Berry geometric phase in oscillatory processes

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Useful physical models that reveal the basic physical properties of the Berry phase are presented. They include Foucault's pendulum, rotation of the plane of polarization of light by three mirrors, and so on. Particular attention is devoted to the phase that arises when the polarization of light is transformed by an anisotropic medium. This is used to introduce modern terminology and then to examine more general situations. Several terminological and methodological questions are examined.

Too much abstraction usually shows up in the use of jargon and the uncritical manipulation of words instead of concepts.
W. L. Burke[1]

1. INTRODUCTION

Nearly ten years have passed since the publication of Berry's celebrated paper "Quantal phase factors accompanying adiabatic changes"[2] that marked the beginning of a stream of publications that has continued to this day. The concept of geometric, or Berry, phase (GP) has been significantly extended during this decade and has become part of a great many branches of physics [see the reviews in Refs. 3 and 4 and popular accounts in Refs. 5 and 6)]. GP is involved in Sagnac, Aharpnov–Bohm, Jahn–Teller, and Hall effects, in certain features of molecular and nuclear spectra, in vortices occurring in superfluid helium, and in chiral anomalies in gauge field theories. Manifestations of GP have recently been found in chemical reactions.[7] The Hannay angle[8,9] is the analog of GP in mechanics.

Many years before the publication of Ref. 2, GP effects were found in optics by Rytov,[10] Vladimirskiĭ,[11] and Pancharatnam[12] (cf. Ref. 12).

Several special experiments have been performed to demonstrate GP effects in optics,[13–19] in NMR,[20] and in neutron interference.[21–22]

We thus have before us an example of the unexpected birth of a universal physical concept. Soon after, Simon[23] pointed out that there was a corresponding concept in modern geometry, namely, the idea of *holonomy*, i.e., the rotation of a tangent vector during its parallel transport along a closed curve on a curved surface, e.g., a sphere. This surface together with the set of its tangent planes is an example of a *fiber bundle*.[24,25] We recall that such mathematical objects lie at the basis of important ideas in modern physics, e.g., Yang–Mills gauge fields, the electroweak interaction, and quantization of the gravitational field. A peculiar gauge "field" and gauge invariance are also found to arise in the formal description of GP (see Appendix).

Several restrictions were introduced in Ref. 2: the analysis was confined to a nondissipative quantum system and only a slowly-varying cyclic Hamiltonian $H(T)=H(0)$ and its nondegenerate stationary states ψ_n were considered, but these restrictions were subsequently lifted. Wilczek and Zee[26] examined the case of degenerate levels, described by a fiber bundle with a non-Abelian structure group, and the corresponding non-Abelian gauge field—the analog of the Yang–Mills field (cf. Ref. 27).

The generalization to the nonadiabatic case and to non-stationary states $\psi(t)$ was made by Aharonov and Anandan[28] who assumed that the Hamiltonian and the state vector $\psi(t)$ had the cyclic property $\psi(T)=\exp(1\gamma)\psi(0)$ where the total phase acquired by the vector in a cycle is equal to the trivial dynamic phase plus the GP, i.e., $\gamma=\alpha+\beta$. Actually, the cyclic functions are eigenfunctions of the evolution operator with eigenvalues $\exp(i\gamma)$ (Refs. 4 and 29).

Jordan[30] started with the quantal analog of the Pancharatnam phase[12] and introduced the GP for partial cycles represented by open trajectories in the phase (projective) space of the fiber bundle. The GP in optical systems with energy dissipation, i.e., polaroids, described by nonunitary evolution operators, was observed in Refs. 15 and 19.

Unfortunately, the mathematics syllabuses of physics departments frequently do not include topics such as group theory, topology, and modern geometry, despite the fact that they are increasingly used in physics. It follows that there is a need for an account of GP at an intermediate level of difficulty that presents fundamental geometric aspects and, at the same time, is accessible to a wide circle of physicists.

This paper is an attempt to fill this gap by examining oscillatory processes in terms of reasonably clear concepts. It seems that the language of optical interference, which allows a direct description of phase relations between 'real' macroscopic oscillations rather than the mysterious quantum state vectors, are more suited to the elucidation of the

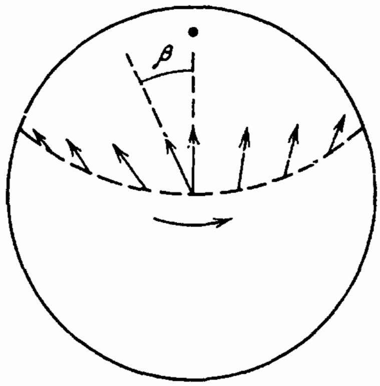

FIG. 1. The plane of oscillation of a Foucault pendulum, shown by the arrows, undergoes slow rotation due to the diurnal rotation of the Earth. After 24 hours, this plane does not return to its original North-South position: the angular difference is β and is a function of latitude.

essence of GP when it is encountered for the first time. It is hoped that clear optical models will help in the assimilation of some of the terminology of modern geometry and will facilitate the removal of the current language barrier between theoretical and experimental physicists. The Pancharatnam effect, i.e., the GP effect observed during the propagation of a beam of polarized radiation in a polarization-transforming medium, was chosen as the basic model serving as a reliable bridgehead to subsequent generalizations.

The phrase *geometric phase* will be used as a generic term covering special cases such as the Berry adiabatic phase, the Aharonov–Anandan geometric phase, the Hannay angle, and so on.

Our presentation begins with some clear models that exhibit GP (Sec. 2). In Sec. 3 we use the example of polarization optics to examine an important new concept, namely, the relative phase of two beams with different polarization **e**. In Sec. 4 we consider what is probably the simplest physical model described by the Hopf fiber bundle (Ref. 24, p. 273) with the Poincaré sphere (PS) as its base manifold and the unitary group U(1) as its structure group represented by the phase factors $e^{i\gamma}$ of the polarization vector **e**. In Sec. 5 we present a direct evaluation of **e** for particular polarization transformers used in optics, i.e., circular and linear phase plates. In Secs. 6 and 7 we show that the formalism used in this presentation describes GP not only for waves with two types of transverse polarization, but also in the case of any system of two or more oscillations. Some methodological and terminological questions are discussed in Sec. 8. The properties of the GP in the quantal description of a field are briefly described in Sec. 9. The Appendix discusses the geometrical meaning of the GP.

2. ELEMENTARY MODELS

Let us now consider a few typical manifestations of the GP. Figure 1 shows the observed rotation of the plane of oscillation of the Foucault pendulum due to the diurnal rotation of the Earth. For small deflections, the harmonic oscillations of the pendulum are linearly polarized in the direction of the vector **e** that slowly rotates relative to the surrounding objects. If we neglect the orbital rotation of the Earth, the pendulum should return to its initial position in space after 24 hours, and the naive expectation is that the vector **e** should have rotated through 2π. However, this occurs only at the poles, whereas at an arbitrary latitude $\tilde{\vartheta}=\pi/2-\vartheta$ (ϑ is the polar angle), the vector **e** revolves with angular velocity $\omega'=\omega\cos\vartheta$ whose modulus is equal to the angular velocity of the Earth, so that an observer detects the 'relative difference'

$$\beta\equiv2\pi\left|1-\frac{\omega'}{\omega}\right|=2\pi|1-\cos\vartheta|. \qquad (2.1)$$

For example, at the equator, $\omega'=0$ and **e** retains its orientation relative to the local coordinate frame, i.e., the pendulum does not react to the rotation of the Earth and the quantity β reaches its maximum value of 2π. We note that, according to (2.1), β is equal to the solid angle Ω subtended at an observer at the center of the Earth during the diurnal displacement of the pendulum.

The rule $\beta=\Omega$ (ignoring the signs of β and Ω) that relates radians and steradians[6] remains valid in the hypothetical case where the Earth is stationary and the pendulum is slowly (compared with the oscillation period) and smoothly transported over the Earth's surface on a close trajectory. Suppose, for example, we start at the North Pole along the Greenwich meridian ($\varphi_0=0$) with **e** parallel to this meridian. On the equator, we turn left toward the meridian φ and then left again, moving toward the Pole along the meridian φ. Throughout this the pendulum will obviously maintain its oscillations in the North-South plane and when we reach the Pole we find that the vector has rotated relative to its initial position by the angle $\beta=\varphi$ which is equal to the subtended solid angle Ω. Although during the transport along the geodesics the orientation of **e** remains the same relative to the local frame, i.e., the latitudes and meridional angles remain constant, there is nevertheless a global effect: $\beta\neq0$. This is an example of *holonomy* generated the parallel transport of the tangent vector over a sphere (see Ref. 24, p. 277).

It is clear that it is also possible to add vertical (adiabatic) transport with a resulting change in the oscillation frequency, but the formula $\beta=\Omega$ would remain valid. Generally speaking, β depends only on the global geometric parameter Ω, but does not depend on the details of experiment, i.e., the velocities and durations of transport. Of course, it is possible to perform a rigorous solution of the equations of motion of the pendulum with the Coriolis force taken into account[31] in which case the formula $\beta=\Omega$ emerges 'automatically' for each special transport path.

In terms of modern geometry, $\beta\neq0$ because the surface of the Earth, i.e., the sphere S^2, has a nontrivial topology ("one can't comb a hedgehog"). The angular orientations of the vector **c** cannot be mapped on the circle S^1 [or the unitary group U(1) consisting of the numbers $\exp(i\beta)$], so that the state of the system is mapped by a point in the fiber bundle that is locally the direct product

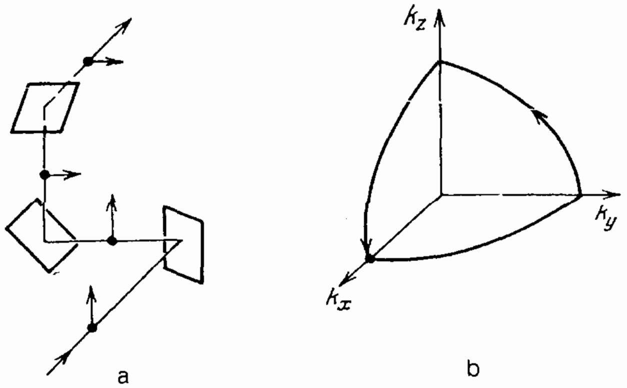

FIG. 2. a—Rotation of the plane of polarization of a beam of light after reflection by three mirrors: at entry the polarization is vertical, at exit it is horizontal. b—Mapping of the variation in the direction of the propagation of the beam into the space of wave vectors k_x, k_y, k_z.

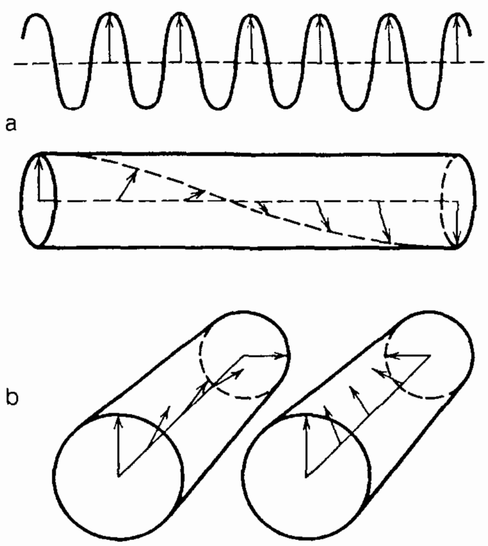

FIG. 3. The geometric phase $\beta = 180°$ produced during the rotation of the polarization vector **e** (arrows) around the direction of propagation of a transverse wave. 'Instantaneous wave pictures' are shown. a—Rotation of **e** by 180° does not result in the transformation of the reference wave (top); b—directions of **e** in both arms of the interferometers rotated by $\pm 90°$ in opposite directions.

$S^2 \times S^1$ (Ref. 24, p. 272), i.e., it is represented by the three numbers (ϑ, φ, β). Now consider a linearly polarized beam of light propagating in the system of mirrors shown in Fig. 2. Suppose that the initial beam is vertically polarized. The direction of **e** relative to the wave vector **k** is preserved on each reflection, but the result of three successive reflections is that the polarization becomes horizontal. This can literally be explained by a hand-waving argument. Let us extend the left hand horizontally with the thumb pointing upward. The hand points in the direction of propagation and the thumb in the direction of polarization. To describe the effect of the *i*th mirror in Fig. 2, we rotate the extended hand to the left through 90° with the thumb still pointing upward. Next, we raise the hand upward and then downward to its original position. We find that the thumb is now horizontal.

We shall represent the direction of the beam of light at any instant by a point on a unit sphere in **k**-space (Fig. 2b). The closed contour describing the effect of three mirrors covers one octant with the solid angle $\Omega = \pi/2$ that is equal to the angle of rotation of the plane of polarization. If we represent the linear polarization vector **e** as the superposition of two vectors $d^{(\pm)}$ with left and right circular polarizations, the rotation of the plane of polarization through an angle β is equivalent to a phase shift of the vectors $d^{(\pm)}$ by $\pm\beta$. We note that the evolution of the system is now represented not in real space but in the space of the parameters (further details relating to a system of mirrors can be found in Refs. 32–36).

An analogous effect is produced when the direction of the light beam is altered by a circular isotropic lightguide[13,37] or by the random variation of the direction of the beam due to fluctuations in the permittivity of a medium.[10,11] We note that the rotation of the plane of polarization in such systems can be determined only if the initial and final directions of propagation are the same, i.e., if the path in **k**-space is closed.

Now consider a simple example in which we observe a change in the phase of a linearly polarized wave with a fixed direction of propagation, which is the result of the rotation of the plane of polarization (this is the special case of the Pancharatnam which will be discussed in greater detail later). We may consider the plane light or transverse acoustic wave, or a wave on a stretched string. Suppose that the plane of polarization is rotated slowly (on the scale of a wavelength) through 180° by some suitable device that does not affect the wavelength. It is clear from Fig. 3a that this is accompanied by a phase shift of 180°: at distances that are multiples of $n\lambda$, the field assumes (at a given time) negative instead of positive values. This can be detected by examining the interference with a reference wave that has transversed the same path but without the rotation of the plane of polarization. This is an example of the influence of geometry on phase. The space that maps the state of polarization is conveniently taken to be the Poincaré sphere (PS), in which case we have $\beta = -\Omega/2$ where now Ω is the solid angle subtended by the orbit on the PS.

The phase jump by π shown in Fig. 3, called a transverse wave jump, is probably the simplest and clearest manifestation of the GP. It was noted in Ref. 38 and was considered in detail in Ref. 39. Figure 3b shows a more symmetric variant of the effect.

3. PANCHARATNAM PHASE

Let us generalize the model of Fig. 3 to an arbitrarily polarized incident wave and arbitrary transformers of polarization **D**. To be specific, we shall consider plane light waves with frequency ω and wave vector **k**. We shall suppose that **E** is a transverse field, in which case polarization is characterized by a two-dimensional complex vector **e** with unit norm: $\mathbf{e}^{*}\cdot\mathbf{e}=1$. It is convenient to include in the vector $\mathbf{e}$ the phase change due to the propagation of the beam along the z-axis:

$$\mathbf{E}(z,t)=E_0\mathrm{Re}(\mathbf{e}(z)e^{-i\omega t}). \tag{3.1}$$

We shall assume henceforth that $E_0=1$.

Consider a pair of orthogonal normalized basis vectors $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ where $\mathbf{d}^{(1)}\cdot\mathbf{d}^{(2)}=0$, so that the polarization state can be defined by two complex numbers $\mathbf{e}=(e_1,e_2)$ where $e_n=\mathbf{d}_n^{*}\cdot\mathbf{e}$ and $|e_1|^2+|e_2|^2=1$. For a complete determination of $\mathbf{e}$ we need to know three real numbers, and the polarization state can be represented by a point on a sphere S^3 in four-dimensional space with coordinates $\mathrm{Re}\, e_1$, $\mathrm{Im}\, e_1$, $\mathrm{Re}\, e_2$, $\mathrm{Im}\, e_2$ (different ways of visualizing this sphere are described in Ref. 40). If on the other hand we are not interested in the phase factor $\exp(i\varepsilon)$ that is common to both components e_1 and e_2 (which is usually the case in optics), then we need only consider two numbers that can be represented by a point on the Poincaré sphere (PS).

In the case of a linear basis, the vectors $\mathbf{d}^{(1)}\equiv\mathbf{d}^{(x)}$ and $\mathbf{d}^{(2)}\equiv\mathbf{d}^{(y)}$ are real and point along the x and y axes that form a right-handed triple with the z axis. It is sometimes convenient to work in a circular basis with complex vectors $\mathbf{d}^{(\pm)}$ describing the right and left handed circular polarizations. We shall take the relation between these main bases in the form

$$\begin{aligned} \mathbf{d}^{(x)}&=\frac{1}{\sqrt{2}}(\mathbf{d}^{(+)}+\mathbf{d}^{(-)}),\\ \mathbf{d}^{(y)}&=\frac{1}{i\sqrt{2}}(\mathbf{d}^{(+)}-\mathbf{d}^{(-)}),\\ \mathbf{d}^{(\pm)}&=\frac{1}{\sqrt{2}}(\mathbf{d}^{(x)}\pm i\mathbf{d}^{(y)}). \end{aligned} \tag{3.2}$$

The new components of the vector $\mathbf{e}$ after the change of basis are as follows:

$$\begin{aligned} \mathbf{e}_{\mathrm{lin}}&=W\cdot\mathbf{e}_{\mathrm{circ}},\\ \mathbf{e}_{\mathrm{circ}}&=W^{-1}\cdot\mathbf{e}_{\mathrm{lin}}.\\ W&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix},\\ W^{-1}&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}, \end{aligned} \tag{3.3}$$

where in accordance with tradition (Ref. 41, p. 53) the signs are chosen so that it follows from $\mathbf{d}^{(+)}$ that $e_y/e_x=i$, i.e., the real fields along the x and y axes are, respectively, equal to $\cos(kz-\omega t)$ and $\cos(kz-\omega t+\pi/2)=\sin(\omega t-kz)$. For an observer looking at the light source we then find that the vector $\mathbf{E}$ rotates in the anticlockwise direction.

Now consider an idealized experiment designed to enable us to observe the polarization GP (Pancharatnam phase) as shown in Fig. 4. This scheme employs a Mach–Zender interferometer with nonpolarizing mirrors and a polarization transformer $\mathbf{D}$ which may consist of a chain of

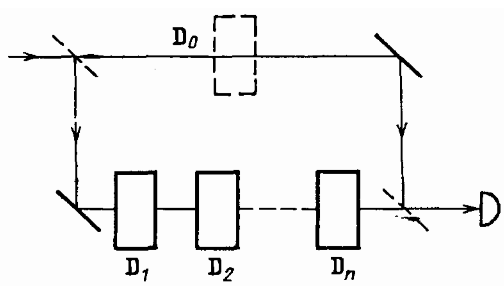

FIG. 4. Interferometer used to demonstrate the geometric phase. The mirrors are assumed to be nonpolarizing. $\mathbf{D}_1,\ldots,\mathbf{D}_n$ are the polarization transformers in the main arm and $\mathbf{D}_0$ is the transformer in the reference arm.

transformers, i.e., $\mathbf{D}=\mathbf{D}_n\ldots\mathbf{D}_2\mathbf{D}_1$ where $\mathbf{D}$ is the Jones matrix describing the transformation of the vector $\mathbf{e}$ by these transformers.

Let $\mathbf{e}$ be the polarization vector at entry to the interferometer, i.e., in the reference beam and at entry into the transformer $\mathbf{D}$, and let $\mathbf{e}'$ be the corresponding vector at exit from the transformer (phase changes in free space are ignored). These fields add vectorially at exit from the transformer: $\mathbf{e}''=(\mathbf{e}+\mathbf{e}')/\sqrt{2}$ so that the intensity at the detector is proportional to

$$I=|\mathbf{e}''|^2=\tfrac{1}{2}[1+\mathrm{Re}(\mathbf{e}^{*}\cdot\mathbf{e}')]=\tfrac{1}{2}(1+V\cos\gamma), \tag{3.4}$$

where

$$V\equiv|(\mathbf{e}^{*}\cdot\mathbf{e}')|=|e_1^{*}e_1'+e_2^{*}e_2'|, \tag{3.5}$$

$$\gamma\equiv\arg(\mathbf{e}^{*}\cdot\mathbf{e}')=\arg(e_1^{*}e_1'+e_2^{*}e_2'). \tag{3.6}$$

Following Pancharatnam,[12] we *define* γ as the relative phase of two beams with polarizations $\mathbf{e}$ and $\mathbf{e}'$ that in general are different (the definition ceases to be valid when $\mathbf{e}$ and $\mathbf{e}'$ are orthogonal). Indeed, if the visibility V is constant and γ is varied, the intensity varies as $\cos\gamma$. The maximum intensity is obtained for $\gamma=0$ in which case the beams are in phase; for $\gamma=\pi$ they are in antiphase.

We note that $\gamma_{ab}\equiv\arg(\mathbf{a}^{*}\cdot\mathbf{b})=-\gamma_{ab}$. The definition given by (3.6) can be used to introduce the running phase $\gamma(z)$ at any point in the channel relative to the initial field:

$$\gamma(z)=\arg[\mathbf{e}^{*}(0)\cdot\mathbf{e}(z)]. \tag{3.7}$$

According to (3.6), the relative phase of vectors $\mathbf{e}$ and $\mathbf{e}'$ is equal to the phase of their dot product. It is clear that this can be generalized to the case of N-component vectors and infinite-dimensional Hilbert space:

$$\gamma=\arg\sum_{n=1}^{N}e_n^{*}\cdot e_n'\to\arg\int dx\psi^{*}(x)\psi'(x), \tag{3.8}$$

where $\mathbf{e}$ can describe oscillations in N coupled classical oscillators (Sec. 7) or the state of an N-level quantum system.

4. POINCARÉ SPHERE AS FIBRATION BASIS

Consider a vector $\mathbf{e}$ at some fixed point in the system in a circular basis in the form $\mathbf{e}=\exp(i\varepsilon)\mathbf{d}$ where

$$
\begin{aligned}
d_+(\vartheta,\varphi) &= e^{-i\varphi/2}\cos\frac{\vartheta}{2}, \\
d_-(\vartheta,\varphi) &= e^{i\varphi/2}\sin\frac{\vartheta}{2}.
\end{aligned}
\tag{4.1}
$$

This type of parametrization is often used in the linear basis, but (4.1) is directly related to the traditional polarization-type mapping on the Poincaré sphere (PS). To see this, we need only identify the parameters ϑ,φ in (4.1) with the usual spherical coordinates on a sphere S^2.

Opposite points on the PS then correspond to orthogonal vectors: $\mathbf{e}^*(\vartheta,\varphi)\cdot\mathbf{e}(\pi-\vartheta,\varphi+\pi)=0$. The significant point is that if the coefficients 1/2 were not introduced in (4.1) in front of ϑ and φ, then opposite points on the PS would correspond to vectors $\mathbf{e}$ and $\mathbf{e}'=-\mathbf{e}$ differing by sign only, i.e., by $\pm\pi$ in phase. Each 'polarization type' that, by definition, does not depend on the common phase factor $\exp(i\varepsilon)$, then corresponds to two points on the PS, i.e., it is not single-valued and the PS is not a projective space (base of fiber bundle). The words 'light is linearly polarized along the x axis' can then be represented by a single point on the PS with coordinates $\vartheta=\pi/2$, $\varphi=0$ and two polarization vectors $\mathbf{e}=\mathbf{d}^{(x)}$ and $\mathbf{e}'=-\mathbf{d}^{(x)}$ (see Fig. 3) (or, in general, $\mathbf{e}'=\exp(i\varepsilon)\mathbf{e}$).

In other words, the projection of vector $\mathbf{e}$ that belongs to the space S^3 on to S^2 is based on the definition of the vectors $\mathbf{e}$ and $\exp i\varepsilon\, \mathbf{e}$ as equivalent vectors (ε is an arbitrary real number here). This definition enables us to split the entire set of points $\mathbf{e}\in S^3$ into *equivalence classes* (*rays* or polarization types). Each ray has its own point on the PS.

We can now use (3.3) and (4.1) to express the Cartesian coordinates of a point on the PS in terms of the linear and circular components of $\mathbf{e}$:

$$
\begin{aligned}
X &= \sin\vartheta\cos\varphi = |e_x|^2-|e_y|^2 = 2\,\mathrm{Re}(e_+^*e_-), \\
Y &= \sin\vartheta\sin\varphi = 2\,\mathrm{Re}(e_x^*e_y) = 2\,\mathrm{Im}(e_+^*e_-), \\
Z &= \cos\vartheta = 2\,\mathrm{Im}(e_x^*e_y) = |e_+|^2-|e_-|^2.
\end{aligned}
\tag{4.2}
$$

The poles $\mathbf{R}=(X,Y,Z)=(0,0,\pm 1)$ thus correspond to circular polarization ($e_\pm\sim 1$) and an equatorial point with longitude $\varphi=2\chi$ corresponds to linear polarization at an angle χ to the x axis ($e_x\sim\cos\chi, e_y\sim\sin\chi$).

The path along the equator that starts and ends at the point (1,0,0) corresponds to a rotation of the plane of polarization, which produces the holonomy $\mathbf{e}\to-\mathbf{e}$, is shown in Fig. 3. For initial light with arbitrary elliptic polarization, a gyrotropic medium shifts the mapping point along a certain latitude that is determined by the initial polarization (with Z=const).

Since (4.2) contains only paired products of components of the form $e_n^*\cdot e_m$, a 'gauge' transformation $\mathbf{e}\to\exp(i\varepsilon)\,\mathbf{e}$ leaves the mapping point in the same place.

When the parameters of the optical transmission system are given, this determines the evolution of the vector $\mathbf{e}(z)$ and we can identify its orbit on S^3. Its projection on to the base manifold (the PS) is a certain curve C that can be described parametrically by $\vartheta=\vartheta(z), \varphi=\varphi(z)$. The third

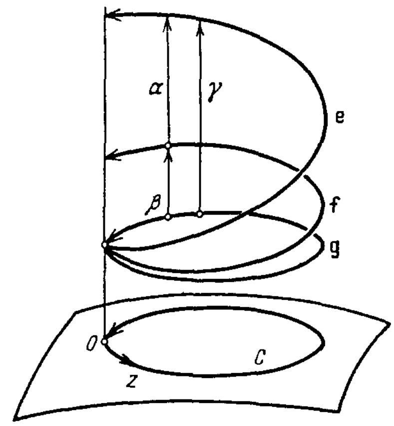

FIG. 5. Illustration of a fiber bundle. Bottom—base space (Poincaré sphere) with closed curve C and curve parameter z. The vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ differ by gauge transformations with phases α, β, γ. The vertical line corresponds to the initial and final values of the polarization vectors.

coordinate that completes the definition of $\mathbf{e}(z)$ can be taken to be the phase $\varepsilon(z)$ defined in accordance with (4.1).

This procedure enables us to determine the trajectory of the system in terms of the coordinates $(\vartheta,\varphi,\varepsilon)$, i.e., the so-called *section* of a fiber bundle. The rule that allows us to calculate ε for different z is called a *connection*. In the case of the Foucault pendulum, the connection is specified physically, i.e., by the conservation of the plane of oscillation in an inertial frame, which leads to the law of parallel transport of the polarization of the pendulum during its displacement. The rule for covariant differentiation in curvilinear coordinates is also an example of a connection (of the Levi–Civita type). The choice of a particular connection determines the internal geometry of a fiber bundle and the holonomy, i.e., the law of transformation of vectors over a closed trajectory C on the base of the fiber bundle.

On the other hand, for given $\mathbf{e}(0)$ and C, we can use (3.7) to associate with each point z a Pancharatnam phase $\gamma(z)$, i.e., specify another section $(\vartheta,\varphi,\gamma)$ that corresponds to the same path C, but is determined by the Pancharatnam connection.

Consider a closed trajectory C on the PS with $\mathbf{R}(L)=\mathbf{R}(0)$. The trajectory that corresponds to it in the common fiber bundle space an be open: $\mathbf{e}(L)=\exp i\gamma(L)\mathbf{e}(0)$. Figure 5 shown schematically a number of trajectories projected on the same contour C, i.e., differing from one another only by a gauge transformation. Rays are represented in Fig. 5 by the vertical lines.

To obtain a single-valued vector $\mathbf{g}(z)$ with a closed orbit, we eliminate $\gamma(z)$ so that $g(z)=\exp[-i\gamma(z)]\mathbf{e}(z)$. As already noted, $\gamma(z)$ is the sum of the dynamic part $\alpha(z)$, which depends on the construction of the transmission system, its length, etc., and the geometric part $\beta(z)$, which depends only on the global properties of the contour C. It is shown in the Appendix that the GP can be written in the form

$$\beta(z)=i\int_0^z dz'\mathbf{g}^*(z')\cdot\dot{\mathbf{g}}(z')=-\tfrac{1}{2}\Omega \tag{4.3}$$

where the dot represents differentiation with respect to z' and Ω is the solid angle subtended by C. We note that the vector $\mathbf{g}$ is normalized so that $\mathrm{Re}\,\mathbf{g}^*\cdot\dot{\mathbf{g}}=0$ and hence $\mathbf{g}^*\cdot\dot{\mathbf{g}}$ is an imaginary number.

The formula given by (4.3) can also be used when the paths on the PS are not closed, provided we close them with a geodesic along the shortest length. Such lines do not contribute to β if they pass through the initial point.

For example, if we move along the equator away from the point $\varphi=0$, we find that β remains equal to zero up to $\varphi=\pi$ because the shortest reverse path runs along the equator in the opposite direction, and this gives $\Omega=0$. At the point $\varphi=\pi$, we find that β jumps up by π because closure occurs in the same direction and encompasses half of the PS. The physical meaning of this jump is explained by Fig. 3.

We note that the coefficient 1/2 appears in (4.3) because it is present in (4.1) [see equation (A11) in the Appendix] where it ensures, as already noted, the single-valued correspondence between rays, i.e., vectors $\mathbf{e}$ differing only in phase, and the points on the PS. In other words, these coefficients reflect the *spinor* character of the vectors $\mathbf{e}$ under the influence of the polarization transformers: two complete circuits must be executed on the PS to return to the initial polarization.

We now use one further gauge transformation to define a vector $\mathbf{f}(z)$ with the dynamic phase excluded (see Fig. 5):

$$\mathbf{f}(z)\equiv e^{-i\alpha(z)}\mathbf{e}(z)=e^{i\beta(z)}\mathbf{g}(z). \tag{4.4}$$

It is readily verified that the so-called parallel transport rule is satisfied by this vector, i.e., $\mathbf{f}^*\cdot\dot{\mathbf{f}}=0$, i.e., the increment $\dot{\mathbf{f}}$ produced by the Hamiltonian is orthogonal to $\mathbf{f}$. We note that, for real vectors, this is a trivial result because it is a consequence of normalization. In geometric language, $\dot{\mathbf{f}}(z)$ is a *tangent vector* to the graph of $\mathbf{f}(z)$ at the point z. The vector $\mathbf{f}$, like the other vectors of the given ray, are assumed to point 'vertically' (Fig. 5), so that the parallel transport condition signifies that the vector $\dot{\mathbf{f}}$ is orthogonal to $\mathbf{f}$ and, consequently, lies in the 'horizontal plane.' Accordingly, the orbit of the vector $\mathbf{f}(z)$ is called the *horizontal lift* of the closed curve C on the base, and its increment on the closure of C is a holonomy (cf. Refs. 24 and 42).

It is shown in the Appendix that, for closed C, the replacement of $\mathbf{g}$ in (4.3) with $\tilde{\mathbf{g}}=\mathbf{g}\exp(i\varepsilon)$, where $\varepsilon(z)$ is an arbitrary function, that does not affect β. All optical systems that give the same contours C on the PS are thus found to introduce the same GPs (although their dynamic phases can be quite different). This is an example of the *gauge invariance* of the GP; it is discussed for arbitrary quantum systems in a number of publications.[42,44]

5. EVALUATION OF THE GP USING JONES MATRICES

Let us now consider the change in the polarization vector $\mathbf{e}$ during the propagation of light in an optical transmission system, ignoring reflections, diffraction, losses, and so on. The individual elements of the optical system, and indeed the system as a whole, perform transformations of the form

$$\mathbf{e}\to\mathbf{e}'=e^{i\sigma}\mathbf{D}\cdot\mathbf{e}, \tag{5.1}$$

where σ is a common phase and $\mathbf{D}$ is a 2×2 matrix called the *Jones matrix*. Since intensity is conserved

$$\mathbf{e}'^*\cdot\mathbf{e}=(\mathbf{D}\cdot\mathbf{e})^*\cdot(\mathbf{D}\cdot\mathbf{e})=\mathbf{e}^*\cdot\mathbf{D}^+\cdot\mathbf{D}\cdot\mathbf{e}=1 \tag{5.2}$$

we obtain $\mathbf{D}^+\cdot\mathbf{D}=I$, i.e., $\mathbf{D}$ belongs to the group U(2) of *unitary* matrices [where $(D^+)_{mn}\equiv(D^+)_{nm}$]. The phase σ can be chosen so that the determinant of $\mathbf{D}$ becomes equal to unity, which is the essential property of the *special* group SU(2). The matrices of this group can in general be written in the form

$$\mathbf{D}=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{5.3}$$

$$|a|^2+|b|^2=1.$$

There are therefore three independent real parameters and any matrix can be mapped by a point on the sphere S^3.

The trivial transformation $\mathbf{e}\to\mathbf{e}\exp(i\omega z/c)$ takes place in empty space, and will not be taken into account here. During propagation in an anisotropic medium, there is a certain basis $\mathbf{d}^{(1)},\mathbf{d}^{(2)}$ in which the two components of $\mathbf{e}$ vary independently (the possible longitudinal component e_z will not be taken into account either; a more rigorous approach is to use the induction vector[43]).

In this particular representation

$$e_m(z)=e^{ik_mz}e_m(0), m=1,2, \tag{5.4}$$

which gives

$$\mathbf{e}(z)=e_1\mathbf{f}^{(1)}e^{ik_1z}+e_2\mathbf{f}^{(2)}e^{ik_2z}=e^{i\sigma}(e_1\mathbf{f}^{(1)}e^{i\delta}+e_2\mathbf{f}^{(2)}e^{-i\delta}).$$

The result is

$$D=\begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}, \tag{5.5}$$

where

$$\varphi\equiv(k_1+k_2)/2, \delta\equiv(k_1-k_2)/2.$$

The evolution of the vector $\mathbf{e}$ can be described by an analog of the Schrödinger equation:

$$i\dot{e}=-\mathbf{H}\cdot\mathbf{e}, \tag{5.6}$$

for which the evolution operator is

$$\mathbf{e}(z)=\mathbf{U}(z)\cdot\mathbf{e}(0), \mathbf{U}(0)=\mathbf{I}. \tag{5.7}$$

For homogeneous materials, $\mathbf{H}$ is independent of z. According to (5.4), we then have in the corresponding representation

$$\mathbf{H}=\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \tag{5.8}$$

where $\mathbf{U}=e^{i\sigma}\mathbf{D}=\exp(i\mathbf{H}z)$.

If, on the other hand, we ignore the phase σ, then it follows from (5.5) that

$$\mathbf{H}=-i\dot{\mathbf{D}}(0)=\frac{1}{2}(k_1-k_2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad (5.9)$$

Hence $\mathbf{U}=\mathbf{D}=\exp(i\mathbf{H}z)$.

Suppose that the eigenvectors of $\mathbf{H}$ have coordinates (ϑ,φ) and $(\pi-\varphi,\varphi+\pi)$ on the PS. We can then readily verify that in the circular basis

$$\mathbf{H}(\vartheta,\varphi)=\frac{1}{2}(k_1-k_2)\begin{pmatrix}\cos\vartheta & \sin\vartheta e^{-i\varphi}\\ \sin\vartheta e^{-i\varphi} & -\cos\vartheta\end{pmatrix}. \qquad (5.10)$$

The displacement of a point on the PS under the influence of this operator can be readily represented by assuming that the PS rotates around an axis passing through the points (ϑ,φ) and $(\pi-\varphi,\varphi+\pi)$.

5.1. The rotator

For a rotator, e.g., a Faraday cell or circular phase plate, the eigenvectors correspond to the poles of the PS. We now transform to the linear basis with the help of (3.3)

$$\mathbf{D}_{\text{lin}}=\mathbf{W}\cdot\mathbf{D}_{\text{circ}}\mathbf{W}^{-1}=\begin{pmatrix}\cos\delta & \sin\delta\\ -\sin\delta & \cos\delta\end{pmatrix}, \qquad (5.11)$$

where now $\delta=\omega z(n_+-n_-)/2c$ and $n_\pm$ are the refractive indices for the circularly polarized waves. The explicit transformation is

$$\begin{aligned} e'_x&=e^{i\sigma}(\cos\delta e_x+\sin\delta e_y),\\ e'_y&=e^{i\sigma}(-\sin\delta e_x+\cos\delta e_y). \end{aligned} \qquad (5.12)$$

The increase in the longitude φ on the PS is -2δ.

Let us now consider an interference experiment involving the rotator (Fig. 4). According to (4.12),

$$\mathbf{e}^*\cdot\mathbf{e}'=e^{i\sigma}(\cos\delta+iZ\sin\delta), \qquad (5.13)$$

where $Z=|e_+|^2-|e_-|^2$ is determined by the original polarization.

Substitution in (3.4) yields

$$I=\tfrac{1}{2}(1+V\cos\gamma), \qquad (5.14)$$

where

$$V=(\cos^2\delta+Z^2\sin^2\delta)^{1/2}, \qquad (5.15)$$

$$\gamma=\sigma+\operatorname{arctg}(Z\operatorname{tg}\delta)\equiv\sigma+\gamma_0. \qquad (5.16)$$

Usually, $\sigma\gg|\delta|$, so that as z varies, (5.14) describes interference beats in intensity with the period $\lambda\equiv4\pi/(k_++k_-)$, in which case the visibility V and the interference phase γ_0 are slowly-varying functions of z with the period $4\pi/(k_++k_-)$. The Pancharatnam phase is thus seen to be a direct observable and (3.6) has an *operational* meaning.

When $Z=\pm1$ (circular polarization), we obtain the trivial result

$$I=\tfrac{1}{2}[1+\cos(\sigma\pm\delta)]=\tfrac{1}{2}[1+\cos(k_\pm z)], \qquad (5.17)$$

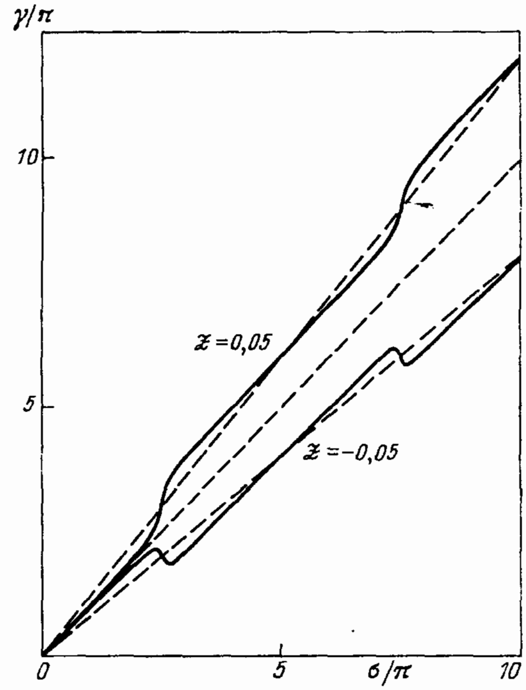

FIG. 6. Resultant phase γ (dynamic plus geometric) as a function of the path length z in a gyrotropic medium according to (5.16). The phase γ divided by π is shown along the vertical axis and the ratio $\sigma/\pi=(k_++k_-)z/2\pi$ is shown along the horizontal axis, where $k_\pm$ are the propagation constants for waves with the indicated circular polarizations. The anisotropy parameter $\delta/\sigma=(k_++k_-)/(k_++k_-)$ is set equal to 0.2. The parameter Z is related to the ellipticity of the wave ($Z=0$-linear polarization, $Z=\pm1$-circular polarization. The dashed lines correspond to $k\pm z$ and $(k_++k_-)z$.

and for $Z=0$ (linear polarization) we find from (5.15)–(5.16) that γ_0 or π, and $V=|\cos\delta|$, so that

$$\begin{aligned} I&=\tfrac{1}{2}(1+\cos\delta\cdot\cos\sigma)\\ &=\tfrac{1}{2}[1+\tfrac{1}{2}\cos(k_+z)+\tfrac{1}{2}\cos(k_-z)]. \end{aligned} \qquad (5.18)$$

The beats stop as we cross the point $\delta=\pi/2$, but they subsequently reappear with phase shifted by π.

The graph of the function $\gamma(z)$ is shown in Fig. 6 for $Z=\pm0.05$. The accompanying phase jumps, discussed above, are smoothed out somewhat, but the nonlinear dependence of the phase on z is still clear (to emphasize the effect, the anisotropy parameter $\delta/\sigma=(n_+-n_-)/(n_++n_+)$ was assigned the value 0.2).

The expression given by (5.16) is the common phase. The question is: how can it be separated from the geometric part? Let us define the difference $\gamma-\beta=\alpha$ as the product of the path length z and the 'mean' wave vector for the given polarization type:

$$\alpha=\langle k\rangle z\equiv(k_+|e_+|^2+k_-|e_-|^2)z=\sigma+Z\delta. \qquad (5.19)$$

For example, when $Z=\pm1$ we have $\langle k\rangle=k_\pm$ and for $Z=0$ we have $\langle k\rangle=(k_++k_-)/2$. Naturally, this definition can be generalized to the case of a spatially-inhomogeneous medium in which the eigenvalues k_n depend on z:

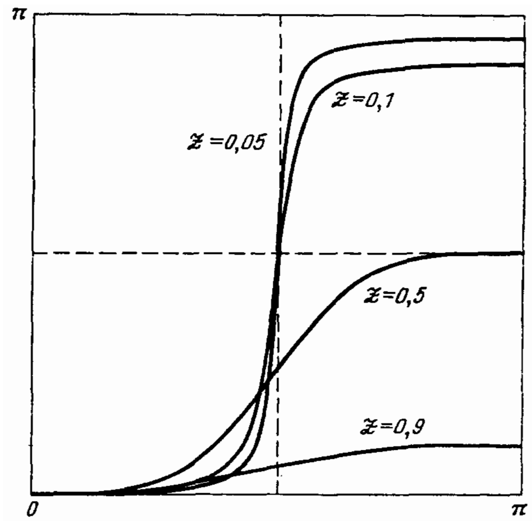

FIG. 7. Geometric phase introduced by a gyrotropic medium as a function of the normalized path length $\delta=(k_+-K_-)z/2$ according to (5.24) for different values of the ellipticity parameter Z.

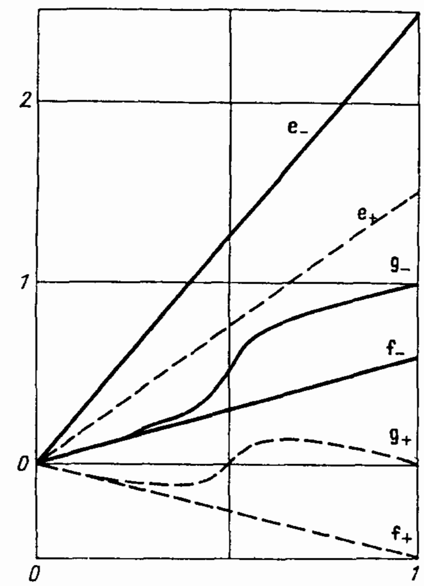

FIG. 8. Phases of the vectors **e**, **f**, **g** as functions of the longitude φ on the PS in the case of a rotator with $Z=0.05$ and anisotropy parameter 0.25. The solid and dashed lines correspond to right- and left-circular components of the vectors. The phases are in units of 2π.

$$\alpha(z)=\int_0^z dz' \sum_{n=1}^{2} k_n(z)|e_n|^2. \tag{5.20}$$

This definition is the analog of the Aharonov–Anandan dynamic phase[28] in a quantum system (the dot now represents differentiation with respect to time):

$$\alpha(T)\equiv-\frac{1}{\hbar}\int_0^T dt\langle\psi|H|\psi\rangle=-\frac{i}{\hbar}\int_0^T dt\langle\psi|\dot{\psi}\rangle. \tag{5.21}$$

In our notation, (5.6) leads to the following expression that is identical with (5.20):

$$\alpha(z)\equiv\int_0^z dz'(\mathbf{e}^*\cdot\mathbf{H}\cdot\mathbf{e})=-i\int_0^z dz(\mathbf{e}^*\cdot\dot{\mathbf{e}}). \tag{5.22}$$

In the above case of a rotator, (5.19) shows that α is simply the linear part of the function $\gamma(z)$:

$$\alpha(z)=z\dot{\gamma}(0)=\sigma+Z\delta. \tag{5.23}$$

Hence[38]

$$\beta(z)=\operatorname{arctg}(Z\operatorname{tg}\delta)-Z\delta. \tag{5.24}$$

The graph of this function is shown in Fig. 7. In the Appendix, we use the formulas of spherical trigonometry to evaluate the solid angle corresponding to motion in latitude with the increase $\varphi=-2\delta$ in longitude. In accordance with (4.3), this angle is equal to twice the expression in (5.24).

For a complete circuit on PS, $\varphi=-2\delta=2\pi$, so that

$$\begin{gathered}\alpha/\pi=(n_++n_-)/(n_--n_+)+Z,\\ \beta/\pi=1-Z,\quad \gamma/\pi=2n_\pm/(n_--n_+).\end{gathered} \tag{5.25}$$

We must now find the explicit form of the vectors **e**,**f**,**g** for motion at constant latitude in which $\vartheta=\text{const}$ and $\varphi=(k_--k_+)z=-2\delta$. According to (5.5)

$$e_\pm(z)=(\sigma\pm\delta)e_\pm(0)=\exp(ik_\pm z)e(0),$$

so that

$$\begin{gathered}\mathbf{e}=e^{i\sigma}\mathbf{h},\ \mathbf{f}=e^{-i\alpha}\mathbf{e}=e^{-i\beta}\mathbf{g},\\ \mathbf{g}=e^{-i\gamma}\mathbf{e}=e^{i({}^*\sigma-\gamma)}\mathbf{h},\end{gathered} \tag{5.26}$$

where

$$h_\pm\equiv e^{\pm i\delta}e_\pm(0),\quad \sigma=\varphi(n_++n_-)/(n_--n_+)$$

and the functions α, β, γ are given by (5.23), (5.24), and (5.16). In particular,

$$\sigma-\gamma=-\beta-Z\delta=-\operatorname{arctg}(Z\operatorname{tg}\delta).$$

Figure 8 illustrates three fiber bundle sections given by the vectors **e**,**f**,**g**, i.e., the evolution operator of the transmission system, the parallel transfer condition $\mathbf{f}^*\cdot\dot{\mathbf{f}}=0$, and the single-valuedness condition $g=\exp(-i\gamma)\mathbf{e}$.

From (5.26) it does actually follow that

$$\mathbf{f}^*\cdot\dot{\mathbf{f}}=i\dot{\delta}[(1-Z)|e_+(0)|^2-(1+Z)|e_-(0)|^2]=0.$$

The vector **g** determines, according to (5.26), the rate of change of the GP [cf. (A.4)]:

$$\mathbf{g}^*\cdot\dot{\mathbf{g}}=\mathbf{g}^*\cdot(i\dot{\beta}\mathbf{g}+e^{i\beta}\dot{\mathbf{f}})=-i\dot{\beta}.$$

The explicit form of $\dot{\beta}$ follows from (5.24):

$$\dot{\beta}=\tfrac{1}{2}(k_+-k_-)Z(1-Z^2)/(Z^2+\operatorname{ctg}^2\delta). \tag{5.27}$$

Hence, for $Z=0$ (equatorial motion), we have $\dot{\beta}=0$ (for $\varphi\neq\pi$). This illustrates the following general rule: the GP does not vary during motion that takes place along a geodesic with arc length less than π and passes through the initial point (the necessity of the last condition is clear from Fig. 8 where the path length along the PS meridian gives the GP).

5.2. Linear plate

Actual experiments[14-18] rely on a sequence of $\lambda/4$ and $\lambda/2$ linear phase plates that are used to vary β without affecting α. Suppose that the axis of symmetry of a uniaxial crystal makes an angle χ with the x axis. The vector $\mathbf{d}^{(e)}$ oriented along the axis of symmetry is then an eigenvector of the matrix of U(2) with eigenvalue exp $e^{ik_e z}$ where the subscript e refers to the extraordinary wave. The second eigenvector $\mathbf{d}^{(0)}$ is at an angle $\chi+\pi/2$ and has the eigenvalue $\exp(ik_0 z)$. Consequently, the crystal transformation matrix is then given by (5.5) [without the factor $\exp(i\sigma)$] where now $k_1=k_e, k_2=k_0$.

The angle χ in real space corresponds to longitude $\varphi=2\chi$ on the PS, so that points on the equator with longitudes 2χ and $2\chi+\pi$ correspond to the vectors $\mathbf{d}^{(e)},\mathbf{d}^{(0)}$. Transforming to the basis rotated through $-\chi$, i.e., to the usual basis $\mathbf{d}^{(x)},\mathbf{d}^{(y)}$, we find the following parameters of the matrix $\mathbf{D}$[see (5.3)]:

$$a=\cos\delta+i\sin\delta\cdot\cos 2\chi,\qquad b=i\sin\delta\cdot\sin 2\chi. \tag{5.28}$$

Hence for plates with $\delta=\pi/4$ and $\delta=\pi/2$ we have

$$a(\lambda/4)=\frac{1}{\sqrt{2}}(1+i\cos 2\chi),\quad b(\lambda/4)=\frac{i}{\sqrt{2}}\sin 2\chi,\quad a(\lambda/2)=i\cos 2\chi,\quad b(\lambda/2)=i\sin 2\chi. \tag{5.29}$$

Suppose that a beam polarized along the x axis is incident on a $\lambda/4$ plate with $\chi=45°$, so that e- and o-waves are excited in the crystal with equal amplitudes. According to (5.29),

$$\mathbf{e}'=\frac{e^{i\sigma}}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\frac{e^{i\sigma}}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}. \tag{5.30}$$

The path on the PS that corresponds to this transformation runs from the equator to the North Pole along the $\varphi=0$ meridian. According to (5.30), the vector has now acquired only the dynamic phase

$$\gamma=\alpha=\sigma=\frac{1}{2}(k_e+k_0)z=\frac{n_e+n_0}{n_e+n_0}\frac{\pi}{4}.$$

The product $\mathbf{e}^*\cdot\mathbf{e}'$ is equal to $D_{11}=\exp(i\sigma)/\sqrt{2}$.

Let us now pass the beam through a second $\lambda/4$ plate with arbitrary orientation χ. The result of this second transformation can be found with the help of (5.29) and (5.30):

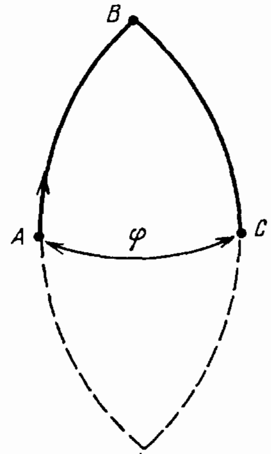

FIG. 9. The path ABC on the Poincaré sphere corresponds to the effect of two $\lambda/4$ plates and gives a geometric phase $\beta=\varphi/2=\chi+45°$ that is a linear function of the angle χ by which the second plate is rotated; the dynamic phase α is independent of χ. Dashed curve—possible closing path for ABC.

$$\mathbf{e}''=\frac{e^{i2\sigma}}{2}\begin{pmatrix}1+i\cos 2\chi & i\sin\chi\\ i\sin\chi & 1-i\cos 2\chi\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix}=e^{i(2\sigma+\beta)}\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}. \tag{5.31}$$

We now find that, in addition to the dynamic phase 2σ, we also have the GP $\beta=\chi+45°$, the field is linearly polarized at an angle β, and the mapping point has returned to the equator to a point with longitude $\varphi=2\beta=2\chi+\pi/2$ (as a result of a rotation by $\pi/2$ around an axis with longitude $2\chi=\varphi-\pi/2$). If we close the trajectory along the equator (Fig. 9), we obtain the solid angle $\Omega=-\varphi=-2\beta$, in agreement with the rule given by (4.3) (we are assuming that $0\leqslant\varphi\leqslant\pi$). We note, by the way, that the GP obtained in this way is smaller by a factor of two than that found for the Foucault pendulum for an analogous circuit on the Earth's surface (see Sec. 2).

We shall now use the Pancharatnam definition (3.6) to compare the field phases at three points in the transmission system (Fig. 9), namely, at entry point A (polarization $\mathbf{e}$), at a point B between the plates (polarization $\mathbf{e}'$, and at an exit point C (polarization $\mathbf{e}''$). If we neglect the phase σ we find from (5.30) and (5.31) that

$$\beta_{AB}=\arg(\mathbf{e}^*\cdot\mathbf{e}')=0,\quad \beta_{BC}=\arg(\mathbf{e}'^*\cdot\mathbf{e}'')=0,\quad \beta_{AC}=\arg(\mathbf{e}^*\cdot\mathbf{e}'')=\chi+\pi/4. \tag{5.32}$$

Thus, although the fields at A and B and at B and C are in phase, the field at A is not in phase with the field at C, i.e., the Pancharatnam 'in phase' binary relation for beams with different polarizations does not have the *transitivity* property and does not therefore enable us to divide the set of fields $\mathbf{e}$ into *equivalence classes*.

If we ignore σ, the resultant effect of the two plates is $\mathbf{e}''=\mathbf{D}_2\cdot\mathbf{D}_1\cdot\mathbf{e}$ where the product of the matrices in (5.30) and (5.31) is given by

$$D=\mathbf{D}_2\cdot\mathbf{D}_1=e^{i\beta}\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}. \tag{5.33}$$

Comparison with (5.11) shows that the effect of the two $\lambda/4$ plates is not equivalent to the effect of a rotator that takes the original point A directly to C along the equator without crossing the pole and without giving the GP (for $\delta<\pi/2$).

The significant experimental point is that the dynamic phase α is independent of the orientation χ of the second plate. From this point of view, it is better to use a closed trajectory on the PS, which can be accomplished with the three plates $\lambda/4$, $\lambda/2$, and $\lambda/4$. A symmetric path in the southern hemisphere (dashed line in Fig. 9)[14–18] is then added to the trajectory indicated by the thick line in Fig. 9, and rotation of the middle plate ($\lambda/2$) produces intensity beats with 100% visibility. Uniform rotation of this plate at the rate $\dot\chi$ produces a linear rise in the beam phase with time, i.e., a frequency shift[17] $\omega/\to\omega+2\dot\chi$.

The closure operation can also be performed in the reference channel (see Figs. 3b and 4). Let $\mathbf{e}_0=\mathbf{D}_0\cdot\mathbf{e}$ be the transformed reference polarization vector, so that the observed interference is determined by the product

$$\mathbf{e}_0^*\cdot\mathbf{e}'=\mathbf{e}^*\cdot\mathbf{D}_0^{-1}\cdot\mathbf{D}\cdot\mathbf{e},$$

and we may consider that the light beam propagates along the reference channel in the reverse direction. The closure condition for a contour on the PS takes the form $\mathbf{D}_0^{-1}\cdot\mathbf{D}=e^{i\gamma}\mathbf{I}$, i.e., $\mathbf{D}_0=e^{-i\gamma}\mathbf{D}$. For example, for the experiment illustrated in Fig. 3b,

$$D=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \mathbf{D}_0=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{5.34}$$

so that $\mathbf{D}_0^{-1}\cdot\mathbf{D}=\mathbf{D}^2=-\mathbf{I}$, $\gamma=\pm\pi$.

To conclude, consider an arbitrary unitary transformation $\mathbf{e}'=\mathbf{U}\cdot\mathbf{e}$, $\mathbf{U}\cdot\mathbf{U}^+=\mathbf{I}$ for which in the characteristic representation of the operator $\mathbf{U}$ we have $e_n'=\exp(i\lambda_n)e_n$, where $\exp(i\lambda n)$ are the eigenvalues of $\mathbf{U}$. Hence

$$\mathbf{e}^*\cdot\mathbf{e}'=\sum_{n=1}^{2}|e_n|^2\exp(i\lambda_n)$$
$$=e^{i\sigma}(\cos\delta+iZ\sin\delta), \tag{5.35}$$

where

$$\sigma=(\lambda_1+\lambda_2)/2,\quad \delta=(L_1-L_2)/2,\quad Z=|e_1|^2-|e_2|^2. \tag{5.36}$$

The expression that we have obtained is identical with (5.13) when the relevant parameters are suitably redefined. The formulas given by (5.14)–(5.16) thus retain their meaning. In general, the dynamic phase is given by (5.22).

For a closed orbit on the PS, the vectors $\mathbf{e}'$ and $\mathbf{e}$ differ only by the phase γ, i.e., $\mathbf{e}$ is identical with one of the eigenvectors of $\mathbf{U}$, and γ is identical with λ_1 or λ_2.

6. GP FOR TWO SCALAR WAVES

The three examples discussed above, namely, the Foucault pendulum, the polarized beam of light, and the vibrations of a stretched spring involve polarized transverse oscillations. The question that arises is whether the transversality of the vibrations is essential to the concept of the geometric phase? Is it possible to observe the GP in the case of, say, longitudinal sound waves?

It is readily verified that the essential feature is not the transversality, but the fact that the transverse oscillations have two degrees of freedom, i.e., we have a set of two degenerate oscillators. These oscillators are independent in free space, but they are coupled (in general) during propagation in an anisotropic materials. In the case of the Foucault pendulum, mode coupling is due to Coriolis forces.[31] The formalism employed above, which employs Jones vectors and matrices and the Poincaré sphere, can be applied to any two-mode systems, including a spin 1/2 particle in a magnetic field.

Consider two scalar waves which, for the sake of clarity we shall take to be two light beams with identical fixed polarizations.

Let e_1 and e_2 be the complex amplitudes of the waves in the two beams whose resultant intensity is $|e_1|^2+|e_2|^2=1$. We now introduce the 'vector' $\mathbf{e}=(e_1,e_2)$, i.e., a set of two numbers specifying the state of the filed in a particular cross section z of a two-channel optical transmission system.

We shall mix the two beams with the help of a perfect semitransparent mirror with a variable transmission coefficient t and reflection coefficient $r=(1-t^2)^{1/2}$. These two coefficients can be expressed in terms of the auxiliary parameter χ, i.e., $t=\cos\chi, r=\sin\chi$, so that the effect of the mirror is described by the transformation $\mathbf{e}'=\mathbf{D}_1\cdot\mathbf{e}$ where

$$\mathbf{D}_1=\begin{pmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{pmatrix}. \tag{6.1}$$

We now point the two beams at the second mirror which differs only by the sign of the reflection coefficient and is described by the matrix $\mathbf{D}_1^{-1}=\mathbf{D}_1^+$. Next, we introduce an additional path difference 2δ into one of the beams between the two mirrors. The effect of this is described by the diagonal matrix

$$\mathbf{D}_2=\begin{pmatrix}e^{i2\delta} & 0\\ 0 & 1\end{pmatrix}=e^{i\delta}\begin{pmatrix}e^{i\delta} & 0\\ 0 & e^{-i\delta}\end{pmatrix}. \tag{6.2}$$

The factor $\exp(i\delta)$ will be omitted henceforth. The device just described is a Mach-Zehnder interferometer with resultant matrix $\mathbf{D}=\mathbf{D}_1^{-1}\mathbf{D}_2\mathbf{D}_1$.

By multiplying these matrices together we can verify that $\mathbf{D}$ is the same as the matrix for the linear phase plate (5.28). The delay 2δ corresponds to the thickness of the phase plate multiplied by k_e-k_0, and the mirror transmission coefficient t corresponds to the cosine of the angle between the axis of symmetry and the x axis. The effect of the interferometer on the two beams with fixed polariza-

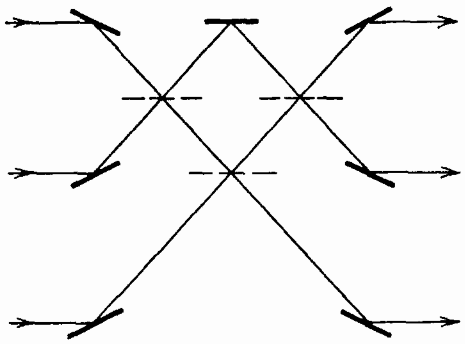

FIG. 10. System of mirrors used to mix three beams of light with fixed polarizations and to realize the unitary group U(3).

tions is thus seen to be isomorphic with the effect of a phase plate with parameters δ,χ on a single beam of polarized light.

To obtain the equivalent of the $\lambda/4$ plate with the orientation χ, we must set the path difference 2δ equal to $2\pi/2$ and the transmission coefficient t of both mirrors equal to $\cos\chi$. Two such interferometers placed in series, one with $t=1/\sqrt{2}$ and the other with the adjustable transmission $t=\cos\chi$ will simulate the system illustrated in Fig. 9. We then have at entry $e_1=1$ and $e_2=0$, i.e., only one channel of the first interferometer is excited. According to (5.30), the amplitudes of the exit beams are, respectively, given by

$$\mathbf{e}_1''=e^{i\beta}\cos\beta,\mathbf{e}_2''=e^{i\beta}\sin\beta \tag{6.3}$$

with a common phase that depends on the transmission of the mirrors in the secondary interferometer: $\beta=2\chi+\pi/2$ $=2\arccos t+\pi/2$.

If we mix one of the exit beams with a third reference beam, we obtain intensity beats with phase and visibility given by (3.5) and (3.6), respectively.

As in the polarization experiments, it is best to use a combination of three interferometers, equivalent to $\lambda/4$, $\lambda/2$, $\lambda/4$ plates and producing a closed trajectory on the PS (see Fig. 9). We then have a single exit beam with phase that depends on the transmission of the middle interferometer that simulates the $\lambda/2$ plate.

7. GP IN MULTIMODE SYSTEMS

We have examined the transformation of two beams that differed either by their polarizations or by the direction of propagation and spatial localization. It is clear that, in the latter case, the number of beams can be greater than two. It is possible to construct an optical system in which N time-independent beams with the same fixed polarizations are displaced by a system of semitransparent mirrors and delays. The use of directional couplers is proposed in Ref. 45. Mixing with the aid of mirrors is discussed in Ref. 46 (Fig. 10).

The state of the N-mode field in a particular section of the transmission system is specified by a point in N-dimensional complex space $\mathbf{C}^N=\mathbf{R}^{2N}$. The transformed states are described by $N\times N$ matrices so that this system is a realization of the linear group GL($2N$,$\mathbf{R}$). If there are no losses, the space of states is bounded by the sphere S^{2N-1} in $2N$-dimensional space. If we ignore the common phase factor $\exp(i\varepsilon)$ of all the N beams, we obtain the projective space S^{2N-2}.

It is clear from the foregoing that, to observe the phase γ, we need a reference beam with amplitude $\mathbf{e}_0$, i.e., the dimensionality of the system has to be increased to $N+1$. The interference system will then allow the observation of the GP associated with the group SU($2N-2$) and defined in terms of the common phase in accordance with (3.8).

Nonlinear optics can be used to realize other types of groups as well. For example, the effect of a two-mode parametric amplifier with a given pump can be described by matrices that belong to the Lorentz group SU(1,1). An amplifier of this kind preserves the intensity difference between the signal and idle modes: $|e_1|^2-|e_2|^2=1$ and the projective space is found to be a hyperboloid.[47–48] The GP introduced by a system of four degenerate (single-mode) parametric amplifiers giving a closed contour in projective space is calculated in Ref. 47 by a group method. Of course, direct calculation gives the same result.[49] We draw attention to the fact that we have here an example of the GP with $N=1$ (without counting the reference channel), which arises as a result of the modulation of the parameters of a linear oscillator by an external 'pump' solution. The general solution of this problem in the case of a quantum oscillator has been discussed by many authors.[58,60,61]

Schemes consisting of several parametric amplifiers and interferometers provide us with new possibilities, e.g., the realization of the group SU(2,1) (Ref. 50), and have been under recent discussion in connection with the EPR-Bell paradox.[51–53] We note that group methods are increasingly used in the description, classification, and evaluation of different models of linear, nonlinear, and quantum optics.

8. BERRY, AHARONOV-ANANDAN, AND PANCHARATNAM PHASES

Let us now consider a few points in relation to terminology which is still evolving and has not reached its final stage.

The phrase *topological phase* is often used instead of *geometrical phase.* This is justified by the importance of the topology of the base space of the fiber bundle, which should be nontrivial (Ref. 24, page 263). The invariance of the GP under arbitrary deformation of the trajectory C in the base space, which preserves the area Ω subtended by it [see (4.3)] may also be regarded as a topological property.

Another phrase that is sometimes encountered is *nonintegrable phase.* This may be due to the fact that the parallel transport rule is not equivalent to the presence of holonomic coupling in mechanical systems (which can be used to reduce the number of degrees of freedom of the system). We also note the therminological conflict between *holonomy* in geometry and *nonholonomic* system in mechanics.[6]

The GP that is produced at the end of the cycle is sometimes described by the words 'global change without local change.' We note, however, that it is possible to define a current, local GP with the help of the Pancharatnam connection [see (3.7) and (5.24)].

It is common to distinguish between a number of modifications of the GP: adiabatic and nonadiabatic Berry phases, the Aharonov–Anandan phase (AA), the Pancharatnam polarization phase, and the Rytov–Vladimirskiĭ quantum and nonquantum GP; the Hannay phase is encountered in the analysis of classical dynamic systems in terms of angle-action variables.

This terminology frequently reflects publishing priorities rather significant differences. A more systematic classification of types of GP is based on the standard notation for the corresponding transformation group such as SU(n) and so on.

The similarities and differences between some of the above types of GP are readily elucidated with the help of the polarization model. The point is that the above description of the transformation of polarization in terms of the Jones vectors and matrices and the Poincaré sphere is isomorphic with the description of the evolution of a two-level quantum system, e.g., a spin 1/2 particle in a magnetic field $\mathbf{B}$. Motion on the PS at constant latitude under the influence of a rotator (Sec. 5) corresponds to the free precession of the particle magnetic moment $\vec{\mu}$ around the direction of $\mathbf{B}$ in real space (or on the Bloch sphere; see, for example, Ref. 54, page 85). The level splitting that occurs in the field $\mathbf{B}$ corresponds to the splitting of the propagation constant $\Delta k = k_1 - k_2$ in an anisotropic medium.

The transmission of light by several phase plates corresponds to the passage of an electron or a neutron through several regions with nonuniform field $\mathbf{B}_n$. This case is actually described by a time-dependent Hamiltonian $H(t)$. A medium with z-dependent anisotropy corresponds to a continuous variation of $H(e)$ (Ref. 43).

The authors of Refs. 4 and 29 have drawn attention to the fact that the adiabatic and nonadiabatic Berry phases can be looked upon as special cases of the AA phase. The latter relies on a cyclicly varying Hamiltonian and is confined to solutions of the Schrödinger equation that also have this property:

$$H(T) = H(0), \quad \psi(T) = e^{i\gamma}\psi(0). \tag{8.1}$$

This condition means that $\psi(T)$ and $\psi(0)$ belong to the same ray (vertical lines in Fig. 5), so that the contour C describing the trajectory of the system in projective space is closed. The closure condition can also be written in the form

$$\psi(T) = U(T\psi(0) = e^{i\gamma}\psi(0), \tag{8.2}$$

where $U(t)$ is the evolution operator corresponding to the given Hamiltonian $H(t)$. The AA phase is therefore the phase acquired by the eigenfunctions of the evolution operator, $\psi(0)$, where $\exp(i\gamma)$ is the corresponding eigenvalue. An analogous situation arises in polarization optics for closed orbits: $\mathbf{e}' = \mathbf{U}\cdot\mathbf{e} = \exp(i\gamma)\mathbf{e}$, i.e., $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{U}$.

In particular, the Hamiltonian can be a constant, in which case all the state vectors of a two-level system are cyclic for $T = n2\pi/\omega$, but the nontrivial phase β arises only in time-dependent vectors, i.e., the parameter Z in (5.24) should differ from ± 1. The formula for β, given by (5.25) for closed cycles is identical with the corresponding formula in Ref. 28 where it was obtained for a spin; the formula for the current phase, given by (2.24), generalizes it to the case of open cycles.

It follows that the Pancharatnam polarization phase is essentially the same as the AA phase for $N=2$.

Berry[2] has examined the eigenfunctions $\psi_n(t)$ of the initial Hamiltonian $H(0)$ that satisfy the cyclic condition (8.1) only for adiabatic changes in $H(t)$ (since otherwise there are 'transitions' to other levels with $m \neq n$).

We note that descriptions of polarization experiments implicitly employ the adiabatic assumption because they neglect reflections from phase plates, i.e., the excitation of additional degrees of freedom (counterpropagating waves).

There is one further formal difference between the Berry and the AA phases: in the former case, the geometric character of the phase is elucidated by mapping the states of the system into the space of the parameters of the Hamiltonian, which are used to accomplish its variation in time [see (A14)] whereas, in the latter case, they are mapped into the projective space relative to the entire Hilbert space of the system.

In polarization optics, the adiabatic GP is observed by slowly (on the scale of λ) varying the anisotropic parameters of the medium along the beam axis (z axis) and by exciting the medium with the eigenwaves for the initial layers of the material.[43] For example, in a uniaxial crystal that gradually twists around the z axis, the ordinary wave remains an ordinary wave, i.e., it 'follows' the rotation of the axis of symmetry of the crystal, but at the same time acquires an additional phase, namely, the Berry adiabatic phase. Obviously, such a twisted crystal acts as a rotator and this takes us to the experiment shown in Fig. 3. If, however, the wave incident on the crystal has arbitrary polarization, the picture becomes more complicated because there are then two characteristic parameters, namely, the length for a twist of 2π and the anisotropic length $2\pi/\Delta k$.

These general solutions are analyzed for a spin 1/2 in Refs. 55–59 and for an oscillator with variable parameters in Refs. 58, 60, and 61.

9. GP IN QUANTUM OPTICS

The quantum aspects of optical GP effects are the subject of relatively few published papers (see, for example, Refs. 38, 48, and 62.

Some discussion was provoked by the question whether the Rytov–Vladimirskiĭ phase observed in lightguides[13] is a classical or a quantum effect.[17,38,63]

It is clear that, in general, the quantum description of optical phenomena is more universal than the classical or semiclassical presentation, so that it may be useful to start with the following definition. The essentially quantal effects are only those that have no classical analogs with the same characteristic features. From this point of view, quantum-optical effects are relatively rare, especially if we exclude from our discussion the detection process, i.e., adopt a semiclassical theory (see the discussion of two-photon interference in Refs. 49, 50, 53, and 64).

It follows from the foregoing discussion that the characteristic features of the GP are associated exclusively with the mathematical structure of the models used: the structure is the same for, say, a spin 1/2 in a magnetic field and for polarized light in an anisotropic medium. The geometric phase $\beta=-\Omega/2$ of the spin state vector $|\phi\rangle$ and of the polarization vector $\mathbf{e}$ of a plane wave arises from the scalar products $\langle\psi|\psi'\rangle$ and $\mathbf{e}^*\cdot\mathbf{e}'$ in precisely the same way, so that there is little point in calling it the quantum GP in the first case and the semiclassical GP in the second.

In the quantum description of a field, the vector $\mathbf{e}(z)$ describes the evolution of two field operators for a plane wave in the Heisenberg representation. The significant point here is that, in the case of linear optical devices, this evolution is exactly the same in form in both quantum and classical phenomenological theories (see, for example, Refs. 64 and 65). The result is that the quantum description of optical experiments that demonstrate GP yields nothing new as compared with the classical description, and the observed phase does not depend on the state $|\psi\rangle$ (Ref. 38).

It is sometimes considered that optical interference experiments reveal the classical Hannay phase and not the quantum Berry phase.[48] This point of view is based on the definition of the optical GP in terms of the product $\langle\psi|\psi'\rangle$ rather than $\mathbf{e}^*\cdot\mathbf{e}'$. However this formal definition has nothing in common with observed effects in quantum optics.[38] In particular, the phase of the product $\langle\psi|\psi'\rangle$ depends significantly on the properties of the state $|\psi\rangle$ and the mean field intensity, and it is only in some cases (single-photon state) that it describes the observed phase.[38,48]

We note in conclusion that GP can also be observed in intensity interference experiments[14,38] which usually employ the photon counting method and the observed interference is interpreted as an essentially nonclassical effect. However, it seems more logical to look upon the interference phenomenon itself as a classical effect and ascribe its only nonclassical characteristic, namely, the high interference visibility, to the nonclassical nature of the nonclassical (two-photon)light used at entry to the interferometer.[38,49,50]

10. CONCLUSION

We have seen that a system of N interacting oscillators with complex amplitudes e_n is characterized by a phase $\gamma=\alpha+\beta$ which, after subtraction of the dynamic part α that is determined by the dynamics of the system, displays specific geometric properties associated with the circuit C that represents the evolution of the system in projective space (i.e., if we ignore the common phase of the amplitude e_n). The evolution of the system in time or in space is described by the linear transformation U: $\mathbf{e}\rightarrow\mathbf{e}'$ where $\mathbf{e}=\{e_n\}$. The phase γ can be observed in an interference experiment employing an additional reference channel with amplitude $\mathbf{e}_0=\mathbf{e}$, which results in interference beats with phase γ. The latter is equal to the phase of the resultant of N complex numbers:

$$\gamma=\mathrm{arctg}(\mathrm{Im}\,u/\mathrm{Re}\,u),$$

where

$$u\equiv\sum_{n=1}^{N}u_n\equiv\sum_{n=1}^{N}\mathbf{e}_n^*\mathbf{e}_n'.$$

The simplest and clearest example of a physical model revealing the geometric phase employs linearly polarized waves traveling, for example, along a stretched string (see Fig. 3). A jet of air can be used to rotate the plane of polarization of such waves. In the case of linear polarization, we have $\beta=\pm\pi$. Generalization to arbitrary elliptic polarization lead to the geometric formula $\beta=-\Omega/2$ where Ω is the solid angle subtended on the Poincaré sphere in the course of the transformation of the polarization.

APPENDIX: A DERIVATION OF THE 'GEOMETRIC' FORMULA $\beta=-\Omega/2$

Consider a vector $\mathbf{g}(z)$ freed from the common phase $\gamma(z)$, so that the polarization vector is

$$e(z)=e^{i\gamma(z)}\mathbf{g}(z) \tag{A1}$$

and let us determine it's the rate of change:

$$\frac{\mathrm{de}}{\mathrm{dz}}\equiv\dot{\mathbf{e}}=i\dot{\gamma}e+e^{i\gamma}\dot{\mathbf{g}}. \tag{A2}$$

If we now multiply by $\mathbf{e}^*$ we obtain the following expression for the rate of change of phase:

$$\dot{\gamma}=i(\mathbf{g}^*\cdot\dot{\mathbf{g}})-i(\mathbf{e}^*\cdot\dot{\mathbf{e}}). \tag{A3}$$

According to (5.22), the second term on the right is equal to the rate of change of the dynamic part of the phase, so that the rate of change of the GP is

$$\dot{\beta}=i(\mathbf{g}^*\cdot\dot{\mathbf{g}})=-\mathrm{Im}(\mathbf{g}^*\cdot\dot{\mathbf{g}}) \tag{A4}$$

which follows from the fact that $|\mathbf{g}|^2=1$. It follows that the GP at the point z takes the form

$$\beta(z)=-\mathrm{Im}\int_0^z \mathrm{d}z'(\mathbf{g}^*\cdot\dot{\mathbf{g}}). \tag{A5}$$

Suppose that the circuit C on the PS is specified parametrically: $\vartheta=\vartheta(z)$, $\varphi=\varphi(z)$, so that on the circuit C we have

$$\dot{\mathbf{g}}=\frac{\partial g}{\partial\vartheta}\dot{\vartheta}+\frac{\partial g}{\partial\varphi}\dot{\varphi}. \tag{A6}$$

Substituting

$$A_\vartheta\equiv\mathrm{Im}\left(\mathbf{g}^*\cdot\frac{\partial\mathbf{g}}{\partial\vartheta}\right),\qquad A_\varphi\equiv\mathrm{Im}\left(\mathbf{g}^*\cdot\frac{\partial\mathbf{g}}{\partial\varphi}\right), \tag{A7}$$

we find that (A5) assumes the form of the curvilinear integral

$$\beta=-\mathrm{Im}\int_0^z dz'\left[\left(\mathbf{g}^*\cdot\frac{\partial\mathbf{g}}{\partial\vartheta}\right)\dot\vartheta+\left(\mathbf{g}^*\cdot\frac{\partial\mathbf{g}}{\partial\varphi}\right)\dot\varphi\right]$$
$$=-\int_C A_\vartheta d\,\vartheta+A_\varphi d\,\varphi. \quad (A8)$$

In the case of a closed circuit, this is the circulation of the 'vector' $\mathbf{A}=(A_\vartheta, A_\varphi)$. We can now use Stokes' formula to express GP in terms of the flux of the 'magnetic field':

$$\beta=-\iint_\Omega Bd\,\vartheta d\,\varphi, \quad (A9)$$

where

$$B\equiv\frac{\partial A_\varphi}{\partial\vartheta}-\frac{\partial A_\vartheta}{\partial\varphi}$$
$$=\frac{\partial\mathbf{g}^*}{\partial\vartheta}\cdot\frac{\partial\mathbf{g}}{\partial\varphi}-\frac{\partial\mathbf{g}^*}{\partial\varphi}\cdot\frac{\partial\mathbf{g}}{\partial\vartheta}$$
$$=2\,\mathrm{Im}\left(\frac{\partial\mathbf{g}^*}{\partial\vartheta}\cdot\frac{\partial\mathbf{g}}{\partial\varphi}\right) \quad (A10)$$

and Ω is the solid angles subtended by the circuit C.

In modern geometry, the integrand in (A8) with the integration variables ϑ,φ is called the differential form of degree 1 or simply the 1-form.[25] Accordingly, $Bd\vartheta d\varphi$ is a 2-form. A commonly used notation is: $\mathbf{g}^*\cdot d\mathbf{g}$ and $d\mathbf{g}^*\wedge d\mathbf{g}$.

If we suppose that the two-component function $\mathbf{A}(\vartheta,\varphi)$ is effectively the vector potential on the PS, the quantity B corresponds to a magnetic field pointing radially. It could be due to the Dirac magnetic monopole placed at the center of the PS. At the same time, there is an analogy with the Aharanov–Bohm effect (cf. Ref. 43).

We note that if we multiply the vector $\mathbf{g}$ by an arbitrary phase factor, i.e., $\mathbf{g}\to\tilde{\mathbf{g}}\equiv\mathbf{g}\exp[i\varepsilon(z)]$, then according to (A7) the functions A_ϑ and A_φ will be incremented by $\partial\varepsilon/\partial\vartheta,\partial\varepsilon$ and $\partial\varphi$, respectively, and similarly for the vector potential of the electromagnetic field under the gauge transformation of the state vector of a charged particle. This transformation may be related to the conservation of the electric charge, which necessarily leads to the existence of an electromagnetic field with known properties (see, for example, Ref. 27).

The essential point is that, according to (A10), B is invariant under an arbitrary gauge transformation. For example, in the case of a closed circuit C, we can use $\mathbf{e}$ or $\mathbf{f}=\exp(-i\alpha)\mathbf{e}$ in (A5) instead of $\mathbf{g}$ [it is $\mathbf{g}$ that appears in the local relation (A4)]. It is convenient to take $\tilde{\mathbf{g}}$ in the form of (4.1), i.e., $\tilde{\mathbf{g}}\equiv\mathbf{d}$. In a cyclic basis, the derivatives of $\tilde{\mathbf{g}}$ are given by

$$\frac{\partial\tilde{g}}{\partial\vartheta}=\frac{1}{2}\left(e^{-i\varphi/2}\sin\frac{\vartheta}{2},e^{i\varphi/2}\cos\frac{\vartheta}{2}\right),$$
$$\frac{\partial\tilde{g}}{\partial\vartheta}=\frac{i}{2}\left(-e^{-i\varphi/2}\cos\frac{\vartheta}{2},-e^{i\varphi/2}\sin\frac{\vartheta}{2}\right), \quad (A11)$$

and if we substitute this in (A7) and (A10) we obtain

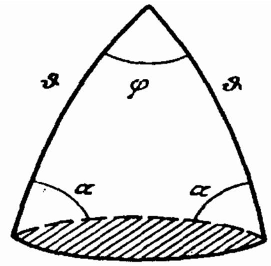

FIG. 11. Calculation of the solid angle (shaded region) on the Poincaré sphere, which corresponds to the geometric phase introduced by a rotator. Dashed curve—geodesic line.

$$A_\vartheta=0,\quad A_\varphi=-\tfrac{1}{2}\cos\vartheta,\quad B=\tfrac{1}{2}\sin\vartheta. \quad (A12)$$

Finally, from (A9) we have

$$\beta=-\frac{1}{2}\iint_\Omega \sin\vartheta d\vartheta d\varphi=-\frac{1}{2}\Omega. \quad (A13)$$

To transform to the case of the adiabatic Berry phase (Sec. 8), we assume that the initial state $\mathbf{e}(0)=\mathbf{g}(0)=\mathbf{g}_n(0)$ is an eigenstate of $H(0)$. For sufficiently slowly varying H, the system remains in eigenstate of the instantaneous Hamiltonian $H(t)$. Suppose that H depends on z (or on t) through the parameters $\vec{\lambda}=\{\lambda_i(z)\}$, so that the vector $\mathbf{g}_n$ is also a function of z through these parameters:

$$\dot{\mathbf{g}}_n=\sum_i\frac{\partial\mathbf{g}_n}{\partial\lambda_i}\dot\lambda_i. \quad (A14)$$

The formula given by (A8) then assumes the form

$$\beta_n=-\int_0^z\sum_i A_i^{(n)}d\lambda_i, \quad (A15)$$

where now

$$A_i^{(n)}\equiv\mathrm{Im}(\mathbf{g}_n^*\cdot\partial\mathbf{g}_n/\partial\lambda_i). \quad (A16)$$

In the case of a two-level system, we can choose the parameters λ_i in accordance with (5.10) to be the spherical coordinates ϑ,φ on the PS (i.e., take the dimensionalities of the projective space and of the parameter space of the Hamiltonian to be the same), which again gives (A8)–(A13), but this does not happen when the number of levels is more than 2 [we have to use the multidimensional Stokes' formula to transform the circulation (A15)].

To conclude, we present a direct derivation of (A13) for the special case of the GP induced by a rotator [cf. (5.24)]. It is clear from Fig. 11 that the required area Ω is equal to the difference $\Omega_1-\Omega_2$ where Ω_1 is the area between two meridians with difference in longitude φ and latitude $\bar\vartheta=90°-\vartheta$, and Ω_2 is the area of the spherical triangle bounded by the same longitudes and the corresponding geodesic. The angle α is found from the formulas of spherical trigonometry (it is assumed to be $<\pi/2$:

$$\sin\vartheta\,\mathrm{ctg}\,\vartheta=\mathrm{ctg}\,\alpha\sin\varphi+\cos\vartheta\cos\varphi. \quad (A17)$$

Substituting $\cos\vartheta = z$, we find that $\cot\alpha = Z\tan 2\varphi$, and replacing $\cot\alpha$ with $\tan(90^\circ - \alpha)$ we obtain

$$\alpha = 90^\circ - \mathrm{arctg}(Z\mathrm{tg}2\varphi). \tag{A18}$$

By the Gauss–Bonnet theorem, the area of a spherical triangle is the difference between the sum of the angles and π:

$$\Omega_2 = \varphi + 2\alpha - \pi = \varphi - 2\,\mathrm{arctg}(Z\mathrm{tg}2\varphi). \tag{A19}$$

The area Ω_1 is readily found by integration: $\Omega_1 = \varphi(1-Z)$. Hence, using the result $\varphi = -2\delta$, we obtain

$$\Omega = -2\,\mathrm{arctg}(Z\,\mathrm{tg}\,\delta) + 2Z\delta.$$

Comparison with (5.24) again gives $\beta = -\Omega/2$.

We assumed above that $\Omega_{1,2} > 0$. When the direction in which the circuit is made is taken into account, the sign of β is found to be opposite to the signs of Ω and $Z\varphi$, but is the same as the sign of $Z\delta$.

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