# Radiation in electrodynamics and in Yang–Mills theory

B. P. Kosyakov

*All-Union Scientific-Research Institute for Experimental Physics, Arzamas, Nizhnegorod District*

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The classical concept of radiation in gauge theories is analyzed. It is concluded from a discussion of three definitions of electromagnetic radiation, i.e., traditional, Dirac's and Teitelboim's definitions, that only the last of these three represents correctly the structure of electromagnetic self-action. Teitelboim's definition is also satisfactory in the non-Abelian case in which the radiation problem is intertwined with that of confinement. The exact solution of the Yang–Mills equations with current formed by an arbitrarily moving color charge is used as a basis for a description of the non-Abelian classical picture. In the confinement phase, the energy of the gauge field is absorbed by the color charge, whereas the deconfinement phase involves the usual emission of radiation, and the color charge (free or accelerated by non-Yang–Mills forces) produces only colorless converging or diverging waves. Certain other fundamental questions concerning classical self-action in Abelian and non-Abelian gauge theories are also examined.

"Mehr Licht!"–*Goethe's dying words*

## 1. INTRODUCTION

Anyone with an education in the physical sciences will know that *radiation* is a "wave process" that propagates with the speed of light and conveys the effects of energy (i.e., a signal) to long distances from the source. Radiation has only "transverse" degrees of freedom of the field and is "dynamically independent" of other degrees of freedom.

A generally acceptable, rigorous definition of radiation has not as yet been formulated. Indeed, three different definitions coexist in electrodynamics. Textbooks (for example, Refs. 1–5) traditionally define electromagnetic radiation as the "long-range" part of the Liénard–Wiechert field that decreases with distance as $1/r$. The energy density of this part of the field varies as $1/r^2$, so that multiplication of this by the area of a sphere, $4\pi r^2$, gives an energy flux that does not vary with distance. This is perceived as indicating the possibility of signal transmission over long distances.

However, the "long-range" part of the field is not a solution of the free-field equations, which means that the criterion of "dynamic independence" of radiation is not satisfied. Dirac was able to rectify this omission by defining[6] radiation as the difference between the retarded and advanced fields. This combination satisfies the wave equation and, under certain particular conditions, decreases as $1/r$.

Dirac's definition is not, however, without its difficulties when one considers the universal concept of radiation. Indeed, in non-Abelian gauge theories it does not reduce to the definition of the free field.

In the definition put forward by Teitelboim,[7] radiation is identified not with a part of the field, but with a part of the energy-momentum density of the field characterized by the $1/r^2$ dependence. It will become clear later that this definition is closest to our intuitive perception of radiation in both Abelian and non-Abelian cases.

Each of these definitions is examined in Sec. 2 in the context of the classical electrodynamics of self-action in the case of a point source.

We should probably explain why it is in general necessary to subdivide the original system into two "dynamically independent" subsystems, one of which is identified with radiation. The point is that we are essentially attempting to uncover the structure of self-action. Even if we ignore fundamental aspects of the problem, we must acknowledge that an understanding of this structure is important for practical purposes. Thus, if we blindly accept the Lorentz–Dirac equation (which includes the finite effects of self-action) we encounter a number of difficulties and paradoxes, the treatment of which and the methods used to overcome which may be significantly different. This sometimes leads to differences between solutions obtained for special cases. The information about the structure of the electromagnetic self-action presented in Sec. 2 leads to the conclusion that the Lorentz-Dirac equation is free from the difficulties ascribed to it; this determines the specific features of this approach to the solution of particular problems.

Section 3 is devoted to the problem of radiation in classical Yang–Mills theory. The discussion is based on Ref. 8 which gives the exact solution of the Yang–Mills equation with current due to a color charge moving along an arbitrary world line. This solution is expressed in terms of the vector potential and contains a term that increases linearly with distance. This behavior of the vector potential is interpreted in chromodynamics as the confinement condition (see, for example, Refs. 9–11). It is assumed that the linear increase in the vector potential is due to the compression of gluon lines of force into a thin string-tube. We note, however, that string-like solutions have not been found directly in chromodynamics and that such solutions are known only for simpler models.[12] In the solution given in Ref. 8, the lines of force of

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the linearly growing term in the vector potential are distributed isotropically, and this does not appear to prevent the possibility of confinement. According to the Wilson criterion, i.e., the law of areas for the contour mean,[13] the linear increase in the vector potential ensures the confinement of immobile quarks independently of other details, e.g., of whether the lines of force are distributed isotropically or are string-like.

If, for the moment, we ignore the quantum-mechanical nature of chromodynamic phenomena and suppose that radiation and confinement are due to the classical Yang–Mills interaction, we obtain a formal classical world that presents us with some interesting questions. Does confinement preclude radiation? If the answer is *yes*, then what is the fundamental difference between an accelerated color charge and an accelerated electric charge as sources of radiation? Is the radiation regime restored during deconfinement? What is the nature of gauge-field waves generated by a color charge accelerated by non-Yang–Mills forces, i.e., are these retarded, advanced, colored, or colorless waves? Is the radiation regime affected by the properties of these waves? Is Gauss' law obeyed in the presence of the color degrees of freedom distributed uniformly in all space? Is the energy of the gluon field finite or infrared-divergent?

These questions are discussed in Sec. 3. We emphasize once again that we are dealing with the classical model whose relation to the chromodynamic reality is not entirely clear. The word "confinement" is used to denote the situation characterized by the presence of a linearly growing term in the vector potential and, as will be shown later, by the absorption of the gluon field, which obviously prevents the detection of an accelerated color charge. On the contrary, "deconfinement" corresponds to the situation in which the growing term is absent from the vector potential, and all the phenomena occur in complete analogy with the electromagnetic situation. On the whole, the model is internally consistent and does not contradict any fundamental physical principles. The color degrees of freedom are distributed uniformly in all space, have no effect on Gauss' law, and do not contribute to integral quantities such as the 4-momentum, so that they are neither radiated nor absorbed.

## 2. ELECTRODYNAMICS

Consider the field due to an electric charge $e$ moving along an arbitrary world line $z^\mu(\tau)$ parametrized by the proper time $\tau$. We shall denote the 4-velocity and 4-acceleration by $v^\mu \equiv \dot{z}^\mu \equiv dz^\mu/d\tau$ and $a^\mu \equiv \dot{v}^\mu$, respectively. We shall take the metric tensor in the form $\eta^{\mu\nu} = \mathrm{diag}(+\,-\,-\,-)$. We shall adopt the Gaussian system of units and set the velocity of light equal to unity. We shall define the projector $b(\perp)$ onto a hyperplane orthogonal to the nonisotropic vector $b^\mu$ by the *formula*

$$b(\perp)_{\mu\nu} = \eta_{\mu\nu} - (b_\nu b_\nu / b^2).$$

The retarded solution of the inhomogeneous wave equation

$$\Box A_\mu(x) = 4\pi e \int v_\mu(\tau)\delta^4(x - z(\tau))d\tau \tag{1}$$

will be taken in the form[6,3–5]

$$A^\mu(x) = e\frac{v^\mu}{(x-z)v},$$

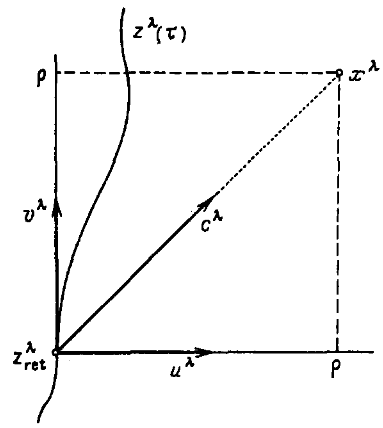


FIG. 1.

where the kinematic quantities refer to the retarded time $\tau_{\rm ret}$ determined from the conditions

$$(x - z(\tau_{\rm ret}))^2 = 0,\ x^0 > z^0(\tau_{\rm ret}).$$

We now recall some of the elements of the technique of covariant retarded variables.[4] Let $R^\mu \equiv x^\mu - z^\mu(\tau_{\rm ret})$. On the plane spanning the vectors $R^\mu$ and $v^\mu$ we construct the imaginary unit vector $u^\mu$ orthogonal to $v^\mu$, and the isotropic vector $c^\mu = v^\mu + u^\mu$ (see Fig. 1). Analytically, this can be described by

$$v^2 = 1,\quad u^2 = -1,\quad c^2 = 0,$$
$$vu = 0,\quad cv = -cu = 1, \tag{2}$$

$$R^\mu = \rho c^\mu, \tag{3}$$

$$\rho = -Ru = Rv. \tag{4}$$

The invariant $\rho$ can be interpreted as the separation between the points of emission and reception of the signal in the reference frame with time arrow $v^\mu$ (see Fig. 1).

As the point $x^\mu$ varies, the point $z^\mu(\tau_{\rm ret})$ will also vary in accordance with the condition $R^2 = 0$. Differentiating $R^\lambda(\eta_{\lambda\mu} - v_\lambda \tau_{,\mu}) = 0$ and using (2)–(4), we obtain

$$\tau_{,\mu} = c_\mu. \tag{5}$$

Hence we find the derivatives of the kinematic quantities, e.g., $v_{\lambda,\mu} = a_\lambda c_\mu$. Differentiating the second equation in (4) and using (2)–(5), we obtain

$$\rho_{,\mu} = -u_\mu + \rho(ac)c_\mu. \tag{6}$$

We are now ready to write down the Liénard–Wiechert equation $F = dA$ in the form

$$F = \frac{e}{\rho^2} c \wedge V, \tag{7}$$

where

$$V_\mu = v_\mu + \rho(u(\perp)a)_\mu. \tag{8}$$

We draw attention to the separability of the 2-form $F$, i.e., the fact that it can be written in the form of a vector product of $c^\mu$ and $V^\mu$. This can also be expressed in a different way. The area of the parallelogram defined by vectors $c^\mu$ and $V^\mu$ is $s = [-V^2(V(\perp)c)^2]^{1/2}$, so that, in view of (8), we have $s = 1$. Hence it follows that (7) is unaffected when

$c^\mu$ and $V^\mu$ are replaced by any two vectors that lie in the same plane and define a parallelogram of unit area. The Liénard–Wiechert field does not depend on $c^\mu$ and $V^\mu$ directly but only on the orientation of the $(c,V)$ plane. In other words, the quantity $F$ is invariant under the group Sp$(1,R)$ of symplectic transformations of the $(c,V)$ plane that preserve the area and the orientation of the parallelogram.

The two terms in (8) correspond to the two parts of the field

$$F_{\rm I}=\frac{e}{\rho^2}c\wedge v \quad \text{и} \quad F_{\rm II}=\frac{e}{\rho}c\wedge(u(\perp)a),$$

which are traditionally interpreted as the Coulomb and radiated component (see, for example, Refs. 1–5).

The quantity $F_{\rm II}$ is 'transverse' in the following sense. It is clear from the equations $F^{\mu\nu}_{\rm II}v_\mu u_\nu=0$, ${}^*F\mu\nu_{\rm II}v_\mu u_\nu=0$ that, in the reference frame with time arrow $v^\mu$ in which $v^\mu=\{1,0,0,0\}$, $u^\mu=\{0,\mathbf{n}\}$, the fields $\mathbf{E}_{\rm II}$ and $\mathbf{B}_{\rm II}$ (the time components of $F^{\mu\nu}_{\rm II}v_\mu$ and ${}^*F^{\mu\nu}_{\rm II}v_\mu$) are oriented across the direction $\mathbf{n}$ of propagation of the wave front. On the contrary, $F_{\rm I}$ contains the 'longitudinal' degrees of freedom: $F^{\mu\nu}_{\rm I}v_\mu u_\nu=e/\rho^2$.

The fact that $F_{\rm II}$ varies as $1/\rho$ is the main evidence that it can be identified with radiation, but we still ask: is this a justifiable conclusion? The Liénard-Wiechert field invariants ${}^*F_{\mu\nu}F^{\mu\nu}=2\mathbf{EB}=0$, $-f_{\mu\nu}F^{\mu\nu}=2(\mathbf{E}^2-\mathbf{B}^2)=2e^2/\rho^4$ show that there is a reference frame (for each point of observation $x^\mu$) in which $\mathbf{B}=0$, $|\mathbf{E}|=e/\rho^2$, i.e., the $1/\rho$ dependence is excluded. This reference frame can be explicitly identified. It follows from the identity $c\wedge V=(V(\perp)c)\wedge V$ that, when $V^2>0$, the field $F$ is a pure Coulomb field in the reference frame with the time arrow $V^\mu$, and that, when $V^\mu<0$, the time arrow can be $(V(\perp)c)^\mu$. The identity $c\wedge V=U\wedge(U(\perp)V)$, where $U=V+c$ and the relations $U^2=2+V^2$ and $(U(\perp)V)^2=-1/U^2$ show that, when $V^2=0$, a suitable time arrow is $U^\mu$.

The presence of the $1/\rho$ dependence in $F$ is an artifact associated with the use of the global Lorentz coordinate frame. Information about the long-range interaction is held not by the 2-form $F$ itself, but by a combination of the 2-form $F$ and the reference frame.

This explains why $F_{\rm I}$ and $F_{\rm II}$ cannot be interpreted as "dynamically independent" quantities. They can be so interpreted if, for example, they are similar to $F$ in that they satisfy the homogeneous Maxwell equations $d\,{}^*F=0$, $dF=0$ outside the world line. Instead, we have $F^{\mu\nu}_{{\rm I},\mu}=-F^{\mu\nu}_{{\rm II},\mu}=2e(ac)c^\nu/\rho^2$.

It follows that we cannot treat $F_{\rm II}$ as radiation because of the symplectic invariance (separability) of the 2-form $F$. This property is in turn due to three factors: the retarded character of the propagating field, the four-dimensional character of the continuum under consideration, and the time-like character of the world line of the source. Actually, these factors [uniquely taken into account in (2)–(6)] enable us to conclude that, for the construction of the 2-form $F$ as a sum of linearly independent outer products, we have at our disposal only the three linearly-independent vectors $u^\mu$, $c^\mu$, and $a^\mu$.

We note that $F$ retains this property when the retarded field condition is replaced by the advanced field condition. For any other condition, e.g., for a linear combination of retarded and advanced conditions, the 2-form $F$ ceases to be separable.

Dirac[6] defined radiation by

$$A_{\rm rad}=A_{\rm ret}-A_{\rm adv}, \tag{9}$$

where $A_{\rm ret}$ and $A_{\rm adv}$ are the retarded and advanced solutions of (1). Since

$$\Box A_{\rm rad}=0,$$

$A_{\rm rad}$ is the free field. This ensures the 'dynamic independence' of the terms in the expansion

$$A_{\rm ret}=\frac{1}{2}(A_{\rm ret}-A_{\rm adv})+\frac{1}{2}(A_{\rm ret}+A_{\rm adv}),$$

interpreted as the "free" and "bound" parts of the retarded field.

The Dirac definition of radiation lies at the basis of the Wheeler–Feynman theory of action at a distance (14) (for a modern review, see Ref. 15 which contains references to other work in this field). This definition claims to describe electromagnetism without the electromagnetic field altogether.

The subdivision into "longitudinal" and "transverse" components is now performed in accordance with a different principle: $A^\mu$ is written in the form of a Fourier integral, i.e., an integral over "waves" $\exp(ikx)$ that are longitudinal and transverse relative to the direction of the "wave vector" $k^\mu$. The transverse components are defined by the Lorentz gauge condition $A^\mu_{,\mu}=0$. However, 'transversality' is often understood in the analogous three-dimensional sense defined by the Coulomb gauge div $\mathbf{A}=0$.

We emphasize that the mathematical properties of $A_{\rm rad}$ that express the properties of the free field are wholly determined by the linearity of the Maxwell equations. In non-Abelian gauge theories, the linear combination of retarded and advanced solutions ceases to play the role of the free field, so that the definition given by (9) is meaningful only within the framework of electrodynamics.

If the source world line consists of two straight lines joined by a curvilinear segment, then as we depart from the curved segment along the generator of the upper light cone, only $F_{\rm rad}$ 'survives' asymptotically and $F_{\rm rad}\to F_{\rm II}$. In this situation, the Dirac and the traditional definitions of radiation become asymptotically equivalent. We note that in the region in which $F_{\rm rad}$ and $F_{\rm II}$ become identical, both can be simulataneously removed by a suitable choice of the reference frame.

The field $F_{\rm rad}/2$ is not singular on the source world line. If we substitute it into the expression for the Lorentz force, we obtain[4–6] the Abraham vector $\Gamma^\mu\equiv(2/3)e^2(\dot a^\mu+v^\mu a^2)$ that appears in the Lorentz–Dirac equation

$$ma^\mu-\frac{2}{3}e^2(\dot a^\mu+v^\mu a^2)-f^\mu=0 \tag{10}$$

as the radiation reaction force. This has served as a powerful argument in favor of the Dirac definition of radiation. However, this interpretation of $\Gamma^\mu$ is in fact erroneous.[7]

Actually, recalling that $av=0$, $\dot a v=-a^2$, let us write (10) in the form

$$v(\perp)(\dot p-f)=0, \tag{11}$$

where

$$p^{\mu} = mv^{\mu} - \frac{2}{3}e^2 a^{\mu}. \quad (12)$$

We note that, when Newton's second law for a neutral particle is written in terms of the geometry of Minkowski space, it takes exactly the form given by (11) with $p^{\mu} = mv^{\mu}$. The presence of $v(\perp)$ in (11) shows that, in any instaneously co-moving inertial frame, Newton's second law is satisfied in its orthodox form $dp/dt = f$.

Mathematically, the necessity for $v(\perp)$ follows from the invariance of the bare action

$$S = -m_0\int (v_{\mu}v^{\mu})^{1/2}d\tau - e\int A_{\mu}dz^{\mu} - \frac{1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}d^4x \quad (13)$$

under the group of reparametrized transformations $\delta\tau = \varepsilon$, $\delta z^{\mu} = v^{\mu}\delta\tau$, where $\varepsilon(\tau)$ is an arbitrary positive infinitesimal function. According to Noether's second theorem,[16] this invariance generates the identity $v_{\mu}\delta S/\delta z^{\mu} = 0$, which shows that the Eulerian $\delta S/\delta z^{\mu}$ contains the factor $v(\perp)$. The fact that the Lorentz–Dirac equation can be written in the form given by (11) shows that the reparametrization invariance was not violated in the regularization-renormalization procedure used to derive this equation.

The Lorentz–Dirac equation is thus an expression of Newton's second law for an object with 4-momentum $p^{\mu}$ and the somewhat unusual dependence on kinematic variables given by (12). The Newtonian character of this object (which we shall call the *electromagnetic complex*) means that it is subject only to the external force $f^{\mu}$.

The more conventional point of view is that equation (10) describes an object with 4-momentum $p^{\mu} = mv^{\mu}$ that is pictured as a 'particle'. The behavior of this 'particle' does not satisfy Newton's second law, which is the origin of many of the misunderstandings and paradoxes.

For example, consider the paradox of motion with uniform acceleration.[2,4] The relativistic condition for uniform acceleration $v(\perp)\dot{a} = 0$, taken together with the equation $\Gamma^{\mu} = (2/3)e^2(v(\perp)\dot{a})^{\mu}$, means that the 'particle' does not experience any radiation reaction during this type of motion. The paradox does not arise for the above complex: the complex does not experience the radiation reaction at all, and the case of uniform acceleration is now no longer special in any way.

The other striking example is the problem of "counter acceleration" which shows that the departure from Newton's second law by the "particle" is not necessarily equivalent to a small correction. For uniform motion $v^{\mu} = \{\cosh\alpha, \sinh\alpha, 0,0\}$, $f^{\mu} = f\{\sinh\alpha, \cosh\alpha, 0,0\}$ and equation (10) reduces to

$$\dot{\alpha} - \tau_0\ddot{\alpha} = f/m,$$

where $\tau_0 = 2e^2/3m$. This equation has the solution

$$\dot{\alpha}(\tau) = e^{\tau/\tau_0}\left(B - \frac{1}{m\tau_0}\int_0^{\tau} e^{-s/\tau_0}f(s)ds\right),$$

where $B$ is an arbitrary initial value of $\dot{\alpha}$ at time $\tau = 0$. Substituting $B = 0$, we find that the acceleration $\dot{\alpha}(\tau)$ and the force $f(\tau)$ point in opposite directions.

If we start with the concept of the complex, we see no particular problem in "counter-acceleration." The behavior of the complex is controlled by Newton's second law, but it does not follow at all that acceleration and force must have the same direction, since the relation $p^{\mu} = mv^{\mu}$ does not hold.

It follows that the argument in favor of the Dirac definition of radiation, which is based on the interpretation of $\Gamma^{\mu}$ as the radiation reaction force, must be acknowledged as unconvincing.

A system with action given by (13) has a symmetric energy-momentum tensor of the form $T^{\mu\nu} = \Theta^{\mu\nu} + t^{\mu\nu}$ where

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F_{\alpha}^{\ \nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \quad (14)$$

$$t^{\mu\nu} = m_0\int v^{\mu}(\tau)v^{\nu}(\tau)\delta^4(x - z(\tau))d\tau. \quad (15)$$

Let us now substitute in (14) the general solution of the field equations $F = F_{\rm in} + F_{\rm ret}$ where $F_{\rm in}$ is the solution of the homogeneous equations with arbitrary initial conditions specified in the distant past and $F_{\rm ret}$ is the retarded Liénard–Wiechert solution. We then obtain $\Theta = \Theta_{\rm in} + \Theta_{\rm mix} + \Theta_{\rm ret}$ where each term satisfies the continuity equation outside the world line:

$$\Theta^{\mu\nu}_{{\rm in},\mu} = 0, \quad \Theta^{\mu\nu}_{{\rm mix},\mu} = 0, \quad \Theta^{\mu\nu}_{{\rm ret},\mu} = 0. \quad (16)$$

Consider $\Theta_{\rm ret}$. Using (7) and (8), we obtain

$$\Theta^{\mu\nu}_{\rm ret} = \frac{e^2}{4\pi\rho^4}\left(c^{\mu}V^{\nu} + c^{\nu}V^{\mu} - V^2c^{\mu}c^{\nu} - \frac{1}{2}\eta^{\mu\nu}\right).$$

This expression is not invariant under Sp$(1,R)$ transformations on the $(c,V)$ plane. It contains information not only about the 2-form $F_{\rm ret}$ but also about the reference frame.[1)] Hence a transformation of the reference frame cannot reduce to zero any of the terms in $\Theta_{\rm ret}$. Since $V^2 = 1 + \rho^2(u(\perp)a)^2$, the quantity $\Theta_{\rm ret}$ splits into the sum of two terms: $\Theta_{\rm ret} = \Theta_{\rm I} + \Theta_{\rm II}$ where

$$\Theta^{\mu\nu}_{\rm I} = \frac{e^2}{4\pi\rho^4}\left(c^{\mu}V^{\nu} + c^{\nu}V^{\mu} - c^{\mu}c^{\nu} - \frac{1}{2}\eta^{\mu\nu}\right), \quad (17)$$

$$\Theta^{\mu\nu}_{\rm II} = \frac{e^2}{4\pi\rho^2}(u(\perp)a)^2c^{\mu}c^{\nu}. \quad (18)$$

According to the Teitelboim definition,[7] $\Theta^{\mu\nu}_{\rm II}$ represents radiation. It decreases with distance as $1/\rho^2$. The flux of $\Theta^{\mu\nu}_{\rm II}$ through the upper light cone (surface area element $d\sigma_{\mu} = c_{\mu}\rho^2 d\rho d\Omega$) is zero. This means that this part of the field energy-momentum leaves the source in the form of a diverging spherical wave whose leading and trailing fronts propagate with the velocity of light. This behavior of $\Theta^{\mu\nu}_{\rm II}$ ensures that the signal is transmitted to long distances from the source.

Since $\Theta^{\mu\nu}_{\rm II}$ is formed entirely from $F^{\mu\nu}_{\rm II}$, and $F^{\mu\nu}_{\rm II}$ is the "transverse" part of the Liénard–Wiechert field, $\Theta^{\mu\nu}_{\rm II}$ contains only the "transverse" degrees of freedom.

The term $\Theta^{\mu\nu}_{\rm I}$ was interpreted by Teitelboim[7] as the part of energy-momentum dragged by the source. It follows from (17) that the flux of $\Theta^{\mu\nu}_{\rm I}$ through the upper half of the light cone is nonzero. This part of the energy-momentum is therefore transported in a time-like direction that is uniquely related to the direction of the source world line.

We can now use (2)–(6) to verify that, outside the world line,

$$\Theta^{\mu\nu}_{{\rm I},\mu} = 0, \quad \Theta^{\mu\nu}_{{\rm II},\mu} = 0. \quad (19)$$

Teitelboim interpreted local conservation laws (16) and (19) as a manifestation of the dynamic independence of $\Theta_{in}$, $\Theta_{mix}$, $\Theta_{I}$, and $\Theta_{II}$. We note that this treatment of "dynamic independence" is unrelated to the requirement that the field from which these energy-momentum densities are formed must be free.

Let us now consider the field 4-momentum, defined as the integral of $\Theta^{\mu\nu}$ over a hypersurface $\Sigma$ that is orthogonal to the world line at the point at which they cross, and contains a small aperture of small radius $\varepsilon$ cut around this point.[4,7] This aperture is necessary for the regularization of the divergent expression, and to ensure that the regularization is Lorentz-invariant, the aperture must be described in a Lorentz-invariant manner. In particular, the hyperplane $\Sigma$ must be specified without referring to the reference frame. The hyperplane was therefore chosen to be rigidly attached to the geometry of the world line.

Since $\Theta_{in}$ is independent of $e$, the integral of $\Theta_{in}$ does not vary along the world line of the charge. Its form is unimportant for our purposes here.

The integral of $\Theta_{mix}$ is readily evaluated:

$$P^{\lambda}_{mix} = \int_{\Sigma} \Theta^{\lambda\mu} d\sigma_{\mu} = -e \int_{-\infty}^{\tau} F^{\lambda\mu}_{in} v_{\mu} d\tau. \qquad (20)$$

This expression can be understood as the 4-momentum absorbed from the external field $F_{in}$ throughout the entire history of the charge between the distant past and the time $\tau$.

Integrals of $\Theta_{I}$ and $\Theta_{II}$ are found in Ref. 7, and are given by

$$P^{\lambda}_{I} = \int_{\Sigma} \Theta^{\lambda\mu}_{I} d\sigma_{\mu} = \frac{e^2}{2\varepsilon} v^{\lambda} - \frac{2}{3} e^2 a^{\lambda}, \qquad (21)$$

$$P^{\lambda}_{II} = \int_{\Sigma} \Theta^{\lambda\mu}_{II} d\sigma_{\mu} = -\frac{2}{3} e^2 \int_{-\infty}^{\tau} a^2 v^{\lambda} d\tau. \qquad (22)$$

It is clear from (21) that the 4-momentum $P^{\lambda}_{I}$ is in fact transported along a path that is close to the world line of the charge. Expression (22) shows that $P^{\lambda}_{II}$ is the 4-momentum of the radiation emitted (in accordance with the well-known Larmor formula) throughout all history up to the time $\tau$.

The tensor $t^{\mu\nu}$, defined in accordance with (15), is also found to have zero divergence outside the world line. Combining $t^{\mu\nu}$ and $\Theta^{\mu\nu}_{I}$, we obtain a new "dynamically independent" quantity. It corresponds to the integral $p^{\lambda} = m_0 v^{\lambda} + p^{\lambda}_{I}$, which is identical with the 4-momentum of the complex (12) if the finite quantity $m$ is defined by

$$m = \lim_{\varepsilon \to 0} \left( m_0(\varepsilon) + \frac{e^2}{2\varepsilon} \right). \qquad (23)$$

The conservation of the 4-momentum of the system is described by

$$\dot{P}^{\lambda}_{in} = 0,$$

$$\dot{p}^{\lambda} + \dot{P}^{\lambda}_{II} + \dot{P}^{\lambda}_{mix} = 0. \qquad (24)$$

On substituting (12), (20), and (22) into the second of these expressions we transform it into (10), where $f^{\lambda} = -\dot{P}^{\lambda}_{mix} eF^{\lambda\mu}_{in} v_{\mu}$. The Lorentz–Dirac equation is thus seen to be an expression of the local balance of the 4-momentum of the renormalized system: the 4-momentum $dP^{\lambda}_{mix}$ absorbed from the external electromagnetic field is spent in producing the increment $dp^{\lambda}$ in the 4-momentum of the complex and the increment $dP^{\lambda}_{II}$ in the radiated 4-momentum.

From the point of view of "particle" dynamics, the Lorentz–Dirac equation is not a representation of the 4-momentum balance. This is prevented by the presence of the term $-(2/3)e^2\dot{a}^{\mu}$. To avoid this, equation (10) can be integrated between infinite limits, subject to the asymptotic conditions $a^{\mu}(\tau) \to 0$, $\tau \to \pm\infty$:

$$mv^{\mu}(\infty) - mv^{\mu}(-\infty) - \frac{2}{3}e^2 \int_{-\infty}^{\infty} a^2 v^{\mu} d\tau = \int_{-\infty}^{\infty} f^{\mu} d\tau.$$

This equation describes the global balance of energy-momentum. Results of this kind lend support to the "nonlocal nature of electromagnetic interactions."[17,18] However, in reality, local balance was disturbed by the artificial and totally unjustified segregation of the term $mv^{\mu}$ from (12). The density corresponding to this term does not satisfy (19) and is not a "dynamically independent" quantity.

We therefore conclude that self-action in classical electrodynamics generates a significant rearrangement of the degrees of freedom as compared with the order that appears in the bare action. Instead of the bare particle and the electromagnetic field, the renormalized objects, i.e., the complex and the radiation, arise in the initial unseparated state. Teitleboim's analysis[7] shows that a correct allowance for this arrangement must be introduced without violating the symmetries of the bare action in local conservation laws. The dynamic equation obtained in this way is an expression of Newton's second law for the complex and, at the same time, a representation of the local balance of 4-momentum of the renomalized system. Other variants of the rearrangement of degrees of freedom, in which the Liénard-Wiechert field is split into "Coulomb" and "long-range" components, or the "free" and "bound" components, and also the segregation of the object with 4-momentum $p^{\mu} = mv^{\mu}$, do not take into account the mathematical structure of the theory and thus lead to different physically absurd results.

### 1. YANG-MILLS THEORY

The classical Yang-Mills field generated by an arbitrarily moving point color charge $Q^a$ can be described by the equations[19]

$$D^{ab}_{\mu} F^{\mu\nu}_{b}(x) = 4\pi \int Q^a(\tau) v^{\nu}(\tau) \delta^4(x - z(\tau)) d\tau, \qquad (25)$$

$$\dot{Q}^a = g f^{abc} Q_b A^{\mu}_c v_{\mu}, \qquad (26)$$

where $D^{ab}_{\mu} = \delta^{ab}\partial_{\mu} - gf^{abc}A^{c}_{\mu}$ is the covariant derivative $F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + gf^{abc}A^{b}_{\mu}A^{c}_{\nu}$ is the Yang–Mills field tensor, and $f^{abc}$ are the structure constants of the gauge group [for the group $O(3)$ to which we confine our attention here, $f^{abc} = \varepsilon^{abc}$].

We now define the unit isovector $\hat{\Gamma}^a_1 = Q^a/\sqrt{Q^2}$. It is possible to choose two other unit vectors $\hat{\Gamma}^a_2$ and $\hat{\Gamma}^a_3$ orthogonal to each other and to $\hat{\Gamma}^a_1$ so that the conditions $\varepsilon_{abc}\hat{\Gamma}^a_i\hat{\Gamma}^b_j = \varepsilon_{ijk}\hat{\Gamma}^k_c$ are satisfied. The basis in isospace is conveniently defined by the set of vectors $\hat{\Gamma}^a_1$, $\hat{\Gamma}^a_+$, $\hat{\Gamma}^a_-$ where $\hat{\Gamma}^a_{\pm} = \hat{\Gamma}^a_3 \pm i\hat{\Gamma}^a_2$.

The retarded solution of (25) and (26) is now written in the form[8]

$$A^a_\mu = \pm \frac{2i\hat{\Gamma}^a_1}{g}\frac{v_\mu}{\rho} + \varkappa\hat{\Gamma}^a_\pm R_\mu, \tag{27}$$

where $\varkappa$ is an arbitrary nonzero real parameter with the dimensions of $l^{-2}$. It is clear that, in addition to the generalized Liénard–Wiechert term, $A^a_\mu$ contains a term that increases linearly with distance. This term could not have arisen in electrodynamics, and since $\Box R^\mu = -2v^\mu/\rho$, this term is specific to the non-Abelian theory.

The Yang–Mills field tensor is now calculated from (27):

$$F = c \wedge W, \tag{28}$$

$$W^a_\mu = \pm \frac{2i\hat{\Gamma}^a_1}{g}\frac{V_\mu}{\rho^2} + \varkappa\hat{\Gamma}^a_\pm v_\mu, \tag{29}$$

where the vector $V_\mu$ is determined in accordance with (8).

We draw attention to the imaginary unit in front of $\hat{\Gamma}^a_1$ in (27) and (29). This arises from the condition

$$g^2Q^2 = -4, \tag{30}$$

which ensures that the equations for the system are compatible for $\varkappa \neq 0$ (see Ref. 8 for further details). However, if $\varkappa = 0$, then condition (30) is absent and instead of (27) we have the real solution $A^a_\mu = Q^a v_\mu/\rho$.

The tensor $F^a_{\mu\nu}$ contains a constant term that ensures that the color degrees of freedom, represented by the factor $\hat{\Gamma}^a_\pm$, are distributed uniformly in space. This may cast doubt on the validity of Gauss' law.[2)] This difficulty can be removed formally relatively simply. If $F^a_{\mu\nu}$ is the solution of the equations given by (25) then substitution of $F^a_{\mu\nu}$ into these equations converts them into identities. Integrating the left hand sides of these identities with respect to the volume containing the source, we obtain $4\pi Q^a$ since this result follows from the integration of the right hand side. We note, however, that the quantity $4\pi Q^a$ arises from one of the divergent terms on the left hand side, and that the remaining terms should mutually cancel out. It is precisely this cancellation that we shall examine now.

Consider an arbitrary point $z^\mu$ on the world line, and let us draw through this point a hyperplane $\Sigma$ with normal $v^\mu$. On $\Sigma$ we define a region $\mathscr{B}$ with boundary $\partial\mathscr{B}$ that is the intersection of $\Sigma$ with the hypersurface $\mathscr{T}$ described by the equation $\rho = L =$ const. According to Gauss' theorem

$$\int_{\mathscr{B}} v_\nu F^{\mu\nu}_{a\,\mu} d^3x = \int_{\partial\mathscr{B}} v_\nu F^{\mu\nu}_a \rho_{,\mu} d^2x, \tag{31}$$

where $\rho_{,\mu}$ is the normal to $\mathscr{T}$. The integrand on the right after the substitution of (8), (28), and (29) becomes

$$v_\mu F^{\mu\nu}_a \rho_{,\mu} = \left(\pm \frac{2i\hat{\Gamma}^a_1}{g\rho^2} + \varkappa\hat{\Gamma}^a_\pm\right)(1 - \rho au). \tag{32}$$

The element of measure on $\partial\mathscr{B}$ is $d^2x = L^2 d\Omega$. Integration with respect to the solid angle $\Omega$ of expressions containing an odd power of the vector $u^\mu$ gives 0, so that the term $pau$ in (32) can be discarded, and (31) finally becomes

$$4\pi\left(\pm \frac{2i}{g}\right)\hat{\Gamma}^a_1 + 4\pi L^2\hat{\Gamma}^a_\pm, \tag{33}$$

where, in view of (30), the first term is equal to $4\pi Q^a$.

To find the integral of $G^a_\nu \equiv g\varepsilon^{abc}A^\mu_b F^c_{\mu\nu} = -2\varkappa\rho^{-1}\Gamma^a_\pm V_\nu$ over $\mathscr{B}$, we use Gauss' theorem. The four-dimensional region bounded by $\Sigma\mathscr{T}$, and the upper light cone $\mathscr{C}$ with apex at $z^\mu$ does not contain any sources, so that the flux of the vector $G^a_\nu$ across $\Sigma$ is equal to the sum of the fluxes across $\mathscr{T}$ and $\mathscr{C}$. The element of measure $d\sigma_\mu$ on $\mathscr{T}$ is $\rho_{,\mu} d\tau L^2 d\Omega$ and, since $V^\nu\rho_{,\nu} = \rho au$, the flux across $\mathscr{T}$, which contains integration of $u^\mu$ over the solid angle, must vanish. The flux across $\mathscr{C}$ is

$$\int_C G^a_\nu d\sigma^\nu = -\int d\Omega \int_0^L d\rho\rho \cdot 2\varkappa\hat{\Gamma}^a_\pm V_\nu c^\nu = -4\pi\varkappa L^2\hat{\Gamma}^a_\pm.$$

This quantity cancels with the second term in (33).

The case $\varkappa \neq 0$ is associated with the confinement phase in chromodynamics. The case $\varkappa = 0$, on the other hand, must be related to deconfinement.

Since $\hat{\Gamma}_1\hat{\Gamma}_\pm = 0$, $\hat{\Gamma}^2_\pm = 0$, only the generalized Liénard–Wiechert term contributes to the energy momentum tensor

$$\Theta^{\lambda\mu} = \frac{1}{4\pi}(F^{a\lambda}{}_\nu F^{\nu\mu}_a + \frac{1}{4}\eta^{\lambda\mu}F^a_{\rho\sigma}F^{\rho\sigma}_a)$$

The confinement is found to require not only the expenditure of energy, but also is not even characterized by any change in energy indicators, contrary to energy considerations underlying string models of quark confinement.[9–11]

The Teitelboim approach can be almost entirely transferred to the Yang–Mills theory with the exception of the portion dealing with the superposition of solutions. We shall not, therefore, repeat the above discussion and merely note the properties of the non-Abelian picture associated with the presence of the imaginary unit in front of $\hat{\Gamma}^a_1$.[3)]

If the color charge is accelerated by forces other than Yang–Mills forces, the radiated intensity

$$\frac{dE}{dt} = \frac{8}{3g^2}a^2 \tag{34}$$

has the "wrong" sign i.e., the energy of the Yang–Mills field is absorbed rather than radiated during confinement. Moreover, this is the absorption of colorless waves because $\hat{\Gamma}^a_i(\tau) =$ const.

The proper energy of the point color charge is negative. If the renormalized mass $m$ is positive, it follows from (23) that the bare mass $m_0$ is also positive.

The equation of motion of the color complex is

$$m[a^\mu + \tau_0(\dot{a}^\mu + v^\mu a^2)] - f^\mu = 0, \tag{35}$$

where $\tau_0 = 8/3mg^2$ and $f^\mu$ is the non-Yang–Mills force that differs from (10) by the sign in front of the parentheses, so that there are no self-accelerating solutions.

We note that $m$ is a poorly defined quantity. There is a number of models that ascribe a small positive mass (of the order of a few MeV) to current quarks. However, in the model proposed in (20), for example, the quark propagator does not have a pole and is described by an entire function. Hence, at this stage, we cannot exclude the possibility that $m < 0$ or $m = 0$. The former leads to self-accelerating solutoins of the familiar type, $|a| \sim \mathrm{ep}(\tau/\tau_0)$, and the latter signifies the presence of uniform self-accelerations $|a| =$ const.

In the first case, $m_0$ may be a positive or a negative divergent quantity, but in the second case $m_0 > 0$.

In the deconfinement phase, all the phenomena are described by the formulas established in electrodynamics except that $e^2$ is replaced with $Q^2$. In particular, the accelerated color charge radiates and, instead of (35), we have the Lorentz–Dirac equation (10).

Equations (25) and (26) have an advanced solution as well. This is obtained from the retarded solution by reversing the sign in front of $\rho(u(\perp)a)_\mu$ in (8) and in front of $i\hat{\Gamma}_2^a$ in (27) and (29). All the final conclusions, including (34) and (35), remain unaltered. This confirms the irrelevance of the "delayed/advanced dilemma" in relation to the question of whether radiation or absorption takes place.

We now turn to the problem of self-acceleration. We note, first, that self-accelerated motion can be accompanied by radiation (deconfinement) or by absorption (confinement with $m \leqslant 0$) of the energy-momentum of the gauge field. At any rate, this does not violate the conservation of energy-momentum. Actually, the equation of motion of a color complex in the absence of external forces can be written in accordance with (24) as the conservation of total 4-momentum of the system:

$$\dot{p}^\lambda + \dot{P}^\lambda_{\rm II} = 0,$$

where $P^\lambda_{\rm II}$ represents radiated or absorbed 4-momentum, depending on the sign. This is, indeed, the case because of the translational invariance that must hold in the absence of external forces.

The real problem with self-acceleration is that integral quantities such as the field 4-momentum are infrared-divergent when world lines corresponding to self-accelerated motion are present. This is clear, for example, from (22) which diverges when the asymptotic condition $a^\mu(\tau) \to 0, \tau \to -\infty$ is not satisfied.

An analogous situation arises in quantum field theory because of the presence of nonequivalent unitary representations of canonical commutation relations and the Haag theorem (see, for example, Ref. 21). If the Hamiltonian $\mathscr{H}$ and the state $\Psi$ are translationally invariant, then

$$\|H\Psi\| = \int (\Psi, \mathscr{H}(\mathrm{x})\mathscr{H}(\mathrm{y})\Psi) \mathrm{d}^3x \mathrm{d}^3y = \begin{cases} 0 & \text{when } H\Psi = 0, \\ \infty & \text{when } H\Psi \neq 0. \end{cases}$$

We usually employ the Fock representation because of its clear physical meaning and technical convenience. Nevertheless, the use of these "strange" representations of commutation relations is wholly consistent with fundamental physical principles.

Uniform Galilean motion is a normal variant of translationally invariant dynamics in the absence of external forces. It would appear that self-acceleration must be regarded as a "strange" variant of this state. It does not contradict Newtonian mechanics because it is described by the solution of (11) which is an expression of Newton's second law. It does not violate any other physical laws or principles. The total energy of the system in this state is constant because the energy of a complex is not a sign-definite quantity: a reduction in this energy compensates exactly the radiated energy and, conversely, a rise in the energy covers exactly the loss of energy by absorption. It would be erroneous to consider (as is often done) that self-acceleration is a "nonphysical state."

In classical Yang–Mills theory, the infrared divergence of integral quantities such as the field 4-momentum arises only during deconfinement as a result of the presence of self-acceleration. As far as confinement is concerned, this problem is not generated by a linearly growing potential nor by self-acceleration (for positive $m$). It is possible that this casts new light on the connection between confinement and the infrared behavior of the gluon propagator defined in Fock vacuum.

## CONCLUSION

The correct formulation of the concept of radiation is hardly a semantic problem. The concept characterizes a form of self-action in which the field system is split into two "dynamically independent" subsystems, namely, the emitted subsystem and the subsystem bound to the source. Before Teitelboim's paper, the "dynamic independence" of radiation used to be reduced to the trivial condition in which radiation was compared with the free field. However, as we have seen, it is more natural to consider the "dynamic independence" of a subsystem means that the equation of continuity is satisfied for the energy-momentum density of the subsystem. The fields forming this subsystem need not be free, which is important from the point of view of non-Abelian gauge theories in which the "free field" satisfies nonlinear equations and is therefore qualitatively indistinguishable from the interacting field.

The subdivision of a system in accordance with an energy criterion enables us to characterize a more general form of self-action as well. This can produce an absorbed rather than radiated subsystem. This form of self-action is realized in a non-Abelian gauge system in the confinement phase. The absorption regime then occurs as a result of the complexification of the Yang–Mills field that, in the final analysis, is due to the nonlinearity of the field equations. The Yang–Mills field remains real in the deconfinement phase, and we have the radiation regime. It is interesting that the color charge (brought into motion by non-Yang–Mills forces) radiates or absorbs only colorless waves.

The asymmetry between diverging and converging wave processes was surprising and gave rise to a debate that has continued throughout the history of physics. The recurring question has been: why is it that an electric charge radiates but does not absorb light waves despite the fact that the Maxwell equations are invariant under time reversal?

This question does not arise in quantum theory: a quantum of light is just as readily emitted as it is absorbed. This is clear, for example, from the description based on the functional integral. In contrast to the classical object with its unique (extremal) behavior, the quantum system has a continuum of variants of behavior characterized by probability amplitudes $\exp(iS/\hbar)$. Emission and absorption are symmetric phenomena precisely because of the extensive behavioral repertoir of quantum mechanical systems.

Nevertheless, the problem has persisted in classical theory. In view of the foregoing discussion, we propose the following solution. We have to consider not diverging or converging (i.e., retarded or advanced) solutions of field equations, but the direction of the energy flux, i.e., whether it is toward the source or away from it. The direction of this flux is not sensitive to the replacement of the retarded condition with the advanced condition. It is determined by the form of the self-action. In Yang–Mills theory, we have self-action that allows either radiation or absorption of energy by a color charge, depending on the phase state. At the same time, in electrodynamics, self-interaction produces an energy flux that can only point away from the source.

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1) This is clear from the definition of the symmetric energy-momentum tensor $T^{\mu\nu} \equiv -2/(-\det g_{\alpha\beta})^{1/2}\,\delta S/\delta g_{\mu\nu}$.

2) I. B. Khriplovich drew the attention of the author to this point.

3) The energy is found to be a negative linearly diverging quantity. This serves as an argument for the stability of the solution given by (27) against small perturbations: a transition to another configuration is energetically unfavorable.


[1] W. Heitler, *Quantum Theory of Radiation*, Oxford University Press, Oxford, 1954 [Russ. transl. Mir, M., 1956].

[2] V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Pergamon Press, Oxford, 1979 [Russ. original, Nauka, M., 1975 (1st ed.), 1981 (2nd ed.), and 1987 (3rd ed.)].

[3] J. D. Jackson, *Classical Electrodynamics*, Wiley, N. Y., 1975 [Russ. transl. of an earlier edition, Mir, M., 1965].

[4] F. Rohrlich, *Classical Charged Particles*, Addison-Welsey, Reading, MA, 1965.

[5] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, Collier-Macmilan, London, 1964.

[6] P. A. M. Dirac, Proc. R. Soc. London **167**, 148, 1938.

[7] C. Teitelboim, Phys. Rev. D **1**, 1572, 1970.

[8] B. P. Kosyakov, Teor. Mat. Fiz. **87**, 422, 1991 [Theor. Math. Phys. (USSR) **87**, 632 (1991)].

[9] M. Bander, Phys. Rep. **75**, 205, 1981.

[10] M. Creutz, *Quarks, Gluons, and Lattices*, Cambridge University Press, Cambridge, 1983 [Russ. transl., Mir, M., 1987].

[11] L. B. Okun', *Elementary Particle Physics* [in Russian], Nauka, M., 1988.

[12] H. B. Nielsen and P. Olesen, Nucl. Phys. B **61**, 45, 1973; Y. Nambu, Phys. Rev. D **10**, 4262, 1974.

[13] K. Wilson, *ibid.*, p. 2445.

[14] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157, 1945.

[15] D. T. Pegg, Rep. Prog. Phys. **38**, 1839, 1975.

[16] E. Noether, *The Variational Principles of Mechanics* [Russian translation from German], Fizmatgiz, M., 1959; N. P. Konopleva and V. N. Popov, *Gauge Fields* [in Russian], Atomizdat, M., 1972.

[17] F. Rohrlich, *Physical Reality and Mathematical Description*, Dortrech, Boston, 1974.

[18] N. P. Klepikov, Usp. Fiz. Nauk **145**, 317 (1985) [Sov. Phys. Usp. **28**, 207, 1985].

[19] S. K. Wong, Nuovo Cimento A **65**, 689, 1970.

[20] G. V. Efimov and M. A. Ivanov, Fiz. Elem. Chastits At. Yadra **12**, 1220, 1981 [Sov. J. Part. Nucl. **12**, 489, 1981].

[21] A. S. Wightman, *Les Problèmes Mathematiques de la Théorie Quantique des Champs*, CNRS, Paris, 1959 [Russ. transl., Nauka, M., 1968].


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