Evolution of the defect structure of a solid during plastic deformation

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A consistent picture of the evolution of the defect structure is presented for a solid undergoing plastic deformation, when defects are multiplied and arbitrary hierarchical structures are formed. The complete problem is reduced to a description of the kinetics of the formation of a new structural level of plastic deformation, a field-theoretic description of a single defect, and a statistical description of ensembles of defects. For the first time a theoretical scheme is constructed for the processes of developed plastic deformation in which a hierarchically connected multilevel structure of defects is formed. A solid with defects is treated as a highly nonequilibrium state of a crystal subjected to an intense external action.

INTRODUCTION

Plastic deformation for a long time was associated with the evolution of a variety of crystal structure defects, which, mutually interacting, and being subject to the action of external fields, take on the aspect of independent structural entities with their own individual properties (their geometric configurations, distribution of elastic fields, etc.).[1-3] Within this conceptual framework the plastic deformation of a sample was treated as the result of the ergodic behavior of a system of defects whose trajectories evolve in time over the entire phase space. On the other hand it has been proposed that there is no hierarchical chain in the behavior of the defects—the behavior is determined only by the strong fields and the action of the constant-temperature reservoir. In this formulation the dependence of the thermodynamic potential of the system on the configuration coordinates would have a regular distribution of minima, the lowest of which would correspond to the stable state and the others to metastable states. As a result, the evolution of a system of defects during plastic deformation was represented as a chain of Debye processes in which the system via thermal fluctuations overcomes the barrier between minima of the thermodynamic potential, with relaxation times given by the Arrhenius relation.

Such a picture is realized for small degrees of plastic deformation. For the opposite case, which is the one ordinarily obtained in practice, the defect density becomes so high that the defects begin to exhibit collective effects.[4-7] This means that in the ensembe of defects coherent connections are established of the type that produces phase transitions[8] and kinetic transitions.[9] While this latter case is characterized by a homogeneous distribution, the establishment of a coherent connection in an ensemble of defects belonging to a single structural level results in a self-localized formation, which assumes the role of the starting structural unit at a higher level.[10] For example, the clustering of vacancies can produce dislocation loops,[2,3] and dislocation pileups can produce misorientation boundaries and disclinations.[6,7] This difference between the collective behavior of defects and the ordinary picture of phase transformations is due to the high degree of nonequilibrium of the ensemble of defects which arises in the development of plastic deformation, so that the transformation is not thermostatic, but kinetic.[11] In particular, the self-localized nature of the products of this transformation (superdefects) is a consequence of the loss of ergodicity in the transformation,[12] and this loss in turn is related to the hierarchical chain. The manifestation of this chain is that the superdefect is formed as a result of the association of defects from the lower level. Since the minima on the graph of the thermodynamic potential as a function of the configuration coordinates are narrower than the minima corresponding to the superdefects, the establishment of the hierarchical chain necessarily implies a fractal structure in the distribution of the thermodynamic potential in configuration space.[13]

Examples of spin glasses,[14] martensitic transformation,[15] polytype structures,[16] and creep of crystals,[17] show that the fractal nature of a system radically alters its thermodynamic and kinetic properties. This is because configuration space is partitioned into a multitude of regions (valleys[18] or components[12]), each of which has its own statistical ensemble. As a result, averaging is done in two stages: first as an average over the pure ensemble of a given valley and then over the ensemble of valleys. The kinetic aspect of the behavior of such a system, which is due to the weak restoration of ergodicity, is determined by the slow joining of valleys into clusters of larger-scale components. This process can conveniently be represented as the motion over the nodes to the trunk of a hierarchic Cayley tree, where the nodes correspond to the valleys, and the magnitude of the plastic deformation ε corresponds to the time (Fig. 1b).

While the above representation of the configuration space of defects by an ultrametric topology is novel,[15-17] in its outward aspects the picture of hierarchical behavior of defects during plastic deformation has had widespread acceptance in recent times.[4-7,10,19] Considering the importance of this topic, let us set forth the main ideas. The overall picture of plastic deformation stems from the evolution of defects that belong to different structural levels. Unlike amorphous systems, the number of such levels is rather small: in terms of the characteristic length scale l_i one generally distinguishes the microscopic level ($a \ll l_1 \ll d$, where a is the interatomic spacing and d is the size of a uniformly oriented system such as a cell or fragment), the mesoscopic level ($d \ll l_2 \ll D$, where D is the size of a weakly oriented region, such as a grain), and the macroscopic level ($D \ll l_3 \leqslant L$, where L is the characteristic size of the sample).[1)] In each of these levels the plastic deformation takes place by the uni-

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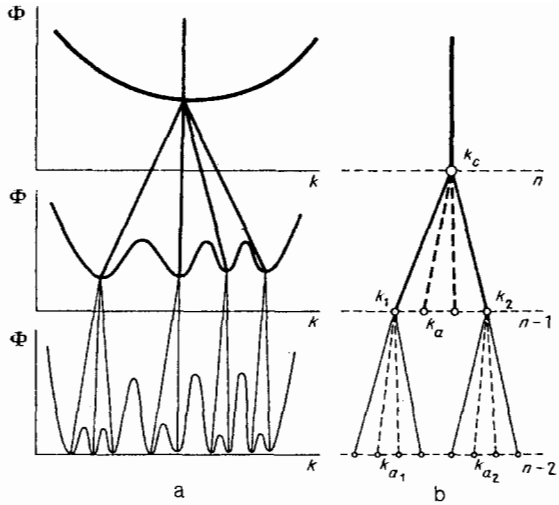

FIG. 1. a) Shape of the potential relief of structural units at various structural levels. b) Corresponding hierarchical tree.

form flow of defects, point defects, dislocations, disclinations, planar defects, and so forth. The microscopic level corresponds to a uniform distribution of point defects, dislocations, disclinations; the mesoscopic level to cells and fragments, and, finally, the macroscopic level corresponds to nonuniformly oriented grains, textured components, etc.[7,19] As the degree of plastic deformation increases each subsequent structural level is "engendered in the womb of the preceding,"[7] when, because of the increase in the uniform density of defects the resources for its evolutionary development are exhausted. Thus, according to Refs. 7 and 19, the first cell boundaries are nucleated when the dislocation density reaches a critical value, partial disclinations (boundaries of fragments) are formed by a decrease in the dimensions of the weakly disordered cells to the limiting dimension ~ 0.2 μm; knife-edge boundaries terminating in disclinations appear only with the formation of a well-developed fragmented structure with large misorientations.

This is because the increase induced by plastic deformation in the characteristic size scale above the critical dimension l_i results in an instability in the uniform distribution of defects at distances $x > l_i$ and self-localized formation of carriers of the plastic deformation at the $(i+1)$th structural level. For example, for $l(\varepsilon) > l_1$ the plastic instability results in the formation of dislocations, disclinations, and their complexes; for $l(\varepsilon) > l_2$ bands of strong shear-rotations are formed, and for $l(\varepsilon) > l_3$, macroscopic rotational shear fields are formed.[7] A very important circumstance is that the switching-on of each successive structural level during plastic deformation does not have an evolutionary character, since it is caused by the spontaneous appearance of new hydrodynamic (translational and rotational) modes at the instant that $l(\varepsilon) = l_i$. Of course, after the structure corresponding to the $(i+1)$th level is nucleated, the volume that it occupies will smoothly increase at the expense of the volumes of the structures corresponding to the levels 1, 2,..., i (here the situation is analogous to phase equilibrium in thermodynamics[8]). Moreover, the function ω_i of the distribution over the levels can change with increasing ε in such a way that a narrow or a wide group of levels are important (for example, in brittle fracture only the levels $i = 1$ and 2 are important[7]).

Despite the progress in understanding the experimental situation,[4-7,10,19] there still is no complete picture of plastic deformation that takes into account all the structural levels. In our view, the construction of such a view has been impeded mainly by the insufficient perception of the hierarchical nature of the defect structure. If it is assumed that to each structure level there corresponds a horizontal level of the Cayley tree or a level of resolution of the minima of the thermodynamic potential in the distribution in configuration space (Fig. 1), then the microscopic description of plastic deformation processes on all axes of the values of ε is given by the usual version of fractal kinetics of hierarchical structures.[14-17] There is, however, an important difference between them. The approach of Refs. 14–17, based on the use of a continuum ultrametric space, assumes that the number of levels (horizontal lines) on the Cayley tree is infinite and the nature of the connection between them does not play any specific role (only the structure of the tree is important). In the present case the number of structural levels, corresponding to dislocations, disclinations, grains, their conglomerates, ..., the sample, is clearly small, and the connection between them is of fundamental importance.

As is shown by an analysis of the experimental data of Refs. 6, 7, 15 and 19, a complete description of developed plastic deformation of a crystal on the basis of the concept of a hierarchically linked ensemble of defects requires the solution of the following problems: 1) the development of a kinetic picture of the nucleation of a new structure level; 2) the description of the mechanism that unites defects at the lower levels into a single superdefect at the upper level; 3) a self-consistent description of elastic fields and crystal geometry that has a well-developed ensemble of defects belonging to a given structural level; 4) a description of a developed structural level that has a large density of defects that belong to it (the measure of non-ergodicity, stability of superdefects against decay into the initial defects, memory effects, etc.). In keeping with the discussion of this section we have prepared an outline of the investigation we are presenting. In each of the cases considered we shall present a brief description of the theoretical framework in which a more complete description is obtained. Then we shall make a comparison between the model and the experimental results, including an analysis of particular cases. We have tried to be complete in presenting the theoretical ideas; this, however, entails inevitable cursoriness in the presentation (especially in the introductory sections of this review).

1. KINETICS OF THE FORMATION OF A STRUCTURAL LEVEL

An adequate mathematical description of a deformed crystal having a defect structure is given by a basis manifold M characterized by internal coordinates x_{i0}; $i = 1, 2, 3$ (Ref. 20). Putting off until Section 3 the geometrical aspects associated with M, we here consider a discrete manifold M, each point of which, $x_{i0}^{(a)}$, $a = 1, 2, ..., N$ is a structural unit at the initial level (atom, dislocation, disclination, etc). A qualitative change in the behavior of an ensemble that leads to the nucleation of a new structural level is reflected in the simplest way by a two-level model.[9,10] In this model M is divided into a direct sum $M_0 \oplus M_1$ of submanifolds corresponding to

unexcited and excited structural units. The difference between them (see Section 2) is that in the submanifold M_0 the coordinate dependence $U(\mathbf{x}_0)$ of the energy of the elementary excitations of the structural type is unique, whereas in M_1 there is an entire ensemble $\{U(\mathbf{x}_0)\}$. To study the kinetics of the excitation of a homogeneous system it is sufficient to specify the fraction $n = \lim_{N\to\infty}(N_1/N)$ of structural units that have gone over from M_0 to M_1. Since under conditions of plastic deformation the degree of excitation n of a system is determined by the hydrostatic component p of the stress tensor $\hat{\sigma}$, the excitation parameter that specifies the degree of qualitative rearrangement of the system can conveniently be taken as the pressure p, which, when it increases to $\sim(10^{-3}\text{–}10^{-1})\mu_0$, where μ_0 is the shear modulus, gives an excitation $n \sim 1$.

Before setting up the theoretical framework, we should mention that plastic deformation can occur both under conditions of active loading, where a constant rate of deformation $\dot{\varepsilon}_{\text{ext}}$ is maintained, and under conditions of creep, where the external load $\hat{\sigma}_{\text{ext}}$ is constant.[1] In the former case the order parameter, which gives the response of the system to the external action, is the amount of plastic deformation ε, and the susceptibility to this action is determined by the effective time $t_\eta = \partial\varepsilon/\partial\dot{\varepsilon}$. During creep the order parameter is the rate of plastic deformation $\dot{\varepsilon}$, and the susceptibility reduces to the inverse of the viscosity. In the phenomenological approach presented below, we shall at first consider the regime of active loading. The transition to the conjugate regime of creep will be made with the standard Legendre transformation.[8]

While the closed description of thermostatic phase transformations is achieved by the separation of the collective mode, whose amplitude is the order parameter,[8] in the present case of a kinetic transformation the introduction of the single quantity $n(p)$ is clearly insufficient.[9] The physical formulation of the problem of plastic deformation itself tells us that the magnitude of the plastic deformation ε is to be taken as the order parameter, while for the conjugate field we should take the corresponding shear component τ of the stress tensor $\hat{\sigma}$. In the simplest case (see below) the rates of change of these quantities, $\dot{p}$, $\dot{\varepsilon}$, $\dot{\tau}$ are given by the Lorentz system[11]

$$\begin{aligned} \dot{p} &= [(p_0 - p)/t_p] - g_p\varepsilon\tau, \\ \dot{\varepsilon} &= -(\varepsilon/t_\varepsilon) + (\tau/\eta_0), \\ \dot{\tau} &= -(\tau/t_\tau) + g_\tau\varepsilon p; \end{aligned} \tag{1.1}$$

where the first terms on the right hand side of each equation represent the autonomous relaxation of these quantities to the steady-state values $\varepsilon = \tau = 0$ and $p = p_0(T,\hat{\sigma}_{\text{ext}})$, where T is the temperature, $\hat{\sigma}_{\text{ext}}$ is the external stress tensor, and $t_{p,\varepsilon,\tau}$ are the characteristic relaxation times of the corresponding quantities. The last terms in Eqs. (1.1) describe, respectively, the decrease in the fraction of excited structural units during plastic deformation in the field of the shear stresses, the flow of the medium under the action of the stresses, and the increase of these stresses as a result of the excitation of the structural units during plastic deformation ($g_{p,\tau}$ and η_0 are positive constants).

The linear equation in (1.1) is the well-known equation for the flow of a viscous-elastic medium, where $t_\varepsilon = \eta_0/\mu_0$, the quantity η_0 is the shear viscosity, and μ_0 is the shear modulus.[21] The minus in front of the energy of plastic deformation in the first equation of (1.1) reflects Le Chatelier's principle and the plus in the last equation means that there is a positive feedback leading to a qualitative reorganization of the system as a result of self-organization in the ensemble of structural units. The nonlinear terms $\varepsilon\tau$ and εp were chosen because they represent the shear and hydrostatic contributions to the energy of the plastic deformation.

If the conditions $t_\varepsilon \gg t_p$ and $t_\varepsilon \gg t_\tau$ are satisfied, we can in the adiabatic approximation set $\dot{p} = \dot{\tau} = 0$ in Eqs. (1.1). As a result, the kinetic description is a single regression equation in the order parameter $Ct_\varepsilon\dot{\varepsilon} = -\partial V/\partial\varepsilon$, where the synergetic potential $V(\varepsilon)$ is

$$\begin{aligned} V &= C\frac{\varepsilon^2}{2}\left\{1 - \frac{p_0}{p_c}\left(\frac{\varepsilon}{\varepsilon_g}\right)^{-2} \ln\left[1 + \left(\frac{\varepsilon}{\varepsilon_g}\right)^2\right]\right\}, \\ p_c &= \frac{\mu_0}{A_\tau}, \quad \varepsilon_g^{-2} = A_p A_\tau; \end{aligned} \tag{1.2}$$

and C is a constant whose meaning will become clear at the end of this section and in Section 4.4. In addition, $A_\sigma = t_\sigma g_\sigma$ and $\sigma = p, \tau$. The components of the stress tensor are given by the equalities

$$p = p_0[1 + (\varepsilon^2/\varepsilon_g^2)]^{-1}, \quad \tau = A_\tau p_0\varepsilon[1 + (\varepsilon^2/\varepsilon_g^2)]^{-1}. \tag{1.3}$$

It can be seen from this description that an increase in the spontaneous plastic deformation ε results in the relaxation of the hydrostatic component of the stress to below the effective value $p_0(T,\hat{\sigma}_{\text{ext}})$, and an increase in the shear component $\tau(\varepsilon)$ according to expression (1.3), which has the form of Hook's law when $\varepsilon \ll \varepsilon_g$ and results in saturation of the limiting value of the plastic deformation $\varepsilon_g \sim 1$. In this connection it might be said that the transition of the system to an excited state, determined by spontaneous plastic deformation $\varepsilon \neq 0$ leads transformation of the high pressure $p \sim (10^{-3}\text{–}10^{-1})\mu_0$ into the shear component τ of the internal stress field. It is easy to see that typical times involved are $A_\tau \sim A_p^{-1} \sim 10 - 10^3$ and $t_p \sim t_\tau \sim 10^{-12}$ s.

For small loads $\hat{\sigma}_{\text{ext}}$, where the effective pressure $p_0(T,\hat{\sigma}_{\text{ext}})$ is less than the critical pressure p_c, the fraction of excited structural units, $n_0(p_0)$, is so small that no coherent connection is established between them. Then expression (1.2) increases monotonically, and the system relaxes to the steady-state values $\varepsilon = 0$, $\tau = 0$, $p = p_0$, $n = n_0$. As the load $\hat{\sigma}_{\text{ext}}$ increases the condition $p_0 > p_c$ (and consequently $n_0 > n_c$) is satisfied at the initial level. Expression (1.2) then has a minimum at the point $\varepsilon_e = \varepsilon_g\sqrt{(p_0/p_c) - 1}$, corresponding to the steady-state value of the deformation at the new level. According to expression (1.2) the effective time t_η to form this level and the partial viscosity $\eta = \mu_0 t_\eta$ of this level, determined by this time; are

$$t_\eta = t_\varepsilon[(p_0/p_c) - 1]^{-1}, \quad \eta = \eta_0[(p_0/p_c) - 1]^{-1}, \tag{1.4}$$

where $\eta_0 = \mu_0 t_\varepsilon$, Since the viscosity is given by the relation $\eta^{-1} = \partial\dot{\varepsilon}/\partial\tau$, and the appearance of a new level contributes an amount $\dot{\varepsilon}_e = \varepsilon_e/t_\varepsilon$ to the rate of plastic deformation, the effective viscosity becomes $\eta_{\text{eff}}^{-1} = \eta_0^{-1} + \eta^{-1}$. Taking into

account (1.4), when $p_0 > p_c$ we obtain $\eta_{\rm eff} = (p_c/p_0)\eta_0 < \eta_0$—the appearance of a new level of plastic deformation reduces the effective viscosity of the medium.

As a result, the synergetic picture of the nucleation of a structural level is as follows. In the absence of a coherent connection of the structural units, which can be characterized by the value of the order parameter ε, we have the initial values of the plastic deformation, ε_0, and of the components of the elastic stresses, p_0 and τ_0. If, when the external conditions $\hat{\sigma}_{\rm ext}$ and T are varied, the effective pressure p_0 exceeds the critical value $p_c \sim (10^{-3} - 10^{-1})\mu_0$, then, as a result of the coherent connection of the structural units, a collective mode appears which supplies an additional amount

$$\varepsilon_e = \varepsilon_g((p_0/p_c) - 1)^{1/2} \tag{1.5}$$

to the initial value ε_0 of the plastic deformation. In addition, a collective component $\tau_e = A_\tau p_0 \varepsilon_e [1 + (\varepsilon_e^2/\varepsilon_g^2)]^{-1} = A_\tau p_c \varepsilon_e$ is added to the initial stress τ_0, while the pressure as well as the fraction of excited structural units, which is determined by the pressure, fall off to the critical values p_c and n_c. As the difference $p_0 - p_c > 0$ increases the values of ε_e, τ_ε, the plastic deformation, and the applied stresses increase, while the pressure p_e, the time of formation of the level, t_η, and the effective viscosity $\eta_{\rm eff}$ decrease. The kinetic transition to the steady state (1.5) results in the production of entropy, the maximum value of which is $C^{-1}|V(\varepsilon_e)|$.

The chosen system of kinetic equations (1.1) is preferable in the sense that the form of its nonrelaxation terms follows from a simple Hamiltonian of the Dicke type. The square of the order parameter is proportional to the number of coherently connected structural units, the pressure is proportional to the number of excited structural units, and the shear stress τ is determined by the excitation-induced polarization of the system of structural units as a result of the subdivision of the manifold M into the submanifolds $M_0 \oplus M_1$ (Ref. 11). The fact that the coupling constant is imaginary reflects the dissipative nature of the hydrodynamic model of plastic deformation (see Section 2).

Although the function we have obtained, Eq. (1.2), corresponds to second-order phase transitions, this scheme is readily generalized to first-order phase transitions. For this purpose it is sufficient to go over to a nonlinear viscous-elastic medium already in the initial state, and assume that the relaxation time t_ε depends on ε (Ref. 11). The synergetic potential $V(\varepsilon)$ as a function of the order parameter surmounts the barrier that separates the initial and the steady state; this is a characteristic of a first-order transition.[8] This situation is in accord with the experiment[7,19] described in the introduction: the formation of a new level on the background of the old level proceeds in a manner like the decomposition of a solid solution,[22] where the role of the average concentration is assumed by the average pressure $\bar{p}$ over the sample. In accordance with Refs. 7 and 19, the increase in the external load (or the multiple plastic deformation in the case of rolling) results in an increase in the average "concentration" $\bar{n} \propto \bar{p}$ of excited structural units, which in accordance with the "lever" rule $\gamma \equiv (\bar{p} - p_e)/(p_0 - p_e) = (\bar{p} - p_c)/(p_0 - p_c)$, causes an increase in the volume fraction γ of the "phase" that corresponds to the new level.

Under conditions of creep $\hat{\sigma}_{\rm ext} = {\rm const}$ it is necessary to go over to the conjugate synergetic potential $\tilde{V} = V + Ct_\varepsilon \dot{\varepsilon}\varepsilon$, which is a function of the rate of plastic deformation ε. Here the deformation itself $\varepsilon \equiv (Ct_\varepsilon)^{-1}\partial V/\partial\dot{\varepsilon}$ acts as a generalized force, which gives the nonequilibrium analog of the thermodynamic potential Φ according to the equation $-t_\varepsilon^{-1}\partial\Phi/\partial\dot{\varepsilon} = C\varepsilon$. Since the left-hand side of this expression represents the shearing stress τ caused by the plastic deformation, we have, using Eqs. (1.3) and (1.5), $C = \tau/\varepsilon = A_\tau p \geqslant \mu$. Thus, the energy density C determines the degree of excitation of the system above the modulus μ. The quantity C does not reduce to an effective pressure $p_0 = p_0(T, \hat{\sigma}_{\rm ext})$. Its physical meaning is established in Section 4.4.

To conclude this section, we present an example of an experimental situation which demonstrates explicitly the formation of a new structural level of deformation. Of course, the nucleation of any type of defect is an example of localization of such a level; however, its properties are most clearly perceived when this happens in a region of macroscopic dimensions. The clearest example of such a situation is the superplastic state, in which the sample under the action of a constant load is able to increase its length many times.[23] It is well known that this state cannot be established instantaneously over the entire volume of the sample, but occurs via the motion of the boundary between the elastic and plastic regions, which is known as the Chernov–Lüders band.[24]

The mechanism for the formation of the superplastic zone through the multiplication of defects has been studied in Refs. 25 and 26, which dealt with the behavior of high-strength alloys containing finely divided precipitates of a nonmetallic phase under the action of intense loads $\sim (10^{-2}–10^{-1})\mu_0$. It was found that the stopping of gliding dislocations near the precipitates results in an influx of vacancies towards these regions, and these vacancies allow the dislocations to climb by overcoming the potential barrier and by virtue of an increase in the dislocation density. The higher the vacancy concentration the greater is the increase in the dislocation density. In turn, the motion of the jogs in the dislocations results in an increase in the vacancy concentration, so that the evolution of the system assumes an autocatalytic type of multiplication. Experiments have shown[25] that this leads to extremely high concentrations of vacancies $\sim 10^{-4}$ and dislocation densities, $\sim 10^{12}$ to 10^{13} cm^{-2}. A coherent connection is thus set up between the defects, and one should speak not of the autonomous behavior of a defect in the field of its neighbors, but of the self-consistent behavior of the entire ensemble. The behavior is described by the system of equations (1.1), where the hydrostatic component p of the stress field determines the defect density, while the critical value of that component, p_c, corresponds to the establishment of a coherent connection between them. As a result, a new structural level of plastic deformation is formed, and is manifested in the experiment as a band of plastic flow localized in a ~ 1-μm region (not to be confused with kink bands, which have a thickness $\geqslant 10$ Å). An S-shaped (sigmoidal) section is thus obtained on the plot of $\dot{\varepsilon}(\hat{\sigma}_{\rm ext})$, the rate of plastic deformation as a function of the external load. It is easy to see that this situation is reminiscent of the formation of electric domains in semiconductors.[27]

2. SPATIAL STRUCTURE OF A NUCLEATING LEVEL

Let us now consider the spatial distribution of the fields that characterize the new level in the steady state. For this purpose it is not sufficient to use the integral characteristics—the fraction of excited structural units. It is necessary to introduce the force fields $U(\mathbf{x}_0)$, $\mathbf{U}(\mathbf{x}_0)$, $\hat{U}(\mathbf{x}_0)$, etc. conjugate to the material fields $\rho(\mathbf{x}_0)$, $\boldsymbol{\rho}(\mathbf{x}_0)$, $\hat{\rho}(\mathbf{x}_0)$, that characterize the distribution of the structural units in the initial level. These fields realize the bases of the irreducible representations of the initial group G_0 of the symmetry of the system. The distribution of the material fields specify the thermodynamic potential $\Phi\{\rho(\mathbf{x}_0),\boldsymbol{\rho}(\mathbf{x}_0),\hat{\rho}(\mathbf{x}_0),...\}$, whose variation gives us the force fields[11]

$$U(\mathbf{x}_0)=\frac{\delta\Phi}{\delta\rho(\mathbf{x}_0)},\quad \mathbf{U}(\mathbf{x}_0)=\frac{\delta\Phi}{\delta\boldsymbol{\rho}(\mathbf{x}_0)},\quad \hat{U}(\mathbf{x}_0)=\frac{\delta\Phi}{\delta\hat{\rho}(\mathbf{x}_0)},\ \ldots. \tag{2.1}$$

The first of these is determined analogously to the energy of the elementary excitations of a many-particle system[28] and represents the potential relief of the structural units in the initial level [for example, for dislocations the distribution of $U(\mathbf{x}_0)$ is the Peierls relief[3]].

A deviation of the system from equilibrium results in the excitation of the ensemble of structural units, which shows up in a "smearing" of the force fields. The probability of realizing specific distributions $U(\mathbf{x}_0)$, $\mathbf{U}(\mathbf{x}_0)$, $\hat{U}(\mathbf{x}_0)$,... is determined by the distribution

$$P\{U(\mathbf{x}_0),\mathbf{U}(\mathbf{x}_0),\hat{U}(\mathbf{x}_0),\ldots\}$$
$$\times\propto\exp[-C^{-1}V\{U(\mathbf{x}_0),\mathbf{U}(x_0),\hat{U}(\mathbf{x}_0),\ldots\}],$$

which is given by the corresponding functional of the synergetic potential and the white-noise correlator C of the temperature type.[9] The set of order parameters that characterize the new level are determined, as usual, by the long-range correlations of the deviations $\delta U(\mathbf{x}_0)=U(\mathbf{x}_0)-\langle U(\mathbf{x}_0)\rangle$, $\delta\mathbf{U}(\mathbf{x}_0)=\mathbf{U}(\mathbf{x}_0)-\langle\mathbf{U}(\mathbf{x}_0)\rangle$, $\delta\hat{U}(\mathbf{x}_0)=\hat{U}(\mathbf{x}_0)-\langle\hat{U}(\mathbf{x}_0)\rangle$,... about the corresponding average values

$$\lim_{|\mathbf{x}_0'-\mathbf{x}_0|\to\infty}\frac{\langle\delta U(\mathbf{x}_0)\delta U(\mathbf{x}_0')\rangle}{\langle|U(\mathbf{x}_0)|^2\rangle}=|\mathbf{E}(\mathbf{x}_0)|^2=\varepsilon^2(\mathbf{X}_0),\ \ldots, \tag{2.2}$$

where the average is taken over the distribution $P\{U(\mathbf{x}_0),...\}$. Here the change in the characteristic length scale at the new level is displayed explicitly: if the value of $U(\mathbf{x}_0)$ changes over distances $x_0\geqslant l_0$, then $\varepsilon(\mathbf{X}_0)\equiv|\mathbf{E}(\mathbf{x}_0)|$ at distances $x_0\geqslant l$, where the scales $l_0\ll l$ correspond to the initial and the new levels (see the Introduction). However, if it is kept in mind that there is a change not only in the modulus ε of the complex order parameter $\mathbf{E}=\varepsilon e^{i\varphi}$, but also in its phase φ, whose oscillations $\varphi(\mathbf{x})$ are observed at small distances $x_0\geqslant l_0$, then one must use the more exact function $\mathbf{E}(\mathbf{x}_0)=\varepsilon(\mathbf{X}_0)\exp[i\varphi(\mathbf{x}_0)]$. The transition from the initial scale l_0 to the new scale l implies spontaneous breaking of the conformal invariance and will be realized below within the framework of a gauge-field scheme. Here it should be noted that the definition (2.2) gives the simplest possible order parameters. The entire set of them $\mathbf{E}^{\langle n\rangle}$, $\mathbf{E}_a^{\langle n\rangle}$, $\mathbf{E}_{ab}^{\langle n\rangle}$, ..., where a, b, ..., are the "internal" polarization indices, corresponds to an expansion in terms of nonequivalent irreducible representations $n=1,2,\ldots$ of the direct product $G_0\otimes G_0$ of the initial group.[29]

Let us consider first the very simple case of a scalar parameter $\mathbf{E}=\varepsilon e^{i\varphi}$, which is characterized by the group $U(1)$. The standard analysis shows[11] that $U(1)$ invariance makes nonunique (for the new level!) the external coordinate $\mathbf{x}_0$. This nonuniqueness, which reflects the appearance of defects at the new structural level, is compensated by the extension of the derivative $\partial_0\equiv\partial/\partial\mathbf{x}_0\to\nabla_0\equiv\partial/\partial\mathbf{X}_0=\partial_0+\boldsymbol{\Gamma}_0$, where $\boldsymbol{\Gamma}_0$ is the potential of the corresponding gauge field;[30] from the geometrical point of view $\boldsymbol{\Gamma}_0$ is the connectivity in the stratification, which is a nontrivial generalization of the direct product of the manifold (base) M of initial structural units to the gauge group $U(1)$.[20,31] In the nucleation of the structural units of a new level it is convenient to use associative stratification, considering at each point $\mathbf{x}_0$ not the transformations of the gauge group, but the corresponding potential $\boldsymbol{\Gamma}_0$, whose value is associated with the displacement $\mathbf{u}_0=\mathbf{X}_0-\mathbf{x}_0$ by the equation $\boldsymbol{\Gamma}_0=-ig_e\mathbf{u}_0$, where g_e is the elastic charge.[11]

Going over to four-dimensional coordinates x_0^μ, X_0^μ, where $\mu=0,1,2,3$; $x_0^0=c_0t$, $X_0^0=C_0t$ (here c_0 and C_0 are the velocities of transverse sound in the initial and the new levels), we shall describe the structural unit of the new level by the distribution of the material field $\mathbf{E}(X_0^\mu)$. Let us consider first the simplest Ginzburg–Landau scheme, which is defined by the Lagrangian[11,32]

$$L_{\mathrm{m}}=\frac{\beta^2}{2}|\nabla_0^\mu\mathbf{E}|^2-V(\mathbf{E})-\nu_\mu(\nabla_0^\mu\varepsilon-w\varepsilon\partial_0^\mu\varphi); \tag{2.3}$$

where $V(\mathbf{E})$ is the synergetic potential defined by a function of the type (1.2), where the role of the order parameter ε is taken over by the complex field $\mathbf{E}(X_0^\mu)$, and β and w are positive constants. The spontaneous breaking of the conformal invariance of the action, corresponding to Eq. (2.3), is obtained by a variation with the Lagrange multiplier ν_μ and leads to the appearance of the scale l. Actually, since the external coordinate x_0^μ corresponds to a change in the phase $\varphi(x_0^\mu)$ at distances $x_0^\mu\geqslant l_0$ while the internal coordinate X_0^μ describes the function $\mathbf{E}(X_0^\mu)$ at distances $X_0^\mu\geqslant l$, we quickly obtain from the equation $\nabla_0^\mu\ln\varepsilon=w\partial_0^\mu\varphi$ the condition $l=l_0/w$, which fixes the scale l by the assignment of the parameter $w\ll1$ (l_0 is given *a priori*). Correspondingly the gauge is specified: $\Gamma_0^\mu=\partial_0^\mu(-\ln\varepsilon+w\varphi)$. Corresponding to this choice of strength of the gauge field there are nonzero values if the phase $\varphi(x_0^\mu)$ is a many-valued function. Then $\oint\Gamma_{0\mu}dx_0^\mu=w\oint d\varphi=2\pi wn$, $n=0,\ \pm1,\ \ldots$. This means that the structural units play the role of the elementary carriers of the gauge fields. Accordingly, the manifold M becomes multiply connected.

The equation for the material field, which follows from Eq. (2.3) has the form

$$\partial_0^\mu\partial_{0\mu}\mathbf{E}=\Gamma_0^\mu\Gamma_{0\mu}^*\mathbf{E}-\beta^{-2}\partial V/\partial\mathbf{E}^*, \tag{2.4}$$

where the gauge condition $\partial_0^\mu\Gamma_{0\mu}=0$ has been taken into account. The nature of the solution of this equation is determined by the value of the parameter[11]

$$\varkappa=\frac{b}{\beta^2g_e}=4\pi\frac{\eta_0b\varepsilon_e}{\rho_0c_0\beta},\quad b^2\equiv4\left.\frac{\partial V}{\partial\varepsilon^4}\right|_{\varepsilon=0}, \tag{2.5}$$

where $g_e = c_0\rho_0/4\pi\eta_0\beta\varepsilon_e$ is the charge of the elastic field, ρ_0 and η_0 are, respectively, the density of the medium and its shear viscosity in the initial level, and ε_e is the maximum steady-state value of the order parameter, expression (1.5). When the conditions are unfavorable for plastic deformation in the initial level, when the shear viscosity $\eta_0 \propto g_e^{-1}$ is so large that $\varkappa > 2^{-1/2}$, then the steady-state dependence has a soliton-like solution that is characterized by a minimum scale $l_m = \beta/b\varepsilon_e$. From this we obtain the maximum value $w_m = l_0\varepsilon_e b/\beta < 1$ for the gauge parameter $w = l_0/l$.

An analysis[11] shows that if a soliton corresponds to the localization of the shear component of the displacement field $\mathbf{u}_0(\mathbf{x}_0) \propto \mathbf{\Gamma}_0(\mathbf{x}_0)$, then its strength falls off as X_0^{-1}. It is obvious that this solution may be related to a dislocation. In the localization of the rotational component of the displacement, the skew-symmetric component of its gradient varies as $\ln X_0$, corresponding to a disclination. To avoid misunderstanding, we note that although we are considering the simplest group U(1), it is a gauge group and not the space-time affine group A(3,**R**), which is studied in Section 3 [its subgroup of translations T(3) corresponds to the ensemble of shear dislocations, the group SL(3,**R**) corresponds to rotational disclinations and dislocations, and the group of dilatations $\Delta(3)$ corresponds to point defects].[32,33] The gauge group U(1) of phase transitions with the order parameter $\mathbf{E}(x_0^\mu) = \varepsilon(X_0^\mu)\exp[i\varphi(x_0^\mu)]$ used here is related to a variation in the external coordinate x_0^μ in the new level. Since the corresponding distances $x_0 \sim l_0$ are much smaller than the scales l of the variation of the internal coordinate X_0^μ, the system is gauge invariant to an accuracy $\sim (l_0/l)^d \ll 1$ (where d is the dimension of the space of M). The introduction of the gauge field compensates for this inaccuracy.

The Ginzburg–Landau scheme does not take explicit account of the behavior of the ensemble of structural units (in Section 1 it was represented by the fraction $n(p)$ of excited structural units). We now lift this restriction, using a field-theoretic approach. Moreover, we keep in mind the more complicated case of the non-Abelian gauge group SU(2), which corresponds to a three-component order parameter $\mathbf{E}_a(x_0^\mu)$, where the index $a = 1, 2, 3$, enumerates the polarization branches of the displacement waves.[34] Here the material fields $\Psi_a(x_0^\mu)$ of the distribution of the structural units that realize the bases of the gauge group will be two-component (the components corresponds to the excited and unexcited states)—in keeping with the dimension of the generators $\hat{\tau}^\mu$ of the group SU(2), which are the Pauli matrices.[20] In this case the material component of the initial Lagrangian has the form[35]

$$L_m = i\beta\overline{\Psi}_a\hat{\tau}^\mu\nabla_{0\mu}^{ab}\Psi_b + \frac{\beta^2}{2}|\nabla_{0\mu}^{ab}\mathbf{E}_b|^2 - \omega j^\mu\mathbf{E}_a - \frac{b^2}{4}|\mathbf{E}_a|^4; \tag{2.6}$$

where β, ω, and b are constant parameters, $\nabla_{0\mu}^{ab} = \partial_{0\mu}\delta_{ab} + \varepsilon_{abc}\Gamma_{0\mu}^c$, $j^a = iC^a_{bc}\overline{\Psi}^b\Psi^c$ is the current of structural units, $C^a_{bc} = \varepsilon_{abc}$ are structural constants, which in the case of SU(2) reduce to the antisymmetric tensor ε_{abc}. As with all non-Abelian models, the Lagrangian (2.6) leads to asymptotic freedom on the one hand and to confinement on the other.[36] In this case this means that as the structural units approach each other the coherence in the distribution over the excited and unexcited states through the exchange of quanta of the gauge field is weakened and as the structural units separate the coherence increases without bound. This fact reflects the long-range order in the distribution of the structural units.

Variation of the action corresponding to Eq. (2.6) over the field $\Psi(x_0^\mu)$ results in an equation of the Weyl type[30]

$$(\beta\hat{\tau}^\mu\partial_{0\mu}\delta_{ab} - i\varepsilon_{abc}\hat{\tau}^\mu\Gamma_{0\mu}^c + \omega\varepsilon_{abc}\hat{I}\mathbf{E}^c)\Psi^b = 0, \tag{2.7}$$

where $\hat{I}$ is the 2×2 unit matrix. It is characteristic that the term that contains the order parameter—the plastic deformation field $\mathbf{E}^c(x_0^\mu)$—plays the role of the mass of the bare "fermion," the structural unit, which is distributed over the excited and unexcited states.[2)] Using Eq. (2.7) to eliminate the field $\Psi_a(x_0^\mu)$ from the Lagrangian (2.6) we bring it in the standard way[30] to the form (2.3), where the function $V(\mathbf{E}^a)$ is given by a Landau-type expansion with a minimum at the point $|\mathbf{E}_e^a| \equiv \varepsilon_e^a = i\omega|\Psi^a|$. This means that the dispersion law (the mass) ω of the bare "fermion" must be of an imaginary nature.[11] As a result of the exchange of Higgs bosons corresponding to the field of the plastic deformation between the fermions, which are the structural units, the gauge symmetry is spontaneously broken. The longitudinal component $\mathbf{E}^1$ of the plastic deformation takes on a fixed value $\mathbf{E}_e^1$, and the two transverse components transform into Goldstone bosons of restoration of SU(2) symmetry, that is they become the elastic components e^2 and e^3 of the strain field. Conversely, for the corresponding components $\Gamma_{0\mu}^2$, and $\Gamma_{0\mu}^3$ of the potential of the stress $\hat{\sigma}$, the dispersion law takes on a plastic character, while for the longitudinal component $\Gamma_{0\mu}^1$ it remains elastic.

Besides the solutions that belong to the Abelian gauge groups, the SU(2) symmetry leads to a new solution related not with the gauge field, but with the topology of the distribution of the plastic deformation field. It is a three-dimensional soliton like a magnetic monopole, where the plastic deformation is localized as a result of the interaction of the various polarizations of the elastic stress field. Since this field falls off as X_0^{-3} far from the soliton, we can conclude that this solution represents a point defect.[35]

Let us consider now the field aspect of the nucleating structural level, described by the Lagrangian

$$L_{0e} = -\frac{1}{g_e^2}\Sigma_{0\mu\nu}^a\Sigma_{0a}^{\mu\nu}; \tag{2.8}$$

where g_e is the elastic coupling constant and the three isonormal components $\Sigma_{0\mu\nu}^a$, $(a = 1,2,3)$ of the elastic stress tensor, which correspond to the different polarizations of the elastic waves, are given by the nonlinear relation[30]

$$\Sigma_{0\mu\nu}^a = \partial_{0\mu}\Gamma_{0\nu}^a - \partial_{0\nu}\Gamma_{0\mu}^a + C_{bc}^a\Gamma_{0\mu}^b\Gamma_{0\nu}^c. \tag{2.9}$$

The corresponding equation for the elastic field of a single structural unit has the form

$$\nabla_{0\mu}^{ab}\Sigma_{0a}^{\mu\nu} = J_{0b}^\nu, \quad J_{0\nu}^a \equiv \beta g_e^2 C_{bc}^a\mathbf{E}^b\nabla_{0\nu}^{bc}\mathbf{E}^c, \tag{2.10}$$

where the second expression determines the current of elementary excitations (the phonons), that create the elastic field. If only a single polarization is considered, then Eq. (2.10) is linear; in the general case a nonlinearity reflects an interaction among the different polarizations. The strength of the elastic field is given by the following quantity averaged over the polarizations[36]

$$\Sigma_{0\mu\nu} \equiv n_a \Sigma^a_{0\mu\nu} - C^a_{bc} n_a (\nabla_{0\mu} n^b)(\nabla_{0\nu} n^c)$$
$$= (\partial_{0\mu}\Gamma_{0\nu} - \partial_{0\nu}\Gamma_{0\mu}) - C^a_{bc} n_a (\partial_{0\mu} n^b)(\partial_{0\nu} n^c), \quad (2.11)$$

where $\Gamma_{0\mu} = n_a \Gamma^a_{0\mu}$ and $n_a = \mathbf{E}_a (\mathbf{E}^b \mathbf{E}_b)^{-1/2}$. It is the quantities (2.11) and not (2.9) that characterize the real elastic field: thus, for a point defect each polarization gives a contribution $\Sigma^a_{0ij} \simeq \varepsilon_{ijk} X^k_0 X^a_0 / X^4_0$, whereas in fact, the field $\Sigma_{0ij} \approx -\varepsilon_{ijk} X^k_0 / X^3_0$ is what is measured.[35]

The scheme described here gives a complete description of an arbitrary defect at the new structural level. The material field $\mathbf{E}$, being the solution of Eq. (2.4), describes the reorganization of the collective behavior of the structural units in the nucleus of the defect. Its characteristic coordinate dependence and the corresponding behavior of the potential relief are shown in Fig. 2. It can be seen that in a region bounded by the correlation length $\xi = \beta C^{-1/2}$, the potential relief becomes smeared out, resulting in a decrease in the shear viscosity η_{eff}. If the Einstein–Smoluchowski relation $D \propto \eta_{\text{eff}}^{-1}$ is taken into account, it is possible to understand the increase by several orders of magnitude in the diffusion coefficient of atoms along dislocation tubes.[2,3] Actually, it can be seen from Fig. 2b that the smearing of the relief that occurs at the nucleus of a dislocation reduces the effective height Q of the relief, and, consequently, increases the diffusion coefficient, $D \propto \exp(-Q/T)$.

In contrast to the distribution of the field of the plastic deformation, the elastic field $\Sigma_{0\mu\nu}(\mathbf{X}_0)$ disappears as a result of viscous flow at distances that are much greater than the correlation length ξ, and are characterized by the length scale $\lambda = g_e^{-1} \varepsilon_e^{-1} \propto \eta_0 \mu_0^{-1/2}$. The use of the field theoretic method allows us to reproduce the nature of the variation of the stress field in the elastic region $X_0 \ll \lambda$ for the principal types of defects. Outside of this region the plastic deformation results in complete smoothing out of the elastic stresses. Because of this circumstance it is possible to avoid the well-known difficulty in the theory of defects,[2,3] which show up, for example, in the calculation of the energy of a dislocation. The localization of the elastic field of a defect in a region of dimensions λ means that the cutoff parameter is to be taken as the dimension of this region. As a result, the standard calculations lead to the following values for the energy of the principal types of defects (in units of $\mu_0 l_m^3$, where $l_m = \beta / b\varepsilon_e$): $\Delta\Phi \sim \varkappa^{-1} \ln \varkappa$ (Ref. 36) for a point defect; $\Delta\Phi \sim \varkappa^{-2} \ln \varkappa$ for a dislocation;[11] $\Delta\Phi \sim \ln \varkappa$ for a disclination of unit length. Here the Ginzburg–Landau parameter (2.5) is related to the characteristic length scales ξ and λ of the variations of the material and elastic fields by the relation $\varkappa = \lambda / \xi$.

A characteristic feature of the field theoretic description of defects is that they are represented as self-localized entities belonging to the nucleating structural level. From the formal point of view the localization of a level is due to the large values of the parameter $\varkappa \sim \eta_0 \mu_0^{-1/2}$ in an elastic medium with a small degree of plasticity. It is noteworthy that the formation of linear defects (dislocations and disclinations) is due to the independent self-localization of each of the polarization branches $a = 1, 2, 3$. In contrast to this picture, the self-localization of the fields that correspond to point defects is due to the linking of the different polarization branches. In this sense Kröner's concept[37] of a point defect as three mutually perpendicular dislocation loops of infinitely small size is justified. In the framework of our approach a point defect is the model of a "Polyakov hedgehog." Here the self-localization is due not to the interaction of the various degrees of freedom, but to the self-action of the gauge field of the elastic stress, brought about by the non-Abelian nature of the SU(2) group. Thus, a point defect is a topological feature of the field $\mathbf{E}_a(\mathbf{X}_a)$ (and not a dynamic one as in the case of a dislocation or a disclination) in which the field changes its sign, while the gauge field $\Gamma^a_{0\mu}(\mathbf{X}_0)$ is absent.

3. SPACE-TIME BEHAVIOR OF A STEADY-STATE ENSEMBLE OF STRUCTURAL UNITS

The use of gauge symmetry in Section 2 allowed us to describe single structural units of a nucleating level. Let us now consider the situation of a well-developed level, where the density of structural units is high (in Section 2 we had $l/l_m \gg 1$; here $l/l_m \sim 1$). In this case we have a change in the geometrical structure of the manifold M of the ensemble of structural units. In particular, it proves to be impossible to separate uniquely the internal coordinate $\mathbf{x}$ into elastic $\mathbf{x}_{(e)}$ and plastic $\mathbf{x}_{(\varepsilon)}$ components. Correspondingly, the separation of the field (elastic) component $\mathbf{u}_{(e)}$ from the total displacement $\mathbf{u} = \mathbf{x} - \mathbf{X}_0$ of the points X_0 of the manifold M in the transition from individual structural units of this level to

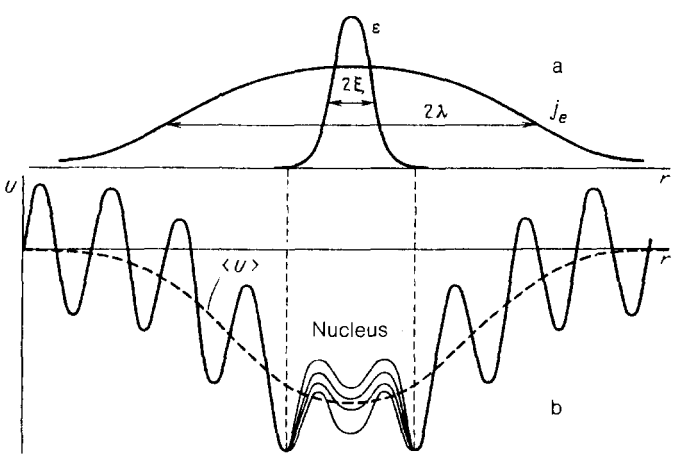

FIG. 2. a) Coordinate dependence of the material deformation field ε and the elastic stress field j_e near a defect. b) Corresponding form of the potential relief.

the ensemble of units is also no longer unique. This is due to the nonintegrability of the 1-forms $e^{\mu} = e_i^{\mu} dX_0^i$ generalizing the concept of the differential $D_0 u^{\mu}_{(e)} = \nabla_{0i} u^{\mu}_{(e)} dX_0^i$, where $\nabla_{0i} = \partial_{0i} + \Gamma_{0i}$, introduced in Section 2 (Ref. 30).

The study of an ensemble of structural units requires an analysis of the physical fields in the principle stratification of the basis M due to the action of the space group G (Ref. 32). The geometry of stratified space is given by the structure equations of Cartan[38]

$$\mathbf{T} \equiv d\mathbf{e} = D_0 \mathbf{e} + \hat{\Gamma} \wedge \mathbf{e}, \quad \hat{\Omega} \equiv d\hat{\Gamma} = D_0 \hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma}, \tag{3.1}$$

which define the 2-forms $\mathbf{T}$ and $\hat{\Omega}$ of the torsion and the curvature as the inner differentials of the initial 1-form $\mathbf{e}(\mathbf{x})$ and the connectivity $\hat{\Gamma}(\mathbf{x})$ (by definition we have $D_0^2 = 0$, but $d^2 \neq 0$). The form of the curvature obeys the Bianchi identity $d\hat{\Omega} = d^2\hat{\Gamma} = 0$ (Ref. 30). In an expansion over the basis $(1/2) e^{\mu} \wedge e^{\nu}$ the 2-form $D_0 \mathbf{e}$ has as components the structural coefficients $\mathbf{C}_{\nu\mu}$, while the forms of the torsion and the curvature have the components $\mathbf{T}_{\mu\nu} = \mathbf{\Gamma}_{\mu\nu} - \mathbf{\Gamma}_{\nu\mu}$ and $\hat{R}_{\mu\nu} = [\partial_{\nu}, \partial_{\mu}]$ (Ref. 20).

In a circuit around the closed contour in the space of M the tangent vector $\mathbf{A}(\mathbf{x})$ turns by an amount $\delta l = l \Omega_{ij} S^{ij}$, and the contour lacks closure by the vector $\xi = -\mathbf{T}_{ij} S^{ij}$, where S^{ij} is the area of the contour, l is its length, $\Omega_{ij} = \partial_{0j} \Gamma^b_{bi} - \partial_{0i} \Gamma^b_{bj}$ is the segmentary curvature tensor, and Γ^b_{ij} are the Christoffel symbols, which are the components of the connectivity form $\hat{\Gamma}^{\alpha}$. Among the three enumerated geometrical objects the following is a natural classification of the structural units at the new level (Fig. 3): A space of absolute parallelism with dislocations corresponds to nonzero $\mathbf{T}_{ij}$; in Riemann space with disclinations $\hat{R}_{ij} \neq 0$; for $\hat{R}_{ij} \neq 0$ and $\mathbf{T}_{ij} \neq 0$ we have a Riemann–Cartan space with dislocations and disclinations; the general case $\hat{R}_{ij} \neq 0$, $\mathbf{T}_{ij} \neq 0$ and $\Omega_{ij} \neq 0$ corresponds to an affine metric space with disclinations, dislocations and point defects.[32] Thus, the tensor $\mathbf{T}_{ij}$ characterizes the density and flux of the shear component of dislocations, and $\hat{R}_{ij}$ corresponds to the space-time distribution of rotational dislocations and disclinations, and finally, Ω_{ij} corresponds to the distribution of point defects.[32,39] The Cartan structure equation (3.1) describes the mutually connected kinematics of ensembles of defects, which comprise the structural units at the given level.[2,33]

The symmetry of the system is described in the general case by the affine group $A(3,\mathbf{R})$, which is isomorphic to the semidirect product $GL(3,\mathbf{R}) \blacktriangleright T(3)$. In turn, $GL(3,\mathbf{R})$ is decomposed into the group of dilatations $\Delta(3)$ and the group $SL(3,\mathbf{R})$ of transformations that conserve volume. Since the matrices of the bases of the Abelian groups $T(3)$ and $\Delta(3)$ commute with the rest of the basis matrices of the original group $A(3,\mathbf{R})$, its representations coincide with the representations of the group $SL(3,\mathbf{R})$, in which the group of rotations $SO(3)$ appears as a subgroup. As a result, the irreducible decomposition of the $A(3,\mathbf{R})$ connectivity has the form[32]

$$\hat{\Gamma}_{ij} = \mathbf{L}_{ij} + \mathbf{K}_{ij} + \frac{1}{3}\mathbf{H}\delta_{ij} \equiv \mathbf{H}_{ij} + \frac{1}{3}\mathbf{H}\delta_{ij}, \tag{3.2}$$

where the antisymmetric component $\mathbf{L}_{ij}$ corresponds to the group of rotations and $\mathbf{K}_{ij} = \mathbf{\Gamma}_{(ij)} - (1/3)\mathbf{H}\delta_{ij}$ is the symmetric traceless component. In the decomposition over the corresponding basis matrices of the groups they have the form

$$\begin{gathered} \hat{L}_{\mu} = l^a_{\mu} \hat{\lambda}_a, \quad \hat{K}_{\mu} = k^a_{\mu} \hat{\varkappa}_a, \quad H_{\mu} = h_{\mu}, \\ \hat{H}_{\mu} = h^{ab}_{\mu} \hat{\gamma}_{ab}, \quad \hat{\gamma}_{ab} \equiv \hat{\lambda}_a \hat{\varkappa}_a, \quad a, b = 1, 2, 3. \end{gathered} \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) we obtain the strengths of the basis fields[33]

$$\begin{gathered} \Omega_{\mu\nu} = h_{[\nu,\mu]}, \quad R^{ab}_{\mu\nu} = h^{ab}_{[\nu,\mu]}, \\ T^i_{\mu\nu} = \varepsilon^i_{[\mu,\nu]} + \gamma^i_j \Omega_{\mu\nu} u^j + \gamma^i_{abj} R^{ab}_{\mu\nu} u^j, \\ e^i_{\mu} = u^i_{,\mu} - \varepsilon^i_{\mu}, \end{gathered} \tag{3.4}$$

where the index after the comma indicates the derivative $\hat{\partial}_{\mu} = (\nabla_{0\mu} + h_{\mu})\hat{I} + \hat{\gamma}_{ab} h^{ab}_{\mu}$ with respect to the internal coordinate $\hat{x}^{\mu}$; antisymmetrization and symmetrization are performed over the indices in the square brackets and parentheses, respectively.

The field component of the Lagrangian is the sum of terms[32,33]

$$\begin{gathered} L_e = \frac{1}{2} e^0_i e^i_0 - \frac{1}{8}\left(\frac{\lambda_0}{\mu_0} e^i_i e^j_j + 2 e_{ij} e^{ij}\right), \\ L_{\varepsilon} = -\frac{1}{2g_{\varepsilon}^2} T^i_{\mu\nu} T^{\mu\nu}_i, \quad L_{\hat{h}} = -\frac{1}{g_{\hat{h}}^2} R^{ab}_{\mu\nu} R^{\mu\nu}_{ab}, \\ L_h = -\frac{1}{g_h^2} \Omega_{\mu\nu} \Omega^{\mu\nu}, \quad L_{\nu} = -\nu^{\mu}_i (u^i_{,\mu} - e^i_{\mu} - \varepsilon^i_{\mu}), \end{gathered} \tag{3.5}$$

where λ_0 and μ_0 are the Lamé parameters at a given level, g_{ε}, g_h, $g_{\hat{h}}$ are the charges of the fields ε^i_{μ}, h_{μ}, h^{ab}_{μ}; ν^{μ}_i is the Lagrange multiplier that takes into account the coupling between these fields, expressed by the last equality in Eqs. (3.4). The covariant notation of Eqs. (3.5) contains implicitly three metric tensors, corresponding to each of these fields. Consequently, the field Lagrangians L_{ε}, L_h, $L_{\hat{h}}$ of the defects contain an additional three velocities, which characterize the ensembles of these defects. As the density of the structural units decreases the elastic component L_e in Eqs. (3.5) reduces to the corresponding contribution L_{0e} [see Eq. (2.8)]. They differ in being expressed in terms of the conjugate quantities e^i_{μ} and $\Sigma^{\alpha}_{0\mu\nu}$, respectively.

The field equations corresponding to Eqs. (3.5) are written in the form[3)] (Refs. 32 and 33)

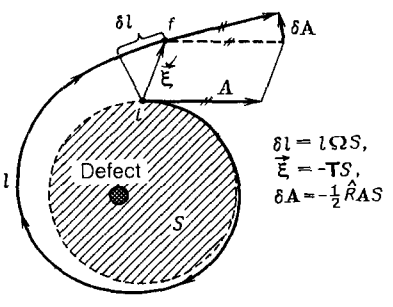

FIG. 3. Variation of the tangent vector A and of the contour (of length l and area S) for a circuit around a defect. (i and f are the initial and final points of the circuit; δl and ξ are the elongation and the deviation of the contour from closure.

$$d\vec{\Sigma} = -F \wedge \hat{R} - F \wedge \Omega;$$
$$dH = J, \quad dF = \Sigma, \quad d\hat{G} = \hat{J}. \tag{3.6}$$

Here d is the differential with respect to the internal coordinate x, and the components of the strengths of H, F, and $\hat{G}$ and of the currents J, Σ, and $\hat{J}$, corresponding to point defects, dislocations, and disclinations, respectively, have the form[33]

$$H^{\mu\nu} = \frac{\partial L_h}{\partial \Omega_{\mu\nu}}, \quad J^{\mu} = \frac{\partial L_h}{\partial h_{\mu}},$$
$$\mathbf{F}^{\mu\nu} = \frac{\partial L_\varepsilon}{\partial \mathbf{T}_{\mu\nu}}, \quad \Sigma^{\mu} = \frac{\partial L_{\mathbf{e}}}{\partial \mathbf{e}_{\mu}}, \tag{3.7}$$
$$\hat{G}^{\mu\nu} = \hat{\gamma}^{ab} C^{cd}_{ab} \frac{\partial L_{\hat{h}}}{\partial R^{cd}_{\mu\nu}}, \quad \hat{J}^{\mu} = \hat{\gamma}^{ab} C^{cd}_{ab} \frac{\partial L_{\hat{h}}}{\partial h^{cd}_{\mu}},$$

where C^{cd}_{ab} are the components of the metric of the Cartan–Killing group SL(3,**R**). The condition of integrability of system (3.6),

$$dJ = 0, \quad d\hat{J} = 0 \tag{3.8}$$

guarantees the equilibrium of the couple stresses for the point defects and disclinations (see Ref. 33). Since the current Σ of the field ε is an elastic stress, the condition of equilibrium for it is given by the first of Eqs. (3.6).

The system of equations (3.6) augmented by conditions (3.8) provide a complete self-consistent description of the distribution of the fields of defects and elastic stresses. It can be seen from Eqs. (3.7) that the strength of H corresponds to the distribution of point defects due to the current J. Correspondingly the field **F** and the current Σ are related to the dislocations and $\hat{G}$ and $\hat{J}$ to the disclinations. These relations refer not only to the static case, but also to the self-consistent time evolution of the fields. For an explicit expression of this evolution one must take into account the form of the time component of the basis $e^0 = ct$, where c is the velocity of sound, $de^0 = 0$, and the corresponding component $dt \wedge \partial/\partial t$ of the operator d in Eqs. (3.1), (3.6) and (3.8). Then each of the 2-forms Ω, **T**, and $\hat{R}$ are given in the form $*\rho + j \wedge dt$, where ρ and j are the density and current of defects of the corresponding type, the asterisk * means the dual conjugation with the use of the unit antisymmetric tensor.[20] The structure equation (3.1) of Cartan establishes the relation between the densities and the currents of the various types of defects of the specified stress fields; i.e., they describe the kinematics of the defect fields. By contrast, system (3.6) also takes into account the self-consistent variation of the force fields, thereby representing the dynamic regime of the evolution of the system of defects.

Because of the complex structure of the components of the Lagrangian (3.5), the analysis of the general case presents considerable difficulties. The various special cases can be studied by analogy with the work of Ref. 33. Here we present examples for only the simplest kinematical situation, which arises in the description of only a single type of defect. If this defect is a dislocation (of the translational type), then in Eq. (3.1) the connectivity is $\Gamma = 0$ and $dT = 0$. From this and the first equality of (3.1) we obtain the well-known equations of dislocation kinematics (see Section 29 in Ref. 21)

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla \times \hat{j} = 0, \quad \nabla \cdot \hat{\rho} = 0, \tag{3.9}$$

$$\frac{\partial \hat{e}}{\partial t} - \hat{j} = 0, \quad \nabla \times \hat{e} = -\hat{\rho}, \tag{3.10}$$

where we have gone over to the standard notation. If the only nonzero component in H in the connectivity (3.2) is that corresponding to point defects, then from (3.1) we obtain the equations

$$d\mathbf{T} = *\mathbf{J}_t, \quad *\mathbf{J}_t = (1/3)\Omega \wedge \mathbf{e}, \quad \Omega = dH, \tag{3.11}$$

that relate the behavior of the dislocations to the field of the point defects, characterized by the current $\mathbf{J}_t$. If the sources of dislocations are also sources of disclinations, then the total current $\mathbf{J}_t = \mathbf{J}_p + \mathbf{J}_d$ on the right-hand side of (3.11) is made up of the components

$$*\mathbf{J}_p = \frac{1}{3}(\Omega \wedge \mathbf{e} - H \wedge \mathbf{T}),$$
$$*\mathbf{J}_d = \hat{\Omega} \wedge \mathbf{e} - \hat{H} \wedge \mathbf{T}, \tag{3.12}$$

which correspond to contributions from each type of defect source. Allowance for self-consistency in the behavior of the dislocations themselves is taken care of by the equation[32]

$$d*\mathbf{T} = *\mathbf{J}, \tag{3.13}$$

in which the current **J** is a direct source of dislocations.

In going to a three-dimensional description Eq. (3.11) reduces to a relation of type (3.9), where, however, the right-hand sides are the nonzero terms $\hat{i}_t$, and $\hat{\alpha}_t$ of the 3-tensors of the current and the density of defect sources. Accordingly, the equation (3.13) of self-consistent behavior of the dislocations is rewritten in the form

$$\frac{\partial \hat{j}}{\partial t} + \nabla \times \hat{\rho} = \hat{i}, \quad \nabla \times \hat{j} = -\hat{\alpha}, \tag{3.14}$$

where the tensors $\hat{i}$ and $\hat{\alpha}$ are the components of the 4-current $*\mathbf{J}$ sources of dislocations.

Above we have presented a scheme for the description of the fields of the density and current of the dislocations for a given distribution of point defects and disclinations, which, in addition to the direct sources **J** of the Frank–Read type play the role of indirect sources. In a similar way, by specifying the distribution of two of the three possible types of defects, one can describe the behavior of point defects and disclinations. A very interesting topic is the study of the mutually consistent behavior of the two types of defects when the distribution of only the third one is specified. Here it is possible to describe from first principles these effects as the autocatalytic multiplication of defects,[26] the onset of self-oscillatory behavior,[40] and so forth. Unfortunately, this program has not yet been realized.

4. DESCRIPTION OF THE HIERARCHICAL MULTILEVEL DEFECT STRUCTURE

4.1. Formation of a hierarchical structure of defects

As can be seen from the previous discussion, the total Lagrangian that represents a given structural level has the form $L = L_m(\mathbf{E},\mathbf{u},\mathbf{e},h,h) + L_e(\mathbf{e}) + L_s(\Omega,\mathbf{T},\hat{R})$, where the structural component is $L_s = L_h + L_\varepsilon + K_{\hat{h}}$. If the density of structural units is low one can neglect the contribution L_s,

assuming that the curvature-torsion of the space of M is zero and the defects are single. A direct calculation then gives an estimate of the minimum values of the intensity of formation of defects (in units of $b\varepsilon_e^2$): $\Omega_{c1} \sim \varkappa^{-1} \ln \varkappa$ for point defects, $T_{c1} \sim \ln \varkappa$ for dislocations, and $R_{c1} \sim \varkappa \ln \varkappa$ for disclinations. Since $\varkappa \gg 1$ we have the hierarchy $\Omega_{c1} \ll T_{c1} \ll R_{c1}$; point defects are the most easily formed, dislocations are more difficult to form, and disclinations are the most difficult.

In the case of a well-developed structural level the density of structural units is so high that one can discard the nonlinear terms in the deformation $\mathbf{E}$. Then the linearized equation (2.4), where Γ_0 is replaced by $\Gamma_0 + \Gamma$ allows us to find the upper critical field for the formation of the next level. Since for all defects this field is given by the same condition for the formation of a localized level in the effective potential of the linearized Schrödinger equation (2.4), the upper critical field for Ω_{c2}, T_{c2}, R_{c2} is $\sim \varkappa$ (in units of $b\varepsilon_e^2$) regardless of the type of defect. For disclinations we have $R_{c1}/R_{c2} \sim \ln \varkappa > 1$, in connection with which they are observed only in the form of disclination dipoles.[4-7]

To ensure a sufficiently wide region of existence of dislocations and point defects at a particular level requires large values of the ratio $\varkappa = \lambda / l_m$ of the damping length of the elastic field $\lambda \propto \eta$ to the correlation length $l_m = l_0/w_m$. A direct check shows that this condition is satisfied best at the atomic (microscopic) level. With the transition to the next mesoscopic level, the formation of the structural units that belong to that level, which act as carriers of the plastic deformation, reduces the effective value of the shear viscosity to $\eta = \eta_0 (1+\chi)^{-1} \ll \eta_0$, and, consequently, decreases the length $\lambda = \lambda_0 (1+\chi)^{-1} \ll \lambda_0$, where χ is the susceptibility to shearing stresses. On the other hand, by virtue of the spontaneous breaking of conformal invariance there is an increase in the scale l_m by a factor of $w_m^{-1} \gg 1$. Therefore, if we have $\varkappa_0 \gg 1$ at the initial level, then at the next level we obtain a much smaller ratio $\varkappa \ll w_m \varkappa_0 \ll \varkappa_0$. As a result, as the structural units become larger the region of existence of equiaxial formations, and, *a fortiori*, of dislocation formations grows smaller (the condition for the formation of disclination complexes, on the other hand, are more favorable). This conclusion is in agreement with experimental data.[4-7]

Within the framework of the approach developed at the beginning of Section 2, let us now explore the pattern of evolution of the defect structure for the deformation of an ideal crystal under conditions of creep ($\hat{\sigma}_{ext} = \text{const}$). Let the external conditions T and $\hat{\sigma}_{ext}$ be such as to make the configuration point of the system fall in the region of existence of the nth level (Fig. 4). In the absence of deformation, a single atomic level is realized. It is represented by a potential energy $U_0(\mathbf{x})$ for a probe atom that is periodic and characterized by an interatomic spacing $a \equiv l_0$. Excitation of the system, which induces the formation of the succeeding level, changes the shape of the potential relief $U(\mathbf{x})$: thermal excitation T smears out the function $U(\mathbf{x})$ into an ensemble of reliefs $\{U(\mathbf{x})\}$, the pressure p changes the height $Q \sim \mu_0 l_0^3$, of the relief, and the shearing load τ changes over the length l the reference level of the potential function $U_0(\mathbf{x})$, which oscillates with a period l_0 (clearly, $l \sim Q/\tau l_0^2 \sim (\mu_0/\tau) l_0$); Ref. 11). The formation of a single structural unit at the new level results in a change in the periodic initial relief $U_0(\mathbf{x})$, of

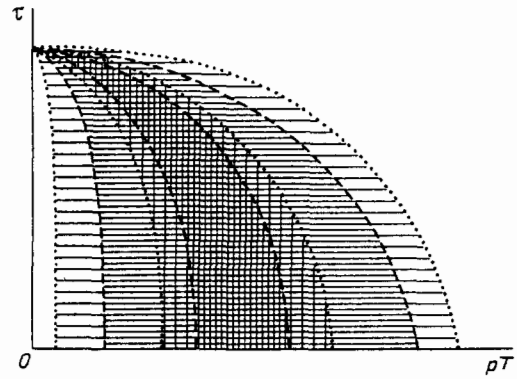

FIG. 4. Schematic diagram of the existence of two structural levels. The dotted line indicates the region of equiaxial formations (point defects); the dashed line indicates the region of existence of linear defects. The horizontal hatching indicates the lower level and the vertical hatching indicates the upper level.

the type shown in Fig. 2b. To determine its coarse structure, not taking into account the rapid oscillations, it is sufficient to describe the change (at large distances l) in the reference level of $\langle U(\mathbf{x}) \rangle$ (the dashed line in Fig. 2b), the height $Q(\mathbf{x})$, and the width of the spreading $\theta(\mathbf{x})$ as functions of the coordinate $\mathbf{x}$. The functions $\langle U(\mathbf{x}) \rangle$, $Q(\mathbf{x})$, $\theta(\mathbf{x})$ determined in this way characterize the structure of the defect of the new level.

As the degree of plastic deformation increases, the number of structural units at a given level increases, and the function $\langle U(x) \rangle$, averaged over the lengths $l_0 \ll x \ll l$, takes on a quasiperiodic character such as is shown in Fig. 2a, except that the period is $l \gg l_0$. A further increase in the plastic deformation can lead to a reorganization of the averaged relief $\langle U(x) \rangle$ at lengths $L \gg l$ in a manner similar to that which occurred in the initial potential function $U_0(\mathbf{x})$. This corresponds to the formation of superdefects at the next structural level, and so forth. As a result, the potential energy of an n-level hierarchical system of structural units is represented in the form of an n-stage functional dependence

$$\begin{aligned} U_1(\mathbf{r}_1) &= U_{\mathbf{r}_1}\{\rho_0(\mathbf{r}_0)\}, \\ U_2(\mathbf{r}_2) &= U_{\mathbf{r}_2}\{\rho_1(\mathbf{r}_1)\}, \ldots, \\ U_n(\mathbf{r}_n) &= U_{\mathbf{r}_n}\{\rho_{n-1}(\mathbf{r}_{n-1})\}; \end{aligned} \tag{4.1}$$

where $\rho_{l-1}(\mathbf{r}_{l-1})$ is the density of structural units at the $(l-1)$th structural level, whose hierarchical reorganization is determined by the energy

$$\mathbf{E}_l(\mathbf{r}_l) = \int_{V_l} U_{l-1}(\mathbf{r}_{l-1}) \rho_{l-1}(\mathbf{r}_{l-1}) d\mathbf{r}_{l-1}, \quad l = 0, 1, \ldots, n-1, \tag{4.2}$$

of a physically small volume V_l whose location is determined by the hydrodynamically determined coordinate $\mathbf{r}_l$ of the next level l.

The multistage functional dependence (4.1) implies that the potential relief $U_l(\mathbf{r})$ has a fractal nature, as shown in Fig. 1. The larger the degree of the external action, which determines the total deformation, the greater is the number of levels that are included in the process of plastic deforma-

tion, and the more complex is the fractal structure of the resulting distribution of the force fields (2.1). This structure is a consequence of the hierarchical chain of the defects, to reflect which it is best to use not the pure spatial dependence, but a mixed dependence: the rapid oscillations at macroscopic distances $r_0 \leqslant l_0$, which characterize the lower level, given as before by the spatial dependence in ordinary geometrical space, and the long-range correlations at lengths l_i, $i = 1, 2, ..., n$ which reflect the hierarchical relation between the defects, which are described in an ultrametric state space. Its metric is constructed so that if two correlated regions of states have even one common point, one of these regions is contained in the other.[41] This means that the points of ultrametric space that correspond to the state of the structural unit of the given level form a cluster of coherently connected states corresponding to a superdefect at the next higher level. To avoid misunderstanding, let us note that the points of the ultrametric space correspond to a cluster of pure states (ensembles) of the defect structure, and not a cell of defects forming a superdefect; the former are distinguished by their level of overlap between the pure ensembles, while the latter are distinguished by the closeness of the values of the fields acting on the defect.[42]

The concept of a hierarchical system of defects of a multilevel sequence of potential reliefs (4.1) is incomplete, since it does not contain an inverse relation between the density $\rho_l(\mathbf{r}_l)$ of the defect distribution at the given level $l = 0, 1, ..., n-1$ and its energy (4.2). To establish this relation we must use the quasi-Gibbs distribution[11]

$$P_l\{U_{l-1}(\mathbf{r}_{l-1})\} \propto \exp(-E_l\{U_{l-1}(\mathbf{r}_{l-1})\}/\theta_{l-1}) \tag{4.3}$$

of the nonequilibrium potential reliefs $U_{l-1}(\mathbf{r}_{l-1})$ for the preceding level, which is characterized by a "temperature" θ_{l-1}. The value of this temperature is determined by the degree of stochasticity of the excitation of the $(l-1)$th structural level, which results in a smoothing out of the uniquely defined function $U_{l-1}(\mathbf{r}_{l-1})$ in the ensemble $\{U_{l-1}(\mathbf{r}_{l-1})\}$. As a result the macroscopic properties of this level are not given by the energy (4.2), but by the thermodynamic potential

$$\Phi_l = -\theta_{l-1}\ln\langle\exp(-E_l/\theta_{l-1})\rangle_{l-1}, \tag{4.4}$$

where the subscript $l-1$ after the angle brackets means that the average is taken over the distribution (4.3) in the ensemble $\{U_{l-1}(\mathbf{r}_{l-1})\}$. Taking into account the quasi-Gibbs nature of the distribution, it is easy to show in the usual way[8] that the value of the thermodynamic potential (4.4) determines the density of structural units at the given level according to the expression

$$\rho_l \propto \exp(-\Phi_l/\theta_{l-1}) = \langle\exp(-\theta_{l-1}^{-1}\int_{V_l} U_{l-1}(\mathbf{r}_{l-1})\rho_{l-1}(\mathbf{r}_{l-1})d\mathbf{r}_{l-1})\rangle_{l-1}. \tag{4.5}$$

In the formation of a hierarchical structure of the level l the quantities ρ_l, and Φ_l in Eq. (4.5) become dependent on the hydrodynamic coordinate $\mathbf{r}_l$. The thermodynamic potential $\Phi_l(\mathbf{r}_l)$ determined in this way will play the role of the potential relief $U_l(\mathbf{r}_l)$, specified by the energy (4.2) of the structural units of the next level. It should be noted that the synergetic potential $V_l\{U_{l-1}(\mathbf{r}_{l-1})\}$ of the lth level, used for the phenomenological description in Section 1 for the case $l = 1$, plays the role of an effective Hamiltonian.[8] It is obtained from the specified energy E_l just as is the thermodynamic potential Eq. (4.4), with the difference that in the averaging over the field $U_{l-1}(\mathbf{r}_{l-1})$ the order parameter (2.2) is considered to be fixed.

In this way, the hierarchical defect structure forms a system of $n > 1$ levels l, each of which is characterized by a potential relief $U_l(\mathbf{r}_l)$ and a defect distribution $\rho_l(\mathbf{r}_l)$. According to Eqs. (4.1) the relief of a particular level l is formed by the distribution of structural units in the previous level $l-1$. In turn, the density of structural units is specified by the statistical distribution (4.5), in which the thermodynamic potential is determined by an average over the stochastic distribution of defects at the lower level. As a result, relations (4.1)–(4.5) give in closed form the relation between the hierarchical distribution of structural units at adjacent levels. The corresponding relation between the potential reliefs is expressed by

$$U_l(\mathbf{r}_l) = -\theta_{l-1}\ln\langle\exp(-\theta_{l-1}^{-1}\int_{V_l} U_{l-1}(\mathbf{r}_{l-1})\rho_{l-1}(\mathbf{r}_{l-1})d\mathbf{r}_{l-1})\rangle_{l-1}, \tag{4.6}$$

which follows from the identity $U_l(\mathbf{r}_l) \equiv \Phi_l$.

Let us point out a characteristic property of hierarchical defect structures, which distinguishes their behavior from the structural relaxation of glass as described in Refs. 43 and 44 (see also below in Section 4.3). This property is due to the difference in the action of the excitation of the system at the different structural levels. In a metallic glass system the degree of excitation, which is fixed by the frozen-in disorder, is so high that during the relaxation considerable stochastic behavior appears at all the structural levels. This behavior stems from the fact that the structure temperature T^*, being a measure of the degree of nonequilibrium of the glass, has a value $T^* \geqslant E_l$, relative to the energies E_l of the levels.[44] In the present case of defect structures the degree of excitation T^* may be so low that the stochastic spread occurs only at the $m < n$ lower structural levels. This means that between the effective temperature T^* at which this structure can be realized as an equilibrium structure, and the "temperatures" θ_l, which characterize the degree of excitation of each level l, the following relation holds: $\theta_l \leqslant T^*$, $l = 0, 1, ..., m$, $m \leqslant n$. As a result, the structural temperatures turn out to be the same, $\theta_l = T^*$, for the lower m levels, while for the upper $n-m$ levels $\theta_l = 0$. Consequently, for a given degree of excitation of the defect structure T^* the lower m levels behave as hierarchical intersubordinate nonequilibrium statistical ensembles, whose stationary regime is determined by the temperature T^*, while the behavior of the upper $n-m$ levels is purely deterministic. This implies that the evolution of these levels has a mechanical nature, which is expressed in the fact that their thermodynamic potentials reduce to the energies E_l. This circumstance is tacitly assumed in the standard theory of defects.[1-7] Thus, for example, in the theory of dislocations[3] it is assumed that the behavior of the dislocations is determined uniquely by the given Peierls relief, whereas the behavior of point defects is treated statistically.

Let us consider in conclusion an example of the formation of a hierarchical defect structure during creep of a solid under a load.[17] For this purpose we first present the general picture of creep.[45] If the stress σ applied to the solid exceeds a critical value $\sigma_k \geqslant 10^{-5}\mu$, determined by the shear modulus μ, then we observe irreversible creep, which is manifested in the smooth increase in the plastic deformation $\mathscr{E}$ over a time t for a constant stress $\sigma > \sigma_k$ and incomplete relaxation of $\mathscr{E}(t)$ to the initial value $\mathscr{E}(0) = 0$ when the load is removed. Depending on the temperature, one can distinguish non-steady-state and steady-state creep, which can be characterized, respectively, by an extremely slow (approximately logarithmic), and a linear increase in the deformation with time. The former type of creep appears at temperatures T less than half the melting temperature T_m and is characterized by a deformation rate $\dot{\mathscr{E}} \equiv d\mathscr{E}/dt$ that goes to zero during the time of the constant load. The latter condition is observed at temperatures $T > T_c$, where $T_c \geqslant 0.5T_m$, and is characterized by a rate $\dot{\mathscr{E}}(t)$ that tends to a finite value $K(\sigma) \neq 0$ as $t \to \infty$. Far from the melting point the variation of K with the applied stress σ is a power-law dependence, $K \propto \sigma^n$, with $3 \leqslant n \leqslant 5$ for values of σ much less than the theoretical strength $\sigma_{max} \sim 10^{-1}\mu$, while it varies exponentially, $K \propto \exp(c\sigma/T)$, with c a constant for $\sigma \leqslant \sigma_{max}$. In the region $T \leqslant T_m$ we observe diffusion creep, which is characterized by a linear variation of $K(\sigma)$ (this happens at small loads, $\sigma < \sigma_c$).

To the present time a satisfactory description has been obtained only for those aspects of the picture that are described by simple models.[45] This situation is most clear near the melting temperature, where diffusion creep is limited by the motion of the simplest type of defect, vacancies, for which the dependence $\mathscr{E}(t) = Kt$, $K(\sigma) \propto \sigma$ has the simplest form. As the temperature is lowered the dislocation mechanism comes into play and the picture becomes more complicated: for example, if an ensemble of dislocations evolves independently from the vacancies, as occurs in alloys, then we have $\mathscr{E}(t) \approx Kt$, $K(\sigma) \propto \sigma^n$, $n \approx 3$. When the behavior of dislocations and of vacancies are interdependent (in pure metals) the exponent n increases to ≈ 5 (Ref. 45). When more carriers of plastic deformation are brought in (disclinations, grain boundaries, etc.) as the temperature is lowered and the stress is increased, there is an increase in the contribution $\mathscr{E}(t) = At^{\alpha}, \alpha < 1$ of non-steady state creep. In a certain range of temperature and stress this power-law creep coexists with the logarithmic behavior $\mathscr{E}(t) = B\ln(1 + \nu t)$. As the temperature is lowered still further and the stress is increased the steady-state and non-steady-state power-law types of creep disappear and only the logarithmic behavior remains. Thus, as additional mechanisms that provide a variety of carriers of plastic deformation are turned on, they slow down the plastic deformation. If only two types of carrier (vacancies and dislocations) are involved in steady-state creep, then the non-steady-state stage is formed by the contribution of a very large number of mechanisms. For this reason a consistent theory of non-steady-state creep has heretofore been lacking.[45]

An analysis of experimental data[23,45] leads to the following picture of creep. At zero temperature and stress, the thermodynamic potential of a solid in configuration state space increases monotonically with a minimum that corresponds to an ideal crystal structure. Application of a load (Fig. 5) produces minima that correspond to defects in the crystal structure: the first of these appears at a minimum value of the stress $\sigma_c^{(1)}$, the next at $\sigma_c^{(2)} > \sigma_c^{(1)}$, and so forth. An increase of the temperature results on the one hand in a decrease in the critical values $\sigma_c^{(n)}$, $n = 1, 2, \ldots$ (Ref. 11) and on the other in a fluctuational concentration of stress up to values $\sigma > \sigma_c^{(n)}$, and, consequently, the formation of metastable defects. Therefore, if a stress $\sigma < \sigma_k(T)$ is applied, where $\sigma_k \equiv \sigma_c^{(1)}$, and then the load is relieved, the system will not be stable in any of the defect states and the relaxation of the internal stress will occur according to the usual Debye law $\sigma(t) \propto \exp(-t/t_0)$, where t_0 is the microscopic relaxation time. This situation is characteristic of reversible creep.[45]

As the stress is increased to $\sigma \geqslant \sigma_k$, isolated defects—vacancies and interstitials—form first, and then for $\sigma \gg \sigma_k$ they unite in clusters and form more complicated entities. Here the hierarchical behavior of the various kinds of defects is characteristic. For example, the point defects are distributed in accordance with the behavior of the linear and surface defects; the linear defects can form grain boundaries, and so forth. This behavior is reflected in the form of the thermodynamic potential in configuration state space in the existence of minima (and, correspondingly, barriers) of various order (Fig. 6). For example, dislocation formations correspond to the potential minimum d, which is separated from the initial (i) minimum by a barrier of height Q_d. However, on its coarse structure there is superposed a finer system of minima, each of which is related, for example, to the nature of the impurity distribution in the Cottrell clouds. In turn, each of these minima can have still finer structure (Fig. 1), associated with the electronic structure of the point defects.

In this sequence we have traced a hierarchical connection going downward from dislocations to point defects and their electronic states. Of course, one can trace out a hierarchy going from the dislocations upward; dislocations unite

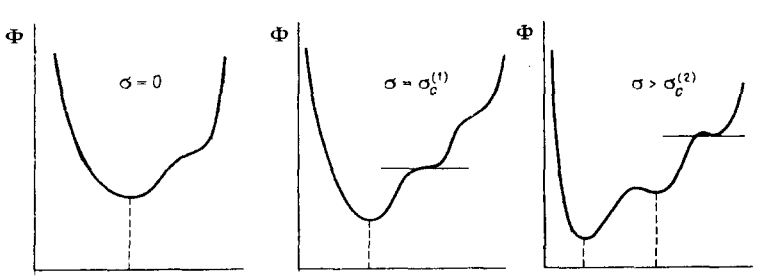

FIG. 5. Change in the thermodynamic potential in configuration space of states as the load is increased ($T = 0$).

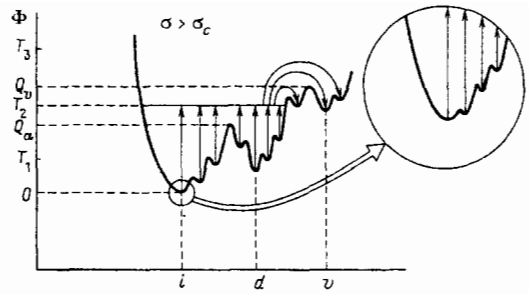

FIG. 6. Shape of the thermodynamic potential in configuration space of states of a solid with defects for a load $\sigma > \sigma_c$.

into small-angle grain boundaries; these in turn form the block structure of grains, and the grains govern the behavior of the entire sample. At the present time it is customary to speak of each step of the hierarchical stairway as being structural levels of the deformation and failure of a solid.[10] Each specific phenomenon is due to processes that take place at a specific structural level:[46] diffusion creep—vacancies; low-temperature steady-state creep—dislocations, and so forth. This approach, however, does not explain phenomena such as non-steady-state creep, with the participation of several structural levels subordinate one to the other. Therefore, in the development of the concept of structural levels of deformation we propose to treat these levels not separately, but in a mutual relationship—as the steps on a hierarchical staircase. The principal argument in favor of this connection between the levels is that at each of them the behavior of the system is determined by the defects, which unite into the clusters that form the behavior of the next level.

4.2. Stochastic theory of hierarchical defect structure

As was mentioned in the introduction, to represent the hierarchical chain in an ensemble of defects we use the geometric image of a discrete ultrametric space—a Cayley tree (see Fig. 1b). Its nodes at each structural level correspond to pure ensembles (states,[42] valleys,[18] or components[12]) of states of defects of the lower levels. We shall label these valleys with Greek letters $\alpha, \beta, \ldots$. Then, in the context of the lattice representation, a structural unit located at node l is characterized by a flux j_l^α in the valley α. The total current gives the rate of plastic deformation

$$\dot{\varepsilon} = \sum_\alpha w_\alpha \dot{\varepsilon}^\alpha, \quad \dot{\varepsilon}^\alpha = \sum_{l=1}^N j_l^\alpha, \tag{4.7}$$

where w_α is the probability of realization of the ensemble α, and N is the number of nodes. According to Rammal *et al.*,[41] hierarchical systems do not pertain to self-averaging systems—for a macroscopically large system the distribution of w_α changes randomly from one sample to another. This circumstance leads to an important conclusion: even under identical macroscopic conditions the total amount of plastic deformation is not reproducible. Physically, this is a result of the random nature of the distribution of the ratio $l_0^\alpha / l^\alpha \sim w_\alpha$ of the characteristic scales l_0^α, l^α at the initial level and the given level. In turn, this nonregularity is related to the fact that in nonergodic systems a fluctuation in the initial conditions and the boundary conditions of the evolution of an ensemble of defects can cause macroscopic changes even though the fluctuation is of a microscopic nature. The reason for this sensitivity of the system to the initial and boundary conditions is the partitioning of configuration space into isolated regions, caused by the loss of ergodicity.[12]

If the contribution of each pure ensemble α to the rate of plastic deformation is characterized by the quantity $\dot{\varepsilon}^\alpha$, then the connection between the ensembles is determined by the Parisi parameter[46]

$$q^{\alpha\beta} = \frac{1}{N\dot{\varepsilon}_{\max}^2} \sum_{l=1}^N j_l^\alpha j_l^\beta \tag{4.8}$$

and its distribution function

$$P(q) = \sum_{\alpha,\beta} w_\alpha w_\beta \delta(q - q^{\alpha\beta}), \tag{4.9}$$

which gives the probability

$$Y(q) = \int_q^1 P(q')\mathrm{d}q' \tag{4.10}$$

for a system of pure ensembles to have an overlap that exceeds a specified value q. By virtue of the stochasticity noted above for a distribution of clusters of ensembles with an overlap $q^{\alpha\beta} > q$, which is given by the sum

$$W = \sum_\alpha w_\alpha \tag{4.11}$$

over these clusters, the functions $P(q)$ and $Y(q)$ and the probability W will also be random. The method of Parisi[47] has made it possible to find the corresponding distribution functions $\Pi_q(Y)$ and $f_q(W)$ for a given value of q.[41] Their form is determined by the average value

$$y(q) = \overline{Y(q)} \equiv \int Y(q)\Pi_q(Y)\mathrm{d}Y, \tag{4.12}$$

of the probability of having an overlap of ensembles $q^{\alpha\beta} \equiv q(y)$ not less than q. The distribution $\Pi_q(Y)$ is bimodal with maxima at $Y \approx 0.5$ and $Y = 1$. The existence of a singularity of $\Pi_q(Y) \propto (1 - Y)^{-y}$ near the second of these maxima causes a difference between the most probable value $Y = 1$ and the average probability $y < 1$. The distribution $f_q(W)$ averaged over the clusters of ensembles with a bounded level of overlap also has a bimodal form

$$\overline{f_q(W)} = \frac{W^{y-2}(1 - W)^{-y}}{\Gamma(y)\Gamma(1 - y)}, \quad y = y(q). \tag{4.13}$$

According to Eq. (4.13) most of the clusters are concentrated near the values $W = 0$ and $W = 1$, especially the former. As the degree of overlap q increases the probability $y(q)$ falls off monotonically, resulting in a concentration of clusters with zero probability W of realization.

The physical meaning of this stochastic scheme is that the hierarchical system of defects should be considered not as a single statistical ensemble, as is usually done,[1-3] but a collection of such ensembles. A feature of this collection is that the ensembles are not independent, but overlap with one another to an extent $q^{\alpha\beta}$, which can be characterized by the Parisi parameter (4.8). In a hierarchical system the stochasticity is not only in the behavior of the defects that belong to

one ensemble, but also in the behavior of the ensembles themselves whose probability of realization is given by the distribution w_α. However, if the distribution of defects in an ensemble is the usual canonical one,[8] then the distribution of ensembles is determined by the random nature of their union into hierarchical clusters; the highest probability is for the union of ensembles with the maximum overlap q, and as q decreases the probability falls off. The nature of the hierarchical stochastization is reflected in the distribution $P(q)$, given by Eq. (4.9). Since this relation itself contains the random quantities w_α and w_β, the distribution function $P(q)$ over the clusters of ensembles is also random. Therefore, the hierarchical nature of the behavior of the defects results in hierarchical stochasticities. The ultimate cause of this situation is the loss of ergodicity in the behavior of the ensemble of defects.[12]

The method developed by Parisi,[47] which is based on the aforementioned fact that the ensembles correspond to points in ultrametric space,[41] has made it possible to hypothesize a hierarchical stochasticity. Its characteristic is that the lowest-order moments $Y(q)$ and W of the distributions $P(q)$ and w_α, respectively, have cutoffs with respect to the overlap parameter q [see Eqs. (4.10) and (4.11)]. The first of these moments gives the probability that the overlap of the ensembles of the level is not lower than a given q, and the second gives the total probability of realization of these ensembles. Clusters with a large overlap q are singled out because ordinary ergodic systems correspond to the limiting case of single ensembles with the maximum overlap $q = 1$. A feature of the hierarchical stochasticity is that in a nonergodic system of defects the most probable value is not only the maximum value $Y(q) = 1$ of the probability of overlap of ensembles, but also the intermediate value, $Y(q) \approx 0.5$. In other words, the distribution $\Pi_q(Y)$ is bimodal, given by the average probability Eq. (4.12). Physically, this means that there is a high probability of realization of ensembles of defects that are only weakly interdependent even in a statistical sense (for example, dislocations that belong to different small-angle walls, belong to almost different, nonintersecting statistical ensembles). On the other hand, if the issue is the number of different clusters of ensembles characterized by the distribution (4.13), then the divergence of the integral of (4.13) over all values of W means that the total number of possible ways in which the ensembles can unite into hierarchical clusters is infinite. However, they are formed mainly of ensembles with a small probability W of realization. This means that infrequently realized structures, belonging to large-scale levels (misorientation boundaries, blocks, grains, etc.), are the most likely to enter the hierarchical chain. On the other hand, the main contribution to the average value of W with the weight $f_q(W)$ comes from the region of large W. Therefore, although most of the defects enter into statistical ensembles that are characterized by large probabilities W, only those that are realized most infrequently form hierarchical structures.

Regarding the distribution of the energies E of the clusters of the states of the defect structure, we note that it has a quasi-Gibbs form, which reflects the independence in the spread in the values of E. However, if for a minimum cluster that corresponds to the maximum overlap $q = 1$ we have a pure Gibbs distribution $n_1(E) = \exp[-(E-\Phi)/T]$, where the energy E is reckoned from the thermodynamic potential Φ, then as the cluster increases in size with a decrease in the overlap $q < 1$ we have a nonequilibrium distribution

$$n_q(E) = \exp[-(1 - y(q))(E - E_q)/T]; \tag{4.14}$$

where the minimum energy of a cluster E_q exceeds its thermodynamic value $E_1 = \Phi$ (Ref. 41).

Having characterized the distribution of clusters of statistical ensembles of defects, let us now turn to a description of the structural units themselves, which form these ensembles during plastic deformation. This means that it will be necessary on the one hand to find the distribution $F(\dot\varepsilon)$ of the rate of deformation $\dot\varepsilon$ and on the other hand, find the distribution $N_q(\tau)$ of the stress field τ acting on the defect in a cluster with an overlap that exceeds the value q (Ref. 42). The nodes l at which the field τ_l takes on values in the range from τ to $\tau + \Delta\tau$ form a cell C_τ, with a volume, for a given overlap q of the ensembles of the structural units, that is equal to $|C_\tau| = NN_q(\tau)\Delta\tau$. In this cell the ensembles with an overlap exceeding a given value q behave independently and are characterized by identical distributions

$$F_q(\dot\varepsilon, \tau) = \frac{1}{|C_\tau|} \sum_{l \in C_\tau} \delta(\dot\varepsilon - f_l'). \tag{4.15}$$

The required distribution of the magnitude of plastic deformation is expressed in terms of those distributions

$$F(\dot\varepsilon) = \int N_q(\tau) F_q(\dot\varepsilon, \tau) d\tau. \tag{4.16}$$

The total value of the rate of plastic deformation $\dot{\mathscr{E}}$ and the overlap parameter q are expressed in terms of the value $\dot\varepsilon_q(\tau)$ of the plastic deformation in a cluster with overlap q and field τ:

$$\dot{\mathscr{E}} = \int F(\dot\varepsilon)\dot\varepsilon d\dot\varepsilon = \int N_q(\tau)\dot\varepsilon_q(\tau) d\tau, \tag{4.17}$$

$$q = \dot\varepsilon_{\max}^{-2} \int N_q(\tau)\dot\varepsilon_q^2(\tau) d\tau, \tag{4.18}$$

$$\dot\varepsilon_q(\tau) = \int F_q(\dot\varepsilon, \tau)\dot\varepsilon d\dot\varepsilon. \tag{4.19}$$

As mentioned before, the conditional distribution (4.15) represents the contribution to the plastic deformation from defects in the field τ in clusters of states with overlap not lower than q. These clusters are singled out because with them one can use the usual concepts of Refs. 1–3 regarding the process of plastic deformation (for example, the relation $\dot\varepsilon^\alpha = b\rho^\alpha v^\alpha$ is valid for the specific rate of plastic deformation in Eq. (4.7) for dislocations with a density ρ^α, a Burgers vector b and a velocity v^α). With equality (4.16) it is possible, knowing the distribution $N_q(\tau)$ of clusters over the values of the field τ, to go from the conditional distribution of the plastic deformation $F_q(\dot\varepsilon,\tau)$ to the total distribution $F(\varepsilon)$. They are different in that the latter, being the first moment of (4.17), gives the total plastic deformation $\dot{\mathscr{E}}$, whereas the former gives the contribution $\dot\varepsilon_q(\tau)$ due to clusters in a field τ with overlap q [see Eq. (4.19)]. With the known distribution $N_q(\tau)$ of the field and the specific rate of plastic deformation $\dot\varepsilon_q(\tau)$ it is possible to find the values of $\dot{\mathscr{E}}$ and q by means of Eqs. (4.17) and (4.18).

As for the function $N_q(\tau)$ itself, for a given spread in the interaction of the structural units can be determined by the method of Ref. 48, developed for amorphous solid solutions. The essence of the method, which is a natural generalization of standard mean field theory, is that the connection is established not between the field τ itself and the interaction parameter of the structural units, but between the distribution functions.[44]

Let us now consider the evolution of a hierarchical ensemble of defects. For this purpose we assume that in the transition to a higher level of overlap $q' > q$ the cell C_τ is split up into parts $C_{\tau\tau'}$. The probability of a node l being in a subcell $C_{\tau\tau'}$

$$\frac{|C_{\tau\tau'}|}{|C_\tau|} = G_{qq'}(\tau, \tau')\Delta\tau' \tag{4.20}$$

gives the relation between the distributions of plastic deformation in a cell and in a subcell:

$$F_q(\dot\varepsilon, \tau) = \int G_{qq'}(\tau, \tau')F_{q'}(\dot\varepsilon, \tau')\mathrm{d}\tau', \tag{4.21}$$

whose volume is

$$|C'_\tau| = N\Delta\tau' \int N_q(\tau)G_{qq'}(\tau, \tau')\mathrm{d}\tau. \tag{4.22}$$

Knowledge of the evolution operator $G_{qq'}(\tau,\tau')$ of the hierarchical defect structure gives complete information on the distribution of the clusters of the states

$$N_q(\tau) = G_{0q}(0, \tau), \tag{4.23}$$

$$F_q(\dot\varepsilon, \tau) = \int G_{q1}(\tau, \tau')\delta[\dot\varepsilon - \dot\varepsilon_{\max}\tanh(\tau'/T)]\mathrm{d}\tau', \tag{4.24}$$

$$\dot\varepsilon_q(\tau) = \dot\varepsilon_{\max}\int G_{q1}(\tau, \tau')\tanh(\tau'/T)\mathrm{d}\tau'. \tag{4.25}$$

According to Eq. (4.23) assigning the distribution function $N_q(\tau)$ of the field in the clusters is equivalent to determining the evolution from a set of isolated ensembles into a cluster that is characterized by overlap q and a field τ. The presence in Eqs. (4.24) and (4.25) of the factor $\tanh(\tau/T)$, which is a feature of usual field theory, means that in a cluster with the maximum overlap $q = 1$ a pure statistical ensemble is realized, in which the rate of plastic deformation is determined by the standard conditions of self-consistency[49]

$$\dot\varepsilon = \dot\varepsilon_{\max}\tanh[(\eta\dot\varepsilon + \tau)/T], \tag{4.26}$$

where η is the shear viscosity, $\dot\varepsilon_{\max}$ is the limiting value of the rate of plastic deformation that can be attained in a given structural level. Relation (4.24) states that the specific distribution function $F_q(\dot\varepsilon,\tau)$ is obtained by the spreading of a δ-distribution corresponding to condition (4.26) in the transition from a pure statistical ensemble to a set of clusters, characterized by an overlap level no lower than q. According to Eq. (4.25) the rate of plastic deformation is transformed from the value (4.26) to the specific value $\dot\varepsilon_q(\tau)$ given by the contribution of this set of clusters.

Thus, knowledge of the operator $G_{qq'}(\tau,\tau')$, which represents the evolution of a hierarchical defect structure, permits a complete description of the process of plastic deformation generated by this evolution. The rearrangement of the structure itself is viewed as a Markovian diffusion process on the Cayley tree.[42] This means, in particular, that the binary evolution operator gives the distribution over an arbitrary number $n \geqslant 2$ of clusters containing $m \geqslant n$ ensembles:

$$F_{q_1q_2\ldots q_n}(\dot\varepsilon^1, \dot\varepsilon^2, \ldots, \dot\varepsilon^m) = \prod_{\alpha=1}^{m}\int \mathrm{d}\tau_\alpha G_{q_{\alpha-1}q_\alpha}(\tau_{\alpha-1}, \tau_\alpha)F_{q_\alpha}(\dot\varepsilon^\alpha, \tau_\alpha), \tag{4.27}$$

where $q_0 = 0$, $G_{qq}(\tau,\tau') = \delta(\tau - \tau')$, and the parameter q_α specifies the overlap of the states α and $\alpha + 1$ (since some pairs of states can have the same overlap q_α, their number n may be less than the number m of ensembles).

If the defect structure is multilevel, then the discrete ultrametric space corresponding to it (the Cayley tree) can be considered a continuum and the transitions between the levels continuous. Then the evolution operator is given by the Fokker–Planck equation[42]

$$C^{-2}\frac{\partial G}{\partial q} = \frac{1}{2}\frac{\partial^2 G}{\partial\tau^2} + \frac{\partial \ln Z}{\partial\tau}\frac{\partial G}{\partial\tau}, \tag{4.28}$$

where C is the effective modulus of the medium with defects, and the roles of the time and the coordinates are taken by the overlap parameter q and the field τ, while the auxiliary function $Z = Z_q(\tau)$ satisfies the equation

$$C^{-2}\frac{\partial Z}{\partial q} = \frac{1}{2}\frac{\partial^2 Z}{\partial\tau^2} + \frac{N_q(\tau)}{1 - y(q)}Z\ln Z \tag{4.29}$$

with the initial condition $Z_1(\tau) = \tanh(\tau/C)$. However, if the number of structural levels is small, then the transitions between them become discrete, and instead of the differential equation (4.28) we must use the more complicated kinetic equation[50]

$$\frac{\partial G_{q_0q}(\tau_0, \tau)}{\partial q} = \int (G_{q_0q}(\tau_0, \tau')w(\tau', \tau) - G_{q_0q}(\tau_0, \tau)w(\tau, \tau'))\mathrm{d}\tau', \tag{4.30}$$

where $w(\tau,\tau')$ is the probability that the field changes from τ to τ' for a unit change in the overlap q. Thus, to describe the process of plastic deformation for a small number of levels it is necessary to know the probability $w(\tau,\tau')$ of their hierarchical union. To determine $w(\tau,\tau')$ it is necessary to consider specific models (such as the alignment of dislocations into walls, formation of disclinations, etc.).[3-6]

The solution of Eqs. (4.28) to (4.30) gives in principle a complete description of the hierarchical chain of a defect structure. However, questions that are still open are those of the defect of the modulus[1-3] and the dependence of the rate of macrodeformation $\dot{\mathscr{E}}$ on the external conditions. In addition, the phenomenon of structural memory due to effects of nonergodicity require study. It is possible to carry out such a program within the framework of a single approach only on the basis of the microscopic theory presented in Section 4.4, which embraces both the thermodynamic and the kinetic behavior of the defect structure. However, if only the processes of structural relaxation are needed, then it is sufficient to use the phenomenological version of fractal theory, given in the next section.

4.3. Phenomenological theory of relaxation of a defect structure

The analysis of Section 4.2 shows that the space-time behavior of the defect structure under conditions of creep is given by the correlator

$$S(\mathbf{r}, t) = \sum_{\alpha\beta} w_\alpha w_\beta S_{\alpha\beta}(\mathbf{r}, t), \quad S_{\alpha\beta}(\mathbf{r}, t) = \langle \dot{\varepsilon}^\alpha(\mathbf{r}, t)\dot{\varepsilon}^\beta(0, 0)\rangle, \tag{4.31}$$

whose form follows from Eqs. (4.7). The correlator $S_{\alpha\beta}(\mathbf{r},t)$ of pure states is made up of a dynamic component $S^0_{\alpha\beta}(\mathbf{r},t)$, which correspond to processes that occur over a microscopic time $t_0 \sim t_{p,\tau}$, and a relaxation contribution $\tilde{S}_{\alpha\beta}(\mathbf{r},t)$, which has a Debye character

$$S_{\alpha\beta}(\mathbf{r}, t) = S^0(\mathbf{r})\delta_{\alpha\beta}\delta(t) + \tilde{S}_\alpha(\mathbf{r})\exp(-t/t_{\alpha\beta}), \tag{4.32}$$

$$S^0(\mathbf{r}) = \dot{\varepsilon}(\mathbf{r}, t_0)\dot{\varepsilon}(0, 0), \quad \tilde{S}_\alpha(\mathbf{r}) = \langle \dot{\varepsilon}^\alpha(\mathbf{r}, t_0)\dot{\varepsilon}^\alpha(0, t_0)\rangle;$$

where it is taken into account that statistical effects do not appear over microscopic times $t \leqslant t_0$, while at time $t = t_0$ the only correlation is between the quantities $\dot{\varepsilon}^\alpha$ that belong to the same pure state. The time $t_{\alpha\beta}$ required for the union of ensembles α and β is determined by the distance $d = d_{\alpha\beta}$ between the states α and β in ultrametric space. Assuming that this space is homogeneous[4)] and going to a Fourier transformation with respect to the coordinate $\mathbf{r}$, we obtain from Eqs. (4.31) and (4.32)

$$\tilde{S}_\mathbf{k}(t) = \tilde{S}_\mathbf{k}\int w(d)\exp(-t/t(d))\mathrm{d}d, \tag{4.33}$$

$$\tilde{S}_\mathbf{k} = \sum_\alpha w_\alpha \tilde{S}_\alpha(\mathbf{k}), \quad \tilde{S}_\alpha(\mathbf{k}) = \int \mathrm{d}^3\mathbf{r}\tilde{S}_\alpha(\mathbf{k})e^{-i\mathbf{kr}}. \tag{4.34}$$

It follows from definition (4.9) that the correlator (4.31) that is used is the first moment of the distribution function $P(q)$ of the ensemble overlap. The stochastic character of the latter means that the correlator $\tilde{S}_\mathbf{k}(t)$ also has a random character. As in spin glasses,[43] this character is due to the nonregularity of the spatial distribution of defects, which leads to a spread in their energies of interaction (frustration of bonds due to the loss of ergodicity).

To find the explicit form of $\tilde{S}_\mathbf{k}(t)$ in Eq. (4.33) it is necessary to find the distributions $w(d)$ and $t(d)$ over the distances d in ultrametric space (the value of d is given by the number of steps to the common ancestor over the levels of the Cayley tree in Fig. 1b, and gives the degree of the hierarchical connection). A determination of $w(d)$ and $t(d)$ leads to a self-consistent problem, and hereafter we shall restrict ourselves to an investigation of the possible forms of the time dependence $\tilde{S}_\mathbf{k}(t)$ for various attainable majorants of the distributions $w(d)$ and $t(d)$.

The relaxation time is given for a particular temperature T by the height of the potential relief $U(d)$ according to the Arrhenius relation

$$t(d) = t_0\exp(U(d)/T), \tag{4.35}$$

where t_0 is the macroscopic (Debye) time. Since an increase in the degree of the hierarchical connection corresponds to a decrease in the distance d on the one hand, and in the field U on the other, one can conclude that the function $U(d)$ must increase monotonically. Its form is easy to determine in the thermodynamic limit $l_{i+1}/l_i \to \infty$, when the number of structural units that form the defect is large enough. Here, one step over the levels of the Cayley tree, corresponding to a single union of m clusters, increases U by a factor of m. The distance d corresponds to n steps to which are related the change in the potential relief $\Delta U \sim Qm^n \propto \exp(n \ln m) = \exp(\mathrm{const}\cdot d)$. Thus, in the thermodynamic limit we find a very strong exponential behavior of the potential relief $U(d)$ in ultrametric space. For small clusters and a large contribution from elastic stress the value of U increases much more slowly as the volume of the cluster increases, and correspondingly the function $U(d)$ will have a weaker exponential behavior. We shall hereafter approximate it by the functions

$$U_l(d) = Q\ln(d/d_0), \quad U_p(d) = Qd^a, \quad U_e(d) = Q\exp(d/d_0), \tag{4.36}$$

where the constants Q, d_0, and a are determined by the state parameters T and $\hat{\sigma}_{\mathrm{ext}}$.

The distribution function $w(d) \equiv w_\alpha$ over the states α that correspond to different points in ultrametric space specifies the depth of the hierarchical connection of the defect structure. Thus, in the case $w(d) = \delta(d)$ the expansion (4.7) contains only a single state, which corresponds to a single ensemble—there is no hierarchical connection. For a function $w(d)$ that falls off rapidly the degree of this connection is small, but if the function falls off slowly, its role is enhanced. Majorizing the weak hierarchical behavior of the exponential function and the strong power-law function, we have

$$w_w(d) \propto \exp(-d/\Delta), \quad w_s(d) \propto d^{-D}, \tag{4.37}$$

where Δ and D are parameters, the first of which represents the depth of the hierarchical connection, and the second the fractal dimension, between 0 and 1.

Substituting expressions (4.36) and (4.37) into (4.33) and (4.35) we find by the method of steepest descents the form of asymptotic behavior as $t \to \infty$, as shown in Table I. For a smooth power-law distribution $w_s(d)$ in ultrametric space the logarithmic increase of $U_l(d)$ of the height of the fractal relief gives a power-law decay for the correlator (4.33), and a power-law increase in the relief $U_p(d)$ leads to a slower logarithmic variation in $\tilde{S}_\mathbf{k}(t)$, and, finally, an exponential distribution of $U_e(d)$ gives a double logarithmic dependence for $\tilde{S}_\mathbf{k}(t)$. Accordingly, for a rapidly falling exponential distribution $w_w(d)$ we have sequentially the law of Kohlrausch,[51] a quasi-power-law, and a logarithmic falloff. According to Ref. 52, this situation does not depend on the degree of branching of the Cayley tree and persists for random branching. It is a characteristic fact that for power-law and exponential increases in the height of the fractal relief the maximum value of the correlator $\tilde{S}_\mathbf{k}(t) \sim 1$ is maintained at temperatures below

$$T_f(t) \leq Q/\ln(t/t_0). \tag{4.38}$$

It must be kept in mind that the integration over d in Eq. (4.33) is bounded from below by the value d_m given by the condition $U(d) = U_m$ for surmounting the minimum potential barrier U_m (in spin glasses $U_m = T$). In finding the

TABLE I. Possible asymptotic forms of the correlator $\widetilde{S}_k(t)$ as $t \to \infty$.

$\widetilde{S}_k(t)$	$U_l(d)$	$U_p(d)$	$U_e(d)$
$w_w(d)$	$e^{-t^\beta}, \beta = \left(1 + \frac{Q}{T}\right)^{-1}$	$\exp\left[-\left(\frac{T}{Q}\ln\frac{t}{t_0}\right)^{1/\Delta}\right]$	$\left(\frac{T}{Q}\ln\frac{t}{t_0}\right)^{-d_0/\Delta}$
$w_s(d)$	$t^{-\gamma}, \gamma = \frac{DT}{Q}$	$\left(\frac{T}{Q}\ln\frac{t}{t_0}\right)^{-D/\Delta}$	$\left[d_0 \ln\left(\frac{T}{Q}\ln\frac{t}{t_0}\right)\right]^{-D}$

asymptotic behavior of the correlator $\widetilde{S}_k(t)$, indicated in Table I, it is understood that the position d^0 of the maximum of the argument of the exponential function in the integral of (4.33) satisfies the condition $d^0 \gg d_m$. Since the value of $d^0(t)$ increases monotonically with time, according to expressions (4.35)–(4.37), this condition leads to the minimum time t_m at which relaxation effects appear: for $t < t_m$ we have steady-state plastic deformation of magnitude $E = \int S^0(t)\,dt = S^0$, and for $t \gg t_m$ the rate of plastic deformation relaxes according to the asymptotic behavior shown in Table I. An analytic expression for the lower threshold for relaxation is possible only for the logarithmic fractal relief $U_l(d)$:

$$\frac{t_m}{t_0} = A \exp\left(B\frac{U_m}{T}\right), \quad A_w = \frac{d_0}{\Delta}\frac{T}{Q}, \quad B_w = 1 + \frac{T}{Q}, \quad A_s = D\frac{T}{Q}, \quad B_s = 1, \tag{4.39}$$

where the subscripts w and s denote a weakly and strongly hierarchical system. In the transition to a more rapidly increasing distribution of $U(d)$ the dependence of the time t_m on the threshold U_m becomes even stronger. This means that if in a microscopic time macroscopic complexes of defects are formed corresponding to a rapid increase in the fractal relief $U(d)$, then even for relatively small values of the minimum height U_m of the relief the time t_m to establish a hierarchical connection of the defect structure will be very long. This situation is found in the evolution of the martensite macrostructure.[15]

It should also be remembered that the critical slowing of the relaxation of the defect structure appears only up to some maximum time t_M, while for $t \gg t_M$ a Debye dependence $S_k(t) \sim \exp(-t/t_M)$ is obtained. The physical reason for this slowing is that after the time t_M the hierarchical connection is broken at a distance d_M, given by the condition $w(d_M) = t_0/t_M$. As a result, the fractal relief limited by the value $U_M = U(d_M)$ is surmounted. If we take into account the specific forms of $U(d)$ and $w(d)$ (4.36) and (4.37), we find the expression given in Table II for the maximum relaxation time t_M for the structure. It is of interest that, other conditions being equal, the value of t_M increases in going to functions $U(d)$ of lower amplitude. Since an exponential increase in $U_e(d)$ corresponds to an increase in U proportional to the volume, and the transition to $U_p(d)$ and $U_l(d)$ is related to taking into account the inhomogeneity and long-range action, this means there is an increase caused by these factors in the time required for stabilization of the defect structure. On the other hand, as is shown by a comparison of the different columns of Table II, a weakening of the hierarchical chain results in a stronger dependence of the time t_M on the external conditions that determine the parameters Q, d_0, and a of the fractal relief. Assuming that their temperature dependence is of the simplest form, $Q \propto T - T_0$, with T_0, d_0, and a constant, it is easy to see that the function $t_M(T)$ takes the form of the Vogel–Fulcher approximation[44] for a logarithmic distribution of the relief $U_l(d)$ in strongly hierarchical systems, while for a power-law distribution it takes that form for weakly hierarchical systems. In the general case the function $t_M(T)$ can also have a power-law or a logarithmic form.

Proceeding now to an interpretation of these data, we note first of all that in the case considered here the fractal nature of the potential relief in state space is due to the presence of volume and thermal effects of the transformation of the structure as well as to the smallness of the energy of inhomogeneity. In fact, under these conditions the realization of a structural transformation in the minimum supercritical volume leads to a local increase in the energy density, which also indicates the presence of a minimum barrier in the potential relief. The smallness of the inhomogeneity energy promotes a step-by-step growth of the clusters of the defect structure: it is more favorable for them to grow not by a displacement of the boundary, as occurs in ordinary first order phase transformations, but by the establishment of a correlation of the individual clusters. Of course, this process is accompanied by large volume and thermal effects, and,

TABLE II. Stabilization time t_M of a defect structure.

t_M	$U_l(d)$	$U_p(d)$	$U_e(d)$
$w_w(d)$	$t_0 \exp\left(\frac{d_0}{\Delta}\exp\frac{U_M}{Q}\right)$	$t_0 \exp\left[\frac{1}{\Delta}\exp\left(\frac{U_M}{Q}\right)^{1/\Delta}\right]$	$t_0\left(\frac{U_M}{Q}\right)^{d_0/\Delta}$
$w_s(d)$	$t_0 d_0^D \exp\frac{DU_M}{Q}$	$t_0\left(\frac{U_M}{Q}\right)^{D/\Delta}$	$t_0 d_0^D\left(\ln\frac{U_M}{Q}\right)^D$

consequently, corresponds to the filling of the deeper minima of the fractal potential relief.

Regarding the effects of the hierarchical chain, it can be seen from a comparison of the different columns of Tables I and II that, other conditions being equal, the strongly hierarchical systems exhibit slower kinetics than do the more weakly hierarchical systems. For example, practically complete stabilization of the defect structure is observed in strongly hierarchical systems with an exponential increase in the height of the relief $U(d)$. If the latter is achieved by virtue of the macroscopic size of the defect cluster, then the strongly hierarchical dependence is realized in the presence of long-range forces. From this discussion it can be seen that the switching on of structural levels that correspond to an increase in the characteristic scale l_i promotes the enhancement of the hierarchical chain in the behavior of the defect structure, and, consequently promotes its stabilization.

If these conditions are not satisfied, then in time the structure will become reorganized. To find these characteristic times we consider the distribution of the heights of the fractal relief, corresponding to the set of elements that are resolvable in a given experiment. Let this set be given by the sequence $U_1 < U_2 < \ldots < U_n$. Then in a microscopic time $\sim t_0$ a structural element characterized by the largest value U_l that satisfies the condition $U_l \leqslant T$ is athermally formed. It will exist until the time $t_{l+1} = t_0 \exp(U_{l+1}/T)$ when thermal fluctuations form the next element of the defect structure in the hierarchical series. As a result, the time of existence $\Delta t_l = t_{l+1} - t_l$ of this structural element is given by

$$\Delta t_l / t_l = \exp[(U_{l+1} - U_l)/T] - 1. \tag{4.40}$$

For a ratio $U_{l+1}/U_l \approx 50$ and $U_l/T \sim 1$ we find that in the first macroscopic level ($t_l \sim t_0 \sim 10^{-13}$ s) the characteristic time of stabilization of the macroscopic structure, Δt_l, is of the order of several tens of years.

Let us now consider the features of this scheme as applied to the description of creep of solids.[17] Here, under certain conditions (see below) some types of defects (dislocations, vacancies, and others) can contribute to creep independently of one another. In the dependence of the thermodynamic potential on the configuration coordinate (Fig. 6) this is expressed by the presence of minima separated from the initial state i by a high barrier Q_l, $l = 1, 2, \ldots$. Correspondingly, the ultrametric space is formed by a highly nonuniform Cayley tree, whose branching vanishes at distances d_l given by the condition $\Phi(d) = Q_l$. This implies that the probability density $w(d)$ of the distribution of distances in ultrametric space becomes a δ-function singularity

$$w(d) = w_0(d) + \sum_l w_l \delta(d - d_l), \tag{4.41}$$

where $w_0(d)$ is a smooth function corresponding to a uniform distribution of type (4.37). At a given temperature T the only singularities that appear athermally are those that satisfy the condition $Q_l < T$. As a result, the athermal component of the correlator of the defect flux becomes

$$S = S_0 + \sum_l{}' S_l, \quad S_0 = \int_0^{d(T)} w_0(d) S(d) \mathrm{d}d, \quad S_l = w_l S(d), \tag{4.42}$$

where the function $S(d)$ corresponds to $\tilde{S}_\alpha(k)$ in Eqs. (4.34) and the prime on the summation sign means that the summation is carried out for $Q_l < T$ [or, equivalently, $d_l < d(T)$]. The monotonically varying component can be rewritten in the following way

$$S_0 = S_0^{(0)} + \sum_l{}' S_0^{(l)}, \; S_0^{(l)} = \int_{Q_l}^{Q_{l+1}} w_0(d) S(d) \mathrm{d}d. \tag{4.43}$$

The zero-order term $S_0^{(0)}$, which corresponds to the initial minimum i in the thermodynamic potential, vanishes by definition for the fluxes, and the rest of them renormalize the quantities S_l. Assuming that this renormalization has been done, we can discard the term S_0 in (4.42). As a result, the correlator of the defect fluxes takes the final form

$$S(t) = \sum_l{}' S_l + \tilde{S}(t), \tag{4.44}$$

where the constant athermal terms S_l are defined by the previous formula (4.42), and the variable component $\tilde{S}(t)$ is defined by expression (4.33).

A characteristic feature of the athermal terms S_l is that they are switched in sequentially as the temperature increases. For example, for $T < Q_1$ all the terms S_l are zero; in the range $Q_l < T < Q_2$ we have only a single nonzero term S_1, and in the range $Q_l < T < Q_{l+1}$ the nonzero terms are those from S_1 to S_l. The renormalization of the terms S_l by the addition of $S_0^{(l)}$ causes them to depend on the quantities T and $\hat{\sigma}_{\text{ext}}$. Since the thermodynamic potential as a function of the configuration coordinate is smoothed out as the temperature is raised and the stress is reduced (see Figs. 1a and 5), the distribution $w_0(d)$ becomes narrower and the terms $S_0^{(l)}$ increase. It can thus be concluded that the athermal terms S_l increase as the temperature is raised and the stress is reduced.

In the interpretation of the time dependence $S(t)$ of the correlator of the defect fluxes (4.31) one must start from the fact that the corresponding rate of macrodeformation $\dot{\mathscr{E}}(t)$ is determined by the conditional probability that the defect flux will be $j(t) = \dot{\mathscr{E}}(t)$ at time t if at $t = 0$ it is equal to $j(0)$. In other words, we assume that the correlator $S(t)$ and the rate of creep $\dot{\mathscr{E}}(t)$ are identical up to some unimportant multiplicative factor. According to Ref. 45, the latter is given by the sum

$$\dot{\mathscr{E}}(t) = K + \delta \dot{\mathscr{E}}(t), \tag{4.45}$$

where the constant term K characterizes steady-state creep and the decaying part $\delta\dot{\mathscr{E}}(t)$ characterizes the non-steady-state creep. Comparing the definition (4.45) with our main result (4.44) for the rate of steady-state creep, we find

$$K = \sum_l K_l, \quad K_l = S_l \theta(T - Q_l), \tag{4.46}$$

where the quantity $S_l = S_l(T, \hat{\sigma}_{\text{ext}})$ increases monotonically with the temperature and the stress (see chapter IV in Ref. 45), $Q_l = Q_l(\hat{\sigma}_{\text{ext}})$ is the decaying dependence (see Section 4.1), $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. The non-steady-state creep $\delta\dot{\mathscr{E}}(t)$ is represented asymptotically by the functions listed in Table I.

This picture allows us to compare the diagram of Fig. 7 to the various regimes of creep. Steady-state creep, which

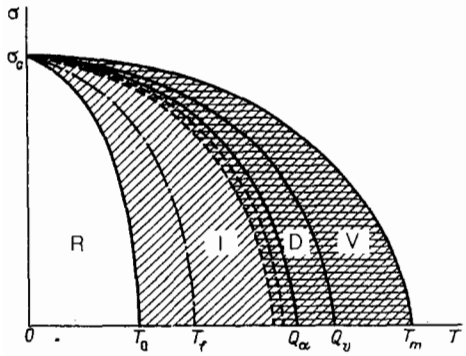

FIG. 7. Creep diagram. The notation R, I, D, and V indicate, respectively, the regions of reversible, irreversible, dislocation, and vacancy creep. The oblique and horizontal hatching denote the regions of non-steady-state and steady-state creep, respectively.

corresponds to athermal surmounting of the barrier Q_l, is realized at high temperatures ($T > Q_1$). Since experimentally[45] the first mechanism to appear is the dislocation mechanism of steady-state creep, and then, near the melting point, the vacancy mechanism comes in, we can conclude[5)] that $Q_d < Q_v$ (see Fig. 5). Moreover, it should not be thought that other mechanisms such as a grain boundary mechanism, the motion of grains as a whole, etc are not involved in steady-state creep. Actually, if the applied stresses are high enough, the most effective creep mechanisms come into play: first, the motion of macroscopic volumes, then conglomerates of grains and their boundaries, disclinations, dislocations, and, finally, point defects.[4–7] Within the framework of our picture, this means that as the temperature is raised the barriers Q_l are surmounted in the indicated sequence; i.e., the barrier is lowest for macrovolumes, the next highest is for conglomerates of grains, then individual grains, and so forth. Correspondingly, in going to the top of the Cayley tree, the branch of the macrovolumes branches out first, then come the branches of the conglomerates of grains, then the individual grains, and so forth. The fact that only two of these have been observed experimentally[45] indicates the large magnitude of their contributions S_l to the steady-state creep: obviously the contribution from macroscopic volumes is less than for conglomerates of grains; correspondingly, the contribution from the conglomerates is less than from the individual grains, and so forth. In other words, in the hierarchy of mechanisms of steady-state creep the following correspondence is observed: the greater the value of Q_l in a sequence of possible barriers the smaller is the corresponding contribution S_l to the rate of creep (the more complex carriers of plastic deformation have to overcome a lower barrier Q_l to become involved in the plastic creep, but they have a low mobility S_l). This dependence has also been verified in a calculation of the value of S_l as applied to a specific mechanism of plastic deformation. As an example, we found the functional dependence $S_1 \propto \sigma^{4.5}$ for the stress dependence for vacancies in the field of "their own" point defects (vacancies); in the case of impurity atoms instead of vacancies we find $S_2 \propto \sigma^3$, and, finally, for a pure vacancy mechanism, $S_3 \propto \sigma$ (Ref. 45). Since these relations represent the first terms in a series in the small parameter σ/μ, we can conclude that for these values of σ we have $S_1 < S_2 < S_3$.

For non-steady-state creep, the functions $\tilde{S}(t)$ in Table I show that the asymptotic behavior $\delta\dot{\mathscr{E}}(t) \propto \tilde{S}(t) \propto t^{-1}$ corresponding to logarithmic creep can be realized only for a logarithmically slow increase in the height of the fractal relief in strongly hierarchical systems, while for a power-law increase in the relief height it can be realized only for weakly hierarchical systems. Evidently a system of point defects is weakly hierarchical: thus, the behavior of a grain as a whole is caused by the behavior of its boundaries, but is almost insensitive to a redistribution of dislocations and point defects, whose action is indirect, through the boundaries.[45] Moreover, it was shown above that an exponential increase in the height of the relief $\Phi_e(d)$ in ultrametric space corresponds to a linear increase in the thermodynamic potential with the volume in real geometric space. Therefore, only when the carrier of the plastic deformation behaves as a thermodynamic phase can one compare the function $\Phi_e(d)$ to it. It is clear that this situation is realized beginning from the grains, while the dislocation and vacancy creep observed in the experiments reported in Ref. 45 are characterized by a power-law function $\Phi_p(d)$. As a result we obtain for the rate of non-steady-state creep

$$\delta\dot{\mathscr{E}}(t) = A \exp\left[-\left(\frac{T}{Q}\ln\frac{t}{t_0}\right)^{1/a}\right], \tag{4.47}$$

where A is a constant that characterizes the height of the relief $Q \equiv Q_1$, and decreases monotonically with an increase in the stress, and the temperature dependence of the quantity a in the exponent is approximated in the form $a - 1 \propto T - T_0$. For temperatures T below T_0, when $a < 1$, the function (4.47) is exponential, and at the point $T = T_0$ it becomes hyperbolic:

$$\delta\dot{\mathscr{E}}(t) = \varepsilon_0 t^{-1}, \quad \varepsilon_0 = A t_0 \exp(-Q/T); \tag{4.48}$$

For temperatures above T_0 we have a quasi-power-law form, $\delta\dot{\mathscr{E}}(t) \sim t^{-\gamma}$ where the exponent γ falls off with the temperature in the interval from 1 to 0.

Thus, the temperature T_0 plays the role of the point at which critical slowing of the creep begins. On the other hand, it was found that above T_0 this slowing is important only up to the time t_M given in Table II. As a result, we obtain the following picture of non-steady-state creep. Up to the temperature T_0 the only mechanisms that are important in creep are those that give an exponentially fast falloff in the rate $\delta\dot{\mathscr{E}}(t)$ (we disregard the difference between the Debye behavior and the Kohlrausch law). Therefore, the value of $T_0(\sigma)$ is an upper limit to the region of reversible creep (Fig. 7). Above T_0 deformation mechanisms become operative that are characterized by an increasing rate of change $\Phi'(d)$ of the fractal relief. Physically, this means that such connected defect complexes that produce a faster increase in the thermodynamic potential than do independent defects become involved in the deformation process. This leads to critical slowing of the rate of creep: exactly at the point $T = T_0$ a slower mechanism comes in prior to the logarithmic behavior of the deformation $\mathscr{E}(t)$ as $T - T_0$ increases. This slowing of the deformation is observable in an experiment as the complete cessation of the deformation at temperatures below the freezing point temperature T_f given by

(4.38). However, the action of these mechanisms is apparent only until the time limited by the time t_M. For $t \gg t_M$ the hierarchical relation in the behavior of the various types of defects is destroyed and the creep is again accelerated.

This picture corresponds to the realization of one of the hierarchical branches, when only the states of the initial minimum i of the thermodynamic potential are activated (Fig. 6). At temperatures $T > Q_1$, in addition to the athermal processes of steady-state creep, thermal fluctuation processes associated with the excitation of states of dislocation complexes, vacancy complexes, and other complexes become important. Since to each of them corresponds its own hierarchical branch on the Cayley tree, and consequently since each has its own law of falloff of the correlator $\tilde{S}(t)$, we obtain a superposition of terms corresponding to the various parameters a, specifically such a situation corresponds to the experimental situation.[45]

In conclusion we note the following important difference between this system of crystal structure defects and a spin glass, by analogy with which we have carried out an investigation of the time dependence of the response to an external mechanical action. This difference lies in the fact that in a spin glass the finest structural units of the hierarchical system are the spins, whose total number is $N_0 \sim 10^{23}$ cm^{-3}, whereas the density of defects is $N \ll N_0$. Since the creep is associated with the evolution of defects, and not of the atoms of the crystal, its behavior is determined by that of the ensemble of defects. However, their contribution to the thermodynamic characteristics is by a factor $N_0/N \gg 1$ smaller than that of the atoms and is essentially unobservable on the background of the atomic contribution. The reason apparently is that the thermal excitation is felt by the entire atomic system of a solid, while the mechanical excitation (more precisely, its plastic component) is felt only by the defects.

4.4. Microscopic theory of a hierarchical defect structure

Let us now introduce a consistent theoretical scheme that presents in a unified way the evolution of the defect structure during plastic deformation. According to the analysis in Section 4.2, the structural states are joined into a hierarchical system of clusters, characterized by the distribution $q(y)$ of overlaps (4.8) over the probabilities (4.12). The function $q(y)$, which is determined by the method of replicas (see Ref. 43), has the form shown in Fig. 8. Its characteristic feature is the presence of a descending part in the formation of a hierarchical connection in the ensemble of defects [in the absence of a hierarchy $q(y) = \text{const} \equiv q_1(\hat{\sigma}_{\text{ext}})$]. The minimum value $q_0 \equiv q(y=0)$, which corresponds to the Edwards–Anderson parameter, determines the degree of stability of the defect structure in the smallest cluster of states. The minimum value of $q_1 \equiv q(y=1)$, which characterizes the stability of the defects structure as a whole, is nonzero only under conditions of external action.[43] This theory on the one hand describes the behavior of the limiting values of q_0 and q_1 as functions of the external conditions and on the other hand gives the time dependence of the parameter (4.8) during structural relaxation.

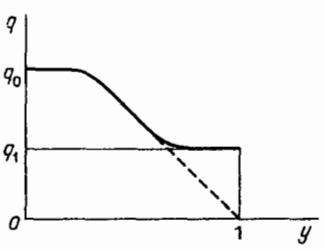

FIG. 8. Dependence of the overlap parameter on the probability of realization of a cluster of states with a given overlap.

However, as already mentioned, the description presented in Section 4.2 is incomplete, since it does not contain the parameters that provide a quantitative description of the nonergodicity of the system. In the regime of creep utilized below, the role of the nonergodicity parameter is taken by the irreversible response

$$\Delta \equiv \lim_{t\to\infty} \delta\gamma(t),$$
$$\delta\gamma(t) = \gamma_1 - \gamma(t), \quad t_0 \ll t < t_M, \tag{4.49}$$

which is defined by analogy with a spin glass[18,43,53,54] as the difference between the dc susceptibility $\gamma_1 \equiv \gamma(\omega=0) = \partial\dot{\mathscr{E}}/\partial\tau \equiv \eta_1^{-1}$, which reduces to the inverse viscosity, and the dynamic susceptibility $\gamma(t)$, which reduces to the time correlator of the rate of plastic deformation[6)] $\dot{\mathscr{E}}(t)$. The fractal structure of the potential relief, studied in Section 4.2, and the partition that it induces in the configuration space of the hierarchical system of clusters are the causes of the difference in these susceptibilities. As does the stability parameter $q(y)$, the irreversible response $\Delta(y)$ falls off slowly with time from the maximum value $\Delta_0 \equiv \Delta(y=0)$, which corresponds to the partition of configuration space into minimal clusters, to the minimum value $\Delta_1 \equiv \Delta(y=1) = 0$, corresponding to the union of all the clusters. The phenomenological theory advanced in Section 4.3 describes the slow relaxation of the stability parameter $q(t)$ of the structure, since the correlator $\tilde{S}(t)$ studied there is the first moment of the distribution $P(q)$, specified by Eq. (4.9). If the fluctuation-dissipation theorem were valid for nonergodic systems,[8] then the function found for $\tilde{S}(t)$ would make it possible to find also the irreversible response (4.49). However, this situation is extremely complicated, and requires a separate investigation.[18,43,53,54]

The effective Hamiltonian of the defect structure in an external field has the form[15,17]

$$\mathscr{H} = \sum_l \tilde{V}(j_l) - \frac{t_\varepsilon}{2} \sum_{l,m} \mathrm{H}_{lm} j_l j_m - t_\varepsilon \sum_l \tau_l j_l, \tag{4.50}$$

$$\tilde{V}(j) = \frac{A}{2} j^2 + \frac{B}{4} j^4. \tag{4.51}$$

The first term in (4.50) is the "intrinsic" contribution to the energy of the plastic deformation of each defect characterized by a flux $\mathbf{j}_l$. It is clear that it reduces to the synergetic potential conjugate to (1.2) and can be approximated by a Landau expansion (4.51). The second term of (4.50) describes the inhomogeneity of the nodal distribution of the defect flux, the third describes the action of the stress fields τ_l, and $j_l = \mathbf{n}\cdot\mathbf{j}_l$ is the component of the flux along the direction of flow $\mathbf{n}$. An important feature of the defect structure is

the nonregularity, which is reflected in the random nature of the parameters $\mathbf{H}_{lm}$, and the field τ_l: their values are distributed randomly over the volume of the sample with the moments

$$\bar{\mathbf{H}} = \overline{\sum_l \mathbf{H}_{lm}}; \quad \mathbf{H}^2 \equiv \overline{\sum_l \mathbf{H}_{lm}^2} = \bar{\mathbf{H}}^2 + \tilde{\mathbf{H}}^2,$$

$$\tilde{\mathbf{H}}^2 \equiv \overline{\sum_l (\mathbf{H}_{lm} - \bar{\mathbf{H}})^2}; \tag{4.52}$$

$$\bar{\tau} \equiv \bar{\tau}_l; \quad \tau^2 \equiv \bar{\tau}^2 = \bar{\tau}^2 + \tilde{\tau}^2, \quad \tilde{\tau}^2 \equiv \overline{(\tau_l - \bar{\tau})^2}, \tag{4.53}$$

where $\bar{\mathbf{H}}$ and $\bar{\tau}$ are the average values of those quantities, $\tilde{\mathbf{H}}$ and $\tilde{\tau}$ are the variances, and the overbar signifies an average over the volume of the sample. For this average to be correct it is necessary that the number of defects N should tend to infinity.

When there is an instantaneous change in the external conditions the evolution of the defect structure proceeds in two stages. At first, in a microscopic time $t_0 \sim t_p, t_\tau$ a specific type of structure is formed, which corresponds to the partition of configuration space into minimal clusters, and then comes a slow relaxation of the structure, associated with the union of the clusters. Assuming the given initial values q_0 and Δ_0 and the final value q_1 of the structure parameters, we first describe the second stage.

As we have mentioned in Section 4.1, an analytic treatment of the problem is possible only for the formation of an infinite number of structural levels. Within the framework of this assumption we begin from the Langenvin equation for the fluxes j_l. Then, following the method of Refs. 18 and 54, we write down the stochastic functional and average it over the spread of values of $\mathbf{H}_{lm}$. Within the framework of mean field theory the standard decoupling is carried out in the effective Hamiltonian and then we obtain the equation of the unique value of $\xi(t) \equiv t_s j(t)$ in the field of the fluctuating forces $f(t)$ and the self-consistent field τ:

$$Ct_\varepsilon \frac{\partial \xi}{\partial t} - \left(\frac{\mathbf{H}}{t_\varepsilon}\right)^2 \int_{-\infty}^{t} G(t-t')\xi(t')\mathrm{d}t' = -\frac{\partial \tilde{V}}{\partial \xi} + \tau(t) + f(t), \tag{4.54}$$

$$\overline{\tau(t)\tau(t')} = (\mathbf{H}/t_\varepsilon)^2 S(t-t'), \quad \overline{f(t)f(t')} = 2C^2 t_\varepsilon \delta(t-t'); \tag{4.55}$$

here C is the characteristic elastic energy (see Section 1) due to the external action, $G(t)$ is the retarded Green's function, which is determined in the usual way,[55] and $S(t) \equiv t_\varepsilon^2 S_l(k=0,t) = \xi_l(t)\xi_l(0)$. The second term on the left hand side of (4.54) is introduced to take into account the memory effects, which lead to the synergetic potential

$$V_1(\xi) = \tilde{V}(\xi) - \frac{\chi}{2}\left(\frac{\mathbf{H}}{t_\varepsilon}\right)^2 \xi^2, \quad \chi \equiv G(\omega = 0) = \frac{1}{C}S(t=0), \tag{4.56}$$

where we have introduced the susceptibility $\chi = \partial\xi/\partial\tau$, which is related to the kinetic coefficients by the equalities $\chi = t_\varepsilon \gamma = t_\varepsilon/\eta$, and reduces to the elastic compliance.

In the nonergodic region the functions $G(t)$, $S(t)$, and $\tau(t)$, acquire singular contributions $\tilde{G}(t)$, $\tilde{S}(t)$, and $\tilde{\tau}(t)$, which vary slowly with time (Table I). On the right hand side of Eq. (4.54) an additional field appears

$$\Theta(t) = \tilde{\tau}(t) + \left(\frac{\mathbf{H}}{t_\varepsilon}\right)^2 \int_{-\infty}^{t} \tilde{G}(t-t')\xi(t')\mathrm{d}t',$$

$$\overline{\tilde{\tau}(t)\tilde{\tau}(t')} = \left(\frac{\mathbf{H}}{t_\varepsilon}\right)^2 \tilde{S}(t-t'). \tag{4.57}$$

In the adiabatic approximation, which corresponds to a field $\Theta(t)$ that changes slowly in comparison to the fluctuations of $\xi(t)$, the probability functional $P\{\xi(t)\} \propto \exp(-\tilde{V}_1[\xi(t),t]/C)$ is determined by the synergetic potential $\tilde{V}_1 = V_1 - \Theta\xi$ for a model with a continuous symmetry and $\tilde{V}_1 = -\Theta\xi$ for a discrete model such as the Ising model. Technically it is more convenient to work with the latter model,[18,53,54] so that is the one we shall use.

Later it will be necessary to average the functional $P\{\xi(t)\}$ over the spread of the stress field τ_l. Since it is a fraction that is being averaged (the denominator is the normalizing constant), we must have recourse to the replica method: the field $\xi(t)$ is replaced by n replicas $\xi_\mu(t)$, with $\mu = 1, 2, ..., n$, and at the end of the calculations we let $n \to 0$ (Ref. 56). As a result the averaged probability functional becomes[54]

$$\bar{P}\{\xi_\mu(t)\} = \exp\left[\frac{1}{2}\left(\frac{\mathbf{H}}{Ct_\varepsilon}\right)^2 \int\int \frac{\mathrm{d}t\mathrm{d}t'}{t_\varepsilon^2} \times \sum_{\mu\nu}(p(t-t') + q(t-t')\delta_{\mu\nu})\xi_\mu(t)\xi_\nu(t')\right], \tag{4.58}$$

where we have introduced two types of overlap parameters of the ensembles

$$p(t-t') = \langle \xi_\mu(t)\xi_\nu(t')\rangle_{\mu\neq\nu} = \overline{\langle\xi(t)\rangle_\xi \langle\xi(t')\rangle_\xi}^{\tau}, \tag{4.59}$$

$$q(t-t') \equiv \frac{Ct_\varepsilon}{2}(\tilde{G}(t-t') + \tilde{G}(t'-t)) = \langle\xi_\mu(t)\xi_\mu(t')\rangle - \langle\xi_\mu(t)\xi_\nu(t')\rangle_{\mu\neq\nu} = \overline{\langle\xi(t)\xi(t')\rangle_\xi - \langle\xi(t)\rangle_\xi\langle\xi(t')\rangle_\xi}^{\tau}, \tag{4.60}$$

where the subscripts on the averaging signs indicate, over which quantity the average is taken. According to the last expressions in formulas (4.59) and (4.60) the parameter $p(t-t')$ characterizes the time correlation of the quantities $\xi(t)$ and $\xi(t')$ taking into account only the spread in the internal field τ_l, and the parameter $q(t-t')$ does so taking into account the spread of the field τ_l and the fluxes ξ_l themselves.

The direct realization of a procedure for averaging with the functional (4.58) is a complicated matter, and is achieved with the use of the concept of the hierarchical structure of defects, discussed in Section 4.2. In particular, the ultrametric topology of the time space allows us to reduce expression (4.58) to the functional integral

$$\bar{P}\{\xi_\mu(t)\} = \int \mathrm{D}\tau(t) F\{\tau(t)\} \exp\left(\frac{1}{C}\int \frac{\mathrm{d}t}{t_\varepsilon} \sum_\mu \tau(t)\xi_\mu(t)\right), \tag{4.61}$$

defined by the distribution (4.15), which in turn is given by Eq. (4.24), which contains the evolution operator $G_{qq'}(\tau,\tau')$ of the hierarchical system of defects in ultrametric space. As for the case of the evolution operator, the parameter q plays the role of the time, and for the functional $F\{\tau(t)\}\equiv F_q(\xi,\tau)$ it reduces to the singular time $z=z(q)=z(y)$, whose metric makes the ordinary time space t ultrametric.[18,54] Therefore, the distribution $F_z(t)$ obeys the Fokker-Planck equation, which follows from Eq. (4.28) with $Z=Z_0\exp[-(\mathbf{H}\Delta'(z)/Cq'(z))\tau\xi]$

$$\frac{\partial F}{\partial z}+\nabla J(F)=0,\quad J(F)=-\mathrm{H}\Delta'(z)F\xi-\frac{q'(z)}{2}\nabla F,\quad \nabla=\frac{\mathrm{H}}{t_\varepsilon}\frac{\partial}{\partial\tau}. \tag{4.62}$$

In turn, the average defect flux $\xi=\xi(z,\tau)$ also obeys an equation of continuity of the type (4.62) with the generalized flux

$$J(\xi)=-\frac{1}{2}\mathrm{H}\Delta'(z)\xi^2+\frac{q'(z)}{2}\nabla\xi \tag{4.63}$$

and the initial condition $\xi(z=0,\tau)=\tanh(\tau/C)$. The functions $\Delta'(z)$ and $q'(z)$ represent the density of the distributions

$$\Delta(y)=-\int_{1-y}^{\infty}\Delta'(z)\mathrm{d}z,\quad q(y)=\int_0^{1-y}q'(z)\mathrm{d}z, \tag{4.64}$$

where y is the probability (4.12). These functions are determined by the equations $D(y)\Delta'(y)=0$ and $D(y)q'(y)=0$ with the kernel[18,54]

$$D(y)=1-\left(\frac{\mathrm{H}}{t_\varepsilon}\right)^2\left[1+18\left(\frac{t_\varepsilon B}{\mathrm{H}}\right)^2 q^2(y)\right](t_\varepsilon\gamma+\Delta(y))^2,$$

$$t_\varepsilon\gamma\equiv\chi(\omega\to 0)=G(\omega=0). \tag{4.65}$$

The condition $D(y)=0$ specifies the point of loss of ergodicity. The resulting equation determines only the relation between the quantities $q(y)$ and $\Delta(y)$, but does not specify them if $q'(y)\neq 0$ or $\Delta'(y)\neq 0$. The internal symmetry corresponding to this arbitrariness is broken by specifying a particular relation between the densities $q'(y)$ and $\Delta'(y)$. Thus, the condition $\mathbf{H}t_\varepsilon^{-1}\Delta'(y)=-q'(y)$ would correspond to the fulfillment of the fluctuation-dissipation theorem, which is violated by the loss of ergodicity. In the present case the relation between the quantities $\Delta'(y)$ and $q'(y)$ is given by the solution of Eq. (4.29). However, it is more convenient in practice to specify this relation by assuming that the probability $0\leqslant y\leqslant 1$ and the singular time $0\leqslant z<\infty$ are related by a specified monotonically increasing function

$$y=y(z), \tag{4.66}$$

taken, e.g., in the form $y=\tanh z$ (Refs. 18, 54). This equality provides a connection between the fractal theory presented in Section 4.2 and the time approach in the present section.

To complete the exposition of this time approach, we must find the time dependence of the singular functions $\widetilde{S}(t)$ and $\widetilde{G}(t)$. In essence, the first of these is defined in Section 4.3 (see Table I), but the ultrametric space is assumed to be homogeneous. In the presence of a field this is not the case, since the function $q(y)$ that characterizes this homogeneity is not linear, but falls off monotonically and saturates near the points $y=0$ and 1 (Fig. 8).[43] This means that the structure of ultrametric space is such that the smallest distances $\sim d_{0;1}$, corresponding to the thermodynamic quantities $q_{0;1}$ are realized with the highest probability. Therefore, we must elucidate the results obtained in Section 4.3.

For this purpose we take the Fourier transforms $\widetilde{S}(\omega)$ and $\widetilde{G}(\omega)$. Since the stabilization of each element of the defect structure shows up as the appearance of a central peak $\propto\delta_l(\omega)=\lim_{\Gamma_l\to 0}\pi^{-1}\Gamma_l(\omega^2+\Gamma_l^2)^{-1}$ in the structure factor, then following Ginzburg,[18] we represent the singular functions as series in the generalized functions

$$\widetilde{S}(\omega)=\sum_{l=0}^{n}\frac{2q_l'\Gamma_l}{\omega^2+\Gamma_l^2},\quad \widetilde{G}(\omega)=\sum_{l=0}^{n}\frac{-i\Delta_l'\Gamma_l}{\omega+i\Gamma_l}, \tag{4.67}$$

where the summation is taken over the n levels of the hierarchy l for which the characteristic relaxation times are $\Gamma_l^{-1}\to\infty$, with $\Gamma_l/\Gamma_{l+1}\to 0$. To go over to a continuous spectrum we let the total number of levels n go to infinity; then $l/n=1-y$, $\Delta_l'\to\Delta'(y)$, and $q_l'\to q'(y)$. Going over to integrals in Eqs. (4.67) we use the method of steepest descents to obtain[18]

$$\widetilde{S}(\omega)=-\frac{\pi\delta}{|\omega|}q'(x),\quad \widetilde{G}(\omega)=\Delta(x)+i\frac{\pi\delta}{2}\Delta'(x); \tag{4.68}$$

$$\widetilde{S}(t)=q(z),\quad \widetilde{G}(t)=\frac{\delta}{t}\Delta'(z)\Theta(t), \tag{4.69}$$

where $x=-\delta\ln(|\omega|t_0)$, and $z=\delta\ln(t/t_0)$ if we take the distribution to be $\Gamma(y)=t_0^{-1}\exp[-(1-y)/\delta]$, $\delta\ll 1$. For other admissible approximations for the function $\Gamma(y)$ we again obtain the functions (4.68) and (4.69) with the difference that $x=\delta(|\omega|t_0)^{-a}$, $z=\delta(t/t_0)^a$, $a>0$ or $x=\delta|\ln|\ln(|\omega|t_0)||$, $z=\delta\ln|\ln(t/t_0)|$, as in the results obtained in Section 4.3. The time dependences (4.69) tend to the asymptotes listed in Table I if we identify the distribution $w(d)$ with $\Delta'(y)$ and majorize them as in Section 4.3. However, in the framework of the self-consistent approach the slow evolution of the parameters q and Δ is given by the equations[54]

$$q(z)=\int F(z,\tau)\xi^2(z,\tau)\mathrm{d}\tau, \tag{4.70}$$

$$\Delta(z)=(q_0-1)+\int F(z,\tau)\nabla\xi(z,\tau)\mathrm{d}\tau, \tag{4.71}$$

where $q_0=q(z=0)=q(y=0)$.

Expressions (4.61)–(4.63), (4.70), and (4.71) form a self-consistent system of equations for finding the distributions $\overline{P}\{\xi_\mu(t)\}$, $F(z,\tau)$, $\xi(z,\tau)$, $q(z)$, and $\Delta(z)$. We shall discover the nature of the solutions starting from the ultrametric topology of the time space: during the slow relaxation of the defect structure the parameter q, which characterizes its microscopic stability, falls off from its initial value $q_0=q(y=0)$ to the thermodynamic equilibrium value $q_1\equiv q(y=1)$. Correspondingly, the irreversible response Δ, which characterizes the nonergodicity, falls off in the interval from $\Delta_0=\Delta(y=0)$ to $\Delta_1\equiv\Delta(y=1)=0$ (Fig. 9).

To determine these parameters q_0, q_1, and Δ_0, we must

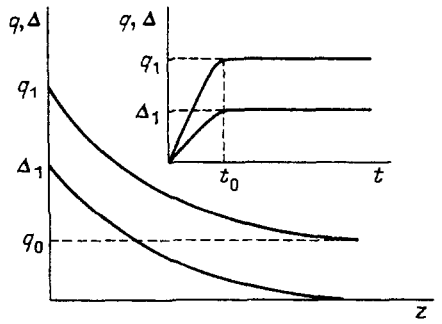

FIG. 9. Time dependence of the overlap parameter q and the nonergodicity Δ on the scales of the singular time z and the ordinary time t.

find, in addition to the susceptibility $\gamma = \gamma_1 - \Delta$, either the thermodynamic potential, and minimize it with respect to q and Δ, or, using a diagrammatic technique, obtain the equations for the Green's functions $S(\omega) + \tilde{S}(\omega)$ and $G(\omega) + \tilde{G}(\omega) \equiv \chi(\omega) = t_\varepsilon \gamma(\omega)$ (in the former case we must have recourse to the method of replicas,[56] and in the latter case we use the supersymmetry approach[57]). For arbitrary values of the probability y (and consequently, for arbitrary values of the parameters γ, Δ, and q, which depend on those probabilities), the equations derived have the following form[18,53]

$$\left\{1 - \left(\frac{\mathrm{H}}{t_\varepsilon}\right)^2 \left[1 + 6\left(\frac{t_\varepsilon B}{\mathrm{H}}\right)^2 q_1^2\right](t_\varepsilon\gamma + \Delta)^2\right\} q_1 = \tau^2 (t_\varepsilon \gamma + \Delta)^2, \tag{4.72}$$

$$\left[\frac{A}{C} + 3\frac{B}{C}(Ct_\varepsilon\gamma + q) - 6\left(\frac{B}{C}\right)^2 ((Ct_\varepsilon\gamma + q)^3 - q^3)\right] Ct_\varepsilon\gamma - (\mathrm{H}\gamma)^2 = 1. \tag{4.73}$$

The complete system of equations is closed by the condition of loss of ergodicity $D(y) = 0$ at $y = 0, y = 1$, where $D(y)$ is defined by Eq. (4.65). This system of nonlinear equations can be solved in analytic form only in the limit of small values of the parameter $\zeta = (2\mathrm{H}/t_\varepsilon A)^2 - 1$. As in the usual model of transitions of the displacement type,[34] the condition $0 < \zeta \ll 1$ means that the long-range elastic forces between the elements of the structure exceed only slightly the corresponding intrinsic contribution and almost cancel it. Then in the first nonvanishing order in $\zeta \ll 1$, $\Theta \equiv C/C_\mathrm{k} - 1 \ll 1$ we obtain[53]

$$C_\mathrm{k} = \frac{\zeta}{6}\frac{A}{B}\frac{\mathrm{H}}{\eta}\mu, \quad \tau_\mathrm{c} = \frac{2}{3}B\left(\frac{A}{2}\frac{\zeta}{B}|\Theta|\right)^{3/2}, \tag{4.74}$$

$$\zeta = \left(\frac{2\mathrm{H}}{At_\varepsilon}\right)^2 - 1, \quad \Theta \equiv \frac{C}{C_\mathrm{k}} - 1, \tag{4.75}$$

$$q_0 = \frac{A\zeta}{6B}|\Theta|, \quad q_1 = 12^{-1/3}\left(\frac{\tau}{B}\right)^{2/3}, \tag{4.76}$$

$$\gamma \equiv \eta^{-1} = \frac{1}{\mathrm{H}}\left[1 - 9\left(\frac{B\eta}{\mu\mathrm{H}}\right)^2\right] q^2,$$

$$\Delta = \frac{9B^2\eta^3}{\mu^3\mathrm{H}^3}\left[1 - \left(\frac{\tau}{\tau_\mathrm{c}}\right)^{2/3}\right] q^2. \tag{4.77}$$

It can be seen from (4.77) that in the absence of a field $\Delta \propto |\Theta|^2$ and for $\tau \neq 0$ we have $\Delta \propto |\Theta|$. Thus, application of the external load results in a more intense increase in the nonergodicity below the critical degree of excitation, C, even though an increase of the quantity $\tau \neq 0$ itself decreases the value of Δ (Fig. 10). This behavior has no analog in ordinary phase transformations. There is also a difference in the form of the phase diagram (Fig. 10): while in our case the curve of the critical field $\tau_c(C)$ as a function of the degree of excitation is concave, for equilibrium transformations it is convex. The functions (4.76) have the usual shape, if we take the quantity $q^{1/2}$ to be the parameter of the transformation. Normalizing its maximum value according to the condition $q_0(C = 0) = 1$, we find the relation $A\zeta/6B = 1$. Here the equations (4.74)–(4.77) take a simple form, which reflects the unimodal nature of the system:

$$\frac{C_\mathrm{c}}{C_\mathrm{k}} = 1 - \left(\frac{\tau}{\tau_\mathrm{k}}\right)^{2/3}, \quad C_\mathrm{k} \equiv \frac{\mathrm{H}}{\eta}\mu, \quad \frac{\tau_\mathrm{c}}{\tau_\mathrm{k}} = |\Theta|^{3/2}, \quad \tau_\mathrm{k} \equiv 2\sqrt{3}B, \tag{4.78}$$

$$q_0 = |\Theta|, \quad q_1 = (\tau/\tau_\mathrm{k})^{2/3}, \tag{4.79}$$

$$\frac{\gamma}{\gamma_\mathrm{k}} \equiv \frac{\mathrm{H}}{\eta} = 1 - \frac{\Delta_\mathrm{k}}{\gamma_\mathrm{k}}q^2, \quad \frac{\Delta}{\Delta_\mathrm{k}} = \left[1 - |\Theta|^{-1}\left(\frac{\tau}{\tau_\mathrm{k}}\right)^{2/3}\right] q^2,$$

$$\gamma_\mathrm{k} \equiv \frac{1}{\mathrm{H}}, \quad \Delta_\mathrm{k} \equiv \frac{3}{4}\left(\frac{\tau_\mathrm{k}}{C_\mathrm{k}}\right)^2 \frac{1}{\mathrm{H}}; \tag{4.80}$$

where we have introduced the scales C_k, τ_k, γ_k, and Δ_k, of measurement of the degree of excitation, the field, the susceptibility, and the irreversible response [as one might expect, under the conditions $\xi \ll 1$ of applicability of these expressions the two last scales are quite different: $\Delta_k/\gamma_k = (3B\eta/\mathrm{H}\mu)^2 \approx \zeta^2 \ll 1$—the defect of the modulus is much smaller than the modulus itself]. It is characteristic

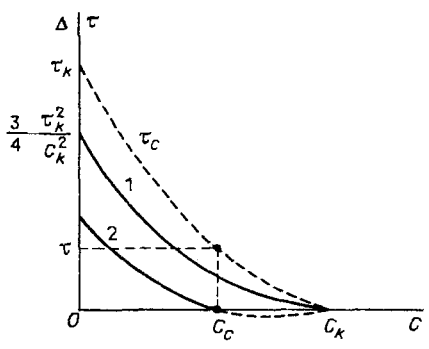

FIG. 10. Irreversible response in the absence of a field (curve *1*) and for $\tau \neq 0$ (curve *2*) as functions of the degree of excitation of the system. The dashed line shows the curve of the critical field that separates the ergodic region from the nonergodic region.

that a field $\tau \neq 0$ leads to a nonzero finite value of q_1, the stability parameter of the macrostructure over the entire range of excitation C. An increase in the stability q in turn increases the viscosity $h = \gamma^{-1}$ and the degree of nonergodicity Δ.

A characteristic feature of this system is that the critical excitation $C_k = (H/\eta)/\mu$ depends on the viscosity of the medium, η, which in turn is determined by the quantities C and τ. Thus, strictly speaking, the system of equations (4.78)–(4.80) gives only an implicit expression for the functions $q(C,\tau)$, $\eta(C,\tau)$, and $\Delta(C,\tau)$ in terms of the phenomenological parameters H, B, and μ. However, since $(\Delta_k/\gamma_k)q^2 \leqslant \Delta_k/\gamma_k \approx \zeta^2 \ll 1$ according to Eqs. (4.80) the effective viscosity η is essentially the same as the characteristic value of H, while the critical excitation C_k reduces to the shear modulus μ. The degree of excitation $\Theta = C/C_k - 1 \approx (C/\mu) - 1$ gives the difference between the stored energy C per unit volume and the elastic limit of μ. An increase in this difference, according to (4.79) and (4.80) results in a decrease in the effective viscosity η, whereas its field τ increases. Finally, the structural relaxation, which leads to a slow falloff in the parameter q, increases the value of η, i.e., it promotes plastic flow.

These effects of nonergodicity should show up experimentally as effects of structural memory. For example, if for a fixed excitation the field τ is turned on at time t_τ, then for the rate of plastic deformation we obtain[18]

$$\dot{\mathscr{E}}(t - t_\tau) = \eta^{-1}(1 + \mu\Delta)\tau, \tag{4.81}$$

$$\partial\dot{\mathscr{E}}/\partial z_\tau = (\mu\Delta'/\eta)\tau, \tag{4.82}$$

where $z_\tau \equiv z(t - t_\tau)$ is the singular time defined after Eqs. (4.69) and $\Delta'(y)$ is the distribution density of the nonergodicity parameter, given by (4.64) and reduces to the probability w_α of realizing a statistical ensemble α [see Eqs. (4.7), (4.9) and (4.11)]. Using relation (4.82) and data on the structural relaxation, we can determine directly the distribution $\Delta'(y)$ and thereby the probability w_α. Formula (4.81) is a direct reflection of the nonergodicity of the defect structure. Actually, if the field τ is turned on with strong excitation ($C \geqslant C_c$), then the nonergodicity does not appear, and Eq. (4.81) takes on the trivial form $(\dot{\mathscr{E}}) = \tau/\eta$. If the field is turned on when $C < C_c$, then in a time t_ε the dynamic value $\dot{\mathscr{E}} = \tau/\eta$ is established, and then the slow structural relaxation brings about an increase in $\varepsilon(z_\tau) = \Delta(z_\tau)\tau$ (Fig. 11).

If a field, oscillating with a frequency $\omega \gg (t - t_\tau)^{-1}$ is turned on, the viscosity acquires the anomalous additional term

$$\frac{\partial\tilde{\eta}(\omega, t)}{\eta} = \frac{3B}{\mu^3}(1 + \mu\Delta(x(\omega)))^2(1 + \mu\Delta(z_\tau))^2\tau^2, \tag{4.83}$$

which shows a slow falloff during the structural relaxation.

It can also be shown that if a field τ is turned on first at time t_1 and then at $t = t_2$, the dependence of the rate of plastic deformation on the singular time z has a kink at the point $z = z_{21}$—the defect structure "remembers" the length of the interval $z_{21} = z_2 - z_1$ between the first and second times the field was turned on. It is characteristic that this kink is observed only on the scale of the singular (e.g., logarithmic) time.

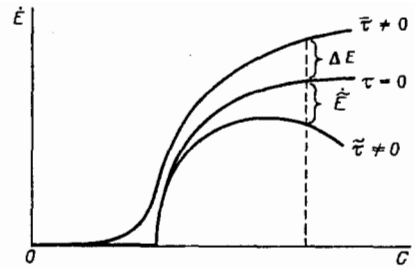

FIG. 11. The dependence of the rate of plastic deformation on the degree of thermal excitation C (the different curves correspond to a total absence of a field ($\tau = 0$) and to nonzero values of its average value $\bar{\tau}$ and variance $\tilde{\tau}$).

CONCLUSIONS

The present investigation reflects the paradoxical situation that has arisen in the physics of plastic deformation. It can be expressed by the widely-held opinion (especially among theorists) that the physics of plastic deformation as such does not exist at all; it is only the mechanics of deformation on the one hand, and the theory of various kinds of defects, on the other that exist. We have tried to follow a scheme that would unite these two poles. It has turned out that to pay for this scheme we have had to use such entities as soliton-like formation of a strongly nonequilibrium crystalline medium, stratified spaces with dilatation, curvature, torsion, and shear, and hierarchical chain structures. The theoretical apparatus that represents these entities requires the introduction of the ideas of stochastic tunable potential relief and gauge fields, which, depending on the situation, can represent both a strong interaction and the defects themselves; the hierarchical structures are described by a set of statistical ensembles, each of which is imaged by a point in ultrametric space.

A characteristic feature of defect structures is that by comparison with spin-glass type systems the number of structural levels is limited and an analytic description, as a rule, is not possible. Numerical schemes must be used in which the values of the hydrodynamic quantities of the upper structural level (for example, the stress field) play the role of boundary conditions for the ensemble of structural units at the lower level. By solving their equations of motion numerically (see Sections 2 and 3), one can find boundary values that provide a specific behavior, and in this way connect two nearest levels. Then the scheme is repeated at a rougher scale level, where the previously found boundary parameters now play the role of the field variables of the structural units of the new level. By repeating this scheme sequentially for each pair of adjacent levels,[58] one can, from the motion of the defects at the microscopic level, determine the change in shape of the entire sample that is, we can completely realize our program.

Of course, all this discussion relates to the arena of the theorist and the calculator, and the experimentalist might well ask what is the use of all this activity. Before answering this question, we should once again point out the extreme complexity of the entire physical picture of the outwardly apparently simple phenomenon of plastic deformation. Ordinarily in an experiment the goal is to abstract from the other mechanisms a particular mechanism and study it by

itself. We, however, have started from another premise—to represent the entire variety of these mechanisms. In this way we have been able, for example, to explain the mode of behavior of the creep of a solid,[17] the fatigue strength of materials,[59] and so forth. Recently it has been discovered that the formation of hierarchical defect structures is observed not only during developed plastic deformation, but also in phase transformations. For example, it turns out that only by taking into account this circumstance can one understand the regularities of the kinetics of the hydrogenation and degasification of palladium,[60] the structural and magnetic relaxation of high-temperature superconducting oxides,[61] the behavior of the macrostructure of alloys, shape memory effects,[15] etc. We suggest that in most experiments related to structural investigations, the "dirt" that experimenters fight against, so as to isolate in its pure form the object or mechanism of interest, is related to the appearance of a more or less developed hierarchical structure. If this is so, then the time has come not to "wash out" this dirt, but to study it (which, by the way, is already being done[6,7,62]).

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[1)] Sometimes within the framework of the last structural level correlated conglomerates of grains etc. are distinguished.[5,6,10] It is also common to distinguish structural levels connected with defect densities (for example, in Ref. 7 the lower, atomic level is determined by the density of dislocations. In Section 2 it will be shown that the latter is, apparently unnecessary.

[2)] The "Fermi-like" properties of the structural units are that a structural unit obeying the exclusion principle can be found in only one of these states. The macroscopic character of the structural units plays no role. An analogous situation occurs in all classical models of the Ising type.[22,34]

[3)] We note that the stresses Ω, $\hat{R}$, H, F, and $\hat{G}$ are 2-forms, and the currents J, Σ, and $\hat{J}$ are 3-forms.[20]

[4)] It is clear that a homogeneous ultrametric space corresponds to a Cayley tree with constant branching and nodal connection only between adjacent levels.

[5)] To avoid confusion, we note that the barriers Q_l characterize the contribution of the defects to the plastic deformation and not the characteristic energy of the defects themselves. Therefore, the magnitude of the barriers Q_l is smaller, the larger is the scale of the defect, that is, the larger its contribution to the deformation of the sample.

[6)] Strictly speaking, in this case, where the rate of plastic deformation plays the role of the order parameter $\mathscr{E}$, the susceptibility reduces to the kinetic coefficient γ. The quantity γ_1 is its steady-state value, and $\gamma(t) \equiv \eta^{-1}(t)$ is the reciprocal of the dynamic viscosity, relaxing to the value $D_1 \equiv \gamma(t=\infty) = C\gamma_1 \equiv C/\eta_1$, where C is the characteristic elastic energy introduced in Section 1. Under conditions of active loading ($\mathscr{E}$ = const), where the order parameter is the magnitude of the plastic deformation $\mathscr{E}$, and the susceptibility reduces to the elastic compliance, the measure of the nonergodicity is the defect of the modulus, which appears in studies of internal friction.[2,3]

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