Statistical and dynamic localization of plane waves in randomly layered media

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This article presents a detailed discussion of the problem of the localization and various methods of describing it on the basis of plane waves multiply scattered in randomly layered media. It is noted that the field of localized waves has a complicated structure, with sharp peaks and extended "dark" regions, where the intensity of the wave is small. It is shown that because of this complicated structure the wave field in a randomly layered medium, the dynamic and statistical characteristics of the wave behave in fundamentally different ways. For example, the statistical moments of the intensity of the wave increase exponentially into the interior of the medium, while the energy of the wave penetrating into randomly inhomogeneous medium can be finite with unity probability. The concept of a majorant curve and of an isoprobability curve, helpful for understanding the phenomenon of localization, are introduced. Also taken into account is the effect of a small regular absorption on the statistical and dynamic properties of the wave, and the localization of space-time pulses in a randomly layered medium is also studied.

1. INTRODUCTION

Recently, animated discussions have taken place concerning the problem of the wave field localization in randomly layered media with or without absorption in the medium (see reviews[1-3]). A unique answer is not always obtained as to the presence of localization in any particular physical situation. This is due to the fact that fields that exist inside the medium are of a complex spatial structure, i.e., the decrease in wave intensity with distance from the source into the medium can alternate with increasingly infrequent but increasingly strong intensity spikes which result from the coherent summation of waves multiply scattered in the medium. Eventually it can happen that in practically every experimentally obtained field realization localization will be observed, whereas the behavior of the statistical averages, for example, that of the average intensity and its higher moments, implies lack of localization. It thus seems expedient to introduce two notions which in general do not coincide: that of *dynamic localization* inherent in individual field realizations and the notion of *statistical energy localization*, i.e., localization of the average wave intensity, expressing the properties of the whole statistical ensemble of realizations.

As an example that clearly illustrates the distinction between statistical and dynamic localization one can consider the problem of normal incidence of a plane wave on a halfspace filled with a randomly layered nonabsorbing medium. The average intensity of the wave inside such a medium is the same everywhere, while the higher intensity moments increase exponentially to infinity into the medium. This shows unambiguously the absence of statistical wave localization.[4,5] Nevertheless, as will be shown later in this article, one can speak in this case about a dynamic localization which manifests itself in that the total energy of the wave penetrating into a randomly inhomogeneous medium is finite with probability 1 (that is in all realizations except for those of zero measure), whereas each intensity realization is bounded from above by a majorant curve that falls off exponentially into the interior of the medium. In this paper we discuss in detail the mutually complementary notions of statistical and dynamic localization, using as an example plane waves radiated within randomly layered media.

The concept of localization originated in the physics of disordered systems (see, e.g., Ref. 8) described by the Schrödinger equation (time independent or time dependent) with a random potential, which is identical in form to the Helmholtz equation with a random refractive index. This coincidence, however, is an entirely superficial mathematical effect, stemming from the formalism used for describing statistical phenomena in the cases considered. In the physics of disordered systems the fundamental entities are the self-averaging values, since they allow one to study the statistical properties of an object using a single realization which is large enough (because of the absence of an ensemble of objects in general), and the main mathematical tool (characteristic of quantum mechanical systems in principle) is a spectral expansion in the eigenfunctions of the corresponding boundary-value problem for the Schrödinger equation. In the problems of wave propagation in randomly inhomogeneous media the most important problems are considered to be those based on the averaging over an ensemble of realizations of the parameters of the medium, and as mathematical tools here one makes use of the usual classical theory of wave processes. Therefore, attempts to solve the problems of wave propagation in random media by the quantum physics methods, as has been demonstrated in some recent works (see, e.g., Ref. 2), arouse a feeling of dissatisfaction and remind one of "using the right hand for scratching the left ear" speaking figuratively. Besides, for the problems of wave propagation in random media a major factor is that of the wave absorption in the medium (even though arbitrarily

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small): in some cases the statistical properties are singular with respect to absorption. In this case by using the classical method of analytic continuation of the solution of the steady state problem into the complex plane of the parameter associated with the absorption, one can obtain a solution to more complex problems, such as nonstationary problems or those concerning waves in layered media in three dimensions. At the same time in the approach based on the quantum-mechanical analogy, there is no dissipation *a priori*. Therefore, when solving the above-mentioned more complex problems one has to start anew taking no advantage of the extensive information at hand, inherent in the solution of stationary problems (see, e.g., Ref. 2). And we have said nothing of the situations where the limits of a vanishingly small absorption and passage to an infinite half-space (or an infinite space) simply do not commute. For these reasons it is not possible to regard the results obtained on the basis of quantum mechanical analogy with due confidence.

An approach based on classic wave analysis is also dealt with in the present paper.

2. THE PROBLEM OF SCATTERING OF WAVES RADIATED IN A RANDOMLY-LAYERED MEDIUM

Let us consider a layer of a randomly layered medium that occupies a portion of space $L_0 \leqslant x \leqslant L$, with a source of plane waves at point x_0 ($L_0 < x_0 < L$) (see Fig. 1). The wave field inside the layer is described by

$$\frac{d^2}{dx^2}G(x, x_0) + k^2(1 + \varepsilon(x)) \times G(x, x_0) = 2ik\delta(x - x_0). \qquad (1)$$

Here $\varepsilon(x)$ describes how random inhomogeneities and absorption in the medium affect the field of the radiated wave. Outside the layer it will be assumed that $\varepsilon(x) \equiv 0$ while inside it $\varepsilon(x) = \varepsilon_1(x) + i\gamma$ where $\varepsilon_1(x)$ accounts for random inhomogeneities of the medium and $\gamma \ll 1$ describes the weak absorption of the wave in the layer.

Outside the layer the field appears as plane waves emanating from the layer:

$$G(x, x_0) = \begin{cases} T_1 \exp[ik(x - L)], & x \geq L, \\ T_2 \exp[-ik(x - L_0)], & x \leq L_0, \end{cases}$$

where $T_{1,2}$ are the complex coefficients of the waves radiated from the layer. As boundary conditions for Eq. (1) we take the continuity conditions of the field and its derivative at the layer boundaries, which reduce to the following two-point boundary conditions

$$G(L, x_0) + \frac{i}{k}\frac{d}{dx}G(x, x_0)\Big|_{x=L} = 0,$$

$$G(L_0, x_0) - \frac{i}{k}\frac{d}{dx}G(x_0, x)\Big|_{x=L_0} = 0. \qquad (2)$$

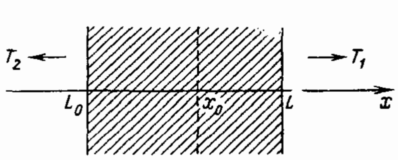

FIG. 1. Propagation pattern of a wave radiated by a source inside the medium layer.

In addition, the derivative of the field has a discontinuity at the source point

$$\frac{d}{dx}G(x, x_0)\Big|_{x=x_0+0} - \frac{d}{dx}G(x, x_0)\Big|_{x=x_0-0} = 2ik, \qquad (3)$$

whereas the source field itself $G(x,x_0)$ is continuous at point x_0. Therefore, the source field in a randomly complex absorbing medium is described by the boundary-value problem (1)–(3), the solution of which is known to be[4]

$$G(x, x_0) = G(x_0, x_0)\begin{cases} \exp\left(ik\int_x^{x_0} d\xi\,\psi_1(\xi)\right), & x \leq x_0, \\ \exp\left(ik\int_{x_0}^{x} d\xi\,\psi_2(\xi)\right), & x \geq x_0, \end{cases}$$

where $G(x_0,x_0) = 2/[\psi_1(x_0) + \psi_2(x_0)]$ and the functions $\psi_j(x)$ satisfy the Riccati equations

$$\frac{d}{dx}\psi_{1,2}(x) = \pm ik(\psi_{1,2}^2(x) - 1 - \varepsilon(x)), \quad \psi_1(L_0) = \psi_2(L) = 1.$$

Instead of functions $\psi_j(x)$ we introduce the auxiliary functions $R_j(x)$.

$$\psi_j(x) = \frac{1 - R_j(x)}{1 + R_j(x)}, \quad j = 1, 2.$$

Then the wave field of the source within the region $x \leqslant x_0$ will take the form

$$G(x, x_0) = \frac{(1 + R_1(x_0))(1 + R_2(x_0))}{1 - R_1(x_0)R_2(x_0)}\exp\left(ik\int_x^{x_0} d\xi\frac{1 - R_1(\xi)}{1 + R_1(\xi)}\right), \qquad (4)$$

where $R_1(x)$ obeys the Riccati equation

$$\frac{d}{dx}R_1(x) = 2ikR_1(x) + \frac{ik}{2}\varepsilon(x)(1 + R_1(x))^2, \quad R_1(L_0) = 0. \qquad (5)$$

The wave equation (1) gives rise to an important relation to be used later:

$$k\gamma I(x, x_0) = \frac{d}{dx}S(x, x_0), \qquad (6)$$

where $I(x,x_0) = |G(x,x_0)|^2$ is the intensity of the wave and $S(x,x_0)$ is the energy flux density defined as

$$S(x, x_0) = \frac{1}{2ik}\left(G(x, x_0)\frac{d}{dx}G^*(x, x_0) - G^*(x, x_0)\frac{d}{dx}G(x, x_0)\right).$$

Making use of equality (4) one can readily obtain the following expression

$$S(x, x_0) = S(x_0, x_0)\exp\left(-k\gamma\int_x^{x_0} d\xi\frac{|1 + R_1(\xi)|^2}{1 - |R_1(\xi)|^2}\right),$$

$$S(x_0, x_0) = \frac{(1 - |R_1(x_0)|^2)|1 + R_2(x_0)|^2}{|1 - R_1(x_0)R_2(x_0)|^2}. \qquad (7)$$

In what follows we shall be mainly concerned with the wave behavior in an unbounded ($L_0 \to -\infty, L \to \infty$) and in a semibounded ($L_0 \to -\infty$) space with vanishing absorption ($\gamma \to 0$). It is clear from Eq. (6) that these limits do not commute in general. If we set $\gamma = 0$, then (6) implies conservation of the wave energy flux $S(x,x_0)$ over the whole half-space $x \leqslant x_0$, whereas if there is a finite but arbitrarily small absorption, then integrating (6) over all $x \leqslant x_0$ we obtain a limit on the value of energy included in this half-space

$$\frac{k\gamma}{D}E = S(x_0, x_0) \quad \left(E = D\int_{-\infty}^{x_0} dx I(x, x_0)\right), \tag{8}$$

where D has the dimension $1/x$ and acts as a diffusion coefficient for this problem (see below). In what follows when calculating the limiting values corresponding, for example, to $L_0 \to -\infty$, $\gamma \to 0$ we shall first calculate a limit for $L_0 \to -\infty$ and then that for $\gamma \to 0$, inasmuch as the presence of an arbitrarily small but finite damping automatically provides satisfaction of the radiation condition for $L_0 = -\infty$.

In addition to the general boundary-value problem Eqs. (1)–(3) of the scattering of a wave when the source is located inside an inhomogeneous layer, some special cases in connection with the problem of localization are of physical interest. For example, if $x_0 = L$ the boundary-value problem Eqs. (1) and (2) along with the jump condition Eq. (3) yield a boundary-value problem for the field $u(x;L) = G(x,L)$:

$$\frac{d^2}{dx^2}u + k^2(1 + \varepsilon(x))u = 0, \tag{9}$$

$$\frac{d}{dx}u(x; L)\Big|_{x=L_0} = -iku(L_0; L),$$

$$\frac{d}{dx}u(x; L)\Big|_{x=L} = ik(u(L; L) - 2),$$

describing the incidence of the plane wave $\exp[-ik(x-L)]$ on a medium layer from region $x > L$. In this case the field outside the layer, for $x > L$, has the following structure

$$u(x; L) = \exp[-ik(x - L)] + R(L)\exp[ik(x - L)],$$

where $R(L)$ is the complex coefficient reflection of the wave from a layer, and is related to the solution of boundary-value problem (9) through equality $R(L) = u(L;L) - 1$. However, within the range $x < L_0$ the field has a structure of an outgoing wave

$$u(x; L) = T(L)\exp[-ik(x - L_0)],$$

where $T(L) = u(L_0;L)$ is the complex transmission coefficient of the wave through a layer of the medium (see Fig. 2). Expression (4) for the field inside the medium, for the case of $x_0 = L$ considered here, goes into

$$u(x; L) = (1 + R_1(L))\exp\left(ik\int_x^L d\xi\frac{1 - R_1(\xi)}{1 + R_1(\xi)}\right). \tag{10}$$

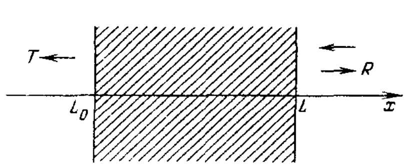

FIG. 2. Propagation pattern of a wave incident on a layered medium.

From this result and from the discussion above it follows that $R_1(L) = R(L)$. Correspondingly, $R_1(x_0)$ in formulas (4) and (7) can be interpreted as the reflection coefficient for a plane wave incident from free space $x > x_0$, reflected from a layer of an inhomogeneous medium (L_0,x_0). Analogously, the quantity $R_2(x_0)$ has the physical meaning of a reflection coefficient of a wave incident from the left on a layer (x_0,L). With (10), the general expression for the field (4) can be written as

$$G(x, x_0) = \frac{1 + R_2(x_0)}{1 + R_1(x_0)R_2(x_0)}u(x; x_0) \quad (x < x_0), \tag{11}$$

where $u(x;x_0)$ is given by equality (10). The effect of the other part of layer (x_0,L) is taken into account only by the coefficient $R_2(x_0)$. In applications the problem of the field of a source located near a reflecting boundary is of interest (see Fig. 3). In particular, for the boundary at point $x = x_0 + 0$ where the condition of total reflection $dG/dx|_{x=x_0} = 0$ is satisfied we have $R_2(x_0) = 1$ and consequently

$$u_{\text{ref}}(x; x_0) = \frac{2}{1 - R_1(x_0)}u(x; x_0). \tag{12}$$

Let us draw our attention to an outstanding fact that *in all physical situations considered here the spatial behaviour of a wave field inside an inhomogeneous medium is described by the same function* $u(x;x_0)$ (10). The fields (10)–(12) differ only in the coefficients for $u(x;x_0)$ describing resonance properties of a stochastic cavity formed by a randomly inhomogeneous layer. In what follows, however, it becomes clear that even though for each realization $\varepsilon_1(x)$ of a randomly layered medium the corresponding field realizations in all problems considered vary in a similar manner in space, the different resonance coefficients for random realizations $u(x;x_0)$ lead to qualitatively different statistical properties of the fields in question.

As follows from the method of embedding[5] the field $u(x;L)$ and $R(L)$ as functions of the parameter L satisfy the problems with initial conditions

$$\frac{\partial}{\partial L}u(x; L) = iku(x; L) + \frac{ik}{2}\varepsilon(L)(1 + R_1(L))u(x; L), \tag{13}$$

$$u(x; x) = 1 + R_1(x),$$

$$\frac{\partial}{\partial L}R_1(L) = 2ikR_1(L) + \frac{ik}{2}\varepsilon(L)(1 + R_1(L))^2, \quad R_1(L_0) = R_0. \tag{14}$$

It should be noted as well that in the above considered case of a layer conforming with a homogeneous space one should use $R_0 = 0$ as an initial condition for Eq. (14). If, however, for $x \leqslant L_0$ the wave number is equal to $k_1 \neq k$ then we should set $R_0 = (k - k_1)/(k + k_1)$ in the initial condition for Eq.

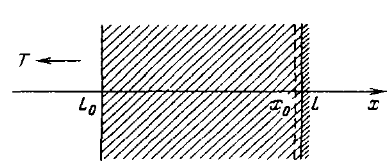

FIG. 3. Propagation pattern of a wave radiated by a source located near the reflecting boundary.

(14). In particular, if the boundary L_0 is totally reflecting then $R(L_0) = \pm 1$.

In the sections that follow we shall not be concerned with the field itself but with its intensity. Therefore, for the wave intensity $J(x;L) = |u(x;L)|^2$ in the auxiliary problem of the wave incident on the medium layer, we write an equation that results from Eq. (13)

$$\frac{\partial J(x;L)}{\partial L} = -\frac{k\gamma}{2}(2 + R_1(L) + R_1^*(L))J(x;L) + \frac{ik}{2}(R_1(L) - R_1^*(L)J(x;L),$$
$$J(x;L) = |1 + R_1(x)|^2. \qquad (15)$$

In the case under consideration $|\varepsilon(x)| \ll 1$ the wave field can be represented as a superposition of counterpropagating waves

$$u(x;L) = a_1(x;L)e^{-ikx} + a_2(x;L)e^{ikx},$$

with complex amplitudes $a_j(x;L)$ which change little on the wavelength scale, while the intensities $Y_j(x;L) = |a_j(x;L)|^2$ satisfy the same Eq. (15) that was used for the full intensity but with initial conditions of their own

$$\frac{\partial}{\partial L}Y_j(x;L) = -\frac{k\gamma}{2}(2 + R_1(L) + R_1^*(L))Y_j(x;L) + \frac{ik}{2}(R_1(L) - R_1^*(L))Y_j(x;L),$$
$$Y(x;x) = 1,\ Y_2(x;x) = |R_1(x)|^2. \qquad (16)$$

3. STATISTICAL DESCRIPTION OF WAVES IN A RANDOMLY LAYERED MEDIUM

Let us proceed now to a discussion of the statistical properties of the problems posed above, assuming for definiteness $\varepsilon_1(x)$ to be a Gaussian field with zero average and a prescribed correlation function

$$\langle \varepsilon_1(x)\varepsilon_1(x')\rangle = B_\varepsilon(x - x').$$

In this case the influence of random inhomogeneities on the statistical properties of the intensity is expressed quantitatively through a diffusion coefficient[5]

$$D = \frac{k^2}{4}\int_{-\infty}^{\infty} ds B_\varepsilon(s)\cos(2ks).$$

Let us analyze first the statistical properties of the reflection coefficient $R(L)$ satisfying the Riccati equation (14). We introduce the quantity $W(L) = |R(L)|^2$ satisfying, as a result of Eq. (14), the following stochastic equation

$$\frac{d}{dL}W = -2k\gamma W - i\frac{k}{2}\varepsilon_1(L)(R(L) - R^*(L))(1 - W),$$
$$W(L_0) = W_0 = |E_0|^2. \qquad (17)$$

In the dissipative term we discard rapidly oscillating terms, which do not make a noticeable contribution to the effects building up with the layer thickness. Passing from Eq. (17) to the Liouville equation with respect to function $\Phi(W;L) = \delta(W(L) - W)$:

$$\frac{\partial}{\partial L}\Phi = 2k\gamma\frac{\partial}{\partial W}(W\Phi) + \frac{ik}{2}\varepsilon_1(L)\frac{\partial}{\partial W}[(1 - W)(R - R^*)\Phi],$$

and averaging the latter over an ensemble of realizations of the random field $\varepsilon_1(L)$ we obtain an equation for probability density of the square of the modulus of the reflection coefficient $P(W;L) = \langle \Phi(W;L)\rangle$:

$$\frac{\partial}{\partial L}P = 2k\gamma\frac{\partial}{\partial W}(WP) - 2D\frac{\partial}{\partial W}[W(1 - W)P] + D\frac{\partial}{\partial W}(1 - W)^2 W\frac{\partial}{\partial W}P,$$
$$P(W;L_0) = \delta(W - W_0). \qquad (18)$$

In deriving this equation and all those to follow we use the diffusion approximation for the statistical averages, which involves the assumption that the influence of $\varepsilon_1(x)$ on the wave field is small on the scale of the correlation length l_0 of the field $\varepsilon_1(x)$.[6] The diffusion approximation holds as long as $Dl_0 \ll 1$. Moreover, Eq. (18) takes into account that $D/k \ll 1$ and the phase $\varphi(L)$ of the reflection coefficient $R(L) = \sqrt{W(L)}\exp[i\varphi(L)]$ oscillates rapidly (on the scale of $\lambda/2$) compared with function $W(L)$, which varies slowly within the wavelength scale. Consequently it is valid to use an auxiliary averaging over the rapid oscillations of $R(L)$ in the derivation of Eq. (18) and others like it.

In the subsequent analysis it is convenient to write the function $W(L)$ as

$$W(L) = \frac{u(L) - 1}{u(L) + 1},\quad u(L) = \frac{1 + W(L)}{1 - W(L)} \qquad (19)$$
$$(u(L) \geq 1).$$

Here the probability density of the random quantity $u(L)$: is $P(u;L) = \langle \delta(u(L) - u)\rangle$ is governed by equation

$$\frac{\partial}{\partial L}P = k\gamma\frac{\partial}{\partial u}[(u^2 - 1)P] + D\frac{\partial}{\partial u}(u^2 - 1)\frac{\partial}{\partial u}P, \qquad (20)$$
$$P(u;L_0) = \delta(u - u_0),$$

where $u_0 = (1 + W_0)/(1 - W_0)$ and for $W_0 = 0$, $u_0 = 1$. If there is no wave absorption by the medium; that is if $\gamma = 0$, then Eq. (20) acquires the form

$$\frac{\partial}{\partial \tau}P(u;\tau) = \frac{\partial}{\partial u}(u^2 - 1)\frac{\partial}{\partial u}P(u;\tau),$$
$$P(u;0) = \delta(u - 1). \qquad (21)$$

Here the dimensionless layer thickness is introduced $\tau = D(L - L_0)$ and we take $u_0 = 1$. A solution of Eq. (21) can be readily found using Mehler-Fock integral transform[7]

$$P(u;\tau) = \int_0^\infty d\mu\mu\,\mathrm{th}(\mu\pi)\exp\left[-\left(\mu^2 + \frac{1}{4}\right)\tau\right]P_{-(1/2)+i\mu}(u), \qquad (22)$$

where $P_{-1/2+i\mu}(u)$ is Legendre function of the first kind (conic function). Using Eq. (22) one can calculate for $\gamma = 0$ the statistical properties of the modulus square of the reflection and transmission coefficient $|T|^2 = 1 - W = 2/(u + 1)$ and, in particular, obtain the expression[4,5]

$$\langle |T|^{2n}\rangle = 2^n\pi\int_0^\infty d\mu\frac{\mu\,\mathrm{sh}(\mu\pi)}{\mathrm{ch}^2(\mu\pi)}K_n(\mu)\exp\left[-\left(\mu^2 + \frac{1}{4}\right)\tau\right], \qquad (23)$$

where

$$K_{n+1}(\mu) = \frac{1}{2n}\left[\mu^2 + \left(n - \frac{1}{2}\right)^2\right]K_n(\mu), \quad K_1(\mu) = 1.$$

The following asymptotic formula results from Eq. (23)

$$\langle |T|^{2n}\rangle \approx \frac{[(2n-3)!!]^2\pi^2\sqrt{\pi}}{2^{(2n-1)}(n-1)!}\tau^{-3/2}\exp\left(-\frac{\tau}{4}\right). \tag{24}$$

Thus, the asymptotic dependence of any moment of the transmission and reflection coefficient on the layer thickness has a universe nature, the only change being that of the numerical coefficient. The vanishing of all the moments of $|T|$ with increase of the layer thickness means that the reflection coefficient modulus $|R| = \sqrt{W} \to 1$ with a probability equal to unity. Therefore a half-space of a randomly layered medium reflects the incident wave completely.

In the presence of absorption a solution of Eq. (20) for a layer of finite thickness τ is supposed to be impossible. For a half-space ($L_0 \to -\infty, \tau \to \infty$), however, there exists a "stationary" probability distribution for $W(L) = |R(L)|^2$ and $u(L)$ which does not depend on L or τ:

$$P_\infty(W) = \frac{2\beta}{(1-W)^2}\exp\left(-\frac{2\beta W}{1-W}\right), \tag{25}$$

$$P_\infty(u) = \beta\exp[-\beta(u-1)],$$

where $\beta = k\gamma/D$ is dimensionless absorption coefficient. The physical meaning of the probability densities Eqs. (25) is obvious: they describe the statistical properties of the coefficient of reflection from a rather extended randomly inhomogeneous layer that cannot be penetrated completely through by an incident wave because of its dynamic absorption by the medium. Using the distributions Eqs. (25) one can calculate all the moments of the quantity $W(L) = |R(L)|^2$. In particular, the following asymptotic formulae are valid

$$\langle W(L)\rangle = \begin{cases} 1 - 2\beta\ln(1/\beta), & \beta \ll 1, \\ 1/2\beta, & \beta \gg 1, \end{cases} \tag{26}$$

as well as the recurrence relation for the higher moments

$$n\langle W^{n+1}\rangle - 2(\beta + n)\langle W^n\rangle + n\langle W^{n-1}\rangle = 0 \tag{26'}$$

$$(n = 1, 2, ...).$$

It should be noted in addition that for the problem under consideration concerning a wave incident on a half-space of a randomly layered medium the average values of the energy flux and intensity of the wave field at the half-space boundary are defined for $\beta \ll 1$ through the asymptotic expressions

$$\langle S(L, L)\rangle = 1 - \langle W(L)\rangle = 2\beta\ln(1/\beta),$$
$$\langle I(L, L)\rangle = 1 + \langle W(L)\rangle = 2. \tag{27}$$

If the plane wave source is inside a half-space occupied by a randomly layered medium, then the wave field and the energy flux density at the source location point are described by formulas (7) and (11). Within the diffusion approximation, the reflection coefficients $R_1(x_0)$ and $R_2(x_0)$ can be viewed as statistically independent, since they are described by stochastic equations in non-intersecting regions of space in which random inhomogeneities $\varepsilon_1(x)$ of the medium are almost statistically independent. Taking this circumstance into account for an unbounded space ($L_0 \to -\infty, L \to \infty$) and using distribution (25) we come to

$$\langle I(x_0, x_0)\rangle = 1 + \beta^{-1}, \quad \langle S(x_0, x_0)\rangle = 1. \tag{28}$$

Hence, it is clear that *the average energy flux density at the source point x_0 is independent of fluctuations of the parameters of the medium and coincides with the case of a source in free space.* An unlimited growth of the average intensity for $\beta \to 0$ testifies, however, to the buildup of wave energy in a randomly layered medium.

In a similar manner, when we have a source located on a totally reflecting boundary $x_0 = L$, if we use Eq. (12) and distribution (25) we obtain

$$\langle I_{\text{ref}}(L; L)\rangle = 4[1 + (2/\beta)], \quad \langle S(L, L)\rangle = 4. \tag{29}$$

Therefore in this case, too, the energy flux density at a reflecting boundary is also independent of parameter fluctuations of the medium and coincides with the flux density for the case of a homogeneous space.

4. STATISTICAL LOCALIZATION OF A WAVE IN A RANDOMLY LAYERED MEDIUM

Let us proceed now to analyze the possibility of statistical localization of a wave in a randomly layered medium. Everywhere in what follows we shall assume that $D = 1$, which corresponds physically to a change to the dimensionless coordinate $\tilde{x} = Dx$. Expressions (27)–(29) derived above define the values of the wave intensity and the energy flux averages only at fixed space points (on the boundary of a randomly layered medium or at the source location point). However, the relation established above between the flux S and the energy of the wave E, according to which

$$\langle E\rangle = \langle S(x_0, x_0)\rangle/\beta, \tag{30}$$

allows one, on the basis of Eqs. (27)–(29), to draw general semi-qualitative conclusions regarding the average intensity inside a randomly inhomogeneous medium as well. Thus, for a wave incident on a half-space $x \leqslant L$, in view of Eq. (27), we obtain for $\beta \ll 1$,

$$\langle E\rangle = 2\ln(1/\beta), \quad \langle I(L; L)\rangle = 2,$$

and, consequently, most of the wave energy is concentrated within a region of space of thickness

$$l_\beta \approx \langle E\rangle/\langle I\rangle = \ln(1/\beta), \tag{31}$$

that is, we have statistical localization of the wave due to the wave absorption in the medium. We note that $l_\beta \ll l_a$, i.e., the localization length l_β for $\beta \ll 1$ is much less than $l_a = 1/\beta$, the wave absorption length in a homogeneous absorbing medium. The latter result can be attributed to the fact that for $\beta \ll 1$ absorption is accompanied by multiple scattering of the wave at random inhomogeneities of the medium. As a result, the wave arriving at point x covers an effective distance within the medium that is much more $(L - x)$ than that in the case of a homogeneous medium and, hence, is absorbed more. For $\beta \to 0$ we have $l_\beta \to \infty$, and for the limiting case of no absorption the statistical localization of the wave disappears.

For a source in an unbounded randomly layered space we obtain from expressions (28) and (30)

$$\langle E\rangle = 1/\beta, \quad \langle I\rangle = 1 + (1/\beta), \tag{32}$$

and, hence, for $\beta \to 0$ statistical localization of the energy takes place within the neighborhood of a source of a size of the order $l \sim 1$. Unlike the previous case, localization here is retained in the absence of absorption as well and the localization length is approximately equal to the thickness of a layer of a randomly layered medium that reflects almost completely the incident wave.

For a source located on a reflecting boundary we have correspondingly

$$\langle E \rangle = 4/\beta, \quad \langle I_{\rm ref} \rangle = 4[1 + (2/\beta)] \tag{33}$$

and, hence, for a small β the wave localization occurs within a region of thickness $l \sim 1/2$.

Let us once again emphasize that such a difference in the behavior of the average intensity for different boundary-value problems is due to statistical averaging over an ensemble, since for each individual realization of random inhomogeneities of the medium $\varepsilon_1(x)$, the corresponding realizations of the wave field possess the same spatial structure according to formulas (11) and (12), differing only in a constant random factor which can, however, be different for different realizations of $\varepsilon_1(x)$. Therefore this fundamental difference between the behavior of the average intensity of the wave incident on a randomly layered half-space and that of the source-excited wave, for example, in an unbounded randomly layered medium is brought about by correlation of these constant factors which are responsible for the resonance properties of a randomly layered medium with the basic spatial structure of the field.

5. A WAVE INCIDENT ON A RANDOMLY LAYERED SPACE

Let us now turn to a more detailed quantitative analysis of the behavior of the average wave intensity in a randomly layered medium and of the problem of statistical wave localization. Consider first the case of a wave incident on a randomly layered half-space (Fig. 2). We introduce the function

$$Q_{\lambda,\mu}(x, L, W) = \langle Y_1^{\lambda-\mu}(x; L) Y_2^{\mu}(x; L) \delta(|R_1^2(L)| - W) \rangle, \tag{34}$$

where the intensities of the counterpropagating waves $Y_{1,2}(x;L)$ and the modulus squared of the reflection coefficient $|R_1^2(L)|$ satisfy stochastic equations (16) and (17). The function $Q_{\lambda,\mu}$ describing the correlation of the counterpropagating wave intensities and the modulus of the reflection coefficient averaged over the rapid oscillations of the reflection coefficient satisfy, to the diffusion approximation, the equation:[4,5]

$$\frac{\partial}{\partial L} Q_{\lambda,\mu}(x, L, W) = -\beta\left(\lambda - 2\frac{\partial}{\partial W} W\right) Q_{\lambda,\mu} - \left[\lambda + \frac{\partial}{\partial W}(1 - W)\right] Q_{\lambda,\mu} + \left[\lambda + \frac{\partial}{\partial W}(1 - W)\right]^2 W Q_{\lambda,\mu}, \tag{35}$$

$$Q_{\lambda,\mu}(x, x, W) = W^{\mu} P_{\infty}(W),$$

where the probability distribution $P_\infty(W)$ is described by formula (25). In particular, without absorption in the medium $\beta = 0$ and for $P_\infty(W) = \delta(W - 1)$ the solution of (35) has the form

$$Q_{\lambda,\mu}(x, L, W) = \delta(W - 1)\exp[\lambda(\lambda - 1)(L - x)],$$

and, consequently,

$$\langle Y_1^{\lambda-\mu}(x; L) Y_2^{\mu}(x; L) \rangle = \exp[\lambda(\lambda - 1)(L - x)]. \tag{36}$$

In view of the arbitrary nature of parameters λ, μ relation (36) shows that, with unity probability $Y_1(x;L) = Y_2(x;L) = Y(x;L)$ and the intensities of counterpropagating waves match and each can be expressed as:[4,5]

$$Y(x; L) = \exp(\eta(L; 1) - \eta(x; 1)), \tag{37}$$

where $\eta(\xi,\alpha)$ is a Gaussian random function given by the equality (A3). Its statistical properties are defined completely by the probability density (A6). Hence, as is pointed out in the Appendix, the opposing wave intensities themselves are distributed according to the log-normal law while the intensity moments corresponding to integral values of the parameters λ and μ grow exponentially with depth into the interior of a randomly-layered medium

$$\langle Y(x; L) \rangle = 1, \quad \langle Y^n(x; L) \rangle = \exp[n(n - 1)(L - x)]. \tag{38}$$

It is worthwhile noting that, as it is indicated in the Appendix, the random function $Y(x;L)$ Eq. (37) is statistically equivalent to the random process

$$y(\xi) = \exp \eta(\xi, 1),$$

where $\xi = L - x$. It follows from the structure of a log-normal process $y(\xi)$, analyzed in the Appendix, that realizations of the counterpropagating wave intensities can have infrequent but strong spikes above the average level $\langle Y \rangle = 1$, which occur against a background of an exponential decrease of the function

$$\exp(\langle \ln Y(x; L) \rangle) = \exp(-\xi), \tag{39}$$

into the depth of the medium. In the physics of disordered systems, this is referred to as a typical realization of the random function $Y(x;L)$. The exponential decrease of the typical realization (39) with ξ is normally identified with the localization property[8,9] in the physics of disordered systems. As is evident from the formulas (A57) and (A26), the term "typical realization," as applied to the function (39), is justified by the fact that (39) is an isoprobability curve of the random function $Y(x;L)$ corresponding to the value $p = 1/2$. In other words, within any interval along the axis $\xi = L - x$ the function $Y(x;L)$ spends, on the average, half of the interval above the typical realization and the other half below it. It follows from (A26) for $\alpha = 1$ that the isoprobability curves of the random function $Y(x;L)$ are given by the equality

$$z(\xi, p) = \exp[-\xi + r(2\xi)^{1/2}], \tag{40}$$

for any $p < 1$ where r is a root of equation $\Phi(r) = p$ and $\Phi(z)$ is defined by formula (A8). Figure 4 depicts isoprobability curves Eq. (40) for $p > 0.5$ (1), $p = 0.5$ (2) and $p < 0.5$ (3). These curves and formula (40) indicate that for any given value of $p < 1$ arbitrarily close to unity the isoprobability curve tends exponentially to zero for a sufficiently large $\xi \gg 1$. Therefore, the behavior of the isoprobability curves gives additional support to the idea that the wave intensity is localized in the vicinity of the boundary of the randomly layered space. It should be noted as well that $\ln Y(x;L)$ is an *additive self-averaging random function formed from the* random intensity $Y(x;L)$,[8,9] which, from another point of view, justifies the term "typical realization" as applied to the exponential function (39).

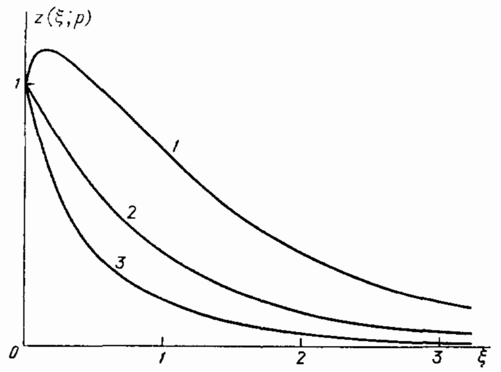

FIG. 4. Plots of isoprobability curves of random function $Y(x;L)$: *1.* $p>0.5$; *2.* $p=0.5$; *3.* $p<0.5$.

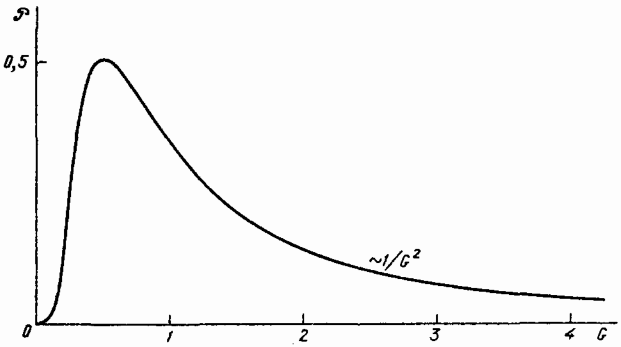

FIG. 5. Probability density of the area under the curve $Y(x;L)$ for all $x<L$.

The statistical properties of the random intensity $Y(x;L)$ of the waves in the randomly layered half-space investigated above lead to contradictory results. On the one hand, an analysis of the isoprobability curves leads to the conclusion that the wave is localized near the boundary of the medium. On the other hand, the behavior of the statistical moments of the wave intensity shows unambiguously that there is no localization. The fact is that the behavior of the wave intensity moments inside a randomly layered medium with increasing ξ is governed mainly by the infrequent but strong intensity spikes. Because of this, the average intensity of the wave inside the medium is the same at any distance from its boundary. It is only with a finite (arbitrarily small) absorption that exponential growth of intensity moments at sufficiently large distances from the boundary ceases and is replaced by an abrupt decay. In particular, for

$$\xi \gg 4[n-(1/2)]\ln(n/\beta)$$

the moments $\langle Y^n(x;L)\rangle$ enter the universal localization relation:[9]

$$\langle Y^n(x;L)\rangle \approx A_n\xi^{-3/2}\exp(-\xi/4)\beta^{-(n-1/2)}\ln(1/\beta), \qquad (41)$$

specified by diffusion operator

$$\frac{\partial}{\partial u}(u^2-1)\frac{\partial}{\partial u}$$

in the corresponding Fokker-Planck equations [see, e.g., Eq. (20)].

It must be emphasized, however, that this behavior of the statistical intensity moments, in particular, their exponential growth inside a non-absorbing randomly layered medium does not reflect the energy distribution of the wave realizations in a randomly layered half-space. The sharp spikes that govern the behavior of the intensity moments contain a relatively small amount of energy and as a result, with any given probability $p<1$, total wave energy inside a randomly layered half-space is bounded. For example, as is demonstrated in the Appendix, the probability distribution $\mathscr{P}(G)$ of the total energy

$$G=\int_{-\infty}^{L} \mathrm{d}x Y(x;L) \qquad (42)$$

of one of the counterpropagating waves in a randomly layered half-space is given by formula (A48). This probability density is shown in Fig. 5.

Note that all the moments of the probability density $\mathscr{P}(G)$ starting from the first are infinite. This conclusion can be drawn beforehand recalling that $\langle Y(x;L)\rangle = 1$ and thus,

$$\langle G\rangle = \int_{-\infty}^{L} \mathrm{d}x\langle Y(x;L)\rangle = \infty.$$

Nevertheless, from the formula Eq. (A48) it follows that the area under the realization $Y(x;L)$ for $x<L$ is bounded by the inequality

$$G<1/\ln(1/p). \qquad (43)$$

for any preassigned probability $p<1$.

The value of G, expression (42), equals the energy stored by a particular realization of the wave over the whole randomly layered half-space. More detailed probabilistic information concerning the spatial redistribution of the energy inside the medium is contained in the integral distribution function of the wave energy stored in the half-space $x'<x(x<L)$:

$$G(x)=\int_{-\infty}^{x} \mathrm{d}x' Y(x';L) \quad (G(L)=G). \qquad (44)$$

As is shown in the Appendix, the integral distributions function, $F(G,\xi)$ of the random energy (44) is described by expression (A55). From this expression it follows that the probability of satisfying inequality $G(x)<G_0$, where G_0 is an energy value specified in advance, tends to unity monotonically with an increase of ξ.

Even though, as was found previously, the wave intensity field inside a randomly layered medium has a rather complicated fine structure, it is convenient, as a rough quantitative characteristic of the region of localization, to introduce an effective wave intensity decay factor inside the layer

$$\mu = \langle Y(L;L)\rangle \left(\int_{-\infty}^{L} \mathrm{d}x Y(x;L)\right) = G^{-1},$$

based on the approximation of the wave behavior within the medium by an exponential function

$$Y(x;L) \sim \langle Y(L;L)\rangle \exp(-\mu\xi) \quad (\xi = L-x>0).$$

As follows from Eq. (A48), the probability density of the random quantity μ has the form

$$\mathscr{P}(\mu) = \exp(-\mu),$$

and its average and variance are equal to $\langle\mu\rangle = \sigma_\mu^2 = 1$. One can use $l = 1/\langle\mu\rangle = 1$ as the effective thickness of the region of wave localization in this case.

As has been mentioned previously, the presence of absorption in a randomly layered medium ($\gamma > 0$) leads to a qualitative change of behavior of the moments of the wave intensity inside the medium. For example, the exponential growth of the intensity moments, predicted above, Eq. (38), is replaced by their abrupt decay formula (41). Nevertheless, for sufficiently weak absorption ($\beta \ll 1$) the wave intensity realizations inside the medium show about the same behavior as that observed in a non-absorbing medium, and their probability properties differ only slightly from those studied above. We shall support these semi-qualitative results with a quantitative calculation. To this end we employ an equation similar to Eq. (8)

$$\beta E = S(L, L), \tag{45}$$

where E, as before, is the energy stored in a randomly layered half-space. It has to be noted that as distinct from G [Eq. (42)], which is equal to the energy of one of the counterpropagating waves in a randomly layered medium E describes total energy contained in the two waves. In what follows it will become evident that statistics of E, in a non-absorbing medium at least, coincides with that of $2G$. That means that even though the intensities of the counterpropagating waves are the same at each point in the medium, $Y_1(x;L) = Y_2(x;L)$, their phases are not correlated, coherent effects are absent, and the total intensity averaged over the phase difference, which varies rapidly in space, is equal to the sum of the counterpropagating wave intensities.

Let us return to equality (45) which relates the total energy of the wave stored in a randomly layered half-space with the intensity flux on its boundary. In the case of a wave incident on a randomly layered half-space we have $S = 1 - W$ so that the equality (45) becomes

$$\beta E = 1 - W.$$

Using the probability distribution (25) of the squared modulus of the coefficient of reflection from a randomly layered absorbing half-space one can readily find probability distribution of E:

$$\mathscr{P}_\beta(E) = \frac{2}{E^2}\exp\left[-2\left(\frac{1}{E} - \beta\right)\right]\theta\left(\frac{1}{\beta} - E\right), \tag{46}$$

which in the limit $\beta \to 0$ becomes

$$\mathscr{P}(E) = \frac{2}{E^2}\exp\left(-\frac{2}{E}\right),$$

which coincides with the probability distribution Eq. (A48) if the equality $E = 2G$ is taken into account. In formula (46) $\theta(z)$ is Heaviside step function, equal to zero for $z < 0$.

It should be noted in conclusion that the foregoing probability analysis of the wave energy in a randomly layered medium for a wave incident on a randomly layered half-space can be applied to the waves excited by a source located in the randomly layered medium. We shall calculate, as an example, the probability energy distribution for a wave excited by a source located near an ideally reflecting boundary (see Fig. 3). Making use of formulas (7) and (8) and taking into account that $R_2 = 1$ we have

$$\beta E/4 = (1 - W)/|1 - R|^2 \tag{47}$$

and consequently

$$\mathscr{P}_\beta(E) = \left(\frac{2}{\pi E^3}\right)^{1/2}\exp\left[-\frac{2}{E}\left(1 - \frac{\beta E}{4}\right)^2\right]. \tag{48}$$

for the wave energy in a randomly inhomogeneous medium. This probability density for the cases $\beta = 1$ and $\beta = 0.1$ is plotted in Fig. 6. As in Eq. (46) this probability distribution admits of a limiting process for $\beta \to 0$:

$$\mathscr{P}(E) = \left(\frac{2}{\pi E^3}\right)^{1/2}\exp\left(-\frac{2}{E}\right). \tag{49}$$

We should point out that even though in a nonabsorbing medium ($\beta \to 0$) the average wave intensity and energy, expressions (33), go to infinity, the total energy of each particular realization of a wave excited by a source in the medium is bounded with unity probability while the main bulk of the probability is concentrated in the energy range $E \sim 1$.

The curves in Fig. 7, representing typical realizations of the wave field, taken from Ref. 11, give a graphic illustration of the behavior revealed by means of the statistical analysis of waves that undergo multiple scattering in a randomly layered medium, namely, the fine structure of the intensity field, which contains alternating dark regions and sharp spikes, as well as the tendency to localization of the wave. This figure shows numerically calculated plots of $I(x;L)$ in rather thick layers $DL = 5$ for a single realization of random inhomogeneities in the medium, $\varepsilon_1(x)$ (white circles) and

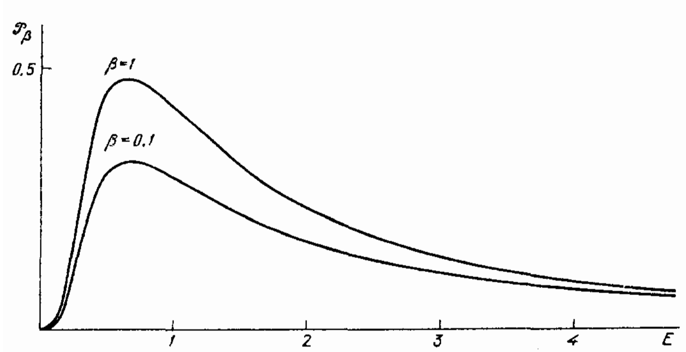

FIG. 6. Probability density of the energy of a wave radiated by a source near the reflecting boundary of a randomly layered half-space.

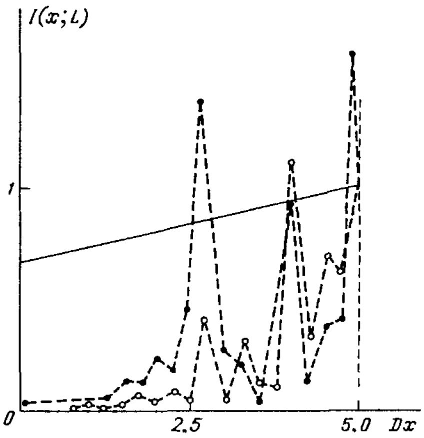

FIG. 7. Plots of the wave intensity realizations $I(x;L)$ in a layer of the medium for $\beta = 0.08$ based on numerical calculation—white circles. Black circles are for the case with $\varepsilon_1(x)$ replaced by $-\varepsilon_1(x)$ over a segment of the order of a wavelength. The continuous curve corresponds to the case without inhomogeneities in the medium.

the parameter value $\beta = 0.08$. The black circles correspond to the case where the function $\varepsilon_1(x)$ is replaced by $-\varepsilon_1(x)$ over a section of the order of a wavelength. The solid line corresponds to the absence of fluctuations over the whole layer, i.e., $\varepsilon_1(x) \equiv 0$. The curves shown here provide semiqualitative information, since the realization values $I(x;L)$ have been selected with relatively large spacings (of the order of ten wavelengths). The true curves of $I(x;L)$ are, however, much more jagged and have a substantially greater number of spikes. Even these smoothed curves cannot hide the change spikes of $I(x;L)$. The amplitudes of the spikes increase with a decrease of the absorption parameter β. Figure 7 also shows that the intensity spikes of the wave field occur against a background of a rapid decrease, which can be identified with the existence of localization for given realizations.

6. A SOURCE IN AN UNBOUNDED RANDOMLY LAYERED SPACE

Let us proceed now to analyze the statistical localization of a wave generated by a source in an unbounded randomly layered medium ($L_0 \to -\infty, L \to \infty$) (see Fig. 1). From formulas (6) and (7) the following relation results for the average wave intensity within the range

$$\beta\langle I(x, x_0)\rangle = \frac{\partial}{\partial x}\langle \psi(x, x_0)\rangle,$$

where $\psi(x,x_0)$ as a function of the parameter x_0 satisfies the stochastic equation

$$\frac{\partial}{\partial x_0}\psi(x, x_0) = -\beta\frac{|1 + \bar{R}_1(x_0)|^2}{1 - |R_1(x_0)|^2}\psi(x, x_0), \tag{50}$$

with

$$\psi(x_0, x_0) = 1.$$

We introduce an auxiliary function

$$\Phi(x, x_0; u) = \langle \psi(x, x_0)\delta(u(x_0) - u)\rangle,$$

where $U(L)$ is defined by equality (19). The function Φ, to the approximations adopted here, satisfies the equation

$$\frac{\partial}{\partial \xi}\Phi(\xi, u) = -\beta u\Phi + \beta\frac{\partial}{\partial u}(u^2 - 1)\Phi + \frac{\partial}{\partial u}(u^2 - 1)\frac{\partial}{\partial u}\Phi,$$
$$\Phi(0, u) = P_\infty(u) = \beta\exp[-\beta(u - 1)], \quad \xi = D|x - x_0|. \tag{51}$$

The average wave intensity that we seek is expressed in terms of Φ by means of the equalities

$$\beta\langle I(x, x_0)\rangle = -\frac{\partial}{\partial \xi}\int_1^\infty du\Phi(\xi, u) = \int_1^\infty du u\Phi(\xi, u). \tag{52}$$

For $\beta \ll 1$ the factor β on the left-hand side of this equality has the role of normalizing to the average wave intensity at the source point Eqs. (28). Therefore, as $\beta \to 0$ it is natural to call the limit curve

$$\Phi_{\rm loc}(\xi) = \lim_{\beta\to 0}(\beta\langle I(x, x_0)\rangle) = \lim_{\beta\to 0}\langle I(x, x_0)\rangle/\langle I(x_0, x_0)\rangle. \tag{53}$$

a localization curve, describing the statistical localization of the wave in a non-absorbing medium. It can be easily demonstrated that the localization curve is given by the expression

$$\Phi_{\rm loc}(\xi) = \int_1^\infty du u\tilde{\Phi}(\xi, u),$$

where $\tilde{\Phi}$ satisfies the equation

$$\frac{\partial}{\partial \xi}\tilde{\Phi}(\xi, u) = -u\tilde{\Phi} + \frac{\partial}{\partial u}(u^2\tilde{\Phi}) + \frac{\partial}{\partial u}\left(u^2\frac{\partial}{\partial u}\tilde{\Phi}\right), \tag{54}$$
$$\tilde{\Phi}(0, u) = e^{-u}.$$

Solving Eq. (54) with the help of the Kantorovich-Lebedev integral transform [7] yields [3]:

$$\Phi_{\rm loc}(\xi) = -2\pi\frac{\partial}{\partial \xi}\int_0^\infty d\tau\,\frac{\tau\,\mathrm{sh}(\pi\tau)}{\mathrm{ch}^2(\pi\tau)}\exp\left[-\left(\tau^2 + \frac{1}{4}\right)\xi\right]$$
$$= -\frac{\partial}{\partial \xi}\langle |T^2(\xi)|\rangle, \tag{55}$$

where $|T(\xi)|^2$ is the squared modulus of the transmission coefficient for a layer of thickness ξ in the case of an incident plane wave [see formula (23)]. For small values of ξ the localization curve falls off very rapidly as $\exp(-2\xi)$, while for large $\xi(\xi \gg \pi^2)$ it does so much slower following the universal law

$$\Phi_{\rm loc}(\xi) \approx \frac{\pi^2\sqrt{\pi}}{8}\xi^{-3/2}\exp\left(-\frac{\xi}{4}\right), \tag{56}$$

such that the total area under the localization curve is $\int_0^\infty d\xi\Phi_{\rm loc}(\xi) = 1$. Figure 8 a localization curve, and the dashed lines indicate the above-mentioned asymptotic curves for comparison. We should stress a fact of fundamental importance, that the localization curve corresponds to the double limit

$$\Phi_{\rm loc}(\xi) = \lim_{\beta\to 0}\lim_{\substack{L_0\to-\infty,\\ L\to\infty}}\langle I(x, x_0)\rangle/\langle I(x_0, x_0)\rangle, \tag{57}$$

and, as can be easily shown in this case, *these limits do not commute*, and the limiting process performed in reverse order yields

$$\lim_{\substack{L_0\to-\infty,\\ L\to\infty}}\lim_{\beta\to 0}\langle I(x, x_0)\rangle/\langle I(x_0, x_0)\rangle = (2/3)(D|x - x_0| \gg 1),$$

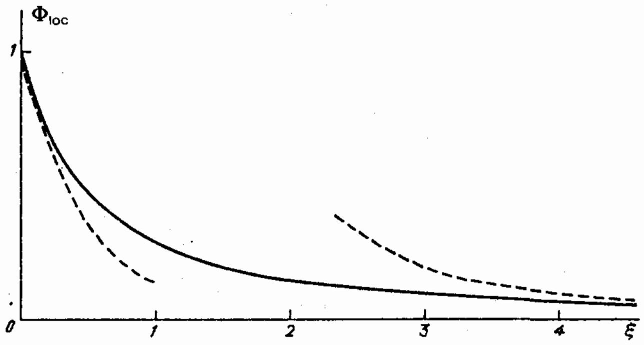

FIG. 8. Localization curves. Dashed lines show the formulas for the asymptotic localization curve valid for $\xi \ll 1$ and $\xi \gg 1$.

which is similar to the case of a plane wave incident on a layer of randomly inhomogeneous medium for which these limits are commutative. Recall that the order of taking the limits adopted in Eq. (57) corresponds to the physical sense of the problem stated, inasmuch as the presence of an arbitrarily small but finite absorption for $L_0 \to -\infty, L \to \infty$ automatically satisfies the radiation conditions.

We should note that a situation of this type can be observed in the case of a source located on a reflecting boundary as well (Fig. 3). Moreover, it can be shown that at distances $\xi = D(L - x) \gtrsim 1/3$ from a reflecting boundary the localization curve is given by

$$\lim_{\beta \to 0} \langle I_{\text{ref}}(x; L)\rangle / \langle I_{\text{ref}}(L; L)\rangle = \frac{1}{2}\Phi_{\text{loc}}(\xi),$$

while at smaller distances from the reflecting boundary $\xi \lesssim 1/3$ the localization curve oscillates because of interference of the incident and reflected waves, which are mutually coherent near the boundary. Recall also that the probability distribution (49) of the energy of this wave, according to which the energy of each particular wave realization is finite, testifies to the localization of a wave excited near a reflecting boundary in a randomly layered half-space. The finite energy implies in addition that in a non-absorbing randomly layered medium bounded with an ideally reflecting mirror the multiply scattered waves efficiently suffer mutual extinction.

7. NONSTATIONARY WAVE PROBLEMS

Until now we have discussed the spatial intensity behavior of a wave radiated by a monochromatic source in a randomly layered medium. More directly relevant to the problem of localization are nonstationary wave problems concerning the radiation or incidence of wave pulses on the medium. In the case of a pulsed source of plane waves located within a layer of the medium at the point x_0 the wave field $u(x,x_0;t)$ satisfies the following boundary-value problem

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2(x)}\left(\frac{\partial}{\partial t} + \tilde{\gamma}\right)^2\right] u(x, x_0; t) = -\frac{2}{c}\delta(x - x_0)\frac{d}{dt}\tilde{\varphi}(t),$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right) u(x, x_0; t)\Big|_{x=L} = 0,$$

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right) u(x, x_0; t)\Big|_{x=L_0} = 0, \tag{58}$$

where the right-hand side describes the generation of a wave pulse. In particular, in a homogeneous ($c(x) = c = \text{const}$) and non-absorbing ($\tilde{\gamma} = 0$) medium the wave pulse $\tilde{\varphi}(t + (x - x_0)/c)(x < x_0)$ is generated on the left of the source. When analyzing a nonstationary wave problem we were able to use the results of the previous sections, applying the spectral approach, according to which the solution of the boundary-value problem Eq. (58) can be given as

$$u(x, x_0; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega G_\omega(x, x_0)\varphi(\omega)e^{-i\omega t}, \tag{59}$$

where $G_\omega(x,x_0)$ is the solution to the boundary-value problem (1), (2) with parameters $k = \omega/c$, $\gamma = \tilde{\gamma}/\omega$, $\varepsilon_1(x) = [c^2(x) - c^2]/c^2$, and $\varphi(\omega)$ is the Fourier transform of the time pulse $\tilde{\varphi}(t)$.

As before, we shall be concerned with the behavior of the wave field intensity in space and time

$$I(x, x_0; t) = u^2(x, x_0; t).$$

Using the spectral representation of the field, Eq. (59), we write the following expression for the intensity

$$I(x, x_0; t) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} d\omega d\psi I_{\omega,\psi}(x, x_0)\varphi\left(\omega + \frac{\psi}{2}\right)\varphi^*\left(\omega - \frac{\psi}{2}\right)e^{-i\psi t}. \tag{60}$$

In this formula a two-frequency analogue of the intensity of plane monochromatic waves is introduced

$$I_{\omega,\psi}(x, x_0) = G_{\omega+(\psi/2)}(x, x_0)G^*_{\omega-(\psi/2)}(x, x_0). \tag{61}$$

The problem of the possible localization of the wave pulse in a randomly layered medium is solved by analyzing the asymptotic behavior of its intensity for $t \to \infty$. Here the behavior of the average intensity is determined by that of the integrand in (6) for small values of ψ while an expression for the average intensity can be given in a simplified form

$$\langle I(x, x_0; t)\rangle = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega |\varphi(\omega)|^2 \int_{-\infty}^{\infty} d\psi \langle I_{\omega,\psi}(x, x_0)\rangle e^{-i\psi t}. \tag{62}$$

Proceeding from Eq. (1) the following relation can be readily obtained for the two-frequency intensity $I_{\omega,\psi}$ for low ψ and $\tilde{\gamma}$ which is similar to Eq. (6)

$$\frac{d}{dx}S_{\omega,\psi}(x, x_0) = \frac{1}{c}(\tilde{\gamma} - i\psi)I_{\omega,\psi}(x, x_0), \tag{63}$$

where $S_{\omega,\psi}(x,x_0)$ is a two-frequency analogue of the energy flux density. Integrating (63) over a randomly layered half-space $-\infty < x < x_0$ and taking Eq. (62) into account we obtain

$$E(t) = \int_{-\infty}^{x_0} dx \langle I(x, x_0; t)\rangle = \frac{c}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega |\varphi(\omega)|^2 \int_{-\infty}^{\infty} \frac{d\psi}{\tilde{\gamma} - i\psi}\langle S_{\omega,\psi}(x_0, x_0)\rangle e^{-i\psi t}. \tag{64}$$

for the energy contained within this half-space.

Let us consider now statistical description of the quantities $S_{\omega,\psi}$ and $I_{\omega,\psi}$. These quantities are related, by definition, with a two-frequency analogue of the modulus squared of the reflection coefficient $W_{\omega,\psi} = R_{\omega+\psi/2} R^*_{\omega-\psi/2}$. Therefore, to calculate the averages (62), (64) knowledge of the statistics of the quantity $W_{\omega,\psi}$ is required. For small $\tilde{\gamma}$ and ψ it is governed by the stochastic equation

$$\frac{d}{dx}W_{\omega,\psi}$$
$$= -\frac{2}{c}(\tilde{\gamma} - i\psi)W_{\omega,\psi} - i\frac{\omega}{2c}\varepsilon_1(x)(R_{\omega+(\psi/2)}$$
$$- R^*_{\omega-(\psi/2)})(1 - W_{\omega,\psi}), \tag{65}$$

where R_ω satisfies the equation

$$\frac{d}{dx}R_\omega = \frac{2i}{c}\left(\omega + i\frac{\tilde{\gamma}}{2}\right)R_\omega + i\frac{\omega}{2c}\varepsilon_1(x)(1 + R_\omega)^2,$$
$$R_\omega(-\infty) = 0.$$

It should be emphasized that Eq. (65) for $W_{\omega,\psi}$ is linear in $\tilde{\gamma}$, ψ and $\varepsilon_1(x)$. From Eqs. (65) and (66) it follows that the moments of $W_{\omega,\psi}: W^{(n)}_{\omega,\psi} = \langle [W_{\omega,\psi}]^n \rangle$ are interrelated through the sequence of equations

$$\frac{d}{dx}W^{(n)}_{\omega,\psi} = -\frac{2n}{c}(\tilde{\gamma} - i\psi)W^{(n)}_{\omega,\psi} + D(\omega)n^2(W^{(n+1)}_{\omega,\psi}$$
$$- 2W^{(n)}_{\omega,\psi} + W^{(n-1)}_{\omega,\psi}),$$

which is transformed, in the case of an unbounded randomly layered medium, into

$$\frac{2}{c}(\tilde{\gamma} - i\psi)W^{(n)}_{\omega,\psi} = D(\omega)n(W^{(n+1)}_{\omega,\psi} - 2W^{(n)}_{\omega,\psi} + W^{(n-1)}_{\omega,\psi}). \tag{66}$$

Here and above the dependence of the diffusion coefficient $D(\omega)$ of the inhomogeneities of the random medium on the frequency ω is taken into account explicitly. In particular, if $\varepsilon_1(x)$ is white noise with the correlation function $\langle \varepsilon_1(x)\varepsilon_1(x') \rangle = 2\sigma^2 l_0 \delta(x - x')$, the diffusion coefficient will be $D(\omega) = \sigma^2 l_0 \omega^2/2c^2$.

Note that recurrence relation (26') resulting from probability distribution $P_\infty(W)$ Eqs. (25) is transformed into the equation sequence (66) after replacing β with $(\tilde{\gamma} - i\psi)/cD$. This means that the statistical properties of $W_{\omega,\psi}$ in an unbounded randomly layered medium with no absorption ($\tilde{\gamma} = 0$) can be found with the help of distribution (25) continued analytically into the complex plane

$$\beta \to (0 - i\psi)/cD. \tag{67}$$

As a result, for $\tilde{\gamma} = 0$ and $t \to \infty$, i.e., for low values of ψ we obtain

$$\langle S_{\omega,\psi}(x_0, x_0) \rangle = 1, \quad \langle I_{\omega,\psi}(x_0, x_0) \rangle = i\frac{D(\omega)c}{\psi + i0},$$

and, consequently, formulas (62) and (64) lead to the expressions

$$\langle I(x_0, x_0; \infty) \rangle = \frac{c}{2\pi}\int_{-\infty}^{\infty} d\omega D(\omega)|\varphi(\omega)|^2,$$
$$E(\infty) = \frac{c}{2\pi}\int_{-\infty}^{\infty} d\omega |\varphi(\omega)|^2. \tag{68}$$

Thus, provided the integrals (68) exist, the average field intensity at the source point and total wave energy in half-space are finite, which implies spatial localization for the average wave field intensity in the medium. The localization length in this case obviously will be defined by the equality

$$l = E/\langle I \rangle = \int_{-\infty}^{\infty} d\omega |\varphi(\omega)|^2 \quad \left(\int_{-\infty}^{\infty} d\omega D(\omega)|\varphi(\omega)|^2\right)^{-1}.$$

By analogy, using the equalities (53), one can show that the form of the localization curve of the wave pulse is

$$\langle I(x, x_0; \infty) \rangle = \frac{c}{2\pi}\int_{-\infty}^{\infty} d\omega D(\omega)|\varphi(\omega)|^2 \Phi_{\text{loc}}(\xi), \tag{69}$$
$$\xi = D(\omega)|x - x_0|,$$

where $\Phi_{\text{loc}}(\xi)$ is the localization curve (55) of the stationary problem. In particular, for the white-noise model and a generated "videopulse" $\varphi(t)$ characterized by only one parameter—the pulse width—we have from Eq. (69)

$$\langle I(x, x_0; \infty) \rangle \sim |x - x_0|^{-3/2}.$$

If, however, a pulse with a high-frequency carrier is generated, with a spectrum concentrated within a narrow band $\Delta\omega$ around the central frequency ω_0 ($\omega_0 \gg \Delta\omega$), then the asymptotic form of the localization curve will be

$$\langle I(x, x_0; \infty) \rangle \sim \Phi_{\text{loc}}(\xi) \quad (\xi = D(\omega_0)|x - x_0|).$$

The asymptotic expression (69) for the localization curve of the wave pulse (69) has a simple physical interpretation. At long enough times the field of a multiply scattered pulse can be represented as a superposition of statistically mutually independent wave packets, each having width $\Delta\omega$. For $t \to \infty$, $\Delta\omega \to 0$ and the field of every wave packet is localized in space according to the laws described for the case of a stationary problem. Figuratively, when $t \to \infty$ a complete randomization of the phases of all time harmonics of the pulse is observed and each harmonic is localized in space independently. As a result, for $t \to \infty$ the localized pulse field can be represented at each point in the region of localization as a stationary random process with a spectral density of the form $\sim D(\omega)|\varphi(\omega)|^2$ which is determined by the shape of the generated pulse spectrum.

In conclusion we observe that when a time pulse is incident on a half-space the following expression[13] is valid for the asymptotic behavior of the back-scattered signal intensity for large t

$$\langle I(L, L; t) \rangle = \frac{c}{\pi}\int_{-\infty}^{\infty} d\omega |\varphi(\omega)|^2 \frac{D(\omega)}{(2 + D(\omega)ct)^2}, \tag{70}$$

from which we obtain for the pulses without a high-frequency carrier and with a high-frequency carrier, respectively,

$$\langle I(L, L; t) \rangle \sim t^{-3/2} \text{ and } \langle I(L, L; t) \rangle \sim t^{-2}.$$

By analogy for the total energy in a half-space we obtain the following expression for large t

$$E(t) = \frac{c}{\pi}\int_{-\infty}^{\infty} d\omega |\varphi(\omega)|^2 \frac{1}{2 + D(\omega)ct}. \tag{71}$$

From expressions (70) and (71) it follows that for $t\to\infty$ radiation is completely emitted from a randomly inhomogeneous medium.

In this article we have presented a statistical description of a wave pulse in a randomly layered medium. In a similar way one can consider a problem of a space-time packet in a randomly inhomogeneous medium. In this case it is clear that the localization property described in this article will also hold true for this situation, which can be interpreted as the existence of a stochastic waveguide in the plane perpendicular to the x axis.[2]

8. CONCLUSIONS

To complete our discussion let us point out that analysis of wave localization in randomly layered media has been essentially reduced to a study of statistical and dynamic properties of realizations of a log-normal process equal to the exponential function of the Wiener process. This fact is of particular importance in that similar log-normal processes are observed in practically all branches of physics where a description of characteristics of positive-definite physical quantities processes, and fields is required. This includes the description of intensity fluctuations of optical and radio waves in turbulent media and the analysis of the behavior of the amplitudes of radiophysical systems subject to fluctuations in the parameter. In these physical problems, among many others, a log-normal process emerges as the simplest adequate model accounting correctly for principal properties of the phenomena under investigation—positive definiteness, conservation laws, parametric instability, alternation of the "fading signal" portions and large sharp spikes in narrow regions.

Therefore, the importance of the analysis of the statistical and dynamic properties of a log-normal process reaches far beyond the bounds of an important but yet relatively narrow physical problem to which the present article is dedicated.

At the same time, undeservedly little attention has been paid to many outstanding features of such processes which afford a deeper insight into the fundamentally important properties of physical phenomena where the log-normal process proves to be the simplest adequate model.

Among the aims of this article, the most important one is, as we understand it, to draw closer attention of researchers in various fields of physics to log-normal processes and to the nontraditional approaches to analyzing such processes suggested in this paper. In particular, we wish to draw attention to a new understanding of isoprobability and majorant curves, to the statistics of the areas under the realizations, the fractal properties, understanding of the dynamical and statistical properties, which necessarily arise in various physical contexts in the most widely diverse physical contexts.

9. APPENDICES

9.1. Statistical and dynamic properties of Wiener and log-normal processes

Consider a random function $\omega(\xi)$ of the argument ξ which will be called time for definiteness. Let $\omega(\xi)$ be a continuous Gaussian random process with independent increments. The latter condition means that if the intervals (ξ_1,ξ_2) and (ξ_3,ξ_4) do not overlap the process increments $\omega(\xi)$ over these intervals

$$\Delta\omega(\xi_i,\xi_{i+1}) = \omega(\xi_{i+1}) - \omega(\xi_i), \quad i = 1, 3,$$

are statistically independent. Just as the process $\omega(\xi)$ itself, its increments have Gaussian statistics while their statistical properties are specified entirely by the first two moments

$$\langle\Delta\omega\rangle = 0, \quad \langle\Delta\omega(\xi,\xi+\Delta))^2\rangle = 2|\Delta|. \tag{A1}$$

It will be assumed without loss of generality that the process $\omega(\xi)$ is "tied to zero"

$$\omega(0) = 0. \tag{A2}$$

The process $\omega(\xi)$ with the aforementioned properties is called a *Wiener process*. A typical realization of a Wiener process is depicted in Fig. 9.

Let us consider specific features of the dynamic behavior of realization of a Wiener process. It is homogeneous in ξ in that realizations of the processes $\omega(\xi)$ and $\Delta\omega(\xi_0,\xi_0+\xi)$ as functions of ξ are statistically equivalent for any given parameter ξ_0. Figuratively speaking, from the realization form of these processes it is not possible to tell to which process it belongs. The processes $\omega(\xi)$ and $\omega(-\xi)$ are statistically equivalent and, hence, the Wiener process is reversible in time in the sense indicated above. Realizations of the Wiener process have another outstanding property, the fractal property. According to this property the realizations of the Wiener process $\omega(a\xi)$ compressed in time (for $a>1$) are statistically equivalent to the vertically extended realizations $\sqrt{a}\,\omega(\xi)$. The fractal property of the Wiener process can be also interpreted as the statistical equivalence of realizations of $\omega(\xi)$ and of the process $\omega(a\xi)/\sqrt{a}$ compressed both in ξ and along the vertical coordinate.

Let us consider a more general process

$$\eta(\xi;\alpha) = \omega(\xi) - \alpha\xi, \tag{A3}$$

satisfying the stochastic differential equation

$$\frac{d\eta}{d\xi} + \alpha = f(\xi), \tag{A4}$$

where $f(\xi)$ is Gaussian white noise with a correlation function

$$\langle f(\xi)f(\xi+s)\rangle = 2\delta(s).$$

Like the Wiener process, the process $\eta(\xi,\alpha)$ is homogeneous in time and has statistically independent increments in ξ. Because of the increment independence the process $\eta(\xi,\alpha)$ is Markovian and its probability density.

$$\mathcal{P}(\eta;\xi,\alpha) = \langle\delta(\eta-\eta(\xi;\alpha))\rangle$$

satisfies the Fokker–Planck equation

$$\frac{\partial}{\partial\xi}\mathcal{P} = \alpha\frac{\partial}{\partial\eta}\mathcal{P} + \frac{\partial^2}{\partial\eta^2}\mathcal{P}, \tag{A5}$$

$$\mathcal{P}(\eta;0,\alpha) = \delta(\eta),$$

which can be derived as a result of stochastic equation (A4). The solution of equation (A5) has the form

$$\mathcal{P}(\eta;\xi,\alpha) = \frac{1}{2\sqrt{\pi\xi}}\exp\left[-\frac{(\eta+\alpha\xi)^2}{4\xi}\right]. \tag{A6}$$

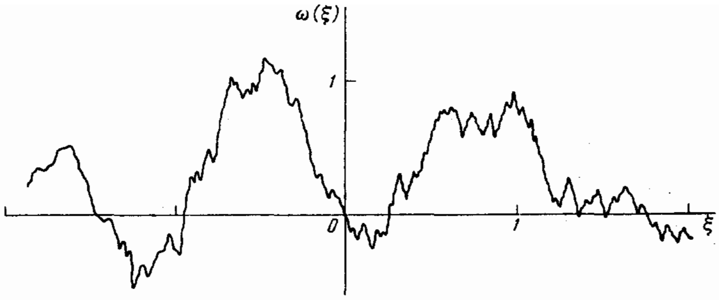

FIG. 9. Typical realization of the Wiener process $\omega(\xi)$.

The corresponding integral distribution function, which is equal to the probability that $\eta(\xi,\alpha) < \eta$ is

$$F(\eta;\xi,\alpha) = P(\eta(\xi,\alpha) < \eta) = \int_{-\infty}^{\eta} d\eta \mathcal{P}(\eta;\xi,\alpha) = \Phi\left(\frac{\eta}{(2\xi)^{1/2}} + \alpha\left(\frac{\xi}{2}\right)^{1/2}\right), \tag{A7}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} dy \exp\left(-\frac{y^2}{2}\right). \tag{A8}$$

In addition to the initial condition, we shall impose the following boundary condition on Eq. (A5)

$$\mathcal{P}(\eta = h;\xi,\alpha) = 0 \quad (\xi > 0), \tag{A9}$$

which terminates the realization of the process $\eta(\xi,\alpha)$ when they reach the boundary h. A solution to the boundary problem (A5) and (A9) denoted as $\mathcal{P}(\eta;\xi,\alpha,h)$ describes, for $\eta < h$, the probability distribution of the values of those realizations of the process $\eta(\xi,\alpha)$ that "survive" to the time ξ; that is, they never reach the boundary h over the whole time interval $(0,\xi)$. Correspondingly, the probability density $\mathcal{P}(\eta;\xi,\alpha,h)$ is not normalized to unity but to the probability that $\xi^* > \xi$ where ξ^* is the time at which the process $\eta(\xi;\alpha)$ reaches the boundary h for the first time

$$\int_{-\infty}^{h} d\eta \mathcal{P}(\eta;\xi,\alpha,h) = P(\xi < \xi^*). \tag{A10}$$

Let us introduce the integral distribution function and probability density of the random time of the first arrival

$$F(\xi;\alpha,h) = 1 - P(\xi < \xi^*) = 1 - \int_{-\infty}^{h} \mathcal{P}(\eta;\xi,\alpha,h) d\eta, \tag{A11}$$

$$\mathcal{P}(\xi;\alpha,h) = \frac{\partial F}{\partial \xi} = -\frac{\partial}{\partial \eta}\mathcal{P}(\eta;\xi,\alpha,h)\Big|_{\eta=h}. \tag{A12}$$

For $\alpha > 0$, where the process $\eta(\xi;\alpha)$ moves away from the boundary h with increasing ξ and for $\xi \to \infty$ the probability $P(\xi < \xi^*)$ (A10) converges to the probability that the process $\eta(\xi;\alpha)$ never reaches the boundary h. In other words, the limit

$$\lim_{\xi\to\infty} \int_{-\infty}^{h} d\eta \mathcal{P}(\eta;\xi,\alpha,h) = P(\eta_M < h) \tag{A13}$$

is equal to the probability that absolute maximum of the process

$$\eta_M(\alpha) = \max_{\xi\in(0,\infty)} \eta(\xi;\alpha) \tag{A14}$$

is less than h. Therefore, it follows from Eqs. (A13) and (A10) that the integral distribution function for the absolute maximum values η_M is

$$F(h,\alpha) = P(\eta_M < h) = \lim_{\xi\to\infty} \int_{-\infty}^{h} d\eta \mathcal{P}(\eta;\xi,\alpha,h). \tag{A15}$$

Solving the boundary-value problem (A5) and (A9), and using the reflection method, we obtain

$$\mathcal{P}(\eta;\xi,\alpha,h) = \frac{1}{2(\pi\xi)^{1/2}}\left\{\exp\left[-\frac{(\eta+\alpha\xi)^2}{4\xi}\right] - \exp\left[-h\alpha - \frac{(\eta - 2h + \alpha\xi)^2}{4\xi}\right]\right\}. \tag{A16}$$

Substituting this expression into (A12) we find the probability density of the time ξ^* at which the process $\eta(\xi,\alpha)$ reaches the boundary h for the first time

$$\mathcal{P}(\xi;\alpha,h) = \frac{h}{2\xi(\pi\xi)^{1/2}}\exp\left[-\frac{(h+\alpha\xi)^2}{4\xi}\right].$$

Finally, integrating (A16) over η and with $\xi \to \infty$ we obtain, according to (A15) the integral distribution function for the absolute maximum η_M:

$$F(h;\alpha) = 1 - \exp(-h\alpha). \tag{A17}$$

Let me turn now to a description of the statistical properties of the log-normal process

$$y(\tau,\xi;\alpha) = \exp(\eta(\tau;\alpha) - \eta(\xi;\alpha)) = \exp[\Delta\omega(\tau,\xi) - \alpha(\tau - \xi)]. \tag{A18}$$

It can be still written as

$$y(\tau,\xi;\alpha) = y(\tau;\alpha)/y(\xi;\alpha), \tag{A19}$$

where

$$y(\tau;\alpha) = e^{\eta(\tau;\alpha)}. \tag{A20}$$

We should bear in mind that the process in the argument of the exponential, $\eta(\tau,\alpha)$, has independent increments. In physics this property is called the additive property of the process $\eta(\tau,\alpha)$. Correspondingly, the process $y(\tau;\alpha)$ has a *multiplicative* feature, according to which the process $y(\tau,\alpha)$ can be expressed as the product of statistically independent processes

$$y(\tau;\alpha)=y(\xi;\alpha)y(\tau,\xi;\alpha), \tag{A21}$$

so that realizations of the process $y(\tau,\xi,\alpha)$ as functions of the argument $\varkappa=\tau-\xi$ are statistically equivalent to realizations of the process $y(\varkappa,\alpha)$. The latter property of the process $y(\tau,\alpha)$ can be naturally called the property of multiplicative homogeneity in time.

Let us discuss in more detail the log-normal process $y(\tau,\alpha)$ (A20). It satisfies the stochastic equation

$$\frac{dy}{d\tau}+\alpha y=f(\tau)y,\quad y(0)=1. \tag{A22}$$

From this result it follows that the probability density of the process

$$\mathcal{P}(y;\tau,\alpha)=\langle\delta(y(\tau;\alpha)-y)\rangle$$

satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial\tau}\mathcal{P}=\alpha\frac{\partial}{\partial y}(y\mathcal{P})+\frac{\partial}{\partial y}[y\frac{\partial}{\partial y}(y\mathcal{P})], \tag{A23}$$
$$\mathcal{P}(y;0,\alpha)=\delta(y-1),$$

whose solution is a log-normal probability density. It can be found by noting that the probability of fulfilling the inequality

$$y(\tau;\alpha)<y \tag{A24}$$

is equal exactly to the probability of satisfying the inequality

$$\eta(\tau;\alpha)<\ln y. \tag{A25}$$

Then using (A7), we see that integral distribution function for the process $y(\tau,\alpha)$ is

$$F(y;\tau,\alpha)$$
$$=\Phi\left(\frac{\ln y}{(2\tau)^{1/2}}+\alpha\left(\frac{\tau}{2}\right)^{1/2}\right)=\Phi\left(\frac{1}{2\tau^{1/2}}\ln(ye^{\alpha\tau})\right). \tag{A26}$$

Differentiating it with respect to y and taking definition (A8) into account, we come to the solution of equation (A23)

$$\mathcal{P}(y;\tau,\alpha)=\frac{1}{2(\pi\tau)^{1/2}y}\exp\left[-\frac{1}{4\tau}\ln^2(ye^{\alpha\tau})\right]. \tag{A27}$$

The log-normal probability density (27) is plotted in Fig. 10 for $\tau=0.5$ and $\alpha=1$. With the help of this probability density or, which is even simpler, directly from Eq. (A23) one can find moments of the process $y(\tau;\alpha)$

$$\langle y^n(\tau;\alpha)\rangle=\exp[n(n-\alpha)\tau]. \tag{A28}$$

In particular, for the process

$$y(\tau)=y(\tau;1)=\exp(\omega(\tau)-\tau), \tag{A29}$$

which plays the most significant role in this paper, the moments are equal to

$$\langle y^n(\tau)\rangle=\exp[n(n-1)\tau] \tag{A30}$$

The average of the process $y(\tau)$: $\langle y(\tau)\rangle=1$ is the same for any τ while all other moments of $y(\tau)$ increase exponentially with τ.

The exponential growth of the higher moments of a log-normal process $y(\tau)$ is attributed to a slow decrease of tails of the probability density Eq. (27) for $y\gg 1$. In terms of realizations, this means that in the realizations of the process $y(\tau)$ we should observe increasingly rare but increasingly high spikes that are responsible for the exponential growth of the moments of $y(\tau)$. At the same time, as seen from Eq. (27) and Fig. 10, the bulk of the probability of the process $y(\tau)$

$$\mathcal{P}(y;\tau)=\mathcal{P}(y;\tau,\alpha=1)$$

is concentrated within the range of small values of y. Indeed, according to Eq. (26) the probability of satisfying inequality $y(\tau)<1$ is

$$P(y(\tau)<1)=F(1;\tau,1)=\Phi((\tau/2)^{1/2})$$

and tends exponentially to unity for $\tau\gg 1$:

$$P(y(\tau)<1)=1-\frac{1}{(\pi\tau)^{1/2}}\exp\left(-\frac{\tau}{4}\right). \tag{A31}$$

Thus, although the statistical moments of the process $y(\tau)$ are mainly determined by its large spikes, during the overwhelming majority of the time of the plot the realization of the plot lies below its average value $\langle y(\tau)\rangle=1$.

The observed discrepancy between the behavior of the statistical moments of the process $y(\tau)$ and its realizations prompts a more detailed investigation of the dynamics of the

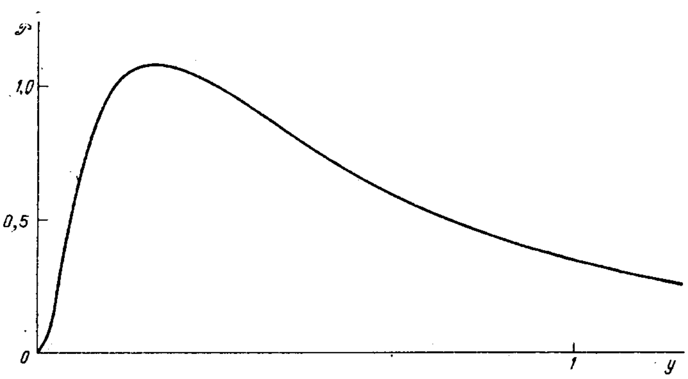

FIG. 10. Log-normal probability density for $\tau=0.5$, $\alpha=1$.

realizations of the process $y(\tau)$ and the more general process $y(\tau;\alpha)$. For this purpose we introduce the idea of majorant curves. We denote a curve as a majorant curve that curve $M(\tau,p,\alpha)$ for which for any τ the inequality

$$y(\tau;\alpha) < M(\tau,p,\alpha). \tag{A32}$$

is satisfied with probability p. In other words, $100\,p$ percent of all realizations of the process $y(\tau;\alpha)$ are located under the majorant curve $M(\tau,p,\alpha)$. The above-studied statistics of an absolute maximum (A14) of the process $\eta(\tau;\alpha)$ permits us to outline a substantially rich class of majorant curves. Let the probability that the absolute maximum $\eta_M(\beta)$ of an auxiliary process $\eta(\tau;\beta)$ with an arbitrary value of the parameter β, lying between $0<\beta<\alpha$ satisfies the inequality

$$\eta_M(\beta) < h = \ln A \tag{A33}$$

be equal to p. Then, obviously, with the same probability p all the realizations of the process $y(\tau;\alpha)$ will lie below the majorant curve

$$M(\tau,p,\alpha,\beta) = A\exp[(\beta-\alpha)\tau]. \tag{A34}$$

As is evident from Eq. (A17), the probability with which the process $y(\tau;\alpha)$ nowhere exceeds majorant curve Eq. (A34) depends on its parameters in the following way

$$p = 1 - A^{-\beta}. \tag{A35}$$

If we apply these ideas to the process $y(\tau)$ (A29) it will become evident that its realizations are bounded from above, with probability given by Eq. (A35), by the majorant curve

$$M = A\exp[(\beta-1)\tau]. \tag{A36}$$

Let us draw attention to an important point, that despite the fact that the statistical average $\langle y(\tau)\rangle = 1$ is a constant and the higher moments of the process $y(\tau)$ increase exponentially, one can always distinguish an exponentially decreasing (for $\beta<1$) majorant curve Eq. (A36), below which are located realizations of the process $y(\tau)$ with any probability $p<1$ prescribed in advance. In particular, one half of the realizations of $y(\tau)$ are located below the exponentially decreasing majorant curve

$$M = 4\exp(-\tau/2). \tag{A37}$$

A typical realization of the process $y(\tau)$ and the majorant curve (A37) are depicted in Fig. 11.

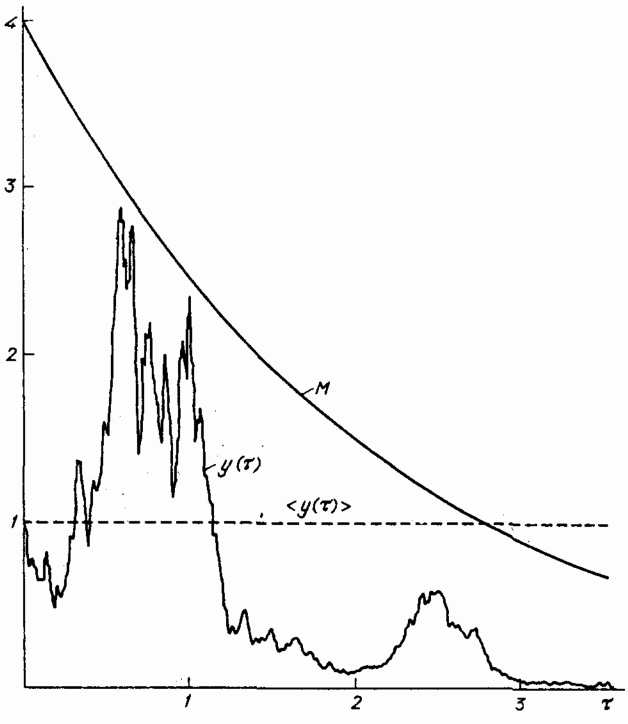

FIG. 11. Typical realization of the process $y(\tau)$ and of the majorant curve under which one half of the realizations of the process $y(\tau)$ are located. Dashed straight line is for the statistical average $\langle y(\tau)\rangle$.

The existence of an exponentially decreasing majorant curve leads to two conclusions useful for understanding the statistics and dynamics of the realizations of the $y(\tau;\alpha)$ process. First, even though the behavior of the higher moments of these processes is governed by the presence of large spikes in their realizations these spikes themselves are not observed in all process realizations. This means, for example, that the constant nature of the average of $\langle y(\tau)\rangle = 1$ and exponential growth of higher moments of $y(\tau)$ are entirely statistical effects attributed to the averaging over the whole ensemble of realizations, among which, together with those dropping rather rapidly, realizations with large spikes are observed. Second, the area under the exponentially decaying majorant curves is finite. Consequently, the large spikes, while causing the exponential growth of the higher moments, do not make a significant contribution to the area under the realizations, which is also finite for almost all realizations of the processes $y(\tau;\alpha)$.

It therefore seems interesting to analyze the statistics of the random area under the realizations of process

$$G(\alpha) = \int_0^\infty d\tau\, y(\tau;\alpha). \tag{A38}$$

Consider an auxiliary random process $G(\tau;\alpha)$ satisfying the stochastic equation

$$\frac{d}{d\tau}G = 1 - \alpha G + f(\tau)G, \quad G(0;\alpha) = 0. \tag{A39}$$

The solution of equation (A39) is

$$G(\tau;\alpha) = \int_0^\tau d\xi\, y(\tau,\xi;\alpha), \tag{A40}$$

where the process to be integrated is given by equality (A19). From the time-inversion invariance of the Wiener process it follows that process $G(\tau;\alpha)$, Eq. (A40), has a single-moment probability density which coincides with that of the random quantity

$$z(\tau;\alpha) = \int_0^\tau d\xi\, y(\xi;\alpha),$$

equal to the area under realization of $y(\xi;\alpha)$ over the interval $(0,\tau)$. Therefore if we find the probability density

$$\mathcal{P}(G;\tau,\alpha) = \langle\delta(G(\tau;\alpha) - G)\rangle, \tag{A41}$$

then in the limit $\tau\to\infty$ it will coincide with the probability *density of the area under all the realizations of the process* $y(\tau;\alpha)$

$$\mathcal{P}(G;\alpha) = \lim_{\tau\to\infty} (G;\tau,\alpha). \tag{A42}$$

Finally, for $\alpha = 1$ the latter probability density coincides with that of the area under realizations of the process $y(\tau)$ (A29)

$$\mathscr{P}(G) = \mathscr{P}(G; \alpha = 1). \qquad (A43)$$

Knowing $\mathscr{P}(G;\alpha)$ one can readily find the probability density of the random integrals

$$G_n = \int_0^\infty d\tau y^n(\tau). \qquad (A44)$$

Indeed, it follows from the fractal property of the Wiener process that the process $y^n(\tau)$ is statistically equivalent to the time-compressed process $y(n^2\tau, 1/n)$. This means that

$$\mathscr{P}_n(G) = n^2\mathscr{P}(n^2G; 1/n). \qquad (A45)$$

The above defined probability density $\mathscr{P}(G;\tau,\alpha)$ satisfies the Fokker-Planck equation resulting from Eq. (A39)

$$\frac{\partial}{\partial\tau}\mathscr{P} + \frac{\partial}{\partial G}\mathscr{P} = \alpha\frac{\partial}{\partial G}(G\mathscr{P}) + \frac{\partial}{\partial G}\left[G\frac{\partial}{\partial G}(G\mathscr{P})\right],$$
$$\mathscr{P}(G; 0, \alpha) = \delta(G). \qquad (A46)$$

For $\tau \to \infty$ the solution of this equation tends to the stationary probability density $\mathscr{P}(G;\alpha)$ (A42) governed by equation

$$\mathscr{P} = \alpha G\mathscr{P} + G\frac{d}{dG}(G\mathscr{P}).$$

Solving it we find

$$\mathscr{P}(G; \alpha) = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{G}\right)^{\alpha+1}\exp\left(-\frac{1}{G}\right). \qquad (A47)$$

Then setting $\alpha = 1$ we obtain the probability density (43) of the random area under the realizations of the process $y(\tau)$

$$\mathscr{P}(G) = \frac{1}{G^2}\exp\left(-\frac{1}{G}\right). \qquad (A48)$$

The corresponding integral distribution function is equal to

$$F(G) = \exp\left(-\frac{1}{G}\right). \qquad (A49)$$

Additional information on the dynamics of the behavior of the realization of the processes $y(\tau,\xi;\alpha)$ and $y(\xi;\alpha)$ in time is contained in the dependence of the probability distribution of the random process

$$\int_{-\infty}^{\tau-\gamma} d\xi y(\tau, \xi; \alpha) \quad (\gamma > 0) \qquad (A50)$$

on the time γ. It can be easily shown that this process is statistically equivalent to the process

$$\int_\gamma^\infty d\xi y(\xi; \alpha). \qquad (A51)$$

In its turn, from the multiplicative property of the process $y(\xi;\alpha)$ (21) it follows that the single-moment probability distribution of the processes (A50) and (A51) coincides with the probability density of the process

$$G(\gamma; \alpha) = y(\gamma; \alpha)G(\alpha), \qquad (A52)$$

where $y(\gamma;\alpha)$ and $G(\alpha)$ are statistically independent and their probability densities are given by expressions (A27) and (A47), respectively. In particular, it follows from this that integral distribution function of the random process

$$G(\tau) = y(\tau)G, \qquad (A53)$$

coinciding with the distribution function for the area under realizations of the process $y(\xi)$ over an infinite interval (τ, ∞), is equal to

$$F(G; \tau) = \frac{1}{(4\pi\tau)^{1/2}}\int_0^\infty \frac{dy}{y}\exp\left[-\frac{y}{G} - \frac{1}{4\tau}\ln^2(ye^\tau)\right], \qquad (A55)$$

and the probability of satisfying the inequality $G(\tau) < G$ with an increase of τ tends monotonically to unity for any given value of G. This confirms once again that each individual realization of the process $y(\tau)$ tends to zero with an increase of τ despite the exponential growth of the higher moments $y(\tau)$ due to the large spikes observed in some realizations of $y(\tau)$.

9.2. Isoprobability curves

Consider an arbitrary random process $y(\tau)$ with an integral distribution function

$$F(y; \tau) = \langle\theta(y - y(\tau))\rangle, \qquad (A56)$$

where $\theta(z)$ is the step function, equal to zero for $z < 0$ and unity for $z > 0$. Let us denote as an isoprobability curve of the process $y(\tau)$ the determinate function $z(\tau;p)$ whose value at each given time τ is obtained from the equation

$$F(z(\tau; p); \tau) = p. \qquad (A57)$$

Integrating this equality over an arbitrary interval yields

$$\int_{\tau_1}^{\tau_2} d\tau F(z(\tau; p); \tau) = p(\tau_2 - \tau_1). \qquad (A58)$$

On the other hand, it follows from definition (A56) of the distribution function that the integral on the left-hand side of this equality is

$$\int_{\tau_1}^{\tau_2} d\tau F(z(\tau; p); \tau) = \langle T(\tau_1, \tau_2)\rangle; \qquad (A59)$$

where

$$T(\tau_1, \tau_2) = \sum_{k=1}^{N}\Delta\tau_k$$

is the total time spent by a realization of the process $y(\tau)$ under the isoprobability curve over the interval (τ_1,τ_2). Correspondingly, $\Delta\tau_k$ are durations of time intervals over which a realization of $y(\tau)$ is located below $z(\tau;p)$ (see Fig. 12) and N is the number of such time lengths over the interval (τ_1,τ_2). Comparing equalities (A58) and (A59) we find that

$$\langle T(\tau_1, \tau_2)\rangle = p(\tau_2 - \tau_1) \qquad (A60)$$

the average time spent by the process $y(\tau)$ within the interval (τ_1,τ_2) under the isoprobability curve $z(\tau;p)$, is proportional to a duration of this interval $\tau_2 - \tau_1$. The coefficient

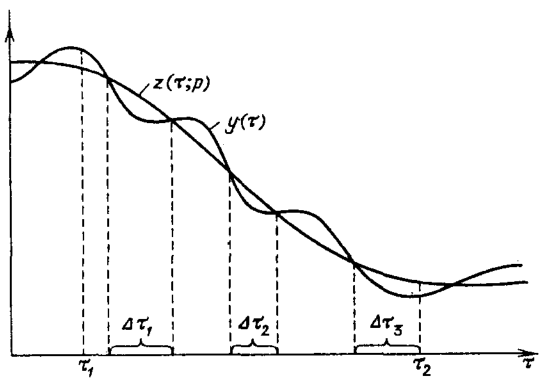

FIG. 12. Plots for the realization of the process $y(\tau)$ and for the corresponding isoprobability curve $z(\tau,p)$.

of proportionality p is equal to the fraction of the time during which inequality $y(\tau) < z(\tau;p)$ is satisfied. Therefore, if p is sufficiently close to unity, the realization plots of the process $y(\tau)$ will lie almost always below the isoprobability curve within any interval (τ_1,τ_2) while for $p = 1/2$, for example, the realization of the process $y(\tau)$ will pass back and forth across the isoprobability curve, spending, on the average, half of the time above the curve and half below it. Because of this behavior, the isoprobability curve $z(\tau;1/2)$ can be naturally called a typical realization of the process $y(\tau)$ although the plot $z(\tau;1/2)$ can, no doubt, differ significantly from that of any particular realization of the process $y(\tau)$. An interpretation of isoprobability curves as typical realizations suggesting the idea of dynamic behavior of the corresponding random process realizations is supported by the limiting property of the isoprobability curves formulated below. As is well known, one can use as a quantitative measure of the randomness of the process $y(\tau)$ its variance $\sigma^2(\tau) = \langle y^2(\tau)\rangle - \langle y(\tau)\rangle^2$. For $\sigma\to 0$ the process $y(\tau)$ tends to some determinate process $y_0(\tau) = \langle y(\tau)\rangle$. It follows from the definition of the isoprobability curves that in the limit $\sigma\to 0$ and for any $p < 1$

$$\lim_{\sigma\to 0} z(\tau;p) = y_0(\tau),$$

that is the isoprobability curve is contracted towards the determinate process.

[1)]Formula (A54) is missing on the Russian original.

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