delta k)} G_{\rm m}(\mathbf{r},\tau)\delta P_{\mu\nu}(\mathbf{r},\tau)\mathrm{d}^3r\mathrm{d}\tau. \quad (5.17)$$

Integrating (5.16) and (5.17) over the entire wave-number spectrum, and transforming to the Fourier representation, we obtain a set of self-consistent equations for the coefficients of viscosity $\eta^{\rm tot}_{\mu\nu}$ and magnetic-field generation $A_{\alpha\mu\nu}$ in the averaged equation (5.5):

$$\eta^{\rm tot}_{\mu\nu}=\int\frac{P_{\mu\nu}(\mathbf{k},\omega)}{i\omega+k_\sigma k_\lambda\eta^{\rm tot}_{\sigma\lambda}}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}+\eta^m_{\mu\nu}, \quad (5.18)$$

$$A_{\alpha\mu\nu}=2\int\frac{k_\nu C_{\alpha\mu}(\mathbf{k},\omega)}{i\omega+k_\sigma k_\lambda\eta^{\rm tot}_{\sigma\lambda}}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}. \quad (5.19)$$

The first step then is to use (5.18) to calculate the magnetic field diffusion tensor and then, from $\eta^{\rm tot}_{\mu\nu}$ found in the above way, determine the tensor $A_{\alpha\mu\nu}$ of rank 3 from (5.19), which is skew-symmetric in the first two indices. Equations (5.6) and (5.7) become simpler for simple gyrotropic turbulence. The diffusion tensor becomes diagonal, $\eta^{\rm tot}_{\mu\nu}=\eta^{\rm tot}\delta_{\mu\nu}$ and the magnetic-field generation tensor is expressed in terms of the pseudoscalar α: $A_{\beta\mu\nu}=\alpha e_{\beta\mu\nu}$. The coefficients $\eta^{\rm tot}$ and α are calculated from the set of equations

$$\eta^{\rm tot}=\eta_{\rm m}+\frac{2}{3}\int\frac{T(k,\omega)}{i\omega+\eta^{\rm tot}k^2}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}, \quad (5.20)$$

$$\alpha=\frac{2}{3}\int\frac{k^2T(k,\omega)}{i\omega+\eta^{\rm tot}k^2}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}. \quad (5.21)$$

These equations differ from the analogous equations obtained in Ref. 68 in that, for a given scale l, they take into account field transport by flows with all other scales, and not only the small-scale flows as in Ref. 68 or Ref. 5. Moreover, our equations (5.20) and (5.21) take into account nonsteady flow which always occurs in practice (integrals over frequencies). Finally, according to Ref. 61, the validity of the renormalization procedure for gyrotropic systems requires more careful justification.

The expression given by (5.14), which relates fluctuations in a small-scale field with turbulent velocity and the mean field, can be used to calculate the correlation tensor for the small-scale magnetic field. In steady state, and assuming that the regular-field scale has its limiting maximum value ($\mathbf{B}=\mathrm{const}$), we obtain

$$\delta\widetilde{B}_\alpha(\mathbf{r},t)=B_\beta\int G_{\rm m}(\mathbf{r}-\mathbf{r}',t-t')\frac{\partial\delta u_\alpha(\mathbf{r}',t')}{\partial r'_\beta}\mathrm{d}^3r'\mathrm{d}t', \quad (5.22)$$

and in the Fourier representation

$$\langle\widetilde{B}_\alpha\widetilde{B}'_\beta\rangle_{k,\omega}=(\mathbf{B}\mathbf{k})^2|\bar{G}_{\rm m}(\mathbf{k},\omega)|^2\langle u_\alpha u'_\beta\rangle_{k,\omega}, \quad (5.23)$$

where according to (5.15) the Fourier transform of the Green's function is

$$\bar{G}_{\rm m}(\mathbf{k},\omega)=(-i\omega+k_\sigma k_\lambda\eta^{\rm tot}_{\sigma\lambda})^{-1}. \quad (5.24)$$

If the turbulence is weak and may be looked upon as a set of quasilinear MHD modes with phase velocities $v_{ph} \gg u$, then $\omega \gg k_\sigma k_\lambda \eta^{tot}_{\sigma\lambda}$ and, according to (5.23) and (5.24)

$$\frac{\langle \widetilde{B}^2 \rangle}{B^2} \approx \frac{\langle u^2 \rangle}{v^2_{ph}} \ll 1. \tag{5.25}$$

For strong turbulence, we have

$$v_{ph} \approx u, \quad \langle \widetilde{B}^2 \rangle \approx B_0^2. \tag{5.26}$$

It is important, however, to remember that, in general, the expression given by (5.23) is not the complete small-scale field correlator. This expression actually describes fluctuations in the large-scale field that is frozen into the plasma and is due to the turbulent motion of the medium. There is also another reason for the appearance of small-scale fields: they grow in the turbulent medium independently of the presence or otherwise of the mean field ('turbulent small-scale field dynamo'). The process is also possible under the influence of reflection-invarariant turbulence, and that gyrotropy is not then essential. This problem has been examined in many papers.[28,32,33,68–71] In the linear (kinematic) approximation, it is often possible to find the criterion for the growth of small-fields, but it is not possible to calculate the steady level of turbulence and (5.23) is valid when there is no spontaneous growth of the small-scaled magnetic field.

It is important to note that the growth of small-scale magnetic fields in a turbulent medium is a difficult problem even in the linear approximation. A relatively simple equation can be obtained for the magnetic energy density, but only on the assumption of zero turbulence correlation time (see, for example, Refs. 69 and 72). This approximation is valid only if the growth time of the magnetic field is greater than the existing finite turbulence correlation time.

Kulsrud and Anderson[72] have noted correctly that the growth times of harmonics with different wave numbers may be different by many orders of magnitude. In particular, the growth time for small-scale fields in the case of Kolmorgorow turbulence is much shorter than the correlation time found for the smallest vortices. These estimates lead to the conclusion that the evaluation of large-scale fields is possible only if we abandon the approximation that relies on zero correlation time, and small-scale fields are correctly taken into account.

6. DIFFUSION AND TURBULENT TRANSPORT OF PARTICLES IN A LARGE-SCALE STOCHASTIC MAGNETIC FIELD

6.1. Qualitative considerations

Magnetic fields often have a decisive influence on charged-particle transport in plasma systems. In this Section, we examine this type of situation when the spatial scales of variation of a random magnetic field $\widetilde{\mathbf{B}}(\mathbf{r},t)$ are much greater than the local range of particles for scattering by small-scale electromagnetic fields (Coulomb or plasma). A quasihomogeneous field $\mathbf{B}_0$ with a variation scale much greater than the random-field scale may also be present in the system. These conditions are typical for the diffusion of different impurities, both thermal and nonequilibrium, in magnetized turbulent plasma with a wide spectrum of fluctuations in the magnetic field and in the velocity $\mathbf{u}(\mathbf{r},t)$ of the medium.

Local diffusion of the particles is highly anisotropic if their Larmor radius is small in comparison with the transport range. The particles travel mostly in the direction of the local magnetic field, with small transverse deviations due to drifts. At the same time, global transport to distances exceeding the random-field correlation length in a time much greater than the large-scale field correlation time may become equivalent to almost isotropic diffusion because of the considerable entanglement of the lines of force and the presence of transverse plasma motion. We thus have to consider the connection between local and global diffusion tensors. The transverse (relative to the quasihomogeneous magnetic field B_0) diffusion coefficient is then particularly important. This problem is important for the transport of heat and of particles in thermonuclear fusion installations[9,10,12,64] and in the analysis of the propagation of cosmic rays and elements synthesized in active processes under astrophysical conditions.[6,14,15,73]

The classical theory of transport yields the following local transverse diffusion coefficient for magnetized particles:

$$\varkappa_\perp \approx \varkappa_\parallel (r_B/\Lambda_\parallel)^2, \tag{6.1}$$

where $\varkappa_\parallel$ is the diffusion coefficient along the magnetic field, r_B is the Larmor radius, and $\Lambda_\parallel$ is the longitudinal transport range of the particles. For example, consider $\varkappa_\perp$ for relativistic particles in the Galaxy. Assuming[6] that $B_0 \approx 3\times 10^{-6}$ G and $\Lambda_\parallel \approx 10^{18}$ cm, we find that the magnetization factor is $\Lambda_\parallel / r_B = \omega_B \tau \approx 5\times 10^6 (1 \text{ GeV}/E)$ where E is the total particle energy and $\tau = 1/v$ is the mean free time between collisions. These estimates suggest that the local diffusion of most cosmic rays ($E \sim 1$ GeV) should be highly anisotropic: $\varkappa_\perp / \varkappa_\parallel \approx 10^{-13}$. On the other hand, there is no experimental evidence for such strong anisotropy in the propagation of cosmic rays in the Galaxy as a whole. Observations suggest isotropic global diffusion of relativistic particles with moderate energies. It may be considered that, both in the Galaxy and in thermonuclear installations, 'anomalous' transverse transport, which may exceed 'classical normal' transport by several orders of magnitude, is the dominant phenomenon.

Both the turbulent velocity field of the medium and the stochastic component of the large-scale magnetic field must be taken into account in calculations of the global diffusion tensor for highly-turbulent systems. The latter effect may have the same order of magnitude as the regular field in spiral arms, e.g., in the Galaxy, and this can probably produce particle transport across $\mathbf{B}_0$ due to deviations of the local field from the mean. The transverse components of the turbulent velocity field (and the associated electric fields in the highly-conducting plasma) play a similar role. Different approaches to transport in large-scale stochastic fields are developed in Refs. 74–77, 9, 14, and 15. Most of them employ perturbation theory ($\widetilde{B} \ll B_0$).

We shall now follow one of our previous papers[15] to calculate the global diffusion tensor (without restricting the amplitude of the random field, i.e., including the case $\tilde{B} \gg B_0$). Turbulent motion will be taken into account in a self-consistent manner and the analysis will be based on the renormalization method presented in Sec. 4 in connection with the drift kinetic equation. In the final analysis, the components of the global diffusion tensor will be given by a set of transcendental equations in terms of pair correlation functions of the turbulent fields.

We shall assume in our calculations that particle transport by percolation is small. It may play an important role in systems in which the longitudinal correlation length of the fluctuations is much greater than the transverse length. Particle transport is then determined by a relatively small number of long lines of force. Percolation is relatively unimportant for systems with isotropic turbulence, the linear size of which is much greater than the fluctuation correlation length. The restriction on the topology of the random magnetic field then reduces to the condition for a random breakup of lines of force for scales greater than the correlation scale. The description of systems with long correlations requires more detailed statistical information than the pair correlation functions that we have used (see Ref. 29 for greater detail).

6.2. Averaging the drift kinetic equation

We start with the drift kinetic equation for the distribution function $f(\mathbf{r},p,\mu,t)$ for magnetized particles, using the approximation of zero gyroradius:

$$\frac{\partial f}{\partial t}+[(\mathbf{v}\cdot\mathbf{b}-\mathbf{u}\cdot\mathbf{b})b_\alpha+u_\alpha]\frac{\partial f}{\partial r_\alpha}=-\nu(f-\bar{f}); \quad (6.2)$$

where $\mathbf{u}(\mathbf{r},t)$ is the turbulent velocity field in the medium, defined by the binary correlation tensor and $\mathbf{b}$ is the unit vector in the direction of the magnetic field

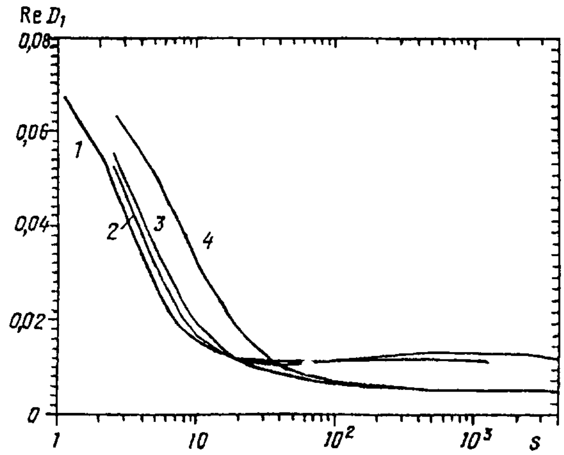

FIG. 3. Fourier transforms of the real part of the kernel D_1 for $\varepsilon=10^{-2}$, $b=10,10^2,10^3,10^4$.

$$\mathbf{b}=\frac{\mathbf{B}_0+\tilde{\mathbf{B}}}{|\mathbf{B}_0+\tilde{\mathbf{B}}|}, \quad (6.3)$$

where $\tilde{\mathbf{B}}(\mathbf{r},t)$ is the turbulent magnetic field with scales l ($\Lambda_\parallel < l \ll L$) over which we perform our averaging. These scales are characteristics for the velocity $\mathbf{u}$. The field $\mathbf{B}_0$ is regular and its scale of variation $R \gg L$ is of the order of the linear dimensions of the plasma system that we are considering. To keep the calculations as simple as possible, we discard all terms in (6.2) that describe changes in the particle energy. These effects are very small if the medium is incompressible, i.e.,

$$\operatorname{div}\mathbf{u}=0. \quad (6.4)$$

If necessary, we may abandon this condition and take into account the acceleration terms as in Sec. 4. To simplify the averaging of the original equation (6.2), we assume that the turbulence is nongyrotropic and take the vectors $\mathbf{u}$ and $\tilde{\mathbf{B}}$ to be perpendicular to $\mathbf{B}_0$. The interaction of particles

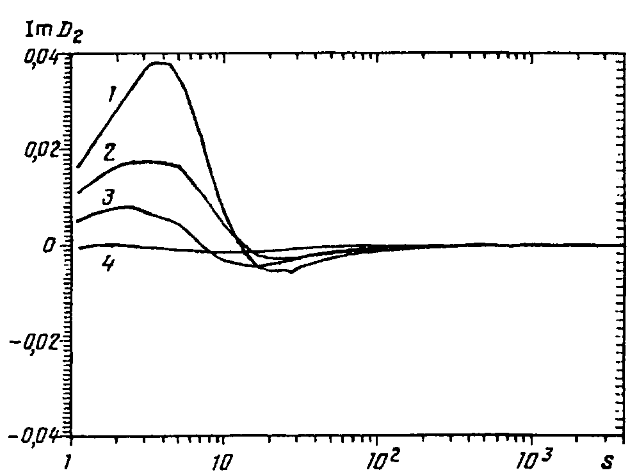

FIG. 2. Fourier transforms of the imaginary part of the kernel D_2 for $\varepsilon=10^{-2}$, $b=10,10^2,10^3,10^4$.

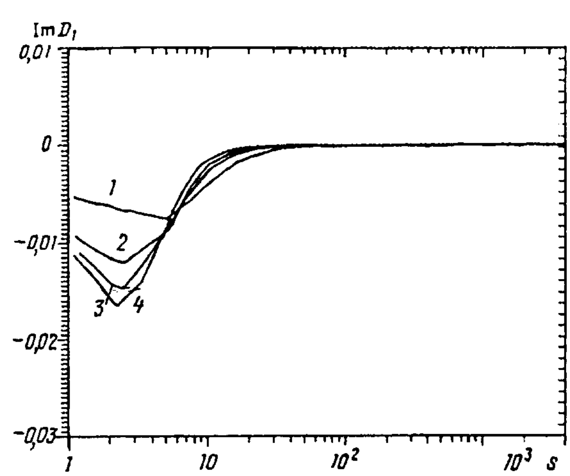

FIG. 4. Fourier transforms of the imaginary part of the kernel D_1 for $\varepsilon=10^{-2}$, $b=10,10^2,10^3,10^4$.

with magnetic fields whose scale is smaller than or of the order of the gyroradius of the particles will be simulated by the right-hand side of (6.2) where ν is the collision frequency for particles interacting with small-scale fields. The bar in $\bar{f}$ represents averaging over the pitch angle $\theta(\mu) = \cos\theta$. The term that describes focusing of particle pitch angles by large-scale field fluctuations is omitted from (6.2) because the collision frequency ν is much greater than the corresponding frequency of variation in the pitch angle $v \operatorname{div} \mathbf{b}$.

We shall seek the average of equation (6.2) over the ensemble of turbulent pulsations in the form

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{r}} - \chi_{\alpha\beta}(\mathbf{p})\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta} = -\nu(F - \bar{F}), \tag{6.5}$$

where $F(\mathbf{r},\mathbf{p},t) = \langle f(\mathbf{r},\mathbf{p},t)\rangle$ is the average distribution function.

We note that $f(\mathbf{r},p,\theta,t)$ and the averaged $F(\mathbf{r},\mathbf{p},t)$ depend on different angles defining the orientation of the momentum vector $\mathbf{p}$. The function f depends on the local pitch angle θ, but does not depend on the local (fast) gyrophase. Averaging over the latter gives (6.2). After averaging over the turbulent fields, the direction of momentum is characterized by the angle θ to the mean magnetic field B_0 and the azimuthal angle φ measured around $\mathbf{B}_0$. The azimuthal angle φ is not a fast variable. The dependence of the distribution function on this angle can be due to the azimuthal anisotropy of the turbulence or by a gradient in the distribution of the particles whose direction is different from that of $\mathbf{B}_0$.

In (6.5),

$$V_\alpha = \langle (\mathbf{v}\cdot\mathbf{b} - \mathbf{u}\cdot\mathbf{b})b_\alpha + u_\alpha\rangle = \langle v_\alpha^{\text{eff}}\rangle \tag{6.6}$$

is the average particle drift velocity. The term containing the second derivative $\chi_{\alpha\beta}\mathbf{p}(\partial^2 F/\partial r_\alpha \partial r_\beta)$ is the result of averaging:

$$\left\langle (v_\alpha^{\text{eff}} - V_\alpha)\frac{\partial f}{\partial r_\alpha}\right\rangle = -\chi_{\alpha\beta}(\mathbf{p})\frac{\partial^2 F}{\partial r_\alpha \partial r_\beta}. \tag{6.7}$$

The tensor $\chi_{\alpha\beta}(\mathbf{p})$ is not the complete diffusion tensor: it describes only part of the contribution of the turbulence to the diffusion. It will be calculated below, using the self-consistent scheme. However, as a preliminary step, we must write down the average velocity V_α in a convenient form. Let

$$\varepsilon = \langle \tilde{B}^2/(B_0^2 + \tilde{B}^2)\rangle, \tag{6.8}$$

so that the correlator of unit vectors at a particular point is

$$\langle b_\alpha b_\beta\rangle = (1-\varepsilon)b_{0\alpha}b_{0\beta} + (\varepsilon/2)\delta^{\perp}_{\alpha\beta}, \tag{6.9}$$

where $b_{0\alpha}$ is the unit vector along the field $\mathbf{B}_0$. From (6.9) we find that

$$V_\alpha = (1-\varepsilon)b_{0\alpha}v_{\parallel} + (\varepsilon/2)v_\alpha^{\perp}, \tag{6.10}$$

where the symbols $\parallel$, $\perp$ refer to the direction of $\mathbf{B}_0$. The parameter ε varies in the range $0 \leqslant \varepsilon \leqslant 1$ and characterizes the contribution of the turbulent component of the magnetic field, $\delta^{\perp}_{\alpha\beta} = \delta_{\alpha\beta} - b_{0\alpha}b_{0\beta}$ where $\delta_{\alpha\beta}$ is the three-dimensional Kronecker symbol.

The total diffusion coefficient must now be expressed in terms of $\mathbf{v}$ and $\chi_{\alpha\beta}(\mathbf{p})$ by transforming from (6.5) to the low anisotropy approximation. If we assume that

$$F = \frac{1}{4\pi}(N(\mathbf{r},p,t) + \delta N(\mathbf{r},\mathbf{p},t)), \tag{6.11}$$

$$\overline{\delta N(\mathbf{r},\mathbf{p},t)} = 0, \quad |\delta N| \ll N,$$

we can write (6.5) in the diffusion form

$$\frac{\partial N}{\partial t} = D_{\alpha\beta}\frac{\partial^2 N}{\partial r_\alpha \partial r_\beta}, \tag{6.12}$$

where

$$D_{\alpha\beta} = \varkappa_{\alpha\beta} + \overline{\chi_{\alpha\beta}(\mathbf{p})}, \quad \varkappa_\perp \delta_{\alpha\beta} = \varkappa^{\perp}_{\alpha\beta} + \varkappa_{\parallel} b_{0\alpha}b_{0\beta},$$

$$\varkappa_\perp = \left(\frac{\varepsilon}{2}\right)^2 \frac{v^2}{3\nu}, \quad \varkappa_{\parallel} = \frac{v^2}{3\nu}(1-\varepsilon)^2. \tag{6.13}$$

We now turn to the evaluation of the transport coefficient $\chi_{\alpha\beta}(\mathbf{p})$ and consider a narrow wave-number interval Δk in the turbulent spectrum. The contributions to the turbulent fields due to harmonics belonging to Δk will be denoted by $\delta\mathbf{u}$ and $\delta\tilde{\mathbf{B}}$. This leads to the expression

$$v_\alpha^{\text{eff}} = v'^{\text{eff}}_\alpha + \delta v_\alpha^{\text{eff}}, \tag{6.14}$$

where

$$\delta v_\alpha^{\text{eff}} = (v\cdot\delta\mathbf{b} - \delta\mathbf{u}\cdot\mathbf{b}' - \mathbf{u}'\delta\mathbf{b})b'_\alpha + (\mathbf{v}\mathbf{b}' - \mathbf{u}'\mathbf{b}')\delta b_\alpha + \delta u_\alpha,$$

$$\delta b_\alpha = \left[\frac{B_0}{B'}\delta_{\alpha\beta} - \frac{B_0}{B'^3}(\mathbf{B}_0 + \tilde{\mathbf{B}}')_\alpha \tilde{B}'_\beta\right]\frac{\delta\tilde{B}_\beta}{B_0}; \tag{6.15}$$

in which primes indicate quantities from whose spectra the interval Δk has been excluded.

We now average the initial equation (6.2) over the entire spectrum of turbulent pulsations with Δk excluded. We shall indicate this average by a prime:

$$\langle f(\mathbf{r},\mathbf{p},t)\rangle' = \tilde{F}(\mathbf{r},\mathbf{p},t),$$

so that we have (details of this calculation can be found in Ref. 15)

$$\frac{\partial\tilde{F}}{\partial t} + \mathbf{v}'\frac{\partial\tilde{F}}{\partial\mathbf{r}} - \chi'_{\alpha\beta}(\mathbf{p})\frac{\partial\tilde{F}^2}{\partial r_\alpha\partial r_\beta} + \nu(\tilde{F} - \bar{\tilde{F}}) = \hat{L}\tilde{F}, \tag{6.16}$$

where $\hat{L}$ is the perturbation operator that is proportional to the small quantities δu_α and $\delta\tilde{B}_\alpha$:

$$\hat{L} = -\delta V_\alpha\frac{\partial}{\partial r_\alpha} - \delta A_{\alpha\beta}\frac{\partial^2}{\partial r_\alpha\partial r_\beta}, \tag{6.17}$$

$$\delta V_\alpha = \langle v_\alpha^{\text{eff}}\rangle'$$

$$= \left(1 - \varepsilon' - \left\langle\frac{B_0^2\tilde{B}'^2}{B'^4}\right\rangle\right)\left(b_{0\alpha}\frac{v\delta\tilde{B}}{B_0} + v_{\parallel}\frac{\partial\tilde{B}_\alpha}{B_0}\right)$$

$$+ \left(1 - \frac{\varepsilon'}{2}\right)\delta u_\alpha. \tag{6.18}$$

The tensor $\delta A_{\alpha\beta}$ is of the same order of magnitude as $(\delta\widetilde{B}/B)\chi_{\alpha\beta}$ and is expressed in terms of a sum of several terms.

The final averaging over the ensemble of realizations of δu_α, $\delta\widetilde{\mathbf{B}}_\alpha$ will be carried out using standard perturbation theory. The fluctuation increment δF on the fully averaged distribution function F will be written in terms of the Green's function

$$\delta F(\mathbf{r},\mathbf{p},t)=\int G(\mathbf{r},\mathbf{p},t;\mathbf{r}',\mathbf{p}',t')\hat{L}'F'\mathrm{d}^3r'\mathrm{d}t'\mathrm{d}\Omega', \qquad (6.19)$$

satisfying the equation

$$\frac{\partial G}{\partial t}+\mathbf{V}\frac{\partial G}{\partial \mathbf{r}}-\chi_{\alpha\beta}\frac{\partial^2 G}{\partial r_\alpha \partial r_\beta}+\nu(G-\bar{G})$$
$$=\delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \qquad (6.20)$$

where $d\Omega'$ is a solid angle element defining the direction of $\mathbf{p}'$.

The calculation of the Green's function in the Fourier representation is made, like all other previous calculations, on the assumption that the wavelength of all the harmonics of this stochastic field are large in comparison with $\Lambda_\parallel=v/\nu$. This leads to the inequality $|\mathbf{k}\cdot\mathbf{u}|\ll\nu$ and enables us to obtain the Green's function in the form

$$G_{\mathbf{k}\omega}(\mathrm{p},\mathrm{p}')=\frac{\nu\nu'}{4\pi(\nu'+ik_\alpha V_\alpha+k_\alpha k_\beta\chi_{\alpha\beta}(\mathbf{p}))(\nu'+\mathbf{i}\mathbf{k}_\alpha\mathbf{V}'_\alpha+\mathbf{k}_\alpha\mathbf{k}_\beta\chi_{\alpha\beta}(\mathbf{p}'))}\,\frac{1}{-i\omega+k_\parallel^2\varkappa_\parallel+k_\perp^2\varkappa_\perp+\overline{k_\mu k_\nu\chi_{\mu\nu}(\mathbf{p})}}$$
$$+\frac{\delta(\cos\vartheta-\cos\vartheta')\delta(\varphi-\varphi')}{\nu'+ik_\alpha V_\alpha+k_\alpha k_\beta\chi_{\alpha\beta}(\mathbf{p})}, \qquad (6.21)$$

where the bar indicates, as before, averaging over the angles θ,φ of the particle momentum and $\nu'=\nu-i\omega$.

Next, if we use the scheme employed in Secs. 4 and 5, we obtain a transcendental algebraic equation for the transverse diffusion coefficient $D_{\alpha\beta}^\perp=D_\perp\,\delta_{\alpha\beta}^\perp$:

$$D_{\alpha\beta}^\perp=\varkappa_\perp\,\delta_{\alpha\beta}^\perp+\left(1-\frac{\varepsilon}{2}\right)^2\int\frac{\langle u_\alpha u'_\beta\rangle_{\mathbf{k},\omega}}{i\omega+k_\parallel^2 D_\parallel+k_\perp^2 D_\perp}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}$$
$$+\varkappa_\parallel\,\frac{1-\varepsilon-\langle B_0^2\widetilde{B}^2/B^4\rangle}{(1-\varepsilon)^2}\int\frac{\langle\widetilde{B}_\alpha\widetilde{B}'_\beta\rangle_{\mathbf{k},\omega}}{B_0^2}$$
$$\times\left(1-\frac{k_\parallel^2\varkappa_\parallel}{i\omega+k_\parallel^2 D_\parallel+k_\perp^2 D_\perp}\right)\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}, \qquad (6.22)$$

where

$$D_\parallel=\varkappa_\parallel\left[1+\frac{\langle\widetilde{B}^2\rangle}{B_0^2}\frac{1-\varepsilon-\langle B_0^2\widetilde{B}^2/B^4\rangle}{(1-\varepsilon)^2}\right]. \qquad (6.23)$$

We emphasize that these relationships are valid for any ratio of regular to random magnetic fields ($0\leqslant\varepsilon\leqslant1$) and also without any restriction on the amplitude of the turbulent-velocity pulsations. The orientation of the vectors that was assumed above ($\mathbf{u}$ and $\widetilde{\mathbf{B}}$ perpendicular to $\mathbf{B}_0$) has ensured that the turbulence correlation tensors appeared only in the equation for the transverse diffusion coefficient.

6.3. Transverse diffusion regimes

We shall now examine particle transport by weak Alfven turbulence: ($\varepsilon=\langle\widetilde{B}^2\rangle/B_0^2=\langle u^2\rangle/v_{\mathrm{ph}}^2\ll1$). When the local transport range is short enough, we have $v_{\mathrm{ph}}L/v\Lambda_\parallel\gg1$. We then have $D_\parallel\approx\varkappa\approx v\Lambda_\parallel/3$ and $\omega=|\mathbf{k}\mathbf{v}_A|\gg k_\parallel^2 D_\parallel+k_\perp^2 D_\perp$, so that, using (5.23), we obtain

$$D_\perp\approx\varepsilon\varkappa_\parallel\ll\varkappa_\parallel\,. \qquad (6.24)$$

In this case, anomalous transport across the uniform magnetic field is proportional to the square of the amplitude of the turbulent component of the field.

When the longitudinal range of the particles is long enough, so that $v\Lambda_\parallel\gg v_{\mathrm{A}}L$, and if we neglect the attenuation of the Alfven modes and integrate with respect to the frequencies and the angles of the vectors $\mathbf{k}$ in the second term on the right-hand side of (33), we obtain

$$\int\frac{\langle u_\alpha u'_\beta\rangle_{\mathbf{k},\omega}}{i\omega+k_\parallel^2 D_\parallel+k_\perp^2 D_\perp}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}$$
$$=\pi\delta_{\alpha\beta}^\perp\int\frac{k^2\mathrm{d}k}{(2\pi)^3}T(k)\frac{1}{k^2(D_\parallel-D_\perp)}\mathrm{Re}$$
$$\times\int_0^1\frac{\mathrm{d}x}{\dfrac{D_\perp}{D_\parallel-D_\perp}+\dfrac{v_{\mathrm{A}}^2}{k^2(D_\parallel-D_\perp)^2}+\widetilde{x}^2}, \qquad (6.25)$$

where $\widetilde{x}=x+[iv_{\mathrm{A}}/2k(D_\parallel-D_\perp)]$. In this calculation, we use the correlation tensor

$$\langle u_\alpha u'_\beta\rangle_{\mathbf{k},\omega}=T(k,\omega)(\delta_{\alpha\beta}^\perp-k_\alpha^\perp k_\beta^\perp k_\perp^{-2}) \qquad (6.26)$$

and the dispersion law $\omega_k=|\mathbf{k}\cdot\mathbf{v}_{\mathrm{A}}|$. For the fluctuations spectra with characteristic scale L, we find from (6.25) that

$$\int\frac{\langle u_\alpha u'_\beta\rangle_{\mathbf{k},\omega}}{i\omega+k_\parallel^2 D_\parallel+k_\perp^2 D_\perp}\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}\approx\varepsilon v_{\mathrm{A}}Lb\delta_{\alpha\beta}^\perp, \qquad (6.27)$$

where b is a numerical factor of the order of unity. Similarly, we can obtain an estimate for the third term on the right-hand side of (6.22). All this finally leads to

$$D_{\perp} \approx \frac{3\pi b}{4} \varepsilon v_{A} L, \tag{6.28}$$

for particles with longitudinal diffusion coefficients in the range $v_A L \ll \varkappa_{\parallel} \ll v_A L \varepsilon^{-1}$ where the constant factor b depends on the shape of the spectrum and is given by

$$bL\langle u^2\rangle = \frac{1}{4\pi^3} \int_0^\infty kT(k)\mathrm{d}k,$$

$$\langle u^2\rangle = \frac{1}{4\pi^3} \int_0^\infty T(k)k^2\mathrm{d}k.$$

If $\varkappa_{\parallel} \gg v_A L \varepsilon^{-1}$, the fluctuations obtained for such particles may be regarded as quasistatic, in which case

$$D_{\perp} \approx \frac{3}{4} \varepsilon^2 \varkappa_{\parallel} + \frac{\varepsilon\pi}{4} (D_{\perp} \varkappa_{\parallel})^{1/2}. \tag{6.29}$$

Hence,

$$D_{\perp} \approx g\varepsilon^2 \varkappa_{\parallel}, \tag{6.30}$$

where the numerical factor is $g \approx 1.8$. These results are in qualitative agreement with the corresponding regimes examined by Ptuskin and Chuvil'gin.[13] The transport of charged particles by small-amplitude quasistatic fluctuations is proportional to the fourth power of the amplitude.

The latter dependence is in agreement with the estimate reported by Kadomtsev and Pogutsee[9] for the transverse thermal conductivity in the presence of a small-amplitude random magnetic field.

The description of the transport of particles by strong turbulence requires a numerical solution of the transcendental algebraic equations (6.22)–(6.23). For typical turbulence spectra, this leads to the conclusion of isotropization of diffusion by strong turbulence: $D_{\perp} \sim D_{\parallel}$. In particular, if the random magnetic field is static, the results do not depend on the particular form of the field correlator. The equation $D_{\perp} = D_{\parallel}$ is attained for $\varepsilon = 0.77$ whereas for $\varepsilon = 1$ we have $D_{\perp} = 0.25v^2/3\nu$ and $D_{\parallel} = 0$. The last result is due to the fact that, when $D_0 = 0$, the lines of force of the large-scale magnetic field become flat and the particles are not displaced at right angle to these planes.

Although the results presented above are valid for any ratio or $\langle \tilde{B}^2\rangle$ and B_0^2, they are restricted by certain other conditions (short local transport range, zero gyroradius, perpendicular $\tilde{\mathbf{B}}$ and $\mathbf{B}_0$, and incompressibility of the medium). Results that are free from some of these limitations, but still assume that the amplitude of the random field is small, can be found in Refs. 11, 13, 41, and 78.

7. PARTICLE KINETICS IN A TURBULENT PLASMA WITH STRONG INTERMITTENCY AND SHOCK FRONTS

7.1. Introduction

Analysis of a number of problems on the evolution of scalar and vector fields in stochastic media has recently led to an understanding of the importance of intermittency in the stochastical description of such fields. Examples involving very different physical systems are cited, for example, in Refs. 79 and 80.

In this Section, we consider intermittent distributions of charged particles that evolve as a result of acceleration and transport of particles in a medium with a large-scale ultrasonic and ultra-Alfven fluctuations. The acceleration of particles by fluctuations in the electric field induced by plasma motion in a magnetic field (Fermi mechanism) is regarded as one of the basic mechanisms responsible for the evolution of the spectra of suprathermal particles, including, cosmic rays.

There is particular interest in the acceleration of particles by shock waves in turbulent media. The universal character of this mechanism was established in Refs. 17–20. Much attention has been devoted to this effect in recent years (see the review paper in Ref. 81) because this type of acceleration can be observed directly near the front of a geomagnetic head shock wave and also in interplanatary space. There is no doubt that particle acceleration processes occur in larger-scale phenomena with shock waves such as supernova explosions and strong stellar wind emanating from stars of early spectral class in the Galaxy.[82]

In active regions of astrophysical objects such as active galatic nuclei, stellar associations of early spectral class, and many others, the presence of numerous sources of powerful perturbation and of strong inhomogeneities suggests that there should be random ensembles of relatively strong shock fronts against a background of large-scale condensation and rarefaction waves and various other smooth perturbations with a very wide spectrum of spatial and state temporal scales. Since shock waves are the main transporters of energy under such conditions, the presence of shock fronts is a characteristics feature of ultrasonic and ultra-Alfven turbulence in compact active regions. Different aspects of the interaction of charged particles with ultrasonic and ultra-Alfven turbulence are examined in Refs. 83–88.

Particles accelerated in the vicinity of shock waves may respond by influencing the shock fronts. This is investigated in many publications, including Refs. 66, 81, and 88–92. A systematic and rigorous theory of this phenomenon essentially reduces to the problem of the structure of a collisionless shock-wave front. Despite the considerable advances in our understanding of the underlying processes, this problem is still far from solved. The so-called two-fluid model[88,89] is the most popular and is based on the idea of two fluids, i.e., thermal plasma and cosmic rays, coupled by fluctuating magnetic fields.

The two-fluid model predicts that the structure of a shock front contains an intrinsically viscous ion discontinuity (in velocity and other quantities) and a region of smooth deceleration of the flow incident on the front (prefront), with significantly different dimensions. The prefront originates from CR effects. For very strong shock waves, the model predicts that the structure of the front may become smeared out until the viscous dissipative discontinuity vanishes altogether (these waves are often referred to in Western literature as 'CR-dominated shock waves'). Jones and Ellison[81] consider that such waves cannot exist and the occurrence of such solutions is a difficulty for the two-fluid model.

On the other hand, the existence of a prefront and of a viscous discontinuity has been confirmed by observations of the interplanatary medium and has long been known in the physics of radiation-dominated shock waves. We shall use these ideas in Sec. 7.2 to show that the nonlinear structure of a shock wave can be approximately taken into account in the theory of acceleration of particles by ensembles of shock waves.

Since particles become accelerated in the vicinity of a MHD shock front, their distribution acquires an inhomogeneity whose spatial scale l is of the order of $\varkappa/u \approx = v\Lambda/u$ where u is the velocity of the front and $\varkappa$ is the local diffusion coefficient along the normal to the front. This diffusion may be due to small-scale fluctuations in macroscopic turbulent fields and, in sufficiently dense media, Coulomb collisions as well. The formation of the spectrum of accelerated particles by an ensemble of shock fronts depends significantly not only on the strength of these fronts, but also on the ratio of the scale l to the mean front separation L (which we shall identify with the maximum size of turbulent cells, i.e., with the principal turbulence scale).

When $\psi = L/l \approx uL/\varkappa \ll I$, a given particle will interact with several fronts within the characteristic time $\varkappa/u^2$, and the inhomogeneity in the particle distribution will be determined by the smaller of the two scales, i.e., by L. The distribution can be averaged over regions with linear dimensions of the order of L. Perturbation theory is valid under these conditions. The corresponding problem of evaluating the transport coefficients and of finding the shape of the accelerated-particle spectrum was solved in Refs. 83 and 86.

The opposite case, $\psi \gg I$, is more complicated and interesting. Here, a strong inhomogeneity in the accelerated-particle distribution is formed near each individual shock front, and the spatial scale of the inhomogeneity is small in comparison with the principal turbulence scale. The particle distribution thus becomes highly intermittent.

The statistical description of intermittent systems requires additional information as compared with systems that are stochastically homogeneous in all their scales. The accelerated-particle distribution will therefore be described by two distribution functions, namely, the mean function corresponding to the distribution of particles in regions between strong fronts and a local distribution function that differs from zero near a given front. The parameter ψ then characterizes the degree of intermittency of the system. For small ψ, the fluctuating part of the distribution function is small in comparison with the mean distribution function. This was demonstrated in Ref. 83 in the course of a derivation of the kinetic equation for the mean distribution function. For large ψ (this usually corresponds to suprathermal particles with low energies), the fluctuations in the particle distribution are stronger.

To take into account more correctly the contribution of shock fronts to acceleration, we shall use the well-known solution[18] for an individual front of arbitrary strength and, on the first stage, average the distribution function over spatial regions of scale l in the vicinity of the front. This results in the appearance in the equation for the mean distribution function of an integral operator describing strong acceleration on individual fronts. Moreover, it is important that this operator can be modified so that the nonlinear distortion of the shock-wave profile can be approximately taken into account.

The next stage is to average the kinetic equation over regions of the order of the principal scale L. Since $\psi \gg I$, perturbation theory is not valid and we must renormalize the transport coefficients. This can be done by the method discussed in Sec. 4 (see also Ref. 87). The transport coefficients cannot be calculated from the pair correlation tensor of the turbulent velocity field alone: it is also necessary to have information on the statistical properties of the fronts and on their correlations with the velocity field between the fronts. This means that higher-order velocity-field correlators must be taken into account.

7.2. Interaction between charged particles and strong shock waves

Suppose that we have a random ensemble of shock fronts separated by a mean distance L and that the Mach numbers are such $M-1 \gg 1$. We shall assume that the ensemble is stochastically homogeneous and isotropic. An inhomogeneous cloud of accelerated particles with a power-type spectrum in a wide range of energies is formed near each front during the mean front collision time L/u where u is the front velocity (not very different from the characteristic velocity of the medium when values $M \gg 1$ have low probability). The inhomogeneity scale of this distribution is of the order of $l \approx v\Lambda/u \gg \Lambda$ where v is the particle velocity and Λ is the transport range in the turbulent medium ahead of the front. The distribution of accelerated particles in space is thus seen to be highly inhomogeneous: near the fronts, there are relatively narrow accelerated-particle peaks which, after front collisions, spread out by turbulent diffusion through the system. However, the appearance of the next shock front of sufficient strength again gives rise to a strong inhomogeneity in the particle distribution. This picture of acceleration is a natural consequence of the intermittency of ultrasonic turbulence, namely, the presence of strong discontinuities within it.

To construct the accelerated-particle distribution function averaged over the ensemble of random fronts, we must correctly take into account these local inhomogeneities. Assuming that $l \ll L$, we isolate scales Δ such that

$$l \ll \Delta \ll L, \tag{7.1}$$

and average the distribution function over such scales. Since $\Lambda \ll L$, we can use the transport equation given by (1.3) in regions between these fronts:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial r_\alpha} \varkappa_{\alpha\beta} \frac{\partial N}{\partial r_\beta} - u_\alpha \frac{\partial N}{\partial r_\alpha} + \frac{p}{3} \frac{\partial N}{\partial p} \frac{\partial u_\alpha}{\partial r_\alpha}, \tag{7.2}$$

and on the front itself we must use the continuity of the distribution functions and differential (with respect to p) particle front densities (see Ref. 25 for further details). We shall use $\bar{N}$ to represent the distribution function averaged over scales Δ. At distances of the order of $\geqslant\Delta$ from the fronts, we have $\bar{N}\approx N$, so that the inhomogeneity scale N in this region is large in comparison with Δ. In the vicinity of a discontinuity, on the other hand, we can use the well-known solution of (7.2) for a plane shock front[18,25,92] and relate the local value of the distribution function to its value $\bar{N}$ well away from the front (at distances of the order Δ)

$$N_i(z,p)=\theta(-z_i)\left\{\bar{N}(p)+\left[(\gamma_i+2)\right.\right.$$
$$\times p^{-(\gamma_i+2)}\int_0^p p'^{(\gamma_i+1)}\bar{N}(p')\mathrm{d}p'$$
$$\left.\left.-\bar{N}(p)\right]\exp\frac{\Delta u_{ni}z_i}{\varkappa_i}\right\}+\theta(z_i)(\gamma_i+2)$$
$$\times p^{-(\gamma_i+2)}\int_0^p p'^{(\gamma_i+1)}\bar{N}(p)\mathrm{d}p'; \qquad (7.3)$$

where z_i is the coordinate measured along the normal to the first front (ahead of the front, $z_i<0$ and after the front, $z_i>0$), $\varkappa_i$ is the diffusion coefficient in the direction of the normal ahead of the front, Δu_{ni} is the change in the normal component of the velocity of the medium across the front, $\gamma=[(\rho_2/\rho_1)+2]/[(\rho_2/\rho_1)-1]$ is the exponent in the 'universal' spectrum on an individual front, and ρ_2/ρ_1 is the relative compression of the medium in the shock wave. The solution given by (7.3) can readily be written with the help of the Green's function constructed for the plane shock front. The quantity $\bar{N}(p)$ that appears in (7.3) will of course remain a function of the large-scale coordinates (determined to within Δ) and time.

The use of the stationary solution in (7.3) is allowed because of the rapid evolution of the power spectrum on the shockfront: when $\varkappa=\text{const}$, the time Δt necessary for it to evolve in the range p_0 and p is given by the approximate expression (see Ref. 25)

$$\Delta t\approx\frac{3\varkappa}{u\Delta u_n}\ln\frac{p_0}{p_0}\approx\frac{l}{\Delta u_n}\ln\frac{p}{p_0}. \qquad (7.4)$$

This time (for $\Delta u\sim u$) is much shorter than the average time between front collisions, which is of the order L/u.

Having evaluated the local inhomogeneities in the distribution function near the fronts (7.3), we perform the direct averaging of (7.2) over the scales Δ. To do this, we extract the singular term from $\partial u_\alpha/\partial x_\alpha$:

$$\frac{\partial u_\alpha}{\partial r_\alpha}=\left(\frac{\partial u_\alpha}{\partial r_\alpha}\right)_0-\sum_i\Delta u_{ni}\delta(z_i); \qquad (7.5)$$

where $(\partial u_\alpha/\partial x_\alpha)_{\text{smooth}}$ is the smooth part of the divergence of the velocity, which is of the order of u/S, and the δ-terms are due to velocity jumps across discontinuities. We now write

$$\overline{\frac{p}{3}\frac{\partial N}{\partial p}\frac{\partial u_\alpha}{\partial r_\alpha}}=\frac{p}{3}\frac{\partial\bar{N}}{\partial p}\frac{\partial\bar{u}_\alpha}{\partial r_\alpha}+\frac{p}{3}\frac{\partial}{\partial p}\overline{(N-\bar{N})\frac{\partial u_\alpha}{\partial r_\alpha}} \qquad (7.6)$$

and evaluate the last term. It is clear that, well away from the front, this term is of the order of $(\Delta/L)(p/3)\times(\partial\bar{N}/\partial p)(\partial u_\alpha/\partial x_\alpha)$ and, since $\Delta/L\ll1$, can be discarded. Near the ith front (in a layer of thickness Δ) we have, using (7.3),

$$(N-\bar{N})_i=-p^{-(\gamma_i+2)}\int_0^p\frac{\partial\bar{N}_i}{\partial p'}(p')p'^{\gamma_i+2}$$
$$\times\left(\theta(z_i)+\theta(-z_i)\exp\frac{\Delta u_{ni}z_i}{\varkappa_i}\right). \qquad (7.7)$$

With the help of (7.7) and (7.5) (in the latter, we take into account only the singular terms), we obtain

$$\frac{p}{3}\frac{\partial}{\partial p}\overline{(N-\bar{N})\frac{\partial u_\alpha}{\partial r_\alpha}}=\sum_i\frac{u_{i\alpha}}{3\Delta}\left(p\frac{\partial\bar{N}_i}{\partial p}-\frac{\gamma_i+2}{3}p^{-(\gamma_i-2)}\right.$$
$$\left.\times\int_0^p p'^{\gamma_i+2}\frac{\partial\bar{N}_i}{\partial p'}\mathrm{d}p'\right). \qquad (7.8)$$

Although the right-hand side is written as the sum over all the fronts, the ith term is actually different from zero only in the vicinity of the ith front.

Next,

$$\overline{u_\alpha\frac{\partial N}{\partial r_\alpha}}=\bar{u}_\alpha\frac{\partial\bar{N}}{\partial r_\alpha}+\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar{N})}. \qquad (7.9)$$

and

$$\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar{N})}=\frac{1}{\Delta}\sum_i\int_{-\Delta/2}^{\Delta/2}u_{i\alpha}\frac{\partial}{\partial r_\alpha}(N-\bar{N})_i\mathrm{d}z_i$$
$$=\sum_i\frac{u_{i\alpha}}{\Delta}(N-\bar{N})\Big|_{-\Delta/2}^{\Delta/2}$$
$$-\sum_i\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2}(N-\bar{N})_i\frac{\partial u_{i\alpha}}{\partial z_i}\mathrm{d}z_i.$$

For distances of the order of $\Delta/2$ from the front, we have $M-\bar{N}\approx0$, so that the term outside the integral must vanish and from (7.5) we finally obtain

$$\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar{N})}=-\sum_i\frac{\Delta u_{ni}}{\Delta}p^{-2-\gamma_i}$$
$$\times\int_0^p p'^{(\gamma_i+2)}\frac{\partial\bar{N}_i}{\partial p'}\mathrm{d}p'. \qquad (7.10)$$

Combining the contributions of (7.8) and (7.10), we have

$$\frac{p}{3}\frac{\partial}{\partial p}\overline{(N-\bar{N})\frac{\partial u_\alpha}{\partial r_\alpha}}-\overline{u_\alpha\frac{\partial}{\partial r_\alpha}(N-\bar{N})}$$
$$=\frac{1}{p^2}\frac{\partial}{\partial p}\sum_i\frac{\Delta u_{ni}}{3\Delta}p^{1-\gamma_i}\int_0^p p'^{\gamma_i+2}\frac{\partial\bar{N}_i}{\partial p'}\mathrm{d}p'. \qquad (7.11)$$

Terms of the form

$$\frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial}{\partial r_\beta}(N-\bar{N})$$

do not provide an additional contribution either near the fronts or away from them after averaging. Hence, taking the average of (7.2) with the help of by (7.6) and (7.9), we find that

$$\frac{\partial \bar{N}}{\partial t}=\frac{\partial}{\partial r_\alpha}\bar{\varkappa}_{\alpha\beta}\frac{\partial \bar{N}}{\partial r_\beta}-\bar{u}_\alpha\frac{\partial \bar{N}}{\partial r_\alpha}+\frac{p}{3}\frac{\partial \bar{N}}{\partial p}\frac{\partial \bar{u}_\alpha}{\partial r_\alpha}+\frac{1}{p^2}\frac{\partial}{\partial p}\sum_i\frac{\Delta u_{ni}}{3\Delta}p^{1-\gamma_i}\int_0^p p'^{\gamma_i+2}\frac{\partial \bar{N}_i}{\partial p'}\,dp'. \quad (7.12)$$

The last equation can be written in a somewhat simpler form. Consider an ensemble of fronts with the same index ($\gamma_i=\gamma$; for sufficiently strong waves, we know that γ is a slow function of M) and let us introduce the function

$$\sum_i \overline{\Delta u_n \delta(z_i)},$$

that assumes the values $\Delta u/\Delta$ near the ith front (in a layer of thickness Δ) and 0 between fronts. Equation (7.12) then takes the form

$$\frac{\partial \bar{N}}{\partial t}=\frac{\partial}{\partial r_\alpha}\bar{\varkappa}_{\alpha\beta}\frac{\partial \bar{N}}{\partial r_\beta}-\bar{u}_\alpha\frac{\partial \bar{N}}{\partial r_\alpha}+\frac{p}{3}\frac{\partial \bar{N}}{\partial p}\frac{\partial \bar{u}_\alpha}{\partial r_\alpha}+\frac{1}{3p^2}\frac{\partial}{\partial p}p^{1-\gamma}\int_0^p p'^{(\gamma+2)}\frac{\partial \bar{N}}{\partial p'}\,dp'\sum_i\overline{\Delta u_n\delta(z_i)}. \quad (7.13)$$

We assumed in the derivation of (7.12) that the fronts were discontinuities in velocity whose width was small in comparison with the gyroradii of the accelerated particles and their transport ranges. On the other hand, it was noted in Sec. 7.1 that the structure of a shock wave could be modified as a result of the retardation of the incident flux by cosmic rays.[66,81,92] Equation (7.12) has a solution that allows us to take approximately into account the presence of prefronts and of a viscous discontinuity.

Qualitatively, the front structure ensures that low-energy particles are accelerated only in the viscous jump, whereas the smooth part is received as an adiabatic perturbation, described by the field $\bar{u}_\alpha(\mathbf{r},t)$ in (7.12). The spectrum of particles with momenta $p<p_{*i}$ and spectral index γ_{ti} is therefore due to the viscous jump Δu_{nti} in which the relative compression is smaller than the total compression of the medium by the entire shock wave. The limiting momentum p_{*i} is determined by the condition that the particle range is equal to the thickness of the prefront. For high-energy particles, $p>p_{*i}$, the spectral index γ_{ti} is formed by the total velocity jump Δu_{nti} across which the compression of the medium is greater. Hence, the spectrum of particles accelerated in the neighborhood of the ith shock wave is harder for $p>p_{*i}$ ($\gamma_{si}<\gamma_{ti}$).

In view of the foregoing, we shall write the transport equation for charged particles in a system with an ensemble of shock fronts in the form

$$\frac{\partial \bar{N}}{\partial t}=\frac{\partial}{\partial r_\alpha}\bar{\varkappa}_{\alpha\beta}\frac{\partial \bar{N}}{\partial r_\beta}-\bar{u}_\alpha\frac{\partial \bar{N}}{\partial r_\alpha}+\frac{p}{3}\frac{\partial \bar{N}}{\partial p}\frac{\partial \bar{u}_\alpha}{\partial r_\alpha}+\frac{1}{p^2}\frac{\partial}{\partial p}\sum_i\Bigg(\theta(p_{*i}-p)\times\frac{\Delta u_{nsi}}{3\Delta_{si}}p^{1-\gamma_{si}}\int_0^p p'^{\gamma_{si}+2}\frac{\partial \bar{N}_i}{\partial p'}\,dp'+\theta(p-p_{*i})\times\frac{\Delta u_{nti}}{3\Delta_{ti}}p^{1-\gamma_{ti}}\int_{p_{*i}}^p p'^{\gamma_{ti}+2}\frac{\partial \bar{N}_i}{\partial p'}\,dp'\Bigg). \quad (7.12a)$$

In writing this equation, we consider a two-stage compression in a shock wave, which means that we might achieve a correct description of the asymptotic behavior of the spectrum (see Ref. 88). This model is simple enough and useful for practical calculations. A more general form of the acceleration term for shock fronts leads to

$$\frac{\partial \bar{N}}{\partial t}=\frac{\partial}{\partial r_\alpha}\bar{\varkappa}_{\alpha\beta}\frac{\partial \bar{N}}{\partial r_\beta}-\bar{u}_\alpha\frac{\partial \bar{N}}{\partial r_\alpha}+\frac{p}{3}\frac{\partial \bar{N}}{\partial p}\frac{\partial \bar{u}_\alpha}{\partial r_\alpha}+\frac{1}{p^2}\frac{\partial}{\partial p}\sum_i\int_0^p G_i(p,p')\bar{N}_i(p')\,dp', \quad (7.12b)$$

where the Green's function $G_i(p,p_i)$ describes the transformation of the particle spectrum on the ith shock front. We now need additional observations and calculations of the structure of collisionless shock waves with a view to establishing the form of the function G in the nonlinear regime. It must also be borne in mind that, in the nonlinear case, p_{*i},γ_{si}, and the Green's function $G_i(\mathbf{p},\mathbf{p}')$ are generally functionals of the mean distribution functions of the accelerated particles. A sufficiently simple working theory can be constructed only by using the corresponding approximation and by parametrizing these quantities.

7.3. Kinetic equation for the distribution function averaged over large-scale motion in the medium

To perform further averaging of (7.13) over regions with dimensions of the order of the principal turbulent scale L, we use the notation

$$\overline{\left\langle\sum_i \Delta u_n\delta(z_i)\right\rangle}=\frac{1}{\tau_{\rm sh}},\quad \frac{\partial \bar{u}_\alpha}{\partial r_\alpha}=\Psi(\mathbf{r},t),\quad \langle\Psi\rangle=0,$$

$$\sum_i \overline{\Delta u_n\delta(z_i)}-\frac{1}{\tau_{\rm sh}}=\varphi(\mathbf{r},t),\quad \langle\varphi\rangle=0, \quad (7.14)$$

where the operators $\hat{L}$ and $\hat{P}$ take the form

$$\hat{L}=\frac{1}{3p^2}\frac{\partial}{\partial p}p^{3-\alpha}\int_0^{p'}dp'p'^{\alpha}\frac{\partial}{\partial p'},\quad \hat{P}=\frac{p}{3}\frac{\partial}{\partial p},$$

in which $\alpha=\gamma+2$ and the angle brackets represent the above averaging. It is readily verified that the above two operators commute. Next, we introduce a new distribution function $\bar{f}(\mathbf{r},p,t)$, related to $\bar{N}$ by

$$\bar{N}=\exp\left(\frac{\hat{L}t}{\tau_{\rm sh}}\right)\bar{f}. \quad (7.15)$$

The equation for $\bar{f}$ is

$$\frac{\partial\bar{f}}{\partial t}-\frac{\partial}{\partial r_\alpha}\bar{\varkappa}_{\alpha\beta}\frac{\partial\bar{f}}{\partial r_\beta}=\hat{Q}\bar{f}, \tag{7.16}$$

where

$$\hat{Q}=-\bar{\mathbf{u}}\nabla+\Psi\hat{P}+\varphi\hat{L} \tag{7.17}$$

is a random operator. The quantity $\bar{\varkappa}_{\alpha\beta}$ is the diffusion tensor due to small-scale turbulence; in general, it undergoes random changes in both space and time. However, since $uL/v\Lambda\ll 1$, the transport of particles in space is largely due to the motion of the medium (turbulent diffusion). Diffusion due to small-scale field plays a minor role under these conditions. We shall therefore assume in (7.16) that the diffusion tensor $\bar{\varkappa}_{\alpha\beta}$ is constant and isotropic and that $\bar{\varkappa}_{\alpha\beta}=\varkappa\delta_{\alpha\beta}$ is independent of the momentum variable p.

We can now apply the general approach developed in Sec. 4 for the averaging of (7.16). We shall assume that only harmonics with sufficiently close wave numbers are correlated in the spectrum of the random quantities $\bar{u}_\alpha$, σ and φ in (7.16). This means that we can extract a macroscopically small wave-number interval Δk and assume that harmonics belonging to this interval do not correlate with all others. The corresponding contributions to the velocity field will be denoted by $\delta\bar{\mathbf{u}},\delta\sigma,\delta\phi$ where

$$\delta\Psi=\int_{-\infty}^{\infty}\mathrm{d}o\int_{\Delta k}\mathrm{d}^3k\Psi_{\mathbf{k},\omega}e^{i(\mathbf{kr}-\omega\mathbf{k})}, \tag{7.18}$$

and so on. During the averaging procedure, the Fourier components of turbulent quantities satisfy the usual relationships for homogeneous turbulence:

$$\langle\Psi_{\mathbf{k},\omega}\Psi_{\mathbf{k}',\omega'}\rangle=|\Psi|^2_{\mathbf{k},\omega}\delta(\mathbf{K}+\mathbf{k}')\delta(\omega+\omega'). \tag{7.19}$$

For a complete description of the acceleration process, we must introduce two further spectral functions[87] in addition to the functions T and S associated with the vortical and potential motions. One of them, $\tilde{\varphi}(\mathbf{k},\omega)$, describes correlations between velocity jumps on shock fronts and the other, $\mu(\mathbf{k},\omega)$ represents mutual correlation between $\varphi(\mathbf{r},t)$ and $\sigma(\mathbf{r}',t')$. The introduction of these spectral functions is dictated by the intermittent character of the particle distribution function, the description of which requires additional statistical information about the random velocity field (cf. statistically homogeneous problems examined in Secs. 4 and 5 where a significantly smaller number of correlators was used).

The mean distribution function $F=\langle\bar{N}\rangle$ satisfies the following kinetic equation when the energy change within the correlation length of the smooth velocity field is small (but the energy change across the shock wave front is large):

$$\frac{\partial F}{\partial t}-\chi_{\alpha\beta}\nabla_\alpha\nabla_\beta F=\left(\frac{1}{\tau_{\mathrm{sh}}}+B\right)\hat{L}F+\frac{1}{p^2}\frac{\partial}{\partial p}p^4D\frac{\partial F}{\partial p}+A\hat{L}^2F$$
$$+2B\hat{L}\hat{P}F. \tag{7.20}$$

The characteristic feature of this equation is the presence of the integral parameter $\hat{L}$ which shows that there is strong acceleration of particles near an individual wave front. At the same time, the acceleration effect may on average be relatively small if the fronts appear relatively infrequently.

The expressions for the transport coefficients have the following form:[87]

$$\chi=\varkappa+\frac{1}{3}\int\frac{\mathrm{d}^3\mathbf{k}\mathrm{d}\omega}{(2\pi)^4}\left[\frac{2T+S}{i\omega+\chi k^2}-\frac{2k^2\chi S}{(i\omega+\chi k^2)^2}\right], \tag{7.21}$$

$$D=\frac{8}{9}\pi\chi\int_0^\infty k^6\mathrm{d}k\int_0^\infty\mathrm{d}\omega\frac{S(k,\omega)}{(2\pi)^4(\omega^2+\chi^2k^4)}, \tag{7.22}$$

$$A=8\pi\chi\int_0^\infty k^4\mathrm{d}k\int_0^\infty\mathrm{d}\omega\frac{\tilde{\varphi}(k,\omega)}{(2\pi)^4(\omega^2+\chi^2k^4)}, \tag{7.23}$$

$$B=8\pi\chi\int_0^\infty k^4\mathrm{d}k\int_0^\infty\mathrm{d}\omega\frac{\mu(k,\omega)}{(2\pi)^4(\omega^2+\chi^2k^4)}. \tag{7.24}$$

These expressions allow for the fact that, for the isotropic turbulence that we are considering, $\chi_{\alpha\beta}=\chi\delta_{\alpha\beta}$. Moreover, we have used the parity property of the spectral functions in ω. To calculate the spatial diffusion coefficient χ we have to solve the transcendental equation given by (7.21), after which the evaluation of the coefficients A,B,D that determine the rate of acceleration reduces to integration.

Strong acceleration

If the change in the particle energy within the turbulent velocity correlation time or length is not small ($\Delta p\gg p$), then acceleration on fronts and between the fronts will be described by an integral operator (see Secs. 4.3 and 4.4). Nevertheless, the renormalized transport coefficients can be calculated even in this case by slightly modifying the above scheme. The case of strong acceleration can be included in the general scheme because the acceleration operators $\hat{P}$ and $\hat{L}$ are homogeneous in the averaged equation (7.15) (they are invariant under the similarity transformation in the momentum variable). After averaging, the kernel of the integral operator can therefore be written as a function of the difference $\eta-\eta'$ where $\eta=\ln(p/p_0)$.

The equation for the total distribution function $F(\mathbf{r},\eta,t)=\langle\bar{N}(\mathbf{r},p,t)\rangle$ will now be sought in the form

$$\frac{\partial F}{\partial t}=\int_{-\infty}^{+\infty}\chi_{\alpha\beta}(\eta-\eta')\nabla_\alpha\nabla_\beta F\mathrm{d}\eta'+\left(\frac{\partial}{\partial\eta}+3\right)$$
$$\times\int_{-\infty}^{+\infty}D(\eta-\eta')F(r,\eta',t)\mathrm{d}\eta', \tag{7.25}$$

where the operator

$$\partial/\partial\eta+3=p^{-2}(\partial/\partial p)p^3 \tag{7.26}$$

ensures that the second term vanishes when both parts of the equation are integrated over all momenta. We shall not go into the details of the averaging procedure (they are described in Ref. 87) and merely present the final result. The Fourier transforms of the kernels of the integral equation (7.25), $\bar{\chi}(s)$ and $\bar{D}(s)$, can be calculated by solving the following set of two transcendental equations:

$$\bar{\chi}(s)=\varkappa+\frac{1}{3}\int\frac{d^3k d\omega}{(2\pi)^4}\left\{\frac{2T+S(k,\omega)}{i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2}\right.$$

$$\left.-\frac{2k^2\bar{\chi}(s)S(k,\omega)}{[i\omega+(is-3)D(s)+\bar{\chi}(s)k^2]^2}\right\}$$

$$+\int\frac{d^3k^2 d\omega}{(2\pi)^4}\bar{\chi}(s)\frac{i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2/3}{[i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2]^2}$$

$$\times\left[k^2\frac{is(is-3)}{9}S(k,\omega)-\frac{is(is-3)}{9(is-2-\gamma)}\mu(k,\omega)\right.$$

$$\left.+\frac{s^2(is-3)^2}{9(is-2-\gamma)}\tilde{\varphi}(k,\omega)\right], \qquad (7.27)$$

$$\bar{D}(s)=\frac{is}{9}\int\frac{d^3k^2 d\omega}{(2\pi)^4}[-k^2S(k,\omega)+(3-2is)\mu(k,\omega)$$

$$\times(is-2-\gamma)^{-1}-(is-3)is\tilde{\varphi}(k,\omega)$$

$$\times(is-2-\gamma)^{-2}][i\omega+(is-3)\bar{D}(s)+\bar{\chi}(s)k^2]^{-1}$$

$$+is[3(is-2-\gamma)\tau_{\rm sh}]^{-1}. \qquad (7.28)$$

In the case of weak acceleration, $\bar{\chi}(s)$ ceases to depend on s and equation (7.25) takes the form of (7.20). Equation (7.28) then becomes

$$\bar{D}(s)=-isD-\frac{is}{3}\frac{is-3}{(is-\gamma-2)^2}\bar{A}+\frac{is}{is-2-\gamma}$$

$$\times\left(1-\frac{2is}{3}\right)B+\frac{1}{\tau_{\rm sh}}\frac{is}{is-2-\gamma}, \qquad (7.29)$$

where D, A, and B are given by (7.21)-(7.24).

7.4. Time-dependent energy spectra of accelerated charged particles

We must now consider solutions of the time-dependent kinetic equation for the average charged-particle distribution function with instantaneous injection of monoenergetic particles into a turbulent medium containing an ensemble of shock waves.[87,93]

The equation for the Green's function is

$$\frac{\partial G}{\partial t}+\frac{G}{\tau_{\rm e}}=\left(\frac{1}{\tau_{\rm sh}}+B\right)\hat{L}G+\frac{1}{p^2}\frac{\partial}{\partial p}p^4D\frac{\partial G}{\partial p}+A\hat{L}^2G$$

$$+2BL\hat{P}\hat{G}+\frac{1}{p_0}\delta(p-p_0)\delta(t-t'). \qquad (7.30)$$

To solve this, it is convenient to perform the Fourier transformation in $\eta=\ln(p/p_0)-s$ and the Laplace transformation in $t-\sigma$. We then find from (7.30) that

$$G(s,\sigma)=\frac{e^{-\sigma t'}}{\sigma+\tau_{\rm e}^{-1}+(is-3)\bar{D}(s)}. \qquad (7.31)$$

The expression for $\bar{D}(s)$ corresponds to (7.29) and the reverse transformation to the variable p gives

$$G(p,\sigma)=\frac{\tau_{\rm sh}e^{-\sigma t'}}{p_0^3 a}\frac{(x_1+\alpha)^2}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}\left(\frac{p}{p_0}\right)^{x_1},$$

$$p<p_0,\quad\left\{\frac{(x_2+\alpha)^2}{(x_1-x_2)(x_2-x_3)(x_2-x_4)}\left(\frac{p}{p_0}\right)^{x_2}\right.$$

$$+\frac{(x_3+\alpha)^2}{(x_1-x_3)(x_3-x_2)(x_3-x_4)}\left(\frac{p}{p_0}\right)^{x_3}$$

$$\left.+\frac{(x_4+\alpha)^2}{(x_1-x_4)(x_4-x_2)(x_4-x_3)}\left(\frac{p}{p_0}\right)^{x_4}\right\},\quad p>p_0, \qquad (7.32)$$

where $\varkappa_i$ are the roots of the fourth-degree polynomial $ax^4+bx^3+cx^2+dx+e$. The coefficients of the polynomial are related to the above renormalized coefficients A,B,D by the following equations:

$$a=\tau_{\rm sh}(A+2B+9D),$$

$$b=\tau_{\rm sh}(6A+9B+27D+2\alpha(B+9D))+3,$$

$$c=\tau_{\rm sh}[9(A+B)+3\alpha(18D+3B+3D\alpha)]$$

$$+9+3\alpha-9(\varepsilon+\sigma\tau_{\rm sh}),$$

$$d=\tau_{\rm sh}9\alpha(B+3\alpha D)+9\alpha[1-2(\varepsilon+\sigma\tau_{\rm sh})],$$

$$e=-9\alpha^2(\varepsilon+\sigma\tau_{\rm sh}),\quad \varepsilon=\tau_{\rm sh}/\tau_{\rm e}. \qquad (7.33)$$

We shall now examine this system for times $t>\tau_{\rm sh}$ where $\sigma\tau_{\rm sh}\ll 1$. We shall determine the roots x_i for this case to within the linear term in $\sigma\tau_{\rm sh}$. Approximate calculations of these roots for the time-independent state $\sigma\tau_{\rm sh}=0$ show that $x_1\approx\alpha\varepsilon, x_2\approx-3, x_3\approx-\alpha, x_4\approx 3/a (a,\varepsilon\ll 1)$ for wide range of turbulence spectral functions that is compatible with the model of particle acceleration in an association of O and B stars (see Ref. 93). It follows that, when $p>p_0$, we need only take into account the first term in the spectrum (7.32):

$$G(p,\sigma)=\frac{\tau_{\rm sh}e^{-\sigma t'}}{p_0^3 a}\frac{x_1+\alpha}{(x_1-x_2)(x_1-x_4)}\left(\frac{p}{p_0}\right)^{x_1},\quad p<p_0,$$

and

$$\frac{x_2+\alpha}{(x_1-x_2)(x_2-x_4)}\left(\frac{p}{p_0}\right)^{x_2},\quad p>p_0. \qquad (7.34)$$

The reverse transformation in the variable t yields

$$G(p,t,t')=-\frac{x_2}{9p_0^3(x_2+2\alpha)}\alpha e^{-\sigma_1(t-t'-t_{p1})}$$

$$\times\theta(t-t'-t_{p1}),\quad p<p_0,$$

and

$$(x_2+\alpha)[e^{-\sigma_1(t-t'-t_{p2})}-e^{-\sigma_2(t-t'-t_{p2})}]\left(\frac{p}{p_0}\right)^{x_2}$$

$$\times\theta(t-t'-t_{p2}),\quad p>p_0, \qquad (7.35)$$

where

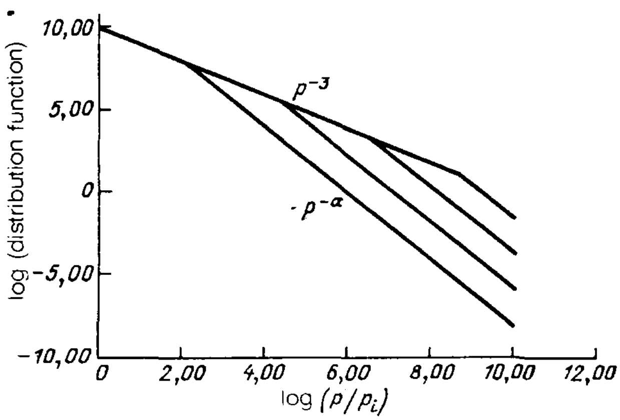

FIG. 5. Spectra of accelerated particles at successive instants of time $t/\tau_{sh}=5,10,15,20$. The curves are plotted on a logarithmic scale: log (intensity of suprathermal particles) as a function of log (p/p_i).

$$\sigma_1=-\frac{a}{9\tau_{sh}}\frac{x_2^2x_4}{(x_2+2\alpha)},\quad \sigma_2=\frac{a}{9\tau_{sh}}\frac{x_2x_4^2}{(2x_2+\alpha)}, \tag{7.36}$$

$$t_{p1}=\frac{9\tau_{sh}}{a}\frac{\alpha}{x_2x_4}\ln\frac{p_0}{p},\quad t_{p2}=\frac{9\tau_{sh}}{a}\frac{(x_2+\alpha)}{x_2x_4}\ln\frac{p}{p_0}. \tag{7.37}$$

For time-independent injection, which is interesting for applications, we must integrate (7.35) with respect to t' between 0 and t. The result is

$$G(p,t)=\frac{\tau_{sh}}{ap_0^3x_2x_4}\alpha[1-e^{-\sigma_1(t-t_{p1})}]\theta(t-t_{p1}),\quad p\leqslant p_0,$$

and

$$(x_2+\alpha)\left(\frac{p}{p_0}\right)^{x_2}[1-e^{-\sigma_2(t-t_{p2})}]\theta(t-t_{p2}),\quad p\geqslant p_0, \tag{7.38}$$

where we have retained only the first term for times $t>\tau_{sh}$.

The time dependence of the Green's function (7.38) has a simple physical interpretation. It describes the evolution of the high-energy part of the particle distribution function. Figure 5 (Ref. 93) illustrates the evolution of the distribution function for power-type particle injection into the acceleration process, which corresponds to the case of injection on shock-wave fronts.

We have thus shown that the description of intermittent particle distributions in systems containing ensembles of shock waves requires detailed statistical information on correlations of shock fronts and their strengths. The determination of the distribution function for suprathermal charged particles can be reduced to the solution of an integral equation for the average distribution function. The intermittent part of the distribution function due to strong deviations from the average distribution function in the neighborhood of shock fronts is then expressed in terms of the average distribution function by means of (7.7).

8. CONCLUSION

We have tried to demonstrate the considerable possibilities of the renormalization method of constructing the equations of transport for impurity particles and the evaluation of the transport coefficients in plasmas with strong turbulence in the presence of an external magnetic field. The advantages of this method include its relative simplicity, possible generalization to compressible media, the inclusion of not only the motion of particles in space but also their acceleration by stochastic electric fields in nonequilibrium systems, and the possibility of including strong intermittency (in the special but important for astrophysics example of interaction of particles with an ensemble of strong shock waves in a turbulent medium). It is important to note that the renormalized equations of transport may not take the Fokker–Planck form, so that they describe regimes that do not reduce to diffusion (see Secs. 4 and 7). The required average equation of transport is then *written in an integral form that is the most general linear* expression which only in special cases, and with the corresponding structure of the kernels, assumes the Fokker-Planck form. The evaluation of the kernels of integral equations for the transport of particles to large distances (greater than the correlation length) is then reduced to the solution of a set of transcendental algebraic (nonintegral) equations, which enables us to express these kernels in terms of the correlators of velocity functions and the magnetic field.

We emphasize that the numerical solutions of such equations does not encounter fundamental difficulties and has been obtained[31,67] for a number of realistic turbulent spectra (see, in particular, Figs. 1–4). We have thus demonstrated the fact that it is possible to take the results of our theory to a numerical stage and to a comparison with experiment.

In our view, these advantages distinguish our method from other approaches to the evaluation of renormalized transport coefficients presented in the literature. For example, the renormalization group method,[49–52] which is being actively developed at present, is apparently capable of describing only the Kolmogorov model of turbulence in an incompressible fluid. We are unaware of any analyses of the effects of compressibility, magnetic field, or particle acceleration performed by this method; this also applies to the theory developed by Phythian and Curtis[4].

At the same time, our method is, of course, subject to certain disadvantages and limitations, which it is convenient to summarize once again. When the renormalized transport coefficients are calculated, we have to employ an approximate description of the motion of particles for small scales (smaller than the correlation length) and consider that the motion is diffusive (see Secs. 4.1–4.3) although this is not strictly correct. Attempts to remove this infelicity (see the end of Sec. 4.1) and to correctly describe transport over short distances involve the evaluation of the kernel of the integral transport equation by solving a certain nonlinear integral equation [such as (4.24)] which is a very difficult but not an insurmountable problem.

The errors introduced by these approximations cannot be estimated theoretically but comparisons with the numerical calculations reported by Drummond *et al.*,[61] suggest that, in most cases, the errors are not large.

Another limitation that has played a very significant role in our method was the assumption that the turbulence was homogeneous [condition (4.4)]. The method will have to be modified to allow for the presence of significant inhomogeneity in the distribution of turbulent fluctuations or background quantities.

The third limitation relates to the approach used to describe the turbulent field. All the information about the turbulent medium enters the calculated transport coefficients exclusively through the pair correlators of velocity, density and magnetic field, which are coupled by the equations of motion of the medium. Since the exact expression for the transport coefficients in terms of the turbulent parameters must in general contain correlators of all orders, this means that some closure procedure is implicit in these calculations, in which the higher-order correlators are expressed in terms of the pair correlator. Consequently, our method can only tackle situations in which the distribution of turbulence is sufficiently homogeneous and the entire significant information about the turbulent field can be described by the lowest-order (pair) correlator. This means, in particular, that intermittency effects that give rise to infrequent but strong inhomogeneities should not play an appreciable role. When this is not so, the method must be modified and we must have more detailed information about the structure of the turbulence. An example of this type of modification is provided by the case of strong intermittency due to the presence of an ensemble of shock fronts in the system, considered in Sec. 7. In addition to the pair correlator, we then also have to use turbulence characteristics such as the distribution of shock fronts over Mach numbers and correlators of the velocity of the medium on and between fronts. Another, more trivial, special case is that of strong static density inhomogeneities (Sec. 4.5).

The method developed above, which is based on the use of pair correlators, presupposes that there is a rapid enough decay of correlations over distances exceeding the correlation length. It ceases to be valid when the system contains an appreciable number of ordered lines of force (or lines of current) that can ensure rapid transport of particles to distances exceeding the dimensions of a turbulent cell. The theory of transport in such systems, which is not included in our review, can be found in the review literature.[29,38,39]

The fourth limitation of our theory is its linearity: we assume that the diffusing impurity does not affect the turbulent medium. When this assumption is not valid, the equations that we have obtained must be augmented by other equations that describe the change in turbulence under the influence of the diffusing substance (accelerated particles, generated magnetic field). Although the linear theory of transport has a relatively wide range of validity, nonlinear effects are often of primary importance. We note particularly two 'hotspots' among the questions touched upon in this review, namely, the generation of a magnetic field by gyrotropic turbulence and the acceleration of particles by an ensemble of sufficiently strong fronts.

In the generation of a magnetic field (Sec. 5), it is important to take in to account the reaction of the field on the turbulent motion and its gyrotropy. The growth times of the field harmonics are very dependent on the spatial scale, and the frequently used approximation in which the correlation time for turbulent velocities is assumed to be zero is not valid for small-scale harmonics. Because of the rapid growth of the latter, a systematic nonlinear theory must evidently rely on a set of nonlinear equations for the large-scale field, the energy density of the small-scale field (the pair correlator), and the equations of motion of the medium, modified significantly by magnetic forces. As far as we are aware, no one has yet formulated a working theory of this kind.

When particles are accelerated by a sufficiently strong shock front, the efficiency of transfer of energy from hydrodynamic motion by accelerated particles may be high and the structure of the front itself may be modified by the accelerated particles (see Section 7.2 and the detailed review in Ref. 81). An analogous phenomenon is naturally expected when particles are accelerated by ensembles of shock fronts.[93] Equations (7.20) and (7.25) must therefore be augmented in the nonlinear theory with relations describing the depletion of ultrasonic turbulence and its modification by particle acceleration.

These basic limitations and disadvantages of our theory also seem to suggest certain future developments, since some of these limitations can be removed within the framework of the method itself.

Finally, let us briefly consider two directions of research that are closely related to our theory and which seem to us to be quite promising. In Sec. 4 we cited the numerical simulation reported by Drummond *et al.*[61] and used it to estimate the precision of the approximate theory. Unfortunately, this simulation was concerned exclusively with turbulent transport in space in an incompressible medium without a magnetic field. Numerical simulations of typical solutions with particle acceleration and the transport of such particles in the presence of a magnetic field could be of major interest. Such studies have become feasible since the advent of the modern supercomputer, and a number of very interesting results has been reported in Refs. 73 and 94. The simulation of particle kinetics and of the magnetic field in a medium with gyrotropic turbulence is particularly important because Drummond *et al.* have shown that there is considerable discrepancy with the self-consistent theory of Phythian and Curtis.[4] Comparisons of the theoretical results with the detailed numerical simulations should establish the precision of the theory and define in greater detail its range of validity.

Another topic is the application of the above method to the evaluation of higher-order (two-particle, etc.) distribution functions. These functions will enable us to obtain a more detailed picture of the behavior of particles in a turbulent medium and to find fluctuations in the distribution function that have already been investigated experimentally for a number of systems (see, for example, Refs. 95 and 96). This problem is particularly topical for the generation of magnetic fields because the large-scale (regular) magnetic field cannot be correctly calculated unless we know the spectral density, i.e., the second correlator, of the turbulent magnetic field.

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Translated by S. Chomet