Law of conservation of energy for the electromagnetic field as applied to radiation by moving charged particles

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If a charged particle radiates electromagnetic waves, then reaction forces due to the radiated field act on the particle. In many problems in the theory of radiation (for example, in the case of Cerenkov radiation, synchrotron radiation, and undulator radiation) the radiation losses of a particle are equal to the work performed by the retardation forces. In the case of Cerenkov radiation this equality holds on any path segment, while in the case of synchrotron and undulator radiation the equality holds for the time-average of the corresponding quantities over a sufficiently long time interval (for example, over one period of revolution): The average work performed by the retardation force is equal to the average radiated energy.

In the general case, however, such an equality does not hold. In the present paper we examine a more general relation that follows from the law of conservation of electromagnetic field energy. This relation relates the radiated energy and the work performed by the retardation forces.

For simplicity, we examine below a field in empty space. At the end of the paper we examine the changes introduced by the refractive properties of a medium.

Suppose that we have found the solution of the Maxwell's equations for a prescribed current density $\mathbf{j}(\mathbf{r},t)$ and charge density $\rho(\mathbf{r},t)$; i.e., we have determined the electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{H}(\mathbf{r},t)$. We choose in space a volume V, which is, generally speaking, arbitrary within wide limits and is bounded by a closed surface S. Then Maxwell's equations yield the following relation for the fields $\mathbf{E}$ and $\mathbf{H}$:

$$\frac{\partial}{\partial t}\int_V \frac{E^2+H^2}{8\pi}\mathrm{d}V = -\int_V \mathbf{jE}\,\mathrm{d}V - \frac{c}{4\pi}\int_S [\mathbf{EH}]\mathrm{d}\mathbf{S}, \tag{1}$$

where $[\mathbf{EH}]$ means the vector (cross) product. Here the symbol V in the integrands indicates that the integral extends over the chosen volume V and the symbol S indicates that the integral extends over the surface S bounding this volume.

The relation (1) is called the law of conservation of electromagnetic field energy. The quantity

$$w = \frac{E^2+H^2}{8\pi} \tag{2}$$

is called the energy density of the electromagnetic field. The integral of this quantity

$$W = \int_V w\,\mathrm{d}V = \int_V \frac{E^2+H^2}{8\pi}\mathrm{d}V, \tag{3}$$

over the volume V, gives the total electromagnetic field energy in the volume V. Thus the left-hand side of Eq. (1) is the rate of change of the electromagnetic field energy in the volume V (or the change in this energy per unit time). This change is produced by the work performed by the electric field $\mathbf{E}$ on the currents $\mathbf{j}$ in the volume V (the first term on the right-hand side of Eq. (1)) and by the flow of energy through the boundary of the volume V, i.e., through the surface S (the second term on the right-hand side of Eq. (1)). The scalar product $\mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)$ is the work performed by the field $\mathbf{E}(\mathbf{r},t)$ on the current $\mathbf{j}(\mathbf{r},t)$ per unit time per unit volume surrounding the point $\mathbf{r}$. Correspondingly, the work performed by the electric field on the currents in the element of volume $\mathrm{d}V$ per unit time is

$$\mathrm{d}a = \mathbf{jE}\,\mathrm{d}V. \tag{4}$$

The second term on the left-hand side of Eq. (1) describes the flow of electromagnetic energy through the surface S bounding the volume V.

We introduce the vector

$$\mathcal{P} = \frac{c}{4\pi}[\mathbf{EH}]. \tag{5}$$

This vector is called the Poynting vector, and it determines the flux of electromagnetic energy through the surface S bounding the volume V. The energy flux through the surface S is written in the form

$$\Pi = \int_S \mathcal{P}\mathrm{d}\mathbf{S} = \frac{c}{4\pi}\int_S [\mathbf{EH}]\mathrm{d}\mathbf{S}. \tag{6}$$

We underscore the fact that the law of conservation of electromagnetic field energy (1) contains the integral of the Poynting vector (5) over the entire surface bounding the volume V. It would be incorrect to identify the expression

$$\mathrm{d}\Pi = \mathcal{P}\mathrm{d}\mathbf{S} \tag{7}$$

as the electromagnetic energy flux through the surface element $\mathrm{d}\mathbf{S}$. Indeed, the conservation law (1) is written in integral form and is obtained from the corresponding local relation

$$\frac{\partial}{\partial t}\frac{E^2+H^2}{8\pi} = -\mathbf{jE} - \frac{c}{4\pi}\mathrm{div}[\mathbf{EH}], \tag{8}$$

which can be considered to be the law of conservation of electromagnetic field energy at a point. The left-hand side of Eq. (8) is the rate of change of the energy density at a given point of the field, and this quantity consists of the work performed by the electric field on the current at the given point and the divergence of the Poynting vector at this point. Thus, the change in the electromagnetic energy density depends not on the Poynting vector itself, but rather on its

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divergence. This means that the solenoidal part of the Poynting vector does not contribute to the electromagnetic energy flux, because the divergence of this part is zero. Correspondingly, the integral over a closed surface (flux) of the solenoidal part of the Poynting vector is equal to zero. But the flux of the solenoidal part of the Poynting vector through an element dS of a closed surface can be different from zero. In this case, the corresponding part of the flux through the surface element dS does not give any real radiation. In the present section we examine the flux of the Poynting vector through the entire surface bounding a prescribed volume, so that the question of the local flux electromagnetic field energy does not arise.

The law of conservation of energy (1) relates the integrated quantities of energy which determine the exchange of energy between the field and the sources. These quantities are the total electromagnetic field energy, the radiation energy losses of the source, and the work performed by the field on the source. Even the form of the conservation law (1) implies that the radiation energy flux (second term on the right-hand side of Eq. (1)) does not equal only the work performed by the field on the source, as one might think *a priori*. Indeed, one would think that if the source radiates electromagnetic fields, then the radiated field exerts reaction forces on the source, and the work of these reaction forces is equal to exactly the radiated energy (with opposite sign). The law (1) shows that this is not the case. It would be the case, if the total electromagnetic field energy remained constant, i.e., if the left-hand side of Eq. (1) were equal to zero. The change in the total field energy must be included in the radiation balance.

We now examine some consequences of the law of conservation of energy (1) in application to the field of a moving charge.

Consider a charged particle moving uniformly in empty space. It is well known that a charge moving with constant velocity in empty space does not radiate. The field of a uniformly moving charge is transported in space with the same velocity as the charge that is the source of the field. The proper field of the charge does not act in the way on the moving charge: It does not retard, deflect, or accelerate the charge. Therefore the field does not perform work on the charge, i.e., in the conservation law (1) $\mathbf{j}\cdot\mathbf{E} = 0$ and the first term on the right-hand side of Eq. (1) is absent. This is evident at least from the fact that in the coordinate system in which the charge is at rest the field of the charge is spherically symmetric and does not accelerate the charge. Then, in accordance with the principle of relativity, the field will also not act on a uniformly moving charge.

We construct a plane through the origin of coordinates and perpendicular to the velocity of the charge. Let the charge move along z-axis in the positive direction. Then the chosen plane is also the (x,y) plane. Consider the half-space $z > 0$. Let it be the volume V. Obviously, the chosen plane can be taken as the surface S bounding the volume V.

We now choose two times t_1 and t_2 as follows. At the time t_1 the charge is located outside the volume V and far away from the boundary S. In other words, the time t_1 lies far in the past. The charge moves towards the surface S, but this charge is located so far away from S that the entire field of the charge is concentrated outside the volume V. Strictly speaking, the field of the charge is also different from zero in the volume V, but the charge is still located at a large distance from the volume V and the field of the charge in the V is so small that it can be neglected.

We choose the time t_2 so that by this time the charge has already crossed the boundary S and is now located far away from it. At the time t_2 the charge is already located in the volume V and, in addition, so far away from the boundary S that the field of the charge outside the volume V is negligibly small.

We now integrate the conservation law (1) over time from t_1 to t_2. As we have already stated, in the case of a charge in uniform motion $\mathbf{j}\cdot\mathbf{E} = 0$, and we obtain

$$\int_V \frac{E^2+H^2}{8\pi}\mathrm{d}V\bigg|_{t=t_2} - \int_V \frac{E^2+H^2}{8\pi}\mathrm{d}V\bigg|_{t=t_1} = \frac{c}{4\pi}\int_{t_1}^{t_2}\mathrm{d}t\int_S [\mathbf{EH}]\mathrm{d}\mathbf{S}. \tag{9}$$

The left-hand side of Eq. (9) is the difference of the values of the total energy of the field generated by the moving charge in the half-space V at the times t_2 and t_1. We use the expression (3) for the total field energy and rewrite Eq. (9) as

$$W(t_2) - W(t_1) = \int_{t_1}^{t_2}\mathrm{d}t\int_S \frac{c}{4\pi}[\mathbf{EH}]\mathrm{d}\mathbf{S}. \tag{10}$$

We have chosen the times t_1 and t_2 so that the quantity $W(t_1)$ can be neglected compared with $W(t_2)$. Thus

$$W(t_2) = \int_{t_1}^{t_2}\mathrm{d}t\int_S \frac{c}{4\pi}[\mathbf{EH}]\mathrm{d}\mathbf{S}. \tag{10$'$}$$

We now extend the time t_1 even farther into the past and the time t_2 even farther into the future. In the limit the quantity $W(t_2)$ will give the total energy of the electromagnetic field generated by the moving charge in unbounded space, because the part of the total field energy residing outside the volume V makes a vanishingly small contribution. The right-hand side of Eq. (10′) gives the energy which has passed through the surface S over all time. We obtain

$$W = \int_{-\infty}^{\infty}\mathrm{d}t\int_S \frac{c}{4\pi}[\mathbf{EH}]\mathrm{d}\mathbf{S}, \tag{11}$$

where

$$W = \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}z\frac{E^2+H^2}{8\pi}. \tag{12}$$

The volume integral extends over all space. The quantity W obtained depends on the velocity of the charge v.

Equation (11) gives a relation between the energy of the electromagnetic field generated by a uniformly moving charged particle and the energy flux of this field.

We now consider another example. Let the charged particle move in the positive direction along the z-axis, approaching the origin of coordinates, so that at large distances from the origin the particle velocity is $\mathbf{v}_1$. In some region with linear dimensions L near the origin the velocity of the particle varies in some manner, which we do not specify (for us it is sufficient to know that the velocity changes), both in magnitude and direction. The particle then emerges from this region and its velocity assumes the value $\mathbf{v}_2$, which thereafter remains constant. For simplicity we assume that the velocity $\mathbf{v}_1$, just as the velocity $\mathbf{v}_2$, is oriented in the positive z direction. This assumption in no way limits the generality of the arguments.

Since the particle is accelerated in a region of dimensions L near the origin of the coordinate system, electromagnetic waves are radiated. We apply to this process the conservation law (1). First, we choose a volume V and surface S bounding it. The choice of volume and surface determines the value of the integrals appearing in Eq. (1). We take for V the volume bounded by two parallel planes P_1 and P_2. The planes P_1 and P_2 are perpendicular to the z-axis and are positioned so that the region L where the charge is accelerated lies between P_1 and P_2. The plane P_1 intersects the z-axis at a large negative value of z and the plane P_2 intersects the z-axis at a large positive value of z. Thus the chosen planes are located on either side of and far away from the region L. The arrangement of the planes P_1 and P_2 and the region L in which the charged particle is accelerated are shown in Fig. 1.

We take for the volume V the space between the planes P_1 and P_2, and we take the planes themselves as the surface S bounding the volume V. This specifies the region of integration for all integrals in the conservation law (1).

Before examining the law of conservation of electromagnetic field energy in the chosen volume V, we shall try to envisage the physical characteristics of the field in the case shown in Fig. 1. Before entering the region V the charge moves with constant velocity v_1. At this time the electromagnetic field is the field of a uniformly moving charge. This field is strong near the charge and rapidly decays away from the moving charge. The electromagnetic field of a charge moving uniformly in empty space is a function of the argument $\mathbf{r} - \mathbf{v}_1 t$, i.e., the field moves in space with the same velocity v_1 as the charge which generates this field. The field at the moment the charge approaches the region is shown schematically in Fig. 2. The shaded region is the region where the field of the charge is strong.

For the time being, these qualitative remarks are sufficient for us. When it enters the region L the charge is accelerated. This acceleration is accompanied by radiation of electromagnetic waves, i.e., a wave packet arises, consisting of

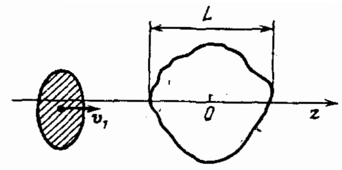

FIG. 2.

waves which diverge in all directions from the region L and escape to infinity with the velocity of light. These waves carry away energy. In addition, the radiated waves act on the moving charge, so that the quantity $\mathbf{j}\cdot\mathbf{E}$ in the conservation law is different from zero. After it leaves the region L the charge moves with constant velocity $\mathbf{v}_2$. The total electromagnetic field consists of the radiation field (i.e., waves moving with the velocity of light away from the region L) and the field of the charge moving uniformly with velocity v_2. The field of a uniformly moving charge is also sometimes called the entrained field. This field depends on the coordinates and time in the combination $\mathbf{r} - \mathbf{v}_2 t$, i.e., it moves as a whole in space with velocity $\mathbf{v}_2$. Since the field of the charge moves in space with the velocity of the charge $\mathbf{v}_2$ and the radiation field propagates with the velocity of light c, the radiation wave packet will sooner or later separate from the field of the charge, i.e., in the regions of space where the field of the uniformly moving charge will be strong, the radiation field will be negligibly small. Conversely, at locations where the radiation field will be appreciable the field of the charge will be negligibly small. In other words, the field of the charge will be spatially separated from the radiation field. The spatial picture of the field after separation is shown schematically in Fig. 3.

We designate by the index 1 the field $(\mathbf{E}_1, \mathbf{H}_1)$ of a charge moving uniformly with velocity v_1 and we designate by $(\mathbf{E}_2, \mathbf{H}_2)$ the field of a charge moving uniformly with velocity $\mathbf{v}_2$. We designate the radiation field with primes: $(\mathbf{E}', \mathbf{H}')$. We now integrate the conservation law (1) over time from t_1 to t_2. We obtain

$$\int_V \frac{E^2 + H^2}{8\pi} \mathrm{d}V\Big|_{t=t_2} - \int_V \frac{E^2 + H^2}{8\pi} \mathrm{d}V\Big|_{t=t_1} = -\int_{t_1}^{t_2} \mathrm{d}t \int_V \mathbf{E}\mathbf{j}\mathrm{d}V - \frac{c}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int_S [\mathbf{E}\mathbf{H}]\mathrm{d}\mathbf{S}. \qquad (13)$$

The left-hand side is the difference of the total energies of the field which are evaluated at the times t_2 and t_1. As one

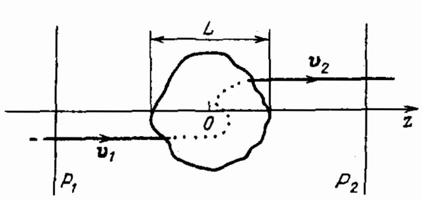

FIG. 1.

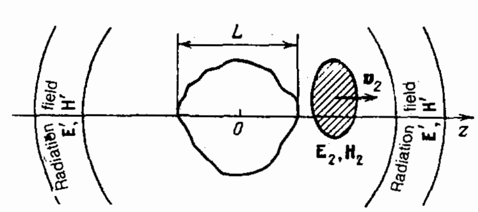

FIG. 3.

can see from the right-hand side of the equation, this difference consists of the work performed by the retardation force exerted by the field on the charge and the electromagnetic energy flux through the surface S bounding the volume V, where the work is the total work performed from t_1 to t_2 and radiated energy is the total energy radiated over the same time interval.

We recall that the volume V and the surface S in the formula (13) have already been chosen (see Fig. 1 and the text referring to it). In addition, we place the plane P_2 in the region where the radiation wave packet has already separated from the field of the charge, and the field of the charge no longer overlaps with the radiation field. Below, we estimate the distance from the region L at which the field of the charge becomes separated from the radiation field. We now determine the times t_1 and t_2, i.e. we determine more accurately the limits of integration over time. We choose the time t_1 far in the past. At the time t_1 the charge is still far away from the plane P_1. Conversely, we choose the time t_2 to be large in absolute magnitude and positive. By the time t_2 the charge has already left the volume V, i.e., it has crossed the plane P_2 and has moved a large distance away from it. For our choice of volume V and surface S the first term on the left-hand side of Eq. (13) is equal to zero, since at the time t_1 the charge is still far from the surface P_1, and the field in the volume V is equal to zero. The second term on the left-hand side of Eq. (8) also vanishes, since by the time t_2 the charge is already far away from the volume V and the radiation field has left the volume V even earlier than the charge. For this reason, at the time t_2 there is no field in the volume V between the planes P_1 and P_2. Thus Eq. (13) assumes the form

$$\int_{t_1}^{t_2} \mathrm{d}t \int \mathbf{jE}\,\mathrm{d}V = -\frac{c}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int_S [\mathbf{EH}]\mathrm{d}\mathbf{S}. \qquad (14)$$

Thus the work performed by the retardation field on the charge is expressed in terms of the energy which has flowed through the surface S over the same time interval.

We now examine the right-hand side of Eq. (14). In the time interval from t_1 to t_2 the energy flux associated with the field of the charge which has entered the volume V with velocity v_1, flows through the surface S into the volume V. We designated this field by $(\mathbf{E}_1, \mathbf{H}_1)$ (see above). Thus the energy flowing into the volume V is

$$W_1 = \frac{c}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int_{P_1} [\mathbf{E}_1\mathbf{H}_1]\mathrm{d}\mathbf{S}. \qquad (15)$$

We now examine the energy flowing out of the volume V in the time interval (t_1, t_2). First, this is the energy of the radiation field $(\mathbf{E}', \mathbf{H}')$

$$W' = \frac{c}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int_S [\mathbf{E}'\mathbf{H}']\mathrm{d}\mathbf{S}. \qquad (16)$$

This energy flows out of the volume V through the two surfaces P_1 and P_2. Second, the energy flux associated with the field of the charge, which has emerged from the volume V with velocity v_2, flows out of the volume V:

$$W_2 = \frac{c}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int_{P_2} [\mathbf{E}_2\mathbf{H}_2]\mathrm{d}\mathbf{S}. \qquad (17)$$

On the basis of Eqs. (15)–(17) we can rewrite the conservation law as follows:

$$\int_{t_1}^{t_2} \mathrm{d}t \int_V \mathbf{Ej}\mathrm{d}V = W_1 - W_2 - W'. \qquad (18)$$

We now extend t_1 even farther into the past and t_2 even farther into the future. In the limits $t_1 \to -\infty$ and $t_2 \to +\infty$ the left-hand side of the last equation gives the total work performed by the field on the charged particle over the entire time of the motion. According to Eqs. (11) and (13), the quantity W_1 becomes the total energy of the field of the charged particle moving uniformly with velocity v_1. The quantity W_2 becomes the total energy of the velocity field of the charged particle moving uniformly with velocity v_2. Finally, the quantity W' gives the total radiation energy over the entire time of the motion. Using the designation (12) for the total energy of the velocity field, we can rewrite the limiting form of the conservation law (18) in the form

$$\frac{c}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}t \int_S [\mathbf{E}'\mathbf{H}']\mathrm{d}\mathbf{S} = -\int_{-\infty}^{\infty} \mathrm{d}t \int_V \mathbf{jE}\mathrm{d}V + W_1 - W_2. \qquad (19)$$

In the formula (19) $\mathbf{E}$ is the total field on the path of the particle and $(\mathbf{E}', \mathbf{H}')$ is the radiation field.

Formula (19) shows that the radiation flux from the volume V is determined not only by the work performed by the forces exerted by the field on the charge (the integral of $\mathbf{j}\cdot\mathbf{E}$) but also by the change in the total energy of the entrained *velocity* field. If the velocity of the charged particle is the same in magnitude before and after the region of acceleration L, then the difference $W_1 - W_2$ in Eq. (19) vanishes. In this case the work performed by the field on the particle is exactly equal to the radiation energy (taken with the opposite sign).

In what follows we shall often work with the fields expanded in a Fourier integral with respect to time. In particular, the Fourier integral expansion of the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ with respect to the time has the form

$$\mathbf{E}(\mathbf{r}, t) = \int \mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}\mathrm{d}\omega,$$

$$\mathbf{H}(\mathbf{r}, t) = \int \mathbf{H}_\omega(\mathbf{r})e^{-i\omega t}\mathrm{d}\omega. \qquad (20)$$

The expansions (20) represent the field as a superposition of monochromatic oscillations. For example, the integrand in the expansion (20) for $\mathbf{E}(\mathbf{r},t)$ has the form $\mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}$. The electric field described by this expression varies harmonically in time with frequency ω, and at a given point r the amplitude of the electric field is equal to $\mathbf{E}_\omega(\mathbf{r})$. The expansion (20) represents the field $\mathbf{E}(\mathbf{r},t)$ as a sum of oscillations of the form $\mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}$ with all possible frequencies. The amplitudes $\mathbf{E}_\omega(\mathbf{r})$ and $\mathbf{H}_\omega(\mathbf{r})$ in the expansion (20) can then be represented as a superposition of plane waves of the form $e^{i\mathbf{k}\cdot\mathbf{r}}$:

$$\mathbf{E}_\omega(\mathbf{r}) = \int \mathbf{E}_{\omega,\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} d\mathbf{k},$$

$$H_\omega(\mathbf{r}) = \int \mathbf{H}_{\mathbf{k},\omega} e^{i\mathbf{k}\mathbf{r}} d\mathbf{k}. \tag{21}$$

For what follows we shall need to express the integrals in the conservation law Eq. (19) in terms of the amplitudes $\mathbf{E}_\omega(\mathbf{r})$ and $\mathbf{H}_\omega(\mathbf{r})$. In order to obtain the corresponding expressions, we now study, for example, the formula (11) for the time integral of the flux of the Poynting vector through the surface S. Expressions with precisely this structure appear on both the right- and left-hand sides of the conservation law (19):

$$W = \int_{-\infty}^{\infty} dt \int_S \frac{c}{4\pi} [\mathbf{E}\mathbf{H}] d\mathbf{S}. \tag{22}$$

We substitute here the expansions (20) for the fields $\mathbf{E}$ and $\mathbf{H}$ and integrate over time. But we first call attention to the following property of the amplitudes $\mathbf{E}_\omega(\mathbf{r})$ and $\mathbf{H}_\omega(\mathbf{r})$. Changing the sign of ω transforms these amplitudes into their complex conjugates:

$$\mathbf{E}_{-\omega}(\mathbf{r}) = \mathbf{E}^*_\omega(\mathbf{r}), \quad \mathbf{H}_{-\omega}(\mathbf{r}) = \mathbf{H}^*_\omega(\mathbf{r}). \tag{23}$$

This property of the amplitudes $\mathbf{E}_\omega$ and $\mathbf{H}_\omega$ follows from the fact that the electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{H}(\mathbf{r},t)$ are real functions of the coordinates and time. The relations (23) can be proved, for example, with the help of the inverse Fourier transform

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{1}{2\pi} \int \mathbf{E}(\mathbf{r}, t) e^{i\omega t} dt,$$

$$\mathbf{H}_\omega(\mathbf{r}) = \frac{1}{2\pi} \int \mathbf{H}(\mathbf{r}, t) e^{i\omega t} dt. \tag{24}$$

Using Euler's well-known formula

$$e^{i\omega t} = \cos \omega t + i \sin \omega t, \tag{25}$$

we can rewrite the expression (24) for $\mathbf{E}_\omega(\mathbf{r})$ as follows

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{1}{2\pi} \left(\int \mathbf{E}(\mathbf{r}, t) \cos \omega t \, dt + i \int \mathbf{E}(\mathbf{r}, t) \sin \omega t \, dt \right). \tag{26}$$

Since $\mathbf{E}(\mathbf{r},t)$ is a real function, the expression (26) for $\mathbf{E}_{-\omega}(\mathbf{r})$ transforms into its complex conjugate since $\sin \omega t$ is odd. The property (23) for $\mathbf{H}_\omega$ can be proved analogously. The amplitude $\mathbf{j}_\omega(\mathbf{r})$, appearing in the Fourier expansion of the current density $\mathbf{j}(\mathbf{r},t)$, and in general the amplitudes f_ω of any real function $f(t)$ have the same property. We now substitute the expansions (20) into the integral expression (22) for the energy flowing through the surface S over an infinite time. We obtain

$$W = \frac{c}{4\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \int_S [\mathbf{E}_\omega \mathbf{H}_{\omega'}] e^{i(\omega+\omega')t} d\mathbf{S}. \tag{27}$$

The integration over time is easily performed with the help of the well-known formula

$$\int_{-\infty}^{\infty} e^{i(\omega+\omega')t} dt = 2\pi\delta(\omega + \omega'). \tag{28}$$

The integration over ω' is then easily performed using a property of the delta function. We obtain finally

$$W = \frac{c}{2} \int_{-\infty}^{\infty} d\omega \int_S [\mathbf{E}_\omega \mathbf{H}_{-\omega}] d\mathbf{S}. \tag{29}$$

Using Eq. (23) we obtain

$$W = \frac{c}{2} \int_{-\infty}^{\infty} d\omega \int_S [\mathbf{E}_\omega \mathbf{H}^*_\omega] d\mathbf{S}. \tag{30}$$

It follows from Eq. (23) that changing the sign of ω changes the integrand in Eq. (30) into its complex conjugate. This means that the imaginary part $\mathrm{Im}[\mathbf{E}_\omega \times \mathbf{H}^*_\omega]$ of the integrand is an odd function of ω and the real part $\mathrm{Re}[\mathbf{E}_\omega \times \mathbf{H}^*_\omega]$ is an even function of ω. The integral of the odd part over infinite limits is equal to zero, and for this reason the expression (30) is real, as should be the case. On the basis of what we have said above, we can rewrite Eq. (30) in the form

$$W = c \int_0^{\infty} d\omega \int_S \mathrm{Re}[\mathbf{E}_\omega \mathbf{H}^*_\omega] d\mathbf{S}. \tag{31}$$

We now recall that the quantity W gives the total energy flowing through the surface S over all time. The formula (31) represents this energy as an integral of the expression

$$W_\omega = \mathrm{Re} \int_S c [\mathbf{E}_\omega \mathbf{H}^*_\omega] d\mathbf{S} \tag{32}$$

over the frequency. It is natural to interpret this expression as the energy of the field (at the frequency ω) which has passed through the surface S.

Expressing all integrals appearing in the conservation law (19) in terms of the Fourier amplitudes, we obtain the law of conservation of energy for the Fourier components corresponding to the frequency ω:

$$\mathrm{Re} \int_S c [\mathbf{E}'_\omega \mathbf{H}'^*_\omega] d\mathbf{S} = -4\pi \, \mathrm{Re} \int_V \mathbf{j}_\omega \mathbf{E}^*_\omega dV + \mathrm{Re} \int_{P_1} c [\mathbf{E}_{1\omega} \mathbf{H}^*_{1\omega}] d\mathbf{S} - \mathrm{Re} \int_{P_2} c [\mathbf{E}_{2\omega} \mathbf{H}^*_{2\omega}] d\mathbf{S}, \tag{33}$$

where $\mathbf{j}_\omega(\mathbf{r})$ is the Fourier component of the current density $\mathbf{j}(\mathbf{r},t)$:

$$\mathbf{j}(\mathbf{r}, t) = \int \mathbf{j}_\omega(\mathbf{r}) e^{-i\omega t} d\omega,$$

$$\mathbf{j}_\omega(\mathbf{r}) = \frac{1}{2\pi} \int \mathbf{j}(\mathbf{r}, t) e^{i\omega t} dt. \tag{34}$$

We recall that the field $(\mathbf{E}',\mathbf{H}')$ is the radiation field, while $(\mathbf{E}_1,\mathbf{H}_2)$ and $(\mathbf{E}_2,\mathbf{H}_2)$ are the fields of a charge moving uniformly with velocity $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. The volume V is included between the two planes P_1 and P_2. These two planes comprise the surface S bounding the volume V. The entire arrangement is shown schematically in Fig. 1.

The preceding analysis, based on the law of conservation of energy, was made for the case of a field in empty space. It is easy to extend the result to the case where the charge moves in a refracting medium. Let the medium be uniform, and assume that the charged particle moves in the medium in the same manner as in the case of empty space analyzed above. This means that before entering the region L the particle moved uniformly with velocity $\mathbf{v}_1$ and in L the particle was accelerated. After leaving L the particle moved uniformly with velocity $\mathbf{v}_2$, which remained constant. We therefore regard the motion as the same as in empty space. The difference lies only in the fact that this time the particle moves in a uniform refracting medium with permittivity ε and magnetic permeability μ. In this case the conservation laws have the same form as Eqs. (19) and (33). The difference lies only in the fact that this time all fields—the radiation field ($\mathbf{E}'$,$\mathbf{H}'$), the entrained field ($\mathbf{E}_1$,$\mathbf{H}_1$) of the approaching charge and the entrained field ($\mathbf{E}_2$,$\mathbf{H}_2$) of the receding charge—are calculated taking the medium into account.

Assume now that the medium is not uniform. We examine the case where the properties of the medium vary along the z-axis, so that the permittivity and magnetic permeability approach some limiting values ε_1 and μ_1 as $z\to -\infty$ and different limiting values ε_2 and μ_2 as $z\to\infty$. We assume that the motion of the charge is of the same character as in the case of empty space, i.e., after leaving the region L the particle moves with uniform velocity $\mathbf{v}_2$. The motion of the charge is shown schematically in Fig. 4. The difference from the case shown in Fig. 1 lies in the fact that the charge leaves a region with constant values of ε_1 and μ_1 and enters a region with different constant values ε_2 and μ_2; thus the permittivity ε and magnetic permeability μ change along the path of the charge from one limiting value to the other. This change can occur abruptly, if a sharp interface is present in the path of the charge.

We draw the planes P_1 and P_2 perpendicular to the z-axis. The plane P_1 lies in a practically uniform medium where the values of ε and μ are equal to the limiting values ε_1 and μ_1. The plane P_2 lies in a practically uniform medium where the values of ε and μ are equal to the limiting values ε_2 and μ_2. Both planes are located at such a large distance from the region L that for the corresponding values of z the radiation field ($\mathbf{E}'$,$\mathbf{H}'$) does not interfere with the entrained fields ($\mathbf{E}_1$,$\mathbf{H}_1$) and ($\mathbf{E}_2$,$\mathbf{H}_2$). In the case studied the entrained field ($\mathbf{E}_1$,$\mathbf{H}_1$) is the field of a charged particle moving uniformly with velocity $\mathbf{v}_1$ in an infinite medium with constant ε_1 and μ_1. Correspondingly, the entrained field ($\mathbf{E}_2$,$\mathbf{H}_2$) is the field of a charged particle moving with uniform velocity $\mathbf{v}_2$ in an infinite medium with constant ε_2 and μ_2. We assume here that there is no absorption. We note that in free space the radiation field eventually separates from the entrained field of the charge. This is explained by the fact that the velocity of the radiation packet is equal to the velocity of light in empty space, i.e., it is always greater than the velocity of the charge. In a medium, however, the radiation field does not always separate from the entrained field, because the phase velocity of the radiation waves is equal to $c/(\varepsilon\mu)^{1/2}$ and can be equal to or even less than the velocity of the charge. In this case there exist directions in which the propagating radiation may not separate from the field of the charge along a path of any length. For the time being we ignore this possibility.

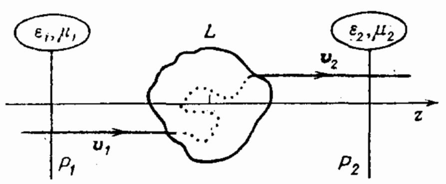

FIG. 4.

Choosing planes P_1 and P_2 satisfying all requirements imposed above, we denote the volume between P_1 and P_2 by V. Then the planes P_1 and P_2 together form the surface S bounding the volume V.

For this case the law of conservation of energy can be written in the form of Eq. (19) or, in terms of Fourier components, in the form (33). It should be remembered, however, that the meaning of some quantities appearing in these relations changes. Thus, for example, in the case of a refracting medium W_1 is the energy of the entrained field of the particle moving with velocity $\mathbf{v}_1$ in the medium ε_1 and μ_1. Similarly, W_2 is the energy of the velocity field of the particle moving with velocity $\mathbf{v}_2$ in the medium ε_2 and μ_2. These velocity fields are themselves ($\mathbf{E}_1$,$\mathbf{H}_1$) and ($\mathbf{E}_2$,$\mathbf{H}_2$), respectively. The quantity W_1 is calculated in one medium and the quantity W_2 is calculated in the other medium. Therefore, these two energies can differ even when $\mathbf{v}_1=\mathbf{v}_2$. A charge of the same magnitude and moving with the same velocity generates different fields in different media. For this reason, the total energy of the entrained field is different in different media, even if the charge is of the same magnitude and has the same velocity. This circumstance can be underscored by denoting as $W_1(v_1)$ the energy of the velocity field in the medium ε_1 and μ_1 and by $W_2(v_2)$ the energy of the velocity field in the medium ε_2 and μ_2. Then the conservation law Eq. (19) assumes the form

$$\frac{c}{4\pi}\int_{-\infty}^{\infty} dt[\mathbf{E}'\mathbf{H}']\,d\mathbf{S} = -\int_{-\infty}^{\infty} dt\int \mathbf{jE}\,dV + W_1(v_1) - W_2(v_2); \tag{35}$$

Here $\mathbf{E}$ is the field in the path of the particle.

If the particle moves uniformly in a nonuniform medium, then the difference $W_1(v) - W_2(v)$, where v is the particle velocity, appears on the right-hand side of Eq. (35). This quantity is often written in terms of the change in the mass of the moving particle. Let

$$W_1(v) - W_2(v) = c^2\Delta m. \tag{36}$$

Then Δm is the change in the mass associated with the entrained field. This change occurs when a uniformly moving charged particle crosses from one medium into another.[1] Garibyan calculated the quantities $W_1(v)$ and $W_2(v)$, i.e., the energy $W_1(v)$ of the field accompanying a particle having charge q and velocity v in the first medium and the energy $W_2(v)$ in the second medium.

The field at high frequencies makes the main contribution to this expression. In both media the permittivity at high frequencies has a similar form:

$$\varepsilon_{1,2}(\omega) = 1 - (\omega_{1,2}^2/\omega^2) \quad (\omega \gg \omega_{1,2}), \tag{37}$$

where for the medium 1 the constant ω_1 is expressed in terms of the number n_1 of electrons per unit volume of the medium 1:

$$\omega_1^2 = 4\pi n_1 e^2/m; \tag{38}$$

where e and m are the electron charge and mass. The constant ω_1 has the dimension of frequency, and it is the natural frequency of oscillations of an electron plasma having density n_1.

Similarly, the constant ω_2 in the expression for the permittivity of the second medium has the form

$$\omega_2^2 = 4\pi n_2 e^2/m, \tag{39}$$

where n_2 is the electron density in the second medium.

For the case of a point charge the quantities W_1 and W_2 are infinite. The field of a point charge grows without bound with decreasing distance from the charge and the energy density of the field increases so rapidly that the volume integral of the energy density diverges. The integral can be made finite by assuming that the charge which generates the field is not point-like but rather extended—for example, distributed uniformly in a small sphere of radius r_0. Then the integral of the energy density is cut off at short distances, and the quantities $W_{1,2}$ become finite. If the electromagnetic field of the moving charge is represented as a collection of waves, then when the point charge is replaced by an extended charge only the waves whose wavelength is greater than the "size" r_0 of the charge need be taken into account. Correspondingly, the wave vectors of the significant waves must be shorter than some limiting value

$$\varkappa_0 \sim 1/r_0, \tag{40}$$

where r_0 is the radius of the charge (or, which is the same thing, the linear size of the volume over which the charge q is distributed). It is obvious that in this case all Fourier integrals must be cut off at the upper limit of integration at $\varkappa_0$. Then the following expressions are obtained for the quantities $W_{1,2}$:

$$W_1 = \frac{q^2}{c[1-(v^2/c^2)]^{1/2}}\left(\frac{1}{2}\varkappa_0 c - \omega_1\right), \quad W_2 = \frac{q^2}{c[1-(v^2/c^2)]^{1/2}}\left(\frac{1}{2}\varkappa_0 c - \omega_2\right). \tag{41}$$

We recall that W_1 is the total energy of the entrained field of the particle in the first medium (the medium is assumed to be infinite) and W_2 is the analogous quantity for the second medium. The properties of the medium enter into the expression (41) through the constants ω_1 (37) and ω_2 (38).

If, for example, the first medium is empty, then $n_1 = 0$, $\omega_1 = 0$, and W_1 assumes the form

$$W_1 = \frac{q^2}{c[-(v^2/c^2)]^{1/2}} \cdot \frac{1}{2}\varkappa_0. \tag{42}$$

It should be remembered that the limit $\varkappa_0 \to \infty$ corresponds to approaching a point charge, and then the quantity (42) diverges and the quantities W_1 and W_2 (41) diverge with it. It is obvious that the character of the divergence for a medium is determined by the singularities of the field in empty space.

Comparing Eqs. (41) and (42) shows that the medium contributes a definite and finite correction and, generally speaking, an indeterminate and infinite value W (in the limit of a point charge). This correction is always negative, perhaps because the medium screens the field of view.

The difference $W_1(v) - W_2(v)$ is finite and does not depend on the cutoff parameter $\varkappa_0$:

$$W_1(v) - W_2(v) = \frac{q^2}{c[-(v^2/c^2)]^{1/2}}(\omega_2 - \omega_1). \tag{43}$$

This quantity determines the difference between the total energy of the transition radiation and the work which the radiation field performs on the particle. We note that the permittivity can change not only in space (nonuniform medium) but also in time (nonstationary medium). In the last case the renormalization must be taken into account in the energy balance.[2]

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[2] V. L. Ginzburg, *Theoretical Physics and Astrophysics* [in Russian], Nauka, M., 1981; V. L. Ginzburg and V. N. Tsytovich, *Transition Radiation and Transition Scattering* [in Russian], Nauka, M., 1984, Secs. 2.5 and 2.6.

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