Anyon superconductivity in strongly-correlated spin systems

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The present state of the gauge theory of high-temperature superconductivity in strongly-correlated two-dimensional spin systems is reviewed. Basic ideas on the statistics of elementary excitations in spatially two-dimensional systems are presented and are used in an analysis of the structure of the energy spectrum and the form of the wave function of a set of anyon quasiparticles in both the long-wave continuous and lattice limits. The continuous and lattice theories are used to classify the phase states, and the hierarchy of phase transitions is described in terms of the topological quantum field theory. Thermodynamic and electrodynamic properties of anyon systems are described, and the experimental consequences of the Chern–Simons theory of high-T_c superconductivity are discussed.

1. INTRODUCTION

Considerable advances have been achieved in our understanding of the properties of planar systems in the relatively short time since the discovery of high-T_c superconductivity.[1,2] Extensive experimental and theoretical studies have lent support to the conviction that the superconducting state and many of the unusual phenomena discovered in the so-called normal state are spatially two-dimensional phenomena that occur in the basal CuO_2 layers. The three-dimensional architecture of the new compounds contains, in addition to the basal planes, some additional elements that serve as a reservoir of dopants.

The other essential feature is that all the "parent" compounds, i.e., compounds free of dopants (La_2CuO_4, YBa_2, Cu_3, O_6, and so on) are good dielectrics insofar as their transport properties are concerned. This means that there are two very different energy scales that refer, respectively, to the band gap and the very much greater separation between bands. When the energy scales in doped compounds are very different, we have the conditions for adiabatically slow motion of carriers against the background of fast spin fluctuations. We shall be interested, above all, in the structure of the ground state, and in the quantum numbers of low-lying excitations. The existence of two very different energy scales ensures, as we shall see later, the basic possibility of a redistribution of spin and charge degrees of freedom of low-energy excitations among different types of quasiparticle. In the standard Fermi liquid of a metal, the spin and charge are carried together by electron-hole excitations of the Fermi ground state.

The third feature is that the original compounds exhibit antiferromagnetic ordering of the 1/2 spins. The removal of the charge and the small spin of 1/2 from a site in the basal plane (in $YBa_2Cu_3O_{6+x}$, the plane containing the chains) into the dopant reservoir leads to strong quantum spin fluctuations. The removal of charge and spin from one of the sites in the unit cell of the basic lattice leads to a charged spin defect and strong correlations between the orientations of the remaining spins on neighboring sites. Moreover, a departure from antiferromagnetic order inside the unit cell produces correlations between neighbors located along a diagonal, i.e., correlations between next-nearest neighbors. Quantum frustrations of spin orientation on neighboring sites lead, on the one hand, to the destruction of antiferromagnetic ordering of spins lying in planes[1)] and, on the other hand, to the establishment of order that results in an average magnetic moment in the direction perpendicular to the plane, which is due to the resulting noncollinearity of particle spins on three neighboring cell sites.

The remarkable properties of the new compounds, namely, the two-dimensional character of the phenomena, the existence of a large energy scale in the initial dielectrics, and the limiting small spin, are jointly reflected in the energy-level systematics and in the structure of low-lying excitations. The recognition of the importance of strong quantum fluctuations underlies the tendency of recent publications to exploit the experience accumulated in quantum field theory in discussions of strong-coupling problems and, in condensed-matter physics, the fractional quantum Hall effect.[3] These theoretical developments tend to emphasize the break between the new approaches and the traditional methods of studying superconductivity in condensed-matter physics. One of our aims in this review of recent publications on the theory of strongly-correlated two-dimensional systems is to bring closer together the attitudes and terminologies of the numerous people working in different areas of the subject.

Since a small parameter is not available in the problem of strong quantum fluctuations, all treatments are necessarily qualitative and the only way of escaping from this dilemma is to use either a numerical experiment or a rigorous solution. This is why we shall throughout confine our attention to qualitative aspects of the topic, always hoping to find some means of formulating the problem in a way capable of rigorous solution. The problem involves additional difficulties associated with the high concentration of different approaches, and also different techniques and samples employed. We shall therefore start with simple methods and very approximate models, and then turn to more sophisticated and less well known techniques.

A state with developed spin fluctuations is called a quantum spin liquid, if we bear in mind one further point in addition to the absence of a small parameter. The character-

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istic size of a region in which the antiferromagnetic disposition of spins has been disturbed is of the order of the separation between such regions. It is determined by the concentration n of spin-texture defects due to doping, and amounts to something of the order of $n^{-1/3} \sim 10^{-7}$ cm. This is greater than the lattice constant in the basal plane by a factor of less than an order of magnitude. It follows that an acceptable theory must necessarily be a theory on a lattice. When a transition is made to the long-wave description with the help of fields that are smoothed on the scale of the lattice, it is essential to retain the basic information obtained at short distances. This legacy of short distances in the presence of strong interactions is also found to contain topological information,[2)] as we shall see elsewhere in this review. The properties of two-dimensional systems will be discussed below in terms of fields defined on a lattice, and also from the standpoint of long distances.

2. STRONG SPIN CORRELATIONS

The transition from the dielectric to the metallic state is described by the Hubbard model[4] with the Hamiltonian

$$H = -t\sum_{\langle i,j\rangle,\sigma}(c^+_{i\sigma}c_{j\sigma} + c^+_{j\sigma}c_{i\sigma}) + U\sum_i n_{i\uparrow}n_{i\downarrow}, \tag{2.1}$$

$$n_{i\sigma} = c^+_{i\sigma}c_{i\sigma}. \tag{2.2}$$

The operator $c_{i\sigma}$ annihilates an electron with spin σ on site i, $n_{i\sigma}$ is the particle-number operator, t is the tunneling amplitude between neighboring sites i and j on a two-dimensional square lattice (this is indicated in the sums by the angle brackets $\langle i,j\rangle$), and U is the energy of repulsion between spin-1/2 fermions on a given site.[3)] In the absence of doping, the mean number of electrons per site is

$$\bar{n}_i = \langle\sum_\sigma c^+_{i\sigma}c_{i\sigma}\rangle = 1$$

and corresponds to a half-filled band.

In a dielectric, the states $|0\rangle$, $|\uparrow\rangle$, $|\downarrow\rangle$ in the lower band, whose width is of the order of t, are separated from the states $c^+_{i\uparrow}c^+_{i\downarrow}|0\rangle = |\uparrow\downarrow\rangle$ in the upper band, which correspond to two-fold filling, by a large energy interval $U \gg T$, while the chemical potential lies at the center of the forbidden band and is equal to $U/2$. Existing estimates suggest that $t \sim 0.1$ eV and $U \sim 1$ eV. If we confine our attention to the space of low-lying singly-occupied states $|0\rangle$, $|\uparrow\rangle$, and $|\downarrow\rangle$, and use the Gutzwiller projection operator[6]

$$P = \prod_i(1 - n_{i\uparrow}n_{i\downarrow}) \tag{2.3}$$

we find in second-order perturbation theory[7,8] that the $t-J$ Hamiltonian is

$$H = -t\sum_{\langle i,j\rangle,\sigma}(a^+_{i\sigma}a_{j\sigma} + a^+_{j\sigma}a_{i\sigma}) + J\sum_{\langle i,j\rangle}(\mathbf{s}_i\mathbf{s}_j - \frac{1}{4}n_in_j), \tag{2.4}$$

$$a_{i\sigma} = c_{i\sigma}(1 - n_{i,-\sigma}), \quad \mathbf{s}_i = c^+_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}c_{i\beta}/2, \tag{2.5}$$

where $J = 4t^2/U$, $\boldsymbol{\sigma}$ are the Pauli matrices, and σ and $\alpha = (\uparrow,\downarrow)$ are the spin indices.

For $t/U \ll 1$ and near half-filling, for which $t(1-\bar{n}) \ll 1$, we can have $t(1-\bar{n}) \ll t^2/U$, so that the first term in (2.4) is negligible in comparison with the second. This limit means that a small number of almost immobile holes[4)] has been placed in the state produced by the second term in (2.4). When we compare this with the canonical physics of metals, in which kinetic and potential energies are of the same order, we see that this is an exceptional situation because the kinetic energy of holes is now small and the Heisenberg Hamiltonian

$$H = J\sum_{\langle i,j\rangle}(c^+_i\boldsymbol{\sigma}c_i)(c^+_j\boldsymbol{\sigma}c_j), \tag{2.6}$$

is the dominant one and is equivalent to the four-fermion Hamiltonian

$$H = -\frac{J}{2}\sum_{\langle i,j\rangle}\left[(c^+_{i\alpha}c_{j\alpha})(c^+_{j\beta}c_{i\beta}) + \frac{1}{2}\right]. \tag{2.7}$$

Strong spin correlations occur because terms that are quadratic in the operators $c_{i\alpha}$ are small, so that we cannot classify states by starting with the states of noninteracting particles.

The Hamiltonians (2.6) and (2.7) are very different in form, so that if we are to use the usual mean-field theory, we have to employ a different order parameter. For example, in the case of the Hamiltonian given by (2.6), we could consider different distributions of the mean spin $\langle s_i\rangle = \langle c^+_{i\alpha}\sigma_{\alpha\beta}c_{i\beta}\rangle$ over the lattice. In particular, $\langle s_i\rangle = (-1)^i\mathbf{m}$ for the antiferromagnetic state, $\langle s_i\rangle = (-1)^{i_x}\mathbf{m}$ for the alternate ordering of moments along the x axis, and so on.

To describe the new possible classifications of states[9–12] when the Hamiltonian is given by (2.7), we introduce the hopping operator[12]

$$\hat{\chi}_{ij} = \frac{J}{2}\sum_\alpha c^+_{i\alpha}c_{j\alpha}. \tag{2.8}$$

Its expectation value can be interpreted as the probability amplitude[8,13] of a hop j to a site i with the parametrization

$$\chi_{ij} = \frac{J}{2}\sum_\alpha\langle c^+_{i\alpha}c_{j\alpha}\rangle = -|\chi_{ij}|\exp(-2\pi i\int_i^j \mathbf{a}d\mathbf{l}), \tag{2.9}$$

which can be used to discuss the properties of phases with different geometrically irregular or chaotic distributions of the complex variables χ_{ij} over the lattice. Some examples are given in Refs. 10, 12, and 15, and are illustrated in Fig. 1.

States with different configurations of the quantities χ_{ij} or $\langle s_i\rangle$ are distributed in some way along the energy scale. It is therefore unclear *a priori* which particular choice of the mean-field theory variables is to be preferred. In general, the mean-field theory may not even have a range of validity. This is a question of the local stability of states with different configurations of χ_{ij} or $\langle s_i\rangle$, and of the structure of states with an absolute energy minimum. The following considerations may provide some justification for preferring the description defined by (2.9) at this stage of our discussion.

We draw attention to the fact that the hopping amplitudes are not invariant under local phase rotations $c_j \to c_j\exp(i\varphi_j)$. In other words, the quantities χ_{ij} are not invariant under the gauge transformations

$$\mathbf{a} \to \mathbf{a}' = \mathbf{a} + \nabla\varphi \tag{2.10}$$

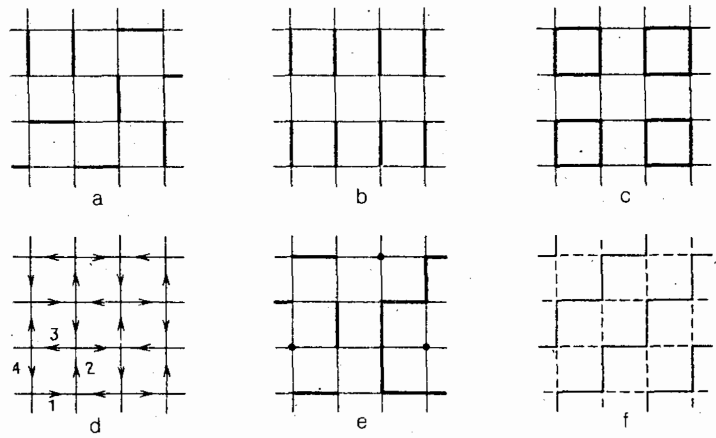

FIG. 1. Distribution of lattice hopping probability amplitudes: a—uniform random distribution, b—dimer phase, c—distribution in the form of cells, d—phase with flux, the link numbers indicate the distribution of indices of the variables χ_i, e—zigzag distribution of amplitudes.

of the gauge field $\mathbf{a}$, which is defined on the link ij and parametrizes the phase in (2.9):

$$\theta_{ij} = 2\pi \int_i^j \mathbf{a} d\mathbf{l}. \tag{2.11}$$

Only gauge-invariant quantities can be "observables." For example, for a cell with corners at the points i, j, k, l, the quantity

$$\langle \chi_{ij}\chi_{jk}\chi_{kl}\chi_{li} \rangle \sim \exp(2\pi i \phi). \tag{2.12}$$

is gauge invariant. It depends on the flux

$$\phi = \frac{1}{2\pi}(\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li}) \tag{2.13}$$

of the statistical magnetic field through the unit cell. Here and henceforth we assume that, unless stated to the contrary, the flux quantum is $\phi_0 = c\hbar/e = 1$ and the system of units is such that $\hbar$, c, and e are all equal to unity.

If the values of the flux $\phi = p/q$ through the unit cell are labelled by mutually primitive numbers p and q, we obtain a series of states usually called[16] the generalized phase with flux. We shall confine our attention to the properties of this wide class of states.[5)]

We start by introducing certain simplifying transformations. First, we integrate over the Hubbard–Stratonovich variables χ_{ij} of the Hamiltonian

$$H = \sum_{\langle i,j \rangle} [(\chi_{ij} c^+_{i\alpha} c_{j\alpha} + \text{h.c.}) + \frac{2}{J}|\chi_{ij}|^2], \tag{2.14}$$

which describes the motion of charged particles in a magnetic field with vector potential $\mathbf{a}$. This takes us to (2.7) once again. This type of mean-field theory works well for the group SU(N) with large N.

The dynamic phases $\exp(-i\theta_{ij})$ are the variables of the effective lattice gauge theory. It will be useful to have basic information about the above problem in the ensuing discussion. In Sec. 6, we shall therefore consider recent publications devoted to the problem defined by (2.14).

For heuristic reasons, we will proceed in the following even more approximate way. We replace the product $c^+_{i\alpha} c_{j\alpha}$ in (2.7) with the modulus $|\chi_{ij}| = |\chi|\delta_{ij}$ of the hopping probability amplitude and, instead of the second pair of operators, we introduce (2.9) with the vector potential given by (2.10), so that (2.7) is replaced with the Hamiltonian

$$H = K \sum_{\langle i,j \rangle} \cos(\varphi_i - \varphi_j - \theta_{ij}) \tag{2.15}$$

of a spin network of random bonds with frustrations. This originally arose[17,18] in the phenomenological description of high-T_c superconductors. A rigorous derivation of the Hamiltonian given by (2.14) and (2.15) is reported in Refs. 12, 19, and 20. The Hamiltonians given by (2.14) and (2.15) complement one another and, as we shall see later, provide us with a complete picture of the distribution of quantum numbers.

To conclude this Section, we consider a way of including the external electromagnetic field with vector potential $\mathbf{a}$ in the Hamiltonian (2.14). The electron hopping operator that is gauge-invariant under transformations of the potential $\mathbf{A}$ assumes the following form instead of (2.8):

$$\sum_\alpha c^+_{i\alpha} c_{j\alpha} \exp(2\pi i \int_i^j \mathbf{A} d\mathbf{l}). \tag{2.16}$$

Averaging as in (2.9), we obtain

$$\chi_{ij} = \sum_\alpha \langle c^+_{i\alpha} c_{j\alpha} \rangle \exp(2\pi i \int_i^j \mathbf{A} d\mathbf{l}) = -|\chi_{ij}| \exp(-2\pi i \int_i^j \mathbf{a} d\mathbf{l}). \tag{2.17}$$

In other words, when the external field is present,[21]

$$\sum_\alpha \langle c^+_{i\alpha} c_{j\alpha} \rangle = -|\chi_{ij}| \exp[-2\pi i \int_i^j (\mathbf{a} + \mathbf{A}) d\mathbf{l}]. \tag{2.18}$$

Consequently, the gauge potential $\mathbf{a}$ that is dynamically generated by this hopping process appears on equal terms with the external-field potential in the sum

$$\vec{\mathscr{A}} = \mathbf{a} + \mathbf{A}, \tag{2.19}$$

and this determines the specific properties of the electromagnetic response of the system.

3. STATISTICS OF EXCITATIONS IN TWO-DIMENSIONAL SYSTEMS

The proximity of the dielectric state means, as we saw in the last Section, that there are strong spin correlations. The question now is: what new quantum possibilities can be associated with the spatial dimensions of the system? The answer to this has long been known.[22–26] It is well known that in

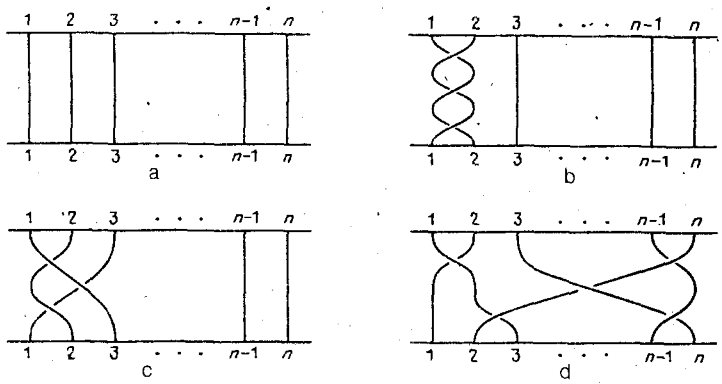

FIG. 2. Quantum braids: a—trivial braids, b—the result g_i^{-3} of the application of the operators g_i from the braid group, c—graphical representation of the longer word $g_1g_2g_1$ consisting of the operators g_i, d—general form of braiding.

systems with a space-time dimensionality 2 + 1, the excitations have fractional or intermediate statistics.[22-26] The concept of fractional statistics arises naturally in the (2 + 1)-dimensional case when we consider a system of particles with the Lagrangian[26]

$$L=\sum_{i=1}^{N}\frac{m}{2}\left(\frac{d\mathbf{r}_i}{dt}\right)^2+\hbar\frac{\vartheta}{\pi}\sum_{i\neq j}^{N}\frac{d}{dt}\varphi_{ij}. \tag{3.1}$$

The position of each particle is defined by the two-dimensional position vector $\mathbf{r}_i$, and the angular separation between particles i and j is defined by the azimuthal angle φ_{ij}.

The last term in (3.1) is the total derivative with respect to time, so that it does not contribute to the classical equation of motion. In quantum mechanics, the presence of the second term in (3.1) ensures that a phase is induced in the wave function under particle interchange. Actually, when particle i covers half the path around particle $j(\varphi_{ij}=\pi)$, which corresponds, after translation, to the interchange of the particles, the wave function acquires the factor $\exp(i\vartheta)$. When $\vartheta=0 \bmod 2\pi$, the particle is a boson, and when $\vartheta=\pi \bmod 2\pi$ it is a fermion; when ϑ is arbitrary, we have[25] an anyon.[6)] For $\vartheta=\pi/2$ this corresponds to the half-way house between a fermion and a boson, and the particle is called a semion (semifermion).[27] The semion is of particular interest in high-T_c superconductivity. In the case of the Kosterlitz–Thouless pairing of excitations in a two-dimensional system (see Sec. 11), a semion pair behaves as a boson under an interchange: $[\exp(i\pi/2)]^4=1$ (Ref. 28).

The existence of intermediate statistics in spatially two-dimensional systems is wholly due to the multiply-connected character of the configuration space of the system of identical particles, i.e., the space in which the state vector is defined. The configuration space of N identical particles is obtained from the coordinate space by identifying the points obtained for any interchange of particle coordinates after the exclusion of singular points at which two or more coordinates coincide. In the three-dimensional case, the configuration space M is doubly connected and the fundamental homotopy group $\pi_1(M)$ is identical with the permutation group S_N with its even (for bosons) and odd (for fermions) representations. In two-dimensional systems, the anyon excitations realize the representations of the braid group (Fig. 2; Ref. 29).[7)]

The simplest way of verifying the multiply-connected character of the two-dimensional configuration space is to consider the canonical momentum which has the following form for (3.1):

$$\mathbf{p}_i=m\frac{d\mathbf{r}_i}{dt}-\hbar\mathbf{a}_i(\mathbf{r}_i), \tag{3.2}$$

$$\mathbf{a}_i=\frac{\vartheta}{\pi}\sum_{k\neq i}\frac{[\mathbf{e}_z(\mathbf{r}_i-\mathbf{r}_k)]}{|\mathbf{r}_i-\mathbf{r}_k|^2}. \tag{3.3}$$

The vector potential given by (3.3) is the sum of the Bohm–Aharanov potentials with flux $\phi=\vartheta/\pi$ in units of the flux φ_0.

The two-dimensional configuration space is thus seen to contain a set of singularities and is therefore multiply connected, and the long-range Dirac–Bohm–Aharanov interaction acts between the particles that convey the unit statistical charge and the flux $\phi=\vartheta/\pi$. The Hamiltonian for the system of such particles is

$$H=\sum_{i=1}^{N}\frac{1}{2m}(\mathbf{p}_i+\hbar\mathbf{a}_i)^2. \tag{3.4}$$

We now know[31-33] that there are three possible equivalent formulations. In the first case, the N-particle wave function of particles in a field with vector potential $\mathbf{a}(\vartheta)$ can be symmetric:

$$\psi(\ldots,\mathbf{r}_i,\ldots,\mathbf{r}_j,\ldots)=\psi(\ldots,\mathbf{r}_j,\ldots,\mathbf{r}_i,\ldots). \tag{3.5}$$

It is clear that the replacement $\vartheta\to\pi-\vartheta$ produces particles in a field with potential $\mathbf{a}(\pi-\vartheta)$ and with a many-particle wave function satisfying the following fermion condition under particle interchanges:

$$\psi(\ldots,\mathbf{r}_i,\ldots,\mathbf{r}_j,\ldots)=-\psi(\ldots,\mathbf{r}_j,\ldots,\mathbf{r}_i,\ldots). \tag{3.6}$$

For the Bohm–Aharanov potential (3.3), the field strength outside sources is $f_{\alpha\beta}=\partial_\alpha a_\beta-\partial_\beta a_\alpha=0$, i.e., the potential is a pure gauge field and can be removed with the help of a singular[8)] gauge transformation:

$$\mathbf{a}_i'=\mathbf{a}_i+\nabla f_i=0, \tag{3.7}$$

$$f_i=\frac{\vartheta}{\pi}\mathrm{Im}\sum_{k\neq i}^{N}\ln(z_i-z_k),\quad z_k=x_k+iy_k. \tag{3.8}$$

The wave function in this third case is multivalued because the substitution $(z_k-z_j)\to e^{i\pi}(z_k-z_j)$ induces the gauge phase:

$$\psi(\{r\}) \to \psi(\{r\})\prod_{k<j}\left(\frac{z_k - z_j}{|z_k - z_j|}\right)^{\vartheta/\pi}, \tag{3.9}$$

and the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} \tag{3.10}$$

describes free any-particles, so that the wave function of the system of N particles in this gauge acquires the factor $\exp(i\vartheta)$ under the interchange of two particles.[9)] Fractional statistics is treated in this representation as providing the boundary conditions for the wave function. We shall elucidate this by the following example. The exclusion of regions from the configuration space of identical particles[23] is equivalent to the boundary condition $j_n(0) \sim \psi \mathrm{d}\psi^*/\mathrm{d}x - \text{c.c.} = 0$ on the identity boundary $x = x_1 - x_2 = 0$. The general solution of the equation $j_n = 0$ is

$$\mathrm{d}\psi/\mathrm{d}x = \eta\psi. \tag{3.11}$$

The case $\eta = 0(\mathrm{d}\psi/\mathrm{d}x = 0,\ \psi$ symmetric) corresponds to bosons and $\eta = \infty$ ($\psi = 0$, ψ antisymmetric) corresponds to fermions. Intermediate values of η correspond to intermediate statistics and, for $\eta < 0$ and $x \to +0$, to edge states $\psi = \exp(\eta x)$.

The difficulty of the problem becomes particularly clear in the singular gauge (3.7), and is due to the multivaluedness[33] of the wave function (3.9). Actually, if $\mathbf{r}_{10}$ and $\mathbf{r}_{20}$ are vectors representing initial positions in the system of two particles, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the final positions, the wave function $\psi(\mathbf{r}_1,\mathbf{r}_2)$ will differ from $\psi(\mathbf{r}_{10},\mathbf{r}_{20})$ by the phase factor $\exp[i(\vartheta/\pi)\varphi]$, where φ is the angle representing the number of revolutions of one particle around the other until the indicated coordinates coincide. In the N-particle system, the phase φ depends on the positions of all the other particles. The Hamiltonian (3.4) which, on the face of it, seems quite simple, is essentially nonlocal because the vector potential $\mathbf{a}$ of one of the particles depends on the position of all the other particles in the system. In another language, we can speak of the formation of knots and links of the world lines that transport the Bohm–Aharanov flux (Fig. 3). The concept of links is well defined since the configuration space of identical particles does not contain points at which the coordinates of two or more particles are the same (this is the so-called hard-core condition).[10)] In the absence of this condition, the configuration space of any dimensionality would be singly-connected, the braids would be untangled, and only bosons would be possible (Ref. 35).[11)]

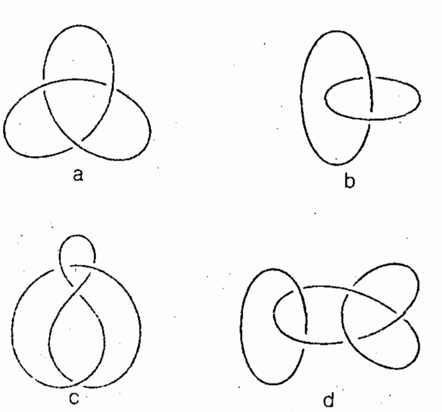

FIG. 3. Knots (a, c, d) and links (b, d). The identification of opposite points with the same indices in Fig. 2b, c gives closed braids, i.e., the knot and link of Fig. 3a, b, respectively.

To elucidate the effects associated with the transmutation of boson statistics in the field of a Bohm–Aharanov vortex, let us consider a system of two charged particles[35,36] with moment of inertia I. The Hamiltonian, the eigenfunctions, and the eigenvalues are given by

$$H = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\varphi}\right)^2, \quad \psi_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}, \tag{3.12}$$

$$E_m = \frac{\hbar^2 m^2}{2I}. \tag{3.13}$$

Since $\psi(\varphi) = \psi(\varphi + 2\pi)$ under boson interchange, the values of m are even and the ground-state energy is zero.

In the presence of the vortex (3.3), the boson Hamiltonian

$$H = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\varphi} + \frac{\vartheta}{\pi}\right)^2 \tag{3.14}$$

has the eigenvalues

$$E_m = \frac{\hbar^2}{2I}\left(m + \frac{\vartheta}{\pi}\right)^2. \tag{3.15}$$

When $\vartheta = \pi$, the numbers $m + 1$ are odd. The eigenvalues (3.15) are then identical with the free-fermion spectrum. The fact that bosons in an external pure gauge field are identical to free fermions for $\vartheta = \pi$ has frequently been discussed in the literature.[37–39] We now draw attention to the existence of angular momentum in the ground state.

The alternative situation is that of three anyons with multivalued wave function or interacting Bose or Fermi particles, which gives rise to the interaction energy[40]

$$V = \sum_{i<j} \frac{V_0(\vartheta)}{(x_i - x_j)^2}. \tag{3.16}$$

The motivation for the x^{-2} potential would also apply in one-dimensional spin systems with the Hamiltonian

$$H = \frac{1}{2}\sum_{i\neq j} \frac{\sigma_i\sigma_j}{(x_i - x_j)^2} \tag{3.17}$$

and the exact solution given in Refs. 41 and 42. It finds support in the two-dimensional case as well. The wave function[43–45] that might have been thought to be the ground state is actually found[46] to be the exact eigenfunction of the Hamiltonian

$$H = \sum_{\alpha}(D_\alpha^+ D_\alpha + \text{h.c.}), \quad D_\alpha = \sum_{\beta\neq\alpha}(z_\alpha - z_\beta)^{-1}\sigma_\alpha\sigma_\beta, \tag{3.18}$$

where $z = x + iy$ is the complex coordinate of the particle in the x,y plane. The potential $|z_i - z_j|^{-2}$ excludes tunneling and ensures that the hard-core condition is satisfied under the braiding operation.[35] In other words, the wave function does not "leak out"[40] (there is no interference) between different sectors of the configuration space of the set of particles.

4. LONG-WAVE DESCRIPTION

It is now convenient to return to the spin Hamiltonian (2.6) in order to explain the mechanism[47] responsible for excitations with intermediate statistics in two-dimensional spin systems. With this in view, we allow the lattice constant to tend to zero and consider the long-wave limit[47–49] of the two-sublattice antiferromagnet (2.6) described by the effective Lagrangian

$$L = (2/g^2)(\partial_\mu \mathbf{n})^2 \tag{4.1}$$

of the O(3)-symmetric nonlinear σ-model. The unit vector can be interpreted as the antiferromagnetic order parameter. The system of units is chosen so that the spin-wave velocity is equal to unity, $\mu = 0, 1, 2$, and $g^2 \sim J^{-1}$ is the coupling constant. Until we reach (4.17), we shall confine our attention to the dynamics of spin degrees of freedom.

Let us write the vector

$$\mathbf{n} = z^+ \sigma z \tag{4.2}$$

in terms of the complex doublet

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{4.3}$$

with the constraint

$$z^+ z = 1, \tag{4.4}$$

since $n^2 = 1$. The Lagrangian (4.1) now takes the form

$$L = \frac{2}{g^2}(\partial_\mu z^+ \partial^\mu z + z^+ \partial_\mu z \cdot z^+ \partial^\mu z). \tag{4.5}$$

The dynamics of a quantum-mechanical system is described by the Feynman integral of $\exp(iS)$ over all paths, where $S = \int d^3x L$. The exponential $\exp(iS)$ does not define the action completely. In particular, the action does not change when the Hopf term

$$\mathscr{H} = (1/4\pi^2)\epsilon^{\mu\nu\lambda}(z^+\partial_\mu z)(\partial_\nu z^+ \partial_\lambda z), \tag{4.6}$$

written in the local form[50–53] is added to the Lagrangian.[12)] This term characterizes the degree of mapping of the compactified space-time[13)] S^3 into the space S^2 of the field z. The Hopf invariant $\int d^3x$ is an integer (homotopy class), so that the addition of the Hopf term with angular factor ϑ that is a multiple of $2\pi k$ to the Lagrangian does not alter the transition amplitude.

We draw attention to the fact that the Lagrangian (4.5) is invariant under local U(1) gauge transformations of the form $z(x) \to \exp(i\alpha(x))z(x)$. Hence, in the continuous limit, just as on the lattice, there is a U(1)-symmetric gauge potential a_μ that enables us to write [47,48] the Lagrangian (4.5) in the following form when (4.6) is taken into account:

$$L = \frac{2}{g^2}|(\partial_\mu + ia_\mu)z|^2 - \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda. \tag{4.7}$$

We can verify that (4.7) and the sum of (4.5) and (4.6) are equivalent, but only in the long-wave limit, by using the equation of motion $a_\mu = iz^+\partial_\mu z$ after substituting it in (4.7). [Note that (4.7) retains only the lowest-order derivatives,[53] which are the leading terms in the long-wave limit.]

The last term in (4.7) is called the Chern–Simons term. It was originally introduced in Refs. 54–65 to provide a topological mass-generation mechanism for gauge fields. It was first used in connection with high-T_c superconductivity theory in Refs. 47 and 48. There is a large number of recent publications devoted to the description of superconducting and normal states in strongly-correlated spin systems and to the fractional quantum Hall effect, which involve the Chern–Simons term. There are several somewhat different ways (which become identical in the long-wave limit) of taking into account the topological terms in the action.[53] For example,

$$L = a_\mu j^\mu - (k/4\pi)\epsilon^{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda, \tag{4.8}$$

where k is an integer and the conserved topological current

$$j^\mu = (1/8\pi)\epsilon^{\mu\nu\lambda}\varepsilon_{abc} n^a \partial_\nu n^b \partial_\lambda n^c \tag{4.9}$$

acts as a source of the auxillary variable, namely, the vector potential a_μ that does not possess an independent dynamics. The zero-order component of the current density, which is identical to the topological charge density of skyrmions,[65] defines, by virtue of the equation of motion

$$(k/4\pi)\epsilon^{\mu\nu\lambda}f_{\nu\lambda} = j^\mu, \tag{4.10}$$

the flux $\int d^2x f_{12}$ of the gauge field $f_{\nu\lambda} = \partial_\nu a_\lambda - \partial_\lambda a_\nu$ (Ref. 37). The gauge potential a_μ plays the role of a Lagrange multiplier in this approach, and (4.10) is the constraint equation. Different ways of taking topological terms into account are reviewed in Ref. 53. Here, we note only the connection between the relation $k = 1/8\pi\beta$ in (4.7) and the duality of the statistical phases: $\vartheta_\beta/\pi = -\pi/\vartheta_\alpha$, $\vartheta_\beta = -1/8\beta$, $\vartheta_\alpha = \pi/k$ (Ref. 53).

There is considerable interest in the fact that $k \neq 0$ is possible in the ground state. The solutions of the classical equations of motion[56] are then the Bohm–Aharanov potentials (3.3) with $\vartheta = \pi/k$ (Refs. 66–69). This means that the Chern–Simons term induces fractional statistics with phase $\exp(i\pi/k)$. When $k = 2$, the resulting semifermions carry the Laughlin–Kalmeyer spin quantum numbers.[43,70–72] The quasiclassical configurations of the field z (Refs. 72–74) then suggest that there are regions of breakdown in antiferromagnetic order with characteristic dimensions $n^{-1/2}$ where n is the two-dimensional dopant concentration. Their existence and stability are determined by the zero-order component of the topological current density (4.9). Actually, as we pass to the lattice, the mean degree of noncollinearity[45,76,77]

$$E_{123} = \langle \mathbf{s}_1 [\mathbf{s}_2 \mathbf{s}_3] \rangle \tag{4.11}$$

of three neighboring spins in the unit cell becomes equal to[74] the topological charge density (4.9) and gives the flux density of the gauge field $\varepsilon^{\mu\nu}\langle \mathbf{s}[\partial_\nu \mathbf{s} \partial_\mu \mathbf{s}] \rangle$ (Fig. 4). The pseudo-

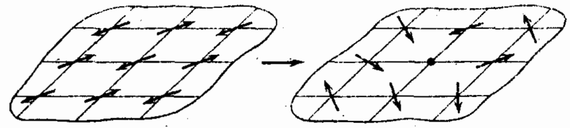

FIG. 4. Appearance of a spin vacancy and noncollinearity of neighboring spins during the destruction of antiferromagnetic order.

scalar order parameter (4.11) was first introduced in Refs. 45, 76, and 77 as a characteristic of the resulting chiral state.

The average (4.11) in the lattice description and the Lagrangians (4.7) and (4.8) in the continuous description are noninvariant under the two-dimensional parity operation $P(x,y\to -x,y)$ and under time reversal T. Spontaneous T- and P- symmetry breaking is possible over distances of the order of the correlation length $m^{-1/2}$, but does not occur in the basal planes or at right angles to them at large distances because of "screening effects." These phenomena will be examined later. Here we merely note that symmetry breaking under T- and P- inversions is an important, but not the only, characteristic of the chiral ground state of the model (4.7).

Another fundamental characteristic of a state is the integer k in the Lagrangian (4.8). This is important, and we must explain it in some detail. We know[11,77,78] that the system given by (2.6) exhibits not only local U(1) symmetry, but is also symmetric[79–81] for $\bar{n}=1$ under local SU(2) transformations. The Chern–Simons terms

$$L=\frac{\vartheta\epsilon^{\mu\nu\lambda}}{8\pi^2}\mathrm{Tr}(a_\mu\partial_\nu a_\lambda+\frac{2}{3}a_\mu a_\nu a_\lambda) \tag{4.12}$$

is then found to contain the gauge potential a_μ raised to the power 3; in general, this term is noninvariant under global gauge transformations. Their invariance under these transformations occurs only[59] for $\vartheta=2\pi k$.

Terms that are of the third degree in a_μ do not appear in the case of the U(1)—symmetric[14)] Chern–Simons term. It would therefore appear that the factor in front of the Chern–Simons term can be arbitrary. However, this is not so. The true answer depends on the compactification of space-time, i.e., on the boundary conditions imposed on the fields **n** or z. For compactification on a sphere S^3, this factor is rigorously equal to zero.[82] And it is only for compactification on the torus $S_2\times S_1$ or $S_1\times S_1\times S_1$ that it is nonzero and equal to an even integer[69,83]

$$k=2\bar{k},\quad \bar{k}=1,2,3,\dots. \tag{4.13}$$

This problem is closely related to that of fermion doubling[84] on a lattice. The point is that the Chern–Simons term is the (2 + 1)-dimensional quantum parity anomaly in a system of noninteracting massive fermions placed in the gauge field. Integration over the fermion fields in the Feynman integral gives the imaginary part of the logarithm of the fermion determinant whose evaluation gives the Chern–Simons term.[60] The result of this calculation[85] is in agreement with the fermion doubling effect on a lattice, but only for even values of k (see Refs. 83, 86, and 87).

We shall neglect the contribution of (4.1) to the total Lagrangian for large coupling constants g^2. It will then contain only the Chern–Simons term and will yield the action[15)]

$$S=\frac{k}{4\pi}\int_M d^3x\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{4.14}$$

of the topological theory[88] that describes low-energy global excitations. Witten has recently shown[89] that the dimensionality of Hilbert space for (4.14) is finite in toroidal compactification M, and is equal to k. Since (4.14) contains only the first derivative with respect to time, the Hamiltonian must vanish and its states have zero energy. This means that the constant k is related to the dimensionality[90–92] of the ground-state degeneracy space of the dynamic system (4.1) that includes the Chern–Simons term. In 1975, Berezin discussed the general rules for the quantization of systems whose phase space is the complex Kähler manifold.[16)] In the special case of a torus, Berezin showed[94] that Planck's constant could assume only the discrete set of values

$$n=4\pi/k \tag{4.15}$$

with dimensionality of Hilbert space equal to k, and the admissible phase-space manifold being a grid on a torus: $(p,q)=(m,n,\hbar/2)$. In this context, the quantum numbers m and n label the magnetic charge (interpreted as the source of vortices) and the electric charge (interpreted as the source of spin-wave excitations), respectively.

The resulting noncollinearity of the remaining spins in the presence of dopants signifies, as already noted, the onset of correlations between next-nearest neighbors on the grid, i.e., diagonal correlations in the unit cell. The magnitude of the correlations is determined by the topological charge density (4.9), (4.11) that produces, in its turn, the flux of the fictitious magnetic field as a consequence of (4.10). The particle-number deficit is rigidly related by this equation to the number of topological spin-texture defects. After the integration of the zero-order component, it follows from (4.10) that the ratio of the topological charge Q to the flux Φ of the statistical magnetic field is

$$Q=(k/2\pi)\Phi. \tag{4.16}$$

The question is: what is the dynamics of the positively-charged vacancies?

The site-filling condition

$$\bar{n}_i=\langle\sum_\alpha c^+_{i\alpha}c_{i\alpha}\rangle<1 \tag{4.17}$$

with

$$\frac{1}{N}\sum_i\bar{n}_i=1-x, \tag{4.18}$$

where N is the total number of sites, will be admissible if we write the Fermi variable

$$c_{i\alpha}=\chi^+_i z_{i\alpha} \tag{4.19}$$

as a product of the neutral spinor z (4.3) and a charged (complex) spinless field

$$\chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$$

of "holes" with $n_i=1-\chi_i^+\chi_i$. The two components of the field χ correspond to different chiralities.

The statistics is determined by the phases acquired as a result of the particle interchange, i.e.,

$$c_{i\alpha}\to e^{-i\pi}c_{i\alpha}, \tag{4.20}$$

$$z_{i\alpha}\to e^{-i\pi/k}z_{i\alpha}. \tag{4.21}$$

Consequently,

$$\chi_i\to e^{i\pi(1-k^{-1})}\chi_i. \tag{4.22}$$

If we use the terminology of the RVB theory, the spin and charge of the electron or hole in strongly correlated systems are distributed between the spinon (field z) and the holon (field χ) while the electron and hole can be looked upon as bound states of these elementary excitations. In the case of strong correlations with $k=2$, the holon and the spinon are both semifermions. For large k, the statistics of the z-quanta of spin degrees of freedom is close to boson statistics, whereas the statistics of charged χ-particles in this limit is close to the statistics of fermions.

Substituting (4.19) in (2.14), we find in the long-wave limit that the Lagrangian describing the dynamics of charged excitations is

$$L = i\chi_\sigma^+ D_0\chi_\sigma + \frac{1}{2m}|D_k\chi|^2 + iz^+(\partial_\mu + iea_0)z + \frac{1}{2m^*}|(\partial_k + iea_k)z|^2 - \frac{ke^2}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda, \tag{4.23}$$

where $D_\mu = \partial_\mu - ie(a_\mu + A_\mu)$ is the covariant derivative and A_μ is the potential of the external electromagnetic field. The statistical potential is renormalized so that the charge e appears as a common factor in the covariant derivative D_μ; the covariant derivative $\partial_\mu + iea_\mu$ contains only the statistical potential a_μ because the field z is electrically neutral (m,m^* are the effective masses). We note once again that the gauge potential a_μ in (4.23) plays the part of the Lagrange multiplier[72] that represents local restrictions on the distributions of charge and spin degrees of freedom.

We thus have two approaches, namely, (3.5), (4.21) and (3.6), (4.22). In the first, i.e., the bosonic, method[33] ($\vartheta = \pi/k$), the excitations take the form of anyons that include the Bohm–Aharanov vortex, and the N-particle wave function (3.5) is symmetric.

In the second case[31] ($\vartheta = \pi(1-k^{-1})$), the N-particle wave function (3.6) is antisymmetric, and the component quasiparticles also contain the Bohm–Aharanov vortex.[96] The parameter $1/k$ in the phase $\vartheta = \pi(1-k^{-1})$ enables us in this approach to perform controlled calculations[97] for $k \gg 1$, beginning with the fermions. Some of the details of operations that begin with the fermion approach will be discussed in Sec. 9. In the next section, we examine common features of the two approaches, and also the particular features of the bosonic picture.

5. QUALITATIVE DESCRIPTION

We shall consider that the number of defects in spin order is so large that the flux of the statistical magnetic field is distributed uniformly over the entire plane. This means that, by virtue of (4.16), the particles find themselves in the average fictitious magnetic field of strength

$$b = (2\pi/k)n, \tag{5.1}$$

which depends on the concentration n of the dopants.[17] Each of the Bohm–Aharanov potentials is a pure gauge potential and therefore yields zero magnetic field. However, P-parity breaking affects the scattering amplitude[98] in such a way that the scattering of one particle by another selects one of the two sides as special. This can be imagined as rotation of the particles in the magnetic field b given by (5.1). The final answer (5.1) is, of course, wholly determined by the topological nontriviality of the Bohm–Aharanov vortices. Actually, as a particle travels on a closed contour C around nS particles, its wave function acquires the phase $\int a dl = (2\pi n/k)S$ which is equal to the flux of the magnetic field (5.1) through the area S inside the contour C.

A typical trajectory of a particle rotating in the magnetic field $b = (2\pi/k)n$ with velocity $v = (4\pi n)^{1/2}/m$ and cyclotron radius $R = mv/B$ contains, on average, k^2 particles:

$$n\pi R^2 = k^2. \tag{5.2}$$

When $k=2$, all the particles in the unit cell participate in this motion. The particle energy spectrum then takes the form of Landau levels

$$\varepsilon_l = (l + 1/2)\omega_c \tag{5.3}$$

with cyclotron frequency $\omega_c = b/m$, $l = 0,1,2,\ldots$ and the following degree of degeneracy per unit area:

$$N = \frac{1}{2\pi l_b^2} = \frac{b}{2\pi} = \frac{n}{k} \tag{5.4}$$

where $l_b = b^{-1/2}$ is the magnetic length. The filling factor

$$\nu = n/N = k \tag{5.5}$$

shows that k of the Landau levels are completely filled.

The cyclotron radius R, the magnetic length l_b, and the mean separation between the particles $n^{-1/2}$ are related by

$$R = (2k)^{1/2} l_b = \frac{k}{\sqrt{\pi}} n^{-1/2}. \tag{5.6}$$

Hence it is clear that the mean-field approximation is valid for $l_b \gg n^{-1/2}$, but only for large k. When $k=2$, we have $R \sim l_b \sim n^{-1/2}$, and the particle orbits intersect. We therefore have, in accordance with the above assumption, a uniform field distribution and a uniform anyon density. The uniform state is stable because the transfer of particles from one orbit to another is accompanied by an increase in the magnetic field and, correspondingly, an increase in ω_c.

We now ask: is this distribution, the distribution of pairs or of single excitations? Qualitative analysis[33] argues in favor of the former. Actually, the motion of two semifermions in a general set of N quasiparticles is a $(N-2)+2$ problem of the motion of two excitations in the field $b = \pi n$ of the remaining $N-2$ quasiparticles. For single excitations, this is a $(N-1)+1$ problem.

The two-particle problem divides into the problem of relative motion plus the motion of the center of mass of the particles. The energy of relative motion $\omega_c/2 = b/2m$ of two identical particles is, of course, equal to the lowest energy $\omega_c/2$ in the single-particle problem (5.3). However, the center of mass in the two problems does not 'feel' the magnetic field b because the phase $\exp[2i(N_1 2\pi)/k]$ acquired when the N_1 particles complete a closed path around the center of mass is equal to unity for $k=2$. Hence, the energy of a pair, which consists of the energy of relative motion plus the energy of the center of mass, remains equal to $\omega_c/2$ which is less than the energy $\omega_c/2 + \omega_c/2$ of the two single excitations in the $(N-1)+1$ problem. We thus see that it is more energetically favorable for the semifermions to combine into pairs and form a boson. The pairing mechanism will be discussed in Sec. 9 from the fermion point of view. It throws additional light on the small-scale spatial structure of the pairs.

We saw in Sec. 3, and will now verify, that intermediate statistics gives nonzero angular velocity and kinetic energy, even in the ground state. For $k = 2$, the kinetic energy of the charged particles is equal to the energy $\omega_c/2 = 2\pi n/k^2 m$ of the first Landau level, and is determined by the density n of the topological defects.

The semifermion character of strongly correlated excitations at $k = 2$ has two manifestations. First, this is in part a Bose ensemble in the sense that, since there is no Fermi energy, there are no excitations such as electron-hole pairs that appear as a result of the crossing of the Fermi surface. Second, when the density is increased at any particular point, particles will transfer from the region of higher density to the region of lower density, and will tend to produce a uniform distribution. This flow arises in part from the fermion exchange pressure and signifies the existence of collective excitations with a linear dispersion law. We shall show this by the method of Ref. 33.

The energy of a system of N particles

$$N\frac{\omega_c}{2} = \frac{\pi N n}{2m} = \frac{\pi}{2m}\int d^2x n^2 \tag{5.7}$$

together with the kinetic energy and the condition for constant total number of particles is[33]

$$F(n) = \int d^2x\left(\frac{mnv^2}{2} + V(n)\right), \tag{5.8}$$

$$V(n) = \mu n + \gamma n^2, \tag{5.9}$$

where $\gamma = \pi/2m$ and μ is the chemical potential. Next, the hydrodynamic equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{mn}\nabla p = -\frac{2\gamma}{m}\nabla n \tag{5.10}$$

and the continuity equation

$$\partial n/\partial t + \mathrm{div}(n\mathbf{v}) = 0 \tag{5.11}$$

yield the following acoustic spectrum (after linearization around the mean density $n_0 = |\mu|/2\gamma$) for the long-wave collective motions with density n and statistical magnetic field b:

$$\omega = v_0 q, \quad v_0^2 = \pi n_0/m^2 . \tag{5.12}$$

The reason for the transition from the dispersion relation $\omega \sim q^2$ to $\omega \sim q$ is, of course, the solid-core condition for bosons or, in fermion language, the finite exchange pressure. The condition $\omega \sim q$ is not sufficient for superfluidity. We know[99] that the necessary condition is a finite limit $\min\lim(\omega(q)/q)$, i.e., the spectrum $\omega(q)$ must contain a roton portion for finite q. In other words, the quantum liquid that is compressed in the long-wave limit must have small-scale structure that ensures the presence of a gap at short wavelengths. This is provided by the vortex distributions associated with the degrees of freedom under consideration. In Sec. 8, we shall examine factors responsible for the appearance of the gap $\omega(0)$ in the collective-excitation spectrum in the nonsuperfluid state even for $q \to 0$, which signifies a transition to a perfectly incompressible quantum liquid. If we start with this state, the phenomenon of compressibility, which is seen as the disappearance of the gap $(\omega(0) \to 0)$ and the onset of collective excitations, i.e., Goldstones with a linear dispersion law (5.12), can be regarded as evidence for the transition of correlated pairs consisting of semifermions to the superfluid state.

If we look upon the energy (5.8) as a Ginzburg-Landau functional, we can use the classical formulas to obtain the expression for the square of the correlation radius

$$\xi^2 = \frac{1}{m|\mu|} = \frac{k}{2\pi n_0}, \tag{5.13}$$

which is the same as the square of the magnetic length l_b^2.

The charge density $n = (k/4\pi)\varepsilon^{ik} f_{ik}$ and the current density $j_i = nv_i = (k/2\pi)\varepsilon^{ik} f_{0k}$ of the anyons determine the gauge field $f_{\nu\lambda}$ by means of (4.10).[18] Hence the expression for the energy given by (5.8) can be rewritten in the long-wave limit by exploiting this local relation in the form of the Maxwell-type Lagrangian for the field $f_{\nu\lambda}$. Actually, since the potential and kinetic energies are equal,

$$\gamma\int n^2 d^2x = \frac{k^2}{4\pi m}\int f_{ij}^2 d^2x, \tag{5.14}$$

$$\int\frac{mnv^2}{2}d^2x = \frac{mk^2}{8\pi^2 n_0}\int f_{0k}^2 d^2x. \tag{5.15}$$

The density of the Lagrangian in the system of units in which the velocity of sound (5.12) is equal to unity is given by[33,100]

$$L = (1/16\pi g^2)f_{\mu\nu}^2 \tag{5.16}$$

with the coupling constant given by $g^2 = m/2k^2$.

The first observation that can be made in connection with this experiment is that (5.16) does not have the Chern–Simons term. This means that the breaking of P- and T-inversion invariance, which is valid for short distances and high frequencies, does not occur for large distances.[101,102] This is an approximate result that will be improved later. Second, semifermions are electrically charged, so that $(e/d)j_\mu$ is the three-dimensional electric current density, where d is the separation between the planes. By calculating the current correlation function we can show for (5.16) that the London relation is valid for current and potential,[33] and the square of the penetration depth is given by

$$\lambda^2 = mc^2 d/4\pi e^2 n_0. \tag{5.17}$$

We note that this expression is a consequence of the Galilean invariance[103] that is valid for different forms[104] of the relation between the current and the potential or field that produces the Meissner effect. The formulas for the correlation and the London lengths [given (5.13) and (5.17)] readily show[71] why the Ginzburg–Landau parameter

$$\varkappa^2 = \frac{\lambda^2}{\xi^2} = \frac{mc^2 d}{2ke^2} \tag{5.18}$$

is so large. When $k = 2$ and $d/4 \sim \hbar^2/me^2$, the Ginzburg-Landau parameter is $\varkappa \sim \alpha^{-1} \sim 10^2$ where $\alpha = e^2/\hbar c$ is the fine-structure constant.

Since we have only the single parameter ω_c with the dimensions of energy at our disposal, it follows that both the gap in the spectrum of single-particle excitations and the superconducting transition temperature

$$T_c \propto n/mk^2 \tag{5.19}$$

will be determined by the topological defect density.[71,105]

6. THE STRUCTURE OF THE ENERGY SPECTRUM

The main property of the chiral state is the appearance of the dynamically generated statistical magnetic field, which in turn affects the dynamics of the particles themselves by modifying the particle spectrum and wave functions. Renewed interest[106] in this problem[107–110] is due to the fact that the energy of the completely filled ground state in a magnetic field and in the periodic potential, obtained in the strong-coupling approximation, is found to be lower than in zero magnetic field.[106,111]

We now return to the Hamiltonian (2.14):

$$H = -\tilde{t}\sum_{\langle i,j\rangle} e^{-i\theta_{ij}} c_i^+ c_j + \text{h.c.} \tag{6.1}$$

which describes a set of spinless charged particles in a uniform magnetic field with flux $\phi = p/q$ (2.13) through the unit cell.

The Hamiltonian (6.1) can be generalized by including in the summation[112,113] the next-nearest neighbors in the cell, by taking into account the different hopping amplitudes along the x and y axes $((\tilde{t}_a)$ and $(\tilde{t}_b)$; Refs. 114 and 115), and by extending the discussion to the spatially three-dimensional case.[116,117]

In the simplest version of (6.1) (with different amplitudes $\tilde{t}_{a,b} \equiv t_{a,b}$, the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ in the k-representation takes the form of the Harper equation[118]

$$-2t_a\cos(k_x + 2\pi\phi j)\psi_j - t_b(e^{-ik_y}\psi_{j-1} + e^{ik_y}\psi_{j+1}) = \varepsilon(k_x, k_y)\psi_j, \tag{6.2}$$

and the wave function assumes the form

$$|\psi\rangle = \sum_{j=1}^{q} \psi_j c^+(k_x + 2\pi\phi j, k_y)|0\rangle \tag{6.3}$$

where the lattice constant is equal to unity and $|0\rangle$ is the vacuum state. In (6.2) we use the condition $\psi_{j+q} = \psi_j$ and choose the following gauge: along the lattice link in the direction of the x axis, $\vartheta_{ij} = 0$, and on the link between lattice sites $i = (n,m)$ and $j = (m,n+1)$ in the direction of the y axis, $\theta_{ij} = 2\pi\phi n$. The lattice position index i has the Cartesian coordinates (n,m) where n and m are integers.[19)] Equation (6.2) is a model of strongly-coupled particles with diagonal modulation due to the first term.

When $\phi = p/q$, equation (6.2) has q eigenvalues for each pair k_x, k_y. In other words, the energy spectrum in the absence of the field

$$\varepsilon(k_x, k_y) = -2t_a\cos k_x - 2t_b\cos k_y \tag{6.4}$$

splits in a magnetic field into q energy bands with wave vectors corresponding to q magnetic Brillouin zones

$$-\pi/q \le k_x \le \pi/q, \quad -\pi \le k_y \le \pi, \tag{6.5}$$

and has an exceedingly rich structure.[107,108] The magnetic Bloch function ψ_j satisfies the following conditions[85] on the boundary (6.5):

$$\begin{aligned}\psi_j(k_x, k_y) &= \psi_j(k_x + 2\pi/q, k_y)\\ &= \psi_j(k_x, k_y + 2\pi)\exp[-i(jk_x + qk_y)].\end{aligned} \tag{6.6}$$

Before we turn to a description of the spectrum, we draw attention to our choice of the gauge. The expression given by (6.2) can be interpreted as describing the dynamics of a particle that hops between q points distributed on a circle. The clockwise hop amplitude is $\exp(-ik_y)$ and the anticlockwise amplitude is $\exp(ik_y)$, where the energy at each point j is $2t_a\cos(k_x + 2\pi\phi j)$. The dual transformation[119]

$$\psi_j = \sum_{l=1}^{q} e^{i2\pi\phi jl} f_l, \tag{6.7}$$

can be treated as the gauge transformation $\mathbf{a} = (-by,0,0) \to \mathbf{a} = (0,bx,0)$, and imposes the replacements $t_a \to t_b$ and $k_x \to k_y$, whereas the energy at each point on the circle transforms into the hop matrix

$$-t_a(e^{-ik_x}f_{l-1} + e^{ik_x}f_{l+1}) - 2t_b\cos(k_y + 2\pi\phi l)f_l = \varepsilon(k_x, k_y)f_l; \tag{6.8}$$

in which $\theta_{ij} = 0$ along the link parallel to the y axis and $\theta_{ij} = 2\pi\phi m$ along the link between sites (n,m) and $(n+1)$ in the direction of the x axis, where

$$-\pi \le k_x \le \pi, \quad -\pi/q \le k_y \le \pi/q. \tag{6.9}$$

This transformation restores the equal importance of the momentum components k_x and k_y, and confines us to the square region $-\pi/q \leqslant k_x \leqslant \pi/q$, $-\pi/q \leqslant k_y \leqslant \pi/q$.

In the very curious gauge employed in Ref. 120, the eigenvalue equation has a near-diagonal modulation. If we take $\theta_{ij} = -\pi\phi(n+m)$ for the link between the sites $i = (n,m)$ and $j = (n+1,m)$ in the direction of the x axis, and $\theta_{ij} = \pi\phi(n+m)$ for the link between $i = (n,m)$ and $j = (n,m+1)$ along the y axis, we obtain the one-dimensional discrete system with the following equation:

$$-t_{j-1}\psi_{j-1} - t_j\psi_{j+1} = \varepsilon\psi_j, \tag{6.10}$$

$$t_j = 2t\cos(K + 2\pi\phi j), \tag{6.11}$$

$$\begin{aligned} K + \pi\phi j &= \frac{k_x + k_y}{2} + \frac{\pi\phi}{2}, \quad 0 \le K \le \pi/q,\\ k &= \frac{k_x - k_y}{2} - \frac{\pi\phi}{2}, \quad 0 \le k \le 2\pi. \end{aligned} \tag{6.12}$$

States with wave vectors K and $K \pm \pi\phi$ are then found to be coupled, but there is no coupling between different k.

The energy spectrum of the problem defined by (6.1) can be obtained by solving the secular equation which, after substituting $\psi_{j'} = \psi_j\exp(ik_y j)$ in (6.2), takes the form

$$\partial\begin{vmatrix} M_1 - \varepsilon & -t_b & & & -t_b e^{-iqk_y}\\ -t_b & M_2 - \varepsilon & 0 & & \\ & & \ddots & & \\ & 0 & & M_{q-1} - \varepsilon & -t_b\\ -t_b e^{iqk_y} & & & -t_b & M_q - \varepsilon \end{vmatrix} = 0, \tag{6.13}$$

where $M_j = -2t_a\cos(k_x + 2\pi\phi j)$. We then readily see from this equation and from the duality relation (6.7) that the secular equation can be written[114]

$$P_{p/q}(\varepsilon) = 2t_a^q\cos(qk_x) + 2t_b^q\cos(qk_y), \tag{6.14}$$

where $P_p/_q(\varepsilon)$ is a polynomial of degree q that has q real roots because the Hamiltonian is Hermitian. We noted in Sec. 4 that even values of q were important for our purposes. We will therefore confine our attention to such values. In the case of even q, the polynomial $P_p/_q(\varepsilon)$ is a symmetric function relative to the band center $\varepsilon = 0$. At this point, it is equal to $2t_a^q + 2t_b^q$. This is readily verified by using the property of duality and the special case $t_a = 0$ in (6.13). The fact that the spectrum is symmetric is a consequence of the hidden supersymmetry of the problem defined[121] by (6.1). The spectrum thus splits into $q/2$ subbands with positive energy and $q/2$ subbands with negative energy. Each subband is then characterized[114] by an integral value of Hall conductivity, for which there is a topological reason since σ_{xy} is the first Chern class of the vector stratification with base in the Brillouin zone.

At $\varepsilon = 0$ and for

$$k_x^{(0)} = \bar{n}\pi/q, \quad k_y^{(0)} = \bar{m}\pi/q, \tag{6.15}$$

where $\bar{n}$ and $\bar{m}$ are integers, the two central subbands are found to touch, and the spectrum exhibits linear dispersion near the points given by (6.15). Each point $f = (\bar{n}, \bar{m})$ in momentum space is characterized by a topological invariant, i.e., the so-called chirality $\gamma_f = \pm 1$ (Ref. 121). The linear dispersion $\varepsilon(\mathbf{k}) = \pm\,\mathrm{const}|\mathbf{k} - \mathbf{k}^{(0)}|$ near the degeneracy points (6.15) means that we are dealing with a set of low-energy excitations that are q-color massless Dirac particles with the Lagrangian[87]

$$L = \sum_{f=1}^{q} \bar{\psi}_f(i\partial\!\!\!/ + a\!\!\!/)\psi_f \tag{6.16}$$

The generalization of (6.16) to the massive case is given in Ref. 85 which also reproduces a more complete analysis of the problem, based on the utilization of the magnetic translation group (see below). The model used in Ref. 85 is

$$H = -\sum_{\langle i,j\rangle} c_i^+ \exp[-2\pi i \int_i^j (\mathbf{a} + \mathbf{A})d\mathbf{l}\,]c_j + \mu(c_i^+c_i - c_j^+c_j) \tag{6.17}$$

and includes the chemical potential μ in order to take into account fluctuations in the particle number in the two sublattices of this square lattice (Fig. 5), and also the total potential $\mathbf{a} + \mathbf{A}$ (see the end of Sec. 2). For the model defined by (6.17) and near the degeneracy points at which subbands

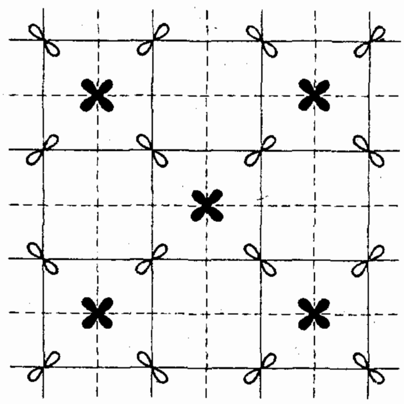

FIG. 5. Distribution of d-states (p-states) over the sites of the dual (basic) lattice in basal planes.

with positive and negative energies are found to cross, we obtain the following expression[85] for the Lagrangian of the (2 + 1)-dimensional theory that describes massive excitations in the external field $a_\mu = \bar{a}_\mu + \delta a_\mu$:

$$L = \sum_{f=1}^{q} \bar{\psi}_f(i\partial\!\!\!/ + a\!\!\!/ + m_f)\psi_f, \tag{6.18}$$

$$m_f = |\mu|\gamma_f, \quad \partial\!\!\!/ = \gamma^\mu\partial_\mu,$$

$$\gamma^\mu = (\sigma_3, \sigma_2, \sigma_1).$$

We now digress slightly from our main theme, in connection with (6.17). In a constant uniform magnetic field with flux $\varphi = p/q$ and potential $\mathbf{A}$, the current of charged particles is the Hall current

$$j_i = \sigma_{xy}\varepsilon_{ij}a_j \tag{6.19}$$

where the potential $\mathbf{a}$ appears on equal terms with $\mathbf{A}$ in the sum $\mathbf{a}+\mathbf{A}$ in (6.17). Since $j_i = -\langle\delta S/\delta a_i\rangle$, this means that, in the long-wave limit, the action S contains the Chern–Simons term (4.12) with $\vartheta = 2\pi\sigma_{xy}$. Since $\vartheta = \pi k = 2\pi\bar{k}$, the Hall conductivity $\sigma_{xy} = \bar{k} = q/2$ is an integer. The total contribution to $\vartheta/2\pi = \Sigma_f\gamma_f = \sigma_{xy}^- - \sigma_{xy}^+$ (Refs. 75 and 85), where the signs ($\pm$) denote two adjacent central bands and the topological invariant, i.e., the first Chern class $\sigma_{xy}^\pm$, can be written[112] in terms of the following integral over the momentum space of the Brillouin zone:

$$\sigma_{xy}^\pm = \frac{1}{2\pi i}\int_0^{2\pi/q} dk_x \int_0^{2\pi} dk_y (\nabla_\mathbf{k} \times \langle\psi_\pm(\mathbf{k})|\nabla_\mathbf{k}|\psi_\pm(\mathbf{k})\rangle)_z = -\sigma_{xy}^\pm. \tag{6.20}$$

If parity $P(k_x \to k_x, k_y \to k_y)$ is conserved in the lattice theory, then the chiralities γf of the points k_f and $\bar{k}_f$ are opposite, i.e., $\gamma_f = -\gamma_{\bar{f}}$ and $\vartheta = 0$. When the "external" flux ϕ is present, all the lattice fermions have the same chirality $\gamma_f = \mathrm{sign}\,\phi$, and the same sign of the mass in (6.1) (Refs. 61, 62, 85, and 123).

We now turn to a discussion of the spectrum. It is clear from (6.14) that the spectrum is invariant under the shifts $k_x \to k_x + 2\pi/q$, $k_y \to k_y + 2\pi/q$. This property is, of course, clear from the explicit expressions for the energy when $q = 2$ and $q = 4$ ($t_a = t_b = 1$):

$$\varepsilon_\pm(k_x, k_y) = \pm 2(\cos^2 k_x + \cos^2 k_y)^{1/2},$$
$$\varepsilon(k_x, k_y) = \pm \{4 \pm 2[3 + \tfrac{1}{2}(\cos 4k_x + \cos 4k_y)]^{1/2}\}^{1/2}. \tag{6.21}$$

The signs of the branches of this spectrum are not interrelated. Degeneracy due to the symmetry of the spectrum under the shifts $k_{x(y)} \to k_{x(y)} + 2\pi/q$ is equivalent to the existence of operators that commute with the Hamiltonian, but not with one another. Actually, if we define matrices A and B by[121]

$$(Af)_j = e^{-2\pi i p/q} f_j, \quad (Bf)_j = f_{j+1}, \tag{6.22}$$

we have

$$AH(k_x, k_y)A^{-1} = H(k_x + 2\pi p/q, k_y), \tag{6.23}$$

$$BH(k_x, k_y)B^{-1} = H(k_x, k_y + 2\pi p/q), \tag{6.24}$$

where

$$AB = BAe^{2\pi ip/q}. \tag{6.25}$$

The Hamiltonian for (6.7) ($t_a = t_b = 1$) is

$$H_{jl}(k_x, k_y) = 2\delta_{jl}^{(q)}\cos(k_y + 2\pi j\phi) + \delta_{jl+1}^{(q)}e^{-ik_x} + \delta_{jl-1}^{(q)}e^{ik_x} \tag{6.26}$$

and will be written in the form[121]

$$H = e^{-ik_y}A + e^{ik_x}B + \text{h.c.} \tag{6.27}$$

Since p and q are not commensurate, there must be a number n that is not commensurate with p and is such that

$$n\frac{p}{q} = \frac{1}{q} + \quad \text{integer} \quad . \tag{6.28}$$

We shall use it in the new operators $\widetilde{A} = A^n$ and $\widetilde{B} = B^n$ that ensure the necessary symmetry

$$\widetilde{A}H(k_x, k_y)\widetilde{A}^{-1} = H(k_x + 2\pi/q, k_y), \tag{6.29}$$

$$\widetilde{B}H(k_x, k_y)\widetilde{B}^{-1} = H(k_x, k_y + 2\pi/q) \tag{6.30}$$

and the commutation relations

$$\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}e^{2\pi ipn^2/q}. \tag{6.31}$$

The algebra of the operators (6.31) corresponds to the magnetic translation group[124] whose representations[20)] are the eigenstates of the Hamiltonian (6.1). The magnetic translation operators $T(\mathbf{a}_j)$ that produce shifts equal to the basis vectors $\mathbf{a}_j$ of a square lattice must satisfy the relations

$$T^n(\mathbf{a}_1)T^m(\mathbf{a}_2) = (-1)^{nm}T(n\mathbf{a}_1 + m\mathbf{a}_2), \tag{6.32}$$

$$[T(\mathbf{a}_j), H] = 0, \tag{6.33}$$

$$T(\mathbf{a}_1)T(\mathbf{a}_2) = T(\mathbf{a}_2)T(\mathbf{a}_1)e^{2\pi ip/q}. \tag{6.34}$$

For example, for the flux $\phi = 1/2 (q = 2)$, the operator $T(\mathbf{a}_1)$ will commute with $T(2\mathbf{a}_2)$, and so on. Thus, in order to number the energy subbands, we have to use the set of orthogonal states $\psi, T(\mathbf{a}_2)\psi, T^2(\mathbf{a}_2)\psi, \ldots, T^{(q-1)}(\mathbf{a}_2)\psi$ generated by magnetic translations from the subgroup T^q.

If we know dispersion relations $\varepsilon_{p/q}(k_x, k_y)$ and the position of the chemical potential μ, which is determined by the number of particles per site,

$$\nu = \int_{\varepsilon_{\min}(\phi)}^{\mu} N(\varepsilon)\mathrm{d}\varepsilon, \tag{6.35}$$

we can write the total energy in the form:[21)]

$$E = \int_{\varepsilon_{\min}(\phi)}^{\mu} \varepsilon N(\varepsilon)\mathrm{d}\varepsilon. \tag{6.36}$$

The density of states is given by the analytic expression[110]

$$N(\varepsilon) = \frac{1}{2\pi^2 q}\left|\frac{\mathrm{d}F}{\mathrm{d}\varepsilon}\right| K'\left(\frac{F}{4}\right), \tag{6.37}$$

where $F(\varepsilon)$ is the determinant (6.13), the upper and lower angles of which contain $+1$, evaluated for $t_a = t_b = 1$ and $k_x = k_y = 0$ where $K'(k) = K((1-k^2)^{1/2})$ is the complete elliptic integral of the first kind. For energies corresponding to the half-filling of each subband, the density of states $N(\varepsilon)$ has logarithmic van Hove singularities, with pagoda-like behavior in their vicinity.

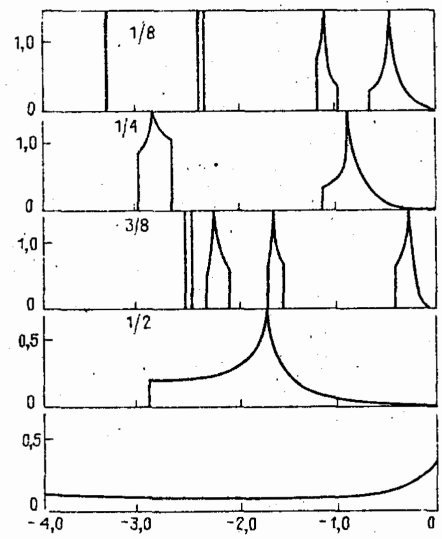

FIG. 6. Density of states as a function of energy for fluxes $\phi = 1/8, 1/4, 3/8, 1/2, 0$ (Ref. 106). The number and position of subbands are rapidly varying functions of site filling for $\nu = \phi$.

Figure 6 shows the density of states as a function of energy for $-4 \leqslant \varepsilon \leqslant 0$ and fluxes $\phi = 0, 1/2, 3/8, 1/4, 1/8$. It is clear that a variation in the flux $\phi = p/q$ is accompanied by sharp changes in the position, width, and total number of magnetic subbands. Numerical calculations of the total energy for fixed filling factors ν and different fluxes were reported in Refs. 111, 115, and 117 (Fig. 7). They confirmed the conclusion drawn in Ref. 109 that there is a set of local energy minima when the filling factor μ is related to the flux ϕ as follows:

$$\nu = M + N\phi \tag{6.38}$$

where M and N are integers. The smallest gap in the energy structure or, in other words, the largest numerical jump in the spectrum (as compared with the case of zero magnetic field) occurs for $M = 0$, $N = 1$, i.e., for

$$\nu = \phi, \tag{6.39}$$

and corresponds to the integral Hall effect with $\bar{k} = 1 (q = 2)$.[22)] Figure 8 shows the position of the absolute

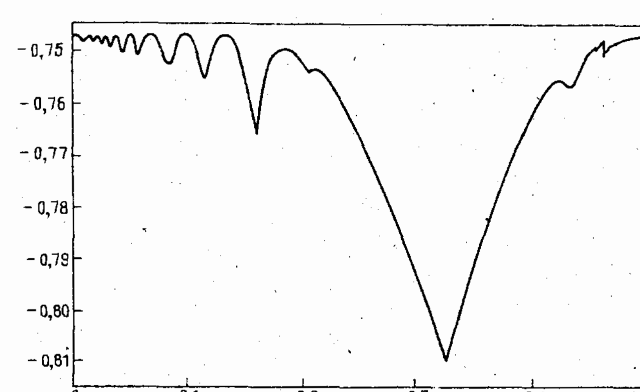

FIG. 7. Total energy as a function of flux for $\nu = 1/3$ (Ref. 111).

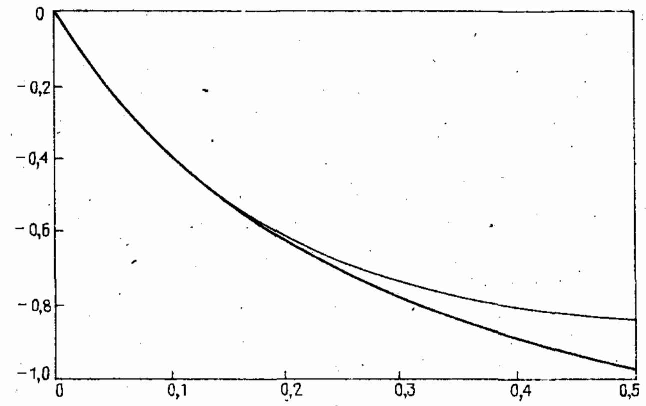

FIG. 8. Absolute total-energy minimum (for $\phi = \nu$) as a function of the band filling factor ν (Ref. 111). The upper (thin) curve shows the total energy as a function of ν for $\phi = 0$.

minimum of $E(\phi)$ as a function of the filling factor ν, subject to the restriction defined by (6.39).[111,117]

It follows from these results that the state of spinless fermions on a lattice in a magnetic field is energetically favorable. The total energy is reduced by opening the gaps and, generally, because of the development of the Peierls-type instability that accompanies the distribution of the magnetic flux over the lattice and the commensurability effect. Calculations show[120] that the result is a modulation of the hopping amplitudes T [see (6.11)] and a distortion of the lattice. In other words, "insertion" into the statistical magnetic field is equivalent to the introduction of lattice dimerization.[120]

We have already noted that each subband carries an integer-valued Hall conductivity (in units of e^2/h). Let us therefore compare (6.38) with the whole-number quantization of σ_{xy} (6.19) (Ref. 114). Suppose that the chemical potential lies int he rth gap, counting from the bottom. The Hall conductivity is then the sum of contributions due to subbands below μ and is equal to t_r (Refs. 125–127), where t_r is the solution of the Diophantine equation

$$r = qs_r + pt_r, \tag{6.40}$$

in which s_r, t_r are integers and $|t_r| \leqslant q/2$, $1 \leqslant r \leqslant q$. The distribution of the numbers t_r is shown in Fig. 9 (Ref. 112). When the chemical potential lies within any of the gaps, and ϕ varies slowly, the numbers r, q, p vary very rapidly, but t_r, s_r remain the same. This means that the global variation of $E(\phi)$ determines the numbers t_r and s_r, i.e., the Hall conductivity is determined by the topological structure of the function $E(\phi)$. At the degeneracy points (6.15), i.e., for $\varepsilon = 0$, $r = q/2$, the Diophantine equation (6.40) has the two solutions $t_r, s_r = (\pm q/2, (1 \mp p)/2)$. The Hall conductivity cannot then be uniquely determined from the Diophantine equation, and requires physical regularization (see below).

Dividing (6.40) by q and setting $s_r = M$ and $t_r = N$, we obtain (6.38)

$$\nu = \frac{r}{q} = s_r + \frac{p}{q} t_r. \tag{6.41}$$

We therefore draw attention to the fact that there are two possibilities: the Hall conductivity may be of the "hole" type ($t_r > 0$), or of the "electron type" ($t_r < 0$), and depends on the position of the chemical potential in the hierarchical structure of subbands in the lower band. This depends on

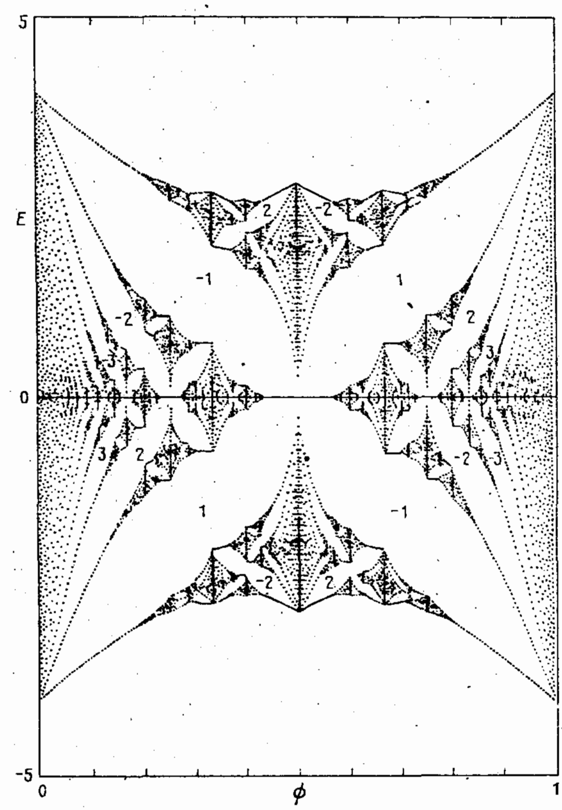

FIG. 9. Energy spectrum in the Hofstadter problem. The values of the integers t_r are shown in the energy gaps.

concentration and temperature. Figure 10 shows the dependence of μ on the filling factor ν for $\phi = 1/4$. When the temperature is reduced, we have the possibility of transition to one set of the numbers (r,q,p,s_r) with $r_r > 0$ to another with $t_r < 0$, which signifies a change in the sign of Hall conductivity.[128] The sign of the carriers is of course determined by the wave nature of the particles [as shown by (6.41)],[125] i.e., by the diffraction of the particles by the lattice.

Doping produces diagonal correlations in the unit cell. The question is: what happens when the next-nearest neighbors are taken into account in (6.1)? The ambiguity in the half-filling then disappears[112,113] and the Hall conductivity is $+q/2$ with $r = q/2$, $\phi = 1/2$ for a positive diagonal-hopping amplitude t_c. The overall picture involves a narrowing

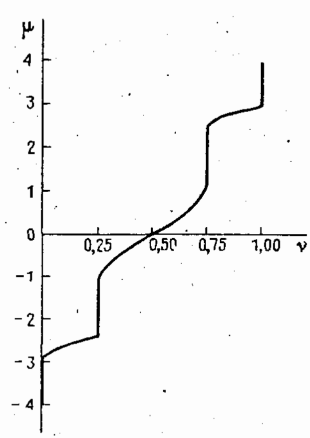

FIG. 10. Chemical potential as a function of μ for $\phi = 1/4$.

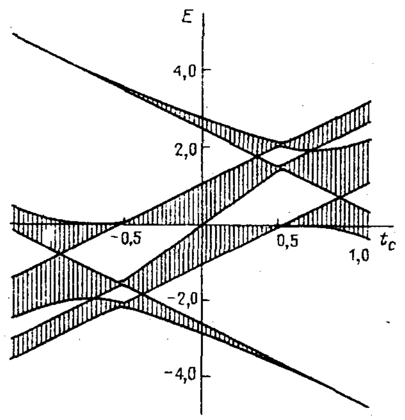

FIG. 11. Band structure as a function of diagonal hopping amplitude in the unit cell for $\phi = 1/4$ (Ref. 112).

of some subbands and the expansion of others, and the opening of new and the closure of old gaps. Figure 11 shows the subband structure as a function of t_c in the special case[112] $\phi = 1/4$.

When the gaps collapse, the global structure of $E(\phi)$ undergoes a radical change near the degeneracy points, and this is accompanied by the appearance of jumps $\Delta s_r = s_r' - s_r$, $\Delta t_r = t_r' - t_r$ in (s_r, t_r) with the conservation of r

$$r = qs_r + pt_r = qs_r' + pt_r', \tag{6.42}$$

which gives rise to jumps in Hall conductivity $\sigma_{xy} = t_r = \bar{k}$ and a change in anyon statistics. The mechanism responsible for this phenomenon can be identified[113,129] as a 'collision' between the spectrum branches with the parameter t_c changes[23)] and, as the subbands touch, q is transferred from the lower to the upper colliding subbands, i.e., we have the transfer of the basis class from the cohomology of the two-dimensional torus of the reciprocal lattice[130] $H^2(T_*^2, Z)$. The quantum numbers t_r' are therefore no longer constrained, as they were earlier, by the condition $|r_r| < q/2$, and the quantity $\sigma_{xy} = t_i'$ is in general completely random.[126,129] This 'collision' between the subbands is accompanied by a change in the parameter $k = 2|\sigma_{xy}|$ in the statistics of anyon excitations.

7. THE WAVE FUNCTION

The complete set of states $\{\psi_l\}$ of particles in the strong magnetic field due to the spin deficit and maintained by the coherent motion of the particles themselves can be taken as the basis for the construction of the many-particle wave function. The unitary transformation

$$\tilde{c}^+_{l\sigma} = \sum_i \psi_l(\mathbf{r}_i) c^+_{i\sigma} \tag{7.1}$$

introduced in Refs. 131 and 132 for the original operators $c_{i\sigma}$ of the Hubbard model, and weighted by the eigenstates ψ_l of the Hamiltonian (6.1), enables us to write the wave function of a set of $2M$ particles on a lattice of N sites in the following form:

$$|\psi\rangle = P\prod_{l\in F} \tilde{c}^+_{l\uparrow}\tilde{c}^+_{l\downarrow}|0\rangle, \tag{7.2}$$

where P is the Gutzwiller projector (2.3), the notation $l\in F$ signifies that the index l lies in the interval $1 < l < M$, and $|0\rangle$ is the vacuum state. The important point is that the transformation defined by (7.1) is diagonal in the spin and the flux, and the gauge potential is the same for spin-up and spin-down particles. Hence, the raising operator $S^+ = \Sigma_i c^+_{i\uparrow} c_{i\downarrow}$ is $\Sigma_l c^+_{l\uparrow}\tilde{c}_{l\downarrow}$ and $s_z|\psi\rangle = S^+|\psi\rangle = 0$, i.e., the state defined by (7.2) is a spin singlet.[133] The wave function (7.2) is hard because of the Hofstadter gaps. The paramagnetic part of the current is therefore small, and the response in the state (7.2) is diamagnetic.[131]

A wave function of different form, but equivalent to (7.2), was proposed in Ref. 134:

$$|\psi\rangle = \sum_{\{|\alpha\rangle\}} \det\psi_l(\mathbf{r}_j^\downarrow)\det\psi_l^*(\bar{\mathbf{r}}_l^\uparrow)|\alpha\rangle, \tag{7.3}$$

$$|\alpha\rangle = S^-_{\mathbf{r}_1^\downarrow}\dots S^-_{\mathbf{r}_M^\downarrow} c_{s_1\uparrow}\dots c_{s_{N-2M}\uparrow}|F\rangle, \tag{7.4}$$

where $S_r^- = c^+_{r\downarrow} c_{r\uparrow}$. This is the wave function of a set of M spin-down particles and M spin-up particles and $N - 2M$ holes; $l\in F$, $\bar{l}\in - F$, $|F\rangle$ is the ferromagnetic state, $\{\bar{r}_j^\uparrow\}$ are the positions of spin-up particles, and $\{c_j\} = \{\bar{r}_j^\uparrow\} - \{r_j^\downarrow\}$ the positions of the holes.

Substituting (7.1) in (7.2), we obtain

$$|\psi\rangle = \sum_{\{|\alpha\rangle\}} \det\psi_l(\mathbf{r}_i^\uparrow)\det\psi_l(\mathbf{r}_j^\downarrow)|\alpha\rangle, \tag{7.5}$$

$$|\alpha\rangle = P\prod_i c^+_{r_i^\uparrow\uparrow} c^+_{r_i^\downarrow\downarrow}|0\rangle. \tag{7.6}$$

The equivalence of (7.2) and (7.3) (Ref. 133) is a consequence of the identity of the states $\Pi_{l\in F}\tilde{c}^+_{l\uparrow}|0\rangle$ and $\Pi_{l\in - F}\tilde{c}_{l\downarrow}|F\rangle$ where $|F\rangle = \Pi_i c^+_{i\uparrow}|0\rangle$, so that $\det\psi_l(r_j^\downarrow) \sim \det\psi_l^*(\bar{r}_j^\uparrow)$. The wave function $|\psi\rangle$ can be written in the explicit singlet form

$$|\psi\rangle = P\Big(\sum_{ij} a(\mathbf{r}_i, \mathbf{r}_j) c^+_{i\uparrow} c^+_{j\downarrow}\Big)^M|0\rangle \tag{7.7}$$

with the symmetric function

$$a(\mathbf{r}_i, \mathbf{r}_j) = \sum_{l\in F}\psi_l(\mathbf{r}_i)\psi_l(\mathbf{r}_j), \tag{7.8}$$

and is used in numerical calculations.[135–137]

This picture is clear, but it does not completely satisfy us. There are doubts about the use of the composite operators (4.19) for electrons and holes that are not the elementary excitations of a system with developed quantum fluctuations (see Sec. 4). At the heuristic level, the replacement of the function (7.2) is well-known.[138] The polynomial (3.9) induced by the gauge transformation and multiplied by the product of the wave functions of the ground-state particles in the magnetic field gives the wave function[24)] (Refs. 44 and 139)

$$\psi(z_1, \dots z_N) = \prod_{j<k}(z_j - z_k)^{\theta/\pi}\prod_i G(z_i) e^{-(1/4)|z_i|^2} \tag{7.9}$$

where the magnetic length is equal to unity, $i = (l,m)$ is the site index, $z_k = x_k + iy_k \equiv l + im$ is the complex spin-down coordinate and the factor $G(z_i) = \pm 1 = (-1)^{l+m+lm+1}$ describes the distribution of the gauge potential $\mathbf{a}$ (Refs. 139 and 140).

The essential difference between (7.9) and (7.2) is contained in the function

$$\prod_{j<k}(z_j - z_k)^{\theta/\pi},$$

which represents the hard-core condition for anyon excitations. A function of the form given by (7.9) was used in Ref. 140 to determine a variety of physical variables. For $\vartheta/\pi\equiv\vartheta_\alpha/\pi = k = 2$, it was found to be very close to the chiral spin state.[45] A very useful result is reported in Ref. 46 in which it is shown that (7.9) is the ground state of the Hamiltonian (3.18). In the dual conjugate case $\vartheta/\pi = \vartheta_\beta/\pi = 1/k$, the function (7.9) for $k = 2$ describes the state of a set of semifermions.[31] We recall that the statistical phases are related by $\vartheta_\alpha/\pi = -\pi/\vartheta_\beta$ (Ref. 53) and that, in terms of the fermion picture (3.4), (3.6) in which each Bohm–Aharanov vortex carries the fraction $1 - k^{-1}$ of the flux quantum, $k = 2$ corresponds to two completely occupied Landau levels (filling factor equal to $1/(1-k^{-1})$). In the Hartree–Fock approximation, the energy of the state (7.9) with $\vartheta/\pi = 1/2$ is somewhat lower than the energy of the two-dimensional Fermi gas in an external magnetic field.[31] For the state defined by (7.9) with $\vartheta/\pi = 1/k$, the characteristic feature is the existence of charged vortex excitations with circulation $(1-k^{-1})h/m$ (Ref. 31) and a collective Goldstone mode with an acoustic spectrum in the long-wave limit, indicating that the medium is compressible. The small-scale rigidity of the anyon liquid is indicated by the roton portion of the spectrum of collective motions.[141]

When the requirements used to construct the functions (7.5) and (7.9) are combined, the result leads to[142]

$$\Phi = \mathscr{A}\Psi[z](\alpha, \alpha, ..., \alpha, \beta, \beta, ..., \beta), \tag{7.10}$$

$$\Psi[z] = \prod_{i<j}(\bar{z}_i - \bar{z}_j)(\bar{z}_{[i]} - \bar{z}_{[j]})e^S, \tag{7.11}$$

$$e^S \equiv \prod_{i<j}|z_i - z_j|^{-1/2}|z_{[i]} - z_{[j]}|^{-1/2}\prod_{k,l}|z_k - z_{[l]}|^{-1/2}, \tag{7.12}$$

which is an exact singlet eigenstate (with zero energy) of the Hamiltonian

$$H = \sum_{j=1}^{N}(\Pi_j^2 - b_j), \tag{7.13}$$

$$b_j \equiv [\nabla_j a_j]_z = 2\vartheta\sum_{k\neq j}\delta^{(2)}(\mathbf{r}_j - \mathbf{r}_k) \tag{7.14}$$

with $\vartheta/\pi = 1/2$ for a set of semifermions. The wave function given by (7.10) was obtained with the help of the supersymmetric representation[142–144] of the Hamiltonian (7.13):

$$H = \sum_{j=1}^{N} Q_j^+ Q_j,$$

$$Q_j = \Pi_j^x - i\Pi_j^y, \quad \Pi_j = -i\nabla + \mathbf{a}_j \tag{7.15}$$

The following notation is used in (7.10)–(7.15): |—antisymmetrizer, $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$ α and β—spinors with spin respectively up and down, and $j = 1, 2, ..., N$, and $[j] = N+1, N+2, ..., 2N$—spatial indices for these spin projection values. The function (7.11) is a solution of the equation $\Delta_j^2 S = -b_j$ and is proportional to the Coulomb energy of the associated classical plasma:

$$S = -(\vartheta/\pi)\sum_{k<j}\ln|\mathbf{r}_k - \mathbf{r}_j|.$$

8. ANYONS ON A LATTICE

The concept of anyons as long-wave excitations was formulated in Sec. 4. There are at present several approaches to the theory of anyons on a lattice.[145–149] We shall confine our attention to certain aspects of this picture, following the results obtained in Ref. 149. The approach from the standpoint of short distances is necessary because, as we have frequently mentioned, the correlation length is only a few times greater than the lattice constant. The details of the distribution of quantum numbers at short distances are found to be essential if we are to settle the question as to which state is realized in macroscopically large volumes.

In the anyon approach (3.10), the Hamiltonian is

$$H = -\tilde{t}\sum_{\mathbf{r},j}\bar{c}^+(\mathbf{r})\bar{c}(\mathbf{r}+\mathbf{e}_j) + \text{h.c.} \tag{8.1}$$

It describes a gas of N_a anyons for which the creation and annihilation operators satisfy the commutation relations[50,149]

$$\{\bar{c}^+(\mathbf{r}), \bar{c}(\mathbf{r}')\}_q = \delta(\mathbf{r}-\mathbf{r}'), \tag{8.2}$$

$$\{\bar{c}(\mathbf{r}), \bar{c}(\mathbf{r}')\}_q = \{\bar{c}^+(\mathbf{r}), \bar{c}^+(\mathbf{r}')\}_q = 0,$$

$$\{a, b\}_q \equiv ab + qba, \quad q = e^{i\delta}.$$

The set of quantum numbers $\{\mathbf{r}, j\}$ in (8.1) labels the links of the lattice and $\mathbf{e}_j$ is a unit vector with $j = 1$ or 2. Instead of the site index, we introduce the position vector $\mathbf{r}$ of the left side of the unit cell. The phase in (8.2) is zero for fermions, $\delta = \pi$ for bosons, and $\delta = \pi/m$ for anyons. The integer m in these expressions and the parameter k previously, e.g., in (4.7), are identical.

In fermion language, the Hamiltonian

$$H_f = -\tilde{t}\sum_{\mathbf{r},j}c^+(\mathbf{r})e^{ia_j(\mathbf{r})}c(\mathbf{r}+\mathbf{e}_j) + \text{h.c.} \tag{8.3}$$

is the same as (6.1), $\{c^+(\mathbf{r}), c(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')$, and the current density $j_0 = c^+(\mathbf{r})c(\mathbf{r})$ of the system of N_f fermions satisfies the local relation (4.10)

$$j_0(\mathbf{r}) = \vartheta B(\mathbf{x}) = \vartheta\varepsilon_{ij}\Delta_i a_j(\mathbf{r}). \tag{8.4}$$

The coordinate $\mathbf{x}$ in the flux function $B(\mathbf{x})$ defines a site in the dual lattice (see Fig. 5) that lies to the right and above the site with coordinate $\mathbf{r}$. The statistical phase ϑ in (8.4) is equal to $m/2\pi$ and the gradient on the lattice is given by

$$\Delta_i a_j(\mathbf{r}) \equiv a_j(\mathbf{r}+\mathbf{e}_i) - a_j(\mathbf{r}). \tag{8.5}$$

The commutation relations that are satisfied by $a_j(\mathbf{r})$ follow from the canonical quantization rules for our system with the Lagrangian

$$L = \sum_{\mathbf{r}} c^+(i\partial_0 + a_0)c - H_f - \frac{\vartheta}{2}\sum_{\mathbf{r}} \varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{8.6}$$

The momentum conjugate of $a_j(\mathbf{r}, t)$ for (8.6) is

$$\Pi_j(\mathbf{r}, t) \equiv \frac{\delta L}{\delta a_j(\mathbf{r}, t)} = \vartheta\varepsilon_{jk}a_k(\mathbf{r}, t). \tag{8.7}$$

It therefore follows from the standard relation

$$[a_j(\mathbf{r}, t), \Pi_k(\mathbf{r}', t)] = i\delta_{jk}\delta(\mathbf{r}-\mathbf{r}') \tag{8.8}$$

that the single-time commutator has the form

$$[a_j(\mathbf{r}), a_k(\mathbf{r}')] = \frac{i}{\vartheta}\varepsilon_{jk}\delta(\mathbf{r}-\mathbf{r}'). \quad (8.9)$$

In the time-like Weyl gauge $a_0 = 0$, the classical equation of motion $\delta L/\delta a_0 = 0$ is the Coulomb constraint (8.4) that defines the Hilbert space of physical states as follows. The generator of time-independent gauge transformations

$$Q = \frac{\delta L}{\delta a_0(\mathbf{r})} = j_0(\mathbf{r}) - \vartheta\varepsilon_{ij}\Delta_i a_j(\mathbf{r}) \quad (8.10)$$

commutes with the Hamiltonian (8.3) and annihilates physical states.

It follows from (8.9) that the momentum conjugate of a_1 is ϑa_2. Their commutator is analogous to the commutation relations between the x and y components of the velocity of a two-dimensional charged particle placed in a perpendicular magnetic field. A detailed account of Chern–Simons topological quantum mechanics based on this comparison is given in Ref. 150.

Let us now consider some of the details of the transition from (8.3) to (8.1) on a lattice. This will be useful not only for methodological but for other purposes as well. We start by finding the solutions of (8.4) by substituting the vector potential $a_j(\mathbf{r})$ in the form

$$a_j(\mathbf{r}) = \varepsilon_{jk}\Delta_k\Phi(\mathbf{x}). \quad (8.11)$$

We then have

$$j_0(\mathbf{r}) = \vartheta\Delta^2\Phi(\mathbf{x}), \quad (8.12)$$

where Δ^2 is the Laplace operator on the lattice. The solution of this equation

$$\Phi(\mathbf{x}) = \frac{1}{\vartheta}\sum_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}')j_0(\mathbf{r}') \quad (8.13)$$

enables us to determine the vector potential

$$a_j(\mathbf{r}) = \frac{1}{\vartheta}\varepsilon_{jk}\Delta_k\sum_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}')j_0(\mathbf{r}'), \quad (8.14)$$

where $\mathbf{x}'$, $\mathbf{r}'$ are dual pairs and G is the Green's function on the lattice.

In terms of the multivalued dual Green's function $\theta(\mathbf{r},\mathbf{x}')$ that satisfies the Cauchy–Riemann equation

$$\Delta_i G(\mathbf{x}, \mathbf{x}') = \varepsilon_{ij}\Delta_j\theta(\mathbf{r}, \mathbf{x}'), \quad (8.15)$$

the required potential

$$a_j(\mathbf{r}) = \Delta_j\phi(\mathbf{r}) \quad (8.16)$$

is equal to the gradient of the multivalued function

$$\phi(\mathbf{r}) = \frac{1}{\vartheta}\sum_{\mathbf{r}'}\theta(\mathbf{r}, \mathbf{x}')j_0(\mathbf{r}') \quad (8.17)$$

[a discussion of (8.17) and of related matters is given in Refs. 151–153].

The function θ satisfies the condition $\Delta\theta = +1$ for a closed contour on the main lattice around the dual site $\mathbf{x}$ which together with (8.17) defines it completely. The multivaluedness of the function $\theta(\mathbf{r},\mathbf{x}')$ is a consequence of the string-section running from the point $\mathbf{x}$ to infinity, and when the function crosses it, it acquires a constant jump. The form of the operator ϕ depends on the gauge[154] and coincides with the Dirac string in the gauge $a_1 = 0$. All these properties are not, of course, an artifact of the theory, but a reflection of the fact that the two-dimensional system is multiply connected.

We shall now determine the operator $\exp(i\phi(\mathbf{r}))$ that creates the coherent state of the gauge field with the $1/\vartheta$ part of the flux quantum attached to each fermion at the point $\mathbf{r}$. The transformation

$$c(\mathbf{r}) \to \bar{c}(\mathbf{r}) = e^{i\phi(\mathbf{r})}c(\mathbf{r}), \quad (8.18)$$

$$c^+(\mathbf{r}) \to \bar{c}^+(\mathbf{r}) = c^+(\mathbf{r})e^{-i\phi(\mathbf{r})}, \quad (8.19)$$

which can be interpreted as the Jordan–Wigner transformation,[155] then leads to operators that satisfy the anyon commutation relations (8.2) with phase δ given by[147–149]

$$\delta = \frac{1}{\vartheta}(\theta(\mathbf{r}, \mathbf{x}') - \theta(\mathbf{r}', \mathbf{x})) = \frac{1}{2\vartheta}. \quad (8.20)$$

The Pauli principle for fermions $(c^+)^2 = c^2 = 0$ then leads to the hard-core condition $\bar{c}^2(\mathbf{r}) = (\bar{c}^+(\mathbf{r}))^2 = 0$ for the anyon operators.

When the ratio $\delta\pi$ is an odd number, the anyons become bosons with a hard core, and the standard Jordan–Wigner transformation

$$\begin{aligned} c_{\mathbf{r}} &\to \bar{c}_{\mathbf{r}} = Q_{\mathbf{r}}c_{\mathbf{r}} = a^+_{\mathbf{r}\downarrow}a_{\mathbf{r}\uparrow} \equiv S^-_{\mathbf{r}}, \\ c^+_{\mathbf{r}} &\to \bar{c}^+_{\mathbf{r}} = c^+_{\mathbf{r}}Q_{\mathbf{r}} = a^+_{\mathbf{r}\uparrow}a_{\mathbf{r}\downarrow} \equiv S^+_{\mathbf{r}}, \end{aligned} \quad (8.21)$$

where

$$Q_i = \exp(i\pi\sum_{j<i} c^+_j c_j), \quad (8.22)$$

gives the Hamiltonian (8.1) that is identical with the Hamiltonian of the XY model of spin half. The relation given by (8.4) then replaces the expression $S_z(\mathbf{r}) = j_0(\mathbf{r}) - 1/2$. Fermions with the Chern–Simons term for δ/π equal to a multiple of an odd number are equivalent to the XY model of spin 1/2 or bosons with a hard core.

We now draw attention to the fact that the nonlocal operator in (8.18) can also be written for arbitrary δ in the form[32,156]

$$Q_k = \exp\left[i\frac{\delta}{\pi}\sum_{j<k}\operatorname{Im}\ln(z_k - z_j)c^+_j c_j\right], \quad (8.23)$$

where $z_k = x_k + iy_k$ is the complex coordinate of a site. This generates a string-section extending behind each particle, and has the significance of the disorder operator.[157–159]

In addition to the spatial distribution of the particle number density and the gauge flux over cells containing the sites of the main and dual lattices, there are also different possible variants of the ratio of quantum numbers that determine the degree of filling of a site and the magnitude of the flux. We shall now examine this in greater detail than was done in Secs. 4 and 6.

The Feynman path integral is

$$Z = \int D\bar{\psi}D\psi Da\,\exp(i\int L\,dt). \quad (8.24)$$

We shall assume that this takes into account the chemical potential and the external electromagnetic field potential A_μ:

$$i\partial_0 + a_0 \to D_0 = i\partial_0 + (a_0 + A_0 + \mu), \quad (8.25)$$

$$e^{ia_j(\mathbf{r},t)} \to e^{i[a_j(\mathbf{r},t)+A_j(\mathbf{r},t)]} \tag{8.26}$$

As usual, the chemical potential shifts the scalar potential A_0.

The Feynman integral (8.24) contains the squares of the Fermi fields. Integration of (8.24) with respect to these fields therefore yields

$$\det(D_0 - H(a+A)) = \int D\bar{\psi}D\psi \exp\{i\int dt \sum_{\mathbf{r},\mathbf{r}'} \bar{\psi}(\mathbf{r},t)[D_0\delta_{\mathbf{r},\mathbf{r}'} - H(a+A)]\psi(\mathbf{r}',t)\}, \tag{8.27}$$

where

$$H = -t\sum_{j=1,2} e^{i(a_j+A_j)}\delta_{\mathbf{r}',\mathbf{r}+\mathbf{e}_j}. \tag{8.28}$$

The result of this is that the effective action for gauge fields takes the form

$$S_{\text{eff}} = -i\,\mathrm{tr}\ln(D_0 - H(a+A)) - iS(a), \tag{8.29}$$

where $S(a)$ is the Chern–Simons contribution due to the last term in (8.6). After the shift of the vector potential $a + A \to a$, under which the measure in (8.24) remains invariant, and after the extraction of the chemical potential from D_0, we obtain

$$S_{\text{eff}} = -i\,\mathrm{tr}\ln(D_0 + \mu - H(a)) - iS(a-A), \tag{8.30}$$

where

$$S(a-A) = S(a) + S(A) - \varepsilon^{\mu\nu\lambda}(a_\mu\partial_\nu A_\lambda + A_\mu\partial_\nu a_\lambda). \tag{8.31}$$

Let us now pause to consider the form of this expression. Its structure is significant for a number of reasons, the most important of which is that there are several published ways[160–162] of including the external electromagnetic field in the Chern–Simons term. In general, the external field A_μ is small in comparison with the static field a_μ, and may be looked upon as a perturbation. A uniform external magnetic field is an exception, so that the correct description[104] of the Meissner effect requires that the two potentials in the sum $\mathbf{a} + \mathbf{A}$ are of equal importance. It is only then that the fluctuation corrections may cancel the bare statistical Chern–Simons term $S(A)$.

We now postpone the case $A_\mu \neq 0$ to Sec. 13 and consider that, at the point of stationary phase $\delta S_{\text{eff}}/\delta a_\mu|_{a_\mu=\bar{a}_\mu} = 0$, the static electric field $\mathbf{e}$ is zero and the static magnetic field $\mathbf{b}$ is uniform. Arguments in favor of the latter were produced in Sec. 5, and the case $\mathbf{e}\neq 0$ is discussed in Refs. 163 and 164. The equation $\delta S_{\text{eff}}/\delta a = 0$ signifies that the fermion current is

$$j^\mu = \vartheta\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda, \tag{8.32}$$

and that the zero-order component gives the fermion density

$$\rho = \vartheta b. \tag{8.33}$$

The number of anyons in the system is

$$N_a = -\frac{i}{Z}\frac{\partial Z}{\partial\mu}. \tag{8.34}$$

Since μ is the shift of the scalar potential a_0, we have

$$N_a = \vartheta\Phi, \tag{8.35}$$

where Φ is the total current through an area with linear dimension L, measured in units of the lattice constant, i.e.,

$$\Phi = bL^2. \tag{8.36}$$

The anyon density is therefore

$$N_a/L^2 = \vartheta b \tag{8.37}$$

and is identical with the fermion density (8.33). It determines the mean statistical magnetic field.[70,71]

Let the density ρ be the ratio of two mutually primitive numbers r and q:

$$\rho = r/q, \tag{8.38}$$

and let the phase δ and the statistical angle ϑ be given by

$$\delta = \pi\frac{n}{m}, \quad \vartheta = \frac{1}{2\delta} = \frac{m}{2\pi n} \tag{8.39}$$

where n and m are mutual primitives.

The mean statistical field b in (8.36)

$$b = 2\pi P/Q \tag{8.40}$$

is the fractional part $[\Phi/L^2]$ of the flux quantum equal to 2π ($\hbar = 1$) Equation (8.37) therefore relates the mutually primitive numbers in the pairs P, Q, r, q, and m,m:

$$2\pi\frac{P}{Q} = \frac{\rho}{\vartheta} = 2\pi\frac{nr}{mq}. \tag{8.41}$$

We recall that the spectrum of single-particle states (see Sec. 6) for magnetic-field values given by (8.40) contains Q subbands where the number of states in each subband is L^2/Q. If f is the fractional or integral part of the filled subbands, we have $f = N_a/(L^2/Q)$ the anyon density is $N_a/L^2 = f/Q$. Taken together with (8.37), this signifies that

$$f = (r/q)Q, \tag{8.42}$$

i.e., f is a whole number if q is a factor of Q.

Let (a,b) be the greatest common factor of the two integers a and b. Let also s and l be two integers satisfying the condition

$$s = (n, q), \quad l = (m, r). \tag{8.43}$$

There exist four numbers $\bar{n},\bar{m},\bar{r},\bar{q}$ such that

$$n = s\bar{n}, \quad q = s\bar{q}, \tag{8.44}$$

$$m = l\bar{m}, \quad r = l\bar{r} \tag{8.45}$$

where

$$(\bar{n},\bar{q}) = 1, \quad (\bar{m},\bar{r}) = 1, \quad (\bar{n},\bar{m}) = 1, \quad (\bar{q},\bar{r}) = 1, \tag{8.46}$$

as a consequence of the mutually primitive numbers in the fractions (8.38) and (8.39).

Equation (8.41) is therefore equivalent to

$$\frac{P}{Q} = \frac{\bar{n}\,\bar{r}}{\bar{m}\,\bar{q}} \tag{8.47}$$

or

$$P = \bar{n}\,\bar{r}, \quad Q = \bar{m}\,\bar{q}, \tag{8.48}$$

and the degree of filling of the subbands is

$$f = \frac{r}{q}Q = \frac{l}{s}\bar{r}\bar{m} = \frac{rm}{(n,q)(m,r)}. \qquad (8.49)$$

Since $s = (n,q)$ does not have a common factor with $l, \bar{r}$, and $\bar{n}$, it follows that f is an irreducible fraction, except when one of the following relations is valid:

$$(n,q) = 1 \quad \left(f = \frac{rm}{(m,r)}\right), \qquad (8.50)$$

$$(n,q) = \frac{m}{(m,r)} \quad (f = r),$$

$$(n,q) = \frac{r}{(m,r)} \quad (f = m).$$

If we consider the stability of a state with respect to fluctuations around the mean field, we find that the preferred situation is that with gaps in the excitation spectrum, which corresponds to integral values of f and, hence, to the implementation of one of the relations in (8.50). The physical properties of the system, e.g., the excitation statistics, will depend on which of the conditions in (8.50) is satisfied. In other words, the properties of the system depend not only on site filling, but also on the commensurability conditions.

The case examined in previous Sections corresponds to $n = 1$ in which case $\vartheta = m/2\pi$. For this case, $s = (n,q) = 1$ and $f = mr/(m,r)$ is the integral number of filled Landau levels in a system with an arbitrary density $\rho = r/q$ and statistical phase $\delta = \pi/m$.

The half-filled site with $\rho = 1/2$ ($r = 1, q = 2$) and $\vartheta = m/(2\pi n)$ with odd n is an exception. It then follows from (8.44)–(8.50) that $f = m$, $P = m$, and $Q = 2m$. This means that the chemical potential at $\varepsilon = 0$ and all states with $\varepsilon > 0$ are filled whereas, in the Brillouin zone, the branches of the energy spectrum for $\varepsilon = 0$ intersect at $Q = 2m = 2\bar{k}$, $\bar{k} = 1,2,\ldots$ points. The standard phase with flux[9,12] would then correspond to $m = 1$ and $\rho = 1/2$, in which case $b = \pi$, which corresponds to half a flux quantum per unit cell.

Quadratic fluctuations around the mean magnetic field in (8.37) provide the following contribution to the action:

$$S = \sum_{x,x'} a_\mu(x)\Pi_{\mu\nu}(x,x')a_\nu(x') - \frac{\vartheta}{2}\sum_x \varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda, \qquad (8.51)$$

where $a_\mu(x)$ is the deviation from the mean potential, $x = (\mathbf{r},t)$, and S contains the sum of the contributions due to the fermion part and to the Chern–Simons term. The first term contains the polarization operator $\Pi_{\mu\nu}(x,x')$ for fermions on the lattice in the presence of the magnetic field b. When the spectrum contains a gap and we perform the gradient expansion of $\Pi_{\mu\nu}$ we can write (8.51) in the following form in the low-energy long-wavelength limit:

$$S = \int d^3x\left[\frac{\varepsilon}{2}e^2 - \frac{\chi}{2}h^2 + \frac{1}{4}(\sigma_{xy} - \vartheta)\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\right] + \ldots \,. \qquad (8.52)$$

If the chemical potential lies in the fth Hofstadter gap, the Hall conductivity σ_{xy} is equal to $t_f/2\pi (\hbar = 1)$, as already noted in Sec. 6, where t_f is the solution of the Diophantine equation $f = Qs_f + Pt_f$. If we now combine (8.37)–(8.42), we obtain the Hall conductivity in the form

$$\sigma_{xy} = \vartheta(1 - s_f\rho^{-1}). \qquad (8.53)$$

It follows from (8.52) and (8.53) that the precise cancellation of the Chern–Simons term in the Gaussian approximation is obtained for solutions of the Diophantine equation with $s_f = 0$. This solution plays an important part because it constitutes a compressible state (see Sec. 5) for which there is a Goldstone collective mode.

Let us first consider the case $n = 1$, $\vartheta = m/2\pi$, and arbitrary density $\rho = r/q$. We then have $P = r/(m,r)$, $Q = mq/(m,r)$ and the number of filled subbands is $f = mr/(m,r)$. The Diophantine equation is then

$$s_f = \begin{cases} 0, & t_f = m, \quad \text{if} \quad |m| < \dfrac{mq}{2(m,r)}, \\ \dfrac{r}{(m,r)}, & t_f = m - \dfrac{mq}{(m,r)}, \quad \text{if} \quad |m| > \dfrac{mq}{2(m,r)}. \end{cases} \qquad (8.54)$$

The degenerate solution and the multivaluedness of t_f arise for $|m| = mq/2(m,r)$, i.e., for even values of $q = 2(m,r)$, which includes the case of half-filling $\rho = 1/2$. The interaction with next-nearest neighbors lifts this degeneracy (see Sec. 6).

Thus, in the absence of degeneracy, and if $q > 2(mr)$, the solution of the Diophantine equation is single-valued and we have $s_f = 0$, $t_f = m$, $\sigma_{xy} = m/2\pi$, which is equal to ϑ! In the opposite case, when $q < 2(m,r)$, the solution is $s_f = r/(m,r) \neq 0$. For this degree of site filling, the Chern-Simons term does not cancel out, there is no Goldstone mode, and the superfluid state is impossible. For $s_f \neq 0$, gauge bosons have a topological mass due to the Chern–Simons term, and the spectrum of collective excitations in the anyon gas has a finite gap in the long-wave limit. In brief, this perfectly incompressible coherent state is analogous to the quantum Hall effect state. For a sequence with $n \neq 1$, it is also impossible to find a solution of the Diophantine equation with $s_f = 0$, and superfluidity is impossible in this series. Such coherent nonsuperfluid states cannot, of course, be found in the continuous limit because they are the result of diffraction by the lattice. There is an extensive current literature[100,102,104,165,166] on fluctuational contributions to the parameter $k \equiv m$ of the statistical phase $\vartheta = k/2\pi$ for different densities and temperatures, including the possibility that it will be annulled and there will be a transition to the superfluid state.

We thus see that the superfluid state of charged anyon excitations is only one of the coherent states of two-dimensional systems with P-parity violation and symmetry breaking under time reversal. The place of the superfluid phase among the many other phases will be examined in Sec. 11.

The distribution of the mutually conjugate degrees of freedom, i.e., the phase φ and the particle number, over the sites of the main and dual lattices, occupies an important place throughout the above discussion. It follows from the commutation relations for the corresponding operators, $[\varphi, N] = i$, that phase and particle-number fluctuations are comparable. If we take the phase and the number of particles as our variables, the "crude" approach (2.15) is formulated in Ref. 167 in terms of the Hamiltonian $H = H_t^{(1)} + V_u^{(1)}$ where

$$H_t^{(1)} = -t^{(1)}\sum_{\mathbf{r},j}\cos(\Delta_j\varphi_{\mathbf{r},j}^{(1)} - a_{\mathbf{r},j}^{(1)}), \qquad (8.55)$$

$$V_u^{(1)} = \frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'}(N_{\mathbf{r}}^{(1)} - \rho^{(1)})U_{\mathbf{r}\mathbf{r}'}^{(1)}(N_{\mathbf{r}'}^{(1)} - \rho^{(1)}) \qquad (8.56)$$

with coupling defined by (8.4)

$$b_{\mathbf{x}} \equiv \varepsilon_{ij}\Delta_i a_{jr} = \frac{\alpha_s}{4}\sum_{\mathbf{r}} N_{\mathbf{r}}^{(1)} \tag{8.57}$$

where the phase factor $\exp(i\pi\alpha_s)$ is parametrized by the angle α_s, $\rho^{(1)}$ is the operator for the density of the compensating background per site, and $U^{(1)}$ is the repulsive potential between the bosons. The sum is evaluated over all the sites of the square cell centered on the site of the dual lattice.

We shall now use the results of Ref. 167 to show how (8.55)–(8.57) lead to a hierarchical sequences of states. The boson Hamiltonian (8.55), (8.56) is isomorphous[168] with the dual $(2+1)$-dimensional scalar quantum electrodynamics in which the vector potential $\mathbf{A}^{(1)}$ is the variable. After the dual transformation, the boson number operator $N^{(1)}$ becomes $\nabla\times\mathbf{A}^{(1)}$ with $\mathbf{A}^{(1)}$ lying on the link of the dual lattice, and the Hamiltonian (8.55), (8.56) becomes

$$H^{(2)} = H_s^{(2)} + H_t^{(2)} + V_u^{(2)}$$

where

$$H_s^{(2)} = \frac{U^{(2)}}{2}\sum_{\mathbf{x}} \pi_{\mathbf{x}}^2(1) + V_u^{(1)}, \quad N_1 = [\nabla\times\mathbf{A}^{(1)}]_z, \tag{8.58}$$

$$H_t^{(2)} = -t^{(2)}\sum_{\mathbf{x},j}\cos(\Delta_j\varphi_{\mathbf{x}}^{(2)} - A_{\mathbf{x},j}^{(1)}), \tag{8.59}$$

$$V_u^{(2)} = \frac{U^{(2)}}{2}\sum_{\mathbf{x},\mathbf{x}'}(N_{\mathbf{x}}^{(2)} - \rho_{\mathbf{x}}^{(2)})G(\mathbf{x}-\mathbf{x}')(N_{\mathbf{x}}^{(2)} - \rho_{\mathbf{x}}^{(2)}), \tag{8.60}$$

the momentum $\pi_{\mathbf{x}}^{(1)}$ is the canonical conjugate of $\mathbf{A}^{(1)}$, $\nabla\cdot\mathbf{A}^{(1)} = 0$, and the Green's function (8.13) is $G(\mathbf{x})\sim\ln|\mathbf{x}|$ for large $|\mathbf{x}|$. The reduced compensating density is

$$\rho_{\mathbf{x}}^{(2)} = -\frac{\alpha_s}{4}\sum_{\mathbf{r}}\varepsilon_{ij}\Delta_i A_{jr}^{(1)}. \tag{8.61}$$

The operator $N^{(2)}$ in (8.60) now represents the vortex of the initial boson field φ in (8.55).

In the second iteration of the model (8.55)–(8.57), the effective Hamiltonian $H^{(3)} = H_s^{(2)} + H_s^{(3)} + H_t^{(3)} + V_u^{(3)}$ is modified as compared to (8.58)–(8.61). The vortex number operator is replaced in accordance with $N^{(2)}\to\mathrm{curl}\,\mathbf{A}^{(2)}$, the sites are interchanged so that $\mathbf{r}\leftrightarrow\mathbf{x}$, and the statistical angle changes so that $\alpha_s^{-1}\to\alpha_s^{-1}+p_1$. The even number p_1 of flux quanta is attached to the vortices $N^{(2)}$ in (8.58)–(8.61) in order to retain the bosonic character of $N^{(2)}$.

The nth step Hamiltonian $H^{(n)} = H_s^{(2)} + \ldots + H_s^{(n)} + H_t^{(n)} + V_u^{(n)}$, depends on the operators $N^{(n)}, \varphi^{(n)}$ and fields $\mathbf{A}^{(i)},\ldots\mathbf{A}^{(n-1)}$. The disorder operators $\exp(i\varphi^{(n)})$ are localized on the sites of the main or dual lattices, and the gauge field $\mathbf{A}^{(n)}$ lies on the links of the dual or main lattices for odd and even n, respectively.

The mean-background neutralization condition for the particles $N^{(3)}$ leads to the second iteration to $\langle\rho^3\rangle = \langle\nabla\times\mathbf{A}^{(1)}\rangle - P_1\langle\nabla\times\mathbf{A}^{(2)}\rangle = 0$. Since the term $H_s^{(3)}$ demands that $\langle\nabla\times\mathbf{A}^{(2)}\rangle = \alpha_s\langle\nabla\times\mathbf{A}^{(1)}\rangle$, it follows that $\alpha_s = 1/p_1$. When this condition is not satisfied, the representation procedure must be continued until the nth iteration produces a background-neutralizing density for particles $N^{(n+1)}$ such that $\langle\rho^{(n+1)}\rangle = 0$. We then have

$$\alpha_s = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cfrac{1}{p_3 + \ldots + 1/p_n}}} \equiv \frac{P}{Q} \tag{8.62}$$

where $p_i = 0, \pm 2, \pm 4, \ldots, i = 1,\ldots,n$.

For example, in a state in which $\{N_r^{(3)}\} = 0$ and $t^{(3)} = 0$, the spectrum of the effective Hamiltonian $\widetilde{H}(\{N^{(3)}\} = 0)$ that is quadratic in $\mathbf{A}^{(1)},\mathbf{A}^{(2)}$ contains the gapless mode (5.12), i.e., an indicator of the phenomenon of superfluidity. The pole of the conductivity

$$\sigma_{xx}(\omega) = \frac{2\pi}{i\omega}\frac{U^{(2)}U^{(3)}}{\alpha_s^2U^{(2)} + U^{(3)}} \tag{8.63}$$

that appears for $\omega\to 0$ shows that the superfluid liquid has nonzero density, and the finite Hall conductivity $\sigma_{xy}(\omega=0) = \alpha_s^3(U^{(2)})^2/(\alpha_s^2U^{(2)} + U^{(3)})^2$ taken together with (8.63) gives zero Hall resistance for $\omega\to 0$.

9. THE FERMION APPROACH

The fermion representation is very convenient for anyons because, as $k\to\infty$, we have the small parameter k^{-1} that can be monitored during the calculations, and the statistical phase $\vartheta = \pi(1-k^{-1})$ is close to the usual fermion value π. When $k\gg 1$, we are, of course, far from the ground state $k=2$ with its strong correlations. States with high k are more akin to metastable states, usually nonequilibrium states, characterized by long-term relaxation because of the fractal anyon energy, cut by valleys, and the fact that excited anyons must cross states separated by energy barriers (see Fig. 7). Nevertheless, even for large values of k, the anyons retain their main property, i.e., their number-theoretical ultrametricity. We shall now follow the results of Refs. 97 and 169 to consider some of the properties of the fermion approach.

In the second-quantization representation, the Hamiltonian corresponding to the fermion picture (3.4), (3.6) takes the form

$$H = \int d^2r\psi^+(\mathbf{r})\cdot\frac{1}{2m}|(\mathbf{p} + \mathbf{a}(\mathbf{r}))|^2\psi(\mathbf{r}), \tag{9.1}$$

where ψ is the field of spinless fermions. We now replace the exact expression for the vector potential

$$\mathbf{a}(\mathbf{r}) = \frac{1}{4}\int d^2r'\,\frac{[\mathbf{e}_z(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^2}\psi^+(\mathbf{r}')\psi(\mathbf{r}'), \tag{9.2}$$

by introducing the mean density ρ and magnetic field b, and obtain the approximate expression

$$\mathbf{a}(\mathbf{r}) = \bar{\mathbf{a}} + \frac{1}{k}\int d^2r'\,\frac{[\mathbf{e}_z(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^2}(\psi^+\psi - \rho), \tag{9.3}$$

where

$$\bar{\mathbf{a}} = \frac{1}{2}b[\mathbf{e}_z\mathbf{r}], \quad b = \frac{2\pi}{k}\rho. \tag{9.4}$$

After substitution in (9.1), we expand the Hamiltonian into a series in powers of $1/k$:

$$H = H_0(\bar{\mathbf{a}}) + \frac{1}{k}H_1(\bar{\mathbf{a}}) + \frac{1}{k^2}H_2(\bar{\mathbf{a}}) + \ldots\ . \tag{9.5}$$

The perturbation theory represented by (9.5) enables us to calculate the reaction to the external electromagnetic field A_μ:

$$J_\mu(\mathbf{q},\omega) = -K_{\mu\nu}(\mathbf{q},\omega)A_\nu(\mathbf{q},\omega). \tag{9.6}$$

It has been shown[33,97,101,169] that, in the long-wave limit, the function $K_{\mu\nu}(\mathbf{q},\omega)$ has a pole at

$$\omega = v_0 q, \quad v_0 = (2\pi\rho)^{1/2}/m. \tag{9.7}$$

The existence of the Goldstone mode (9.7) signifies that the system is translationally invariant.

Let us examine this in greater detail. The Goldstone mode usually appears after a phase transition and is due to continuous-symmetry breaking. However, in the two-dimensional case, the Elitzur theorem[170] states that the usual local order parameter does not exist in lattice systems with continuous symmetry. The assumption made in Ref. 169 is therefore that (9.7) must be looked upon as the result of the restoration of translational invariance that was lost for commuting Hamiltonian and translational generators, i.e., the restoration of the commutator $[P_x, P_y]$ subject to $[P_i, H] = 0$. The translation generators do not commute in a uniform magnetic field with the Hamiltonian, and the magnetic translation generators (6.34) commute with the Hamiltonian, but not with one another. This is why, at the "macroscopic" quasiparticle level, it may be considered that in the initial state for the phenomenon (9.7), the commutator of the translation operators P_i is not zero, i.e.,

$$[P_k, P_j] = ib\epsilon_{kj} Q \tag{9.8}$$

and is determined by the number Q of particles and the statistical magnetic field $b = \varepsilon_{ij}\partial_i a_j$. The latter plays the part of the order parameter. In effect, this is the same order parameter as (4.11). The tensor ε_{kj} and i in (9.8) represent, respectively, the breaking of the two-dimensional parity P and time reversal T, and their product conserves the PT-invariance of the anyon system. We must now consider the cancellation of Q in (9.8) in the superfluid state.[25)]

First, for the sake of simplicity, consider a single particle in a given constant magnetic field, so that the Hamiltonian is

$$H = (-1/2m) D^2, \tag{9.9}$$

and the covariant derivatives satisfy the commutation relation

$$[D_i, D_j] = i\epsilon_{ij} b. \tag{9.10}$$

Since $[D_i, H] \neq 0$, the translation operators are different from D_i, and are given by

$$P_i = -iD_i + b\epsilon_{ij} x^j. \tag{9.11}$$

We then have

$$[P_i, H] = 0, \quad [P_k, P_l] = ib\epsilon_{kl}. \tag{9.12}$$

In terms of the second-quantized quasiparticle field χ, the Hamiltonian

$$H = \frac{1}{2m}\int d^2x |D_k \chi|^2 \tag{9.13}$$

and the corresponding translation operator

$$P_k^{(\chi)} = \int d^2x [\chi^*(-iD_k)\chi + b\epsilon_{kj} x^j \chi^* \chi] \tag{9.14}$$

commute with one another, but

$$[P_i^{(\chi)}, P_k^{(\chi)}] = ib\epsilon_{ik} Q, \tag{9.15}$$

where

$$Q = \int d^2x \chi^* \chi \tag{9.16}$$

is the charge operator.

We now introduce the massless boson in addition to the set of quasiparticles χ which, for high values of k, are close to fermions and occupy k Landau levels. This ensures that the 'macroscopic' equation $[P_i, P_j] = 0$ is satisfied. We shall represent the boson by the scalar field ϕ with the Lagrangian

$$L = \frac{1}{2}\int d^2x [(\partial_0 \phi)^2 - v^2(\partial_i \phi)^2]. \tag{9.17}$$

Next, by analogy with (5.16), we rewrite the Lagrangian for the field ϕ in terms of the gauge field a_i of strength $h_{ij} = \partial_i a_j - \partial_j a_i$. If we then change the variables so that $\partial_0 \phi = h_{12}$, $v^2 \partial_i \phi = \varepsilon_{ij} h_{0j}$, we obtain

$$L = \frac{1}{2}\int d^3x (h_{0i}^2 - v^2 h_{12}^2). \tag{9.18}$$

The translation operator for (9.18) has the form

$$P_i^{(a)} = \int d^2x T_{0i}^{(a)}, \tag{9.19}$$

where the momentum density is

$$T_{0i}^{(a)} = -h_{0j} h_{ij} \tag{9.20}$$

and, of course, $[P_i, P_j] = 0$. However, after the operation defined by (9.11),

$$T_{0i}^{(a)} = -h_{0j} h_{ij} + \varepsilon_{ij} b h_{0j}, \tag{9.21}$$

$$[P_i^{(a)}, P_j^{(a)}] = ib\epsilon_{ij} \oint dl n^k h_{0k}. \tag{9.22}$$

The integral in (9.22) is evaluated over a contour at infinity and n^k is the unit normal to the contour.

Combining the two systems, we obtain the total momentum operator $P_i = P_i^{(\chi)} + P_i^{(a)}$ and the commutator

$$[P_i, P_j] = ib\epsilon_{ij}(Q - \oint dl n^k h_{0k}). \tag{9.23}$$

Thus, if we limit the space of states by the condition

$$Q = \oint dl n^k h_{0k}, \tag{9.24}$$

we obtain $[P_i, P_j] = 0$. In other words, the translation operators will commute only when the surface terms are taken into account.

We would not have obtained (9.23) without introducing the additions in (9.14) and (9.21). This means that, to obtain (9.23), we must modify the original Lagrangians so as to produce a dynamic gauge field a_μ. The required Lagrangian is

$$L = \int d^2x \left[\frac{1}{2}(h_{0i}^2 - v^2 h_{12}^2) + i\chi^* D_0 \chi - \frac{1}{2m}|D_k \chi|^2\right], \tag{9.25}$$

where $D_\mu = \partial_\mu + ia_\mu$. The role of the field a_μ (or ϕ) is to restore the commutation of the translation operators. The fact that the right-hand side of the commutator (9.23) is zero, ensures that the Cherns–Simons term is absent from (9.25). In terms of the field ϕ, the expression given by (9.21) is equivalent to

$$T'_{0i} = -\partial_0 \phi \partial_i \phi + b\partial_i \phi, \tag{9.26}$$

and the surface term

$$\int d l n^k h_{0k} = v^2 \oint d l^i \partial_i \phi \tag{9.27}$$

is equal to the number of vortices Φ. Hence, the cancellation condition

$$Q = \Phi \tag{9.28}$$

signifies that the vortical excitations are charged. In other words, vorticity and charge cannot be separated and must be assigned to the same particles.

The following observations will conclude this Section. It follows from (9.27) that, in any state with $\Phi \neq 0$, the scalar field must be singular at large distances. This can occur when ϕ is the argument of a complex scalar field with zeros, for example, on the boundary of an inhomogeneity in the distribution of quantum numbers. To put it simply, edge states and nontrivial boundary conditions are important for our purposes.

Since the number of vortices is conserved in two-dimensional space, the electric charge is also conserved in the superfluid state as a consequence of (9.28). We therefore conclude that the usual order parameter (such as is encountered, for example, in three-dimensional BCS superconductors, in which the charge conservation is spontaneously broken) does not exist in our case.

10. CLASSIFICATION OF PHASE STATES

The order parameter can be used to classify the possible phase states of a system with developed quantum fluctuations. A broader and more substantive choice of this parameter as compared with (9.8), which would reflect the changes occuring in the system, is therefore exceedingly important. We are aided in the solution of this problem by recalling a similar situation in the case of the fractional quantum Hall effect.[171,172] For the state described by the wave function (7.9) with $\vartheta = \pi/k$, this analogy has been used[137,173,174] to show that the k-particle density matrix decreases with the distance $|\xi - \eta|$ as follows:

$$\rho(\{\xi_k\}, \{\eta_k\}) \sim |\eta - \xi|^{-k/2}. \tag{10.1}$$

Similar suggestions about the nonlocal order parameter in anyon systems and a reduction concerning the power-law falling off of the correlation functions are reported also in Ref. 32.

The quasi-long-range order (10.1) is typical of conformal $(1+1)$D theories and we shall later discuss the possibility of a theory with topological $(2+1)$D action of the Chern–Simons type as a $(1+1)$D conformal theory. For the moment, we note that our problems are actually identical with the strong-coupling problems in quantum chromodynamics (possibly, with the exception of dimensionality). Actually, in the presence of doping we are dealing with SU(2)-symmetric gauge theory on a lattice. Doping reduces the local symmetry to U(1), and the physical observables of the gauge theory are gauge-invariant quantities. They include the Wilson loop operator and the t'Hooft disorder operator[175] whose expectation values can be used to characterize different phase states. Of course, the theory always contains the gauge-invariant fermion propagator

$$G(x, x') = \langle c^+(x') \exp(i \int_x^{x'} \mathbf{a} \cdot d\mathbf{l}) c(x) \rangle. \tag{10.2}$$

Let us now consider the expectation value of the Wilson loop operator for the statistical vector potential, ordered along a closed contour C:

$$\langle W(C) \rangle = \langle \mathrm{tr}\, P \exp(i \oint_C \mathbf{a}\, d\mathbf{l}) \rangle. \tag{10.3}$$

The expectation value in this expression must be interpreted in the sense of Feynman integration and the trace tr takes into account the case of the SU(2) group. The other significant feature is the P-ordering along the contour C, since the component a_2 is canonically conjugate to a_1 [See (8.9)].

If the contour in (10.3) is space-like, the Wilson average for an arbitrarily closed loop on a quadratic lattice describes, by virtue of (8.4), the fluctuations in the number of anyons inside the contour C:

$$\langle W(C) \rangle = \langle \exp\left(\frac{i}{\vartheta} \int_\sigma j_0 d^2x \right) \rangle = \langle \exp\left(\frac{2\pi i n}{k} N_a(\sigma) \right) \rangle. \tag{10.4}$$

We note that (10.4) is a measure of the number of links (windings)[77,176] of the vortex around the contour C. When the loop C is time-like, as it is in QCD, the expectation value (10.3) determines the energy necessary to add a particle at one point and to remove a particle from another.

The disorder operator $V(C)$ (Refs. 76 and 75) can be defined by its action on the state vector $|a_\mu\rangle$, as follows:

$$V(C) | a_\mu \rangle = | a_\mu^{g(c)} \rangle. \tag{10.5}$$

The disorder operator takes $|a_\mu\rangle$ to a state determined relative to the gauge-transformed field

$$a_\mu^{g(c)} = g^{-1} a_\mu g + i g^{-1} \partial_\mu g \tag{10.6}$$

with a singular element on the curve C, which is such that another loop C' winds itself n times around C. If C' is parametrized by the variable $\varphi \in [0, 2\pi]$, we have $g^{(c)}(\varphi = 2\pi) = g^{(c)}(\varphi = 0) \exp(2\pi i n/N)$ where N is the rank of the group.[175,176] The disorder operator can be determined with the help of the commutation relations

$$W(C) V(C') = V(C') W(C) \exp(2\pi i n/N), \tag{10.7}$$

where C' winds itself n times around C

$$[W(C), W(C')] = 0, \quad [V(C), V(C')] = 0. \tag{10.8}$$

The expression given by (10.7) was used in Refs. 175–177 to show that a system with strong gauge fluctuations can be found only in one of the following states:

1. Higgs phase, characterized by the following behavior of expectation values:

$$\begin{aligned} \langle W(C) \rangle &\sim \exp(-\alpha L(C)), \\ \langle V(C) \rangle &\sim \exp(-\gamma S(C)), \end{aligned} \tag{10.9}$$

where $L(C)$ and $S(C)$ are, respectively, the perimeter of and the area bounded by C.

2. Confinement phase, characterized by the expectation values

$$\begin{aligned} \langle W(C) \rangle &\sim \exp(-\alpha S(C)), \\ \langle V(C) \rangle &\sim \exp(-\gamma L(C)). \end{aligned} \tag{10.10}$$

3. Partial Higgs phase, together with confinement, signifying that

$$\langle W(C)\rangle \sim \exp(-\alpha S(C)),$$
$$\langle V(C)\rangle \sim \exp(-\gamma S(C)). \qquad (10.11)$$

4. The state of massless particles

$$\langle W(C)\rangle \sim \exp(-\alpha L(C)),$$
$$\langle V(C)\rangle \sim \exp(-\gamma L(C)) \qquad (10.12)$$

that is in conflict with (10.7) and can be looked upon as intermediate between (10.9) and (10.10).

The confinement phase occurs, for example, in a system with a random Gaussian distribution of the gauge field flux[178]

$$P(\phi) = (\pi cS)^{-1/2}\exp(-\phi^2/cS) \qquad (10.13)$$

with mean $\langle\phi\rangle = 0$ and variance

$$\langle\phi^2\rangle_S = \sum_i \langle\phi^2\rangle_{S_i} = cS, \qquad (10.14)$$

that is proportional to the total area S. We then have

$$\langle W\rangle = \int d\phi e^{i\phi}P(\phi) \sim e^{-aS}, \qquad (10.15)$$

i.e., the system is in the confinement phase (10.10). We must now consider what happens in the distribution of dual degrees of freedom under a change in the coupling constant and a transition from confinement (10.10) to the Higgs phase (10.9).

11. HIERARCHY OF PHASE TRANSITIONS

The composition of anyon quantum numbers and the special features of their distribution on the respective planes impose a considerable limitation on the choice of the lattice model for the description of the phase properties of systems with developed quantum fluctuations. In particular, the requirement of gauge invariance forces us to consider the gauge invariant Wilson loop variables (2.12) and (10.3) and the correlation functions of these fields. The model defined by (8.55)–(8.57), even if we ignore the absence of sources of spin-wave excitations and the incomplete allowance for symmetry, is not entirely satisfactory for the further reason that it is dominated by the nonlinear function $\cos(\Delta_i\phi_{r,j} - a_{r,j})$ that is unimportant at low energies. The property of compactness, which is important for charge quantization,[26)] can be expressed in the simpler form of a periodically continued quadratic function of the same variables. When we pass to the periodically continued Gaussian model, we shall consider a more extended symmetry as compared with (8.55)–(8.57), and will fully take into account the topological limitations. Apart from convenience, the Gaussian model arises from the common desire to formulate a theory of strongly-coupled particles in terms of noninteracting elementary excitations, having taken on board the information about the maximum possible number of channels for the variation of quantum numbers.

Before we turn to the description of the model, we make one further observation with regard to dimensional reduction. In real $(3+1)$D space-time, the leading low-energy term in gauge-theory action is the topological charge density $f_{\mu\nu}\tilde{f}^{\mu\nu} = \varepsilon^{\mu\nu\lambda\sigma}f_{\mu\nu}f_{\lambda\sigma}$, whose $(2+1)$D boundary value gives the Chern–Simons action and whose projection onto the 2D space of a plane perpendicular to the z and t axes forms the corresponding 2D topological charge density. (A more detailed discussion will be given in the next Section).

We shall take the topology of the four-manifold as $T^2\times S^1\times S_2$, where T^2 is a 2D plane compactified into a torus or, in the presence of defects, into a Riemann space of genus g, and a large number of handles;[27)] S^1 and S_1 are, respectively, z- and t-cyclic directions, of which we shall need small uniform pieces $R_{1,2}$ lying near perpendicular sections. The analytic projection onto 2D space can be performed for the kinetic term in the action (similarly to the mapping of a topological charge). After projection, and as a natural result of it, the model will describe the dynamics of traces on a 2D lattice on which "helical" quasiparticle world-line linkages appear in the form of frustrated units. Since periodicity along the z axis is analogous to the Matsubara cycle, the quasiparticle world-line linkage can be looked upon as the spatial linking of vortices along the z direction as we pass from one plane to another[28)] (see, for example, Refs. 179 and 180).

We now turn to the discussion of the model. Consider a set of quasiparticles covering the main lattice, and suppose that the system is in the liquid phase. This means that, similarly to the fractional quantum Hall effect, the contribution of a contour with N_1 particles to the vacuum transition amplitude has the form $\exp(iN_1S)$ with Im S of the order of unity.[181] Here we also take into account the contribution of intersecting contours that renormalize the imaginary part of the action. The contiguity of contours along links traversed in opposite directions gives rise to frustrations in the space of the contours. In order to take into account the contribution to the real part of the action due to coherent circular exchange,[181] we must consider the symmetry of the phase variables in (2.15) or (8.55). Until now, these were the parameters of the group U(1) of local transformations. However, the greatest interest attaches to the global symmetry of the variables, which reflects the topology of nontrivial boundary conditions. The conclusion that there is a hidden topological symmetry[29)] Z_N emerges from a variety of considerations, including constructions involving the insertion of contours into one another,[182] gauge invariance on a torus and representations of the braid group,[183] and the condition that the theory is defined on a lattice[169] or associations connected with the commutation relations (6.31). The phases $\{\exp(2\pi i\varphi/N)\}$ in this group are distributed discretely on the unit circle: $\varphi = 0,1,\ldots,N-1$. For the moment, we shall allow the parameter N to be free.

The reader should now be ready to accept each term in the partition function

$$Z = \mathrm{Tr}\exp\Big[-\frac{1}{2g^2}\sum_{\mu,a}(\Delta_\mu\varphi_a - 2\pi s_{\mu a})^2 + iN\sum_a n_a\varphi_a + \frac{iN\vartheta}{32\pi^2}\sum \varepsilon_{\mu\nu}\varepsilon_{ab}(\Delta_\mu\varphi_a - 2\pi s_{\mu a})(\Delta_\nu\varphi_b - 2\pi s_{\nu b})\Big] \qquad (11.1)$$

of the Z_N-model with the ϑ term that was investigated in Refs. 185 and 186. We shall, nevertheless explain the structure of (11.1) once again. The first term is the periodically continued kinetic term, the second contains the source n_a of the fields φ_a, and the third is the lattice approximation to the topological charge density. The discrete fields $\varphi_a(x_1,x_2) = 0,1,\ldots,N-1$ that parametrize the phase of the Wilson variables can be interpreted as the fluxes of the statistical magnetic field, measured in units of the flux quantum. The variables $n_a(x_1,x_2)$ and $s_{\mu a}$ that "reside" on the sites and on the links of the lattice, respectively, are integers, and the lattice indices are μ, $\nu = 1,2$. The indices $a,b = 3,4$ are the byproducts of the section $z = \text{const}$, $t = \text{const}$ in the $(3+1)$D theory, i.e., they are the internal indices of the $(2+0)$D model. The coupling constant is $g^{-2} = J/T$ in (11.1) and the vacuum angle ϑ in the continuous limit as a factor of the topological term

$$\frac{\vartheta}{16\pi^2}\int d^4x\varepsilon^{\mu\nu\lambda\sigma}f_{\mu\nu}f_{\lambda\sigma} \tag{11.2}$$

in the action of $(3+1)$D theory prior to the projection operation.

The variables n_a in (11.1) are the electric charges that now appear as sources of spin-wave motion of the field φ. A detailed discussion of the correct definition of the electric charge in strongly-correlated systems is given in Ref. 187. the monopole charges $m_\mu = (1/2)\varepsilon_{\mu\nu\lambda\sigma}\Delta_\nu s_{\lambda\sigma}$ that represent the Berezinskiĭ effect have the nonzero components

$$\begin{aligned} -m_3 &= \Delta_1 s_{24} - \Delta_2 s_{14}, \\ m_4 &= \Delta_1 s_{23} + \Delta_2 s_{13} \end{aligned} \tag{11.3}$$

and represent, respectively, the field vortices φ_4 and φ_3. The variables φ_4 and m_4 are defined on the sites of the dual lattice. The last term in (11.1) therefore describes the local interaction

$$\frac{iN\vartheta}{2\pi}\Big(\sum_{\mathbf{r}} m_3\varphi_3 + \sum_{\mathbf{x}} m_4\varphi_4\Big) \tag{11.4}$$

between the field vortex φ_3 and the field φ_4, and vice versa. This interaction is completely analogous to the coupling between the electric charges n_a and the fields φ_a in (11.1).

The action in (11.1) is a quadratic function of φ. It follows that, after integration with respect to it, we can rewrite (11.1) in the representation of a Coulomb gas of interacting electric and magnetic charges:[185,186]

$$\begin{aligned} Z = \operatorname{Tr}\exp\Bigg[&-\frac{2\pi^2}{g^2}\sum_{\left\{\substack{r,r',\,\mu=3,\\ r\to x,r'\to x',\,\mu=4}\right\}} m_\mu(\mathbf{r})G(\mathbf{r}-\mathbf{r}')m_\mu(\mathbf{r}') \\ &-\frac{1}{2}N^2g^2\sum_{\left\{\substack{r,r',\,\mu=3,\\ r\to x,r'\to x',\,\mu=4}\right\}} \tilde{n}_\mu(\mathbf{r})G(\mathbf{r}-\mathbf{r}')\tilde{n}_\mu(\mathbf{r}') \\ &+ iN\sum_{\mathbf{r},\mathbf{x}}(m_3(\mathbf{r})n_4(\mathbf{x}) - m_4(\mathbf{x})n_3(\mathbf{r}))\theta(\mathbf{r}-\mathbf{x}) \\ &-\frac{N^2g^2}{8}\Big(\sum_{\mathbf{r}} n_3^2(\mathbf{r}) - \sum_{\mathbf{x}} n_4^2(\mathbf{x})\Big) - \frac{\pi^2}{2g^2}\Big(\sum_{\mathbf{r}} m_3^2(\mathbf{r}) + \sum_{\mathbf{x}} m_4^2(\mathbf{x})\Big)\Bigg] \end{aligned} \tag{11.5}$$

where $G = -(1/2)\ln r$ is the Coulomb Green function, $\theta(\mathbf{r}-\mathbf{x}) = -\operatorname{arctg}[(y-x_2)/(x-x_1)]$ is its dual partner, and

$$\tilde{n}_\mu = n_\mu + (\vartheta/2\pi)m_\mu \tag{11.6}$$

represents the fact that the electric charge contains a monopole contribution that is proportional to the vacuum angle.[188]

The advantage of the representation of the model (11.1) by (11.5) is that it expresses the dynamics directly in terms of the quantum numbers of its elementary topological excitations. The first two terms describe the Coulomb interaction in a gas of magnetic and electric charges, and the third represents the Bohm–Aharanov effect, i.e., the linking of the electric current Ngn_a to a Dirac string transporting a current $2\pi n_a/g$ of the magnetic charge. We emphasize once again that the magnetic charge has to be introduced because the field φ_a is regarded in (11.1) as an angular variable. The result of this is that, in order to take into account the multivaluedness of the gauge configurations, we must sum over all the numbers $s_{\mu a}$, i.e., over all the topologically nontrivial configurations with magnetic monopoles (cf. Ref. 189 where it is shown how the Chern–Simons term "releases" electric charges by acting against the monopoles and confinement).

We shall now describe the phase diagram of the model defined by (11.1), following the argument[190] of Kosterlitz and Thoules.[30] In a system of the form of (11.5) with characteristic dimension L, i.e., and entropy $\ln L^2$, the energy of excitations with quantum numbers[31] $n_4 = m_4 = 0$, $n_3 \equiv n$ and $m_3 \equiv m$ takes the form

$$\varepsilon_{n,m}(g,\vartheta) = \left[\frac{\pi^2}{2g^2} + \frac{N^2g^2}{8}\Big(n + \frac{\vartheta}{2\pi}m\Big)^2\right]\ln L. \tag{11.7}$$

Comparison of this with entropy shows that, in the ground state, there are quasiparticles whose quantum numbers satisfy the condition

$$\min_{\{n,m\}}\left[\frac{m^2}{T_N} + T_N\Big(n + \frac{\vartheta}{2\pi}m\Big)^2\right] < \frac{4}{N}. \tag{11.8}$$

The ratio of the axes of the ellipse in (11.8) is $T_N = NT/2\pi J$, its slope and therefore the slope of the lattice of charges in Fig. 12 is $\tan\alpha = \vartheta/2\pi$, and the area is $4\pi/N$. This means that different phases are possible for different temperatures T and different vacuum angles ϑ, and also different values of N.

For a given $\vartheta \neq 0$ and small coupling constants $g^2 = T/J$, there is always a low-temperature phase iwth charges $(n,m) = (1,0)$ that satisfies (11.8), i.e., the usual Higgs phase. It is followed by the Coulomb phase whose particular feature is that the ellipse (11.8) does not contain any of the sites on the charge lattice. As N increases, i.e., as we pass to a

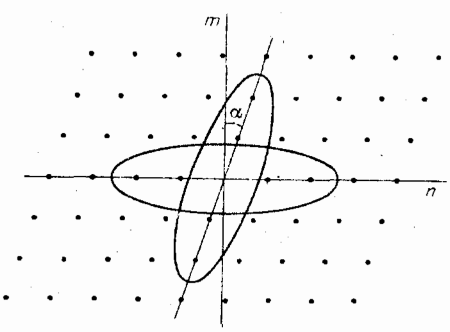

FIG. 12. The charge lattice with tilt $\alpha = \arctan(\vartheta/2\pi)$ (Ref. 185). The figure shows ellipses for different coupling constants and the charges inside these ellipses that become part of the condensate.

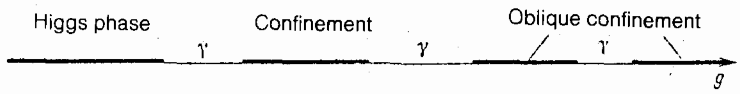

FIG. 13. Typical phase diagram. Coulomb phases with massless gauge bosons are represented by γ.

state of greater isotropization, e.g., because of the applied external field,[193] the region in which the Coulomb boundary phase with long-range correlations is present is found to expand. For large values of g^2, we have the usual confinement phase with monopole condensation, separated by the Coulomb phase from the state of oblique confinement in which the ground state contains condensed current loops carrying the electric and magnetic charges. The phase sequence is illustrated in Fig. 13. Figure 14 shows the phase diagram[185,186] for $N<4$ and $N=4$.

The number of oblique-confinement phases is very critical in relation to the vacuum angle ϑ. When $\vartheta/2\pi = 1/q$ and q is an integer, there is only one such phase. When $\vartheta/2\pi = p/q$ and q is odd, the number of oblique-confinement phases is equal to p. For an irrational $\vartheta/2\pi$, the number of phase states is infinite.

In any phase, excitations with electric and magnetic charges that are multiples of particle charges in the condensate behave as physical particles. All other particles are held by the confinement potential. For example, in the state of oblique confinement, the electric charge and the magnetic monopole are trapped by the linear potential, but the bound states of these charges can be the states of free quasiparticles. Calculations[185] of the Wilson order parameter (10.3) show that, in the phase with condensed charges (n,m), the Wilson correlator for particles with quantum numbers (n',m') satisfies the law of areas with string tension proportional to $(mn'-m'n)^2$. This is the "distance" on the charge lattice from the quantum numbers of particles in the condensate. Particles in the condensate are characterized by the perimeter law for loop correlation functions and, since they are not held by the trapping potential, they have a short-range interaction with scale inversely proportional to the gap in the spectrum: $M\sim[(2\pi^2/g^2)\cos^2\alpha + (N^2g^2/2)\sin^2\alpha]^{1/2}$ where $\mathrm{tg}\,\alpha = \tilde{n}/m$ for the charges $(n+(\vartheta/2\pi)m,m)$ from the condensate.

It follows from this and from the previous Section that the theory contains a rich and very beautiful picture of dual replacement of the phenomena of confinement and Higgs phase in two orthogonal channels of variation of the quantum numbers of "electrically" and "magnetically" charged particles. Self-duality is discussed in Ref. 191 from the $(2+1)$D point of view.

The duality relations[192] *are the equivalence relation for* statistical models at high and low temperatures. In our case, for $\vartheta=0$ (Ref. 193), the model defined by (11.5) is self-dual[194–196] if

$$g\to 2\pi/Ng. \tag{11.9}$$

This replacement interchanges the electric and magnetic charges. When $\vartheta/2\pi = 1/q$, the phase diagram of Fig. 14 is inversion-symmetric with respect to the self-dual value[185] $g^2 = 2\pi q/N$.

These symmetries of the parametric space of coupling constants can be generalized as follows.[185] We first introduce the complex parameter

$$\zeta = \frac{\vartheta}{2\pi} + i\frac{2\pi}{Ng^2}. \tag{11.10}$$

The dual symmetry which is hidden in the theory that leaves the Hamiltonian $H\{m\},\{n\},g,\vartheta) = H(\{n\}, -\{m\},g',\vartheta')$ $(Z=\mathrm{Tr}\exp(-H))$ unaltered then corresponds to the inversion

$$\zeta\to\zeta' = -\zeta^{-1}. \tag{11.11}$$

The noncommuting operations of duality S (11.11) and translation T in the vacuum angle $\vartheta(\zeta\to\zeta+1, m\to m, n\to n-m)$ generate an infinite discrete symmetry group with group element

$$T^{a_n}ST^{a_n-1}S\ldots ST^{a_1}. \tag{11.12}$$

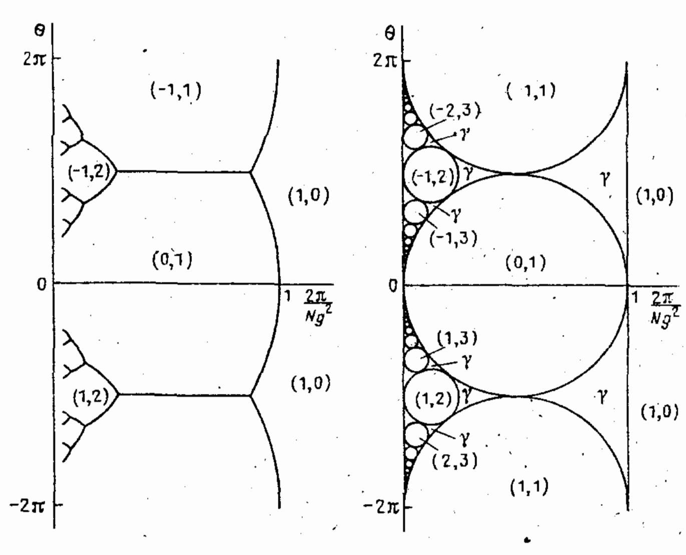

FIG. 14. Phase diagram for $N<4$ (a) and $N=4$ (b) (Refs. 185 and 186). The parentheses (n,m) show the electric and magnetic charge of particles in the condensate.

The integers a_i in (11.12) define the "word" $\{a_n, a_{n-1}, \ldots, a_1\}$ that determines the transformation of ζ in the form of the continued fraction

$$\zeta \to a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{\ldots - \cfrac{1}{a_1 - \zeta}}}. \qquad (11.13)$$

It is clear from this expression that the group of all such transformations is isomorphous to the group SL(2,Z) of 2×2 matrices with unit determinant and integer elements

$$\zeta \to \frac{a\zeta + b}{c\zeta + d}, \quad ad - bc = 1, \quad \{a, b, c, d\} \in Z. \qquad (11.14)$$

The existence of this large symmetry group links the nonhierarchical sequences (8.62) of ground states to the phase diagram of the model. In the limit as $g \to \infty$, the phase diagram becomes infinitely complicated, since there is always a phase of condensed electric n and magnetic m charges with the following ratio:

$$\frac{n}{m} = a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{\ldots - 1/a_1}}, \qquad (11.15)$$

which corresponds to[186] $\vartheta/2\pi = n/m$. On the other hand, if we fix ϑ and change g, then Fig. 14 shows that we obtian a finite number of phase transitions. The critical temperature and the critical indices of the correlation functions[191] of the Coulomb gas will depend on the vacuum angle ϑ and the parameter N of the group center, i.e., they will be determined by the excitation statistics.

The source of this hierarchy of states in strongly correlated systems is universal. It resides in the operations of adiabatic localization of a given cover (distribution of the flux of the statistical magnetic field over the lattice) and also the particle-hole conjugation which can be written[184,197] in the form of the elements of congruence subgroups of the group of SL(2,Z):

$$\alpha \to \pm \alpha + 2 \quad (\nu^{-1} \to \pm \nu^{-1} + 2), \qquad (11.16)$$
$$\alpha \to 1/\alpha \qquad (\nu \to 1 - \nu),$$

where $\alpha = \nu^{-1} - 1 = (\vartheta/2\pi)^{-1}$ and ν is the filling factor. In the fractional quantum Hall effect the SL(2,Z) operations (11.16) on the toruses[89,198] produce hierarchic sequences of states with filling factor[198] $\nu = r/(2r \pm 1)$.

12. ANYONS FROM THE (3+1)D and (1+1)D POINTS OF VIEW

The question now is: does the spatially three-dimensional point of view introduce anything new into the anyon picture? Comparison of fields at neighboring points in the kinetic part of the action shows that they have the same form in all dimensionalities and produce no difficulties. However, when $g^2 \gg 1$, the main contribution to the action is provided not by the kinetic but by the topological Chern-Simons term. The Chern–Simons term was originally introduced[56–59] as a correction to the standard expression, but becomes the dominant term for large coupling constants. In this limit, excitations with energy of the order of the mass $m = g^2k^2/4\pi^2$ detach themselves from the vacuum sector of the theory, where g is the coupling constant in the usual term $(1/4g^2)F_{\mu\nu}^2$ in the Lagrangian and m is the coefficient in the equation of motion $(\square + m)^*F_\alpha = 0$ in which[62] $^*F_\alpha = (1/2)\varepsilon_{\alpha\beta\gamma}F^{\beta\gamma}$. The Chern–Simons term is also related to changes in the structure of phase space[89,94] and, as we already know, to the multiple-connectedness of $(2+1)$D systems.

The Chern–Simons term on its own can be looked upon as a contribution due to the boundary M_3 of the enclosing $(3+1)$D space M_4. Actually, for the contribution

$$S_{(4)} = \int_{M_4} d^4x \varepsilon^{\mu\nu\lambda\sigma} f_{\mu\nu} f_{\lambda\sigma} = \int_{M_4} d^4x \partial_\mu j^\mu = \int_{M_3} d\Omega_\mu j^\mu \qquad (12.1)$$

to the action in four-dimensional, theory where

$$j^\mu = \varepsilon^{\mu\nu\lambda\sigma} a_\nu \partial_\lambda a_\sigma, \qquad (12.2)$$

integration over the hypersurface perpendicular to the z axis yields

$$S_{(4)} = S_{(3)}\big|_{z=L_z} - S_{(3)}\big|_{z=0}. \qquad (12.3)$$

The topological charge $S_{(4)}$ in the case of U(1) symmetry can be reduced to zero[32)] by a continuous gauge transformation. Hence it follows that violation of two-dimensional parity and symmetry under time reversal may be masked by "antiferromagnetic" ordering of the signs of the factor k in the Chern–Simons term $S_{(3)}$ from layer to layer along the z axis. (We draw attention to footnote 28 and the assumption of the "ferromagnetic" ordering of the signs of k in Ref. 32.) The alternation of oriented vortex filaments with charges at the ends. The braiding of the vortex lines along the z axis then provides us with a clear spatial picture. The distribution of magnetic charges among the lattice sites between the basal planes is discussed in Refs. 140 and 200.

It is clear from (12.1) that, in addition to the usual Chern–Simons term $S_{xy+\tau}$, there are also the contributions $S_{xy+z}, S_{xz+\tau}$ due to other sections of the four-dimensional manifold. The choice is between S_{xy+z} and $S_{xy+\tau}$, where τ is the Matsubara variable, since their simultaneous utilization depends on the architecture of the lattice[33)] and the temperature. The relative importance of topological terms is reduced as we pass to the isotropic case (for example, the cubic lattice) because of mutual cancellations on different sections of the space M_4.

Let us now consider the representation of the $(2+1)$D Chern–Simons theory in a model defined in a space of lower dimensionality. In contrast to Sec. 11, we shall be interested in the vacuum sector of the theory. It is shown in Ref. 89 that the Chern-Simons gauge theory in three-dimensional space is directly related to a rationally conformal theory in two dimensions.[203] We shall consider a three-dimensional manifold with $\Sigma \times R$ topology, where R is a small homogeneous part of the z axis or the τ axis near a section Σ perpendicular to them. If the manifold Σ is compact, the dimensionality of the Hilbert space of states in the Chern–Simons theory is the same as that of the vector space of conformal field theory on Σ., If, on the other hand, Σ has a boundary, then the Hilbert space appears as an infinite-dimensional space of representations of the chiral algebra of conformal field theory.

Witten's paper[89] has stimulated the publication of a large number of papers that establish new relationships between different and apparently unconnected regions. In particular, order was introduced into the conformal zoo,[204] a relation was found with the theory of knots and links,[205] and new relationships were established between quantum groups, the braid group, and the exact solutions of conformal theory.[206–208] All this led to the discovery of some deep properties of the topological Chern-Simons theory.[83,105,209–215] Unfortunately, we are unable within the confines of this review to give a sensible presentation of the results published in these and similar papers[216–218] that could be relevant to the problem considered here. In order to draw the attention of the reader to new possibilities, we therefore turn to the analysis reported in some of these papers.[105,213–215] This will provide some useful information about the structure of the vacuum sector of the theory.

Consider the limit $g^2 \to \infty$ which leaves the Chern–Simons term in the total Lagrangian. The Lagrangian for the group U(1) in the gauge $A_0 = 0$ has the following form in this limit:

$$L = \frac{k}{4\pi}(A_1 \dot{A}_2 - A_2 \dot{A}_1) \tag{12.4}$$

and the equation of motion is

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = 0, \tag{12.5}$$

and constitutes a constraint on the state vector $|\ \rangle$ of quantum theory (Gauss' law): $F_{12}|\ \rangle = 0$. We recall that the canonically conjugate variables A_1 and A_2 in the phase space of the system satisfy the commutation relation

$$[A_1(\mathbf{x}), A_2(\mathbf{y})] = i(2\pi/k)\delta^{(2)}(\mathbf{x} - \mathbf{y}). \tag{12.6}$$

Since, as is readily seen from (12.4), the Hamiltonian is equal to zero, we shall be interested in the properties of the vacuum sector of the theory and, in particular, its dimensionality.[34)] Suppose that the space with suitably chosen boundary conditions has been compactified to an arbitrary manifold Σ. In topological theory, the physical variables are the Wilson gauge variables that are defined on Σ and are characterized by the contour C, charge n, and representation R_k:

$$W_{R_k}(C, n) = \mathrm{Tr}_{R_k} P \exp(in \oint_C \mathbf{A}\, d\mathbf{l}). \tag{12.7}$$

In the Abelian case U(1) one can work directly with the argument *in* $\phi(C)$, of the Wilson loop exponential if we put $n = 1$. At each instant of time (or on each section $z = \text{const}$), the set of all the loops $\{\phi(C)\}$ on Σ forms a complete set of variables. The meaning of this is as follows.

The constraint (12.5) shows that the potential is a pure gauge potential on a section $t = \text{const}$. For such potentials, the Wilson variables can be transformed one into another by continuous deformation. Hence, only homotopic classes of mutually undeformable loops will be meaningful, and the number of independent nontrival loops on Σ will be the only important quantity. For a Riemann surface of genus g (with g handles) there are $2g$ such loops. They constitute a complete set of variables in the phase space of the Chern–Simons theory.

The commutation relations given by (12.6) can be rewritten in terms of $\phi(C)$ as follows:[140]

$$[\phi(C_i), \phi(C_j)] = i\frac{2\pi}{k}\int_{C_i}\int_{C_j} d^2x\,\delta^{(2)}(x_i - x_j) = i\cdot\frac{2\pi}{k}\sum_{P_{ij}}(-1)^{P_{ij}}, \tag{12.8}$$

where P_{ij} are the points of intersection of the curves C_i and C_j, and P_{ij} is the parity of these intersections. In some suitable basis of the loops $\phi(C_1)$, $\phi(C_2)$ on the surface of genus g, where they correspond to two nontrivial cycles a_i, b_i for each handle with $i = 1, \ldots, g$ and winding numbers $m^i_{a,b}, n^i_{a,b}$ around cycles a,b of the ith handle, the commutator (12.8) is

$$[\phi(C_1), \phi(C_2)] = i\cdot\frac{2\pi}{k}\sum_i (m^i_a n^i_b - m^i_b n^i_a). \tag{12.9}$$

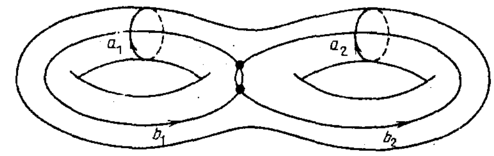

FIG. 15. Nontrivial cycles on Riemann surface with two handles.

Since the loops from each pair (a_i, b_i) intersect an odd number of times (Fig. 15), we have

$$[\phi(a_i), \phi(b_j)] = i(2\pi/k)\delta_{ij}, \tag{12.10}$$

and all the other operators commute.

In the case of a torus, for which $g = 1$, and if we choose the polarization[214] in the form $\phi_a = Q$, $(k/2\pi)\phi_b = P$, we obtain the standard relation $(Q,p) = i$ and we can use the representation $P = -i\partial_Q$. For the compact group U(1), the variables ϕ are pure phases. Consequently, the phase space is compact and the states are invariant under the shifts $Q \to Q + 2\pi$ and $P \to P + k$. The wave function must therefore satisfy the quasiperiodic condition

$$\psi(Q + 2\pi) = e^{i\beta}\psi(Q). \tag{12.11}$$

This is also valid for the Fourier transform $\tilde{\psi}$ of the wave function $\psi(Q)$:

$$\tilde{\psi}(P + k) = e^{i\alpha}\tilde{\psi}(P); \tag{12.12}$$

where α and β are arbitrary constant phases that can be interpreted as the vacuum angles ϑ_1 and ϑ_2.

It follows from (12.11) and (12.12) that, when k is an integer, there are k linearly independent states in the Hilbert space of the δ-vector of state, so that the expansion over the eigenstates is

$$\psi(\phi_a) = \sum_n A_{l,n}\delta(\phi_a - \frac{\alpha}{k} - \frac{2\pi}{k}n)e^{in\beta/k} \tag{12.13}$$

or

$$\tilde{\psi}(\phi_b) = \sum_n \tilde{A}_{l,n}\delta(\phi_b - \frac{\beta}{k} - \frac{2\pi n}{k})e^{-in\alpha/k} \tag{12.14}$$

with the periodic condition for the coefficients

$$A_{l,n+k} = A_{l,n}, \quad \tilde{A}_{l,n+k} = \tilde{A}_{l,n}, \tag{12.15}$$

that are related by the Fourier transformation

$$A_{l,n} = \sum_{m=0}^{\infty} e^{i(2\pi/k)mn}\tilde{A}_{l,m}. \tag{12.16}$$

The complex numbers $\{A_l\}$ exhaust the k-dimensional Hilbert space. This conclusion is, of course, unaffected if we use the Wilson variables $W_{a,b} = \exp(i\phi_{a,b})$, since the commutation relations

$$W_a W_b = W_b W_a e^{i(2\pi/k)} \tag{12.17}$$

have representations with k independent states. Comparison with (10.7) suggests that $N = 2k$. Hence, the Z_2-theory corresponds to a set of three bosons, whereas the Z_4-gauge theory describes semifermions.[218] We also note that the number N that determines the center $U(1)^{N-1}$ of the groups $SU(N)$ can be interpreted as the number of sites in the unit cell of a square lattice with independently transforming phase parameters of the group $U(1)\otimes U(1)\otimes U(1)\otimes U(1)$ or $U(1)\otimes U(1)$.

In the general case of a surface Σ with g handles, the Hilbert space can be parametrized with the help of $2g$ vacuum angles, and has k^g linearly independent states. This result is valid for integral k for which the wave function is invariant, to within the phase, under global gauge transformations.

When $k = 2(k_1/k_2)$, where k_1 and k_2 are mutually primitive numbers, invariance under $SL(2,Z)$ transformations in the space of the variables ϕ_a and ϕ_b enables us to establish[214] that the dimensionality of the Hilbert space is $(2k_1k_2)^g$. In these expressions, k_1 and k_2 replace the numbers q and p of the previous Sections. The requirement of modular invariance, which can be interpreted as general covariance during the parametrization of the handles, is an additional condition imposed on k and on the values of the vacuum angles.

We now return to the Lagrangian (12.4) and the relation between the Hilbert space of the Chern–Simons theory and the space of the moduli of two-dimensional conformal field theory. Let $M_3 = D_2\times R$, where D_2 is a disk and $A_i = \partial_i \Phi$. Substituting into the Lagrangian, and putting $\Phi = k^{-1/2}\varphi$, we obtain the action

$$S = (1/2\pi)\int ds\, dt \partial_s\varphi\partial_t\varphi \tag{12.18}$$

for the free boson field φ in flat two-dimensional Minkowski space. The argument s in (12.18) parametrizes the boundary of the disk D_2. Since the field Φ is periodic, the field φ is compactified on a circle whose radius r is given by $r^2 = k = 2(k_1/k_2)$. For all these values, the Hilbert space of the original theory corresponds to the space of conformal blocks of the conformal field theory (12.18) with central charge $c = 1$.

When $M_3 = S^2\times S^1$, the orthogonal basis of the Hilbert space consists of the functions[210]

$$\psi_p = \exp(-2k\gamma(U))\exp[\pi ka(\mathrm{Im}\,\tau)^{-1}a]\eta^{-1}(\tau)\vartheta\begin{bmatrix} p/2k \\ 0 \end{bmatrix}(2ka|2k\tau)$$
$$p\in \mathbf{Z}_{2k}, \tag{12.19}$$

that are identical to the characters of the rational conformal theory.[83,209,219] If we use the properties of the Jacobi theta-function and the Dedekind function $\eta(\tau)$, we can show[83,209,210] that the generators of the modular group S: $a\to a/\tau$, $\tau\to -1/\tau$ and T: $a\to a$, $\tau\to\tau+1$ transform (12.19) as follows:

$$\psi_p|_S = \frac{1}{(2k)^{1/2}}\sum_{q=0}^{2k-1}\exp\left(-\frac{i\pi pq}{k}\right)\psi_q, \tag{12.20}$$

$$\psi_p|_T = \exp\left[2\pi i\left(\frac{p^2}{4k}-\frac{1}{24}\right)\right]\psi_p. \tag{12.21}$$

Comparison of (12.21) with the standard transformation $\psi_p|_T = \exp 2\pi i[h_p - (c/24)]\psi_p$ yields the conformal dimensionality $h_p = p^2/4k$ and the central charge $c = 1$. We know[203] that the conformal dimensionality determines the exponent in the power-type reduction in correlation functions of primary fields and the value of conformal spin, whereas the central charge gives the number of degrees of freedom per unit cell of momentum space[220,221] and determines their contribution to the thermal capacity.

We now draw attention to the result of applying[209,210] the Wilson operator to the state (12.19):

$$W(C,n)\psi_p = \psi_{p+n}, \tag{12.22}$$

from which we have the Verlinde[222] Z_n-rule $W(n)\times W(q) = W(n+q)$ for these operators and the possibility of identification with primary rules of conformal theory. The physical states (correlation functions) satisfy[223] the Knizhnik–Zamolodchikov equation[105,210,212,214,215,222] that describes transport in the space of the moduli of the $(1+1)$D Wess–Zumino–Witten–Novikov $SU(2)$ model with the solution[210]

$$\mathcal{F}(\{P\},\{Q\}) = \left(\prod_{i<j}^{N}(z_{P_i}-z_{P_j})(z_{Q_i}-z_{Q_j})\right)^{P_iP_j/2k}$$
$$\times\left(\prod_{i,j=1}(z_{P_i}-z_{Q_j})\right)^{-P_iP_j/2k}.$$

The relation between the states (7.9) and (7.11) of the chiral liquid and the correlation functions of the Wess–Zumino–Witten–Novikov chiral $SU(2)$ model is discussed in Refs. 225 for $k = 1$ and $P_i = 1$.

The inclusion of the topological term[226] $i\alpha_0 R\phi$ in the Lagrangian of the $(1+1)$D theory (12.18) for an arbitrary metric,[35] where R is the two-dimensional curvature of the surface spanning the contour C, i.e., the addition of two-dimensional gravity to the theory (12.18), is equivalent to introducing a charge at infinity,[227] which shifts the value of conformal dimensionality.

The inclusion in the Lagrangian

$$L = \frac{k}{4\pi}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma + A_\mu j^\mu \tag{12.23}$$

of the conserved external current $j^\mu = (\rho,\mathbf{j})$ with charge

$$Q = \int\rho(\mathbf{r},t)d^2x \tag{12.24}$$

modifies the above results in the following ways.[213] The Hamiltonian is no longer zero and takes the form

$$H = \int A_k j^k d^2x, \tag{12.25}$$

and (12.5) is replaced with the familiar expression

$$\varepsilon_{ij}\partial_i A_j = b = (2\pi/k)\rho. \tag{12.26}$$

We now divide the vector potential

$$A_i = \partial_i\phi + \varepsilon^{ij}\partial_j^{-1}b,\quad \partial_j^{-1} = \partial_j/\nabla^2 \tag{12.27}$$

into longitudinal and transverse components, and rewrite the commutation relation (12.6) for the canonical pair (ϕ,b) in the form

$$[\phi(\mathbf{x}), b(\mathbf{y})] = i(2\pi/k)\delta(\mathbf{x}-\mathbf{y}). \tag{12.28}$$

If we choose a rotationally-invariant polarization,[36] the magnetic field d is realized as the derivative with respect to the coordinate ϕ, i.e., $b = i(2\pi/k)\partial_\phi$. The Gaussian constraint on the state vectors, given by (12.26), therefore takes the form of the equation

$$(-i\frac{\partial}{\partial\phi} + \rho)\psi(\phi, t) = 0 \tag{12.29}$$

with the solution

$$\psi(\phi, t) = N(t)\exp\left(i\int\rho\phi\, d^2x\right),$$

$$N(t) = \exp\left(\frac{2\pi i}{k}\int_0^t \rho(t', \mathbf{x})j(t', \mathbf{x})d^2x\, dt'\right). \tag{12.30}$$

The statistical state $\psi(\phi)$ is an eigenstate of the Hamiltonian (12.25) with $j_i = -\partial_i^{-1}\dot{\rho} + \varepsilon_{ik}\partial_k j$ and energy

$$E = \frac{2\pi}{k}\int\rho j\, d^2x. \tag{12.31}$$

The Wilson operator $W(C)$ in (10.3), when it acts on the state (12.30), produces the eigenvalue[213]

$$W(C) = \exp[i(\gamma(C) + \frac{2\pi}{k}Q(C))], \tag{12.32}$$

that depends on the total charge $Q(C)$ within the contour C and the phase $\gamma(C)$.

If the loop C cuts itself ν times, the contributions to $Q(C)$ appear with the corresponding signs, and the vacuum holonomy $\exp[i\gamma(C)]$ is determined[208,213] by the phase

$$\gamma = 2\pi m/k, \quad m = -\nu/2, -(\nu/2) + 1, ..., (\nu/2) - 1, \nu/2. \tag{12.33}$$

In other words, for the class of self-crossing loops, vacuum holonomy corresponds to representations of the group SU(2) with spin $\nu/2$, dimensionality $\nu + 1$, and minimal set of SU(2) representations with $\nu/2 = 0, 1, ..., k$. For these values of ν and $Q(C) = 0$, the eigenvalues of the Wilson operator $\exp(2\pi im/k)$ are identical with the value of the parameter q in minimal models of conformal field theory. This parameter appears in the quadratic constraint $g_i^2 = (q-1)g_i + q$ and, together with the standard conditions $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, $1 \leqslant i \leqslant n-1$, $g_i g_j = g_j g_i$ for $|i-j| \geqslant 2$ on the elements g_i of the braid group, determines the relations of the Hecke algebra $H_n(q)$ (Ref. 217) and also the argument of the Jones polynomials $V_L(t)$ (Ref. 205), where $t = -q^{-1}$. The representations $H_n(q)$ are identical[228] with the monodromies of N-particle correlation functions satisfying the Knizhnik–Zamolodchikov equation. The nonzero charge $Q(c)$ in (12.32) deforms the values of m.

We draw attention to the fact that the statistics of charges, proportional to $1/k$, is a dual conjugate of the statistics of fluxes for which it is of the order of k. In the theory (12.33) of a charged current interacting with a U(1) statistical gauge field, the charge Q, the magnetic flux $2\pi\Phi$ (in units of $\hbar = 1$), and the spin s of an elementary flux tube are related by[214,229,230] $Q = k\Phi$, $s = k\Phi^2/2 = Q^2/2k$. All that remains is to identify the elementary excitations of the theory and their quantum numbers. We saw in Sec. 11 that this depends on the coupling constant. If these are flux quanta, then by substituting $\Phi = 1$, we obtain $s = k/2$, i.e., spin proportional to k. If, on the other hand, these are charge quanta, then by putting $Q = 1$ we obtain $s = 1/2k$. The latter point of view was adopted throughout this Section, since the Wilson operator in (12.22) generated units of charge.

13. THERMODYNAMICS AND ELECTRODYNAMICS OF ANYON SYSTEMS

The changes in the thermodynamic parameters of sets of particles with intermediate statistics, as compared with standard cases, are most simply seen by considering the example of (3.14). The partition function

$$Z = \sum_{m=-\infty}^{+\infty}\exp[-(m+\alpha)^2T^{-1}] = (\pi T)^{1/2}\theta_3(\alpha, i\pi T) \tag{13.1}$$

is expressed[231] in terms of the theta-function θ_3, T is the temperature normalized to $h^2/2I$, $\alpha = \vartheta/2\pi$, $\alpha = 0$ and $\alpha = 1/2$ correspond to Bose and Fermi statistics, and $\alpha = 1/4$ corresponds to semifermions.

The temperature dependence of thermal capacity is shown[231] in Fig. 16 for (13.1). At high temperatures and for $0 < \alpha < 1/4$, c_V tends from above to the classical value of 1/2. If $1/4 < \alpha < 1/2$, we have $c_V \to 1/2 - 0$. We note that the thermal capacity is a maximum when $\alpha \neq 1/2$.

Another significant thermodynamic quantity is the second virial coefficient $B(T)$. The striking result reported in Refs. 232 and 143, namely,

$$B(T, \alpha) = (\lambda^2/4)\times[1 - 2(1-2\alpha)^2], \tag{13.2}$$

may have been for many people the starting point of their exploration of anyon theory. The quantity $\lambda^2 = h^2/mT$ in (13.2) is the square of the thermal de Broglie wavelength. The deep valleys at $\alpha = 0 \bmod 2\pi$ with $B < 0$ and the maximum at $\alpha = 1/2$ with $B > 0$ in (13.2) represent attraction and repulsion for Bose and Fermi systems, respectively. Similar valleys appear on the vacuum energy density

$$E = \frac{1}{12} - \left(\frac{1}{2} - \alpha\right)^2 \tag{13.3}$$

of a complex field with phase defined to within the replacement $\exp(2\pi i\alpha) = \exp(i\vartheta)$ on the manifold $T \times S^1$ in the presence of an elementary solenoid with flux.[233] We also note that the result given by (13.2) is also valid[93] for a gas of two-skyrmion configurations of the z-field of the CP^1-representation (4.5), (4.6) of our problem.

The evaluation of the distribution function is more difficult. The extension of the standard method of evaluating the distribution function to the case of finite degeneracy of order

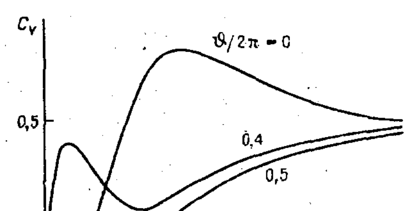

FIG. 16. Specific heat as a function of temperature for three values of the static parameter $\vartheta/2\pi$ (Ref. 231) for the ideal gas of rotators.

k ($k = 1$ and $k = \infty$ for Fermi and Bose particles, respectively) is incorrect because the determination of the distribution function is part of the rigorous solution of the problem. To illustrate the possibilities that arise in this type of problem, we reproduce the result for the distribution function

$$n(\varepsilon) = \frac{1}{1 + \sigma^{-1}\exp(\varepsilon/T)}, \tag{13.4}$$

$$\sigma = \frac{\mathrm{sh}(NH/2T)}{\mathrm{sh}(H/2T)} \frac{\sin[\pi N/(N+M)]}{\sin[\pi/(N+M)]} \tag{13.5}$$

in the sector of massive excitations with spectrum $\varepsilon_j = (c^{+2}p^2 + \Delta_j^2)^{1/2}$, $\Delta_j = \Delta_0 \sin(\pi j/N)$, $j = 1, ..., N-1$ in the exact solution[234] for the $SU(N) \otimes SU(M)$ symmetric $(1+1)$D fermion model. We note that (13.4) differs from the Fermi distribution function. There are reasons to suspect that a similar structure occurs in our case as well. The point is that the thermodynamic potential

$$\Omega = -T \ln \sum_i \sum_{N=0}^{1} g_N e^{\beta(\mu-\varepsilon_i)N} = -T \ln \sum_i (1 + g_1 e^{\beta(\mu-\varepsilon_i)}), \tag{13.6}$$

that gives the mean filling $\bar{n}_i = -\partial\Omega/\partial\mu = [1 + g_1^{-1} e^{\beta(\varepsilon_i-\mu)}]^{-1}$ of the i-th fermion state contains in the general case the statistical weights g_N. For standard fermions, $g_1 = 1$. In the case of a degenerate ground state, g_1 depends on the q-dimensionality[217] $[2j+1]_q$ of the degeneracy space, as shown in (13.4). We note that, for the $SU_q(M)$ group we have, $q = \exp[2\pi i/(k+M)]$ and $j = (k-1)/2$ (Ref. 208).

In the absence of a rigorous solution, we turn to a qualitative analysis. For example, the important point for any estimate of the superconducting transition temperature $T_c \sim \hbar^2 n/mk^2$ in (5.19) is that its form is typical of a Bose condensate: it is inversely proportional to the square of the statistical parameter k. If we suppose that the condensation energy is the ζ th part of the energy of the first Landau level, i.e., $\zeta\pi\hbar^2 n/k^2 n$ with $0 < \zeta \ll 1$, then estimates of the ratio Δ/T_c, which is actively discussed[33] by experimentalists, suggest that it depends on k and, hence, on the experimental method used to prepare the state. We recall that, in equilibrium, $k = 2$.

Let us now examine some of the electrodynamic properties of anyon systems. Suppose that the temperature is zero and that an external electromagnetic field is present. If $\sigma_{xy} = \vartheta = k/2\pi$ and if we use (8.31), we find that the action (8.52) takes the form

$$S = \int d^3x \left[\frac{1}{2}\varepsilon e^2 - \frac{1}{2}\chi b^2 + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda} a_\mu F_{\nu\lambda} - \frac{\vartheta}{4}\epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}\right], \tag{13.7}$$

where a_μ, $\mathbf{e}$ and $\mathbf{b}$ are the statistical fields that fluctuate around the mean field $\bar{b}$ and A_μ is the potential of the external field of intensity $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The dual formulation equivalent to (13.7) is achieved[149] with the help of the antisymmetric field $\Lambda_{\mu\nu}$ with components $\Lambda_{0i} = \tilde{e}_i$, $\Lambda_{ij} = \varepsilon_{ij}\tilde{b}$ and action

$$S' = \int d^3x \left[-\frac{1}{2\varepsilon}\tilde{e}^2 + \frac{1}{2\chi}\tilde{b}^2 + \frac{1}{2}\Lambda_{\mu\nu} f^{\mu\nu} - \frac{\vartheta}{4}(f_{\nu\lambda} + F_{\nu\lambda})\epsilon^{\mu\nu\lambda} A_\mu\right]. \tag{13.8}$$

Integration over α_μ gives the constraint

$$\partial_\mu(\Lambda_{\mu\nu} - \vartheta\varepsilon_{\mu\nu\lambda}A^\lambda) = 0 \tag{13.9}$$

for the field $\Lambda_{\mu\nu}$ with the solution ($\vartheta = k/2\pi$)

$$\Lambda_{\mu\nu} = (1/2\pi)\epsilon_{\mu\nu\lambda}(\partial^\lambda\alpha + kA^\lambda). \tag{13.10}$$

Substitution in (13.8) and the addition of the Lagrangian for the electromagnetic field $(-1/4e^2)F_{\mu\nu}F^{\mu\nu}$, gives the Lagrangian

$$L = \frac{1}{8\pi^2\chi}(\partial_0\alpha + kA_0)^2 - \frac{1}{8\pi^2\varepsilon}(\partial_i\alpha + kA_i)^2 - \frac{\vartheta}{4}\epsilon_{\mu\nu\lambda}A^\mu F^{\nu\lambda} - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}, \tag{13.11}$$

which is written in terms of the fields $\alpha(\mathbf{x},t)$ and $A_\mu(\mathbf{x},t)$.

The first two terms constitute the Ginzburg–Landau functional for the order parameter $\exp\{i\alpha(\mathbf{x},t)\}$ in the London limit. The third term describes fluctuation-induced effects due to P- and T-parity breaking that occur at large distances in the superconducting state, but only in the presence of an external electromagnetic field.[100]

The classical equations of motion that follow from (13.11) enable us[149] to find the screening length for the static charge $\xi = M_-^{-1}$ and also the dispersion relations for the transverse and longitudinal plasma oscillations. In the long-wave limit the frequency of the transverse (M_+) and longitudinal (M_-) waves is finite:

$$M_\pm^2 = \frac{\vartheta^2 e^2}{2\chi} \times \left\{\left(1 + \frac{\chi}{e} + e^2\chi\right) \pm \left[\left(1 - \frac{\chi}{\varepsilon}\right)^2 + e^4\chi^2 + 2e^2\chi\left(1 + \frac{\chi}{e}\right)\right]^{1/2}\right\}. \tag{13.12}$$

This difference between the plasma frequencies derives from the electromagnetic Chern–Simons term and has long been known in the literature.[235,236] It is readily seen that, in the long-wavelength limit, the finite frequency in the dispersion relation for the transverse oscillations arises from the coupling between the gapless transverse Goldstone mode (9.7) and the oscillations in the transverse components of the electromagnetic field. The Goldstone oscillations of the phase variable $\alpha(\mathbf{x},t)$ are "consumed" by the electromagnetic field and become massive. The difference between the squares of the frequencies $\delta M^2 = (k^2e^2/4\pi^2)[1 + (4\varepsilon/e^2)]^{1/2}$ is independent of the sign of the integer $k = 2, 4, ...$, and is probably the best expression for the experimental verification of P- and T-parity violation within the body of the sample.

In the superconducting state, the London superfluid current and the Hall current of the condensate form the sum

$$J_i = -\frac{\delta L}{\delta A_i} = \frac{k}{4\pi^2\varepsilon}(\partial_i\alpha + kA_i) - \frac{k}{4\pi}\epsilon_{ij}E_j, \tag{13.13}$$

whose terms depend on the sign of k. Because of the change in the sign of this coefficient between planes, it is exceedingly difficult to observe the Hall contribution to the total current (13.13). It is interesting that, because of its vacuum origin, the Hall effect occurs under the conditions prevailing in the Meissner effect. We draw attention to the fact that the latter is proportional to k^2.

If we use the charge density

$$J_0 = \frac{\delta L}{\delta A_0} = \frac{\vartheta}{2\pi\chi}(\partial_0\alpha + kA_0) - \vartheta B, \quad (13.14)$$

in the static case with $A_0 = 0$ to calculate the total charge

$$Q = \int d^2x J_0 = -\vartheta\int d^2xB = -\vartheta\oint \mathbf{A}\,d\mathbf{l}, \quad (13.15)$$

we again see[149,236] that the vortices in (13.11) are charged. Actually, the vortex configurations with winding number s of the "order parameter" phase ($\Delta\alpha = 2\pi s$) in the absence of both external field E_i and current ($\partial_i\alpha + kA_i = 0$) have the integral total charge equal to s units of vorticity:

$$Q = \vartheta\Delta\alpha/k = s. \quad (13.16)$$

Naturally, effects associated with the rotation of the plane of polarization of light[121,162,169] on reflection by the superconductor occur in the system in which a state with P- and T-parity violation has developed. This effect does not vanish even in the nonsuperconducting state because the Lagrangian describing the properties of the system for both $T<T_c$ and $T>T_c$ contains the Chern–Simons terms that depend only on the external electromagnetic potentials for $T<T_c$. The dependence of the angle of rotation φ on the frequency ω is[21]

$$\varphi(\omega) \sim \omega_p\omega^2/(\omega_p^2 - \omega^2)^{3/2}, \quad (13.17)$$

where ω_p is the plasma frequency. As temperature increases, the thermally excited states with high k are found to reduce the angle of rotation to zero.

To examine the electrodynamic properties of the anyon system at finite temperatures,[237] we turn to the Lagrangian (4.23), retaining only the variables that describe the dynamics of the charge, and including in our discussion the external electromagnetic field. The magnetic properties of the system were investigated in Refs. 104 and 237 by including the following terms in the Lagrangian:

$$L = -\frac{1}{4}F_{\mu\nu}^2 - \frac{e^2\bar{k}}{2\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + en_eA_0 + i\psi^+D_0\psi$$
$$-\frac{1}{2m}|D_k\psi|^2 + \frac{e}{2m}B\psi^+\sigma_3\psi, \quad (13.18)$$

where $D_0 = \partial_0 + ie(A_0 + a_0)$, $D_k = \partial_k + ie(A_k + a_k)$, $B = \partial_1A_2 - \partial_2A_1$, $b = \partial_1a_2 - \partial_2a_1$ and n_e is the neutralizing background density. In these expressions, ψ is the fermion field (vacuum angle $\vartheta = \pi[1-(2\bar{k})^{-1}]$) and the statistical potential a_μ is normalized so that it has the same coupling constant, i.e., the charge e, as the potential A_μ. When the factor in front of the Chern–Simons term is normalized to 1/2, the coupling constant between the field a_μ and the fermions is $(\pi/|k|)^{1/2}$, and the limit $\bar{k}\to\infty$ corresponds to the absence of the Chern–Simons interaction.

The equations of motion for the gauge fields take the form

$$\partial_\nu F^{\nu\lambda} = \langle J^\lambda\rangle - en_e\delta^{\lambda 0},$$
$$-(e^2\bar{k}/2\pi)\varepsilon^{\lambda\nu\rho}f_{\nu\rho} = \langle j^\lambda\rangle, \quad (13.19)$$

where

$$j^k = -\frac{ie}{2m}(\psi^+D_k\psi - (D_k\psi^+)\psi) = J^k - \frac{e}{2m}\varepsilon^{kj}\partial_j(\psi^+\sigma_3\psi), \quad (13.20)$$

and the field ψ satisfies the equation

$$i\partial_0\psi = \left[-\frac{1}{2m}(D_k^2 + eB\sigma_3) + e(a_0 + A_0)\right]\psi. \quad (13.21)$$

The mean current $\langle j^\mu\rangle$ in (13.19) is understood to be evaluated over the ground state and the ensemble.[237] For example,

$$\langle j^0(\mathbf{x})\rangle = \mathrm{Tr}(j_0(\mathbf{x})\exp\{\beta F_e[A,a] - H_e[A,a]\}) = \frac{\delta F_e[A,a]}{\delta a_0}, \quad (13.22)$$

$$j_0(\mathbf{x}) = \frac{\delta H_e[A,a]}{\delta a_0}, \quad (13.23)$$

where F_e is the free energy in $\exp(-\beta F_e) = \mathrm{Tr}\exp(-\beta h_e)$ and the electron part of the Hamiltonian is given by

$$H_e[A,a] = \int d^3x\left\{\frac{1}{2m}|[\partial_k - ie(A_k + a_k)]\psi|^2\right.$$
$$\left.-\frac{e}{2m}B\psi^+\sigma_3\psi + e(a_0 + A_0)\psi^+\psi\right\}. \quad (13.24)$$

The main results of this approach[237] may be summarized as follows. When $T=0$, there is a critical field $H_c' = en_e/2m$ of the order of 10 G above which there is partial external-field penetration and the field is uniformly distributed in space. As the temperature increases, $H_c'(T)$ decreases and becomes infinitesimal at a temperature T_c'. This temperature is of the order 100 K according to the estimate[237]

$$\frac{2\pi^2n_e}{\bar{k}^3e^2T_c'}\exp\left(-\frac{\pi n_e}{2\pi\bar{k}T_c'}\right) \sim 1, \quad (13.25)$$

which follows from the temperature dependence of the London penetration depth[237]

$$\lambda^{-2}(T) = \lambda^{-2}(0)\left[1 + \frac{2\pi^2n_e}{\bar{k}^3e^2T}\exp\left(-\frac{\pi n_e}{2m\bar{k}T}\right)\right]. \quad (13.26)$$

When $T<T_c'$, the magnetic field near the boundary falls from $B_{\rm ext}$ to $B_{\rm in} = B_{\rm ext} - H_c'$ for $B_{\rm ext} > H_c'$ in accordance with the exponential expression $\exp(-x/\lambda(T))$. For $T<70$ K, the length $\lambda(T)$ is almost temperature-independent and rapidly tends to zero for $T=100$ K without exhibiting the BCS singularity $(T_c - T)^{-1/2}$.

In practice, the measured effective depth is determined by the magnetic field gradient

$$\lambda_{\rm eff}(T,d)^{-1} = -\frac{1}{d}\int_0^d dx\frac{1}{B}\frac{dB}{dx} = \frac{1}{d}\ln\frac{B(0)}{B(d)} \quad (13.27)$$

and indicates the rate at which the magnetic field changes within a distance d inside the superconductor. The effective length $\lambda_{\rm eff}$ is identical in the Ginzburg–Landau theory with $\lambda(T)$, but the two are now different and are given by the following formulas[237] for $T\sim T_c'$ and $T=0$:

$$\lambda_{\rm eff}^{-1}(T,d) = d^{-1}\ln\frac{B_{\rm ext}}{B_{\rm ext} + M(B_{\rm in})[1-\exp(-d/\lambda(0))]}, \quad (13.28)$$

$$\lambda_{\rm eff}^{-1}(0,d) = \begin{cases}\lambda^{-1}(0), & B_{\rm ext} < H_c',\\ d^{-1}\ln\dfrac{B_{\rm ext}}{B_{\rm ext} - H_c'[1-\exp(-d/\lambda(0))]}, & B_{\rm ext} > H_c'.\end{cases} \quad (13.29)$$

We also draw attention to the fact that the free energy and the magnetization are functions of the magnetic induction for different temperatures, and that magnetization is a function of temperature for different values of the external field.[237] These results were obtained in the self-consistent field approximation without taking vortex excitations into account. Some suggestions relating to vortices and their contribution to the observed characteristics are presented in Ref. 33.

The question now is: how does the anyon system react to the application of variable electromagnetic fields? The total Hamiltonian of the system[101]

$$H = H_{\rm holon} + H_{\rm spinon}, \tag{13.30}$$

$$H_{\rm holon} = \sum_{a=1,2} \left[\frac{1}{2m_{\rm h}} \varphi_a^+ (i\nabla - \mathbf{a} - \mathbf{A})^2 \varphi_a + \varphi_a^+ (a_0 + A_0 - \mu_{\rm h}) \varphi_a \right], \tag{13.31}$$

$$H_{\rm spinon} = \sum_{\sigma,a=1,2} \left[\frac{1}{2m_s} \chi_{\sigma a}^+ (i\nabla + a)^2 \chi_{\sigma a} + \chi_{\sigma a}^+ (a_0 - \mu_s) \chi_{\sigma a} \right] \tag{13.32}$$

contains both the holon and spinon parts. When the wave vector $\mathbf{k}$ and the frequency ω are small, the fluctuations of the gauge field a_μ around its mean value $\bar{a}$, which is a solution of (5.1), are small and may be discussed in terms of the Gaussian approximation.

After integration with respect to φ and χ, the effective action

$$S_{\rm eff} = (a + A)_\mu \pi_{\rm h}^{\mu\nu} (a + A)_\nu + a_\mu \pi_{\rm s}^{\mu\nu} a_\nu \tag{13.33}$$

is expressed in terms of polarization operators that are equivalent to the current correlation functions for two-dimensional particles in the strong magnetic field $\bar{b}$ in the ground state of the system. Integrating over the fluctuations in a_μ as in Ref. 20, we obtain[101] the electromagnetic response

$$K_{\mu\nu} = -\langle j_\mu, j_\nu \rangle = (\pi_{\rm s}^{-1} + \pi_{\rm h}^{-1})^{-1}, \tag{13.34}$$

where $\pi_{\rm s,h}^{-1}$ are operators that are the reciprocals of the transverse operators $\pi_{\rm s,h}$. The latter can be written in the form

$$A_\mu \pi_{\mu\nu} A_\nu = -\varepsilon_\| E_\|^2 - \varepsilon_\perp E_\perp^2 + \chi B^2 + i\sigma_{xy} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{13.35}$$

where $\mathbf{E} = \omega\mathbf{A} - \mathbf{k}A_0$, $E_\|$ and $E_\perp$ are the longitudinal and transverse components of the electric field and $\mathbf{B} = [\mathbf{kA}]$ is the magnetic field. The permittivity $\varepsilon(\mathbf{k},\omega)$, the magnetic susceptibility $\chi(\mathbf{k},\omega)$, and the Hall conductivity $\sigma_{xy}(\mathbf{k},\omega)$ of the spinons and holons are analytic functions of $\mathbf{k}$ since the single-particle excitations have a gap ω_c and display the following properties.[101] In the special case where $\omega = 0$, we have $\sigma_{\rm h}(\mathbf{k},0) = \sigma_{\rm s}(\mathbf{k},0)$. When $\mathbf{k} = 0$, we have $\omega_c^2 \varepsilon_\| \varepsilon_\perp = 4\sigma_{xy}^2(0,\omega)$. The components of the electromagnetic response $K_{\mu\nu}$ for $\mathbf{k} \neq 0$ and $\omega \neq 0$ are

$$K_{00} = -\langle \rho(\mathbf{k},\omega) \rho(-\mathbf{k},-\omega) \rangle = d^{-1} \mathbf{k}^2 (\varepsilon_{\rm h}^\| d_{\rm s} + \varepsilon_{\rm s}^\| d_{\rm h}), \tag{13.36}$$

$$K_\perp = -\langle \mathbf{j}(\mathbf{k},\omega) j(-\mathbf{k},-\omega) \rangle = d^{-1} (\Delta_{\rm h} d_{\rm s} + \Delta_{\rm s} d_{\rm h}), \tag{13.37}$$

$$K = \frac{\langle \rho, [\mathbf{kj}] \rangle}{ik^2} = \frac{\langle [\mathbf{jj}] \rangle}{i\omega} = d^{-1} (\sigma_{\rm h} d_{\rm s} + \sigma_{\rm s} d_{\rm h}), \tag{13.38}$$

where

$$\Delta_{\rm h,s} = -\varepsilon_{\rm h,s}^\perp \omega^2 + \chi_{\rm h,s} k^2, \tag{13.39}$$

$$d_{\rm h,s} = -\varepsilon_{\rm h,s}^\| \Delta_{\rm h,s} - \sigma_{\rm h,s}^2, \tag{13.40}$$

$$d = -(\varepsilon_{\rm h}^\| + \varepsilon_{\rm s}^\|)[-(\varepsilon_{\rm h}^\perp + \varepsilon_{\rm s}^\perp)\omega^2 + (\chi_{\rm h} + \chi_{\rm s})k^2] - (\sigma_{\rm h} + \sigma_{\rm s})^2. \tag{13.41}$$

When $\omega = k = 0$, we have $\sigma_{\rm h} = -\sigma_{\rm s} = \sigma_{xy}(0) = \vartheta/2\pi = 2\bar{k}$. It follows from these expressions that

$$K_{00}(\mathbf{k},\omega) = \frac{1}{4\pi\lambda^2} \frac{k^2}{\omega^2 - v_0^2 k^2} \tag{13.42}$$

which we already know,

$$K_\perp(0) = 1/4\pi^2\lambda^2, \tag{13.43}$$

i.e., the Meissner effect, and

$$K(\mathbf{k},\omega) \sim \frac{\omega^2 k^2}{\omega^2 - v_0^2 k^2}, \tag{13.44}$$

$K(\mathbf{k},0) = K(0,\omega) = 0$, i.e., we have the restoration of broken parity in the long-wavelength and static limits. This is a general feature of the anyon system, i.e., the breaking of discrete symmetries occurs only for the shorter time scales for which $K \to \vartheta$ (Ref. 101).

14. EXPERIMENTAL CONSEQUENCES OF THE CHIRAL STATE

One of the characteristic properties of two-dimensional systems with developed quantum mechanical fluctuations is the rigidity of the coherent states of these systems. The long-wavelength softening of the collective mode during a transition to the superconducting state has little effect in this respect. As before, the main events develop over a scale of the order of a few lattice constants. Whereas at low temperatures, and in the absence of the external electromagnetic field, symmetry breaking under P- and T-inversions is compensated for these scales by hopping, no such compensation occurs for $T > T_c$ for which the static gauge field has a finite correlation scale. Moreover, the weakening of symmetry breaking under P and T-inversions occurs because of the antiferromagnetic distribution of the sign of the factor k of the Chern–Simons term over the basal planes.

It is therefore exceedingly difficult to find experimental support for the anyon picture. The μ^+SR experiments[238] suggest that local deviations from the mean magnetic field are small. In this situation, the formulas containing the square of the factor k in front of the Chern-Simons term would appear to have a better chance of success from the point of view of an experimental verification. An example is provided by (5.19) which was confirmed experimentally in Ref. 239. Another suggestion,[149] which is also quadratic in k and therefore refers to phenomena within the body of the sample, relates to the difference between the plasma frequencies of longitudinal and transverse oscillations [see (13.12)]. However, this has not as yet attracted the attention of experimentalists.

The chiralilty of a superconducting state in an external electromagnetic field[240] [see (13.11)] is the reason for the predicted rotation by an angle φ (13.17) of the plane of polarization of electromagnetic waves on reflection by the surface of the sample.[21] This effect does not accompany the propagation of waves in thin films because of the alternation of the sign of k (Ref. 162). These predictions were confirmed by early experiments,[24] but a definitive experimental answer has not been forthcoming (cf. Ref. 242).

There are several published programs of experimental investigation of the consequences of anyon quantum dynamics.[32,169] Here we merely wish to draw attention to some points that are common to these proposals.

Consider the reaction of the system in the so-called "normal" region to an external agency characterized by frequency and wave vector **k**. At finite temperatures, transitions within the magnetic sidebands under the influence of the external field will be accompanied by transitions between subbands in which the energy spectrum is distinguished by the sign in the "relativistic" dispersion relation such as (6.21). The conservation of energy in such processes takes the form $\varepsilon_i + \omega = \varepsilon_f$ where $\varepsilon_i = -\varepsilon$, $\varepsilon_f = +\varepsilon$, i.e., $\varepsilon = \omega/2$, and also $n(-\varepsilon) = 1 - n(\varepsilon)$ for the distribution function $n(\varepsilon)$. Absorption is proportional to the difference between the distribution functions $n(\varepsilon_f) - n(\varepsilon_i) = 2n(\varepsilon) - 1 = -\tanh(\varepsilon/2T) = -\tanh(\omega/4T)$. In these expressions, we have used the homogeneous limit $k = 0$ and the law of conservation of energy. Moreover, because of the inseparability of spin and charge degrees of freedom in bound states in anyon multiplets [see (4.19)] that exist for $t > T_c$ in a wide range of wave vectors, the contributions of spin and charge excitations to the imaginary part of the charge and spin susceptibility, which is proportional to the difference $n(\varepsilon_f) - n(\varepsilon_i)$, must have the same form: $\operatorname{Im} P_{c,s}(\mathbf{k},\omega) \sim N(0)\tanh(\omega/4T)$, where $n(0)$ is the density of single-particle states. This elucidates the suggestion made in Ref. 243 that $\operatorname{Im} P_{c,s} \sim -\tanh(\omega/4T)$, which is the basis for the existence of the Fermi liquid with the spectral weight of the single-particle Green's function that vanishes at $\varepsilon = 0$. This type of medium has many unusual properties. They manifest themselves, for example, in the linear dependence of resistance on temperature, the voltage dependence of the channel conductivity of the SIN contact, discovered experimentally in Ref. 224, and in transport phenomena.[245] We note that the current-voltage characteristics of SIN contacts are asymmetric under the replacement $V \to -V$, which may be an indication of a departure from the symmetry of the ground state under P- and T-inversions. This may be regarded as a partial realization of the general asymmetry of transport coefficients[32] due to P- and T-parity violation effects in the scattering amplitude.[98] Detailed analysis of the current-voltage characteristics, which have a number of features at voltages corresponding to the 'normal' region, deserve special attention. They are usually ascribed to phonons. However, because of the rich structure of the energy spectrum in the Hofstadter problem, a detailed analysis would be particularly desirable. The point is that channeling, like photoemission, is one of the few ways of investigating the detailed structure of states because it relies on processes involving a change in the total charge of the system. A symmetry analysis of the states of different types of vortex and their contribution to the Josephson current is reported in Ref. 246 and is very significant from this point of view.

Real compounds contain a variety of structure imperfections that act as pinning centers and induce a variety of hysteresis phenomena and effects associated with flux creep. It should be clear from the last few Sections that the reason for the quantum memory of the system resides in this intense masking background. It is entirely due to the coherent motion of quasiparticles in the ground state, and is related to commensurability-incommensurability phenomena when the main lattice is covered with cells having nonzero flux of the statistical magnetic field, which leads to hierarchical structures and fractal dependence of total energy on magnetic field (see Fig. 7). The energy scale of these phenomena is of the order of 10^{-4}–10^{-3} eV.

Let us imagine that radiation or an external magnetic field has disturbed the system out of its equilibrium state with a certain filling factor $\nu_0 = \phi$ and total energy $E_{\nu_0}(\phi)$, to some other state with energy $E_\nu(\phi)$. Each of the final states is characterized by a radically altered function $E_\nu(\phi)$. The nonequilibrium macroscopic state that arises after thermalization is one of the hierarchical system of states distributed along the energy scale with local total-energy minima that are separated from the equilibrium ground state by a sequence of energy barriers. Downward relaxation along the energy scale by tunneling through these barriers can therefore appear as a long-term flux flow, and any new rise will depend on past history and will be of the thermal activation type. Long-term relaxation of the magnetic moment, and the closely allied properties of photoinduced phenomena, are effects that involve different sets of quantum numbers of quasiparticle metastable states. This applies equally to bound states of holons and spinons for $T > T_c$ and to nonequilibrium anyon-holon excitations for $T < T_c$.

The existence of a set of relaxation times associated with tunneling by nonequilibrium anyons through barriers and the P- and T-asymmetry of the scattering amplitude,[98] i.e., the 'two-level' character of the system,[247] may be the reason for the $1/f$ noise in the system of charged quantum vortices. The fact that for $T \ll T_c$ the external magnetic field visualizes the existing vortical elementary excitations suggests that the external field strength for which the vortex lattice is formed is exceedingly low.[248] It would be exceedingly important to verify the predicted electric charge of the vortices under the conditions prevailing in such experiments. This would not be difficult to do because the screening length is short, but is still outside the range of current experiments. Studies of vortex streets are also of interest in connection with the theoretically predicted[120] spatially one-dimensional clusterization.

The instrumental zero of resistance during the superconducting transition defines a temperature that is very close to the Kosterlitz–Thouless phase transition discovered in Refs. 249–252.[37)] The Kosterlitz–Nelson jump[254] is observed at the temperature of the Kosterlitz–Thouless transition.[253] It is found[255] that the theory[256] has a very wide range of applicability in layered superconducting compounds, whereas it is usually valid only near the critical point. All these results may be looked upon as experimental support for the Kosterlitz-Thouless mechanism of semifermion pairing with the formation of a molecular bosonic phase (cf. Sec. 11 and Ref. 218). From the standpoint of the lattice gauge theory of Sec. 11, for large values of the parameter N of the group Z_N, the "atomic" phase of dissociated Kosterlitz–Thouless semifermion pairs (Coulomb phase with long-range gauge interaction) approaches from above the critical point T_c. The isotropization ($Z_N \to U(1)$) will therefore extend the range of existence of the Coulomb phase (see Fig. 14) due to the superconducting condensate, which has indeed been observed in a magnetic field.[259]

To conclude this Section, we note that the interpretation of experimental results in terms of the anyon mechanism has the necessary degree of universality and completeness that is essential for the understanding of a wide range of phenomena in superconducting and "normal" phases from a unified point of view.

The extensive range of current experimental problems includes studies of the condensate of holon-spinon bound states against the background of oblique confinement, adjacent phase transitions, and the effect of the magnetic field upon them; a detailed analysis of the dependence of critical indices on the magnetic field for the Coulomb gas of Kosterlitz–Thouless vortices;[257] and also studies of the intriguing details of the dependence of resistance on temperature in an external magnetic field. It also includes the temperature dependence of the Hall voltage which changes sign near the temperature T_c' (Refs. 128, 258, 259) after which, for $T > T_c'$, the external magnetic field has no effect on the resistivity $\rho_{xx}(T, H)$ as a function of temperature. We also draw attention to the fact that, near the superconducting transition point $T < T_c'$, the Hall voltage and resistivity ρ_{xx} are described by the same[128] thermal activation law, i.e., the Arrhenius law[260] $\exp(-T_0/T)$ with $T_0 \approx \omega_c$.

15. CONCLUSION

The properties of the ground state and of low-lying excitations were described in this review by terminologies and technical devices that often were outwardly dissimilar. This led to repetition, which is probably unavoidable from the pedagogic as well as other points of view. In particular, Gauss' law, i.e., the cancellation condition given by (4.10), was frequently repeated [cf. (4.16), (5.1), (6.39), (6.41), (7.14), (8.4), (8.37), (8.41), (8.53), (8.57), (9.28), and (12.29)]. An aspect of this relation was discussed each time. On the whole, and apart from the multiple-connectedness of the two-dimensional system, we tried to place the main emphasis on the strongest quantum fluctuations for which the uncertainty in the phase is of the order of the uncertainty in its canonical conjugate, i.e., the paticle number. We now turn to some general aspects of the problem, which we shall tackle from the standpoint of hidden internal symmetries.

One of the results of the mean-field theory[261] was the discovery of local spin SU(2) symmetry.[79–81] Doping reduces the local gauge symmetry down to U(1). It is interesting that the continuing breaking of gauge invariance occurs in almost the same form[262] as in BCS theory.[38] Although it would appear that the main role of the U(1) group should be to assist the fermion in turning itself into a boson and vice-versa, the condition $\vartheta = \pi$ does not produce this transformation.

Indeed, let us consider the dynamics of liberating the square-lattice vacancies from spin. They are surrounded by contours that are oriented because of the particular dimensionality of the space, and the coherent transport of spin degrees of freedom takes place over these contours. Splicing, i.e., the identification of links and diagonals of these contours, results in topological spaces of which the simplest is the torus. Global gauge transformations of gauge fields that parametrize the phases of the hopping amplitudes must under these conditions be assigned to different topological manifolds. The Chern–Simons theory on the torus[263,264] gives a multicomponent wave function[183] and is determined by non-Abelian representations of the braid group.[265] In the general case, the topology of the lens space $L_{k,p}$ that determines the statistics is the fundamental group $\pi_1(L_{k,p}) = Z_k$ (Ref. 266), which was used in Sec. 11. The existence of the discrete group Z_k [see (12.33)] and, in the more general case, $Z_{k_1} \times \ldots \times Z_{k_l}$ for the symplectic modular group $Sp(2k, \mathbf{Z})$ (Ref. 184), is due to lenses that are stratification spaces for the monopole with topological charge k (Refs. 267 and 268).

Studies of discrete hidden symmetries[269–277] are important for many reasons, above all, the natural appearance of root lattices that lead to the generalization of hierarchical sequences in anyon systems.[273] Indeed, the extension of the action (12.18) by the root lattice ($k \to k_{ij}$), the replacement of the scalar field φ with the matrix variable, and the addition of the term $i\alpha R$ (Ref. 226), i.e., the Jacobian for the transformation from the geometric quantization variables to the conformal theory variables, which amounts to the most complete implementation of gauge and diffeomorphic invariance, leads to the action of the two-dimensional quantum gravity. It may be considered that the fractal geometry[274] of this theory is at the basis of the stochastic behavior of observables in high-T_c superconductivity. This proposition relies on the fact that the two theories have the same action. Indeed, the values of k are found to be important on the very first step toward (12.18), namely, in the reparametrization of the field $\Phi \to \varphi / k^{1/2}$. The series $k = 2k_1$, $k_1 = 1, 2, 3, \ldots$ and $k = 2k_1/k_2$ with $k_2 \neq 1$ belong to different compactification radii with different critical behavior,[219] e.g., of the Kosterlitz–Thouless or Ising type. We recall that the superconducting state is absent in the sequence with $k_2 \neq 1$ (see Sec. 8).

The collective motion induced by the spin deficit of mutually dual degrees of freedom over multiply-linked contours that are the generators of topological manifolds signifies using another language the covering of the main lattice by triangular cells containing the flux of the statistical gauge field. In contrast to classical and quantum mechanics, the states of this system are determined by the elements of a non-Abelian group and the observables, e.g., the values of the Wilson loop, are functions of the elements of the groups.[275] This situation is typical for quantum groups (Hopf algebras).[275,276] Its particular feature in our case is that the deformation parameter is a root of unity and a generator of the cyclic group Z_N that determines the statistics of 'flux' excitations. The nonsingular representations[277] of quantum groups for such values of the deformation parameter are identical with the representations of the Hecke algebra of the braid group and are specified by monodromies[217,228] of conformal blocks of the $(1+1)$D space of WZWN theory. Detailed examination of quantum groups and advances in noncommutative geometry[278] enable us to assign internal symmetries to their proper places and to identify those cases in which the correlator behaves *á la* Wilson and those in which the perimeter law applies and the Higgs condensate appears. The preliminary answer reached in Sec. 11 suggests that, for a given set of quantum numbers n, m, a change in the regime and a phase transition occur at the self-duality point $T_c^{(k)}$, and as $k \to \infty$ we have $T_c^{(k)} \to 0$.

The new situation that has arisen in condensed-state physics in connection with strong-coupling has resulted in a need not only for numerical experiments[154,279] with few-particle systems, but it has also stimulated the application of analytic current-algebra techniques from quantum field theory to the description[280,281] of edge states.[282,283] The point is that particular boundary conditions are needed in the theory of strong correlations during compactification into some kind of a topological manifold. Hence experimental studies of edge states in condensed-state physics at low Coulomb energies enable us to approach new phenomena in topological quantum field theory from a new point of view.

There are several published reviews that touch upon these topics.[284,285] They overlap to some extent some of the material presented here and were published as our review was completed. I hope that the greater thematic range of our review will be useful to both theorists and experimenters studying strongly correlated systems.

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[1)] We are neglecting the small departure of the spins from the basal plane with alternating direction of deflection as we pass from one plane to another.

[2)] This conclusion is valid when the boundary conditions are taken into account on the two-dimensional lattice.

[3)] The parameters t and U in (2.1) must be looked upon as the effective parameters related to the hopping repulsion amplitudes of the d- and p-states of copper and oxygen:[5] $t = t_{dp}^2/U_p$, $U = t_{dp}$ for $U_d \gg U_p$.

[4)] To avoid misunderstanding, we emphasize once again that the word "hole" signifies in this case the formation of a spin texture defect and does not have its more usual meaning associated with the excitation of the ground state of electrons in metals.

[5)] The class that we are considering is a set of mutually primitive simple numbers p and q, the ratio of which lies in the range $0 < p/q < 1$.

[6)] The term "anyon" was coined by Wilczek[25] as a derivative of the word *any*.

[7)] In this review, we confine our attention to basic statements, and refer the reader to the extensive literature now available (see, for example, Ref. 30).

[8)] The singularity of the gauge transformation is a consequence of the aperiodicity of the function f under rotation by 2π and signifies a transition on the next sheet of the Riemann surface to another component of the wave function.

[9)] The prefix "any" represents in this case the stage in our discussion when the parameter ϑ has not as yet been determined.

[10)] In terms of the boson creation operators, the hard-core condition[34] $(b_i^+)^2 = 0$ excludes the occupation of the state by two bosons.

[11)] The existence of singular points is related in the case of doping to toroidal compatification.

[12)] We are considering the possibility of adding further terms to the action. The necessity for such terms arises because the states of excitation in two-dimensional systems realize the representations of the braid group.

[13)] The method of compactification depends on the boundary conditions imposed on the field **n** or z.

[14)] In the presence of doping, the symmetry group SU(2) reduces to U(1).

[15)] The expression given by (4.14) may be looked upon as representing a fixed point (in the sense of the renormalization group) in the space of the $(2+1)$ D Lagrangians.

[16)] The Kähler manifold is defined as a complex differentiable manifold with a Hermitian metric and a potential. The explicit expression for it is given in Ref. 93 for the case of a system of CP^1 skyrmions.

[17)] For the concentration $n = 10^{14}\ \mathrm{cm}^{-2}$ and $\phi_0 = 10^{-7}\ \mathrm{G\,cm}^2$, the statistical magnetic field is $b = 10^7$ G.

[18)] Strictly speaking, the fields a_μ in (4.10) and (4.7) are mutually dual. The property of duality is discussed in detail in Secs. 8 and 11.

[19)] The use of the same notation for different quantities should not, we hope, lead to confusion, because the meaning of the quantities is different in the different contexts and is individually defined.

[20)] The representations of the magnetic translation groups are the same as the projective representations of the lattice translation group.

[21)] Anyons are regarded here as fermions in the 'external' statistical magnetic field.

[22)] The biggest gap occurs for $\phi/\nu = 1/2$ on the triangular lattice and for $\phi/\nu = 2$ on the hexagonal lattice. (Ref. 106).

[23)] We draw attention to the fact that the diagonal hopping amplitude on the square lattice is replaced by the hopping amplitude in the direction perpendicular to the layers.[116]

[24)] The ground-state function of a particle in a doubly-periodic lattice potential and in an external magnetic field was investigated in Ref. 91.

[25)] The electric charge Q in the commutator (9.8) does not cancel in the "state of fractional quantum Hall effect" for $T > T_c$.

[26)] In the case, we are dealing with the magnetic charge that arises as a consequence of the compactification of the lattice theory. It can be used to take into account the Berezinskiĭ–Kosterlitz–Thouless effect.

[27)] Accordingly, there is also an increase in the number of nontrivial loop cycles in the expression for the expectation value of the Wilson operator (10.3).

[28)] One of the arguments that closure along the z axis occurs not on this plane but on neighboring planes is as follows. We know that when the anisotropy of thin films is small and the films are thin in comparison with the thickness of the Bloch wall, domains do not form in the film. Precisely the same relationship between spatial scales is encountered in this situation (see, however, Ref. 32).

[29)] We are considering the simple case $l = 1$ of possible $Z_{N_1} \times \ldots \times Z_{N_l}$ symmetry on the lattice R_1^1 (Ref. 184).

[30)] The picture under consideration can be improved by renormalization-group analysis.[191]

[31)] The quantum number sets (n_4, m_4) and (n_3, m_3) correspond to the cycles S_1 and S^1 and two uncoupled clock models.

[32)] This is valid for the open manifold M_4. For a closed manifold and in the case of U(1)SO(3), the product kn is an integer that is a multiple of 4, where $n = 1/4$ is the topological charge.[199]

[33)] In particular, when the fractional quantum Hall effect is considered from the spatial three-dimensional point of view, the statistical Hall conductivity depends on the period of the reciprocal lattice along the z axis.[202]

[34)] The Hamiltonian vanishes in all topological theories that are independent of the metric. Hence the contribution to the partition function is determined by the dimensionality of the space of states.

[35)] The depenence of the flat-space metric in (12.18) and the departure from diffeomorphic invariance arose because of the imposition of the gauge $A_0 = 0$.

[36)] Polarization is meant to be a definite set of variables, one of which we have called "coordinate" and the other "momentum," the two being mutual canonical conjugates.

[37)] There are very considerable difficulties in experimental verification of the existence of states with broken symmetry under P- and T-inversion. For example, the arguments against the conclusion that the experiment[128] has confirmed the Kosterlitz–Thousless correlation mechanism for vortices are based on the experimental detection of the transition only in quite thin films. Hence it may be considered that it is the two-dimensional space of the film and not of an individual plane that gives the Kosterlitz–Thouless transition. This is a typical example and is a good illustration of the difficulties that arise in the interpretation of experimental results.

[38)] We note that the symmetry remaining in BCS theory is Z_2 (Ref. 262).

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