# Propagation and transformation of electromagnetic waves in one-dimensional periodic structures 

S. Yu. Karpov and S. N. Stolyarov<br>A. F. Ioffe Physicotechnical Institute, Russian Academy of Sciences, St. Petersburg; "Polyus"Scientific Research Institute, Moscow

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#### Abstract

Different analytic methods (perturbation theory in the Born approximation and under Bragg reflection, as well as coupled-wave theory and its modifications) are used to derive and discuss approximate analytic expressions for electromagnetic wave fields in bounded one-dimensional periodic dielectric structures and the corresponding reflection coefficients. The range of validity of each of the analytic solutions is established and it is shown that the modified coupled-wave method, which is valid simultaneously for large and small modulation periods and appreciable modulation depths, has the widest range of validity. The method is used to calculate the reflection coefficients of such structures as functions of the incident-wave frequency, taking into account the finite size of the structures, the properties of the ambient media, absorption, and small nonlinearity and aperiodicity.


## INTRODUCTION

One-dimensional problems have always been popular among physicists. The mathematical formalism that is necessary for their solution is usually particularly simple in the one-dimensional case. It is thus often possible to obtain an exact solution of a problem, which can subsequently serve as a standing point and testing ground for approximate methods describing more complicated physical situations.

Wave propagation in one-dimensional periodic media has gone through two well-defined "highs" in its history. The first involved studies of the structure of energy bands in one-dimensional crystals, and was initiated by the well known paper by Krönig and Penney. ${ }^{1}$ At first, the electron theory of one-dimensional crystals was merely a convenient physical model, but it subsequently found its real experimental basis in semiconducting superlattices. ${ }^{2,3,49}$ Another "high" in the study of wave propagation in one-dimensional periodic structures was the rapid advance in optical holography, ${ }^{4,5}$ acousto-optics, ${ }^{6,7}$ integrated and optical electronics, ${ }^{8-10} \mathrm{X}$-ray diffractometry, ${ }^{11,12}$ and so on in the middle 60 s and early 70 s . The renewed interest in one-dimensional problems was dictated by the desire to investigate wave propagation and scattering in one-dimensional structures at a new level (defined by the then current problems), and to use new tools (such as computer technology), with a view to practical applications. And although the one-dimensional model of a periodic medium was invalid more often than not in practical cases, the results obtained in this way were useful in the qualitative analysis and in approximate descriptions of the physics of these processes. The one-dimensional problem was thus the source of a key or basic model whose function was, on the one hand, to improve our understanding of processes occurring in periodic structures and, on the other, to enable us to develop and test quantitative methods for their analysis.

The aim of this paper is to review from a unified stand-
point the methods now available for the analysis of wave processes in one-dimensional periodic structures, to expose similarities and differences between these methods, to compare their ranges of validity, and to provide a brief description of the basic physical effects encountered in periodic media with weak aperiodicity and small nonlinearity in structure parameters.

It has not been our intention to provide an exhaustive review of these questions, since this has already been done to some extent elsewhere (we recall, for example, the monograph by Brillouin and Parodi ${ }^{13}$ and the review paper by Elachi. ${ }^{14}$ We have concentrated our attention on universal approximate methods of analysis that lead to analytic solutions. Such methods enable us to describe from a unified standpoint the different processes that occur in periodic structures of arbitrary configuration. They can usually be extended to two-dimensional problems and to problems with a large number of degrees of freedom. In this sense, analysis of one-dimensional periodic structures provides us with a methodological basis for the solution of more complicated problems. Finally, universal approximate methods allow us to perform rapid estimates and qualitative analyses of special cases, which is important from the practical point of view.

We begin our review with classical methods of solving Maxwell's equations in periodic media. They include the Floquet-Bloch method, the multiwave and two-wave dynamic theory of diffraction (using the perturbation-theory expansion with the permittivity modulation depth as the small parameter), and the method of integral equations. We continue by considering the approximate analytic theory of coupled waves due to Kogelnik. All these methods lead to simple approximate analytic solutions that are valid for small permittivity modulation depths. A more detailed account is then given of the modified coupled-wave theory that is valid not only for small modulation depths but, particularly, when the radiation wavelength is much smaller than the
modulation period. This approximate analytic theory is used as a basis for the derivation of the corresponding dispersion relations and the wavelength dependence of the wave transformation coefficients for an arbitrary modulation profile and arbitrary thickness of the periodic layer, taking its absorption and the properties of ambient media into account. It has been shown ${ }^{57.59}$ that there is good agreement (for small modulations depths) between the results obtained in this analytic theory and the exact numerical calculations performed near a Bragg resonance and for large detuning from this resonance. Small aperiodicity in the layer and weak nonlinearity of the media are taken in account within the framework of this unified approach.

Our entire discussion is based on the simple example of wave propagation in the direction of a periodic variation in the properties of a medium with scalar and frequency-independent permittivity. There has been considerable recent progress in theoretical and experimental studies of the optics of anisotropic and gyrotropic periodic media. ${ }^{48,50-52}$ The most interesting results have been obtained in the optics of liquid crystals, including the effect of polarization, frequency, and amplitude on reflection properties. ${ }^{48,50,52}$ The propagation and transformation of waves in such media will not be discussed here for lack of space.

## 1. ONE-DIMENSIONAL PERIODIC MEDIA

In general, a one-dimensional periodic medium consists of a layer of thickness $L$ (Fig. 1), filled with a medium whose permittivity $\varepsilon(z)=\varepsilon(z+a)$ varies periodically in the direction of the $z$ axis ( $a$ is the period of the structure). If the medium has absorbing or amplifying properties, its permittivity becomes complex: $\varepsilon=\varepsilon^{\prime}+i \varepsilon^{\prime \prime}$. In practice, most known media have $\varepsilon^{\prime \prime} \ll \varepsilon^{\prime}$ where the real part $\varepsilon^{\prime}$ is related to the refractive index by the formula $n=\left(\varepsilon^{\prime}\right)^{1 / 2}$ and the imaginary part $\varepsilon^{\prime \prime}$ is related to the intensity absorption coefficient $\alpha$ by $\varepsilon^{\prime \prime}=\alpha k^{-1}\left(\varepsilon^{\prime}\right)^{1 / 2}$ where $k=\omega / c=2 \pi / \lambda$ and $\lambda$ is the wavelength of light in vacuum. For the sake of simplicity, we consider a plane light wave, incident normally on a one-dimensional periodic structure (OPS). For arbitrarily polarized light, the equation for the electric field $E(z)$ inside the layer then takes the form

$$
\begin{equation*}
\mathrm{d}^{2} E(z) / \mathrm{d} z^{2}+k^{2} \varepsilon(z) E(z)=0 \tag{1}
\end{equation*}
$$

where

$$
E(z, t)=E(z) \exp (-i \omega t)
$$

Equation (1) is called Hill's equation when $\varepsilon(z)$ is a periodic function. We shall take it as our basic equation for


FIG. 1. Layer with one-dimensional perodicity: $L$-layer thickness, $\varepsilon_{1}$ and $\varepsilon_{2}$-permittivities of ambient homogeneous media, a-period of the function $\varepsilon(z)$.
the description of light propagation in OPS.
Since Hill's equation is linear, its general solution is a superposition of two independent special solutions $E_{1}(z)$ and $E_{2}(z)$ :

$$
\begin{equation*}
E(z)=C_{1} E_{1}(z)+C_{2} E_{2}(z) \tag{2}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants. For a periodic medium, Floquet's theorem ${ }^{13,15,16}$ shows that a special solution of (1) can be written in the form

$$
\begin{equation*}
E_{1}(z)=\Phi_{1}(z) \exp (i \mu z) \tag{3}
\end{equation*}
$$

where $\mu$ is the so-called characteristic index that is generally complex ( $\mu=\mu^{\prime}+i \mu^{\prime \prime}$ ) and $\Phi_{1}(z)$ is a periodic function with period $a$. A solution such as (3) gives the following expression for the resultant field:

$$
E_{1}(z, t)=\Phi_{1}(z) \exp \left(-\mu^{\prime \prime} z\right) \exp \left[i\left(\mu^{\prime} z-\omega t\right)\right]
$$

This is a spatially modulated [ $\Phi_{1}(z)$ is periodic], inhomogeneous ( $\mu^{\prime \prime} \neq 0$ ) electromagnetic wave, propagating (for $\mu^{\prime} \neq 0$ ) along the $z$ axis with phase velocity $v_{\mathrm{ph}}=\omega /\left|\mu^{\prime}\right|$.

There are usually two standard electrodynamic problems that are of interest. The first involves the determination of the dispersion relation $\mu=\mu(k)$ and the use of this relation as a means of establishing the regions of instability and stability of solutions such as ( $3^{\prime}$ ) with real ( $\mu^{\prime \prime}=0$ ) and complex values of $\mu\left(\mu=\mu^{\prime}(k)+i \mu^{\prime \prime}(k)\right)$. The next step is to determine the amplitudes of the Fourier components of the functions $\Phi_{1,2}(z)$ for an unbounded periodic medium. In the second problem, we have to find the reflection and transmission coefficients for a plane wave incident on a layer of a periodic medium of thickness $L$. These coefficients, and also the constants $C_{1}$ and $C_{2}$, are determined from the field continuity conditions at the layer boundaries, i.e., at $z=0$ and $z=L$. The dispersion relation $\mu(k)$ and the form of the special solutions $E_{1}(z)$ and $E_{2}(z)$ are assumed known. The first problem is therefore a preliminary stage to the solution of the second, and this is indeed how we shall proceed below.

## 2. CLASSICAL METHODS OF FINDING SOLUTIONS FOR UNBOUNDED PERIODIC STRUCTURES

In practice, the two most frequent one-dimensional periodic structures are: layered media with (1) a step (piecewise constant) variation in permittivity $\varepsilon(z)$ and (2) with harmonically varying $\varepsilon(z)$. These periodic media are discussed in detail in a number of monographs ${ }^{13,17,18,53}$ and we shall therefore confine our attention to a brief review of the methods available for their analysis. For layered media, there are effective matrix methods ${ }^{15,17,53}$ that lead to a transcendental dispersion relation for the characteristic index $\mu(k)$. In simple cases such as, for example, the Krönig-Penney model, ${ }^{1,13,19}$ the dispersion relation can be expressed in terms of trigonometric functions and its solution can be obtained approximately by a graphical method or numerically to a given precision. For harmonically modulated permittivity, the solution of (1) can be expressed in terms of Mathieu functions. ${ }^{18}$ The dispersion relation $\mu(k)$ and the field configurations are then most frequently calculated with the help of rapidly converging series. The number of terms that must be taken into account in the series is determined by the required precision of the final result. In each of these cases, therefore, specific calculations sooner or later involve a nu-
merical procedure. Moreover, the computational schemes developed for OPS with piecewise $\varepsilon(z)$ and harmonically modulated permittivity are not universal, but are designed for the analysis of these particular structures.

This is why subsequent studies were concerned with creating a unified numerical method for the solution of (1) that was independent of the particular form of $\varepsilon(z)$ and was based on the expansion of $\varepsilon(z)$ and of the solutions $\Phi_{1,2}(z)$ into infinite Fourier series. Hill's classical papers served as the starting point for all this work (see Refs. 13 and 15). In more recent times, the tendency toward the unification of numerical calculations has been represented by the so-called Floquet method, widely used in the analysis of one-dimensional periodic structures. ${ }^{20-26}$

### 2.1. The Floquet-Bloch method

The essence of this method is as follows (see, for example, Ref. 25). In accordance with (3), we write the solution of ( 1 ) in the form

$$
\begin{equation*}
E_{1}(z)=\exp \left(i \mu_{z}\right) \sum_{l=-\infty}^{l=+\infty} A_{l} \exp \left(i \frac{2 \pi}{a} l z\right) \tag{4}
\end{equation*}
$$

where $A_{l}$ are unknown coefficients that determine the form of the periodic function $\Phi_{1}(z)$. We also make a Fourier expansion of the periodic permittivity $\varepsilon(z)$ where $a$ is the structure period:

$$
\begin{equation*}
\varepsilon(z)=\sum_{m=-\infty}^{m=+\infty} \varepsilon_{m} \exp \left(i \frac{2 \pi}{a} m z\right) \tag{5}
\end{equation*}
$$

in which

$$
\begin{equation*}
\varepsilon_{m}=\frac{1}{a} \int_{0}^{a} \varepsilon(z) \exp \left(-i \frac{2 \pi}{a} m z\right) \mathrm{d} z \tag{5'}
\end{equation*}
$$

Substituting (4) and (5) in (1), separating out the $m=0$ term, and replacing in the double infinite sums over $m$ and $l$ the summation index $l+m$ with $l$, and in the sum over $m$ the summation index $m$ with $l-m$, we obtain the following infinite set of equations for the coefficients $A_{t}$ :

$$
\begin{align*}
& {\left[k^{2} \varepsilon_{0}-\left(\mu+\frac{2 \pi}{a} l\right)^{2}\right] A_{l}} \\
& \quad=-k^{2} \sum_{m=-\infty} \varepsilon_{l-m} A_{m}\left(1-\delta_{l m}\right) \tag{6}
\end{align*}
$$

where $l=0, \pm 1, \pm 2, \ldots, \delta_{m l}$ is the Kronecker symbol, and the factor $\left(1-\delta_{m l}\right)$ annuls the $m=l$ term. The set of equations given by (6) is exact. If we equate its determinant to zero, we obtain the dispersion relaxation for the characteristic index $\mu$, and the unknown coefficients $A_{l}$ can be expressed in terms of $A_{0}$ either by the method of continued fractions ${ }^{13,15}$ or by the well-known methods available for the evaluation of infinite matrices. ${ }^{13,15}$ In practice, instead of the infinite set of equations, we solve a finite set that is obtained from (6) by discarding the higher-order harmonics. The order of the approximate set of equations is determined by the required precision of the final result. We note that, if the permittivity is not modulated, and $\varepsilon_{m}=0$, equation (6) has nonzero solutions, $A_{l} \neq 0$, only if the wave vector $k$ is such that

$$
\begin{align*}
& k l^{ \pm)}\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}= \pm\left(\mu^{\prime}+\frac{2 \pi}{a} l\right. \\
& l=0, \pm 1, \pm 2, \ldots
\end{align*}
$$

where $k_{l}^{(+)}$and $k_{l}^{(-)}$refer to waves traveling in the positive and negative directions of the $z$ axis, respectively [see (3')].

It is clear from (6) and from the solutions given by (4) that the Floquet-Bloch method is particularly suitable for computer evaluations: the numerical calculation follows directly the formulation of the problem. It is therefore much simpler to use the Floquet-Bloch method in specific calculations rather than derive general properties of wave transformations in one-dimensional periodic media. The latter problem is usually solved by approximate analytic methods.

### 2.2. Integral-equation method

Approximate analytic solutions of equation (1) can be found by starting with the equivalent integral equation (see, for example, Ref. 27)

$$
\begin{equation*}
E(z)=k^{2} \int_{-\infty}^{+\infty} \mathrm{d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) G\left(z-z^{\prime}\right) E\left(z^{\prime}\right) \tag{7}
\end{equation*}
$$

in which the periodic perturbation $\varepsilon_{p}(z)=\varepsilon(z)-\varepsilon_{0}$ has a zero mean and $G\left(z-z^{\prime}\right)$ is the Green's function of the unperturbed equation (1) with $\varepsilon(z)=\varepsilon_{0}$ and right-hand side equal to $-\delta\left(z-z^{\prime}\right)$. According to Ref. 27, $G\left(z-z^{\prime}\right)$ is given by (see Appendix I)

$$
\begin{equation*}
G\left(z-z^{\prime}\right)=\frac{i}{2 k \varepsilon_{0}^{1 / 2}} \exp \left(i k \varepsilon_{0}^{1 / 2}\left|z-z^{\prime}\right|\right) \tag{8}
\end{equation*}
$$

where $k=\omega / c=2 \pi / \lambda$ and $\varepsilon_{0}=\varepsilon_{0}^{\prime}+i \varepsilon_{0}^{\prime \prime}$. For an infinite periodic structure, the integral in (7) is conveniently evaluated over the period $a$ instead of the entire $z$ axis. If we use the periodicity relations $\varepsilon_{p}(z \pm m a)=\varepsilon_{p}(z)$ and $E(z \pm a)$ $=E(z) \exp ( \pm i \mu m a)$, which follow from (3) and (7), we find that instead of (7) we have

$$
\begin{equation*}
E(z)=k^{2} \int_{0}^{a} \mathrm{~d} z^{\prime} \varepsilon_{\mathrm{p}} g\left(z-z^{\prime}\right) E\left(z^{\prime}\right) \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
g(z) & =\sum_{n=-\infty}^{n=+\infty} G(z-n a) \exp (\dot{\mu} n a) \\
& =-\frac{1}{a} \sum_{n=-\infty}^{n=+\infty} \frac{\exp \left(i \beta_{n^{2}}\right)}{k^{2} \varepsilon_{0}-\beta_{n}^{2}}, \beta_{n}=\mu+\frac{2 \pi}{a} n .
\end{align*}
$$

A detailed analysis of (9) and (10) and a proof of the equivalence of the integral equation (9) and the system given by (6) in the Floquet-Bloch method are given in Appendix I.

## 3. PERTURBATION THEORY

### 3.1. Born approximation and Raman-Nath multiwave dififraction

When approximate analytic formulas are derived in the Born approximation, we start not with the integral equation (7) [or (9)] and its subsequent solution by the iteration method, but with the equivalent set of equations of the dynamic diffraction theory, given by (6). To construct an ap-
proximate solution of (6) for $E_{1}(z)$ in (4), we consider its right-hand side as a perturbation. This is possible, for example, for small permittivity modulation amplitudes, i.e., for $\varepsilon_{m}<\varepsilon_{0}$ and $m \neq 0$. If, at the same time, we suppose that $A_{m}<A_{0}$, and if we solve (6) by the method of successive approximations, we obtain a perturbation-theory series for the characteristic index $\mu$ and the amplitudes $A_{m}$ (see, for example, Ref. 25). This gives

$$
\begin{align*}
& \mu \approx k \varepsilon_{0}^{1 / 2}\left[1+\left(\frac{a}{\lambda \varepsilon_{0}^{1 / 2}}\right)^{2} \sum_{m=1}^{m=\infty} \frac{\left|\varepsilon_{m}\right|^{2}}{m^{2}-\left(4 a^{2} \varepsilon_{0} / \lambda^{2}\right)}\right], \\
& A_{m} \simeq\left(\frac{a}{\lambda}\right)^{2} \frac{\varepsilon_{m}}{m+(2 a / \lambda) \varepsilon_{0}^{1 / 2}} A_{0}, \tag{11}
\end{align*}
$$

where $\lambda=2 \pi / k=2 \pi c / \omega$ is the wavelength in vacuum.
This solution procedure corresponds to the Born approximation that is widely used in quantum theory of scattering ${ }^{62}$ and in calculations on the scattering of electromagnetic waves by ultrasound. ${ }^{17}$ It is based on the expansion of the field in powers of a small parameter that involves the perturbation and the ratio of the size of the scatterer to the wavelength. It is readily seen that, in the case of (11), the small parameter is

$$
\begin{equation*}
\zeta=\left(k a /\left|\varepsilon_{0}\right|^{1 / 2}\right) \Delta \varepsilon=k_{0} a \Delta \varepsilon /\left|\varepsilon_{0}\right|, \tag{12}
\end{equation*}
$$

where $\Delta \varepsilon$ is the permittivity modulation amplitude, i.e., $\Delta \varepsilon=\left|\varepsilon_{m}\right|_{\text {max }}$ and $k_{0}=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$ is the wave vector of light propagating in the homogeneous medium.

The resultant field $E_{1}(z)$ in (4) is a superposition of plane electromagnetic waves with different amplitudes $A_{i}$ and wave vectors

$$
k_{l}=\left(\mu^{\prime}+\frac{2 \pi}{a} l\right), l=0, \pm 1, \pm 2, \ldots
$$

When the permittivity modulation depth $\Delta \varepsilon$ is small and the wavelength $\lambda$ is comparable with the structure period $a$, in which case the Born-approximation parameter $\zeta$ in (12) is also small, all the amplitudes $A_{l}(l \neq 0)$ are small in comparison with the amplitude $A_{0}$ of the zero-order approximation. This means that, when $\zeta \ll 1$, the total field $E_{1}(z)$ in (4) consists, in the first approximation, essentially of one wave with amplitude $A_{0}$ and propagation constant $\mu_{0}^{\prime}=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$, which corresponds to phase velocity $v_{\text {ph }}=\omega / \mu_{0}^{\prime}$ $=c /\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$. The secondary waves with small amplitudes $A_{1}<A_{0}$ that are excited by this wave on permittivity inhomogeneities produce a weak wave "background." This takes the form of a set of waves traveling with different velocities $\omega / k_{l}$ (different in both magnitude and direction) and small amplitudes $A_{l}$ that can be calculated from the approximate formulas given by (11).

### 3.2. Two-wave dynamic diffraction theory; Bragg diffraction

The Born approximation considered in the last Section is valid when the wave amplitudes $A_{m}$ other than $A_{0}$ are all small, i.e., when we are dealing with essentially single-wave propagation. On the other hand, it is clear from the solutions given by (11) and obtained in the Born approximation that the condition $A_{m}<A_{0}$ ceases to be valid for numbers $m=-n$ for which the radiation wavelength $\lambda=2 \pi c / \omega$ satisfies the Bragg condition

$$
\begin{equation*}
k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=\pi n / a, \text { or } n \lambda /\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=2 a, \tag{13}
\end{equation*}
$$

where $n=1,2,3, \ldots$ define the Bragg resonances. The condition given by (13) refers to the $n$th Bragg resonance for which the amplitude $A_{-n}$ of the ( $-n$ )-th harmonic can become equal to or greater than the amplitude $A_{0}$ of the leading incident wave for which, according to (6') $k_{0}^{(+)}=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=\mu_{0}^{\prime}=\pi n / a$. It is then clear from (11) that the amplitude $A_{m}$ of the remaining harmonics with $m \neq(0,-n)$ are small, as before, and this means that the propagation of light in this case is essentially of the two-wave kind. For $x$-rays in crystals, this is known as the two-wave dynamic theory of diffraction (see, for example, Refs. 11 and 48).

If we solve (6) with allowance for two-wave diffraction, we can neglect in the first approximation all the other harmonics except for the $(-n)$-th harmonic with amplitude $A_{-n}$ and the fundamental harmonic with amplitude $A_{0}$ if as before $\zeta \ll 1$. The approximate set of equations then assumes the form (see, for example, Refs. 24 and 25)

$$
\begin{align*}
& \left(k^{2} \varepsilon_{0}-\mu^{2}\right) A_{0}+k^{2} \varepsilon_{n} A_{-n} \approx 0  \tag{14}\\
& {\left[k^{2} \varepsilon_{0}-\left(\mu-\frac{2 \pi}{a} n\right)^{2}\right] A_{-n}+k^{2} \varepsilon_{-n} A_{0} \approx 0}
\end{align*}
$$

where $\varepsilon_{0}, \varepsilon_{ \pm n}$, and $\varepsilon_{m}$ are complex quantities defined by ( $5^{\prime}$ ). The condition that this set of equations has a solution gives the following dispersion relation for the characteristic index $\mu(k)$ :

$$
\begin{align*}
\left(\mu^{2}-k^{2} \varepsilon_{0}\right)\left[\left(\mu-\frac{2 \pi}{a} n\right)^{2}-k^{2} \varepsilon_{0}\right] & \\
& -k^{4} \varepsilon_{n} \varepsilon_{-n}=0 \tag{15}
\end{align*}
$$

Near a Bragg resonance (13), and when the permittivity modulation amplitude is small $\left(\left|\varepsilon_{ \pm n}\right|<\left|\varepsilon_{0}\right|\right)$, equation (15) has the following approximate analytic solution (see, for example, Refs. 25 or 54):

$$
\begin{equation*}
\mu=\frac{\pi}{a} n+i \gamma, \gamma \approx\left[x_{n} x_{-n}+\left(\frac{\alpha}{2}-i \delta\right)^{2}\right]^{1 / 2} \tag{16}
\end{equation*}
$$

where

$$
\begin{align*}
& \delta=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}-(\pi / a) n, x_{ \pm n}=k \varepsilon_{ \pm n} / 2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2} \\
& \alpha=k \varepsilon_{0}^{\prime} /\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}
\end{align*}
$$

The expression for $\gamma$ given by (16) was obtained on the assumption that the detuning $\delta$ in (16') from the Bragg resonance (13) was small for $|\delta|<\pi n / a$. The quantities $\varkappa_{ \pm n}$ are usually called the coupling constants. They determine the strength of the diffraction coupling between the leading harmonic of amplitude $A_{0}$ and the ( $-n$ )-th harmonic of amplitude $A_{-n}$. The coefficient $\alpha$ describes the absorption of light by the homogeneous medium. The quantity $\gamma$ in (16) is defined so that $\operatorname{Re} \gamma \leqslant 0$ and $\delta \operatorname{Im} \gamma \leqslant 0$. It is now a relatively simple matter to obtain the relation between the amplitudes $A_{-n}$ and $A_{0}$, using (14) and the solution given by (16). The result is

$$
\begin{equation*}
A_{-n}=r_{\mathrm{B}}^{\infty} A_{0}, r_{\mathrm{B}}^{\infty}=\frac{i e_{-n}}{\gamma+(\alpha / 2)-i \delta} \tag{17}
\end{equation*}
$$

As the wave number $k=\omega / c$ departs from the Bragg resonance (13), i.e., when $|\delta|>\left|x_{m}\right|$ and $\alpha$, but as before
$|\delta| \ll k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$, the quantity $\mu$ tends to $k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$, and the expression for the amplitude $A_{-n}$ assumes the form given by (11), which agrees with the solutions found earlier in the Born approximation.

The accuracy of these calculations can be increased if, as in (14), we retain terms containing $A_{0}$ and $A_{-n}$ on the left-hand side of the equations and the other nonresonant terms in the sums on the right-hand side of (6) are taken according to perturbation theory (see, for example, Ref. 54). This is essentially the procedure used in Ref. 28. It leads to the renormalization of the coupling constants $x_{n}$ and $\varkappa_{-n}$, and also to the wavelength shift of the Bragg resonance relative to its position given by (13). The result ${ }^{54}$ is that the shift of the center of the Bragg resonance in the presence of absorption $\alpha(\alpha \neq 0)$ is proportional to the absorption coefficient $\alpha$, whereas in a transparent medium $(\alpha=0)$ it depends on the third power of the small coupling constants $\varkappa_{m}$. The change in the coupling constants is then proportional to the first power of $\chi_{m}$ (see Ref. 54).

The main result that follows both from the general theory ${ }^{13.18,48}$ and from particular formulas such as (16) is that a range of forbidden frequencies $k=\omega / c$ appears on the dispersion curv $k=\omega / c=k(\mu)$ near the $n$th Bragg resonance in nonabsorbing media $\left(\alpha=0, \quad x_{-n}=x_{n}^{*}, \quad\right.$ and $x_{n} x_{-n}=\left|x_{n}\right|^{2}=x^{2}$ ). The solution of (16) is then complex ( $\gamma \neq 0$ ). In this forbidden frequency band, the electromagnetic waves ( $3^{\prime}$ ) are found to be damped (for periodic nonstationary media these solutions become unstable), but outside the forbidden frequency band we have $\gamma=0, \mu$ in (16) is purely real, and we have propagating electromagnetic waves. It follows from (16) that the width of the forbidden frequency band is equal to twice the coupling constant $x$, and the function $\gamma(k)$ is a parabola with $\gamma_{\text {max }}=\varkappa$ (see Fig. 2). When $\alpha \neq 0$, the characteristic index $\mu$ of an absorbing medium is always complex and waves of all frequencies are damped in the medium to a greater or lesser extent. If the medium is not subject to periodic modulation, so that $\kappa=0$, the dispersion relation $k(\mu)$ degenerates to a straight line that passes through the origin in Fig. 2. This corresponds to a constant velocity of light in a homogeneous medium. In addition to the dispersion relation $k=\omega / c=k(\mu)$, we are interested in the dependence of the quantity $r_{B}^{\infty}=A_{-n} / A_{0}$ on the frequency $\omega=c k$ [see (17)], which determines the relative amplitude of the $(-n)$-th harmonic. For the Bragg resonance (13), this ( $-n$ )-th harmonic has, according to


FIG. 2. Dispersion relation $k(\mu)$ near the $n$th forbidden band (other bands not shown) for transparent media: solid curve- $k=k(\operatorname{Re} \mu)$, dot-dash curve- $\operatorname{Im} \mu=\gamma=\gamma(k)$, dashed curve-wave propagation in a homogeneous medium.


FIG. 3. Spectral dependence of the power reflection coefficient $R_{\infty}(\delta /$ $x)=\left|r_{B}^{\infty}(\delta / x)\right|^{2}$ and the phase $\chi(\delta / \kappa)$ of $r_{B}^{\infty}$ for a semi-infinite medium: $\delta$-frequency detuning for the $n$th Bragg resonance, $x$-coupling coefficient of counterpropagating waves, $\varphi$-initial phase for reflection with $\delta=-\chi$.
( $6^{\prime}$ ), the wave vector $k_{-n}=-\pi n / a$ whose magnitude is equal to that of the wave vector $k_{0}=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=\pi n / a$ of the leading incident wave propagating in the opposite direction. Since ${ }^{54}$ the amplitudes $A_{l}$ of the remaining waves with $l \neq(0,-n)$ are small in comparison with $A_{0}$ and $A_{-n}$, the quantity $r_{B}^{\infty}$ must be the Bragg reflection coefficient of an infinite periodic medium (OPS). Figure 3 shows the square of the modulus of the Bragg reflection coefficient $R_{\infty}=\left|r_{B}^{\infty}\right|^{2}$ and its phase $\chi=\arg r_{B}^{\infty}$ as functions of the normalized detuning, calculated for a lossless OPS. The phase $\varphi$ from which $\chi$ is measured is related to the initial modulation phase ( for $z=0$ ) of the permittivity: $\varkappa_{-n}=\varkappa \exp (i \varphi)$. It is clear that $R_{\infty}=1$ in the forbidden frequency band. This means that the amplitude of the $(-n)$-th harmonic becomes equal at Bragg resonance to the amplitude of the original zeroth harmonic. We also note that at the shortwave ( $\delta / \kappa=+1$ ) and long-wave ( $\delta / \varkappa=-1$ ) edges of the forbidden frequency band, the reflection coefficient $r_{B}^{\infty}$ has opposite signs because there is a phase change of $\pi$ when $\delta / \chi$ crosses the forbidden band.

The expression given by (4) with the characteristic index (16) is one of the independent solutions of (1). Another independent solution for an even function $\varepsilon(z)$ is

$$
\begin{equation*}
E_{2}(z)=\Phi_{2}(z) \exp (-\dot{\mu} z), \Phi_{2}(z)=\Phi_{1}(-z) . \tag{18}
\end{equation*}
$$

For arbitrary $\varepsilon(z)$, the second independent solution of (1) can be obtained by changing the sign of the characteristic index and repeating from the beginning the procedure for finding its value while at the same time determining the quantities $A_{m}$. If we know the solutions $E_{1}(z)$ and $E_{2}(z)$, we can calculate the characteristics of a bounded OPS by demanding field continuity across its boundaries. We shall discuss this procedure later in relation to the coupled-wave equations

## 4. STANDARD COUPLED-WAVE THEORY (CWT)

This approach to the solution of (1) with periodic permittivity $\varepsilon(z)$ was first used by Kogelnik to analyze light scattering by phase holograms in the case of small harmonic permittivity modulation. ${ }^{14,29}$ We shall illustrate this approach for the more general case of a medium with arbitrary periodic permittivity $\varepsilon(z)$, confining ourselves to the basic assumption made by Kogelnik, namely, that the wave transformation in one structure period is small. This assumption is satisfactory when the incident wave frequency is close to a Bragg resonance frequency.

### 4.1. Derlvation of coupled-wave equations

The essence of the method that we shall use is as follows. ${ }^{26,55}$ The solution of (1) is taken in the form of a superposition of two counterpropagating waves

$$
\begin{align*}
E(z)= & A^{(+)}(z) \exp \left(i k \varepsilon_{0}^{1 / 2} z\right)+ \\
& +A^{(-)}(z) \exp \left(-i k \varepsilon_{0}^{1 / 2} z\right) \tag{19}
\end{align*}
$$

with variable amplitudes $A^{( \pm)}(z)$ and wave vector $k \varepsilon_{0}^{1 / 2}=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}+(i \alpha / 2)$ that corresponds to wave propagation in a homogeneous absorbing medium with complex permittivity $\varepsilon_{0}=\varepsilon_{0}^{\prime}+i \varepsilon_{0}^{\prime \prime}$ where $\varepsilon_{0}^{\prime \prime}=\alpha\left(\varepsilon_{0}^{\prime}\right)^{1 / 2} / k$ and $\alpha$ is the absorption coefficient. Substituting (19) in (1), using the expansion given by (5) for the periodic perturbation

$$
\varepsilon_{p}(z)=\varepsilon(z)-\varepsilon_{0}=\sum_{\substack{m=-\infty \\ m=0}}^{m=+\infty} \varepsilon_{m} \exp \left(i \frac{2 \pi}{a} m z\right)
$$

and substituting $\delta=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}-(\pi / a n)$ for the detuning from the $n$th Bragg resonance, we obtain an exact solution that involves the amplitudes of the forward, $A^{(+)}(z)$, and backward, $A^{(-)}(z)$, waves

$$
\begin{align*}
& {\left[\frac{\mathrm{d}^{2} A^{(+)}}{\mathrm{d} z^{2}}+2 i k \varepsilon_{0}^{1 / 2} \frac{\mathrm{~d} A^{(+)}}{\mathrm{d} z}+k^{2} A^{(+)} \sum_{\substack{m=-\infty \\
m \neq 0}}^{m=+\infty} \varepsilon_{m}\right.} \\
& \left.\times \exp \left(i \frac{2 \pi}{a} m z\right)\right] \exp \left(i \frac{\pi}{a} n z-\frac{a-2 i \delta}{2} z\right) \\
& +\left[\frac{\mathrm{d}^{2} A^{(-)}}{\mathrm{d} z^{2}}-2 i k \varepsilon^{1 / 2} \frac{\mathrm{~d} A^{(-)}}{\mathrm{d} z}+{k^{2} A^{(-)} \sum_{\substack{m=-\infty \\
m=0}} \varepsilon_{m}}_{\substack{2 \\
2}}^{2} z\right)=0 .
\end{align*}
$$

If, as assumed by Kogelnik, ${ }^{29}$ the transformation of forward into backward waves is small within the structure period $a$, the functions $A^{( \pm)}(z)$ and their derivatives, and also the factors $\exp \pm[(\alpha-2 i \delta) / 2 z]$, can be regarded as constant within the structure period $a$. If this is so, we can multiply (20) by

$$
\exp \left(-i \frac{\pi}{a} n z+\frac{a-2 i \delta}{2} z\right)
$$

and, by

$$
\exp \left(i \frac{\pi}{a} n z-\frac{\alpha-2 i \delta}{2} z\right)
$$

and then, by averaging over the structure period $a$, we obtain the approximate set of coupled equations for the forward and backward wave amplitudes (see also Refs. 26 and 55)

$$
\begin{align*}
& \frac{\mathrm{d}^{2} A^{(+)}(z)}{\mathrm{d} z^{2}}+2 i k \varepsilon_{0}^{1 / 2} \frac{\mathrm{~d} A^{(+)}(z)}{\mathrm{d} z} \\
& \quad+k^{2} \varepsilon_{n} \exp [(\alpha-2 i \delta) z] A^{(-)}(z)=0,  \tag{21}\\
& \frac{\mathrm{~d}^{2} A^{(-)}(z)}{\mathrm{d} z^{2}}-2 i k \varepsilon_{0}^{1 / 2} \frac{\mathrm{~d} A^{(-)}(z)}{\mathrm{d} z} \\
& \quad+k^{2} \varepsilon_{-n} \exp [-(\alpha-2 i \delta) z] A^{(+)}(z)=0 .
\end{align*}
$$

When we average over the period in the sums over $m$, all the
terms vanish with the exception of the resonant terms with $m=n$ and $m=-n$; expressions containing $A^{( \pm)}(z)$ and their derivatives multiplied by $\exp ( \pm i 2 \pi n z / a)$ also vanish. The condition that the amplitudes $A^{( \pm)}(z)$ and their derivatives, and also the factors $\exp [ \pm(a-2 i \delta) z]$, are constant over the period $a$ is satisfied when

$$
\begin{equation*}
a a \ll 1,|\delta| a \ll 1,\left|x_{ \pm n}\right| a \ll 1 \tag{2la}
\end{equation*}
$$

or

$$
\frac{\pi}{2} n\left|\varepsilon_{ \pm n}\right| /\left|\varepsilon_{0}\right| \ll 1
$$

where

$$
\begin{align*}
& \delta=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}-\frac{\pi}{a} n, x_{ \pm n}^{-}=k \varepsilon_{ \pm n} / 2 \varepsilon_{0}^{1 / 2},  \tag{21b}\\
& k=\frac{\omega}{c}=\frac{2 \pi}{\lambda} .
\end{align*}
$$

The last inequality in (21a) was derived using the Bragg resonance condition given by (13).

The set of equations with constant coefficients given by (21), which relates the amplitudes $A^{(+)}(z)$ and $A^{(-)}(z)$, can be solved by standard exponential substitution. ${ }^{26,55}$ However, in the widely-used Kogelnik approach ${ }^{14,29}$ one usually neglects the second order derivatives of $A^{( \pm)}(z)$ in (21). These equations then assume the form of the standard set of Kogelnik equations for coupled waves:

$$
\begin{align*}
& \frac{\mathrm{d} A^{(+)(z)}}{\mathrm{d} z}=\dot{i} x_{n} \exp [(\alpha-2 i \delta) z] A^{(-)}(z)  \tag{22}\\
& \frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z}=-\dot{\delta}{ }_{-n} \exp [-(\alpha-2 i \delta) z] A^{(+)}(z)
\end{align*}
$$

where $\kappa_{ \pm}=k \varepsilon_{ \pm n} / 2 \varepsilon_{0}^{1 / 2}, \alpha$ is the absorption coefficient, and $\delta$ is the detuning from the $n$th Bragg resonance.

A similar set of approximate equations is obtained from the integral equation given by (7) by substituting into it the solution given by (19), dividing the range of integration with respect to $z^{\prime}$ by 2 (in which $z-z^{\prime}$ has a particular sign), and then equating the coefficients of $\exp \left( \pm i k \varepsilon_{0}^{1 / 2} z\right)$ on the left and on the right of (7). The result of all this is the following exact set of integral equations:

$$
\begin{gather*}
A^{(+)}(z)=\frac{i k}{2 \varepsilon_{0}^{1 / 2}} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} \varepsilon_{p}\left(z^{\prime}\right)\left[A^{(+)}\left(z^{\prime}\right)\right. \\
\left.+A^{(-)}\left(z^{\prime}\right) \exp \left(-2 i k \varepsilon_{0}^{1 / 2} z^{\prime}\right)\right] \\
A^{(-)}(z)=\frac{i k}{2 \varepsilon_{0}^{1 / 2}} \int_{z}^{\infty} \mathrm{d} z^{\prime} \varepsilon_{p}\left(z^{\prime}\right)\left[A^{(-)}\left(z^{\prime}\right)\right. \\
\left.\quad+A^{(+)}\left(z^{\prime}\right) \exp \left(2 i k \varepsilon_{0}^{1 / 2} z^{\prime}\right)\right]
\end{gather*}
$$

If we again use the conditions given by (21a), which ensure that we are close to the $n$th Bragg resonance and that the change in the amplitude $A^{( \pm)}(z)$ per period is small, we can readily reduce ( $22^{\prime}$ ) to the set of coupled wave equations given by (22) (see Appendix II). However, the set of exact integral equations given by ( $22^{\prime}$ ) is convenient because it allows us to use a simple iteration method to estimate the next approximation for $A^{\left({ }^{\prime \prime}\right)}(z)$ (see Appendix II).

The general solution of the approximate equations given by (22) is

$$
\begin{align*}
& A^{(+)}(z)=\left[C_{1} \exp (-\gamma z)+C_{2} \exp (\gamma z)\right] \\
& \quad \times \exp \left[\left(\frac{\alpha}{2}-i \delta\right) z\right], \\
& A^{(-)}(z)=\left[C_{1} r_{B}^{\infty} \exp (-\gamma z)+\frac{x_{-n}}{x_{n}} \frac{1}{r_{B}^{\infty}} C_{2} \exp (\gamma z)\right] \\
& \quad \times \exp \left[-\left(\frac{\alpha}{2}-i \delta\right) z\right], \tag{23}
\end{align*}
$$

where $\gamma$ and $r_{B}^{\infty}$ are given by (16) and (17), respectively. The constants $C_{1}$ and $C_{2}$ are determined from the boundary conditions imposed on the fields at $z=0$ and $z=L$.

### 4.2. Reflection from a matched layer

We shall now calculate the reflection coefficient for a light wave incident on a layer of a one-dimensional periodic medium (OPS) matched to the ambient media and having a thickness $L$. This means that the permittivity of the layer defined by (5) satisfies the continuity conditions $\varepsilon(0)=\varepsilon_{1}$ and $\varepsilon(L)=\varepsilon_{2}$ at $z=0$ and $z=L$, respectively, where, in the notation of Fig. l, $\varepsilon_{1}$ is the permittivity of the homogeneous medium behind the layer, for $z<L$ (the light wave travels from the former to the latter). When the modulation depth of the permittivity $\varepsilon_{z}$ in (5) is small, and the media on either side of the layer are the same, i.e., $\varepsilon_{1}=\varepsilon_{2}$, the approximate matching condition can be written in the form $\varepsilon_{1}=\varepsilon_{2}=\varepsilon_{0}$ where $\varepsilon_{0}$ is the homogeneous part of the function $\varepsilon(z)$ in (5), i.e., its zero-order Fourier harmonic.

For a matched layer, the refractive index is continuous across the separation boundaries at $z=0$ and $z=L$, so that there is also no Fresnel wave reflection from the separation boundaries. This means that, on the other hand, reflection by the entire layer is determined exclusively by Bragg reflection from the periodic structure and, on the other, the electromagnetic fields on either side of these boundaries $(z=0$ and $z=L$ ) are identical. The quantity $A_{0}=A^{(+)}(z)$ in the solution given by (19) is then identical to the amplitude of the wave incident on the layer, and $A^{(-)}(0)$ is the amplitude of the wave reflected by the layer. Similarly, $A^{(+)}(L)$ is equal to the amplitude of the wave transmitted by the layer into the region $z \geqslant L$. By virtue of the radiation principle (see, for example, Refs. 33 and 38), the incident wave is absent from the region $z>L$ behind the layer, $A^{(-)}(L)=0$. This gives the boundary conditions in the form

$$
\begin{equation*}
A^{(-)}(L)=0, A^{(+)}(0)=A_{0} \tag{24}
\end{equation*}
$$

If we use the first of these conditions for solutions such as (23), we obtain

$$
\begin{align*}
\frac{C_{2}}{C_{1}} & =-\frac{x_{n}}{x_{-n}}\left(r_{\mathrm{B}}^{\infty}\right)^{2} \exp (-2 \gamma L) \\
& =\frac{\gamma-(\alpha / 2)-i \delta}{\gamma+(\alpha / 2)+i \delta} \exp (-2 \gamma L) \tag{25}
\end{align*}
$$

where $\operatorname{Re} \gamma \geqslant 0$. Hence, and from the solution given by (23), in which we use the second equation in (24), we obtain the following expression for the Bragg reflection coefficient of the matched layer of thickness $L$ :

$$
\begin{align*}
r_{\mathrm{B}}^{L} & =\frac{A^{(-)}(0)}{A_{0}} \\
& =\frac{\dot{i} e_{-n} \operatorname{sh}(\gamma L)}{\gamma \operatorname{ch}(\gamma L)+[(\alpha / 2)-i \delta] \operatorname{sh}(\gamma L)}, \tag{26}
\end{align*}
$$

and also the transmission coefficient

$$
\begin{align*}
t_{\mathrm{B}}^{L} & =\frac{E(L)}{A_{0}}=\frac{A^{(+)}(L)}{A_{0}} \exp \left(i k \varepsilon_{0}^{1 / 2} L\right) \\
& =\frac{\gamma \exp (i \pi n L / a)}{\gamma \operatorname{ch}(\gamma L)+[(\alpha / 2)-i \delta] \operatorname{sh}(\gamma L)} . \tag{27}
\end{align*}
$$

These formulas can be interpreted as follows. Since, in the real world, there are no lossless layers ( $\alpha>0$ ), it follows that $\operatorname{Re} \gamma>0$ and, in the limit of the semi-infinite space ( $L \rightarrow \infty$ ), the formula given by (26) gives the quantity $r_{B}^{\infty}$ obtained in (17) from dynamic diffraction theory. Naturally, we then have $t_{B}^{\infty}=0$. If we now substitute the solutions given by (23) for the field $E(z)$ in (19), we find that $C_{1}$ and $C_{2}$ refer to the forward Bloch wave in (2) and the backward Bloch wave, respectively. It is clear from (25) that, for a matched layer of finite thickness, these two counterpropagating Bloch waves are always coupled to one another. This coupling vanishes only in a semi-infinite medium because we then have $C_{2} \rightarrow \infty$.

Figure 4 shows the spectral dependence of the energy reflection coefficient $R_{L}=\left|r_{B}^{L}\right|^{2}$ as a function of the dimensionless detuning $\delta \varkappa$ of the wave frequency from the Bragg resonance for different values of absorption, characterized by the parameter $\alpha L$. It is clear from Fig. 4 that, when $\alpha L$ is small and lies outside the forbidden frequency band, so that $|\delta| \geqslant|\mathcal{\chi}|$, the function $R_{L}(\delta)$ oscillates as a result of interference between waves reflected from the more distant boundary of the layer of thickness of $L$. The fraction of light reaching this more distant boundary of the mirror decreases with increasing $\alpha L$. This means that, as $\alpha L$ increases, the amplitude of the oscillation decreases and vanishes altogether when $\alpha L \gg 1$. We then find that $R_{B}^{L}$ tends to $r_{B}^{\infty}$ in (17), and the shape of the $R_{L}(\delta)$ curve approaches $R_{\infty}(\delta)=\left|r_{B}^{\infty}\right|^{2}$ in the semi-infinite periodic medium.

The reflection curve $R_{L}(\delta)$ of a layer of finite thickness $L$ will also cease to oscillate when the layer is illuminated by a nonmonochromatic radiation with wave-packet width $\Delta \omega$ such that its width $\Delta k=\Delta \omega / c$ in wave number space satisfies the condition $\Delta k \geqslant L^{-1}$.


FIG. 4. Spectral dependence of the power reflection coefficient $R_{L}$ ( $\delta /$ $x$ ) for a matched layer ( $\alpha / x=0.3$ and $x L=3$ ) for $\alpha L=0.9$ (a), $1.8(\mathrm{~b})$, and $3.6(\mathrm{c})$.

### 4.3. Connection between the dynamic theory of diffraction and the coupled-wave theory, and the conditions for the validity of the latter

All the results obtained in coupled-wave theory for the dispersion relations, the field distribution, the reflection and transmission coefficients, and so on, can also be derived in the dynamic theory of diffraction. Comparison of (16) and (17), on the one hand, and the solutions given by (23), on the other, shows that the two approaches yield the same result near a Bragg resonance, but the computational scheme in the dynamic theory of diffraction is more laborious.

We must now examine the ranges of validity of the cou-pled-wave theory, which can be deduced from (21a) and (21b). The coupled-wave equations (22) demand that the absorption coefficient $\alpha$ remains small within the structure period $a$, and so does the detuning $\delta$ as compared with $a^{-1}$, but they also demand that the change in the amplitudes $A^{( \pm)}(z)$ per structure period must also be small (see Appendix II). This change is characterized by the value of the coupling constant $x$ in (21b), which can be estimated from the expression $k \approx k \Delta \varepsilon /\left|\varepsilon_{0}\right|^{1 / 2}$ where $\Delta \varepsilon$ is the maximum deviation of $\varepsilon(z)$. The basic condition for the applicability of the coupled-wave theory is thus the condition

$$
x \approx \frac{k \Delta \varepsilon}{\left|\varepsilon_{0}\right|^{1 / 2}} \ll a^{-1}
$$

or [see (12)]

$$
\begin{equation*}
\zeta=\frac{k a}{\left|\varepsilon_{0}\right|^{1 / 2}} \Delta \varepsilon=2 \pi \frac{a}{\lambda} \frac{\Delta \varepsilon}{\left|\varepsilon_{0}\right|^{1 / 2}} \ll 1 . \tag{27'}
\end{equation*}
$$

On the other hand, the dynamic theory of diffraction is valid when the parameter $\zeta$ in (11) is small. This means that the coupled-wave theory (CWT) and the dynamic theory of diffraction (DTD) become identical in this range. All the results obtained in these theories are then identical as well.

It is clear from ( $27^{\prime}$ ) that both theories provide a reasonable description of the propagation of light for low-order Bragg resonances for which $n=1,2,3$ in (13) and the structure period $a$ is of the order of the radiation wavelength $\lambda$. For a given $\lambda$, an increase in the structure period $a$, i.e., in the Bragg resonance number in (13), or in the modulation depth $\Delta \varepsilon /\left|\varepsilon_{0}\right|$, is accompanied by the onset of an increase in $\zeta$ in (12), or in (27'), and the analytic formulas deduced from the two theories (CWT and DTD) become less accurate. The physical reason for this is the increasing importance of multiwave and multiple diffraction. On the one hand, the amplitudes $A_{n}$ in (11) increase with increasing $\zeta$ in (12), so that there is also an increase in the resulting contribution of the wave 'background' to the wave field in (4). On the other hand, the contribution of the successive multiple scatterings, i.e., of the next perturbation orders, becomes significant (see, for example, Refs. 26 or 54). According to (11), the amplitudes $A_{m}$ (terms with $m= \pm 1$ ) of singly-scattered (diffracted) waves are proportional to the parameter $\zeta$ in (12). These singly-diffracted scattered waves can then be diffracted again by inhomogeneities of $\varepsilon(z)$, producing secondary waves whose amplitudes are proportional to $\zeta^{2}$. These secondary waves can, in turn, produce tertiary waves with amplitudes proportional to $\zeta^{3}$, and so on. The amplitudes of $n$-fold scattered (diffracted) waves are then proportional to $\zeta^{n}$. The contribution of first-order perturbation
theory for the $n$th Bragg resonance is then proportional to the same quantity [see (17) in which $\varkappa_{ \pm n} \sim \zeta^{n}$ for harmonic modulation]. This means that, near the $n$th Bragg resonance (13), the resultant amplitude of Bragg-diffracted waves can contain comparable contributions due to (1) nonresonant $n$-fold scattering [i.e., inhomogeneities in $\varepsilon(z)$ with dimensions of the order of the wavelength], primary diffracted waves, ( $n-1$ )-fold scattered secondary waves, ( $n-2$ )-fold scattered tertiary waves, and so on, and (2) singularly scattered waves corresponding to the $n$th Bragg resonance. We thus have to sum the contributions due to all these $n$ scattering channels in order to determine the resultant amplitude of the diffracted waves corresponding to the $n$th Bragg resonance, which is proportional to $\xi^{n}$. We find that this is not at all easy to do. Attempts to construct this type of multiwave and multiple diffraction theory ${ }^{21,23}$ have forced us to use numerical procedures that significantly reduce the typical simplicity and clarity of approximate analytic formulas. We therefore turn to a different approximate analysis of wave transformation near the $n$th Bragg resonance, i.e., large structure periods $a$, in which multiple and multiwave diffraction is partially taken into account by a modified coupled-wave theory (MCWT).

## 5. MODIFIED COUPLED-WAVE THEORY (MCWT)

The principles of the modified coupled-wave theory were formulated in Refs. 30 and 60 and were later used to examine wave transformation in periodic corrugated waveguides. ${ }^{31,37,60}$ Subsequent comparison of these calculations with exact numerical results ${ }^{57,59}$ showed good agreement between them near Bragg resonances and well away from them (which includes periodic media with a small number of periods and an appreciable modulation depth ). This confirms that the modified coupled-wave theory evidently takes partially into account the multiwave and multiple diffraction of waves by periodic inhomogeneities in $\varepsilon(z)$, especially for high-order Bragg resonances.

### 5.1. Derivation of the MCWT equations

The essence of the method is as follows. We start by substituting in (1) the solution for $E(z)$ in the form of counterpropagating waves with variable amplitudes $A^{(+)}(z)$, $A^{(-)}(z)$ and geometric-optics phases

$$
\begin{align*}
\psi(z) & =k \int_{0}^{z}\left(\varepsilon\left(z^{\prime}\right)\right)^{1 / 2} \mathrm{dz}^{\prime}, \\
E(z) & =\frac{1}{(\varepsilon(z))^{1 / 4}}\left[A^{(+)}(z) \exp (i \psi(z))\right. \\
+ & \left.A^{(-)}(z) \exp (-i \psi(z))\right] . \tag{28}
\end{align*}
$$

Equation (1) then becomes an identity (see, for example, Ref. 33), and the problem thus reduces to the solution of (29), if the amplitudes $A^{(+)}(z), A^{(-)}(z)$ satisfy the equations

$$
\begin{align*}
& \frac{\mathrm{d} A^{(+)}(z)}{\mathrm{d} z}=S^{(-)}(z) A^{(-)}(z)  \tag{29}\\
& \frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z}=S^{(+)}(z) A^{(+)}(z)
\end{align*}
$$

where

$$
\begin{equation*}
S^{( \pm)}(z)=\frac{1}{4 \varepsilon(z)} \frac{d \varepsilon(z)}{d z} \exp ( \pm 2 i \psi(z)) \tag{30}
\end{equation*}
$$

We can now average over rapid oscillations to obtain from the exact equations (29) an approximate and simpler set of equations. ${ }^{24,35}$ In practice, this means that the main contribution to the exact solutions of (29) is provided by the slowly-varying components of the coefficients $S^{( \pm)}$. It is precisely these components that are taken into account when the approximate solution is constructed. Since $\varepsilon(z)$ is periodic, we have $\psi(z+a)=\psi(z)+\psi(a)$, because

$$
\begin{aligned}
& \int_{z}^{z+a}\left(\varepsilon\left(z^{\prime}\right)\right)^{1 / 2} \mathrm{~d} z^{\prime}=\int_{z}^{a} \varepsilon^{1 / 2} \mathrm{~d} z^{\prime}+\int_{a}^{z+a} \varepsilon^{1 / 2} \mathrm{~d} z^{\prime} \\
& \quad=\int_{0}^{a} \varepsilon^{1 / 2} \mathrm{~d} z^{\prime}=\psi(a)
\end{aligned}
$$

or, because of periodicity,

$$
\int_{a}^{z+a}\left(\varepsilon\left(z^{\prime}\right)\right)^{1 / 2} \mathrm{~d} z^{\prime}=\int_{0}^{z}\left(\varepsilon\left(z^{\prime}\right)\right)^{1 / 2} \mathrm{~d} z .
$$

If we now introduce the average permittivity $\tilde{\varepsilon}_{0}^{1 / 2}=\psi(a) /$ $k a$, which is related to the change in the phase $\psi(a)$ per OPS period, we find that the quantities $S^{( \pm)}(z) \exp \left( \pm 2 i k \tilde{\varepsilon}_{0}^{1 / 2} z\right)$ are periodic functions that can be expanded into Fourier series. The final result is

$$
\begin{align*}
& S^{( \pm)}(z)=\exp \left( \pm 2 i k \varepsilon_{0}^{1 / 2} z\right) \sum_{m=-\infty}^{m=+\infty} x_{m}^{( \pm)} \\
& \quad \times \exp \left(\frac{2 \pi i}{a} m z\right), \tag{31}
\end{align*}
$$

where

$$
\begin{align*}
x_{m}^{( \pm)}= & \frac{1}{a}\left\{f_{0}^{a} \frac{\mathrm{~d} z}{4 \varepsilon(z)} \frac{\mathrm{d} \varepsilon(z)}{\mathrm{d} z}\right. \\
& \times \exp \left[i\left( \pm 2 \psi(z) \mp 2 k \widetilde{\varepsilon}_{0}^{1 / 2} z-\frac{2 \pi}{a} m z\right)\right] \\
& +\frac{1}{4} \sum_{j=1}^{j=v} \ln \frac{\varepsilon\left(z_{j}+0\right)}{\varepsilon\left(z_{j}-0\right)} \\
& \left.\times \exp \left[i\left( \pm 2 \psi\left(z_{j}\right) \mp 2 k \widetilde{\varepsilon}_{0}^{1 / 2} z_{j}-\frac{2 \pi}{a} m z_{j}\right)\right]\right\} . \tag{32}
\end{align*}
$$

The integral in (32) represents the principal value, and the sum over $j=1,2 \ldots, p$ takes into account the contribution to $\varkappa_{m}^{( \pm)}$of jumps in the permittivity $\varepsilon(z)$ at the points of discontinuity $z_{j}$ within the period (Refs. 30, 57, and 60). If a discontinuity in $\varepsilon(z)$ occurs at the beginning or the end of the period, it is, of course, taken into account only once, e.g., at the beginning of the period. The quantities $\varepsilon\left(z_{j} \pm 0\right)$ are the limiting values of the permittivity $\varepsilon(z)$ to the right, $\left(z_{j}+0\right)$, and to the left, $\left(z_{j}-0\right)$, of a point of discontinuity $\boldsymbol{z}_{j}$.

Near the $n$th Bragg resonance (13), and for small detuning from resonance, i.e., when $|\delta| \ll \pi n / a$, we have

$$
\begin{align*}
& k\left(\tilde{\varepsilon}_{0}^{\prime}\right)^{1 / 2}=\frac{\pi}{a} n+\delta, \varepsilon_{0}^{\prime \prime}=\alpha\left(\tilde{\varepsilon}_{0}^{\prime}\right)^{1 / 2} / k, \\
& 2 k \tilde{\varepsilon}_{0}^{1 / 2}=\frac{2 \pi}{a} n+i(\alpha-2 i \delta), \\
& S^{(+)}(z)=x_{-n}^{(+)} \exp [-(\alpha-2 i \delta) z] \\
& m=+\infty  \tag{33}\\
& +\sum_{m=-\infty}^{\prime} x_{m}^{(+)} \exp \left[\frac{2 \pi i}{a}(n+m) z-(\alpha-2 i \delta) z\right], \\
& S^{(-)}(z)=x_{n}^{(-)} \exp [(\alpha-2 i \delta) z] \\
& m=+\infty \\
& +\sum_{m=-\infty}^{\prime} x_{m}^{(-)} \exp \left[\frac{2 \pi i}{a}(m-n) z+(\alpha-2 i \delta) z\right],
\end{align*}
$$

where we have separated out the slowly-varying terms in $S^{( \pm)}(z)$ and have discarded terms with $m= \pm n$ in $\Sigma_{m}^{\prime}$. Averaging over the fast oscillations is then equivalent to the neglect of $\Sigma_{m}^{\prime}$ in the second term in $S^{( \pm)}(z)$ in (33). The set of equations given by (29) then takes the form of (22) with $x_{n}$ replaced with $-i x_{n}^{(-)}$and $\varkappa_{-n}$ replaced with $i x_{-n}^{(+)}$. Its solutions are constructed by analogy with the solution of the Kogelnik coupled-wave equations.

### 5.2. Comparison of standard and modified coupled-wave theories

It is clear from the foregoing comparison of the formulas for $\varkappa_{ \pm n}$ in (16'), (21a), and (32) that the modified coupled-wave theory differs from the standard Kogelnik theory, first, by the value of $\tilde{\varepsilon}_{0} \neq \varepsilon_{0}$ and, second, by the magnitude of the coupling constants $\mathcal{\varkappa}_{ \pm n}$. The first factor leads to a more accurate (as compared with the Kogelnik theory) determination of the position of the Bragg resonance on the wavelength axis. In the usual dynamic theory of diffraction, the shift of the Bragg resonance is taken into account only via the second or higher order perturbation-theory terms. ${ }^{14,21,54}$ Here, on the other hand it is obtained immediately in the first approximation. Secondly, the coupling constants $x_{m}^{( \pm)}$calculated from the MCWT formulas in (32) also describe multiwave diffraction and, in this sense, are more accurate than the coupling constants $\chi_{ \pm n}$ in the Kogelnik formulas given by ( $16^{\prime}$ ). This is so because the periodically modulated function $\varepsilon(z)$ makes its appearance under the integral sign in the expressions for $x_{m}^{( \pm)}$given by (32) in a complicated way, so that when these functions are expanded in powers of the small modulation depth $\Delta \varepsilon /\left|\varepsilon_{0}\right|$, the coupling constants $x_{m}^{( \pm)}$involve all the expansion orders in this small quantity. This actually means that multiwave and multiple wave diffraction by periodic inhomogeneities of $\varepsilon(z)$ has been taken into account.

For example, for a harmonically modulated permittivity, the formulas given by ( $5^{\prime}$ ), and ( $16^{\prime}$ ) yield the following expressions for the coupling constant in the Kogelnik theory (for $m \neq 0$ ):

$$
\begin{equation*}
x_{m}=\frac{k \varepsilon_{m}}{2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}}=\frac{k \Delta \varepsilon}{4\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}}\left(\delta_{1, m}+\delta_{-1, m}\right) \tag{33'}
\end{equation*}
$$

where we have put $\varepsilon(z)=\varepsilon_{0}+\Delta \varepsilon \cos (2 \pi z / a)$ and $\delta_{ \pm 1, m}$ are the Kronecker symbols that are equal to 0 for $m \neq \pm 1$ and to 1 for $m= \pm 1$.

It follows from (33') that, for harmonically modulated $\varepsilon(z)$, the Kogelnik coupled-wave theory confines Bragg diffraction to first-order resonances for which $n=1$. There is
no Bragg diffraction in the Kogelnik theory for the higherorder resonances with $n=2,3, \ldots$ in the case of harmonically modulated $\varepsilon(z)$, since the coupling coefficients $\chi_{ \pm n}$ are all zero for such resonances. At the same time, in the modified coupled-wave theory, with harmonically modulated $\varepsilon(z)$, the coupling constants $x_{n}^{( \pm)}$in (32) are nonzero for all Bragg-diffraction orders $n$. The same result follows from exact numerical calculations (see, for example, Refs. 13, 15, and 18 ).

We now turn to the expression given by (32) for a more accurate comparison between the usual (CWT) and modified (MCWT) coupled-wave theories. This comparison shows that, as the modulation depth $\Delta \varepsilon$ of the function $\varepsilon(z)=\varepsilon_{0}+\Delta \varepsilon f(z)$ tends to zero, the coupling constants given by (32) automatically become identical with the coupling constants $x_{ \pm n}$ in (16), which appear both in diffraction theory and the Kogelnik coupled-wave theory. Actually, since $d \varepsilon(z) / d z$ in (32) is proportional to $\Delta \varepsilon$, we can put $\varepsilon(z) \approx \varepsilon_{0}$ in the other integrands for $\Delta \varepsilon \rightarrow 0$. We then have $\psi(z) \approx \psi_{0}(z)=k \varepsilon_{0}^{1 / 2} \approx k \tilde{\varepsilon}_{0}^{2} z$ and

$$
\begin{align*}
x_{m}^{( \pm)} & \approx \frac{1}{4 a \varepsilon_{0}} \int_{0}^{a} \mathrm{~d} z \frac{\mathrm{~d} \varepsilon(z)}{\mathrm{d} z} \exp \left(-\frac{2 \pi i}{a} m z\right) \\
& \approx i \frac{\pi}{a} m \frac{\varepsilon_{m}}{2 \varepsilon_{0}} \tag{34}
\end{align*}
$$

In the derivation of the last equation, we first integrated by parts and then used the periodicity of $\varepsilon(z)$ and of the expression for $\varepsilon_{m}$ given by ( $5^{\prime}$ ). From (34) with the condition (13) for the $m$ th Bragg resonance and small $\varepsilon_{0}^{\prime \prime}\left(\varepsilon_{0}^{\prime \prime} \ll \varepsilon_{0}^{\prime}\right.$ and $\left.\varepsilon_{0}^{\prime \prime} \approx \varepsilon_{0}^{\prime}\right)$, we obtain expressions for $\chi_{n}^{( \pm)}$that are identical with the coupling constants in Kogelnik's theory because $x_{n}=-i x_{n}^{(-)}$and $x_{-n}=i x_{-n}^{(+)}$and $x_{ \pm n}=k \varepsilon_{ \pm n} /$ $2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$.

We shall now use the special case of small ( $\Delta \varepsilon \ll \varepsilon_{0}$ ) harmonic modulation, $\left[\varepsilon(z)=\varepsilon_{0}+\Delta \varepsilon \cos 2 \pi z / a\right]$ to examine in greater detail the effect of multiwave diffraction in MCWT. We shall do this by comparing in greater detail the expressions for the coupling constants $\chi_{n}^{( \pm)}$in (32) with the analogous expressions for $\chi_{ \pm n}$ in the Kogelnik theory [see (21b) and (22)]. When we substitute the harmonically modulated $\varepsilon(z)$ in (32), we shall neglect the function $\varepsilon(z)$ in the amplitude of the integrand because $\Delta \varepsilon \varangle \varepsilon_{0}$, but, in contrast to (34), we shall additionally take into account the contributions due to the first order in $\Delta \varepsilon_{0}$ to the phase $\left[\psi(z) \pm k \tilde{\varepsilon}_{0}^{1 / 2} z\right]$ of the integrand. If we then use the integral representation of the Bessel function $J_{n}(x)$ (see, for example, Ref. 36) and its properties, we obtain the following approximate analytic expression for the coupling contants in (32):

$$
\begin{align*}
x_{n}^{( \pm)} & =-\frac{\varepsilon_{0}^{-1} \Delta \varepsilon}{4 \pi a} \int_{0}^{2 \pi} \mathrm{~d} x \sin x \cdot \exp ( \pm i \Omega \sin x-i n x) \\
& =( \pm 1)^{n+1} \frac{i \pi}{2 a} \frac{\Delta \varepsilon}{\varepsilon_{0}} \frac{\partial J_{n}(\Omega)}{\partial \Omega}  \tag{35}\\
\Omega & =\frac{k a \Delta \varepsilon}{2 \pi\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}}=\frac{\zeta}{2 \pi}
\end{align*}
$$

where $x=2 \pi z / a$ and $\zeta=2 \pi \Omega$ is the parameter in (12), which is small both in the dynamic theory of diffraction and in the Kogelnik coupled-wave theory [see condition (27')].

### 5.3. Conditions for the validity of MCWT and CWT

The MCWT expression for the coupling constants given by (35) for media with small permittivity modulation depth ( $\Delta \varepsilon \ll \varepsilon_{0}$ ) can be analyzed in the following two limiting cases: (1) wavelength $\lambda$ greater than or comparable with the structure period $a$, i.e., $k a \leqslant 1$ or $\lambda \geqslant 2 \pi a$ and (2) small wavelength $\lambda$ and large period $a$, i.e., $k a \geqslant 1$ or $\lambda \leqslant 2 \pi a$. In the former case, the parameter $\Omega$ in (35) is found to be small for $\Delta \varepsilon \ll \varepsilon_{0}^{\prime}$ and $k a \leqslant 1$. We can then use the expansion $J_{n}(x) \approx x^{n} 2^{n} n!$ (see Ref. 36) and the Bragg resonance relation (13), so that (35) gives

$$
\begin{align*}
x_{n}^{( \pm)} & \approx( \pm 1)^{n+1} \frac{i \pi n}{2 a} \frac{\Delta \varepsilon}{\varepsilon_{0}^{\prime}} \frac{\Omega^{n-1}}{2^{n} \cdot n!} \\
& =( \pm 1)^{n+1} \frac{i k \Delta \varepsilon}{2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}} \frac{\Omega^{n-1}}{2^{n} \cdot n!} \\
& =( \pm 1)^{n+1} \frac{\zeta^{n}}{a \pi^{n} \cdot 2^{2 n+1} \cdot n!} \tag{35a}
\end{align*}
$$

Hence it is clear that, for the first Bragg resonance ( $n=1$ ), the coupling constants $\chi_{1}^{ \pm}$are equal in magnitude to the constants $x_{ \pm}$, of the standard Kogelnik coupled-wave theory [cf. (32)]. When $n>1$, the coupling constants are found to vanish both in the dynamic theory of diffraction (16') and the Kogelnik coupled-wave theory (21b) [cf. (33')]. At the same time, the quantities $x_{n}^{( \pm)}$in (35a), i.e., in MCWT, are nonzero for any $n$, and are proportional to the $n$th power of the small parameter $\zeta=2 \pi \Omega$. This has already been explained qualitatively as a manifestation of multiwave and multiple diffraction.

The MCWT formulas given by (35) can also be analyzed in the opposite limiting case when the structure period $a$ is much greater than the wavelength, and the parameter $\Omega$ in the (35) is much greater than unity. We can then use the asymptotic properties of Bessel functions ${ }^{36}$

$$
J_{n}(x), \approx\left(\frac{2}{\pi x}\right)^{1 / 2} \cos \left(x-\frac{\pi n}{2}-\frac{\pi}{4}\right)
$$

to show the approximate analytic estimate for the coupling constants is

$$
\begin{align*}
& \left|x_{n}^{( \pm)}\right| \leq \frac{\pi \Delta \varepsilon}{2 a \varepsilon_{0}^{\prime}}\left(\frac{2}{\pi|\Omega|}\right)^{1 / 2} \\
& \quad=\frac{\pi}{a}\left[\frac{1}{k a\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}} \frac{\Delta \varepsilon}{\varepsilon_{0}^{\prime}}\right]^{1 / 2} \tag{35b}
\end{align*}
$$

The conditions for the validity of the modified coupledwave method, given by (21a), which are a consequence of the averaging procedure applied to the fast oscillations within the structure period $a$, signify that $\left|x_{n}^{( \pm)}\right| \ll \pi / a$. This immediately yields an estimate for the small parameter in MCWT for $\lambda \ll a$ :

$$
\begin{equation*}
\xi=\frac{\left|x_{n}^{( \pm)}\right|}{\pi / a} \approx\left[\frac{1}{k a\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}} \frac{\Delta \varepsilon}{\varepsilon_{0}^{\prime}}\right]^{1 / 2} \ll 1 \tag{36}
\end{equation*}
$$

This estimate indicates the validity of (32) in MCWT for short waves with $k a>1$. The last proposition will be illustrated by the example of harmonically modulated permittivity $\varepsilon(z)$. It is also valid for arbitrary smooth modulations. Actually, for large-period one-dimensional structures ( $a>\lambda$ or $k a>1$ ), a large number of wavelengths $\lambda$ will fit into each
structure period. If the change in $\varepsilon(z)$ within the structure period $a$ is $\Delta \varepsilon$, the permittivity change for short wavelengths $\lambda$ is smaller by the factor $\lambda / a$, i.e., it is equal to $\Delta \varepsilon \lambda / a$. It is this quantity that is the small parameter of the theory [see (36)] in the case of approximate goeometric-optics type solutions (28). Hence, it follows that condition (36) is more readily satisfied as the Bragg-resonance number $n \approx a / \lambda$ increases. The geometric-optics solution is then closer to the true solution.

It is thus clear that the modified coupled-wave theory has a wider range of validity than the usual Kogelnik cou-pled-wave theory, and includes the latter as a special case.

Among the other advantages of MCWT we note that the theory is convenient for layered periodic media with power-type variation of $\varepsilon(z)$. It is shown in Ref. 30 that the direct solution of the set of coupled equations in (22) for $x_{ \pm n}= \pm i x_{ \pm n}^{(\mp)}$ and $x_{ \pm n}^{(\mp)}$, described by (32), then leads to the well-known exact solutions for the reflection coefficients of a number of familiar one-dimensional periodic structures. ${ }^{17}$ For example, for the two-layer periodic structure in which $\varepsilon(z)$ has the constant value $\varepsilon_{b}$ over a length $b$ within the period, and is equal to $\varepsilon_{d}$ within $d=a-b$, we obtain the following expression from (32) [since $d \varepsilon / d z=0$ in each layer, the contribution to $\mathcal{X}_{ \pm n}^{(\mp)}$ is due entirely to the points of discontinuity of $\varepsilon(z)$ ]:

$$
\begin{align*}
& x_{ \pm n}^{(\mp)}=\mp \frac{i}{2} \ln \frac{\varepsilon_{d}}{\varepsilon_{b}} \sin \varphi_{n} \cdot \exp \left(\mp i \varphi_{n}\right) \\
& \varphi_{n}=\frac{b}{a}\left[\pi n-k d\left(\varepsilon_{d}^{1 / 2}-\varepsilon_{b}^{1 / 2}\right)\right] \tag{37}
\end{align*}
$$

where we must use either all the upper or all the lower signs. The difference between $\varepsilon_{0}$ and $\tilde{\varepsilon}_{0}$ in such media produces a shift of the center of the Bragg resonance (13) that is proportional to $(\Delta \varepsilon)^{2}$ :

$$
\varepsilon_{0}-\widetilde{\varepsilon_{0}}=b d(\Delta \varepsilon)^{2} /\left[a^{2}\left(\varepsilon_{b}^{1 / 2}+\varepsilon_{d}^{1 / 2}\right)^{2}\right]
$$

where $\Delta \varepsilon=\varepsilon_{d}-\varepsilon_{b}$ and the coupling constants differ from zero for all numbers $n$ of the Bragg resonances. For equal optical thicknesses ( $k b \varepsilon_{b}^{1 / 2}=k d \varepsilon_{d}^{1 / 2}$ ) and exact Bragg resonances ( $k \tilde{\varepsilon}_{0}^{1 / 2}=\pi n / a, \delta=0$ ), we have $k b \varepsilon_{d}^{1 / 2}=k d \varepsilon_{1 / 2}^{d}$ $=\pi n / 2$ and $\varphi_{n}=\pi n / 2$. For odd Bragg resonances ( $n=2 p-1, p=1,2, \ldots$ ), all the $x_{ \pm n}^{(\mp)}$ are then equal to $\ln \left(\varepsilon_{b} / \varepsilon_{d}\right)$ and the coupling constants for even Bragg resonances ( $n=2 p, p=1,2, \ldots$ ) are all zero. As a result, the reflection coefficients $r_{B}^{L}$ from this type of layer of thickness $L=M a(M=1,2, \ldots$; see Fig. 1), calculated from this MCWT, ${ }^{30}$ are the same as those obtained from the exact Born-Wolf formulas ${ }^{17}$

$$
\begin{align*}
& r_{\mathrm{B}}^{L}=\left[1-\frac{\varepsilon_{2}^{1 / 2}}{\varepsilon_{1}^{1 / 2}}\left(\frac{\varepsilon_{b}}{\varepsilon_{d}}\right)^{M}\right]\left[1+\frac{\varepsilon_{2}^{1 / 2}}{\varepsilon_{1}^{1 / 2}}\left(\frac{\varepsilon_{b}}{\varepsilon_{d}}\right)^{M}\right]^{-1} \\
& \text { for } n=2 p-1 \text { and }  \tag{38}\\
& r_{\mathrm{B}}^{L}=\frac{\varepsilon_{1}^{1 / 2}-\varepsilon_{2}^{1 / 2}}{\varepsilon_{1}^{1 / 2}+\varepsilon_{2}^{1 / 2}} \text { for } n=2 p, p=1,2, \ldots
\end{align*}
$$

The first of these expressions describes the reflection of waves by a set of quarter-wave layers with equal optical thicknesses, and the second refers to a system of semiconducting layers. These expressions do not involve the thickness $L=M a$ or the properties of the periodic layers because the insertion of any number of half-wave layers between the


FIG. 5. Range of validity of different approximate methods used to calculate the diffraction properties of periodic media in terms of the variables $x=\Delta \varepsilon / \varepsilon_{0}^{\prime}, y=k_{0} a / \pi, k_{0}=k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}, k=\omega / c$ : region $I$ between the $x, y$ axes and the dashed hyperbola-dynamic diffraction theory (DDT), CWT, MCWT (for $\zeta \ll 1$ ), region 2 between dashed hyperbola and the dot-dash line-MCWT, region 3 between the dot-dash line and the solid curve-MCWT for harmonic modulation, region 4 between solid curve and the dashed hyperbola-requires numerical calculations.
homogeneous media with permittivities $\varepsilon_{1}$ and $\varepsilon_{2}$ does not affect the reflection coefficient of the structure (see Ref. 17 or 53 or 61). For other layered systems, the approximate MCWT expressions allow relatively simple numerical calculations (see Refs. 57 and 59).

For a clearer comparison between different methods, Fig. 5 plots $y=k_{0} a / \pi=k a\left(\varepsilon_{0}^{\prime}\right)^{1 / 2} / \pi$ as a function of $x=\Delta \varepsilon / \varepsilon_{0}^{\prime}$. This shows the regions of validity of the Kogelnik coupled-wave theory (CWT) (or the dynamic diffraction theory) and the modified coupled-wave theory (MCWT). To define the boundaries between these regions, we have followed Rytov's example ${ }^{37}$ and replaced strong inequalities such as $x \ll a^{-1}$ with the weaker inequality $x \leqslant 0.25 a^{-1}$, and then used (13) to express the Bragg resonance number $n$ in terms of $k_{0}=a$ [see, for example, the inequality given by (12)]. The dynamic diffraction theory and the Kogelnik CWT described by (27') are valid in region $l$ of Fig. 5 , which lies between the $x, y$ coordinate axes and the dashed hyperbola. In region 2, which lies between the hyperbola and the dot-dash straight line on which $k_{0} a \approx \Delta \varepsilon / \varepsilon_{0}^{\prime}$, the MCWT inequality given by (36) is satisfied. At the same time, the modified coupled-wave theory (MCWT) is also valid in region I in which ( $27^{\prime}$ ) is satisfied. In other words, MWCT has a wider range of validity in regions $I$ and 2 of Fig. 5 than the Kogelnik theory, and includes the latter as a special case. Figure 5 also shows region 3 which lies between the dot-dash line $k_{0} a \approx \Delta / \varepsilon_{0}^{\prime}$ and the solid curves. In this region, $\left|\mathcal{\chi}_{ \pm n}^{(\mp)}\right| \leqslant 0.25 a^{-1}$, which ensures the validity of MCWT for harmonically modulated $\varepsilon(z)$ if we numerically evaluate the expressions for the coupling constants in (35). In region 4, we can use neither of these approximations, and must employ numerical methods such as, for example, the Floquet-Bloch method discussed above and involving the numerical solution of (6) [Ref. 14] or the immersion method ${ }^{38}$ which reduces the problem to a numerical integration of a nonlinear first-order differential equation.

We therefore conclude that the modified coupled-wave theory (MCWT) discussed in this section is valid in a wide range of parameter values with the exception of the small region 4 of Fig. 5, i.e., when the more relaxed inequalities (27') and (36) are satisfied.

The MCWT formulas can be used to calculate the reflection and transmission coefficients of periodic media with any variation of the real and imaginary parts of permittivity $\varepsilon(z)=\varepsilon^{\prime}(z)+i \varepsilon^{\prime \prime}(z)$ within the structure period. In particular, an analytic expression was obtained in Ref. 56 for the reflection and transmission coefficients of a medium with periodic step changes in the refractive index and absorption coefficient. The latter is largely concentrated near the beginning and the end of the period, and is zero on the remaining part of the absorption period. It is shown in Ref. 56 that, when the permittivity modulation depth is less than the absorption modulation depth, we obtain the one-dimensional analog of the Bormann effect (see Ref. 11): the wave penetrates the absorbing medium to a distance appreciably greater than the distance in a medium with an equivalent mean absorption period because the absorption maximum occurs at the minimum of the resultant-field amplitude. In the opposite case, when the refractive-index modulation depth is greater than the absorption modulation depth, the attenuation of the field in the medium is largely due to Bragg reflection. Both effects are significant in intermediate cases.

## 6. REFLECTION OF LIGHTBY A LAYER WITH PERIODICALLY-VARYING PERMITTIVITY

We now return to the reflection coefficient of a layer (see Fig. 1) of a uniform periodic medium, taking into account possible permittivity jumps at the separation boundaries: $\varepsilon(z) \neq \varepsilon_{1}$ and $\varepsilon(L) \neq \varepsilon_{2}$. Approximate solutions such as (28), which involve $A^{( \pm)}(z)$ found for the interior of the layer and given by (23), must then be matched at $z=0$ and $z=L$ to the solutions for the exterior of the layer. We then obtain the following expressions instead of (24) (see Appendix III):

$$
\begin{align*}
& A^{(+)}(0)+r_{1} A^{(-)}(0)=\left(1+r_{1}\right) A_{0}(\varepsilon(0))^{1 / 4}, \\
& A^{(-)}(L)=r_{2} A^{(+)}(L) \exp (2 \dot{\psi} \psi(L)), \tag{39}
\end{align*}
$$

where

$$
\psi(L)=k \int_{0}^{L}(\varepsilon(z))^{1 / 2} \mathrm{~d} z
$$

and the Fresnel reflection coefficients of the layer boundaries are given by

$$
\begin{equation*}
r_{1}=\frac{\varepsilon_{1}^{1 / 2}-(\varepsilon(0))^{1 / 2}}{\varepsilon_{1}^{1 / 2}+(\varepsilon(0))^{1 / 2}}, r_{2}=\frac{(\varepsilon(L))^{1 / 2}-\varepsilon_{2}^{1 / 2}}{(\varepsilon(L))^{1 / 2}+\varepsilon_{2}^{1 / 2}} \tag{40}
\end{equation*}
$$

where we note that, in the Kogelnik theory, $\varepsilon(z)$ $=\varepsilon(L)=\varepsilon_{0}$ and $\psi(L)=k \varepsilon_{0}^{1 / 2} L$.

Substituting for $A^{( \pm)}(z)$ from (23) into the second condition in (39), we obtain the following expression for the ratio $C_{2} / C_{1}$ instead of (25):

$$
\begin{align*}
\frac{C_{2}}{C_{1}} & =-\frac{x_{n}}{x_{-n}} r_{B}^{\infty} \exp (-2 \gamma L) \\
& \times \frac{r_{B}^{\infty}-r_{2} \exp \left(2 i \varphi_{n}(L)\right)}{1-\frac{x_{n}}{x_{-n}} r_{B}^{\infty} r_{2} \exp \left(2 i \varphi_{n}(L)\right)} \tag{41}
\end{align*}
$$

where, if we assume small losses $\left(\varepsilon_{0}^{\prime \prime}<\varepsilon_{0}^{\prime}\right)$ and adopt the resonance condition $k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=(\pi n / a)+\delta$, we obtain

$$
\varphi_{n}(L)=\psi(L)-i\left(\frac{\alpha}{2}-i \delta\right) L=\frac{\pi}{a} n L+\int_{0}^{L} x_{p}(z) \mathrm{d}_{z}
$$

in which $\chi_{p}(x)=k \varepsilon_{p}(z) / 2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$ and $\varepsilon_{p}(z)=\varepsilon(z)-\varepsilon_{0}$. When $r_{2}=0$, i.e., when the layer is matched for $z \geqslant L$, the ratio given by (41) becomes identical with (25).

The reflection coefficient $r_{\Sigma}^{L}$ of the entire structure can be expressed in terms of the amplitude of the forward, $A^{(+)}(0)$, and backward, $A^{(-)}(0)$, waves by using the continuity equation at $z=0$ (see Appendix III):

$$
\begin{align*}
& r_{\Sigma}^{L}=\frac{A_{1}}{A_{0}}=\frac{r_{1}+\tilde{r}}{1+r_{1} \tilde{r}} \\
& \tilde{r}=\frac{A^{(-)}(0)}{A^{(+)}(0)}=\frac{r_{\mathrm{B}}^{\infty}+\left(x_{-n} / x_{n}\right)\left(1 / r_{\mathrm{B}}^{\infty}\right) C_{2} / C_{1}}{1+\left(C_{2} / C_{1}\right)} \tag{42}
\end{align*}
$$

where we have used the solutions given by (23), which are valid near a Bragg resonance. The coupling constants $\chi_{ \pm n}$ of the Kogelnik theory are related to the coupling contants $x_{-n}^{( \pm)}$of MCWT [see (32)] as follows: $x_{n}=-i x_{n}^{(-)}$and $x_{-n}=i x_{-n}^{(+)}$so that $x_{n} / x_{-n}=-x_{n}^{(-)} / x_{-n}^{( \pm)}$and $x_{n} x_{-n}=x_{n}^{(-)} x_{-n}^{(+)}$. Since the ratio $C_{2} / C_{1}$ is given by (41), the reflection coefficient $\tilde{r}$ of the entire periodic structure assumes the following final form when the second boundary at $z=L$ is taken into account (see Appendix III):

$$
\begin{align*}
\tilde{r}= & \left\{r_{\mathrm{B}}^{(L)}+\frac{\exp (-2 \gamma L)+\left(x_{n}^{(-)} / x_{-n}^{(+)}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2}}{1+\left(x_{n}^{(-)} / x_{-n}^{(+)}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2} \exp (-2 \gamma L)}\right. \\
& \left.\times r_{2} \exp \left(2 i \varphi_{n}(L)\right)\right\} \\
& \times\left\{1+\frac{x_{n}^{(-)}}{x_{-n}^{(+)}} r_{\mathrm{B}}^{L} r_{2} \exp \left(2 i \varphi_{n}(L)\right)\right\}^{-1}, \tag{43}
\end{align*}
$$

where

$$
\begin{align*}
& r_{\mathrm{B}}^{(L)}=\frac{r_{\mathrm{B}}^{\infty}[1-\exp (-2 \gamma L)]}{1+\left(x_{n}^{(-)} / x_{-n}^{(+)}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2} \exp (-2 \gamma L)} \\
& r_{\mathrm{B}}^{\infty}=\frac{-x_{-n}^{(+)}}{\gamma+(\alpha / 2)-i \delta}  \tag{44}\\
& \gamma=\left[x_{n}^{(-)} x_{-n}^{(+)}+\left(\frac{\alpha}{2}-i \delta\right)^{2}\right]^{1 / 2} \\
& \varphi_{n}(L)=\frac{\pi n}{a} L+\int_{0}^{L} x_{p}(z) \mathrm{d} z
\end{align*}
$$

It is clear from these formulas that the backward-traveling wave is formed both as a result of scattering by the localized boundaries of the OPS $(z=0$ and $z=L)$ and by the distributed reflection within the periodic medium itself, and that this is essentially an interference effect. The result is a very complicated spectral dependence of the modulus and phase of the reflection coefficient $r_{\Sigma}^{L}$ in (42). We must therefore consider the simple case of a semi-infinite periodic structure ${ }^{39,40}$ for which $\alpha L \gg 1$ and $\exp \left(-2 \gamma^{\prime} L\right) \rightarrow 0$ where $\gamma=\gamma^{\prime}+i \gamma^{\prime \prime}$ and $\gamma^{\prime}>0$. It then follows from (44) and (43) that ${r_{B}^{L} \rightarrow r_{B}^{\infty} \text { and } \tilde{r} \rightarrow r_{B}^{\infty} \text {, and the reflection coefficient in (42) }}_{\text {(4) }}$ assumes the much simpler form

$$
\begin{equation*}
r_{\Sigma}^{\infty}=\frac{r_{1}+r_{\mathrm{B}}^{\infty}}{1+r_{1} r_{\mathrm{B}}^{\infty}} \tag{45}
\end{equation*}
$$

which describes the successive reflection from the front face


FIG. 6. Spectral dependence of the power reflection coefficient $R_{\Sigma}(\delta / \varkappa)$ for an unmatched ( $r_{1}=0.6, \varphi=\pi / 3$ ) semi-infinite periodic medium with the following absorption coefficients: $1-\alpha / \varkappa=0,2-\alpha / \kappa=0.1$, $3-\alpha / \varkappa=0.25,4-\alpha / \varkappa=0.5,5-\alpha / \varkappa=1.0$.
of the semi-infinite medium and from its most peripheral structure.

Figure 6 shows typical spectral distributions of the energy reflection coefficient $R_{\Sigma}=\left|r_{\Sigma}^{\infty}\right|^{2}$ for different values of the absorption coefficient and the initial permittivity modulation phase $\varphi=\arg \left(i x_{-n}^{(+)}\right)$in the OPS. It is clear that the function $R_{\Sigma}(\delta)$ is highly asymmetric. The reason for this can be readily understood considering the spectral dependence of the phase of the Bragg reflection $r_{B}^{\infty}$ (see Fig. 3). It is clear from this figure that, as we pass from the long-wave $(\delta / x \leqslant-1)$ to the short-wave $(\delta / x \geqslant+1)$ edge of the Bragg reflection band, $\arg r_{B}^{\infty}$ changes by $\pi$, i.e., the sign of $r_{B}^{\infty}$ is reversed. This means that the amplitude of the wave reflected by the periodic structure will be added to or subtracted from the amplitude of the wave scattered directly by the OPS boundary. This interference can lead to a substantial reduction in $r_{\Sigma}^{\infty}$ as compared with $r_{1}$ and $r_{B}^{\infty}$, and $r_{B}^{\infty}$ may actually vanish. The effect of the OPS boundary is even stronger in the case of the spectral dependence of the phase $\chi=\arg r_{\Sigma}^{\infty}$ of the reflection coefficient. ${ }^{41}$ This is seen, in particular, in the very high sensitivity of the phase $\chi$ of a change in the OPS parameters such as the absorption coefficient $\alpha$ and the initial modulation phase $\varphi$. For example, a small change in the range $\alpha / \varkappa$ often leads to a qualitative change in the character of the function $\chi(\delta)$, whereas $R_{\Sigma}(\delta)$ changes only slightly (Fig. 7). Moreover, since the phase of the Bragg reflection coefficient $r_{D}^{\infty}$ of a semi-infinite structure changes by $\pi$ as we cross the Bragg-resonance frequency


FIG. 7. Spectral dependence of the power reflection coefficient $R_{\Sigma}$ ( $\delta /$ $x)=\left|r_{\Sigma}(\delta / x)\right|^{2}$ and the phase $\chi(\delta / x)$ of the reflection coefficient $r_{\Sigma}$ for an unmatched layer ( $r_{1}=0.6, \varphi=\pi / 4$ ) for different losses: curves 1 ' and $2^{\prime}-R_{\Sigma}(\delta / \chi)$ for $\alpha / \chi=0.7$ and 0.9 , respectively; curves 1 and $2-\chi(\delta /$ $x$ ) with the same values of $\alpha / \chi$, respectively.
band (Fig. 3), and the phase of the Fresnel reflection coefficient $r_{1}$ in (40) is $\pi$ when $\varepsilon(0)>\varepsilon_{1}$, it follows that as we cross the Bragg resonance in the coefficient $r_{\Sigma}^{\infty}$ in (45), the interference between $r_{B}^{\infty}$ and $r_{1}$ ensures that the resultant phase of the coefficient $r_{\Sigma}^{\infty}$ can vary in the range between 0 and $2 \pi$, which is clearly seen in Fig. 7.

The boundaries of a finite layer of a uniformly periodic medium can thus significantly affect the spectral dependence of the Bragg reflection coefficient as compared with the matched perodic structure of finite thickness (see Section 4.2). It then follows from Refs. 39-41 that the effects of the separation boundary can be observed even for the relatively low values $\left|r_{1}\right|^{2} \gtrsim 0.05$ of the Fresnel reflection coefficient of the boundary.

The reflection coefficients calculated in MCWT are compared in Refs. 57 and 59 with direct numerical computations based on the differential equations for the layered periodic media and media with a harmonically modulated refractive index. This shows that there is good agreement (within the given modulation depth) between the reflection coefficients as functions of frequency and the number of periods in the structure for different modulation depths (up to $50 \%$ modulation depth ) and different numbers of the Bragg resonances. This applies to frequencies both inside and outside the Bragg reflection region, so that the spectral dependence of reflection coefficients obtained by the two methods (exact and approximate MCWT) are in good agreement with one another for all frequencies between the first and third resonance, i.e., for a continuous transition from the Bragg reflection zone to transparency zones and back again. The higher the number of a Bragg resonance the better the agreement. The agreement was found to hold not only for the magnitude of the reflection coefficient, but also for the positions of the maxima and minima on the frequency axis. Moreover, the agreement was found to be satisfactory even for media consisting of only 1,2 , or more periods.

The efficacy of MCWT will now be demonstrated by considering the example of wave propagation in almost periodic and nonlinear periodic media.

## 7. APPLICATION OF MCWT TO ALMOST PERIODIC AND NONLINEAR PERIODIC MEDIA

The approximate modified coupled-wave theory is obtained from the exact set of equations for the amplitudes of the counter-propagating waves, given by (29). This set of equations is outwardly similar to the standard coupled-wave equations except that its coefficients $S^{( \pm)}(z)$ depend on the coordinates $z$. It is clear from (30) that these coefficients contain explicitly in analytic form the permittivity $\varepsilon(z)$ with arbitrary (not merely periodic) dependence on the coordinate $z$. Hence, in the general formulas for $S^{( \pm)}(z)$ given by (30), small deviations from periodicity of the function $\varepsilon(z)$ can be taken into account. In this Section, we illustrate typical examples of such calculations by considering approximate analytic solutions of (29) for almost periodic and nonlinear periodic media. Problems of this kind were solved earlier in Refs. 43-45 (see also Ref. 48). However, in these earlier publications, each of the problems was solved by a particular method typical for the particular type of deviation from periodicity. We shall show below that MCWT will enable us to solve different problems by the same approach.

### 7.1. Almost-perlodic medla

A periodic medium often contains optical inhomogeneities that give rise to additional regular variation in the phase of the propagating waves, i.e., the permittivity $\varepsilon(z)$ exhibits an additional regular variation. A Bragg resonance is then found to occur not over the entire length of a uniformly periodic medium, but only in certain portions of it. The main objective of the calculation reproduced below is to determine the reflection coefficient of this type of structure and to estimate quantitatively the distance over which the above change of phase takes place.

For the sake of simplicity, we shall consider this problem in the special case of a medium whose permittivity is subject to periodic modulation, but has in addition a small linear component

$$
\begin{align*}
& \varepsilon(z)=\varepsilon_{0}+\sum_{\substack{m=-\infty \\
m \neq 0}}^{m=+\infty} \varepsilon_{m} \exp \left(\frac{2 \pi i}{a} m z\right)+\xi z, \\
& \xi L \ll \varepsilon_{0}, \tag{46}
\end{align*}
$$

where $L$ is the thickness of the layer with periodic permittivity variation. We shall use the modified coupled-wave method described in Sec. 6. For this, we substitute (46) for $\varepsilon(z)$ in the expression for the coefficients $S^{( \pm)}(z)$ in (30), and then substitute the resulting $S^{( \pm)}(z)$ in (29) before averaging over fast oscillations in precisely the same way as in Appendix II. Instead of the coupled-wave equations of the form given by (22) we thus obtain a more complicated set of equations relating the amplitudes of the forward, $A^{(+)}(z)$, and backward, $A^{(-)}(z)$, waves:

$$
\begin{align*}
& \frac{\mathrm{d} A^{(+)}(z)}{\mathrm{d} z} \\
& =\dot{x} e_{n} A^{(-)}(z) \exp [(\alpha-2 i \delta) z] \exp \left(-i \frac{\sigma}{2} z^{2}\right)  \tag{47}\\
& \frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z} \\
& =-\dot{i x_{-n}} A^{(+)}(z) \exp [-(\alpha-2 i \delta) z] \exp \left(i \frac{\sigma}{2} z^{2}\right)
\end{align*}
$$

where, for example, $\sigma=k \xi / \varepsilon_{0}^{1 / 2}$ has the dimensions of the inverse square of length. The two equations in (47) are written on the assumption that the permittivity modulation depth is small, i.e., $\Delta \varepsilon \ll \varepsilon_{0}$ for $\Delta \varepsilon=\left|\varepsilon_{m}\right|_{\max }$ and for $\xi L \ll \varepsilon_{0}$. Instead of the coupling coefficients $x_{m}^{( \pm)}$of (32), MCWT involves the coupling coefficients $x_{( \pm n)}$ of the Kogelnik theory, given by (21b) (cf. Sec. 5.2).

The set of equations given by (47) can be readily reduced to independent second-order equations for the functions $A^{( \pm)}(z)$ whose solutions are related (for $\alpha=0$ and $\delta=0$ ) to the parabolic cylinder functions ${ }^{42}$ of the form

$$
V\left( \pm i p \pm \frac{1}{2}, 2 \sigma^{1 / 2} \exp \left( \pm i \frac{\pi}{4}\right)\right)
$$

where $\rho=\left|\varkappa_{n}\right|^{2} / \sigma=\varkappa^{2} / \sigma$ (see Appendix IV). The general solutions of (47) can then be written as superpositions of the corresponding parabolic cylinder functions

$$
\begin{align*}
& A^{(+)}(z)=\left[C_{1} V\left(i \varphi-\frac{1}{2}, z \sigma^{1 / 2} \exp \left(-\frac{i \pi}{4}\right)\right)\right. \\
& \left.+C_{2} V\left(-i \rho+\frac{1}{2}, z \sigma^{1 / 2} \exp \left(\frac{i \pi}{4}\right)\right)\right] \exp \left(-\frac{i \sigma}{4} z^{2}\right) \\
& A^{(-)}(z)=\frac{x_{-n}}{\sigma^{1 / 2}}\left[C _ { 1 } V \left(i \rho+\frac{1}{2}\right.\right. \\
& \left.z \sigma^{1 / 2} \exp \left(-i \frac{\pi}{4}\right)\right)+C_{2} \frac{1}{\rho} V\left(-i \varphi-\frac{1}{2}\right. \\
& \left.\left.z \sigma^{1 / 2} \exp \left(\frac{\pi}{4}\right)\right)\right] \exp \left(i \frac{\sigma}{4} z^{2}+i \frac{3 \pi}{4}\right) \tag{48}
\end{align*}
$$

where $\rho=\left|\chi_{n}\right|^{2} / \sigma=\chi^{2} / \sigma$ is a dimensionless parameter and the constants $C_{1,2}$ are determined from the boundary conditions. As $\sigma \rightarrow 0$, the set of equations given by (48) becomes identical with (23) except for the definition of the constants $C_{1,2}$. Let us determine the power reflection coefficient $R_{\infty}$ of a matched semi-infinite medium with $\varepsilon(z)$ given by (46) in the case of an exact Bragg resonance ( $\delta=0$ ) on the separation boundary at $z=0$. We must then put $C_{p}=0$ in (48) and obtain (see Appendix IV)

$$
\begin{align*}
& R_{\infty}=\left|\frac{A^{(-)}(0)}{A^{(+)}(0)}\right|^{2}=\operatorname{th}\left(\frac{\pi}{2} \frac{x^{2}}{\sigma}\right)=\operatorname{th}\left(\frac{L_{\mathrm{c}}}{L_{\mathrm{B}}}\right)^{2} \\
& L_{\mathrm{B}}=\frac{1}{x}, L_{\mathrm{c}}=\left(\frac{\pi}{2 \sigma}\right)^{1 / 2} \tag{49}
\end{align*}
$$

An analogous result was obtained in a different way in Ref. 43. The quantities $L_{B}$ and $L_{c}$ in (49), which have the dimensions of length, have the following interpretation. The length $L_{B}=1 / \varkappa$ defines the distance within which the Bragg reflection from a purely periodic medium is produced. Actually, it is clear from (26), which gives the expression for $r_{B}^{L}$ for a purely periodic medium with $a=0$ and $\delta=0$, but $\gamma=\varkappa$, that the Bragg reflection coefficient $\left|r_{B}^{L}\right|$ is close to unity for $L \geqslant L_{B}=1 / x$. At the same time, the length $L_{c}=(\pi / 2 \sigma)^{1 / 2}$ determines the distance over which the phase change in the solutions given by (48) becomes comparable with the Bragg reflection phase because of the linear increase in the permittivity (46). This occurs for $L_{c} \approx L_{B}$ or $\varkappa^{2} \approx 2 \sigma / \pi$. Actually, it is clear from the original set of equations given by (47) that the presence in (46) of the increment that increases linearly with $z$ signifies the appearance in (47) of additional detuning from Bragg resonance, which rises linearly with distance $z$, so that the total detuning is $\delta(z)=\delta+(1 / 4) \sigma z$. When $\delta=0$, which is the case in this example, we have $\delta(z)=\sigma z / 4$. This detuning then appears in the effective constant $\gamma(z)$ given by (15), i.e., $\gamma(z)=\left[\varkappa^{2}-\delta^{2}(z)\right]^{1 / 2}$, which determines the effective Bragg reflection coefficient in the form given by (26). It is readily seen that $\gamma(z)$ decreases with increasing $z$, and there is a corresponding reduction in the effective reflection coefficient of the form given by (26). When $x \approx \sigma z / 4$, i.e., when $z \approx 4 \chi / \sigma \approx L_{c} \approx L_{B} \gamma\left(L_{c}\right)$ and, consequently, the reflection coefficient $r_{B}^{L}$ vanishes, i.e., there is no Bragg reflection. This means that portions of the pe: iodic medium that lie at a distance $z$ from the separation boundary $L_{c}$ do not participate in Bragg reflection for $z>L_{c}$. This follows formally from the expression for $R_{\infty}$ in (49). Thus, as $\sigma \rightarrow \infty$, i.e., $L_{c} \gg L_{B}$, we have $R_{\infty} \approx 1$ and Bragg reflection occurs in the semi-infinite medium. In the reverse case, when $L_{c} \ll L_{B}$ or $\sigma \geqslant(\pi / 2) \mathcal{K}^{2}$ we have $R_{\infty} \approx\left(L_{c} / L_{B}\right)^{2} \ll 1$ and Bragg reflection vanishes almost entirely. When $L_{c} \approx L_{B}$, we have $R_{\infty} \approx \tanh 1 \approx 0.76$ and the
linear inhomogeneity does not as yet affect the Bragg reflection. It is precisely for such conditions that we have to estimate the reflection characteristics of real periodic structures.

### 7.2. Nonlinear periodic structures

Several interesting effects are also observed in periodic media with a nonlinear permittivity (see, for example, Ref. 48). A strong light field induces a diffraction grating in such structures, which acts like a stationary periodic inhomogeneity and provides an additional contribution to the Bragg reflection of light. It is clear that the resultant reflection depends significantly on the phase shift between the stationary and induced gratings. In some cases, this leads to a nonlinear 'transmission' by the one-dimensional medium (OPS) in the neighborhood of a Bragg reflection. Thus, reflection and transmission by the OPS at the Bragg frequency then assumes a bistable character that depends on the light-field intensity.

To be specific, consider the propagation of light in a periodic medium with permittivity given by

$$
\begin{equation*}
\varepsilon(z)=\varepsilon_{0}+\sum_{\substack{m=-\infty \\ m \neq 0}}^{m=+\infty} \varepsilon_{m} \exp \left(\frac{2 \pi i}{a} m z\right) 4 \pi \chi|E(z)|^{2} \tag{50}
\end{equation*}
$$

which contains a nonlinear cubic term ( $\chi$ is the cubic nonlinear susceptibility). To obtain the equations for the transformation of light in the nonlinear and periodic media, we substitute (as in the last section) the expression given by (50) into the coefficients $S^{( \pm)}(z)$ in (30), and then substitute the result in (29). These equations are then simplified by assuming that the periodic modulation depth is small ( $\Delta \varepsilon \ll \varepsilon_{0}$ ) and that the change in $\varepsilon(z)$ in (50) due to the nonlinear increment is small, i.e., $4 \pi \chi|E(z)|^{2} \ll \varepsilon_{0}$. The resulting equations for $A^{( \pm)}(z)$ are then averaged over fast oscillations (see Appendix II). At the same time, if we separate out in the phase $\psi(z)$ in (28) the term that depends on the lightfield intensity, we obtain the following set of coupled equations for the slowly-varying amplitudes $A^{( \pm)}(z)$ :

$$
\begin{align*}
& \frac{\mathrm{d} A^{(+)}(z)}{\mathrm{d} z}=i x_{n} A^{(-)}(z) \exp (-2 i \delta z) \\
& \quad+i \xi\left(\left|A^{(+)}(z)\right|^{2}+\left|A^{(-)}(z)\right|^{2}\right) A^{(+)}(z) \\
& \frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z}=-i x_{-n^{\prime} A^{(+)}(z) \exp (2 i \delta z)} \quad-i \xi\left(\left|A^{(+)}(z)\right|^{2}+\left|A^{(-)}(z)\right|^{2}\right) A^{(-)}(z),
\end{align*}
$$

where $\xi=2 \pi k \chi / \varepsilon_{0}^{1 / 2}$. When $\xi=0$, i.e., in the absence of nonlinearity ( $\chi=0$ ), the set of equations given by (51) becomes identical with the standard coupled-wave equations given by (22).

The above set of nonlinear equations for the complex amplitudes of the forward and backward waves is solved by considering the real amplitudes and phases of these waves, i.e.,

$$
\begin{align*}
& A^{(+)}(z)=a_{\mathrm{F}}(z) \exp \left(i \varphi_{\mathrm{F}}(z)\right), \\
& A^{(-)}(z)=a_{\mathrm{B}}(z) \exp \left(i \varphi_{\mathrm{B}}(z)\right) . \tag{52}
\end{align*}
$$

Substituting these expressions in (51) and separating real and imaginary parts, we obtain the following nonlinear set of coupled equations for the real functions $A_{F}(z), A_{B}(z)$ and
$\varphi(z):$

$$
\begin{align*}
& \frac{\mathrm{d} a_{\mathrm{F}}(z)}{\mathrm{d} z}=x a_{\mathrm{B}}(z) \sin \varphi(z), \frac{\mathrm{d} a_{\mathrm{B}}(z)}{\mathrm{d} z}=x a_{\mathrm{F}}(z) \sin \varphi(z), \\
& \frac{\mathrm{d} \varphi(z)}{\mathrm{d} z}=2 \delta+3 \xi\left(a_{\mathrm{F}}^{2}(z)+a_{\mathrm{B}}^{2}(z)\right) \\
& \quad+x\left(\frac{a_{\mathrm{F}}(z)}{a_{\mathrm{B}}(z)}+\frac{a_{\mathrm{B}}(z)}{a_{\mathrm{F}}(z)}\right) \cos \varphi(z) \tag{53}
\end{align*}
$$

where

$$
\begin{aligned}
& x=\left|x_{n}\right|=\left|x_{-n}\right| \\
& \varphi(z)=\varphi_{\mathrm{F}}(z)-\varphi_{\mathrm{B}}(z)+2 \delta z-\operatorname{arctg} x_{n} .
\end{aligned}
$$

This set of equations has two integrals, namely,

$$
\begin{align*}
\Gamma_{1} & =\left(a_{\mathrm{F}}^{2}(z)-a_{\mathrm{B}}^{2}(z)\right) \\
\Gamma_{2} & =a_{\mathrm{F}}(z) a_{\mathrm{B}}(z) \cos \varphi(z)+ \\
& +\frac{\delta}{x} a_{\mathrm{F}}^{2}(z)+\frac{3 \xi}{2 x} a_{\mathrm{F}}^{2}(z) a_{\mathrm{B}}^{2}(z) . \tag{54}
\end{align*}
$$

For a matched layer, for which the conditions in (24) are satisfied and $a_{B}^{2}(L)=0$, the integral $\Gamma_{1}=a_{F}^{2}(L)$ is proportional to the light intensity leaving the structure at $z=L$, whereas the second integral is $\Gamma_{2}=\delta \Gamma_{1} / x$, i.e., it is proportional to the first integral. The solution of (53) with allowance for the integrals $\Gamma_{1,2}$ at the exact Bragg resonance for which $\delta=0$ leads to the following expression for the Bragg reflection coefficient (see Appendix IV):

$$
\begin{align*}
R_{L} & =\frac{a_{\mathrm{B}}^{2}(0)}{a_{\mathrm{F}}^{2}(0)}=\left(1-\frac{\Gamma_{1}}{a_{\mathrm{F}}^{2}(0)}\right) \\
& =\frac{\mathrm{nd}\left(2 a x a / a^{-2}\right)-1}{\operatorname{nd}\left(2 a x L / a^{-2}\right)+1} \tag{55}
\end{align*}
$$

where $\operatorname{nd}(x / q)$ is one of the Jacobi functions ${ }^{42}$ for which $0 \leqslant q \leqslant 1$. The parameter

$$
a=\left(1+\frac{1}{4} J_{L}\right)^{1 / 2}
$$

is related to the dimensionless light intensity at exit $J_{L}=a_{F}^{2}(L) / E_{c}^{2}$. It follows from the solution of (53) that $J_{L}$ depends on the dimensionless intensity at exit, $J_{0}=a_{F}^{2}(0) / E_{c}^{2}$, through the implicit relation (see Appendix IV)

$$
\begin{equation*}
I \equiv J_{0}=\frac{1}{2}\left[1+\operatorname{nd}\left(2 a x L / a^{-2}\right)\right] J_{L} \tag{56}
\end{equation*}
$$

where

$$
\begin{aligned}
& J_{0} \equiv I=a_{\mathrm{F}}^{2}(0) / E_{\mathrm{c}}^{2}, J_{L}=a_{\mathrm{F}}^{2}(L) / E_{\mathrm{c}}^{2} \\
& a=\left(1+\frac{1}{4} J_{L}\right)^{1 / 2}, E_{\mathrm{c}}=(2 x / 3 \xi)^{1 / 2}
\end{aligned}
$$

The critical field $E_{c}$ corresponds to the amplitude of the light-induced diffraction grating that is comparable with the amplitude of the stationary permittivity modulation. We now use the implicit relation (56) to find $J_{L}$ as a function of $J_{0} \equiv I$ for specific values of $\varkappa L$ and substitute these expressions in (55). We thus calculate the Bragg reflection coefficient in (55) as a function of the light intensity $J_{0}$ leaving the structure. The results of these calculations are shown in Fig. 8 for $x L=2$ and $x L=4$. The figure clearly shows the bistable behavior of the function $R_{L}=R_{L}\left(I \equiv J_{0}\right)$ with well-de-


FIG. 8. Power reflection coefficient $R_{L}\left(J_{0}\right)$ of a nonlinear periodic medium for incident light with normalized intensity $J_{0} \equiv I$ for different coupling of counterpropagating waves: $1-x L=2,2-x L=4$.
fined hysteresis that is typical for nonlinear systems. To obtain the function $R_{L}(\delta)$ for different values of the initial intensity $J_{0} \equiv I$, we have to solve the set of equations given by (53) for $\delta \neq 0$, which is a relatively complex task. However, if we compare (53) with the set of coupled-wave equations given by (22) and written for the functions $a^{( \pm)}(z)$ $=A^{( \pm)}(z) \exp ( \pm i \delta z)$ with $\alpha=0$, we can readily show that the influence of the nonlinearity in (53) can be described in terms of the effective additional detuning

$$
\delta_{n}^{( \pm)}(z)=\xi\left(\left|A^{( \pm)}(z)\right|^{2}+\left|A^{(\mp)}(z)\right|^{2}\right),
$$

which depends on the forward and backward wave intensities. Since the quantities $\delta_{m}^{( \pm)}(z)$ are always positive, the spectral dependence of the Bragg reflection coefficient on the nonlinear periodic medium is asymmetric with the maximum shifted at low intensities toward the longer wavelengths.

For an infinite periodic structure with a nonlinear filling, the set of equations given by (53) has a 'soliton' solution at exact Bragg resonance ( $\delta=0$ ) that is of the following form (see also Refs. 44 and 45):

$$
\begin{align*}
& a_{\mathrm{F}}^{2}(z)=a_{\mathrm{B}}^{2}(z)=\frac{E_{\mathrm{c}}}{\operatorname{ch}\left[2 x\left(z-z_{0}\right)\right]} \\
& \varphi(z)=-\operatorname{sign}\left(z-z_{0}\right) \arccos \frac{1}{\operatorname{ch}\left[2 x\left(z-z_{0}\right)\right]} \tag{57}
\end{align*}
$$

where sign $x=x /|x|$ and $z_{0}$ is an arbitrary coordinate on which this solitary wave is centered. It is noted in Ref. 45 that the wave can propagate with a given velocity along the periodic structure, but analysis of the conditions for the excitations of such solitary waves involves the solution of the time-dependent electrodynamic problem. We note at this point that nonlinear periodic structures have not been investigated in the same detail as the linear structures. For example, there is undoubted interest in the generation of harmonics in a periodic medium in which diffraction by a periodic inhomogeneity produces phase locking (see Ref. 48 for further details).

## CONCLUSION

We now conclude our review as follows. The method based on the modified-coupled-wave theory provides the most convenient procedure for the derivation of reasonably
simple approximate analytic solutions for waves propagating in one-dimensional periodic structures. When the modulation depth is small, this includes as a special case the Kogelnik coupled-wave method and the dynamic diffraction theory. On the other hand, the formulas obtained by the modified coupled-wave theory are valid both for small structure periods, when $k a \Delta \varepsilon / \varepsilon_{0} \& 1$, and for large periods for which $\Delta \varepsilon /\left(k a \varepsilon_{0}\right) \ll 1$. The latter condition applies to high Bragg resonance numbers and modulation depths $\Delta \varepsilon / \varepsilon_{0}$ that are not too low.

We have examined in some detail the analytic methods available for the solution of problems involving the propagation and transformation of waves in the special case of isotropic media with uniform periodicity, when the waves propagate in the direction of this periodicity. This occurs for waves incident normally on the separation boundary with a periodic medium, when the separation boundary is perpendicular to the direction of the periodicity. The permittivity of the isotropic medium is then independent of position in the plane of the separation boundary. The wave reflection and transmission problem in the case of oblique incidence can then be reduced by a simple transformation to the onedimensional problem with an effective permittivity that is a function of the angle of incidence (see, for example, Refs. 33 and 53). If the direction of the one-dimensional periodicity is at an angle to the normal on the separation boundary (see, for example, Ref. 29), then a periodic structure appears on the separation boundary and we have to take into account the entire angular spectrum of propagating harmonics that arise from the diffraction of the incident wave by the surface periodicity, even in the case of normal incidence. When the periodic medium is anisotropic or gyrotropic, the solution of the oblique-incidence problem is significantly different from the normal-incidence problem even when the separation boundary is perpendicular to the direction of the periodicity. This difference is particularly significant for gyrotropic liquid crystals. Polarization effects then ensure that instead of the simple system of two first-order equations, we have to deal with a set of four or more first-order equations, which significantly complicates the process of fining simple analytic solutions.

Methods that are valid only for infinite periodic structures of a particular type were used in Refs. 46 and 47 to find analytic solutions. However, these methods of solution differ from the modified coupled-wave theory because they do not allow us to take into account absorption by the medium and the finite periodic structure; they are even less effective for arbitrary modulation or in the presence of small aperiodicity and nonlinearity.

We have presented the results of a completed study of the propagation and transformation of waves in isotropic media with one-dimensional perodicity. The Kogelnik cou-pled-wave theory that we have examined together with its modifications enables us to consider more complicated wave propagation and transformation problems, e.g., multiperiodic media, periodic media with statistical properties, and periodic media with local anisotropy and gyrotropy. All these techniques and approaches can also be applied to the propagation of electron waves in periodic quantum superlattices.

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## APPENDIXI. DERIVATION OF THE INTEGRAL EQUATION AND PROOF OF ITS EQUIVALENCE TO THE FLOQUETBLOCHSYSTEM (6)

1. If we put $\varepsilon(z)=\varepsilon_{0}+\varepsilon_{\rho}(z)$ and use the properties of $\delta\left(z-z^{\prime}\right)$ we can write (1) in the form

$$
\begin{align*}
& \frac{\mathrm{d}^{2} E(z)}{d z^{2}}+k^{2} \varepsilon_{0} E(z)=-k^{2} \varepsilon_{\mathrm{p}}(z) E(z) \\
& \quad=-k^{2} \int_{-\infty}^{+\infty} \mathrm{dz}^{\prime} e_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) \delta\left(z-z^{\prime}\right) \tag{I.I}
\end{align*}
$$

We now substitute

$$
\begin{equation*}
E(z)=k^{2} \int_{-\infty}^{+\infty} \mathrm{d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) G\left(z-z^{\prime}\right) E\left(z^{\prime}\right) \tag{I.2}
\end{equation*}
$$

and obtain the following equation for the Green's function $G\left(z-z^{\prime}\right)$ :

$$
\begin{equation*}
\frac{\mathrm{d}^{2} G\left(z-z^{\prime}\right)}{\mathrm{d} z^{2}}+k^{2} \varepsilon_{0} G\left(z-z^{\prime}\right)=-\delta\left(z-z^{\prime}\right) \tag{I.3}
\end{equation*}
$$

Expanding $G(z)$ and $\delta(z)$ into Fourier integrals of the form

$$
\begin{align*}
& G(z)=\int_{-\infty}^{+\infty} d x G(x) \exp (\dot{i} x z), \\
& \delta(z)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} d x \exp (\dot{i} x z), \tag{I.4}
\end{align*}
$$

substituting these expressions in (I.3), and evaluating the resulting integrals with the help of the theory of residues on a complex plane ( $\varepsilon_{0}=\varepsilon_{0}^{\prime}+i \varepsilon_{0}^{\prime \prime}, \varepsilon_{0}^{\prime \prime}>0$ ), we obtain

$$
\begin{align*}
G(z) & =\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \frac{\exp (\dot{x} z z) d x}{x^{2}-k^{2} \varepsilon_{0}} \\
& =\frac{i}{2 k \varepsilon_{0}^{1 / 2}} \exp \left(i k \varepsilon_{0}^{1 / 2}|z|\right) . \tag{I.5}
\end{align*}
$$

2. We shall now show how to obtain (9) from (7). We divide the integral with respect to $z^{\prime}$ between $-\infty$ and $+\infty$ into two integrals, namely, one between 0 and $+\infty$ and the other between $-\infty$ and 0 . Each of these integrals is now split into an infinite sum of intergals over the period $a$ :

$$
\begin{align*}
& \int_{0}^{\infty} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}=\int_{0}^{a} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}+\int_{a}^{2 a} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}+\ldots \\
& (n+1) a \\
& +\int_{n a}^{n=\infty} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}+\ldots=\sum_{n=0}^{n} \int_{n a}^{-a} f\left(z^{\prime}\right) \mathrm{d} z^{\prime},  \tag{I.6}\\
& \int_{-\infty}^{0} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}=\int_{-a}^{0} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}+\int_{-2 a}^{-a} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}+\ldots \\
& -n a \\
& +\int_{n=+\infty}^{n a} f\left(z^{\prime}\right) \mathrm{d} z^{\prime}+\ldots=\sum_{n=0}^{n a} \int_{-(n+1) a} f\left(z^{\prime}\right) \mathrm{d} z^{\prime},
\end{align*}
$$

where $f\left(z^{\prime}\right)=\varepsilon_{p}\left(z^{\prime}\right) G\left(z-z^{\prime}\right) E\left(z^{\prime}\right)$. We shall show that since $\varepsilon_{p}\left(z^{\prime}\right)$ is periodic and Floquet's theorem (3) applies, we have

$$
\begin{aligned}
& \int_{n a}^{(n+1) a} \mathrm{~d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}\right) \\
& \quad=\exp (\dot{\psi} n a) \int_{0}^{a} \mathrm{~d} z^{\prime \prime} \varepsilon_{\mathrm{p}}\left(z^{\prime \prime}\right) E\left(z^{\prime \prime}\right) G\left(z-z^{\prime \prime}-n a\right), \\
& \int_{-(n+1) a}^{-n a} \mathrm{~d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}\right)=\exp (-\dot{\mu} n a) \\
& \quad \times \int_{-a}^{0} \mathrm{~d} z^{\prime \prime} \varepsilon_{\mathrm{p}}\left(z^{\prime \prime}\right) E\left(z^{\prime \prime}\right) G\left(z-z^{\prime \prime}+n a\right) .
\end{aligned}
$$

The first equation is obtained by substituting $z^{\prime}=z^{\prime \prime}+n a$ and the conditions

$$
\begin{aligned}
& \varepsilon_{\mathrm{p}}\left(z^{\prime}\right)=\varepsilon_{\mathrm{p}}\left(z^{\prime \prime}+n a\right)=\varepsilon_{\mathrm{p}}\left(z^{\prime \prime}\right) \\
& E\left(z^{\prime}\right)=E\left(z^{\prime \prime}+n a\right)=\exp (\dot{\mu} r t a) E\left(z^{\prime \prime}\right)
\end{aligned}
$$

In the second equation in (I.7), we can substitute $z^{\prime}=z^{\prime \prime}-n a$. Since

$$
\begin{aligned}
& \int_{-a}^{0} \mathrm{~d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}\right) \\
& \quad=\int_{0}^{a} \mathrm{~d} x \varepsilon_{\mathrm{p}}(x) E(x) \exp (-\dot{\psi} a) G(z-x+a),
\end{aligned}
$$

for $z^{\prime}=x-a$, we find that these relations yield

$$
\begin{align*}
& \int_{0}^{\infty} \mathrm{d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}\right) \\
& \quad=\sum_{n=0}^{n=+\infty} \exp (\dot{\mu} n a) \int_{0}^{a} \mathrm{~d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}-n a\right), \\
& \int_{-\infty}^{\infty} \mathrm{d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}\right)=\sum_{n=1}^{n=+\infty} \exp (-\dot{\psi} n a) \\
& \quad \times \int_{0}^{a} \mathrm{~d} z^{\prime} \varepsilon_{\mathrm{p}}\left(z^{\prime}\right) E\left(z^{\prime}\right) G\left(z-z^{\prime}+n a\right) . \tag{I.8}
\end{align*}
$$

This gives (9) with

$$
g\left(z-z^{\prime}\right)=\sum_{n=-\infty}^{n=+\infty} G\left(z-z^{\prime}-n a\right) \exp (\dot{\mu} n a)
$$

The expression for $g\left(z-z^{\prime}\right)$ in (10) is obtained as follows.
3. Consider the function

$$
g(z)=\sum_{n=-\infty}^{n=+\infty} G(z-n a) \exp (\dot{\mu} n a)
$$

By changing the variables in the sum, we can show that $g(z+a)=g(z) \exp (i \mu a)$. This means that function $f(z)=g(z) \exp (-i \mu z)$ is periodic, i.e., $f(z+a)=f(z)$. We now expand into a Fourier series:

$$
\begin{gather*}
f(z)=g(z) \exp (-i \mu z)=\sum_{s=-\infty}^{s=+\infty} f_{s} \exp \left(\frac{2 \pi i}{a} s z\right), \\
f_{s}=\frac{1}{a} \int_{0}^{a} \mathrm{~d} z f(z) \exp \left(\frac{-2 \pi i}{a} s z\right) . \tag{I.9}
\end{gather*}
$$

Transforming $f_{s}$ by substituting $f(z)=g(z) \exp (-i \mu z)$ and using

$$
g(z)=\sum_{n=-\infty}^{n=+\infty} G(z-n a) \exp (\dot{\psi} n a)
$$

we obtain for $z^{\prime}=z-n a$

$$
\begin{align*}
f_{s}= & \frac{1}{a} \sum_{n=-\infty}^{n=+\infty} \int_{0}^{a} \mathrm{~d} z G(z-n a) \\
& \times \exp [-i \mu(z-n a)] \exp \left(-\frac{2 \pi i}{a} s z\right) \\
& =\frac{1}{a} \sum_{n=-\infty}^{n=+\infty} \int_{-n a}^{a-n a} d z^{\prime} G\left(z^{\prime}\right) \exp \left(-i \beta_{s^{\prime}} z^{\prime}\right) \\
& =\frac{1}{a} \int_{-\infty}^{+\infty} \mathrm{d} z^{\prime} G\left(z^{\prime}\right) \exp \left(-i \beta_{s^{\prime}} z^{\prime}\right) \tag{I.10}
\end{align*}
$$

where

$$
\beta_{s}=\mu+\frac{2 \pi}{a} s, \exp (-2 \pi i n s)=1
$$

Substituting (I.10) in (I.9), we obtain

$$
\begin{align*}
g(z) & =f(z) \exp (i \mu z) \\
& =\frac{1}{a} \sum_{n=-\infty}^{n=+\infty} \exp \left(i \beta_{n^{2}}\right) \int_{-\infty}^{+\infty} d y G(y) \exp \left(-i \beta_{n} y\right), \tag{I.11}
\end{align*}
$$

where the summation index $s$ in (1.9) is replaced with $n$ and $z^{\prime}=y$. When

$$
G(y)=\frac{i}{2 \alpha} \exp (i \alpha|y|)
$$

in which $\alpha=k \varepsilon_{0}^{1 / 2}$, the last integral can be evaluated since $\varepsilon_{0}=\varepsilon_{0}^{\prime}+i \varepsilon_{0}^{\prime \prime}$ and $\varepsilon^{\prime \prime}>0$ :

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} d y G(y) \exp \left(-i \beta_{n} y\right)=\int_{0}^{\infty} d y G(y) \exp \left(-i \beta_{n} y\right) \\
& \quad+\int_{-\infty}^{0} d y G(y) \exp \left(-i \beta_{n} y\right)=\frac{(-1)}{\alpha^{2}-\beta_{n}^{2}}
\end{aligned}
$$

Substituting in (I.11), we obtain the required formula given by (10).
4. We shall now show that the integral equation (9) with $g\left(z-z^{\prime}\right)$ given by (10) leads to the Floquet-Bloch system (6). This can be done by substituting the following Fourier expansions in (9):

$$
\begin{align*}
& E\left(z^{\prime}\right)=\exp \left(i \mu z^{\prime}\right) \sum_{l=-\infty}^{l=+\infty} A_{l} \exp \left(\frac{2 \pi i}{a} l z\right), \\
& \varepsilon_{\mathrm{p}}\left(z^{\prime}\right)=\sum_{m=-\infty}^{m=+\infty} \varepsilon_{m}\left(1-\delta_{m 0}\right) \exp \left(\frac{2 \pi i}{a} m z^{\prime}\right),  \tag{I.12}\\
& g\left(z-z^{\prime}\right)=-\frac{1}{a} \sum_{n=-\infty}^{n=+\infty} \frac{\exp \left[i \beta_{n}\left(z-z^{\prime}\right)\right]}{k^{2} \varepsilon_{0}-\beta_{n}^{2}}, \\
& \beta_{n}=\mu+\frac{2 \pi}{a} n .
\end{align*}
$$

Canceling $\exp (i \mu z)$ and using the fact that

$$
\frac{1}{a} \int_{0}^{a} \mathrm{~d} z^{\prime} \exp \left[\frac{2 \pi i}{a}\left(m-n+l^{\prime}\right) z\right]=\delta_{n, m+l^{\prime}}
$$

we obtain

$$
\begin{align*}
& \sum_{l=-\infty}^{l=+\infty} A_{l} \exp \left(\frac{2 \pi i}{a} l z\right)=-k^{2} \sum_{m=-\infty}^{m=+\infty} \sum_{l^{\prime}=-\infty}^{l^{\prime}=+\infty} \varepsilon_{m}\left(1-\delta_{m 0}\right) \\
& \times \frac{A_{l^{\prime}} \exp \left[(2 \pi i / a)\left(m+l^{\prime}\right) z\right]}{k^{2} \varepsilon_{0}-\beta_{m+l^{\prime}}^{2}}
\end{align*}
$$

where

$$
\beta_{m+l^{\prime}}=\mu+\frac{2 \pi}{a}\left(m+l^{\prime}\right)
$$

We now change the summation indices so that $m+l^{\prime}=1$ and obtain

$$
\begin{equation*}
A_{l}=-\frac{k^{2}}{k^{2} \varepsilon_{0}-\beta_{l}^{2}} \sum_{m=-\infty}^{m=+\infty} \varepsilon_{m}\left(1-\delta_{m 0}\right) A_{l-m} \tag{I.14}
\end{equation*}
$$

Multiplying by $k^{2} \varepsilon_{0}-\beta_{l}^{2}$ and replacing $l-m$ with $m^{\prime}$, i.e., $m=l-m^{\prime}$, we obtain

$$
\begin{align*}
& {\left[k^{2} \varepsilon_{0}-\left(\mu+\frac{2 \pi}{a} l\right)^{2}\right] A_{l}} \\
& \quad=-k^{2} \sum_{m^{\prime}=-\infty}^{m^{\prime}=+\infty} \varepsilon_{l-m^{\prime}} A_{m^{\prime}}\left(1-\delta_{m^{\prime}}\right) \tag{I.15}
\end{align*}
$$

which is identical with the Floquet-Bloch equations (6).

## APPENDIX II. DERIVATION OF THE APPROXIMATE FORMULAS OF COUPLED-WAVE THEORY AND DISCUSSION OF THEIR VALIDITY

By differentiating ( $22^{\prime}$ ) with respect to $z$, we obtain the following first-order differential equations for $A^{( \pm)}(z)$ :

$$
\begin{align*}
& \frac{\mathrm{d} A^{(+)}(z)}{\mathrm{d} z}=\operatorname{be}(z) A^{(+)}(z) \\
& \quad+b z(z) A^{(-)}(z) \exp \left(-2 i k \varepsilon_{0}^{1 / 2} z\right)  \tag{II.1}\\
& \frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z}=-\dot{b e}(z) A^{(-)}(z) \\
& \quad-b z(z) A^{(+)}(z) \exp \left(2 i k \varepsilon_{0}^{1 / 2} z\right)
\end{align*}
$$

where $\chi(z)=k \varepsilon_{p}(z) / 2 \varepsilon_{0}^{1 / 2}$. It is readily seen that

$$
\begin{aligned}
& {\left[\frac{\mathrm{d} A^{(+)}(z)}{\mathrm{d} z} \exp \left(i k \varepsilon_{0}^{1 / 2} z\right)\right.} \\
& \left.\quad+\frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z} \exp \left(-i k \varepsilon_{0}^{1 / 2} z\right)\right]=0 .
\end{aligned}
$$

Hence, it follows from (II.1) that the field $E(z)$ given by (19) satisfies the wave equation (1). This confirms once again its equivalence to the integral equation (7).

When $\varepsilon_{p}(z)$ is a periodic function, and if we consider the neighborhood of the $n$th Bragg resonance, we can solve (22') or (II.1) by the averaging method, which yields a rapidly converging asymptotic perturbation-theory series. Actually, substituting (5) for $\varepsilon_{p}(z)$ in (22'), and separating out the term with the $n$th resonance, we obtain the exact set of equations

$$
\begin{align*}
& A^{(+)}(z)=\dot{x} \int_{n}^{z} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} A^{(-)}(z) \exp \left[(\alpha-2 i \delta) z^{\prime}\right] \\
& +\sum_{m=0}^{\prime} \dot{x} x_{-m} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} A^{(+)}\left(z^{\prime}\right) \exp \left(\frac{2 \pi i}{a} m z^{\prime}\right) \\
& +\sum_{m=(0, n)}^{\prime \prime} \dot{x} x_{m} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} A^{(-)}\left(z^{\prime}\right) \exp \left[\frac{2 \pi i}{a}(m-n) z^{\prime}\right. \\
& \left.+(a-2 i \delta) z^{\prime}\right] ; \\
& A^{(-)}(z)=i x_{-n} \int_{z}^{\infty} d z^{\prime} A^{(+)}\left(z^{\prime}\right) \exp \left[-(\alpha-2 i \delta) z^{\prime}\right] \\
& +\sum_{m \neq 0}^{\prime} \dot{x}_{m} \int_{z}^{\infty} \mathrm{d} z^{\prime} A^{(-)}\left(z^{\prime}\right) \exp \left(\frac{2 \pi i}{a} m z^{\prime}\right)  \tag{II.2}\\
& +\sum^{\prime \prime} \dot{x} x_{m}^{\infty} d z^{\prime} A^{(+)}\left(z^{\prime}\right) \\
& \times \exp \left[\frac{2 \pi i}{a}(m+n) z^{\prime}-(a-2 i \delta) z^{\prime}\right],
\end{align*}
$$

or the following exact differential equations equivalent to (II.1):

$$
\begin{align*}
& \frac{\mathrm{d} A^{(+)}(z)}{\mathrm{d} z}=x_{n} A^{(-)}(z) \exp [(\alpha-2 i \delta) z] \\
& \quad+A^{(+)}(z) \sum_{m \neq 0}^{\prime} x_{m} \exp \left(\frac{2 \pi i}{a} m z\right) \\
& \quad+A^{(-)}(z) \exp [(\alpha-2 i \delta) z] \sum_{m \neq(0, n)}^{\prime \prime} x_{m} \\
& \times \exp \left[\frac{2 \pi i}{a}(m-n) z\right], \\
& \frac{\mathrm{d} A^{(-)}(z)}{\mathrm{d} z}=-x_{-n^{\prime}} A^{(+)}(z) \exp [-(\alpha-2 i \delta) z] \\
& \quad-A^{(-)}(z) \sum_{m \neq 0}^{\prime} x_{m} \exp \left(\frac{2 \pi i}{a} m z\right) \\
& \quad-A^{(+)}(z) \exp [-(\alpha-2 i \delta) z] \\
& \times \sum_{m=(0,-n)}^{\prime \prime} x_{m} \exp \left[\frac{2 \pi i}{a}(m+n) z\right], \tag{II.3}
\end{align*}
$$

where $\varkappa_{m}=k \varepsilon_{m} / 2 \varepsilon_{0}^{1 / 2}, \varepsilon_{0}$ and $\varepsilon_{m}$ are complex quantities, and the sum over $m$ is evaluated between $-\infty$ and $+\infty$ with the exception of the special values $m \neq(0, \pm n)$.

If the functions $A^{( \pm)}(z)$ and $\exp [ \pm(\alpha-2 i \delta) z]$ vary slowly within the structure period $a$, i.e., if

$$
\begin{equation*}
\left\{\left|x_{n}\right| a,|\delta| a, \alpha a\right\} \ll 1 \tag{II.4}
\end{equation*}
$$

where $a=\pi n / k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=\lambda n / 2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}$, the solutions can be sought in the form of the asymptotic perturbation-theory series

$$
\begin{equation*}
A^{( \pm)}(z)=A_{0}^{( \pm)}(z)+A_{1}^{( \pm)}(z)+\ldots \tag{II.5}
\end{equation*}
$$

where the functions $A_{0}^{\left({ }^{ \pm)}\right.}(z)$ can be regarded as constants within the structure period $a$. We thus obtain the following approximate set of equations from (II.2):

$$
\begin{align*}
& A_{\delta}^{(+)}(z)=\dot{x x_{n}} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} A_{0}^{(-)}\left(z^{\prime}\right) \exp \left[(\alpha-2 i \delta) z^{\prime}\right]  \tag{II.6}\\
& A_{\delta^{(-)}}(z)=\dot{x} x_{-n} \int_{z}^{\infty} \mathrm{d} z^{\prime} A_{\delta}^{(+)}\left(z^{\prime}\right) \exp \left[-(\alpha-2 i \delta) z^{\prime}\right]
\end{align*}
$$

since the integrals of the rapidly oscillating functions in the sums over $m$ in (II.2) are all zero. Differentiating these two equations with respect to $z$, we obtain

$$
\begin{align*}
& \frac{\mathrm{d} A_{0}^{(+)}(z)}{\mathrm{d} z}=x_{n} A_{\delta}^{(-)}(z) \exp [(\alpha-2 i \delta) z]  \tag{II.7}\\
& \frac{\mathrm{d} A_{0}^{(-)}(z)}{\mathrm{d} z}=-x_{-n} A_{\delta}^{(+)}(z) \exp [-(\alpha-2 i \delta) z]
\end{align*}
$$

The same set of equations is obtained from (II.3) by averaging these equations over the structure period $a$. It is identical with the set of coupled-wave equations given by (22).

The higher-order approximations to the exact solutions $A^{( \pm)}(z)$ in (II.5) are most simply obtained from the exact set of integral equations given by (II.2). This is done by replacing $A^{( \pm)}(z)$ with $A_{0}^{( \pm)}(z)$ in the discarded second and third terms that contain sums over $m$.

This yields

$$
\begin{align*}
& A Y^{+}(z)=\sum_{m \neq(0, n)}^{\prime \prime} x_{m} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} A_{O^{\prime}}^{(-)}\left(z^{\prime}\right) \\
& \times \exp \left\{\frac{2 \pi i}{a}(m-n) z^{\prime}+(\alpha-2 i \delta) z^{\prime}\right\} \\
& \quad+\sum_{m \neq 0}^{\prime} x_{m} \int_{-\infty}^{z} \mathrm{~d} z^{\prime} A_{0}^{(+)}\left(z^{\prime}\right) \exp \left(\frac{2 \pi i}{a} z^{\prime} m\right),  \tag{II.8}\\
& A_{1}^{(-)}(z)=\sum_{m \neq(0,-n)}^{\prime \prime} x_{m} \int_{z}^{\infty} \mathrm{d} z^{\prime} A_{\delta}^{(+)}\left(z^{\prime}\right) \\
& \times \exp \left[\frac{2 \pi i}{a}(m+n) z^{\prime}-(a-2 i \delta) z^{\prime}\right] \\
& \quad+\sum_{m=0}^{\prime} x_{m} \int_{z}^{\infty} \mathrm{d} z^{\prime} A_{0}^{(-)}\left(z^{\prime}\right) \exp \left(\frac{2 \pi i}{a} m z^{\prime}\right) .
\end{align*}
$$

The convergence condition for the functional series (II.5) i.e., $\left|A_{1}^{( \pm)}(z)\right| \ll\left|A_{o}^{( \pm)}(z)\right|$, enables us to specify more precisely the original conditions (II.4) for the validity of the coupled-wave equations.

## APPENDIX III. DERIVATION OF BOUNDARY CONDITIONS AND OF THE EXPRESSIONS FOR $r_{\Sigma}$ IN THE GENERAL CASE OF UNMATCHED LAYER

## 1.Boundary conditions and their consequences

To solve the boundary-value problem, we must write down the solutions for $E(z)$ and $H(z)=(i / k) d E(z) / d z$ for $z \leqslant 0,0 \leqslant z \leqslant L$, and $z \geqslant L$, and then join them at $z=0$ and $z=L:$

$$
\begin{gather*}
z \leq 0: E(z)=\left[A_{0} \exp \left(i k n_{1} z\right)+A_{1} \exp \left(-i k n_{1} z\right)\right] \\
H(z)=-n_{1}\left[A_{0} \exp \left(i k n_{1} z\right)\right. \\
\left.-A_{1} \exp \left(-i k n_{1} z\right)\right]  \tag{III.1}\\
0 \leq z \leq L: E(z)=\frac{1}{\sqrt{n(z)}}\left[A^{(+)}(z) \exp (i \psi(z))\right. \\
\left.+A^{(-)}(z) \exp (-i \psi(z))\right] \\
H(z)=-\sqrt{n(z)}\left[A^{(+)}(z) \exp (i \psi(z))\right. \\
\left.\quad-A^{(-)}(z) \exp (-i \psi(z))\right] \\
z \geq L: E(z)=A_{2} \exp \left(i k n_{2} z\right) \\
H(z)=-n_{2} A_{2} \exp \left(i k n_{2} z\right)
\end{gather*}
$$

where

$$
\psi(z)=k \int_{0}^{z} n\left(z^{\prime}\right) d z^{\prime}
$$

and $n(z)=(\varepsilon(z))^{1 / 2}$ is a complex function.
The continuity condition for $E(z)$ and $H(z)$ at $z=0$ gives

$$
\begin{align*}
& A_{0}+A_{1}=\frac{1}{(n(0))^{1 / 2}}\left(A^{(+)}(0)+A^{(-)}(0)\right) \\
& n_{1}\left(A_{0}-A_{1}\right)=(n(0))^{1 / 2}  \tag{III.2}\\
& \quad \times\left(A^{(+)}(0)-A^{(-)}(0)\right)
\end{align*}
$$

Hence, we can readily show that

$$
\begin{align*}
& \left(A^{(+)}(0)+r_{1} A^{(-)}(0)\right)=\left(1+r_{1}\right) A_{0}(n(0))^{1 / 2}, \\
& r_{1}=\frac{n_{1}-n(0)}{n_{1}+n(0)}, \\
& r_{\Sigma}^{L}=\frac{A_{1}}{A_{0}}=\frac{r_{1} A^{(+)}(0)+A^{(-)}(0)}{A^{(+)}(0)+r_{1} A^{(-)}(0)}=\frac{r_{1}+\tilde{r}}{1+r_{1} \tilde{r}}, \\
& \tilde{r}=\frac{A^{(-)}(0)}{A^{(+)}(0)} . \tag{III.3}
\end{align*}
$$

The difference by $(n(0))^{1 / 2}$ in the first boundary condition is due to the different form of the solutions in the Kogelnik coupled-wave theory where

$$
E(z)=A^{(+)}(z) \exp \left(i \psi_{k}(z)\right)+A^{(-)}(z) \exp \left(-i \psi \psi_{k}(z)\right)
$$

for $\psi_{k}=k \varepsilon_{0}^{1 / 2}$ and a similar expression in the modified cou-pled-wave theory

$$
\begin{aligned}
& E(z)=\frac{A^{(+)}(z)}{(n(z))^{1 / 2}} \exp (i \psi(z)) \\
& \quad+\frac{A^{(-)}(z)}{(n(z))^{1 / 2}} \exp (-i \psi(z))
\end{aligned}
$$

where

$$
\psi(z)=k \int_{0}^{z} n\left(z^{\prime}\right) \mathrm{d} z^{\prime}
$$

It is clear that, in the last case, $A^{( \pm)}(z) /(n(z))^{1 / 2}$ are similar to the functions $A^{( \pm)}(z)$ in Kcgelnik's theory. The difference vanishes where there is a relationship between $A^{(-)}(z)$ and $A^{(+)}(z)$.

The continuity condition for $E(z)$ and $H(z)$ at $z=L$ gives

$$
\begin{align*}
& \frac{1}{(n(L))^{1 / 2}}\left[A^{(+)}(L) \exp (i \psi(L))\right. \\
& \left.+A^{(-)}(L) \exp (-i \psi(L))\right]=A_{2} \exp \left(i k n_{2} L\right),  \tag{III.4}\\
& (n(L))^{1 / 2}\left[A^{(+)}(L) \exp (i \psi(L))\right. \\
& \left.\quad-A^{(-)}(L) \exp (-i \psi(L))\right]=n_{2} A_{2} \exp \left(i k n_{2} L\right) .
\end{align*}
$$

Hence, the boundary condition at $z=L$ assumes the form

$$
\begin{align*}
& A^{(-)}(L)=r_{2} A^{(+)}(L) \exp (2 i \psi(L)), \\
& r_{2}=\left(n(L)-n_{2}\right) /\left(n(L)+n_{2}\right) . \tag{III.5}
\end{align*}
$$

## 2. Derivation of the formula for $\boldsymbol{R}_{\Sigma}^{\mathbf{2}}$

It follows from (23), which gives the expressions for $A^{( \pm)}(z)$, and from the formula for $\tilde{r}$ given by (III.3), that

$$
\begin{align*}
\tilde{r} & =\frac{A^{(-)}(0)}{A^{(+)}(0)} \\
& =\left(r_{\mathrm{B}}^{\infty}+\frac{x_{-n}}{x_{n}} \frac{1}{r_{\mathrm{B}}^{\infty}} \frac{C_{2}}{C_{1}}\right)\left(1+\frac{C_{2}}{C_{1}}\right)^{-1} \tag{III.6}
\end{align*}
$$

The ratio $C_{2} / C_{1}$ can be determined from the boundary condition given by (III.5) by substituting for $A^{( \pm)}(z L)$ from (23):

$$
\begin{align*}
\frac{C_{2}}{C_{1}} & =-\frac{x_{n}}{x_{-n}} r_{\mathrm{B}}^{\infty} \exp (-2 \gamma L) \\
& \times \frac{r_{\mathrm{B}}^{\infty}-r_{2} \exp \left(2 i \varphi_{n}(L)\right)}{1-\left(x_{n} / x_{-n}\right) r_{\mathrm{B}}^{\infty} r_{2} \exp \left(2 i \varphi_{n}(L)\right)}, \tag{III.7}
\end{align*}
$$

where

$$
\begin{aligned}
& \varphi_{n}(L)=\psi(L)-i\left(\frac{\alpha}{2}-i \delta\right)=\frac{\pi}{a} n L+\int_{0}^{L} x_{p}(z) \mathrm{d} z \\
& x_{p}(z)=k \varepsilon_{p}(z) / 2\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}, \varepsilon_{p}=\varepsilon(z)-\varepsilon_{0}
\end{aligned}
$$

and we have assumed the condition for the Bragg resonance $k\left(\varepsilon_{0}^{\prime}\right)^{1 / 2}=\pi / a(n+\delta)$ and that the losses were small ( $\varepsilon_{0}^{\prime} \ll \varepsilon_{0}^{\prime}$ ). When $r_{2}=0$ (layer matched at the rear), the expression for $C_{2} / C_{1}$ given by (III.7) becomes identical with (25) in the main text. Substituting (III.7) in (III.6), and rearranging, we obtain

$$
\begin{align*}
\tilde{r}= & \frac{A^{(-)}(0)}{A^{(+)}(0)} \\
= & \frac{\tau_{\mathrm{B}}^{L}+\frac{\exp (-2 \gamma L)-\left(x_{n} / x_{-n}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2}}{1-\left(x_{n} / x_{-n}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2} \exp (-2 \gamma L)} r_{2} \exp \left(2 i \varphi_{n}^{\prime}(L)\right)}{1+\left(-x_{n} / x_{-n}\right) r_{\mathrm{B}}^{L} r_{2} \exp \left(2 i \varphi_{n}(L)\right)}, \tag{III.8}
\end{align*}
$$

where

$$
\begin{align*}
r_{\mathrm{B}}^{L} & =r_{\mathrm{B}}^{\infty} \frac{1-\exp (-2 \gamma L)}{1-\left(x_{n} / x_{-n}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2} \exp (-2 \gamma l)}, \\
r_{\mathrm{B}}^{\infty} & =\frac{i x_{-n}}{\gamma+(\alpha / 2)-i \delta}, \tag{III.9}
\end{align*}
$$

and in the modified coupled-wave theory we must introduce the following replacements: $\quad x_{n}=-i x_{n}^{(-)}$, $x_{-n}=-i x_{-n}^{(+)}$, so that $x_{n} / x_{-n}=-x_{n}^{(-)} / x_{n}^{(+)}$and $x_{n} x_{-n}=\chi_{n}^{(-)} x_{-n}^{+}$.

For a layer matched at the rear, for which $\varepsilon(L)=\varepsilon_{2}$ we have $r_{2}=0$ in which case $\tilde{r}=r_{B}^{L}$, i.e., $\bar{r}$ becomes identical with the Bragg reflection coefficient in (26). For reflection by a semi-infinite periodic structure, we have $\tilde{r}=r_{B}^{\infty}$. Actually, when the absorption coefficient $\alpha$ is as small as desired, the quantity

$$
\gamma=\left[x_{n} x_{-n}+\left(\frac{\alpha}{2}-i \delta\right)^{2}\right]=\gamma^{\prime}+i \gamma^{\prime \prime}
$$

is always complex and

$$
\begin{aligned}
\gamma^{\prime} & =\operatorname{Re} \gamma=\frac{1}{\sqrt{2}}\left\{\left[\left(x^{2}-\delta^{2}+\frac{\alpha^{2}}{4}\right)^{2}\right.\right. \\
& \left.\left.+\alpha^{2} \delta^{2}\right]^{1 / 2}+\left(x^{2}-\delta^{2}+\frac{\alpha^{2}}{4}\right)\right\}
\end{aligned}
$$

is always positive. As $L \rightarrow \infty$ and $\exp (-2 \gamma L) \rightarrow 0$, we then obtain

$$
\begin{aligned}
\tilde{r}_{\infty} & =\lim _{L \rightarrow \infty} \tilde{r}= \\
& =\frac{r_{\mathrm{B}}^{\infty}-\left(x_{n} / x_{-n}\right)\left(r_{\mathrm{B}}^{\infty}\right)^{2} r_{2} \exp \left(2 i \varphi_{n}(\infty)\right)}{1-\left(x_{n} / x_{-n}\right) r_{\mathrm{B}}^{\infty} r_{2} \exp \left(2 i \varphi_{n}(\infty)\right)}=r_{\mathrm{B}}^{\infty}
\end{aligned}
$$

for any $r_{2}$.

## APPENDIX IV. THE COEFFICIENTS $\boldsymbol{R}_{\infty}$ and $\boldsymbol{R}_{L}$ in (49) and (55)

1. For an almost-periodic medium, we can substitute

$$
\begin{equation*}
A^{( \pm)}(z)=U^{( \pm)}(z) \exp \left(\mp i \frac{\sigma}{4} z^{2}\right) \tag{IV.1}
\end{equation*}
$$

in (47), so that these equations reduce to the form

$$
\begin{align*}
& \frac{\mathrm{d}^{2} U^{( \pm)}(z)}{\mathrm{d} z^{2}}+\left[\frac{\sigma^{2}}{4} z^{2}-x_{n} x^{x}-n\right. \\
& \left.\quad-\left(\frac{\alpha}{2}-i \delta\right)^{2} \mp i \frac{\sigma}{2}\right] U^{( \pm)}(z)=0 . \tag{IV.2}
\end{align*}
$$

The solutions of this equation take the form of the parabolic cylinder functions. ${ }^{42}$ When $\alpha=0$ and $\delta=0$, we can take the independent solutions to be the Whittaker functions

$$
\begin{aligned}
& V\left(i \rho \mp \frac{1}{2}, z \sigma^{1 / 2} \exp \left(-i \frac{\pi}{4}\right)\right), \\
& V\left(-i \varphi \pm \frac{1}{2}, z \sigma^{1 / 2} \exp \left(i \frac{\pi}{4}\right)\right),
\end{aligned}
$$

where $\rho=\left|\chi_{n}\right|^{2} / \sigma$, which for $z \rightarrow \infty$ take the form of waves propagating in the positive and negative directions of the $z$ axis. The general solution of (47) with (IV.1) taken into account can then be written in the form of the superposition of these waves, given by (48).

For a semi-infinite periodic medium for which the Bragg resonance, i.e., $\delta=0$, occurs directly on the separation boundary at $z=0$, we can put $C_{2}=0$ in (48). To show this, we must apply the boundary conditions (24) to the general solutions in (48), and then by introducing a small negative increment into $\rho$, pass to the limit as $L \rightarrow \infty$. If $C_{2}=0$, then (48) with (IV.1) taken into account immedi-
ately yields the expression for the reflection coefficient for a semi-infinite OPS:

$$
\begin{align*}
r_{\mathrm{B}}^{\infty}= & \frac{A^{(-)}(0)}{A^{(+)}(0)}=\frac{x_{-n}}{\sigma^{1 / 2}} \exp \left(i \frac{3 \pi}{4}\right) \\
& \times \frac{V(i \rho-(1 / 2), 0)}{V(i \rho+(1 / 2), 0)} \tag{IV.3}
\end{align*}
$$

Since, according to Ref. 42,

$$
\begin{equation*}
V(q, 0)=\sqrt{\pi} 2^{-\frac{1}{4}-\frac{q}{2}}\left(\Gamma\left(q+\frac{3}{4}\right)\right)^{-1} \tag{IV.4}
\end{equation*}
$$

and if we recall the properties of the gamma functions $\Gamma(x)$, we immediately obtain the formula given by (49) for the Bragg intensity relfection coefficient $R_{\infty}=\left|r_{B}^{\infty}\right|^{2}$.
2. The expressions for $R_{L}$ given by (55) and the relation given by (56) are obtained from (53) in the following way. If we introduce the dimensionless function $y=a_{F}^{2} / E_{c}^{2}$ and the dimensionless variable $x=2 \varkappa z$, where $E_{c}=(2 \varkappa / 3 \psi)^{1 / 2}$, then (53) and (54) yield

$$
\begin{align*}
& \frac{\mathrm{d} y(x)}{\mathrm{d} x}=\left\{\left(y(x)-J_{L}\right)\right. \\
& \left.\times\left[y(x)-\left(y(x)-J_{L}\right)\left(y(x)+\frac{\delta}{x}\right)^{2}\right]\right\}^{1 / 2}, \tag{IV.5}
\end{align*}
$$

where $J_{L}=\Gamma_{1} / E_{c}^{2} \equiv a_{F}^{2}(L) / E_{c}^{2}$. The solution of this equation can be expressed in terms of the Jacobi elliptic functions. ${ }^{42}$ In the simplest case $\delta=0$ and we introduce the new function $v(x)=y(x)-\left(J_{1} / 2\right)$ and the parameters $A^{2}=1+\left(J_{L} / 4\right)$ and $b^{2}=J_{L} / 4$, so that equation (IV.5) can be written in the form

$$
\begin{equation*}
\frac{\mathrm{d} v(x)}{\mathrm{d} x}=\left(v^{2}(x)-b^{2}\right)^{1 / 2}\left(a^{2}-v^{2}(x)\right)^{1 / 2} \tag{IV.6}
\end{equation*}
$$

Integrating this equation and transforming to the variables $y(z)$ and $z$, we obtain

$$
\begin{equation*}
y(z)=J_{L}\left[1+\operatorname{nd}\left(2 \operatorname{axs}(z-L) / a^{-2}\right)\right] / 2 \tag{IV.7}
\end{equation*}
$$

where $\mathrm{nd}(x / q)$ is one of the Jacobi functions ${ }^{42} 0 \leqslant q \leqslant 1$. Substituting $z=0$ in (IV.7), and introducing $J_{0} \equiv I=a_{\mathrm{F}}^{2}$ $(0) / E_{c}^{2}$, we finally obtain (56). Hence, it is readily shown that the reflection coefficient $R_{L}$ is given by (55).
${ }^{1}$ R. Krönig and W. G. Penney, Proc. R. Soc. London A13, 499 (1930).
${ }^{2}$ A. Ya. Shik, Fiz. Tekh. Poluprovodn. 8, 1841 (1974) [Sov. Phys. Semicond. 8, 1195 (1974)]; L. V. Golubev and E. I. Leonov, Superlattices [in Russian], M., 1977.
${ }^{3}$ A. P. Silin, Usp. Fiz. Nauk 147, 485 (1985) [Sov. Fiz. Usp. 28, 972 (1985) ]; E. A. Andryushin and A. A. Bykov, Sov. Fiz. Usp. 154, 122 (1988) [Sov. Fiz. Usp. 31, 68 (1988)].
${ }^{4}$ R. J. Collier, C. B. Burckhardt, and L. H. Lin, Optical Holography, Academic Press, N.Y., 1971 [Russ. transl., Mir, M., 1973].
${ }^{5}$ M. Miler, Holography: Theory, Experiment, and Applications [in Russian ], Mashinostroeniye, M., 1979.
${ }^{6}$ Yu. V. Gulyaev, V. V. Proklov, and G. N. Shkerdin, Usp. Fiz. Nauk 124, 61 (1978) [Sov. Fiz. Usp. 21, 29 (1978)].
'I. B. Yakovkin and D. V. Petrov, Diffraction of Light by Surface Acoustic Waves [in Russian], Nauka, Siberian Division, Novosibirsk, 1979.
${ }^{8}$ T. Tamir (ed.), Integrated Optics, Springer-Verlag, Berlin 1979 [Russ. transl., Mir, M., 1978].
${ }^{9}$ M. K. Barnoski, Fundamentals of Optical Fiber Communications, Academic Press, N.Y., 1976 [Russ. transl., Mir, M., 1980].
${ }^{10}$ R. G. Hunsperger, Integrated Optics: Theory and Technology, SpringerVerlag, Berlin, 1984 [Russ. transl., Mir, M., 1985].
"Z. G. Pinsker, Dynamical Scattering of X-rays in Crystals, Springer-Verlag, Berlin, 1978 [Russ. original, Nauka, M., 1974].
${ }^{12}$ A. N. Andreev, Usp. Fiz. Nauk 145, 113 (1985) [Sov. Fiz. Usp. 28, 70 (1985)].
${ }^{13}$ L. Brillouin, Wave Propagation in Periodic Structures, Dover, N.Y., 1953 [Russ. transl., IL, M., 1959].
${ }^{14}$ C. Elachi, Proc. IEEE 64, 1666 (1976) [TIIÉR 64(12), 22 (1976)].
${ }^{15}$ E. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, Cambridge, 1927 [Russ. transl., GIFML, M., 1963, Part 2, Chapter 19].
${ }^{16}$ M. I. Rabinovich and D. I. Trubetskov, Introduction to the Theory of Oscillations and Waves [in Russian ], Nauka, M., 1984, Chapter 11.
${ }^{17}$ M. Born and E. Wolf, Principles of Optics, Pergamon Press, Oxford, 1970 and 1980 [Russ. transl. of an earlier ed., Nauka, M., 1973].
${ }^{16}$ N. W. McLachlan, Theory and Application of Mathieu Functions, Clarendon Press, Oxford, 1947 [Russ. transl., IL, M., 1953].
${ }^{19}$ M. B. Vinogradova, O. V. Rudenko, and A. P. Sukhorukov, Theory of Waves [in Russian], Nauka, M., 1979, Chapter IV.
${ }^{20}$ T. Tamir, H. C. Wang, and A. A. Oliner IEEE Trans. MTT-12, 324 (1964).
${ }^{21}$ D. L. Jaggard and C. Elachi, J. Opt. Soc. Am. 66, 674 (1976).
${ }^{22}$ S. F. Su and T. K. Gaylord, ibid., 65, 59 (1975).
${ }^{23}$ B. Beularbi, D. J. Cooke, and L. Solimar, Optica Acta 27, 885 (1980).
${ }^{24}$ A. Yariv and Rochi Yeh, Optical Waves in Crystals, Wiley, N. Y., 1984 [Russ. transl., Mir, M., 1987].
${ }^{25}$ S. N. Stolyarov, Kratk. Soobshch. Fiz. (FIAN) No. 11, 12 (1987) [Sov. Phys. Lebedev Inst. Rep. No. 11 (1987)].
${ }^{26}$ T. K. Gaylord and M. G. Moharam, Proc. IEEE 73, 894 (1985). [TIIÉR, 73(5), 53 (1985)].
${ }^{27}$ R. B. Vaganov and B. Z. Katsenelenbaum, Fundamentals of the Theory of Diffraction [in Russian], Nauka, M., 1982, IV.
${ }^{28}$ S. Yu. Karpov, O. V. Konstantinov, and M. E. Raîkh, Fiz. Tverd. Tela (Leningrad) 22, 3402 (1980) [Sov. Phys. Solid State 22, 1991 (1980)].
${ }^{29}$ H. Kogelnik, Bell Syst. Tech. J. 48 (9), 2909 (1969).
${ }^{30}$ N. N. Martynov and S. N. Stolyarov, Kvant. Elektron. (Moscow) 5, 1853 (1978) [Sov. J. Quantum Electron. 1056 (1978)].
${ }^{31}$ N. N. Martynov, Kvant. Elektron. (Moscow) 6, 1798 (1979) [Sov. J. Quantum Electron. 9, 1062 (1979)].
${ }^{32}$ N. N. Martynov, Radiotekh. Elektron. 25, 1851 (1980) [Radio. Eng. Electron. Phys. (USSR) 25 (1980)].
${ }^{33}$ V. L. Ginzburg, The Propagation of Electromagnetic Waves in Plasmas, Pergamon Press, Oxford, 1970 [Russ. original, Nauka, M., 1967)].
${ }^{34}$ N. N. Bogoliubov and Yu. A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, N. Y., 1964 and 1985 [Russ. original, Nauka, M., 1963 and 1974].
${ }^{35}$ A. H. Nayfeh, Introduction to Perturbation Techniques, Wiley-Interscience, N. Y., 1981 [Russ. transl., Mir, M., 1984].
${ }^{36}$ G. A. Kom and T. M. Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, N. Y., 1961 and 1968 [Russ. transl., Nauka, M., 1968 and 1970].
${ }^{37}$ S. M. Rytov, Izv. Vyssh. Uchebn. Zaved., Fiz. No. 2, 223 (1937).
${ }^{38}$ V. I. Klyatskin and K. V. Koshel', Zh. Eksp. Teor. Fiz. 84, 2092 (1983) [Sov. Phys. JETP 57, 1217 (1983)].
${ }^{39}$ Yu. V. Zhilyaev, O. V. Konstantinov, and M. M. Panakhov, Fiz. Tverd.

Tela (Leningrad) 19, 1798 (1977) [Sov. Phys. Solid State 19, 1049 (1977)].
${ }^{40}$ S. A. Gurevich, S. Yu. Karpov, and E. L. Porshnoŭ, Pis'ma Zh. Tekh. Fiz. 10, 945 (1984) [Sov. Tech. Phys. Lett. 10, 396 (1984)].
${ }^{4}$ 'S. A. Gurevich, S. Yu. Karpov, and E. L. Porshnor, ibid., 11, 989 (1985) [11, 409 (1985)].
${ }^{42}$ M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, 1965 [Russ. original, Nauka, M., 1979].
${ }^{43}$ O. V. Konstantinov and M. E. Raikh, Zh. Tekh. Fiz. 49, 703 (1979) [Sov. Phys. Tech. Phys. 24, 408 (1979)].
${ }^{44}$ H. G. Winful, J. H. Marburger, and E. Garmire, Appl. Phys. 35, 379 (1979).
${ }^{45} \mathrm{Yu}$. Voloshchenko, Yu. N. Ryzhov, and V. E. Sotin, Zh. Tekh. Fiz. 51, 902 (1981) [Sov. Phys. Tech. Phys. 26, 541 (1981)].
${ }^{46}$ S. M. Wu and C. C. Shin, Phys. Rev. A 32, 3736 (1985).
${ }^{47}$ V. K. Ignatovich, Usp. Fiz. Nauk 150, 145 (1986) [Sov. Fiz. Usp. 29, 880 (1986)].
${ }^{48}$ V. A. Belyakov, Diffraction Optics of Periodic Media with Complex Structure [in Russian], Nauka, M., 1988.
${ }^{49}$ F. G. Bass, A. A. Bulgakov, and A. P. Tetervov, High-Frequency Properties of Semiconductors with Superlattices [in Russian], Nuaka, M., 1989.
${ }^{50}$ V. A. Belyakov and A. S. Sonin, Optics of Cholesteric Liquid Crystals [in Russian], Nauka, M., 1982.
${ }^{51}$ O. S. Eritsyan, Optics of Gyrotropic Media and Cholesteric Liquid Crystals [in Russian], Al̆astan, Erevan, 1988.
${ }^{52}$ V. A. Belyakov and V. E. Dmitrienko, Soviet Physics Reviews 13, part. 1 (1989).
${ }^{53}$ L. M. Brekhovskikh, Waves in Layered Media, Academic Press, N.Y., 1960 [Russ. original, Nauka, M., 1957 and 1973].
${ }^{54}$ S. N. Stolyarov, Bragg Wave Transformation in One-Dimensional Periodic Structures, Including Higher Perturbation-Theory Orders [in Russian], MFTI, M., 1989, pp. 37-41.
${ }^{5 s}$ S. N. Stolyarov, in Pulsed Lasers and their Applications [in Russian], MFTI, M., 1988, pp. 120-122.
${ }^{56}$ S. N. Stolyarov, Optical and Electronic Methods of Data Processing [in Russian], MFTI, M., 1990, pp. 85-90.
${ }^{57}$ B. M. Bolotovskiī, V. E. Volovel'skiĭ, N. N. Martynov, and S. N. Stolyarov, Preprint FIAN [in Russian], No. 101, M., 1989.
${ }^{58}$ B. M. Bolotovskiĭ and S. N. Stolyarov, Usp. Fiz. Nauk 159, 155 (1989) [Sov. Phys. Usp. 32, 813 (1989)].
${ }^{59}$ V. E. Volovel'skii, Thesis [in Russian], MGTU, M., 1990.
${ }^{60}$ N. N. Martynov, Thesis [in Russian], MFTI, M., 1979.
${ }^{61}$ S. N. Stolyarov, Kratk. Soobshch. Fiz. No. 6, 21 (1989) [Sov. Phys. Lebedev Inst. Rep. No. 6 (1987) ]; Kvant. Elektron. (Moscow) 15, 1637 (1988) [Sov. J. Quantum Electron. 15, 1021 (1988)].
${ }^{62}$ L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Pergamon Press, Oxford, 1977 [Russ. original, GIFML, M., 1963].

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