Finite-dimensional spatial disorder[1)]

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The article discusses a new branch of the theory of disorder: the origin, stability and diagnostics of deterministic spatial chaos, i.e., the disorder that can be generated by a dynamical system. Problems are posed, some of which are solved.

Literal translation: "Take care of order,
And disorder will take care of itself"
(A piece of advice for everyday use)

1. ON THE SUBJECT OF THE ARTICLE[2)]

When speaking about "disorder" even physicists have quite different associations in their minds. Some think of disordered arrangement of molecules of gas or liquid at an arbitrary moment of time, others, for example specialists in solid state physics, recall long-range order in magnetic domains, spin orientation, and so on. Being a complex, irregular distribution of different elements (or structures) in space, disorder is a quite traditional object of investigation that has been analysed by employing, in particular methods of statistical phsyics. We can refer our reader, for example, to the monographs of Refs. 1,2,3,4 including J. M. Ziman's brilliant book "Models of Disorder." The author of this book considers in terms of theoretical physics not only models for disordered crystal and nearly crystal structures but also solid–liquid and liquid–gas transitions and many other problems.

What we are planning to consider in this paper is, to a certain extent, a new branch of the theory of disorder. We are making attempts to describe irregular spatial field distributions, for instance, the distribution of electron density in crystals or density distribution in galaxies in terms of nonlinear dynamics. We are trying to understand whether spatially irregular distributions of physically meaningful fields may have a dynamical origin, in other words, whether there exist irregular distributions that can be described within dynamical models, for example, partial differential equations or a little more specific equations for dynamical systems with several "times" (here spatial coordinates stand for the "times").

Even the simplest spatio-temporal analogy shows that "finite-dimensional disorder," i.e., irregular spatial field distribution described by a dynamical system possessing a finite number of degrees of freedom must exist in Nature (and it must be no less typical than finite-dimensional temporal chaos). We can give many examples, in particular, stationary chaotic waves that are observed in different situations.[5,6] There arises a natural impulse to use results known from the theory of dynamical (temporal) chaos for the description of spatial dynamical disorder. This is trivial to a certain extent if we are concerned with one spatial coordinate and a static (or steady-state) regime. However, even in the case of one-dimensional disorder there arise very difficult questions. The main one is: Is such a finite-dimensional disorder evolutionary, i.e., does disorder evolve (in time) out of order remaining, at the same time, deterministic? When we think about two-dimensional, the more so about three-dimensional disorder of dynamical origin qualitatively new problems arise.

A pictorial example of how the difficulties are building up in a multidimensional geometry is the transition to chaos (or disorder) through quasiperiodicity. The "commensurability–incommensurability (quasiperiodicity)–chaos" transition is well known within one-dimensional differential equations. Is such a sequence possible with the variation of the governing parameter for a two- or three-dimensional field? What kind of dynamical system may describe such disorder? We have no definite answer to this question. However, such a transition is highly probable. The more so that its first stage—the "crystal–quasicrystal" transition—is currently being widely discussed.

It is well known that the main requirement on the dynamics that results in stochasticity is the instability of individual motions. A similar property must also be the basis for the description of spatial disorder. In order to formalize this property we must as a minimum introduce for the description of the space series notions such as dynamical system, phase space, trajectories, and the distance between them. Then irregular spatial patterns, that are also referred to as snapshots, will be described by notions such as stochastic set, correlation dimension, Kolmogorov entropy, ergodicity and others. We hope that this approach may help us understand the difference between complete disorder that is usually considered in thermal dynamics and finite-dimensional disorder. The definition of the term "finite-dimensional disorder" will be given below.

We would add to it that, physically, the formation of finite-dimensional disorder may be interpreted as emergence of irregular structures. This results from the breaking of the symmetry of the initial state, as in the case of ordered struc-

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tures. Such a symmetry breaking is possible both in strongly nonequilibrium media (e.g., hydrodynamic flows, autocatalytic chemical reactions, etc.) and in systems with thermodynamic equilibrium (e.g., disorder of atoms in crystals, of magnetic domains in ferromagnetics, of molecular axes in liquid crystals, etc.). When we speak about disorder that is established as $t \to \infty$, then both equilibrium and nonequilibrium media must be described by the same models. In this sense, disorder may be classified according to Ziman (see Fig. 1).

Most of the problems related to dynamical disorder have as yet no rigorous mathematical formulation. Therefore the aim of our article is to outline and formulate particularly attractive and promising problems as well as to bring together various examples that look optimistic.

Now a few words about the architecture of the paper. The problem of finite-dimensional disorder consists of three fundamental aspects. The first one: What is the reason for disorder of this type to be realized in nature? The second problem are methods for the description of finite-dimensional disorder. And, finally, the third problem is the "identification" of dynamical disorder. Imagine that we have a definite spatial picture: a snapshot of the field or density distribution of molecules on a polymer film. How shall we find out whether this picture is generated by a dynamical system and which equations may be used for its description? In order to answer the latter question it appears promising to generalize the approach proposed by Takens and other researchers for the treatment of the corresponding series, in our case space series, in terms of nonlinear dynamics. For this purpose we must, in particular, introduce the notions of the correlation dimension of the space series, embedding space, Lyapunov exponents, etc.

At present, it is still difficult to realize the proposed program completely, however we will attempt to answer these questions making the most of the knowledge we have today.

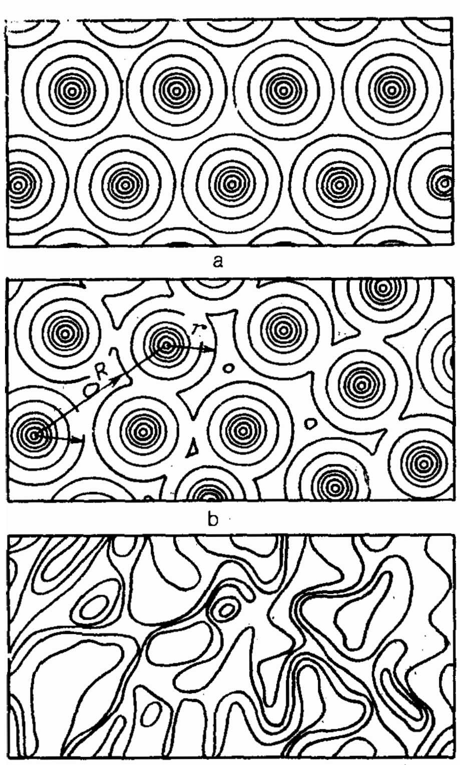

FIG. 1. a—Lattice order. b—Disorder of structures (topological disorder). c—Continuous disorder.

2. QUASICRYSTALS

We all have become accustomed to extremes: when speaking about order we inevitably think about a crystal and complete periodicity, while disorder is identified with a chaotic picture that allows only a statistical description. Therefore the discovery of quasicrystals in the middle of the eighties[7,8] was so unexpected and challenging. A typical feature of these media is the presence of several incommensurate harmonics in the spectrum of Bragg scattering, which is indicative of a quasiperiodic spatial structure. Analogous spatial pictures are also realized in some wave systems.[9] Strictly speaking, these are, of course, ordered media, but order is rather unusual here. In particular, the spatial length after which the picture repeats itself may be so large that externally this medium looks very much like a disordered one.[2)]

It is expedient to recall here the well-known Landau-Hopf model of turbulence within which the turbulence (at least near its excitation threshold) is a quasiperiodic motion with a large number of degrees of freedom, i.e., a quasiperiodic "winding" on a multidimensional torus.[10,11]

No significant difficulties occur when we have one spatial coordinate. We take a combination of two periodic structures and represent the field as a set of harmonics with incommensurate spatial periods. However, when we pass over to a two-dimensional problem there arise some serious questions: What will the cells of the periodic structure having such a discrete spectrum look like? In other words, how can one fill the plane, leaving no "holes" and preserving the spectrum as a set of a few incommensurate vectors? This question is quite nontrivial in terms of topology.

Figure 2 depicts a two-dimensional quasicrystal structure—Penrose tiling.[8] It possesses certain properties of order. Here, in particular, you may observe arbitrarily large fragments with fifth-order symmetry. It is also evident that this is a quasiperiodic structure: arbitrarily large portions of the structure are repeated at rather large distances in any direction. Note, finally, that this pattern possesses scaling symmetry: changing the scale by $\chi = (\sqrt{5} + 1)/2$ times (golden section) in the neighborhood of each structure we will be able to find a structure of the same shape but with the scale increased χ times (cf. the boldface stars in Fig. 2). The symmetry properties enumerated above can be found for arbitrary large spatial scales.

The existence of a quasicrystal is a precursor of dynamical (finite-dimensional) disorder. Actually, open "windings on a torus" in the phase space of a dynamical system are, as a rule, structurally unstable and the topology of the trajectories changes qualitatively with arbitrarily small parameter variations. Either a periodic (closed) trajectory or a stochastic set is formed. Spatio-temporal analogy indicates that the first case corresponds to a "crystal" while the second one, to spatial disorder.

It is only natural to suppose that there exist, between complete order and absolute disorder, numerous states of

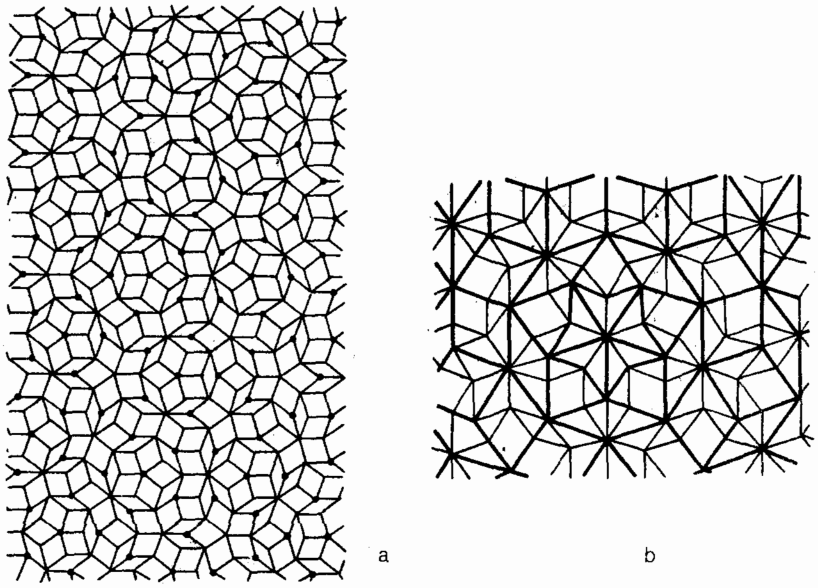

FIG. 2. Penrose tiling: a—location of atoms in a two-dimensional quasicrystal; b—example of self-similar structures.

medium or field possessing different degrees of order. Such a finite-dimensional disorder must be quite typical. Imagine that we have a governing parameter, for instance, the amplitude of external spatially periodic field. If this parameter is increased, then it is very likely that the transition from quasiperiodicity to finite-dimensional spatial disorder will occur, as in a one-dimensional case, because the open winding on the torus may break in a certain phase space.

No attempts have been made to solve such a two-dimensional problem, moreover no one has even formulated it yet. We believe that this problem of the generation of spatial disorder with an increase of the governing parameter will be solved in the near future, although, of course, it will need subtle mathematical notions when we pass over to two- or three-dimensional space. For example, it is necessary to introduce the notion of an open winding on a torus in some matrix space.

We believe that there is a simpler formulation of this problem. We know that many experiments, for instance on liquid crystals or in Bénard convection, reveal intriguing pictures of irregularly arranged defects. The defects themselves can often be fundamentally non-one-dimensional for example, "rosette"-like (see Fig. 3) but one can also observe less complicated, wave-type defects like those shown in Fig. 4. Such defects can be analysed within a nearly one-dimensional problem taking a few modes along one coordinate (e.g. in the case of magnetohydrodynamic convection in a narrow band of a liquid crysta) and considering the second coordinate to be unbounded. Then we obtain a system of ordinary differential equations along this unbounded direction. Following this route one can construct a periodic street of defects[3)] from a small number of modes. This was done by Eckmann and Procaccia[12] but it was not dynamical disorder yet. If we apply a controlled external periodic field (here we are concerned with an open winding on a torus in conventional (vector) phase space of a system of ordinary differential equations), we will obtain also spatial disorder of such defects. The picture obtained in this fashion looks very much like a two-dimensional one and resembles the picture observed in experiments on liquid crystals (Fig. 5).

There arises an interesting question: Why are simple crystal structures with a relatively simple symmetry so often encountered in the Nature with the decrease of temperature? Possibly, this is explained by thermal fluctuations that are substantial in the process of cooling and "force" the system out of numerous metastable states with a finite but narrow stability region. If this hypothesis is true, then, with a rather fast cooling, very different substances may end us in nontrivial quasiperiodic or "finite-ordered' states.

3. FINITE-DIMENSIONAL DISORDER. EXAMPLES AND DEFINITIONS

There are very many attributes to disorder: topological, continual, thermodynamical, etc. But these adjectives do not, actually, describe the essence of disorder, its quality, whether it is true or not.

Let us recall here the evolution of our concepts of temporal randomness. Discovery of the dynamical systems capable of generating the time series that look like true randomness was accompanied with many sceptical publications such as "Dynamical chaos—reality or fiction?" or "Can there exist randomness produced by a simple dynamical sys-

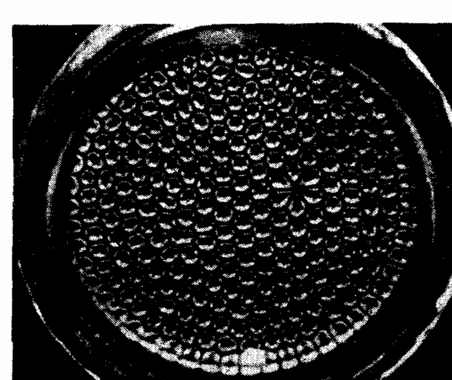

FIG. 3. "Rosette" defect against the background of hexagonal convective cells.

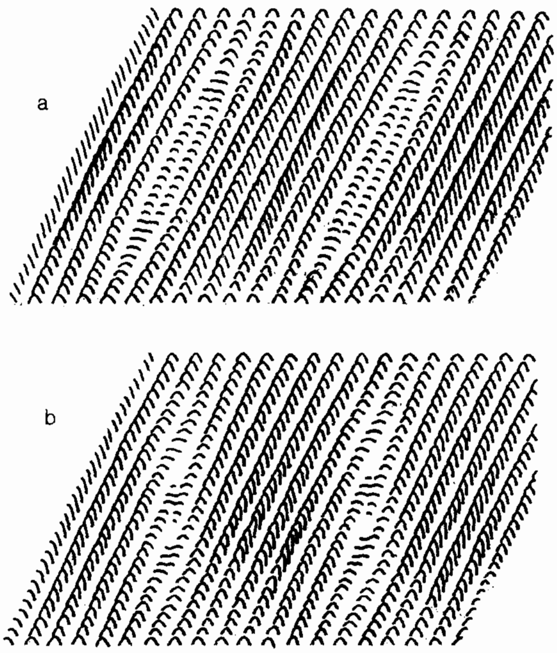

FIG. 4. "Wave" defects[12] produced by functions of the form: a—$\cos x + 1.3 \cos y \cdot \cos(\sqrt{1.26x})$; b—$\cos x + \cos y \cdot \cos(1.123x + 1.134y)$, for $y \in [-\delta/2, \delta/2]$. (Positive values of the functions are taken.)

tem?" It was asserted that "dynamical chaos is not true chaos." There existed a widespread opinion that real randomness has the following typical features: there are only absolutely unpredictable fluctuations i.e., there are no simple equations and no "laws of behavior." But the dynamical chaos follows rather simple rules and regularities. so there can be no true randomness."

Nevertheless, analysis of the time series, referred to as dynamical chaos, produced by employing definite laws or rules showed that the process looked absolutely random: decaying correlation function, continuous temporal Fourier spectrum. It was, actually, indistinguishable from a truly random process, for example, from pictures produced by shot noise or temperature pulsations in the atmosphere. A little later new characteristics of random series appeared, the most important among them being dimensionality (Lyapunov, Hausdorff or fractal dimensionality).[14]

It became clear that any random signal that is produced by following specified rules, i.e., that is generated by a dynamical system, has finite dimension. Sinaĭ referred to such signals as finite-dimensional randomness. While signals that are generated by fluctuations, i.e., truly random signals, typically have a very high or even infinite dimensionality. In this sense white noise has an infinite dimensionality.

Today we can easily remove the scepticism concerning the origin of dynamical randomness. Experimental results on hydrodynamic flows,[15] on the processing of electrocardiograms[16] and electroencephalograms[14] which appeared already in the beginning of the eighties showed that the random time series observed in experiments often have finite dimension and, consequently, may be produced by a dynamical system.

For a better understanding of the peculiarities of the disordered systems under discussion will now compare two photos presented in Figs. 6 and 7. One of them depicts an irregular structure produced by an avant-garde painter. The other photograph shows soap-suds (or, to be more exact, what we have when washing in a bath with shampoo). Of course, we could take many other examples, in particular, a picture of irregular domains in a thin magnetic garnet film.[17] All these pictures are, essentially, a great amount of cells of different kinds that are distributed, at first sight, absolutely at random. What is the difference between disorder in the two pictures? All classical characteristics will be the same for both photos if we perform a quantitative investigation by the usual methods of statistical physics for the description of random fields. In both cases we will have a continuous spatial Fourier spectrum, and the correlation function will decrease with distance. Nevertheless, it seems that the first picture must be close to the so-called speckle-noise[18] because in principle the circles on the canvas may be arranged absolutely at random. This will be true disorder, absolutely "irregular disorder."

The situation is quite different in the picture of bubbles. It is very likely that the spatial disorder observed in this case will have a different quality. Bubble formation follows definite regularities: they are closely connected with surface tension, some typical cluster structures are formed, and so on, the same is true for magnetic domains. Therefore there is every reason to believe that this disorder will be not completely irregular. This is an example of what we will define below more rigorously as finite-dimensional disorder. So as to make this definition more clear we will start with a spatio-temporal analogy and assume, for beginning, that the field is one-dimensional, i.e., it depends on one spatial coordinate.

It is instructive to recall here come peculiarities of the procedure for the reconstruction of the dynamical system characterizing a given stochastic series.[14,19] To be more exact, we mean here the reconstruction of a model dynamical system that is capable of generating the observed signal rather than the actual dynamical system that produces this series.

The reconstruction procedure is as follows. A contin-

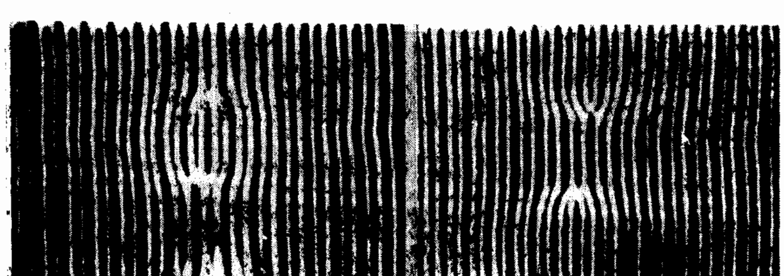

FIG. 5. Single "wave" defects observed for electrohydrodynamic convection in a liquid crystal.[13]

FIG. 6. Example of hand-made disorder (from the canvas of Mary Bovermeister 1964, The Whitney Museum of American Art. N.Y.).

uous series $u(x)$ is made discrete by choosing its values at the points $x_i: u_i = u(x_i)$ and then forming from close points a cluster

$$U = \{u(x_i), u(x_i + k), u(x_i + 2k), ..., u(x_i + k(m-1))\},$$

that determines the coordinates of an m-dimensional vector in some space. For convenience, this space will be referred to as phase space. Then the shift of the cluster along the trajectory (i.e., the variation of the subscript i) will correspond to the vector drawing in this space a set of points or a trajectory. Since we assume that the motion is steady in time, we have no preferential direction, i.e., we may move both along the increasing or the decreasing x-coordinate. The same is true for motion along the trajectory in our phase space.

The procedure for the reconstruction of the trajectory that is the image of our space series in m-dimensional "phase" space may have different details. But it should be emphasized that we reconstruct not the initial but a model system that gives a correct description only for the one trajectory corresponding to our space series. This will be of particular importance when we shall proceed to the description of two- and three-dimensional space series. This factor is often neglected.

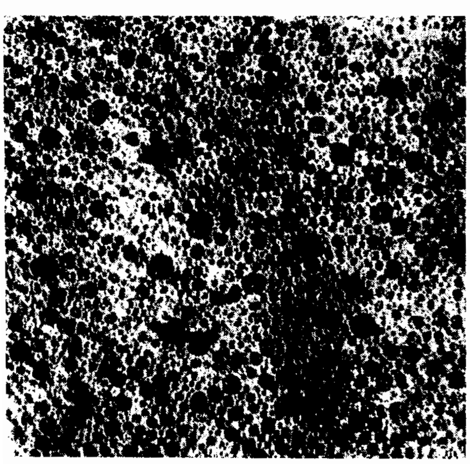

FIG. 7. Bubbles in soap-suds.

If, for instance, we have constructed a system of differential equations that reproduces an attractor of the same type and with the same properties as our space series, it does not at all mean that this sytem will give a correct description of transition processes. The transition processes in an initial dynamical system may be absolutely different, moreover, they may have a quite different dimension and even be infinite-dimensional. A typical example are low-dimensional hydrodynamic flows in a steady regime. In particular, experiments on the Couette-Taylor turbulent flow or on convection[14,15] demonstrated that immediately beyond the point of onset of turbulence as the characteristic parameter (the Taylor number or Rayleigh number) increases, the dimensionality grows up to the value of order 4–5. However, the corresponding model dynamical system that can, in principle, be reconstructed by the space series describes only the motion on a certain inertial manifold. We will speak about it in more detail. It was shown in the Refs. 20 and 21 that an infinite-dimensional (functional) space of an initial dynamical system (Navier-Stokes or Ginzburg-Landau equation) contains an inertial manifold the motion on which is described by a finite-dimensional dynamical system. This manifold may contain a strange attractor and other trajectories. If the observed trajectory belongs to a strange attractor, then one can reconstruct the dynamical system to the phase space of which this attractor belongs. This sytem generates a space series but it gives no information about the other trajectories of the original infinite-dimensional system.

We have already said that a spatio-temporal analogy holds for one-dimensional random spatial field distributions. This analogy is undoubtedly true for formal description using the theory of dynamical systems. But many new physical problems arise here. Indeed, let us take a "random" function of the x-coordinate whose statistical properties do not change when $-x$ is substituted for x. Experiments show that processing of such a space series gives a stochastic set with a fractional dimension! How can this happen? The change of sign of the x-coordinate does not change the properties of the space series, consequently, the properties of the dynamical system generating the space series must not change either. Such a dynamical system must be reversible along the coordinate, like a conservative system. Then, the dimension of the invariant stochastic set of our system may be some integrals less than the dimension of phase space, but it must be an integer in any event.

Here we come to a very interesting question. Suppose we have many snapshots of the space field that depends only on one coordinate. The snapshots are taken at different moments of time, like frames of a movie film. There arises a logical question: do evolutionary mature, i.e., established static spatial pictures actually differ from evolutionary "immature," i.e., intermediate (in time) pictures (literally, from snapshots)? Frankly speaking, we have no comprehensive answer to this question. But even now we can say that if this spatial distribution is "evolutionary mature," i.e., if it corresponds to the established as $t\to\infty$, attractor of, for example, a gradient system, then the dimension of the picture must be an integer, because the relevant dynamical system (that is obtained from initial equations for $\partial/\partial t=0$) is autonomous, conserves its phase space, and is reversible along the spatial coordinate. Otherwise, if the picture is only an "instant" in the evolution and the next moment of time will be different, then the dimension of the picture, indeed, may be fractional! Actually, the structure of a non-steady-state solution is determined not only by the interaction of fields at different points in space but also by the background processes. The terms with time derivatives in a non-steady-state equation play the role similar to that of an external force in a nonautonomous system. And a nonautonomous system may have a fractional dimension.

Let us now come back to the two-dimensional pictures that were considered above: soap films and hand-made disorder. How can we find out whether one or the other picture depicts speckle-noise, i.e., the analog of white noise in the time series, or is this a picture of finite-dimensional disorder?

Taking a snapshot unbounded in space we will determine the dynamical system that is capable, in principle, of generating it. This spatially unbounded snapshot will be called a single space series of the dynamical system. Since we have here at least two coordinates, the phase space in which the dynamical system evolves will, apparently, be different. In our preceding analysis we operated with a vector space in which each position of the image point corresponded to a definite position of the radius vector and the image point in phase space was specified by a set of numbers which were the values of our variables. We will follow the same route. But now each point will correspond to a matrix rather than to a vector. We will obtain a matrix space, which is rather conventional for a mathematician but not quite usual for a physicist. So physical intuition has not been developed here yet.

We will now introduce the notion of phase space for a two-dimensional space series $u(x,y)$ (for d-dimensional one analogously). As in a one-dimensional case, for a two-dimensional space series $u(x,y)$ we will make the snapshot discrete, i.e., we will represent it as a grid with the x_i, y_j-nodes and consider the values of the space series for each of them: $u_{i,j}=u(x_i,y_j)$. Each node has two indices, it has neighbors on the left and on the right, as well as on top and at the bottom. We will introduce a discrete finite cluster $A_{K,L}^{(m)}=\{(u_{i,j}),\ i=K,...,K+m-1,\ j=L,...,L+m-1\}$ that is determined by $(m\times m)$ points. By shifting this cluster (which is now not a set of numbers as in a one-dimensional case but a square matrix) along the space series we obtain a trajectory in a matrix space that can also be called phase space. Actually, in defining the notion of phase space it is essential to define the term "closeness" of states. To this end we must introduce the distance between the points mapping different states.

Using this approach, as in the case of one-dimensional systems, we can determine the correlation dimension and all the characteristics that were formerly used for the description of stochastic sets in an ordinary phase space: entropy, Lyapunov exponents, etc. If the stochastic set determined in this fashion has a finite dimension, then the snapshot of interest may be referred to as finite-dimensional disorder.

We will now consider this program for d-dimensional series using a formal mathematical language.[22]

Consider a set of continuous (vector) functions $u(x)$, $x\in\mathbf{R}^d$, $u\in\mathbf{R}^p$ employing conventional procedures of summation and scaling up. Introducing into this set a distance we obtain a metric space B that will be referred to as the phase space of the system. With each d-dimensional vector $\alpha=(\alpha_1,...,\alpha_d)\in\mathbf{R}^d$ we associate the translation map T^α: $B\to B$ that is determined by the expression $T^\alpha u(x)=u(x+\alpha)$. Thus we determine the action of the group $\mathbf{R}^d$ on B or, in other words, we have a dynamical system with d times that will be referred to as a translational dynamical system.

If the process under study is such that knowing the initial state (initial field distribution) one can unambiguously determine the subsequent states at any moment of time, then a semigroup of evolution operators $\{S^t\}_{t>0}$ also acts on B, i.e., an evolutional dynamical system is determined as well. The behavior of the trajectories of translational and evolutional dynamical systems in the common phase space B gives a full mathematical description of the spatio-temporal properties of the nonequilibrium medium of interest. Under the supposition of translation invariance, the characteristics of the snapshot $u(x)$ will, evidently, be independent of the coordinate system in $\mathbf{R}^d$. In other words, these characteristics must describe an invariant set of points along the trajectory of a translational dynamical system: $\{T^{\alpha} u(x)\}_{\alpha\in\mathbf{R}^d} \equiv \mathbf{A}_{u(x)}$.

Accordingly, the value $C(\mathbf{A}_{u(x)})$ [where $C(M)$ is the limiting capacity of the set M] will be referred to as the limiting capacity, or the fractal dimension of the snapshot $u(x)$. The Hausdorff dimension of the snapshot and other measure-independent characteristics are determined in a similar fashion. If a measure μ that is invariant with respect to T^{α} is defined on $\mathbf{A}_{u(x)}$, then the μ-dependent characteristics (e.g., pointwise or correlation dimensions) will also be referred to as the characteristics of the snapshot.

Note that if, for example, a two-dimensional snapshot is periodic along x_1 and x_2, then the set $\mathbf{A}_{u(x)}$ is merely a two-dimensional torus; if the snapshot has a quasi-periodically repeating structure, then $\mathbf{A}_{u(x)}$ is also a torus but now of a higher dimension; while for the patterns chaotically distributed over the plane, $\mathbf{A}_{u(x)}$ will be a fractal set. The time evolution of the snapshots (space series) corresponds to the motion of the set $\mathbf{A}_u$ in the space B: $\mathbf{A}_{u(x)} \overset{S'}{\to} A_{S'u(x)}$.

For the sake of simplicity, we will consider the space and time to be discrete by analogy with ordinary systems, i.e., we will take Z^d instead of $\mathbf{R}^d$ and $\mathbf{Z}_+$ instead of $\mathbf{R}_+$. Now we are in a position to give a rigorous definition of finite-dimensional disorder.

The snapshot $\mathbf{u} = \{u(j), j = (j_1,...,j_d) \in \mathbf{Z}^d\}$ will be called a finitely generated one if there exists 1) a dynamical system with d-dimensional time and finite phase space M, and 2) a Lipschitz-continuous, conjugate, one-to-one map h: $\mathbf{A}_u \to M$ such that the inverse map h^{-1} is also Lipschitz-continuous. We will provide the space of the "sequences" $B\{\mathbf{u}\}$ with the following norm

$$\|\mu\| = \sum_j |u(j)|/2^{|j|}, \quad \mathbf{u} = \{u(j), j \in \mathbf{Z}^d, u(j) \in \mathbf{R}^p\}, \tag{1}$$

where $|j| = |j_1| + ... + |j_d|$ and $|u(j)| = \sqrt{|u_1(j)|^2 + ... + |u_p(j)|^2}$.

It can be readily verified that B is a Banach space. Let for a fixed snapshot u the fractal dimension of the set $\mathbf{A}_u$ be finite: $C(\mathbf{A}_u) < \infty$ and $\mathbf{A}_u$ be a compact set. Let $m > 0$ be an integer such that $m^d \geqslant 2\cdot C(\mathbf{A}_u) + 1$. By C^d_m we will denote an integral d-dimensional cube with the side m, i.e., $C^d_m = \{j = (j_1,...,j_d) \in \mathbf{Z}^d: 0 \leqslant j_i \leqslant m\}$. Let M_m be the m^d-dimensional subspace of B (e.g., take $M_m = \{\mathbf{u} \in B: u(j) = 0 \text{ for } j \in \mathbf{Z}^d/C^d_m\}$). Let $\Pi_m: B \to M_m$ be a natural projection. According to the Mañe theorem, one-to-one (and bicontinuous) projections are typical for $B \to M_m$ on the set $\mathbf{A}_u$.

Assum[illegible]at Π_m is a typical natural projection. Then a dynamical system with d-dimensional time that is generated by the map $\widetilde{T}^{\alpha} = \Pi_m \cdot T^{\alpha} \cdot \Pi_m^{-1}, \alpha \in \mathbf{Z}^d$ is determined on the subpace $E = \Pi_m(\mathbf{A}_u)$ of set M and the snapshot u will be finitely generated, provided that $\Pi_m/\mathbf{A}_u)$ is a Lipschitz-continuous map.

Now, generalizing the algorithms presented in Ref. 24, we can propose algorithms for the calculation of correlation and pointwide dimensions of the snapshots. Let us take a two-dimensional snapshot u in the form of an array $\{u_{i,j}, i,j \in \mathbf{Z}_+\}$. In practice, the array, naturally, has a limited size: $i \leqslant N_1$, $j \leqslant N_2$, but N_1 and N_2 are supposed to be sufficiently large. For each integer $m \geqslant 1$ we will construct from the array $\{u(i,j)\}$ $(m\times m)$ matrices: $\mathbf{A}^{(m)}_{K,L} = \{(u_{k,l}), k = K,...,K+m-1, l = L,...,L+m-1\}$. Let us define the correlation integral in the form

$$C^{(m)}(\varepsilon) = \frac{R^{(m)}(\varepsilon)}{[(N_1 - m)(N_2 - m)]^2}, \tag{2}$$

$$R^{(m)}(\varepsilon) = \#\{((K, L), (K', L')):\mathrm{dist}(A^{(m)}_{K,L}, A^{(m)}_{K',L'}) \le \varepsilon\}, \tag{3}$$

where $\#(E)$ is the number of elements in the set E.

Then, for sufficiently small ε, the $\log C^{(m)}(\varepsilon)/\log\varepsilon$ ratio will be approximately equal to the correlation dimension D_s of the two-dimensional snapshot in m-dimensional embedding space.

Following Ref. 25 we will estimate the minimal size of the array $(u_{i,j})_{N_1\times N_2}$ that is needed for a correct calculation of the dimension on the interval $[\varepsilon_1, \varepsilon_2]$. Because

$$D_s \approx \frac{\log_2 C^{(m)}(\varepsilon'') - \log_2 C^{(m)}(\varepsilon')}{\log_2\varepsilon'' - \log_2\varepsilon'}, \tag{4}$$

where $C^{(m)}(\varepsilon') \geqslant \frac{1}{N_1^2 \cdot N_2^2} \cdot C^{(m)}(\varepsilon'') \leqslant 1$, then, assuming $\varepsilon'' = 2^k \cdot \varepsilon'$, we will estimate the dimension:

$$D_s \le \frac{2}{K}\log_2(N_1 N_2). \tag{5}$$

Note that for d-dimensional snapshots such an estimate has the form

$$D_s \le \frac{2}{K}\sum_{l=1}^{d} \log_2 N_l. \tag{6}$$

Thus, when determining the correlation dimension of a multidimensional snapshot one must bear in mind that the number of discretization points along each time coordinate may be much smaller than in the case of one-dimensional time.

Since the construction of the correlation integral needs a great number of calculations of the distance between the matrices, we save time and effort calculating the distance in the form:

$$\mathrm{dist}(A^{(m)}_{K,L}, A^{(m)}_{K',L'}) = \max\{|u_{k,l}| - |u_{k',l'}|),$$
$$k = K, ..., K+m-1, \quad k' = K', ..., K'+m-1, \tag{7}$$
$$l = L, ..., L+m-1; l' = L', ..., L'+m-1\}.$$

This method for determination of distances allows us to eliminate operations of multiplication, instead we compare the contents of the corresponding computer cells.

The number of calculations reduces significantly if all the matrices $A^{(m)}_{K,L}$ are compared to the array of reference matrices

$$\{A^{(m)}_{K_i L_j}, i \le i_{\mathrm{ref}}, j \le j_{\mathrm{ref}}\}, \tag{8}$$

where i_{ref} and j_{ref} are relatively small numbers. In this case the accuracy of calculation of the correlation dimension of a two-dimensional snapshot is determined by the estimate similar to (5):

$$D_s \le \frac{1}{K}\log_2(N_1 N_2 i_{\mathrm{ref}} j_{\mathrm{ref}}). \tag{9}$$

The testing of the algorithm (see Ref. 22) showed that the behavior of the correlation integrals does not, in fact, depend on turning the snapshot by an arbitrary angle, which indicates that the algorithm is robust. Figure 8 shows the plots of $\log_2 C^{(m)}$ against $\log_2 r$ (where $r = 2^{10}\cdot\varepsilon/\varepsilon_{\max}$) for a two-dimensional snapshot $U(x,y) = \sin(x)\cdot\sin(\sqrt{3}/2\cdot y)$ which corresponds to a two-dimensional torus in phase space. The correlation dimension was calculated to a good accuracy to be $D_s \in [1.96;2.03]$ even when $N_1,N_2 = 256$, $i_{\text{ref}},j_{\text{ref}} = 4$.

Thus we arrive at the following definition: finite-dimensional disorder is the disorder that can be considered as a trajectory of a certain dynamical system. In a one-dimensional case, this is an ordinary dynamical system, for example, a system of ordinary differential equations while in a two-dimensional problem we have a dynamical system with two times. Systems of this type have not, in fact, been considered in physics. Although it should be noted that a broad class of such systems (with two times) are, for example, Lie groups that have widely been used by physicists and there are a few good books on this topic.[26,27] Lie groups are used, as a rule, for the description of symmetries of different types and for the derivation of solutions, which follow from other solutions, by means of different group transforms. Here we set, in a sense, a converse goal, i.e., we will attempt to find irregular solutions.

So as to avoid ambiguity in definitions, we will make an important comment. The fractual dimensional of a space series calculated by the generalized Takens methods should not be confused with the fractal dimension of a two-dimensional picture that is produced, for example, by a chaotic trajectory of a particle in an alternating field.[28] In the first case, we mean the dimension of a stochastic set in the phase space of a translational dynamical system, while in the second case we are concerned with the dimension of the snapshot that can be "drawn," for instance, by one line densely filling certain regions on a plane. Generally, these two characteristics are irrelevant to one another. Actually, the fractal dimension, for example, of a "quasicrystal tiling" is the dimension of the net (boundaries of the cells), i.e., it is equal to unity. At the same time, such a picture corresponds in the phase space of a translational dynamical system to an open winding on an n-dimensional torus and the dimension of this set is equal to n. While the dimension of a fractual picture on a plane cannot be higher than two in any event.

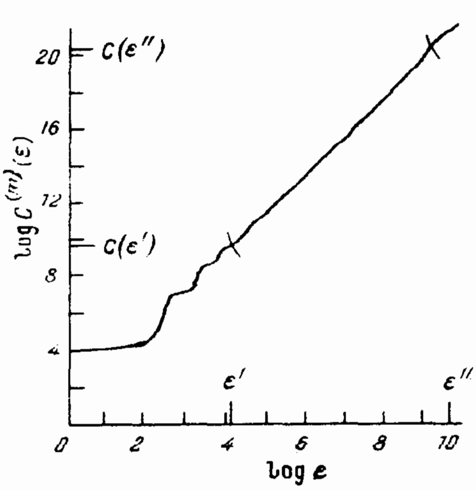

FIG. 8. $\log_2 C(\varepsilon)$ as a function of $\log_2 \varepsilon$ for a two-dimensional field $U(x,y) = \sin x \cdot \sin(\sqrt{3}/2y)$.

We would like to emphasize that while our disordered field is described by partial differential equations, dynamical systems with several times cannot describe the entire class of solutions of these equations but are capable of describing a definite particular class of solutions.

4. MODEL EQUATIONS

> "In what way must we treat equations of mathematical physics? Shall we simply draw conclusions from them and consider the equations as an imperceptible reality? No, not this way! Equations must teach us, primarily, what we can and what we should change in them."
>
> *Henri Poincaré*

What we see now in the theory of dynamical chaos and in the theory of finite-dimensional disorder that is now only taking shape reminds us of a well-known situation in the theory of critical phenomena and in other developed branches of nonlinear physics. Namely, many phenomena are being investigated by means of equations and models that do not appear, at first sight, to be relevant to the experiment described. For example, the basic notion in the theory of critical phenomena is the model Hamiltonian. Formally, it may be irrelevant to any particular physical situation. But this model is universal in that it describes effects typical for a broad class of critical phenomena. Universality and easy handling make model equations a good tool in the construction of a qualitative theory.

What do we understand as "qualitative theory?" When we are concerned with complex objects such as nonlinear fields, which are moreover varying in time, to obtain exact solutions, is merely a freak of chance rather than success or even a piece of luck. And we must be aware of the fact that we shall not often have such a chance. Therefore a qualitative theory is handy for a more or less self-consistent understanding of the phenomena in such a situation. The term "qualitative" does not indicate the absence of rigorousness, it has the same meaning as, for example, in the qualitative theory of differential equations.

Such a qualitative theory implies the combination of the following components. First, selection or construction of basic nonlinear models relevant to phenomena of different physical or other origins and, moreover, design of key experiments for verification of *a priori* hypotheses that these models are universal. As far as we know there is no other way to validate the models. We will adopt this approach in our investigation and analyze it on an example of the well-known Swift–Hohenberg model.[30]

The second component needed in the construction of a qualitative theory is the development of approximate methods, primarily, asymptotic ones using various small parameters. We would like to note here that a small parameter may not immediately enter basic equations, instead it may be determined by the properties of the solution. For example, it may be connected with a rather fast field decay at the periphery of some structures so that the interaction of different structures that is determined by the value of their potential in the maximal field of other structures may be considered as a small parameter.

Finally, the third component is, of course, computer experiment without which it is often impossible to put two and two together in a sufficiently complicated problem of finite-dimensional disorder. But it is to be not a mere finding of particular solutions, but rather the investigation of the phenomenon as a whole. We would like to underline here the word "phenomenon" and not system or model. We are interested not in the model *per se* but in the effects it studies. For example, we investigate the effect of stochastic scattering of particles or structures of soliton fields or the formation of finite-dimensional disorder as a result of temporal evolution and other effects. What kind of model do we choose for this purpose? Must it correspond exactly to a particular physical system or may it differ a little from this system? This is often of minor importance for the investigation of the characteristic features of the phenomenon if we are sure that it is typical enough. Moreover, with a broad variety of models, qualitatively different phenomena appear to be not so great in number. This allows us sometimes to "violate" the rigorous theory and to draw on some "truncated" models. If, however, we advance rather far in understanding the qualiative meaning of the processes within these models, we later might be able to complete them, if necessary, and obtain a more exact quantitative description. Actually, this is the manifestation of the difference between physical and formally mathematical approaches. A qualitative physical approach allows for definite conclusions even for problems that at present cannot be solved rigorously enough. Such problems are the ones most often encountered in practice.

We would like to add that the tendency to construct a qualitative theory is typical for fundamental branches of physics. In particular, R. Feynman[29] emphasized (may be a little too pompously): "The forthcoming great era of awakening of human intellect will bring about a method for the understanding of qualitative contents of equations. Today we are not able to do that. Today we cannot see in the equations for water flows things such as the spiral structure of turbulence... . We cannot say whether anything beyond the equations is needed."

Consider as an example illustrating the possibility of constructing a qualitative theory of spatial disorder (that is static when $t \to \infty$) a one-dimensional variety of the generalized Swift-Hohenberg equation (SHE):

$$\frac{\partial u}{\partial t} = \varepsilon u + \beta u^2 - u^3 + \left(k_0^2 + \frac{\partial^2}{\partial x^2}\right)^2 u, \qquad (10)$$

that describes ordinary supercriticality when $\varepsilon > 0$ and subcritical bifurcation when $\varepsilon < 0$. The parameter β determines the instability threshold for $\varepsilon < 0$. Only perturbations with finite amplitude

$$u \ge u_0 = \frac{\beta}{2} + \left(\frac{\beta^2}{4} - (\varepsilon - k_0^4)\right)^{1/2},$$

(where $1/k_0$ is the characteristic spatial scale of the field) increase. This equation is as natural and universal as, for example, the famous Van der Pol's equation in the theory of self-excited oscillations. The SHE is popular for the following reasons.

First of all, within certain approximations, this equation can be obtained from the Oberbeck-Boussinesq equation near the threshold of linear instability in the problem of Rayleigh-Bénard convection. This equation was derived in the papers of Refs. 30, 31 and in the review of Ref. 32. We must admit, that if we are absolutely precise and neglect no details, then in the derivation of the equation there appear additional terms, in particular, squared gradients or their derivatives, etc. which we have omitted. Mathematically, of course, this is not justified. But the Swift-Hohenberg equation describes such interesting and fine details of real experiments that it can be used for a qualitative description of phenomena in a sufficiently broad range.

Another, no less important circumstance is that the SHE possesses a number of remarkable properties and is easy to analyse. It is a gradient equation with a free energy functional (also referred to as the Lyapunov functional):

$$F = \int_\Omega \left[-\frac{\varepsilon}{2}u^2 - \frac{\beta}{3}u^3 + \frac{u^4}{4} + \frac{1}{2}\left(\left(K^{0^2} + \frac{\partial^2}{\partial x^2}\right)u\right)^2 \right] dx. \qquad (11)$$

It may be represented in a gradient form

$$\frac{\partial u}{\partial t} = -\frac{\delta F}{\delta u}. \qquad (12)$$

Because the functional F can only decrease along the trajectory:

$$\frac{dF}{dt} = -\int \left(\frac{\partial u}{\partial t}\right)^2 dx \le 0, \qquad (13)$$

the gradient system is able to demonstrate behavior only of two types at $t \to \infty$. If the free energy functional has no minima, then in a large-box problem the front propagation will be observed as, for example, in the equations describing the reaction of burning. In this case, the free energy functional will be continuosly decreasing until the front approaches the boundary of the medium if it is bounded. An alternative possibility is realized when the free energy functional has minima. There may be many such minima. Each minimum corresponds to an equilibium state in time (multistability). Thus, the limit behavior for gradient systems is always either a static attractor or propagating fronts. We will consider the case when the free energy functional has minima. Spatial field distributions corresponding to different minima may qualitatively differ from one another: they may be periodic or quasiperiodic, they may also include many states of the type of stable finite-dimensional disorder which will be described in the next section.

Consider only one example illustrating the potentialities of SHE for the description of a real situation, for instance, in experiments on Rayleigh-Bénard convection. Computer investigation of the Swift-Hohenberg equation with different initial conditions yields in a two-dimensional geometry, for $\varepsilon \geqslant 0$ and $\beta > \beta_{cr}$, hexagonal structures like the ones shown in Fig. 9 (Ref. 33). The following interesting phenomenon was observed in these experiments. If a single hexagonal cell is taken as the initial condition, it rapidly "accumulates" neighbors and then the entire "crystal lattice" gradually builds up. It is remarkable that exactly the same structural growth of a crystal hexagonal lattice was observed in the experiments on real convection performed by Ahlers and his group.[34] The pictures on a computer display and those observed in a laboratory experiment (see Fig. 10 a and b) coincide completely! Moreover, the same coinci-

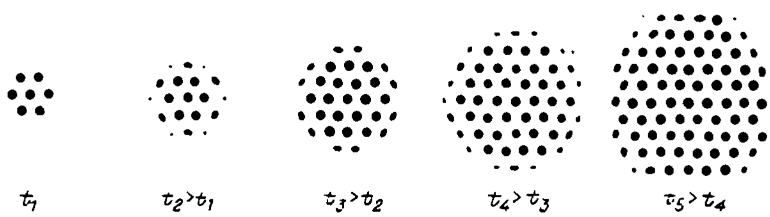

FIG. 9. Growth of hexagonal crystal structure described by the two-dimensional Swift-Hohenberg equation[33] ($\varepsilon>0$).

dence occurs when the medium is close to the excitation threshold of convection but is a little lower than this threshold (this corresponds to the case of subcritical bifurcation). A finite initial perturbation is needed to pass across the excitation threshold. Computer experiment shows that this perturbation grows, transforms into a universal structure and exists as a localized state. This phenomenon was also observed by Ahlers and his group[34] (see Fig. 10d).

We believe that even these briefly outlined examples will convince the reader that the Swift-Hohenberg equation is not merely a convenient and popular model but it does describe real physical processes, at least, for Rayleigh-Bénard convection near the linear instability threshold.

Of course, the Swift-Hohenberg equation will not be the only model in our research. We considered it here as an illustration of the efficiency of model approach. No less significant and, perhaps, even more universal (because it describes non-steady-state processes as $t\to\infty$ is the complex Ginzburg-Landau equation (CGLE):

$$\partial a/\partial t = Ra - (1+i\beta)a|a|^2 + (1+i\alpha)\Delta a, \qquad (14)$$

and its various generalizations (see, e.g., Ref. 32). This equation is often cited in the literature, perhaps, because it can be obtained using a standard procedure as an asymptotic approximation of initial equations in different branches of physics and not only physics.

As distinct from the SHE that describes the field itself, the CGLE refers to the class of the so-called amplitude equations. In particular, substituting into the Swift-Hohenberg equation $u = a\cdot\exp(i\mathbf{kr})$ yields the Newell-Whitehead-Segel equation[35]:

$$\partial a/\partial t = a + [(\partial/\partial x - (i/2k_0)\partial^2/\partial y^2]^2 a - a|a|^2, \qquad (15)$$

that is a particular case of the CGLE with real coefficients. The CGLE is a gradient equation only when it has real coefficients: $\alpha=\beta=0$. And it is completely integrable in the converse limiting case, $\alpha,\beta\to\infty$, when it transforms into the well-known nonlinear Schrödinger equation.

5. THE EVOLUTION OF ONE-DIMENSIONAL DISORDER

When we speak about evolution we are interested in what happens to the object under study in time. A typical example are discontinuities in gas dynamics. There exists a great variety of discontinuities but not all of them are evolutionary, i.e., not all of them result from the transformation of

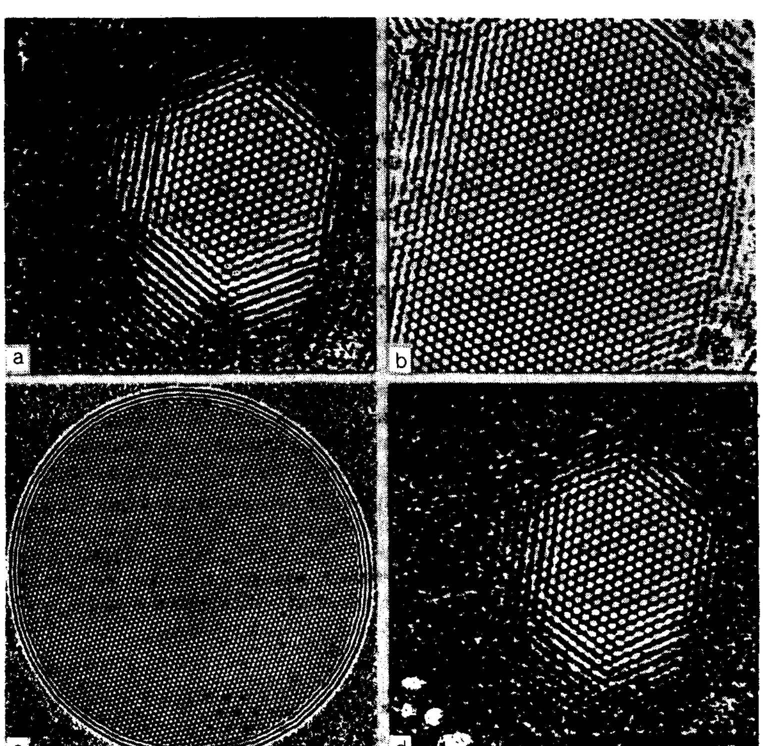

FIG. 10. Thermoconvection in a thin layer of gaseous CO_2 (Ref. 34): a,b—growth of hexagonal crystal lattice for $\varepsilon>0$ (b corresponds to later time); c—completely regular lattice; d—stable localized state for $\varepsilon<0$ ($\varepsilon = -1.92\cdot10^{-3}$).

a simple wave. A similar question arises about spatial disorder. We have "good," i.e., simple enough regular initial field distributions. Are regular initial conditions, for example, periodic along two directions likely to give rise to disorder? It is a very difficult and unconventional question. Researchers consider, as a rule, the problem of generation of order from disorder. It might seem to be a quite independent issue. Actually, these two problems are closely connected.

Let us take a completely random initial distribution—inifinte-dimensional disorder. How will it evolve? Will it also transform into finite-dimensional disorder? Researchers have always been interested in the limiting case of this problem: how does a well-organized order (crystal order, structures, etc.) evolve from initial disorder? This is a problem of self-organization. Here we are concerned with significantly more complicated—stochastic objects which are self-organized from completely random initial conditions. Does there emerge a disorder, not completely irregular but of dynamical origin? We will make an attempt (to the best of our ability) to answer these questions.

An interesting result was obtained in computer experiment with CGLE in a very long one-dimensional system.[36] A random number generator was used to produce initial disorder and then the spatial dimension, D_s, was measured as a function of time. The dimension first descreased rapidly and then acquired a constant value (Fig. 11). When initial conditions were specified to be nearly sinusoidal, which corresponds to $D_s \approx 1$, the dimension acquired, after a transition process, the same constant value! The regime of spatio-temporal chaos whose snapshots are finite-dimensional disorder with a definite value of dimension is established for quite different initial conditions.

It is a striking result. Of course, we find nothing remarkable in the emergence of a stable periodic structure—"crystal"—because we have gotten accustomed to that. But now, when something has settled in an unbounded space where we produced complete disorder, why must this "something" be finite-dimensional? The number of degrees of freedom must be infinite in an unbounded medium and it might appear that individual distant regions do not affect one another. At first sight, it looks more natural that disorder will be infinite-dimensional and its properties will be determined merely be temperature, as in thermodynamics; besides there must be no connection between the fragments spaced far apart. Nevertheless, we have established a fascinating fact: the disorder that emerges in an unbounded system can be described within a finite-dimensional dynamical

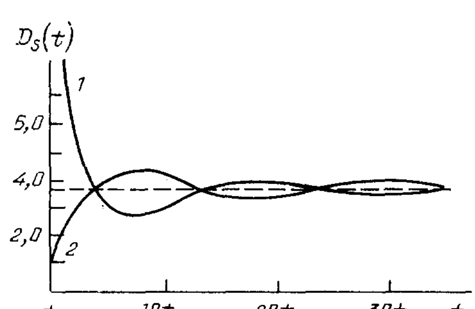

FIG. 11. Time dependence of the correlation dimension, D_s, of the space series for CGLE under chaotic (*1*) and periodic (*2*) initial conditions.[36]

system, for example, within a finite number of differential equations.

Now we come back to convection. We take a large-box system with convection and consider a situation when the supercriticality is a little higher than the excitation threshold of a purely periodic ("crystal") convective structure. First, we will make the system chaotic, i.e., we will produce absolutely arbitrary initial conditions, for example, by means of usual mechanical mixing. The resulting picture will be complete disorder as in Fig. 12a. How will this disorder change in the evolution of the system? The answer is not at all clear. In the simplest case, a regular hexagonal structure may appear as the one in Ahlers' experiments (see Fig. 10c). But if the supercriticality is sufficiently large, such a lattice will, apparently, be unstable and more complicated non-steady-state regimes will emerge, including trubulent convection. Disorder sets in this case too. But it is still unclear whether it will be finite-dimensional disorder or not.

Consider now a simpler situation. Assume that the supercriticality is small and, as time grows, a disordered picture like the one depicted in Fig. 12c,d (Ref. 37) is established. We can see "crystal" domains separated by boundaries ("cracks"). This disorder into which the original irregular picture, extraordinary in its irregularity, has transformed is undoubtedly a finite-dimensional one. Computer and laboratory experiments suggest that finite-dimensional disorder has, to a certain extent, a higher or a little more complicated degree of order and like order itself may emerge from background chaos as $t \to \infty$. It is an example of self-organization of stochastic structures.

Regular initial conditions is an alternative limiting case. If the initial conditions are absolutely regular and there are no perturbations in the system, nothing new will appear of course. But this is not a physical formulation of the problem. In practice we must always consider some small volume rather than a point in phase space. This means that we must analyse an ensemble of similar initial conditions, for example, of the type $\sin kx + \mu\psi(x)$, where $\mu \ll 1$. Such an experiment using for instance, model (10), gives an amazing result. The initial conditions evolve into finite-dimensional disorder.[38]

In this connection consider a mechanism of the generation of spatial disorder through the emergence of localized static states of the field. This mechanism was discovered in Ref. 38 and seems to be quite typical. It is realized, not only in the Swift-Hohenberg model and is essentially as follows. A nearly harmonic initial field distribution evolves into a chain of "particles"—solitons—which diverge to an arbitrary distance from one another, because of instability, and form, as $t \to \infty$, a static irregular sequence.

The existence of finite-dimensional disorder within the Swift-Hohenberg model is almost an evident fact. Indeed, because it is a gradient system, only static attractors may exist in its phase space. The number of attractors may be arbitrary large, as can be easily verified. All static solutions, that are established as $t \to \infty$ satisfy an ordinary fourth order equation along the x-coordinate, that can be derived from (10):

$$u_{xxxx} + 2k_0^2 u_{xx} + f(u) = 0, \tag{16}$$

where $f(u) = (k_0^4 + \varepsilon)u - \beta u^2 + u^3$.

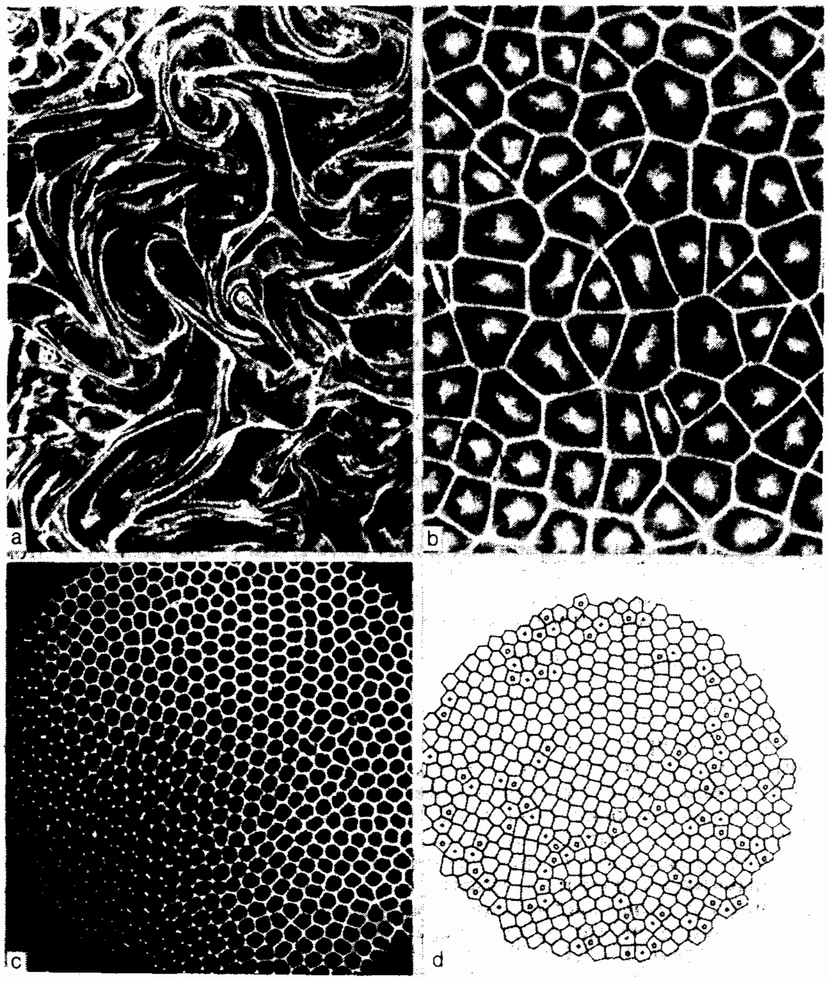

FIG. 12. a) Mixed initial state of a liquid convective layer. b) "Half-ordered" lattice observed against the background of convective crystal structure. c) "Domain" disorder in Marangoni convection. d) Reconstruction of "domain" disorder.[37]

A homoclinic structure (see, e.g., Ref. 39) as well as periodic and quasiperiodic trajectories exist in the phase space of this dynamical system in a broad range of parameters. The homoclinic structures themselves tend to equilibrium states as $x \to \pm \infty$ and, in this sense, they are not an example of finite-dimensional disorder. However, the homoclinic structure contains also a continuous set of nearly homoclinic open finite trajectories which correspond to chaotic spatial distribution of the field.

Naturally, not all these trajectories are attractors for the evolution system (11)–(12). The trajectories that *are* attractors, evidently, have intricately intertwining basins of attraction in the space of initial conditions. Therefore it is very difficult to predict the finite state of the field described by (10) even under trivial initial conditions. Representing the initial regular field state as a point in the phase space of the gradient system of interest, we can formulate the problem in the form of a simple question. Is this point contained in the attraction basin of the attractor corresponding to disordered field distribution for $t \to \infty$? But we must speak about a set of close initial field distributions rather than about a concrete distribution. This set corresponds not to a point in phase space but, instead, to some initial phase volume with the characteritic size μ. Then the question will have a different formulation: Will an irregular field state whose statistical characteristics do not depend on μ (including $\mu \to 0$) be established $t \to \infty$? A positive answer to this question was obtained in a numerical experiment.

We would like to emphasize the following. The presence of a set of homoclinic and nearly homoclinic complex trajectories in the phase space of the system describing static solutions does not at all indicate that they are stable (or, to be more exact, that they may be attractors). An answer to this question may be found by considering the evolution of *different*(!) initial distributions. An approach to the solution of this problem is proposed in Ref. 38 and is, essentially, as follows.

Computer experiments show that the evolution of a smooth initial perturbation is divided into two stages (see Fig. 13a). The first (fast) one is completed by the formation of a chain of localized states the number of which per unit length is determined by the period of the initial perturbation. These solitons have decaying oscillatory "tails" (Fig. 13b), which is important for our furhter consideration. The second (slow) stage ends by the onset of a static chain of periodic or quasiperiodic structures. The second, slow stage of the evolution is determined by the stability of the soliton chain that has been formed at the first stage.

Naturally, initial distributions with different periods must correspond to different steady distributions—the effect of multistability. Because of instability, the neighboring solitons may either move "one minimum" apart or draw nearer to one another by the same "unit length" or their tails. Then it seems evident that the degree of disorder may

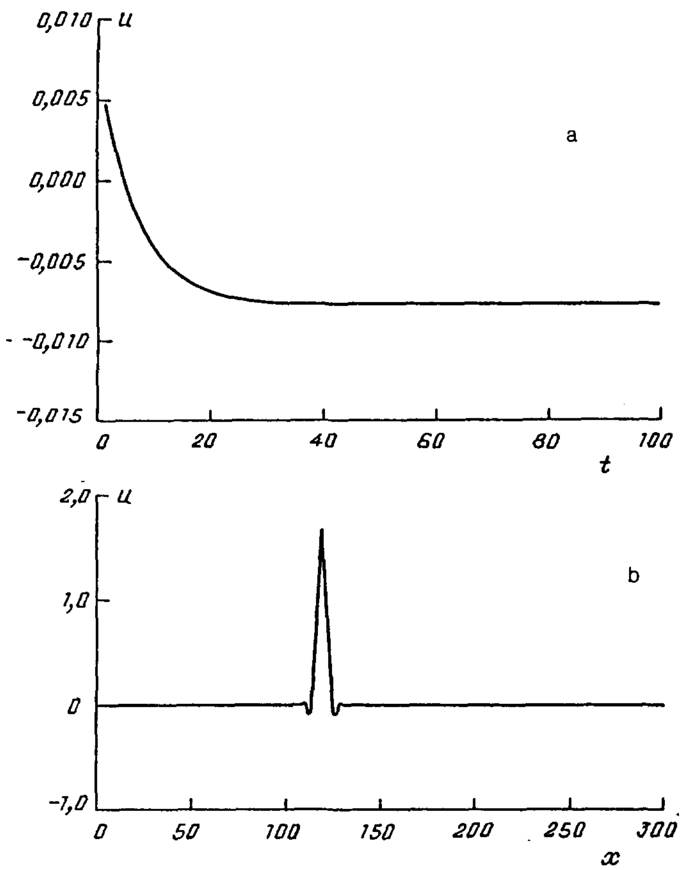

FIG. 13. a) Evolution of free energy in model (10) under initial conditions of the form $\sin kx + \mu\psi(x)$ as a function of time: $\beta = 2.3, \varepsilon = 0.3, k_0 = 0.5$. b) Distribution of the field of a localized state for the same parameters.

only *decrease*(!) with the growth of the period of the initial periodic distribution. This guess was wonderfully confirmed by a direct computer experiment (Fig. 14): the Kolmogorov entropy decreases and tends to zero as l grows.

These phenomena may be described as follows. If the distances between the neighboring solitons are not too small (equal to or larger than their characteristic size), the asymptotic method[40] allows us to describe soliton motion as the dynamics of a chain of particles with a specific interaction potential that is determined by the structure of the "tails" of localized states:[4)]

$$M\frac{dx_i}{dt} = \frac{\partial}{\partial x_i}u. \tag{17}$$

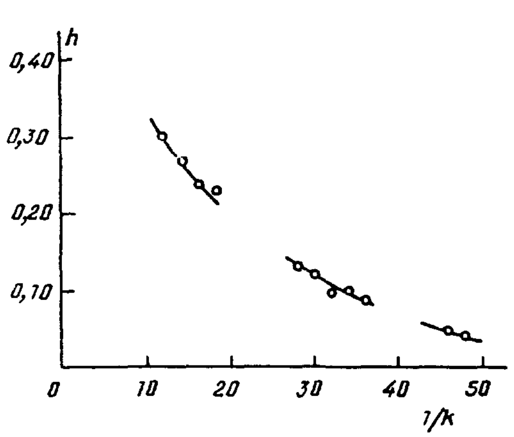

FIG. 14. Dependence of Kolmogorov–Sinaĭ entropy, h, for the space series (described by (10) at $t\to\infty$) on the period of initial distribution $l = 2\pi/k$.

Here x_i is the coordinate of the center of the ith soliton, $M = \int (u^{(0)})^2 dx$ is the mobility of the soliton having the structure $u(x)$ (see Fig. 13b), and u_i is the potential produced by all solitons except the ith one at the point x_i:

$$u_i = \sum_{j\neq i} e^{-\vartheta|x_j - x_i|}\cos(\varkappa|x_j - x_i| + \varphi_0), \tag{18}$$

where $\upsilon = \mathrm{Re}\sqrt{i - k_0^2}$, $\varkappa = \mathrm{Im}\sqrt{i - k_0^2}$, and φ_0 is a numerical constant. Equidistant distribution of particles along the x-axis with an arbitrary period l ($x = i\cdot l$) corresponds to the equilibrium state of system (12).

The instability conditions

$$2\pi n < \varkappa l + \varphi_0 < (2n+1)\pi, \quad n = 1, 2, \ldots \tag{19}$$

for such a periodic chain of particles were found in Ref. 38.

What happens if the period, l, of the initial distribution is taken within the interval (19) and, consequently, unstable periodic structure evolves? Under the action of arbitrary small disturbance, as $t\to\infty$, two situations are possible, either regular distribution of particles or chaotic sequences of particles. Both cases may be realized in experiments (fragments of pictures are shown in Fig. 15).

The considered scenario of the generation of spatial disorder through the formation of a periodic lattice of localized states with a subsequent decrease of the degree of order is, evidently, quite general. The same scenario must be observed in the two-dimensional analog of (10):

$$\frac{\partial u}{\partial t} = -u + \beta u^2 - u^3 + (k_0^2 + \nabla^2)^2 u. \tag{20}$$

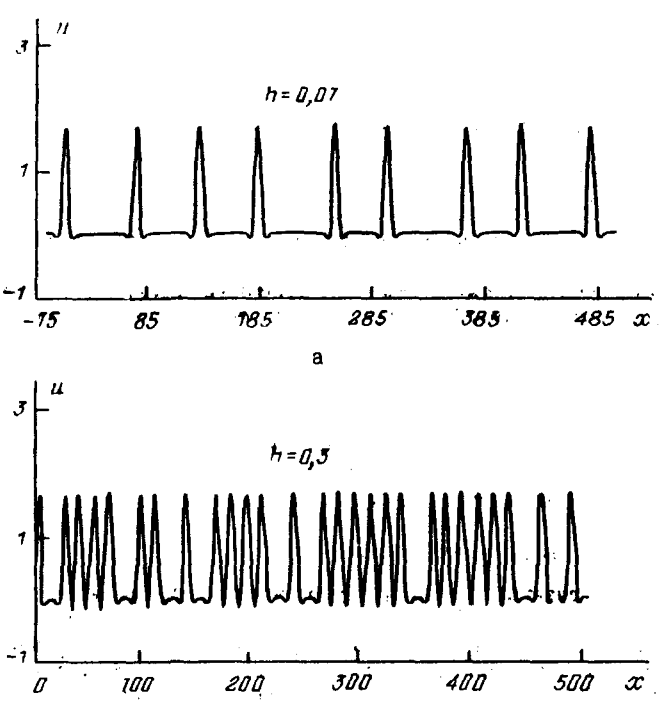

FIG. 15. Examples of nearly periodic (a) and chaotic (b) spatial distribution becoming established within model (10) for $\beta = 2.3$, $\varepsilon = 0.3$, $k_0 = 0.5$ and different k (see Fig. 13).

Indeed, as was shown in Ref. 41, there exist within (20) stable localized structures with exponetially decaying oscillating tails. The interaction of structures is described by a system of equations which was obtained by means of an asymptotic method[40] in the following form

$$M\frac{dR_j}{dt} = \sum_{i\neq j} F(R_i - R_j), \tag{21}$$

where R_j is the coordinate of the jth structure on the x,y-plane. This system has a set of equilibrium states which correspond to stable location of "particles" in the form of a random lattice. Preliminary experiments (see also Ref. 42) indicate that the formation of such a statical spatial disorder may occur as it was described above.

It should also be emphasized that the found steady-state irregular solutions of the gradient model (12) are, at the same time, steady-state solutions of its conservative (Hamiltonian) analog

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta F}{\delta u}. \tag{22}$$

However, while the stability of an irregular chain within (12) follows from the fact that the functional F has a local minimum on the solution of interest, the problem of stability for such a chain within (22) is much more complicated and challenging. It reminds one of the problem of the excitation spectrum in a one-dimensional model of a liquid proposed by Krönig-Penney (see, e.g., Ref. 1). This analogy (including the phenomenon of localization of eigenfunctions) may be very helpful if there are no resonances between collective excitations of the chain and eigenmodes of the localized structures. When these resonances are substantial, the evolution of instability may result in a completely different state of the system, perhaps without localized structures.

6. DISORDER OF DEFECTS. EXAMPLE

Disorder of defects randomly scattered against the background of a periodic lattice (a crystal lattice of atoms or a lattice of convective cells) is the type of spatial disorder that is often encountered in nature. An example of such disorder is depicted in Fig. 16. It was observed in the experiment on magnetohydrodynamic convection in a liquid crystal performed by Kramer's group.[43] Extended transverse defects are well pronounced against the background of roll convection. A single defect of this type may look like two identical lattices of rolls that fill the upper and lower half-spaces and are displaced half a period relative to one another. Such defects are also observed in computer experiments on coupled Ginzburg-Landau equations (see, e.g., Fig. 17). These transverse defects are, as a rule, repeated irregularly along the y-axis.

The disorder of such transverse defects reminds one of the disorder of localized structures that was discussed above. The similarity will be even more complete if we filter the initial periodic lattice. This can be done by representing the initial field in the form $u(x,y,t) = A(y,t)e^{ix} + \text{c.c.}$ (the lattice period is supposed to be equal to 2π). Substituting this solution into the initial equations (we will take the Swift-Hohenberg equation again) we obtain a fourth-order Newell–Whitehead–Segel (NWS) equation for complex amplitude along y [cf. (15)]:

$$\frac{\partial A}{\partial t} = A + \gamma A^* - \frac{\partial^4 A}{\partial y^4} - A|A|^2. \tag{23}$$

When $\gamma \neq 0$, this equation takes into account periodic (with the period π) inhomogeneity of the medium. In the case of convection, this is, for example, a periodic inhomogeneous temperature distribution at the lower boundary of the layer (see Fig. 18).

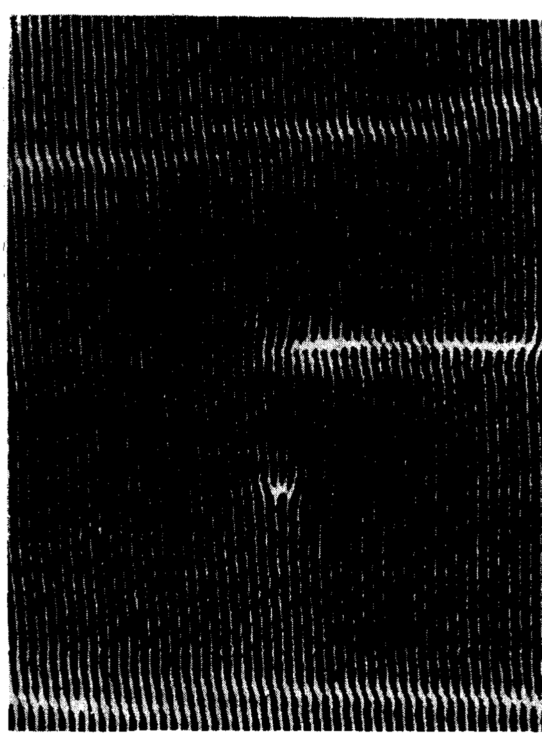

FIG. 16. Transverse defects against the background of roll structure observed in magnetohydrodynamic convection in liquid crystal (Ref. 46).

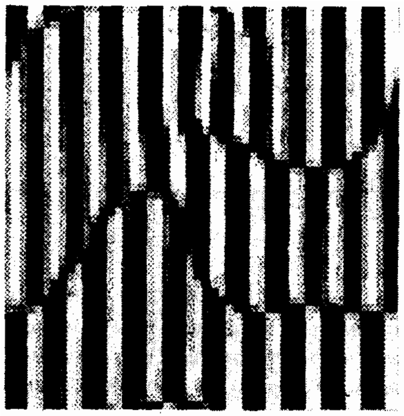

FIG. 17. Transverse defects in a computer experiment with coupled Ginzburg–Landau equations (Ref. 46).

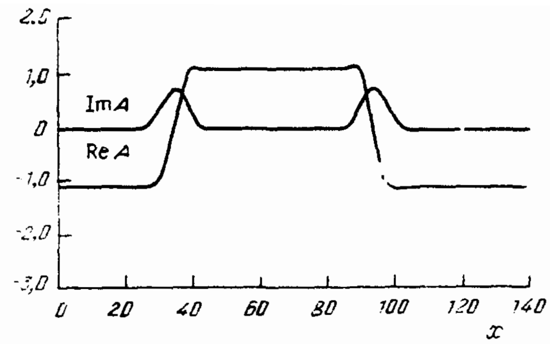

FIG. 19. Bloch wall.

The solutions of (23) that describe jumps of the phase $A(y)$ by π correspond to defects. The absolute value of $A(y)$ then tends to constant A_0 as $y \to \pm\infty$. Equation (23) is also a gradient model.[5)] Therefore all regimes that are established in our system as $t \to \infty$ (also including the ones with defects—jumps) must be described by the ordinary differential equation

$$A_{yyyy} - A(1 - |A|^2) + \gamma A^* = 0. \qquad (24)$$

Using Devaney's theorem[39] we can show that there exists in the phase space of this dynamical system a countable set of homoclinic trajectories. These trajectories are the ones that correspond to localized structures of phase and amplitude.

Our model describes transverse defects of two types: 1) the absolute value of amplitude turns to zero (Fig. 18) along the line of the phase jump and 2) the phase simply turns (as in Fig. 19) while $|A|$ remains finite. These defects are called Ising wall and Bloch wall (Refs. 44, 45) by analogy with the walls between ferromagnetic domains. The defects are stable for $\gamma > \gamma_{cr}$ and for $\gamma < \gamma_{cr}$, respectively.[44]

Single defects correspond only to the simplest homoclinic trajectory in the phase space of (24).[6)] We have already seen on the example of the analogous equation (10) that (24) must describe, besides simplest localized states, more complicated combinations of "solitons," including their chaotic sequences like the ones presented in Fig. 15.

Note that such chaotic distributions of transverse defects also exist in the absence of periodic inhomogeneity along x, i.e., when $\gamma = 0$. They agree qualitatively with the picture observed in experiments (see Fig. 16).

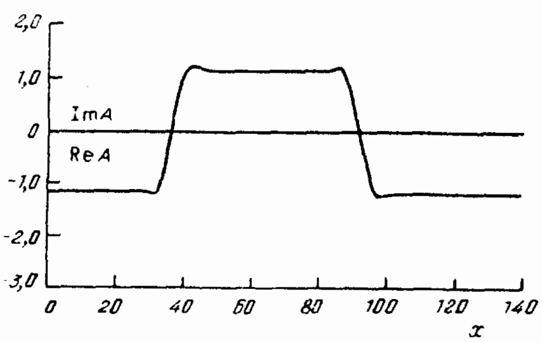

FIG. 18. Ising wall.

Of course, the one-dimensional model considered above has many weak points: the transverse defects are not infinitely extended along x, they are encountered, as a rule, in combination with other defects (e.g., pointwise defects), and they are not always at rest, etc. However, we would like to emphasize that this model has been a handy tool in understanding and qualitative description of a new phenomenon—finite-dimensional disorder of defects against the background of a regular lattice.

7. FINITE-DIMENSIONAL DISORDER IN THE $x-y$ AND $x-t$ SPACES

What is more astonishing: the existence in nature of finite-dimensional spatial disorder or true irregularity to which, in our language, corresponds a very high (formally infinite) correlation dimension? Frankly speaking, both these phenomena are amazing.

The examples of the generation of finite-dimensional disorder within one-dimensional gradient models would seem to be an obvious confirmation of the existence of such disorder. Indeed, what else may it be if the field distribution along x is described, for $t \to \infty$, by ordinary differential equations [e.g., (24)]. But, when we take into account the second spatial dimension, the field distribution for $t \to \infty$ is described by partial nonlinear differential equations, even in the case of a gradient model. In principle, the fields $u(x,y)$ that are solutions to these equations may also have infinite dimension. However, this is not likely in real media. Let us again turn to experiment.

What dimension, D_s, can the irregular field distribution shown in Fig. 20 have? In the figure you can see the snapshot of turbulent capillary ripples (Faraday ripples) in a very large cuvette.[46,47] We remind our reader that here we speak about parametrically excited capillary waves on the surface of a horizontal layer of fluid in an oscillating gravity field (technically, this means that the layer of fluid is on a plane surface that oscillates with the pump frequency ω_p and amplitude A_p). When $A_p > A_{cr}$, the regular lattice of capillary cell is destroyed—turbulence is generated. Results of processing a series of snapshots of turbulent capillary ripples, by empoying the procedure considered in section 3, are presented in Fig. 21. The dimension, D_s, of this two-dimensional spatial disorder is finite are relatively low.

It is significant that as supercriticality, A_p/A_{cr}, in-

FIG. 20. Turbulent capillary ripples.[46,47]

creases, the dimension, D_s, grows too. However, preliminary experiments show that this growth is rapidly saturated, which is, evidently, explained by the cell structure of the "capillary medium."

This is, no doubt, a useful example. But maybe this is a unique case? Perhaps, most real two-dimensional irregular field distribution have a very high (infinite) dimension? For instance, what is the dimension of a disordered density distribution of magnetic fields in the Universe where the characteristic spatial scale of inhomogeneity is about $3\cdot 10^4$ km (Ref. 48)? The same question arises about a two-dimensional picture of microcracks that emerge in a material before failure, etc. We are not in a position to answer these questions now. But it is highly probable that even "purely chaotic" initial field distributions evolve into finite-dimensional disorder for most fields described by partial differential equations. In other words, various disturbances of the medium cease to be independent, and this is what permits us to describes the snapshot by a finite-dimensional system (with three or two times, correspondingly).

Let us consider again the one-dimensional spatial disorder but now changing in time. In this case too we will have a two-dimensional irregular picture on the x,t-plane. Can we find the dimension of this disorder knowing the dimension of the time series (dynamical chaos) and the dimension of the space series? Apparently, we cannot do it in a general case. We can only say, using, for instance, results of the investigations of a one-dimensional CGLE model

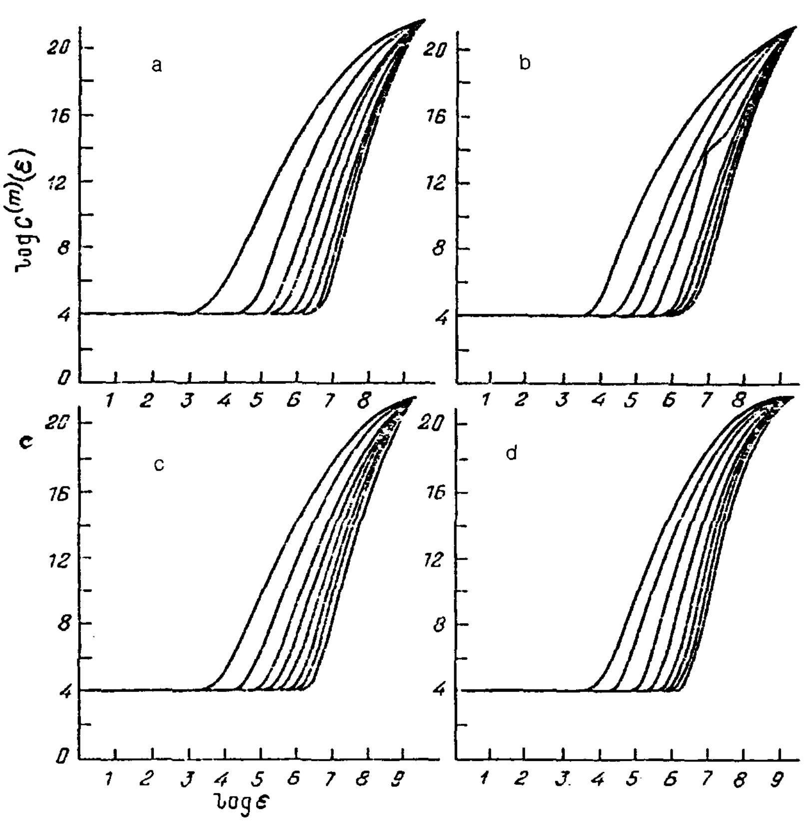

FIG. 21. Determination of D_s for snapshots of capillary ripples with increasing supercriticality; a—$D_s = 6.2$; b—$D_s = 6.3$; c—$D_s = 7.2$; d—$D_s = 7.8$.

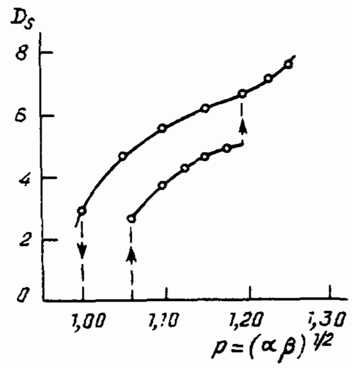

FIG. 22. "Chaos–chaos" phase transition and hysteresis on the D_s, p-plane in model (25) ($s=\sqrt{\beta/\alpha}=1.15$ and $R=10^4$).

$$\frac{\partial A}{\partial t}=A-A|A|^2+\frac{\partial^2 A}{\partial x^2}+i\alpha\frac{\partial^2 A}{\partial x^2}+i\beta A|A|^2. \qquad (25)$$

that there is an (unknown) relation between D_s and D_t.

The dependence of the space series dimension, D_s, on $p=\sqrt{\alpha\beta}$ is shown in Fig. 22. This dependence has two remarkable peculiarities: a stepwise increase by almost one and a half times of D_s near the critical point p'' and hysteresis.[49] Note that the dimension of the time series, D_t, also changes in a jump at the same value of the parameter $p=p''$. Thus, in this case there apparently is a relation between D_s and D_t. This relation, however, does not seem to be robust and can be represented in the form of an inequality $D_s \leqslant D_t$.[7)]

The jump in the spatial dimension in Fig. 22 has the following explanation. The field described by (25) may have two, qualitatively different irregular states. One of them is "phase turbulence" when the phase changes chaotically and the amplitude weakly pulsates near its average value (see Fig. 23). The other state is "strong" turbulence when both amplitude and phase are strongly irregular (Fig. 24). In the course of formation of spatial disorder, new spatial perturbations (amplitude pulsations) emerge in the transition across the point p'' and this transition can be considered, in a sense, to be critical phenomenon.

The onset of one or another chaotic regime in the region of the parameters p' and p'' depends on initial conditions [in this case two strange attractors co-exist in the phase space of the dynamical system (25)]. This explains the hysteresis phenomenon: in the transition through the critical point p'' from right to left, we remain in the attraction domain of a multidimensional attractor. While starting from regular initial conditions on the interval $[p',p'']$, we enter the regime of low-dimensional chaos.

The corresponding two-dimensional pictures of disorder on the x,t-plane are given in Figs. 25 and 26. Such pictures contain information on the prehistory of one-dimensional spatial disorder and on the degree of its homogeneity in time. Note that such a time evolution of spatial disorder is readily observed in laboratory when the wake behind a long inhomogeneous cylinder placed across the flow is visualized by smoke.[52,53] The y-coordinate (Fig. 27), along which the vortices generated near the cylinder drift, corresponds to the time axis directed from right to left. The spatio-temporal disorder that is established in this case can also be described within a CGLE model but now it wil have the coefficients varying along x.[52]

8. DIAGNOSTIC OF FINITE-DIMENSIONAL DISORDER AGAINST THE BACKGROUND OF SPECKLE NOISE

The consideration presented above was concerned with an ideal situation: finite-dimensional disorder, if any, existed "all by itself," i.e., no uncontrolled inhomogeneities were taken into account. Whereas in real snapshots, there always is present some kind of noise whose spatial Fourier spectrum is rather broad and the origin of which is unknown in most cases. The noises may be produced by technical peculiarities of the diagnostics, by noisy channel through which the signal is transmitted, by the inhomogeneities in the photographic material, and so on. It is extremely difficult to detect finite-dimensional order against the background of spatial noise employing traditional methods. However, our experience in the investigation of space series shows that the use even of simplest forms of dynamical treatment of pictures (calculation of D_s, spatial entropy, etc.) makes solution of this problem quite promising.

Let us explain how it may be solved. Spatial dimensions, D_s, and entropy, H_s, are formally, defined by the expressions

$$D_s=\lim_{m\to\infty}\lim_{\varepsilon\to 0} d_s(m,\varepsilon),\quad H_s=\lim_{\varepsilon\to 0}\lim_{m\to\infty} h_s(m,\varepsilon), \qquad (26)$$

where ε is the diameter of the "spheres" filling the phase space (possibly a matrix one) to be reconstructed and m is the number of points in the space series which determine the dimension of this space (see also Section 3).

Of course it makes not sense to take the ($m\to\infty$, $\varepsilon\to 0$) limits when processing real time and space series. We proceed in an alternative way: we plot C_ε^m against ε and d_s against m and analyse the graphs. We must bear in mind that the $d_t(m)$ is usually saturated at some m^* for the time series

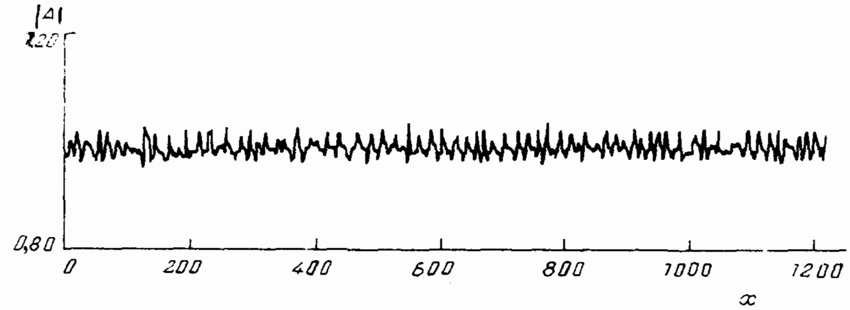

FIG. 23. Amplitude distribution along x at "phase turbulence" (solution of (10) for the values of the parameters $R=10^4$, $\alpha=1.0$, and $\beta=1.3$).

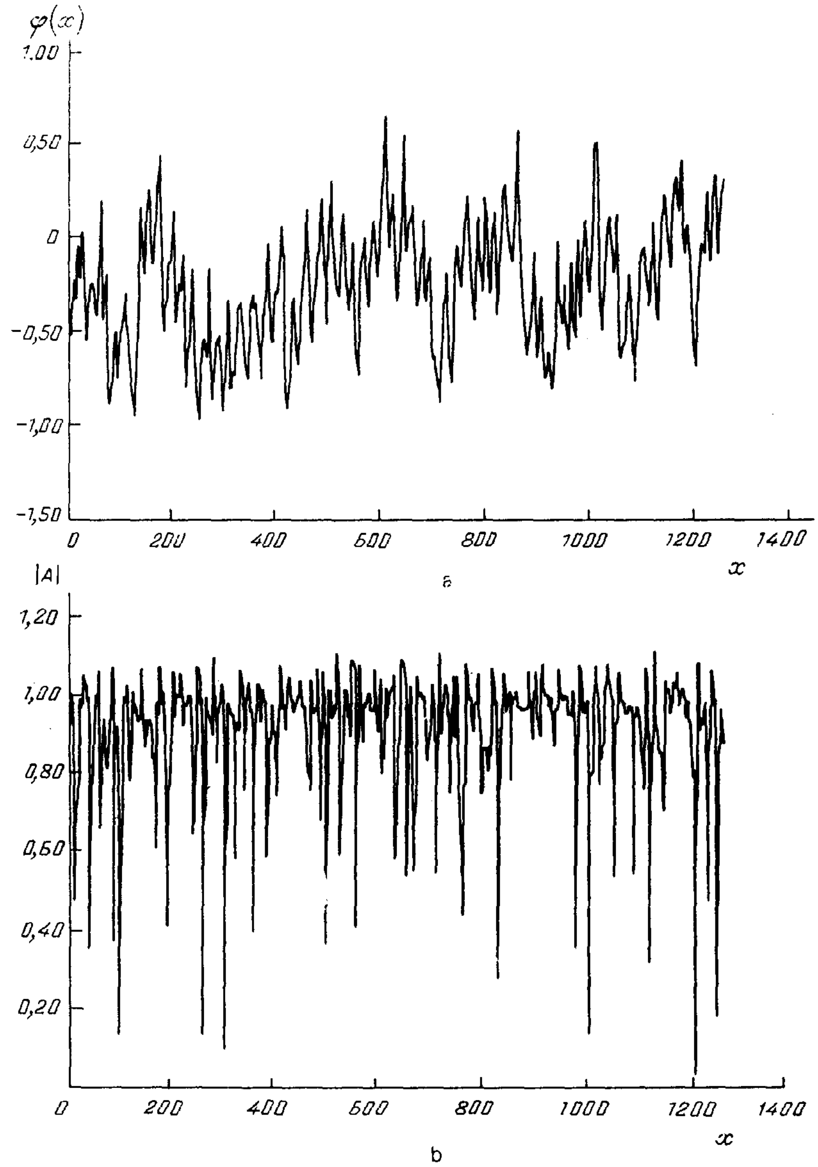

FIG. 24. Phase (a) and amplitude (b) distributions in the regime of strong turbulence [model (10), $\alpha = 1.1$, $\beta = 1.44$, and $R = 10^4$].

of dynamical origin. The value m^* is referred to as the dimension of embedding space of a stochastic set (it can also be determined using a different algorithm). When $m \geqslant m^*$, $d_t \approx D_t$. The relation $D_t \leqslant m^* \leqslant 2D_t + 1$ holds here (Mañe theorem). The closeness of m^* to the left and right limits of this inequality indicates the complexity of internal structure of the stochastic set. To be more exact, it shows the excess (as compared to the dimension of the set) of the number of geometrical variables needed for its structure to be constant for $m \to \infty$. What is said above must also be valid for dynami-

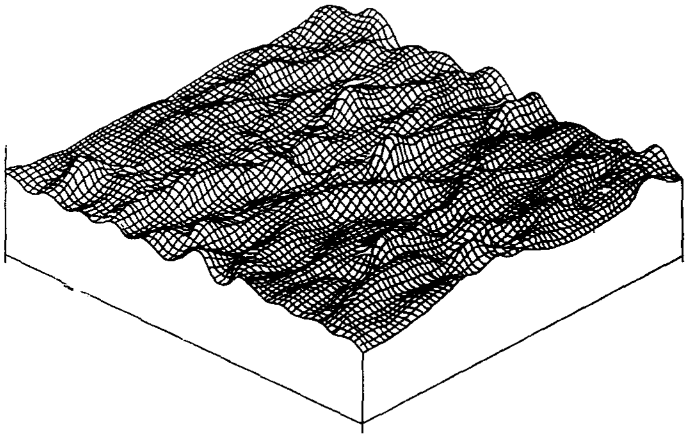

FIG. 25. Amplitude distribution at phase turbulence on the x, t-plane within model (10) the parameters are the same as in Fig. 23).

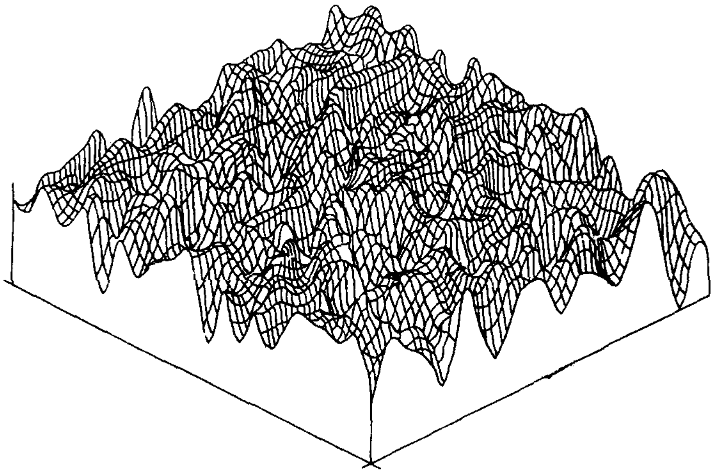

FIG. 26. Amplitude distribution at strong turbulence on the x, t-plane (the parameters are the same as in Fig. 24).

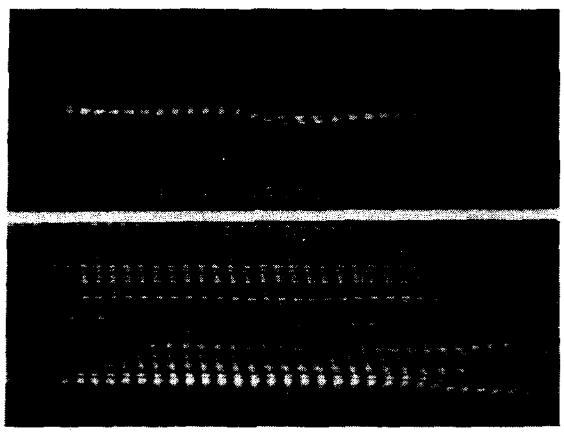

FIG. 27. Defects in the wake behind a cylinder (see Ref. 51 for details).

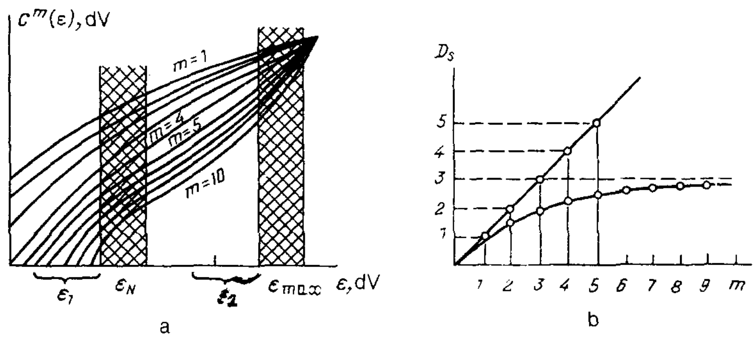

FIG. 28. a) Correlation integral $C^m(\varepsilon)$. (ε_N is the level of speckle noise and ε_{max} is the maximal amplitude of the signal. Transition regions are hatched.) b) The dependence of correlation dimension D_s on embedding dimension m.

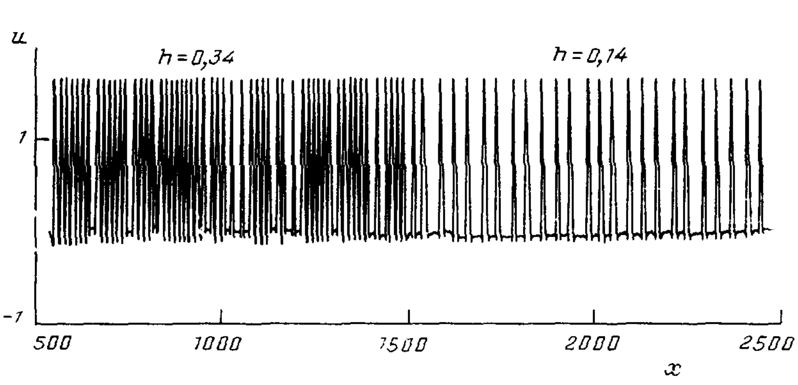

FIG. 29. Stable co-existence of regimes with different finite-dimensional disorder within model (10). The values for the Kolmogorov–Sinaĭ entropy are indicated on top (cf. Fig. 15, the parameters are the same).

cal systems with several times, i.e. as applied to multidimensional space series.

When processing the space series corresponding to finite-dimensional disorder, we can observe on the $\log C^n$ $(\log \varepsilon)$ plot (cf. Fig. 20) several linear sections with different slopes on different intervals of the $(\varepsilon_1,\varepsilon_2)\in(0,\varepsilon_{\max})$ scales. The curves break sharply at the joints of different regions.

The situation is quite different when the space series corresponds to speckle-noise. In the reconstructed phase space, with the finiteness of the spatial spectrum band due to inevitable filtration taken into account, the distribution of points will no longer be "structured", instead, it will be uniform at any embedding dimension. In this case, $d_s \approx m$ and weakly depends on ε!

What happens if our space series is a mixture of finite-dimensional and "idea" (speckle) disorder? We will calculate for such a space series a family of correlation integrals $C^m(\varepsilon)$ for successive values of $m = 1,...,m_0$ and construct cross-sections of this family at different ε (see Fig. 28). Apparently, the behavior of $d(m,\varepsilon)$ at sufficiently small ε_N will be a signal[8)] of the presence of a noise component for any real disorder, including a finite-dimensional one. The value ε_N may be considered to be an estimate (although a not too accurate one) of the amplitude of the noise component of disorder. The $\varepsilon\in(\varepsilon_N,\varepsilon_{\max})$ range determines the domain of finite-dimensional disorder. A hierarchy of irregular patterns having different values of D_s may, in principle, exist within this domain.

This procedure has been developed in ample detail for ordinary dynamical systems (see, e.g., Refs. 19, 54). We believe that there is a need, and necessary knowledge too, to generalize this method to dynamical systems with several times (see, in particular, Ref. 55) where results of experiments on the effect of stochasticity on the formation of different types of structures in Rayleigh-Bénard convection are presented).

9. INSTEAD OF A CONCLUSION

Finishing a lecture on finite-dimensional disorder it is highly tempting to speak about its significance in physics and on the extent to which it occurs in nature. However, we will refrain from that and just formulate several problems which may, possibly, attract the attention of young (and, perhaps, not only young) researchers to this fascinating and very promising field of knowledge.

1. Any settled disorder bears imprints of its past. For example, what can we say looking at the one-dimensional disorder shown in Fig. 29? We know that these patterns that are chaotic along x result from the evolution of a certain initial field distribution within the Swift–Hohenberg model (10). Analysis shows that the Kolmogorov entropy is different at different sections of x (the sections are long, see, e.g., Fig. 29) and varies stepwise at different points. We know also (see Sec. 5) that the value of entropy depends on the density of localized states, which are determined by the spatial period of the initial field distribution. Thus, we can say that our disorder has evolved from "pieces" of distribution with different periods. Jumps of entropy are observed at the boundries of these regions.

This is an amazing fact. Actually, we have discovered

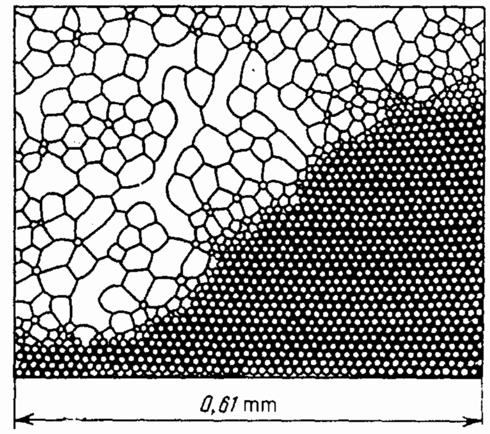

FIG. 30. Moving front between disordered and ordered phases of magnetic domains in a thin film (observed experimentally in Ref. 17).

an absolutely new phenomenonon: stable existence of domains with different kinds of finite-dimensional disorder! The sharp boundary between the domains is also a defect. In the two-dimensional case, the boundary between such domains may be very complicated.

There arise the questions: To what extent can we diminish the size of the domain for the mosaic disorder to remain stable? and Does the stability factor of the mosaic depend on the shape of the domain? Similar questions arise also with reference to defects at the interface of finte-dimensional disorder and crystal order (for example, in magnetic films (see Fig. 30)).

2. We have almost neglected the problem of finte-dimensional disorder in "oscillating media" (we have only mentioned it in the discussion of the "phase–amplitude" turbulence transition within CGLE). In this case also there arise very unexpected problems related to the effect of temporal dynamics on spatial disorder. For instance, different regions of disorder may be partially or completely synchronized in the course of formation of a steady-state spatiotemporal picture. This leads to the change of the boundary between domains to which different values of D_t corresond. It is not clear yet whether such a stochastic synchronization may result in stable static boundaries between such spatial domains having different chaotic dynamics (see, e.g., Refs. 56, 57, 58.

3. Computer experiments performed using different models show that the degree of disorder of the field increases with the growing number of localized states. Apparently, if such states do not interact with one another, the dimension D_s will be infinite, as for complete thermodynamic disorder. The question arises: What kind of particle interaction transform infinite-dimensional disorder into a finite-dimensional one? Finally, is there any relation between thermodynamical and dynamical characteristics for finite-dimensional disorder?

The authors are grateful to H. Abarbanel, V. S. Afraĭmovich, A. V. Gaponov–Grekhov, K. A. Gorshkov, and Ya. G. Sinaĭ for helpful discussion of many problems examined in this article.

[1)]This is an expanded version of M. I. Rabinovich's lecture (prepared together with A. L. Fabrikant and L. Sh. Tsimring) for the International

School "Teaching Modern Physics-Statistical Physics" (Badajors, Spain 1992).

[2)]We would like to remind our reader that according to the Poincaré return theorem, the return period grows exponentially with the increase in the number of incommensurate harmonics.

[3)]One can easily observe single defects of this type (see, e.g., Fig. 5, Ref. 13).

[4)]The interaction force in gradient systems determines the velocity of localized states and not acceleration (see also[5)]).

[5)]Averaged equations for the function $A(y,t)$ are gradient by virtue of the gradient initial system for $u(x,y,t)$. The free energy functional describing the field A follows from the free energy fuctional for u as a result of averaging along x (Ref. 44). The same is true for equations for the centers of localized states.

[6)]Because this fourth-order equation has the energy functional ,

$$H = \int_{-\infty}^{\infty} \Big[-|A|^2 + \frac{1}{2}|A|^4 - \gamma((\mathrm{Re}\,A)^2 - (\mathrm{Im}\,A)^2) - \Big(\frac{\partial^2}{\partial y^2}\mathrm{A}\Big)^2\Big] dy$$

this dynamical system has a three-dimensional phase space.

[7)]It is a very interesting and significant question and it will be easier to answer it by considering a discrete model of the medium, for example, in the form of a chain of coupled nonlinear elements. An approach to the solution of this problem for a chain of Feigenbaum maps can be found in Ref. 50. For related results concerning cellular automata see Ref. 51.

[8)]Because a few points will enter the ε-neighborhood and the asymptotic behavior of the correlation integral will change.

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