The Skyrme model and strong interactions (On the 30th anniversary of the creation of the Skyrme model)

V. G. Makhan'kov, Yu. P. Rybakov, and V. I. Sanyuk

Joint Institute for Nuclear Research, Dubna; P. Lumumba Friendship of Nations University, Moscow

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The review summarizes the development of Skyrme's soliton approach to the description of baryon structure. In contrast to existing literature on this subject, the principal attention is devoted not to the pragmatic aspects of the model, but rather to its initial ideas, to its deep topological content and to such subtle problems as the existence of solutions, the attainability of the absolute energy minimum based on the hedgehog ansatz, and so on. It is exactly these features of the Skyrme approach that are, in the authors' opinion, the main advantages compared to other schemes used in strong-interaction physics. The material assembled in the review will, first, enable the reader to gain a deeper understanding of the structure and special features of the Skyrme model and, second, will serve as an adequate base for further modifications and for development of more realistic scenarios of processes in low-energy QCD, the need for which is beyond any doubt.

1. INTRODUCTION

The distinguished English physicist Tony Hilton Royle Skyrme (1922–1987) is now widely recognized for his outstanding and far-reaching contributions to modern nuclear physics. Until 1982 he was chiefly known among nuclear physicists as one of the authors of the Bethe–Rose–Elliott–Skyrme theorem in nuclear shell theory and of phenomenological Skyrme forces in the theory of nuclear matter. But his main creation, that attracted all his thoughts, was the model of baryons as topological solitons. The approach suggested by Skyrme was based on deep topological ideas, to which physicists had been so unaccustomed at that time. Possibly, this explains the fact, that for more than two decades this direction was developed mainly by Skyrme himself and by his few followers. The situation changed drastically in the early 1980's after the realization that the Skyrme model could be considered as the possible low-energy limit of QCD. This circumstance initiated an increased interest in the model. It turned out that in the limit of a large number of colors Quantum Chromodynamics is equivalent to an effective meson theory, well approximated by a nonlinear σ-model at low energies. The term "Skyrmion" became a colloquial one. It symbolizes an image of an extended baryon, being regarded as a topological soliton, made up of bosons but possessing fermion features. The topological charge was interpreted by Skyrme as the baryon number.

The Skyrme model turned out to be a rather successful image in the low-energy physics of strong interactions. In its framework one succeeds to describe the nucleon-nucleon interaction and the main statistic characteristics of baryons. Exceptional interest in the Skyrme model can be explained as follows: this is the first realistic model, which being relatively simple provides on the whole an accurate reflection of the symmetry and structural properties of hadrons.

There is a vast literature concerning the practical implementation of the Skyrme model to describe the properties of baryons and their interactions, including some papers of a review character (Refs. 60, 72, 111, 120, 121 and 124). Therefore we mainly concentrate on the background ideas of the Skyrme approach as well as on some methods, which might be found applicable in other branches of nonlinear physics.

2. PHYSICAL AND MATHEMATICAL FOUNDATIONS OF THE SKYRME MODEL

As noted in the introduction, the approach suggested by T. Skyrme to describe nuclear matter differed strikingly from the schemes, generally adopted in the early 50s. Therefore it seems appropriate to focus first of all on the physical ideas, taken by Skyrme as the basis for his nucleon model, and as far as possible to elucidate their sources. Then we follow the evolution of Skyrme's initial notions up to the final version of the model. Along the way we shall note the original hypotheses, ideas and conjectures, contained in Skyrme's earlier papers (Refs. 1–6) and which have become part of the "arsenal" of present-day elementary particle physics. Among them one can find "nuclear democracy" and "super-democracy" hypotheses, the soliton mechanism idea, which allows one to construct massive fermion states out of boson fields, etc. In existing reviews on the Skyrme model this material has not been reflected to the extent it merits, so we hope to fill this gap.

2.1. Kelvin's "vortex atoms" and the "pion fluid" model in nuclear physics

One can deduce from Skyrme's papers[1,2] that he turned to model notions on nuclear structure in connection with the problem that arose in nuclear physics at the beginning of 1950s. The calculations on the basis of α-decay and heavy particle scattering experimental data for the radius of the nucleus gave the value $R = 1.5A^{1/3}$ fm (where A is the mass number), while the fast electron scattering data led to a

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much smaller value, i.e., $R' = 1.2A^{1/3}$ fm. In Ref. 1 Skyrme succeeded to give a qualitative explanation of the observed difference, using the following model concepts:

—the nucleus is considered to be a drop on incompressible electrically neutral "pion fluid," which occupies a region of radius R, i.e., at any point of the nucleus, the mean densities of π^+ and π^- components would be equal to each other. In the standard hydrodynamical manner the state of this fluid at any point can be characterized by some density and a vector in isospace;

—the nucleons are immersed in the "pion fluid" and occupy in the nucleus a region of smaller radius R'.

Therefore in the first kind of experiments, where the pion interaction is substantial, one obtains R for the radius of the nucleus. In turn, in experiments with electrically charged particles when only the region occupied by sources is essential *one obtains the mean square radius* R'.

Skyrme in Ref. 2 describes the "pion fluid" model dynamics by the Lagrangian density:

$$L = \frac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 - \frac{k^2}{2}\boldsymbol{\phi}^2 + \overline{\Psi}(i\gamma^\mu\partial_\mu + ig\gamma_5\boldsymbol{\tau}\boldsymbol{\phi})\Psi, \qquad (2.1.1)$$

where $\boldsymbol{\phi}$ is the isovector pseudoscalar pion field; $\boldsymbol{\tau}$ are the isospin Pauli matrices, g is the pion-nucleon coupling constant and the spinor-isospinor Ψ describes nucleon fields. Note that in contrast to standard Lagrangians of pseudoscalar meson theories,[7] there is no mass term of "bare" nucleons in (2.1.1). This difference is fundamental, as according to Skyrme the nucleon mass is of pion origin, i.e., it arises as a result of "pion fluid" density fluctuations. Essentially, this suggestion is one of the first mentions of the idea of the "soliton mechanism," expressed in a clearer form more than 20 years later by L. D. Faddeev:[8]

—strongly interacting massive particles could be described in the framework of a nonlinear field theory as collective excitations in the system of weakly coupled fundamental fields;

—the theory should admit the existence of particlelike (soliton) solutions, describing the entire hierarchy of observed particles.

The Skyrme model was designed as an attempt to realize these requirements. But let us turn back to the ideas, which, according to his own words, inspired the author of the model of Ref. 9.

First, Skyrme handled the notion of a fermion state with great care, as a concept which does not have a clear analogy in classical physics. In particular, he regarded fermions only as a convenient tool of mathematical description. In Skyrme's opinion a successful search for a method of constructing fermion states (nucleons) out of boson fields (pions) would result in an alternative scheme to the de Broglie–Heisenberg scheme of coalescence (bosons out of fermions). Secondly, Skyrme regarded as inadmissible any pointlike description of particles and considered the renormalization theory as a temporary and forced concession, enabling us to live with our ignorance of processes which actually go on at short distances. As is well known, a description of the aforementioned particles as extended objects is possible only in the framework of a nonlinear field theory.

Apparently the decisive role in the process of realization of the above ideas was played by Skyrme turning to W. Thomson's (Lord Kelvin) work on the "vortex model" of atoms.[10] Recall that Kelvin, following H. Helmholtz's ideas, considered atoms as "vortex rings" in the ether, which fills the Universe and has the property of a "perfect fluid." He was one of the first to introduce topological concepts into physics, explaining distinctions between different sorts of atoms by the different number of intersections of vortex rings. There even exists an opinion,[11] that Kelvin was the first who made attempts to construct a "soliton (in current terminology) theory of particles." It is curious, that he also for the first time introduced the term "chirality," using it for description of vortex orientations (Kelvin's "ovals"). In what follows we will show, how these ideas and discoveries of Kelvin were used by Skyrme in the process of modification of the "pion fluid" model and will explain in what sense Skyrme's baryon can be understood as a vortex in a "pion fluid."

2.2. Chiral invariance

To realize the idea of the pion origin of the nucleon mass, Skyrme[2] used the unitary transformation of fields Ψ in the form

$$\Psi' = (1/\sqrt{2})(1 + i\gamma_5\boldsymbol{\tau}\cdot\mathbf{n})\Psi, \qquad (2.2.1)$$

where $\mathbf{n} = \boldsymbol{\phi}/|\boldsymbol{\phi}|$ is a unit pseudovector in isotopic space. According to Skyrme's idea, taking pion fluctuations into account should effectively lead to just such a transformation. As a result in the nucleon part of the Lagrangian density (2.1.1) an additional "mass" term arises in the form $g\phi\overline{\Psi}\Psi$:

$$L_N = \overline{\Psi}[i\gamma^\mu\partial_\mu - g\phi + \frac{1}{2}(\gamma_5 + i\boldsymbol{\tau}\mathbf{n})\gamma^\mu\partial_\mu(\boldsymbol{\tau}\mathbf{n})]\Psi. \qquad (2.2.2)$$

It is also possible to regard the chiral transformation (2.2.1) as the massless form of Foldy's transformation, or as the strong-coupling method transformation, which diagonalizes the interaction.

The discoveries (1956,1957) of "strange" K-mesons (the "$\theta - \tau$" puzzle resolution) and of the parity nonconservation in weak interactions led to realization that the isotopic internal symmetry in hadron physics had to be enlarged. In addition to the three SU(2) generators of isotopic rotations $\mathbf{T}_i$, which do not change the parity of states, an enlarged group of internal symmetries has to incorporate transformations which do mix up states with different parities (one can find more details in Appendix A of the lecture notes of Ref. 12). Such an enlargement has been discussed in studies of W. Pauli and C. N. Yang, J. Schwinger, R. Feynman and M. Gell–Mann (Refs. 13–15) and the appropriate symmetry acquired the name of "chiral symmetry." In his paper[4] Skyrme proposed the chiral-invariant modification of the "pion fluid" model, which turned to be one of the first nonlinear realizations of the chiral $SU(2) \otimes SU(2)$ group, known as the nonlinear σ-model.

As $G = SU(2) \otimes SU(2)$ is a 6-parameter group [three generators of isotopic rotations $\mathbf{T}_i$ and three generators of chiral rotations (boosts) $\mathbf{K}_j$], and there is no linear realization of such a group in the 3-dimensional isotopic space, one can either extend the isotopic space to be 4-dimensional or construct a nonlinear realization of G in 3-space. Choosing the first possibility, one can proceed by analogy with the extension of the SO(3) rotation group to the homogeneous Lorentz group (see Ref. 16, Ch. 5). To the 3-isovector $\boldsymbol{\phi}$ one can add a fourth component ϕ_0 and consider $(\phi_0,\boldsymbol{\phi})$ as a vector in 4-isospace. The generators of isorotations $\mathbf{T}_i$ will mix components of $\boldsymbol{\phi}$ only and will not affect ϕ_0. At the same time the chiral boost generators $\mathbf{K}_j$ will mix ϕ_0 with components of $\boldsymbol{\phi}$. The algebra of generators is specified by the relations:

$$[T_i, T_j] = i\varepsilon_{ijk}T_k, \quad [T_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = i\varepsilon_{ijk}T_k \tag{2.2.3}$$

and is locally isomorphic to the Lie algebra of the O(4) group. Introducing the left and right generators

$$L_i = \frac{1}{2}(T_i - K_i), \quad R_i = \frac{1}{2}(T_i + K_i), \tag{2.2.4}$$

we obtain the commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, \quad [R_i, R_j] = i\varepsilon_{ijk}R_k, \quad [L_i, R_j] = 0, \tag{2.2.5}$$

indicating that the initial algebra (2.2.3) splits into two independent SU(2) subalgebras (whence the notation $SU(2)_L \otimes SU(2)_R$ originates for the chiral group).

The generators of isorotations $\mathbf{T}_i$ commute with the parity operator $\mathbf{P}$, and the chiral rotations generators $\mathbf{K}_j$ anticommute:

$$[P, T_i] = 0, \quad [P, K_i]_+ = 0. \tag{2.2.6}$$

Let us also exhibit the commutation relations between the $SU(2) \otimes Su(2)$ generators in the vector representation and the components of the 4-isovector $(\phi_0,\boldsymbol{\phi})$:

$$[T_i, \phi_j] = i\varepsilon_{ijk}\phi_k, \quad [T_i, \phi_0] = 0, \tag{2.2.7a}$$

$$[K_i, \phi_j] = -i\delta_{ij}\phi_0, \quad [K_j, \phi_0] = i\phi_j, \tag{2.2.7b}$$

and correspondingly for left and right chiral generators

$$[L_j, \phi_0] = -\frac{i}{2}\phi_j, \quad [L_i, \phi_j] = \frac{i}{2}(\delta_{ij}\phi_0 + \varepsilon_{ijk}\phi_k), \tag{2.2.8a}$$

$$[R_j, \phi_0] = \frac{i}{2}\phi_j, \quad [R_i, \phi_j] = -\frac{i}{2}(\delta_{ij}\phi_0 - \varepsilon_{ijk}\phi_k); \tag{2.2.8b}$$

Here in classical theory one has to understand the bracket [,] as the Poisson bracket, multiplied by $(-i)$, and in quantum theory as the commutator. $\mathbf{T}_i$ and $\mathbf{K}_j$ are integrals of motion, i.e., the functionals of fields and canonical momenta, with the structure defined by the Noether theorem.

The relations (2.2.7) and (2.2.8) can be rewritten in a compact form, using the quaternion representation of the 4-isovector

$$U = \phi_0 + i\boldsymbol{\varphi}\cdot\boldsymbol{\tau} \tag{2.2.9}$$

for example

$$[L_j, U] = -\frac{1}{2}\tau_j U, \quad [R_j, U] = \frac{1}{2}U\tau_j. \tag{2.2.10}$$

It proves to be convenient first to divide the isospinor fields $\boldsymbol{\Psi}$ into the left and right components

$$\Psi_L = \frac{1}{2}(1 - \gamma_5)\Psi, \quad \Psi_R = \frac{1}{2}(1 + \gamma_5)\Psi, \tag{2.2.11}$$

transforming according to the fundamental representations of the $SU(2)_L \otimes Su(2)_R$ group. Acting on them by left and right generators, we have

$$[\mathbf{L}_i,\Psi_L] = -\frac{1}{2}\tau_i\Psi_L, \quad [\mathbf{R}_i,\Psi_L] = 0, \tag{2.2.12a}$$

$$[\mathbf{R}_i,\Psi_R] = -\frac{1}{2}\tau_i\Psi_R, \quad [\mathbf{L}_i,\Psi_R] = 0. \tag{2.2.12b}$$

When constructing a nonlinear realization of the $SU(2)_L \otimes SU(2)_R$ chiral symmetry we impose the following additional constraint:

$$\phi_0^2 + \boldsymbol{\phi}^2 = 1, \tag{2.2.13}$$

on the components of the 4-isovector $\phi_\alpha = (\phi_0,\boldsymbol{\phi})$, thereby leaving only three components ϕ_i independent. According to S. Weinberg[17] under chiral boosts they transform in the following manner:

$$[\mathbf{K}_i,\phi_j] = -i\delta_{ij}f(\boldsymbol{\phi}^2) - i\phi_i\phi_j g(\boldsymbol{\phi}^2); \tag{2.2.14}$$

Here $f(\boldsymbol{\phi}^2)$ is an arbitrary regular function, and $g(\boldsymbol{\phi}^2)$ is given in terms of $f(\boldsymbol{\phi}^2)$ due to the Jacobi identity for commutators:

$$g(\boldsymbol{\phi}^2) = \frac{1 + 2f(\boldsymbol{\phi}^2)f'(\boldsymbol{\phi}^2)}{f(\boldsymbol{\phi}^2) - 2\boldsymbol{\phi}^2 f'(\boldsymbol{\phi}^2)}. \tag{2.2.15}$$

The relations (2.2.3) and (2.2.7a) remain unaltered, and spinor fields commute with $\mathbf{K}_i$ according to

$$[K_i, \Psi] = \frac{1}{2}\frac{\varepsilon_{ijk}\tau_j\phi_k}{f(\boldsymbol{\phi}^2) - (f^2 + \boldsymbol{\phi}^2)^{1/2}}\Psi. \tag{2.2.16}$$

Here we touched upon only the algebraic aspects of chiral symmetry, without any connection with its physical content. We find it more convenient to do that in Sec. 4 taking into account modern notions of the quark structure of hadrons.

Now we are ready to come back to the chiral-invariant modification of the "pion fluid" model. As already noted, in his paper[4] Skyrme chose the nonlinear realization variant, and imposed on ϕ_α the constraint (2.2.13). But under such a restriction the natural generalization of the pion mass term in the Lagrangian (2.1.1):

$$\frac{k^2}{2}\sum_{i=1}^{3}\phi_i^2 \to \frac{k^2}{2}\sum_{\alpha=0}^{3}\phi_\alpha^2 = \text{const.}$$

appears to be meaningless.

To avoid the difficulty that has arisen there are two possibilities: either to introduce into the Lagrangian a fourth order term $(1/4)\gamma^2\Sigma_\alpha\phi_\alpha^4, \gamma = \text{const}$, or to assume that the pion mass is generated through pion-nucleon interactions. This idea, considered by Skyrme in Ref. 4, together with the previous assumption concerning the pion generation of the nucleon mass, has been formulated later by other authors as the hypothesis of "nuclear democracy:" fluctuations of pion fields give rise to the nucleon mass, and in turn pions obtain their mass through their interaction with nucleons.

In Refs. 4, 5 Skyrme dwells on the introduction into the Lagrangian of the fourth order term and arrives at the modified Lagrangian in the form:

$$L = \frac{1}{2}\sum_{\alpha=0}^{3}[(\partial_\mu\phi_\alpha)^2 + \frac{1}{2}\gamma^2\phi_\alpha^4] + \bar{\Psi}[i\gamma^\mu\partial_\mu + g(\phi_0 + i\gamma_5\boldsymbol{\tau}\boldsymbol{\phi})]\Psi, \tag{2.2.17}$$

which produces qualitative agreement with data on pion-nucleon interaction. Next Skyrme proceeded to study possible simplifications of the model. If one restricts oneself to only two field components ϕ_0 and ϕ_1, which depend only on the single spatial variable x and the time t, the resulting "two-dimensionalized" model turns out to be a rather interesting one and in the next subsection 2.3 we will convince ourselves of this.

2.3. The two-dimensional simplification, sine–Gordon and the idea of topological charge

In the case of only two field components ϕ_α the analog of the constraint (2.2.13) is fulfilled by the transition to the angular variable $\theta(x,t)$:

$$\phi_0 = \cos\theta, \quad \phi_1 = \sin\theta. \tag{2.3.1}$$

The meson part of the Lagrangian density (2.2.17) after some redefinition of variables takes the form of

$$L_M = \frac{\varepsilon}{2}[(\partial_t\theta)^2 - (\partial_x\theta)^2] - \varepsilon k^2(1-\cos\theta), \tag{2.3.2}$$

where ε is a constant setting the energy scale, and k is a reciprocal length.

The corresponding Euler–Lagrange equation:

$$\partial^2_{xx}\theta - \partial^2_{tt}\theta = k^2\sin\theta, \tag{2.3.3}$$

is known by the name "the sine-Gordon equation," and is completely integrable. The equation (2.3.3) describes numerous phenomena of nonlinear physics. Methods of solving and applications of this equation are well described in the literature.[18-22] We not here only Skyrme's contribution to the investigation of its properties, together with the conclusions he made. Not being acquainted with the geometrical papers by Bäclund, Steurwald *et al.*, Skyrme in Refs. 4, 6 and 23 found independently three kinds of solutions of this equation (in modern terms these solutions correspond to 2π-king, 4π-kink and breathers). Moreover, in Ref. 23 Skyrme and Perring considered the head-on collision of two kinks, moving towards each other with equal velocities and discovered their "particle-like" behavior, i.e., kinks were neither destroyed nor scattered, but instead they passed one through another without any alternation in their forms or velocities. It is remarkable, that this result predated by several years the famous Zabusky and Kruskal paper,[24] which had launched the "soliton boom."

In the framework of a two-dimensional model[6] Skyrme made his first step towards an understanding of a mechanism for constructing fermion states out of boson ones. He succeeded in proving that the sine-Gordon quantum solitons, arising in an essentially boson field model, might behave in the same way as fermions participating in a four-fermion interaction. About 14 years later S. Coleman[25] in the framework of perturbation theory rigorously demonstrated the equivalence between the quantum sine-Gordon model and the zero fermion charge sector of the massive Thirring model.

Still one more, and may be the most impressive, disclosure was the discovery by Skyrme of a conserved current j^μ ($\mu = 0;1$) with components:

$$j^0 = \frac{1}{2\pi}\partial_x\theta, \quad j^1 = -\frac{1}{2\pi}\partial_t\theta. \tag{2.3.4}$$

and with the conservation law $\partial_\mu j^\mu = 0$, which holds independently of the equation of motion (2.3.3), only because of the continuity of the angular variable $\theta(x,t)$. Skyrme proposed to interpret the integral conserved quantity

$$Q = \int j^0 dx = \frac{1}{2\pi}\int \partial_x\theta dx = \frac{1}{2\pi}(\theta(\infty,t) - \theta(-\infty,t)) \tag{2.3.5}$$

as the "number of particles" by analogy with classical mechanics, where one can relate the conservation of the number of particles with the continuity of particle trajectories. Almost simultaneously with D. Finkelstein and Ch. Misner,[26] Skyrme introduced in physics the topological classification of solutions of the field equations, together with a new kind of conservation laws, which acquired the name "topological" or "homotopical." The conserved quantity (2.3.5) was called the *topological charge*.

The meaning of conserved topological characteristics (charges) is easy to understand either in the framework of an algebraic approach[22] or in a geometrical way, using the principal concepts of homotopy theory.[27-30] Choosing the second possibility, one can regard the sine-Gordon model as the theory of scalar fields $\phi(x,t) = \exp[i\theta(x,t)]$, which due to condition (2.2.13) take on values on the circle S^1—the field manifold of the model. At any fixed moment of time one can regard these fields as a map

$$\phi(x):\ \mathbb{R}^1 \to S^1. \tag{2.3.6}$$

If one requires the field energy, corresponding to the Lagrangian (2.3.2), to be finite-valued, this leads to the boundary condition

$$\theta(x) \to 0 (\text{mod } 2\pi) \quad \text{as} \quad |x| \to \infty. \tag{2.3.7}$$

In this condition, first, the multivaluedness in the definition of the angle field variable $\theta(x)$ is reflected. To obviate it one should deem identical angles, which differ by $2\pi n$ in value, where n is an integer. Secondly, under condition (2.3.7) the real $\mathbb{R}^1$ axis is naturally compactified, as the points at infinity ($x = \pm\infty$) are mapped into the "north pole" of the circle S^1. Thus the mapping (2.3.6) with the boundary condition (2.3.7) due to the fact that $\mathbb{R}^1 \cup \{\infty\} = S^1$ can be replaced by the equivalent mapping of circles:

$$\phi(\alpha(x)) = \exp(i\theta(\alpha(x))):\ S^1 \to S^1, \tag{2.3.8}$$

where $\alpha(x)$ is the inverse stereographic projection $\mathbb{R}^1 \to S^1$, which parametrizes the "spatial" circle S^1.

Two subsequent states of the system at moments of time t_1 and t_2, as described by fields $\phi_1 = \phi(x,t_1)$ and $\phi_2 = \phi(xt_2)$, respectively, are to be linked by the time evolution, i.e., by a continuous solution $\phi(x,t)$ of the field equation. Formally, this obvious physical fact in the language of homotopy theory reads as follows: two continuous maps ϕ_1 and ϕ_2 are homotopic with each other ($\phi_1 \sim \phi_2$), when there exists a continuous function $\phi(x,t)$—the homotopy, which takes on the value ϕ_1 at the moment $t = t_1$ and ϕ_2 at the moment $t = t_2$. Since an assignment of a homotopy relation on a set of maps is equivalent to an assignment of an equivalence relation, this face means that the entire space of maps (2.3.6). Map $(\mathbb{R}^1,S^1)$ can be divided into disjoint classes of equivalent or mutually homotopic maps, i.e., Map $(\mathbb{R}^1,S^1) = \cup_i [\mathbb{R}^1,S^1]_i$, where for the ith homotopic class the standard notation $[\mathbb{R}^1,S^1]_i$ is adopted. If such a decomposition exists, (this can be established through the well-developed methods of homotopy theory) and the initial state function of the system belongs to one of the homotopy classes, then at any subsequent moment of time this function

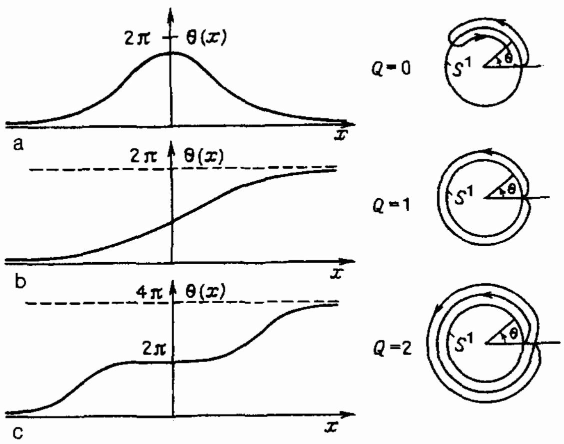

FIG. 1.

has to stay in the same class. In this sense the problem of finding a homotopy is equivalent to that of finding evolutionary solutions of the field equation.

Among all possible mappings from the space Map $(\mathbb{R}^1,S^1)$ we shall be interested only in those which fulfil the condition (2.3.7), i.e., we restrict ourselves to the subspace $\mathrm{Map}^0(\mathbb{R}^1,S^1)$, and will give a descriptive picture of the splitting of the latter into homotopy classes. If the angle variable $\theta(x)$ takes on the value zero at points $x = \pm\infty$, then the corresponding solution of the sine-Gordon equation is a traveling wave or a breather, and the image of $\mathbb{R}^1$ on S^1 under the map (2.3.6) will be a closed loop, which does not cover the whole circle S^1, and which could be shrunk in a continuous way to a point on S^1 (see Fig. 1a).

In the case $\theta(-\infty) = 0, \theta(+\infty) = 2\pi$ the image of $\mathbb{R}^1$ will be the loop, which covers the whole circle S^1 and cannot be shrunk in any continuous way to a point (see Fig. 1b). These maps correspond to the 2π-kink solution. Under the choice of conditions in the form: $\theta(-\infty) = 0, \theta(+\infty) = 4\pi$, the image of $\mathbb{R}^1$ the axis winds around the circle S^1 twice and the corresponding map cannot be continuously deformed into either of the two former ones. This situation is displayed in Fig. 1c and corresponds to two separated kinks (the bion solution).

It is clear that the considered solutions belong to different homotopy classes. As the characteristic **Q** (topological charge) of these classes it seems natural to take the number of times the image of $\mathbb{R}^1$ winds around the circle S^1 under the map (2.3.6). Taking this map in the form (2.3.8) it is not difficult to rewrite the expression (2.3.5) for the topological charge in another way:

$$\mathbf{Q} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\theta}{\partial\alpha} d\alpha, \tag{2.3.9}$$

where the integrand is just the Jacobian of the passage from the coordinate $\alpha(x)$ on the "spatial" circle S^1 to the coordinate $\theta(x)$ on the field manifold S^1. It is easy to recognize in (2.3.9) a particular case of the well-known mathematical concept called the degree of mapping or the Brauer degree, defined for a smooth map $f{:}M^n \to N^n$ between two connected oriented manifolds at a regular point $q \in N^n$. Under these conditions the full preimage $f^{-1}(q)$ consists of a finite number of points p_i, with the degree of mapping defined as

$$Q = \deg_q(f) = \sum_{p_i \in f^{-1}(q)} \operatorname{sgn}\det \frac{\partial x^\alpha}{\partial y_i{}^\beta}. \tag{2.3.10}$$

(see Ref. 31 for details). The topological charge **Q** in each homotopy class takes on a definite integer value, and for the sine-Gordon model this fact is almost an obvious one. However, in view of further applications in more general cases, this concept of a topological characteristic can be further formalized, by endowing the set of homotopy classes $\{[S^1,S^1]_i\}$ with a group structure. To do that it is enough to define an algebraic composition law (see, for example, Appendix B in lecture notes of Ref. 12), which converts the homotopy classes into elements of the Poincaré fundamental group $\pi_1(S^1)$ for the manifold S^1 (the first homotopy group). To compute this group means to establish an isomorphism between $\pi_1(S^1)$ and the group (or a subgroup) of integers $\mathbb{Z}$. The topological charge **Q** of Eqs. (2.3.5) or (2.3.9) is just what realizes this isomorphism $\mathbf{Q}{:}\pi_1(S^1) \to \mathbb{Z}$.

2.4. The structure of the Skyrme model

The appearance of the quantity **Q** is the two-dimensionalized model, which is conserved independently of the model dynamics and is interpreted as the number of "particles" minus "antiparticles," prompted Skyrme to further modifications of the $(3+1)$-dimensional model (2.2.17). If one were to be lucky, then the analogous quantity **Q** might be interpreted as the baryon charge and one could suggest an explanation of the experimentally established fact of conservation of the baryon-antibaryon number difference in all observed processes. The baryon number conservation law, which has been empirically introduced by E. Wigner and E. Stückelberg in analogy with the electric charge conservation law, is usually related with the U(1)-symmetry of the Lagrangian. The formality of such an approach evidently follows from the physical difference between the electric charge and the baryon number. While the electric charge e does determine the electromagnetic coupling constant $e^2/\hbar c$, in contrast the coupling constants of strong interactions $g_{\pi NN}, g_{\pi N\Delta}, \ldots$ do not depend on the value of the baryon number. This means that the baryon number does not determine the dynamics of baryons. The requirement of the baryon number conservation only restricts the possible types of reactions in strong interactions.

To generalize the concept of topological charge in the $(3+1)$-dimensional case Skyrme in Ref. 32 used the fact that due to the condition (2.2.13) the field ϕ_α takes on values on the manifold of the 3-dimensional sphere S^3, which is isomorphic to the group SU(2). Therefore to perform the analogous change to "angle"-variables it proves convenient to make use of the quaternion representation of the group SU(2) with elements in the form (2.2.9), for which the condition (2.2.13) reduces to the equality $\mathbf{U}\cdot\mathbf{U}^+ = I$. An analog of the "angular" variable $\theta(x,t)$ in the $(3+1)$-dimensional case will be the quantities $\mathbf{L}^a_\mu$ defined by the relations

$$\partial_\mu U = iU\tau_a L^a{}_\mu, \quad a = 1, 2, 3, \tag{2.4.1}$$

in virtue of which we have

$$L^a{}_\mu = \frac{1}{2i}\mathrm{tr}(\tau^a U^+ \partial_\mu U) = \phi_0\partial_\mu\phi^a - \phi^a\partial_\mu\phi_0 + \varepsilon^{abc}\phi_b\partial_\mu\phi_c.$$

The quantities $\mathbf{L}^a_\mu$ are components of the vector field $\mathbf{L}_\mu$ with values in the Lie algebra of the group SU(2), which has been named the left chiral current:

$$L_\mu = U^{-1}\partial_\mu U = i\tau^a L^a{}_\mu. \tag{2.4.2}$$

The compatibility condition for the definitions (2.4.1) and (2.4.2) requires the equality of mixed derivatives $\partial_\mu\partial_\nu \mathbf{U} = \partial_\nu\partial_\mu \mathbf{U}$, whence

$$\partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0, \tag{2.4.3}$$

or in the components $\mathbf{L}^a_\mu$:

$$\partial_\mu L^a{}_\nu - \partial_\nu L^a{}_\mu - 2\varepsilon^{abc}L^b{}_\mu L^c{}_\nu = 0. \tag{2.4.4}$$

By analogy with the two-dimensional model one can write down the conserved topological current:

$$J^\mu = -\frac{1}{12\pi^2}\varepsilon^{\mu\nu\lambda\rho}\varepsilon^{abc}L^a{}_\nu L^b{}_\lambda L^c{}_\rho$$

$$= \frac{1}{12\pi^2}\varepsilon^{\mu\nu\lambda\rho}\varepsilon^{\alpha\beta\gamma\delta}\phi_\alpha\partial_\nu\phi_\beta\partial_\lambda\phi_\gamma\partial_\rho\phi_\delta. \tag{2.4.5}$$

Its time component defines the density of the conserved topological charge

$$Q = \int J^0 d^3x = -\frac{1}{12\pi^2}\int \det L^a{}_i\, d^3x. \tag{2.4.6}$$

Skyrme chose the Lagrangian density in the form:

$$L = -\frac{1}{4\lambda^2}\mathrm{tr}(L_\mu L^\mu) + \frac{\varepsilon^2}{16}\mathrm{tr}([L_\mu, L_\nu][L^\mu, L^\nu]), \tag{2.4.7}$$

where ε and λ are scale parameters. Note, that in contrast to previous variants (2.1.1) and (2.2.17) the Lagrangian is expressed through meson fields only, i.e., nucleons have to appear in accord with the above stated hypotheses, as collective excitations or solitons. A naive scenario of appearance of these excitations can be imagined as follows.[34,35] In the framework of the hydrodynamical analogy the chiral current components $\mathbf{L}^a_\mu$ can be treated as those of the "pion fluid" generalized velocity, and the commutator term in (2.4.7) as a squared generalized vorticity. Then an appearance of a collective excitation, say of a nucleon, can be regarded as an appearance of a "vortex" in the pion fluid. At an intuitive level, the problem of the existence of these "vortices" in nontrivial homotopy classes can be clarified by means of graphic topological reasoning. As the currents $\mathbf{L}_\mu$ define a vector field on a sphere in isotopic space specified by the condition (2.2.13), then according to the "hairy ball" theorem such a field has to contain at least one singularity (in the sense that its direction is ill defined at this point). In a colloquial manner the latter statement means, that the "hairy ball" cannot be "combed" without a "top" in its "hairdo." In a sense one can identify the nucleon source in the Skyrme model with this "top" on the sphere in isotopic space. In this way Kelvin's "vortex atoms" ideas have been transfigured in the Skyrme model.

For the model with the Lagrangian (2.4.7) we write down the Euler–Lagrange equation

$$\partial_\mu(2L^\mu - \varepsilon^2\lambda^2[L_\nu, [L^\mu, L^\nu]]) = 0, \tag{2.4.8}$$

which has the form of a local conservation law. Since the Lagrangian (2.4.7) is invariant under the chiral $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{Su}(2)_\mathrm{R}$ transformations, there exist conserved Noether currents:

$$I^a{}_{\mu,\mathrm{R}} = \mathrm{tr}\left\{i\tau^a\left(\frac{1}{2\lambda^2}L_\mu - \frac{\varepsilon^2}{4}[L^\nu, [L_\mu, L_\nu]]\right)\right\}, \tag{2.4.9a}$$

$$I^a{}_{\mu,\mathrm{L}} = \mathrm{tr}\left\{i\tau^a\left(\frac{1}{2\lambda^2}R_\mu - \frac{\varepsilon^2}{4}[R^\nu, [R_\mu, R_\nu]]\right)\right\}, \tag{2.4.9b}$$

where $\mathbf{R}_\mu = \partial_\mu \mathbf{U}\cdot\mathbf{U}^{-1}$ is the right chiral current. The conserved isovector current, related with rotations in isotopic space in accordance with (2.2.4) has the form

$$I^a{}_\mu = I^a{}_{\mu,\mathrm{R}} + I^a{}_{\mu,\mathrm{L}} \equiv V^a{}_\mu. \tag{2.4.10}$$

The presented version of the model of a baryon as a soliton was proposed by Skyrme in his papers of Refs. 32, 33. Before passing to a more detailed treatment of the model and an explanation of its connection with modern concepts in the physics of strong interactions, based on ideas of Quantum Chromodynamics (QCD), we conclude this historical survey by listing here the significant results, obtained initially by Skyrme and rediscovered later by other investigators:

1. To describe a nucleon as a soliton Skyrme suggested the "hedgehog" ansatz

$$\phi_0 = \cos\theta(r), \quad \phi_i = \frac{x_i}{r}\sin\theta(r), \quad \theta(0) = N\pi, \tag{2.4.11}$$

which corresponds to the topological charge $\mathbf{Q} = N$, if $\theta(\infty) = 0$. As will be shown below, based on this ansatz the absolute minimum of energy is realized in the class of fields with unit topological charge $\mathbf{Q} = \pm 1$ (skyrmion).

2. A lower bound was obtained of the energy functional in terms of the topological charge $\mathbf{Q}$, which leads one to the conclusion of skyrmion stability.

3. In his paper[33] Skyrme considered the two-skyrmion interaction problem and proposed to describe two-skyrmion states by the "product-ansatz" (for details see Sec. 4.7).

However, there remained an open question, which has troubled Skyrme from the very beginning. This is the problem of the fermion properties of the skyrmion. After 10 years Skyrme returned to this problem in Ref. 36 and proposed the method of specifying collective coordinates to describe the skyrmion motion in a quasi-classical approximation. He succeeded to show that in lower orders of perturbation theory the skyrmion motion is governed by an equation of the Dirac type. We will acquaint the reader in section 4.5 with the current state of this problem as well as with an answer to the question in what cases may a skyrmion be regarded as a fermion.

3. SOLITONS IN THE SKYRME MODEL

3.1. Chiral solitons and topological stability

In accordance with Skyrme's main idea, a nucleon can be regarded as a vortex in the "pion fluid," the existence of which is prompted by topological considerations of the structure of vector fields on a sphere, as has been outlined earlier. The image of a nucleon as a chiral soliton proved to be attractive from the point of view of stability theory. The point is that, broadly speaking, multi-dimensional solitons, i.e., defined in $D \geqslant 2$ dimensional space, are unstable.[37,38]

Let us say, that the field $\phi = u(r): \mathbb{R}^3 \to \mathbb{R}^n$ has a soliton behavior, i.e., for $r \to \infty$

$$|\nabla u| = O[r^{-(3/2+\alpha)}], \quad \alpha > 0, \tag{3.1.1}$$

and the function $u(r)$ is the critical point of the Hamiltonian

$$H = \int F(\phi, \nabla\phi)\, d^3x, \tag{3.1.2}$$

i.e., it satisfies the Euler–Lagrange equations

$$F_s - \partial_i F^i{}_s = 0, \quad s = \overline{1,n}; \quad i = 1, 2, 3, \tag{3.1.3}$$

where the notation is used

$$F_s = \frac{\partial F}{\partial \phi^s}, \quad F^i{}_s = \frac{\partial F}{\partial(\partial_i \phi^s)}.$$

Let us construct the second variation of the Hamiltonian

$$\delta^2 H = \int (F_{sr}\xi^s\xi^r + F^{ik}{}_{sr}\partial_i\xi^s\partial_k\xi^r + 2F^i{}_{rs}\partial_i\xi^r\xi^s)\, d^3x,$$

where $\xi^s = \delta\phi_s$, and take a particular excitation in the form $\xi^s = f^j(r)\partial_j u^s$. Then we shall find

$$\delta^2 H = \int [\partial_i f^l A^{ik}{}_{lj}\partial_k f^j + (\partial_i f^l f^j - \partial_i f^j f^l) B^i{}_{jl}]\, d^3x, \tag{3.1.4}$$

where the notation is used

$$A^{ik}{}_{lj} = \partial_l u^r F^{ik}{}_{rs}\partial_j u^s, \quad 2B^i{}_{jl} = -2B^i{}_{lj} = \partial_{[j} F^i{}_r \partial_{l]} u^r. \tag{3.1.5}$$

Note, that the second term in (3.1.4) can change sign, and by virtue of equation (3.1.3) the following equality

$$\partial_i B^i{}_{jl} = 0,$$

holds, and it implies

$$2B^i{}_{jl} = \varepsilon^{ikm}\partial_k a_{mjl}, \tag{3.1.6}$$

where, in turn,

$$a_{mjl} = \frac{1}{2\pi}\varepsilon_{mki}\partial^k \int B^i_{jl}(\mathbf{r}')|\mathbf{r} - \mathbf{r}'|^{-1}\, d^3x'. \tag{3.1.7}$$

Substituting (3.1.6) into (3.1.4) we find after integration by parts

$$\delta^2 H = \int \partial_i f^l (A^{ik}_{lj} + \varepsilon^{ikm} a_{mjl})\partial_k f^j\, d^3x. \tag{3.1.8}$$

To study the integrand sign-definiteness in (3.1.8) consider the asymptotical region $r \to \infty$. Then from (3.1.1) and (3.1.5) it follows, that

$$A^{ik}_{lj} = O(r^{-(3+2\alpha)}), \quad B^i_{jl} = O(r^{-(3+2\alpha)}),$$

and from (3.1.7) it is easy to deduce the estimate

$$a_{mjl} = O(r^{-3}).$$

Thus, it has been proved, that the Hamiltonian (3.1.2) has a sign alternating second variation in the neighborhood of a soliton solution. This proof can be extended with ease to the case of space dimension $D \geqslant 2$, as well as to soliton solutions with harmonic dependence on time. The obtained result can be regarded as a generalization of the well-known Hobart–Derrick criterion[39,40] and it proves that only conditionally stable solitons are possible, i.e., they can be stable under some restrictions on admissible perturbations. Such restrictions emerge in the case of chiral solitons, endowed with the topological charge $\mathbf{Q}$.

Recalling, that

$$Q = -\frac{1}{48\pi^2}\varepsilon^{ijk}\int \mathrm{tr}(L_i[L_j, L_k])\, d^3x, \tag{3.1.9}$$

and the static Hamiltonian in the Skyrme model

$$H = -\int \left\{\frac{1}{4\lambda^2}\mathrm{tr}\, L_i^2 + \frac{\varepsilon^2}{16}\mathrm{tr}[L_k, L_j]^2\right\} d^3x$$
$$= -\int \left\{\frac{1}{4\lambda^2}\mathrm{tr}\, L_i^2 + \frac{\varepsilon^2}{32}\mathrm{tr}(\varepsilon^{ikj}[L_k, L_j])^2\right\} d^3x,$$

we easily find the estimate

$$H \geq \int \left|\mathrm{tr}\left(\frac{\varepsilon}{4\lambda\sqrt{2}}\varepsilon^{ikj}L_i[L_k, L_j]\right)\right| d^3x,$$

or in accordance with (3.1.9)

$$H \geq 6\pi^2\sqrt{2}\frac{\varepsilon}{\lambda}|Q|. \tag{3.1.10}$$

The estimate (3.1.10) means, that in a given homotopy class, i.e., for $\mathbf{Q} = N \neq 0$ the energy of system has a nontrivial lower bound. If this bound is attainable at a soliton solution, the latter would be stable in the Lyapunov sense, since $\delta^2 H \geqslant 0$. Note, that the equality in (3.1.10) never holds in reality, as it requires that

$$\frac{1}{2\lambda}L_i = \pm\frac{\varepsilon}{4\sqrt{2}}\varepsilon_{ikj}[L_k, L_j] = \mp\frac{\varepsilon}{2\sqrt{2}}\varepsilon_{ikj}\partial_k L_j.$$

But here the latter relation is definitely broken, since it implies $\partial_i \mathbf{L}_i = 0$, which contradicts the equations of motion (2.4.8).

3.2. The Coleman–Palais theorem and the dimensional reduction

In the search for multi-dimensional solitons a possibility to separate out angular variables, i.e., to perform a dimensional reduction in the equations of motion is very important. In a favorable case, one can obtain ordinary differential equations for the radial functions. In many physical problems to perform such a dimensional reduction the symmetry principle, with the most clear formulation due to S. Coleman (Refs. 12, 41), is frequently exploited.

Let the Hamiltonian $H[\phi]$ be invariant with respect to the action of some group G. Then we introduce a concept of invariant (more precisely, equivariant or covariantly constant) field $\phi_0(x)$, given by the condition

$$\phi^0(x) = T_g\phi^0(g^{-1}x), \tag{3.2.1}$$

here T_g is an operator in a representation of the group $G \ni g$. Coleman proposed to look for the extremum of the Hamiltonian in the class of invariant fields $\Phi_0\{\phi_0(x)\}$, and after doing that to check whether the obtained invariant configuration is the solution of the equations of motion, i.e., the true extremal of the functional $H[\phi]$. It turns out that for almost all cases, which are interesting in view of physics applications, i.e., when we deal with symmetry transformations from compact groups, semisimple groups, and with unitary representations of noncompact groups the Coleman principle is valid, in a sense that invariant extremals are true extremals. This statement was proved by Palais[42] and later on generalized by many authors.[43]

To elucidate the idea of Palais' proof, we start by introducing the notation $X \equiv \delta H/\delta\Phi[\phi_0]$ for the variational derivative, and write down the condition for the Hamiltonian to have an extremum in the class of invariant fields [or invariant set]

$$\langle X, \delta\phi^0\rangle = 0 \text{ for any } \delta\phi^0 \in \Phi_0, \tag{3.2.2}$$

where the angular brackets have been used to denote a linear functional. On the other hand the G-invariance of the Hamiltonian means, that

$$\delta H = \langle X, \delta\phi\rangle = \langle X_g, \delta\phi_g\rangle, \qquad (3.2.3)$$

where X_g and $\delta\phi_g$ denote the quantities X and $\delta\phi$, as transformed under the action of the element $g\in G$. At the same time

$$\delta H = H[\phi^0 + \delta\phi_g] - H[\phi^0] = \langle X, \delta\phi_g\rangle. \qquad (3.2.4)$$

Taking into account the arbitrariness of $\delta\phi_g$ and comparing (3.2.3) and (3.2.4) we find that

$$X_g = X. \qquad (3.2.5)$$

Let us denote by $\widetilde{\Phi}_0$ the subset of fields from X, which satisfy the invariance condition (3.2.5). Let us stress that in general the set Φ_0 and its dual set $\widetilde{\Phi}_0$ are different. The condition (3.2.2) means that X simultaneously belongs to the annihilator of the set Φ_0, i.e., $X\in\Phi_0^*$. Now it is not difficult to conclude that ϕ_0 would be the true extremal, if $\widetilde{\Phi}_0\cap\Phi_0^* = \varnothing$ (the empty set), since in this case $X = 0$ or $\delta H = 0$. Thus the following theorem is shown to be valid.

Theorem 3.2.1. (Coleman–Palais)

Let $H[\phi]$ be a functional, invariant under the action of the group G. Φ_0 is the set of invariant fields, Φ_0^* is the annihilator set of Φ_0, and $\widetilde{\Phi}_0$ is the set dual to Φ_0. Then the field $\phi_0\in\Phi_0$ being the extremal of H on the invariant set, is simultaneously the true extremal, i.e., the extremal with respect to noninvariant perturbations, if the Palais condition

$$\widetilde{\Phi}_0 \cap \Phi_0^* = \varnothing. \qquad (3.2.6)$$

holds.

Let us illustrate the significance of condition (3.2.6) by the example due to O. Ladyzhenskaya.[43] Consider on the plane $\mathbb{R}^2 = \{x^1,x^2\}$ a function $H = f(x^2)$, which is invariant under the action of group G: $\{x^1,x^2\}\to\{x^1 + \tau x^2,x^2\}$, with a parameter $\tau\in\mathbb{R}^1$. The invariant set in this case if $\Phi_0 = \{x^1,0\}$, so it coincides with the axis x^1. To construct the set $\widetilde{\Phi}_0$ we have to determine how the group G acts in the dual set $\{X_1,X_2\} = \{\delta H/\delta x^1,\delta H/\delta x^2\}$. To achieve this we write the invariance condition (3.2.3)

$$X_1\delta x^1 + X_2\delta x^2 = X_1'\delta x'^1 + X_2'\delta x'^2,$$

where $\delta x'^1 = \delta x^1 + \tau\delta x^2, \delta x'^2 = \delta x^2$, whence we deduce the transformation law $\{X_1,X_2\}\to\{X_1,X_2 - \tau X_1\}$ and the structure of the set $\widetilde{\Phi}_0 = \{0,X_2\}$. Finally, one finds the annihilator Φ_0^* from the condition (3.2.2):

$$X_1\delta x^1 = 0,$$

so that $\Phi_0^* = \{0,X_2\} = \widetilde{\Phi}_0$. Thus the condition (3.2.6) does not hold and the Coleman principle does not work in this case. Indeed, if we put $f'(0)\neq 0$, then $df(0) = f'(0)dx^2\neq 0$, although on the invariant set $\{x^1,0\}$ we have $df = \partial_1 f dx^1 \equiv 0$.

Now we apply the Coleman–Palais theorem to the Skyrme model. We take the principal chiral field $U\in SU(2)$ in the form

$$U(\mathbf{r}) = \exp[i(\mathbf{n}\boldsymbol{\tau})\theta(\mathbf{r})], \qquad (3.2.7)$$

where $\mathbf{n}(\mathbf{r})$ is a unit vector, $\theta(\mathbf{r})$ is the chiral angle, satisfying the condition at the spatial infinite $\theta(\infty) = 0$, as a result of which all the fields (3.2.7) are divided into homotopy classes, specified by the value $\mathbf{Q} = N$ of the topological charge (3.1.9). The Hamiltonian in the Skyrme model admits the group of spatial rotations $SO(3)_S$ and the group of isotopic rotations $SO(3)_I$, corresponding to transformations of the form $U\to V\cdot U\cdot V^{-1}$, $V\in SU(2)$. Thus, the invariance group of the Hamiltonian is

$$G = SO(3)_s \otimes SO(3)_I. \qquad (3.2.8)$$

However, it is clear, that the fields U, which would be invariant with respect to transformations from the group (3.2.8), do not exist. Therefore we have to consider its subgroups:

$$G_1 = \mathrm{diag}[SO(3)_s\otimes SO(3)_I], \qquad (3.2.9)$$

$$G_2 = \mathrm{diag}[SO(2)_s\otimes SO(2)_I], \qquad (3.2.10)$$

where diag means that the parameters of the multiplied groups either coincide or are proportional to each other, and the groups $SO(2)_S$ and $SO(2)_I$ correspond to rotations around the third axis in space or isospace, respectively.

To find the structure of G_1-invariant fields, let us write the condition (3.2.1) for infinitesimal transformations

$$-i[\mathbf{r}\nabla]U + \frac{1}{2}[\boldsymbol{\tau},U] = 0. \qquad (3.2.11)$$

Taking the scalar product of the vector relation (3.2.11) and $\mathbf{r}$, we obtain $[\tau_r,U] = 0$, where $\tau_r = (\boldsymbol{\tau}\cdot\mathbf{r})/r$. Hence, it follows that in (3.2.7) $\mathbf{n} = \mathbf{r}/r$. Then taking the trace of the expression (3.2.11) one finds

$$[\mathbf{r}\nabla]\cos\theta = 0,$$

whence $\theta = \theta(r)$ and this leads to the "hedgehog" ansatz

$$U_0(r) = \exp(i\tau_r\theta(r)) = \cos\theta(r) + i\tau_r\sin\theta(r), \qquad (3.2.12)$$

suggested by Skyrme in Ref. 33. As will become clear in a moment, the configuration (3.2.12) realizes the absolute minimum of the energy in the first homotopy class, i.e., among fields with $|\mathbf{Q}| = \mathbf{1}$.

Finally, for G_2-invariant fields it is convenient to write the condition (3.2.1) in terms of spherical coordinates (r,ϑ,α):

$$-i\partial_\alpha U + \frac{k}{2}[\tau_3, U] = 0, \qquad (3.2.13)$$

where k is an integer, with its value determined from the requirement on U to be a periodic function in α. Taking the trace of (3.2.13) we find $\partial_\alpha\theta = 0$, or $\theta = \theta(r,\vartheta)$. As a result the equation (3.2.13) is simplified:

$$-\partial_\alpha(\mathbf{n}\boldsymbol{\tau}) + \frac{k}{2}[\tau_3,(\mathbf{n}\boldsymbol{\tau})] = 0. \qquad (3.2.14)$$

Making use of the relation $[\tau_i,\tau_k] = 2i\varepsilon_{ikj}\tau_j$, from (3.2.14) we derive the equations for n_i:

$$\partial_\alpha n_3 = 0, \quad \partial_\alpha n_1 = -kn_2, \quad \partial_\alpha n_2 = kn_1.$$

Introducing the polar coordinates β,γ of the vector $\mathbf{n}$ by setting

$$n_3 = \cos\beta, \quad n_1 + in_2 = \sin\beta\cdot e^{i\gamma}, \qquad (3.2.15)$$

we get the following structure for the G_2-invariant chiral fields:

$$\theta = \theta(r,\vartheta), \quad \beta = \beta(r,\vartheta), \quad \gamma = k\alpha, \quad k\in\mathbf{Z}. \qquad (3.2.16)$$

In physical literature G_1-invariant fields are frequently named as spherically-symmetric or "hedgehog" fields, and G_2-invariant fields are called axially-symmetric fields. As will be shown below, G_2-invariant fields realize the energy minimum in higher homotopy classes, that is for $|\mathbf{Q}| > 1$.

3.3. Skyrme's "hedgehog" ansatz (Skyrmion) and absolute minimum of the energy

Now we are going to look for the minimum of the Hamiltonian in the Skyrme model in a given homotopy class, that is, for $\mathbf{Q} = \mathbf{N}$. Then the method of minimization in the extended phase space proves to be useful (see Refs. 45, 46 and in detail Ref. 12). The method is based on the following obvious property, that the minimum of a function of several variables can only increase after imposing some constraints on these variables. In particular, one might first minimize the energy density, regarding there the derivatives of the field functions $\partial_i \phi$ and the fields themselves as mutually independent variables. This is the quintessence of extending the phase space. Naturally, after performing such a procedure of minimization one should verify whether the obtained limiting configurations are solutions of the equations of motion.

If one introduces the auxiliary quantities:

$$\mathbf{X} = \nabla\theta, \quad \mathbf{Y} = \sin\theta\cdot\nabla\beta, \quad \mathbf{Z} = \sin\theta\cdot\sin\beta\cdot\nabla\gamma, \qquad (3.3.1)$$

then the static Hamiltonian H, shifted by a constant, can be represented in the form

$$H - 6\pi^2\sqrt{2}\frac{\varepsilon}{\lambda}|Q| = \int d^3x\left\{\left(\frac{1}{\lambda\sqrt{2}}\mathbf{X} + \varepsilon'\,[\mathbf{YZ}]\right)^2 + \left(\frac{1}{\lambda\sqrt{2}}\mathbf{Y} + \varepsilon'\,[\mathbf{ZX}]\right)^2 + \left(\frac{1}{\lambda\sqrt{2}}\mathbf{Z} + \varepsilon'\,[\mathbf{XY}]\right)^2\right\}, \qquad (3.3.2)$$

where $\varepsilon' = \varepsilon\,\mathrm{sgn}\mathbf{Q}$. For the sake of definiteness we choose $\mathbf{Q} > 0$, i.e., $\varepsilon' = \varepsilon$, and go over to dimensionless variables $\mathbf{r} \to \varepsilon\lambda\mathbf{r}$. Then it is not difficult to deduce from (3.3.2), that the minimum of H is attained in the case when the following pairs of vectors: $\mathbf{X}$ and $(\mathbf{Y}\times\mathbf{Z})$, $\mathbf{Y}$ and $(\mathbf{Z}\times\mathbf{X})$, $\mathbf{Z}$ and $(\mathbf{X}\times\mathbf{Y})$ turn out to be antiparallel. This means that the vectors (3.3.1) are orthogonal or, equivalently,

$$(\nabla\theta\;\nabla\beta) = (\nabla\theta\;\nabla\gamma) = (\nabla\beta\;\nabla\gamma) = 0. \qquad (3.3.3)$$

Then in virtue of the condition (3.3.3) H takes the form:

$$H = \frac{\varepsilon}{\lambda}\int d^3x\left\{(\nabla\theta)^2\left[\frac{1}{2} + \sin^2\theta\cdot\left((\nabla\beta)^2 + \sin^2\beta\cdot(\nabla\gamma)^2\right)\right] + \frac{1}{2}\sin^2\theta\cdot((\nabla\beta)^2 + \sin^2\beta\cdot(\nabla\gamma)^2) + \sin^4\theta\cdot\sin^2\beta\cdot(\nabla\beta)^2(\nabla\gamma)^2\right\}. \qquad (3.3.4)$$

Noting, that now the expression for the charge $\mathbf{Q}$ is

$$\mathbf{Q} = -\frac{1}{2\pi^2}\int \sin^2\theta\sin\beta(\nabla\theta[\nabla\beta\;\nabla\gamma])d^3x, \qquad (3.3.5)$$

one can minimize H with respect to the quantity $\delta = |\nabla\beta| - \sin\beta\,|\nabla\gamma|$, assuming that $\sin\beta\,|\nabla\beta|\,|\nabla\gamma|$ is fixed. As a result from (3.3.4) one finds, that $\delta = 0$, or in another form

$$(\nabla\beta)^2 = \sin^2\beta\cdot(\nabla\gamma)^2. \qquad (3.3.6)$$

Taking (3.3.6) into account the Hamiltonian simplifies:

$$H = \frac{\varepsilon}{\lambda}\int [(\nabla\theta)^2\left(\frac{1}{2} + 2\sin^2\theta\cdot(\nabla\beta)^2\right) + \sin^2\theta\cdot(\nabla\beta)^2(1 + \sin^2\theta)]d^3x. \qquad (3.3.7)$$

Let us turn now to formula (3.3.5), which expresses the topological charge $\mathbf{Q}$ as the degree of mapping $\mathbb{R}^3 \to \mathbb{S}^3$, i.e., an integer equal to the number of times one passes over the manifold $\mathbb{S}^3$, which is parametrized in our case by polar angles θ,β,γ, when the point $\mathbf{r}$ covers the whole space $\mathbb{R}^3$ once. In terms of spherical coordinates (r,ϑ,α), in order to insure that the field manifold $\mathbb{S}^3$ has been covered at least once, the following boundary conditions are to be imposed:

$$\beta|_{\vartheta=\pi} = \pi, \quad \beta|_{\vartheta=0} = 0, \quad \gamma|_{\alpha=2\pi} = 2\pi k + \gamma|_{\alpha=0}, \qquad (3.3.8)$$

where $k\in\mathbb{Z}$. From (3.3.8) it follows, that

$$\beta = \tilde{\beta} + \vartheta, \quad \gamma = k\alpha + \tilde{\gamma}, \quad k \in \mathbf{Z}, \qquad (3.3.9)$$

where the function $\tilde{\beta}$ is periodic in ϑ with the period π, and the function $\tilde{\gamma}$ is periodic in α with the period 2π. It is not difficult to see, that due to its periodicity the function $\tilde{\gamma}$ does not contribute to $\mathbf{Q}$, and therefore from (3.3.4) one finds that the Hamiltonian depends on this function only through the combination $(k\nabla\alpha + \nabla\tilde{\gamma})^2 = (\nabla\gamma)^2$ and attains its minimum if the vectors $k\nabla\alpha$ and $\nabla\tilde{\gamma}$ are antiparallel. This means, that $\gamma = \gamma(\alpha)$. But in that case from (3.3.3) it follows, that the functions θ and β do not depend on α and as a consequence the Euler-Lagrange equation for the function γ has the form $\partial_\alpha^2\gamma = 0$. Hence, in accordance with (3.3.9), we find

$$\gamma = k\alpha. \qquad (3.3.10)$$

Substituting (3.3.10) into the condition (3.3.6) we obtain the following equation for β:

$$k^2\frac{\sin^2\beta}{\sin^2\vartheta} = (r\partial_r\beta)^2 + (\partial_\vartheta\beta)^2. \qquad (3.3.11)$$

The equation (3.3.11) has an obvious integral

$$r\partial_r\beta = \mathrm{const}. \qquad (3.3.12)$$

The unique finite solution of equations (3.3.11) and (3.3.12), in accordance with (3.3.9), is

$$\beta = \vartheta, \qquad (3.3.13)$$

which corresponds to $k = 1$ in (3.3.11) and (3.3.10). Finally, from (3.3.3), (3.3.10) and (3.3.13) we deduce the following structure of the field configuration

$$\theta = \theta(r), \quad \beta = \vartheta, \quad \gamma = \alpha. \qquad (3.3.14)$$

Thus we have come to Skyrme's "hedgehog" ansatz (3.2.12). However, if $\sin\theta(0) \neq 0$, the mapping (3.3.12) is not single-valued at the point $r = 0$. To improve this it is necessary to impose the boundary conditions

$$\theta(0) = N\pi, \quad N \in \mathbf{Z}, \qquad (3.3.15)$$

which corresponds to the value of topological charge $\mathbf{Q} = N$, in accordance with (3.3.5).

For $N = \pm 1$, the obtained configuration (3.3.14) is unique and for this reason realizes the absolute minimum of the energy (Refs. 12, 44–47), but it is not the case for $N \neq \pm 1$, when states with lower energy are possible. Intuitively it is clear, that if one considers a system of N far separated skyrmions with unit topological charge, then its energy appears to be lower, i.e.,

$$E_N > |N| E_1, \tag{3.3.16}$$

where E_N is the energy of the "hedgehog" configuration with $\mathbf{Q} = N$.

In order to prove the inequality (3.3.16), we write down the Hamiltonian (3.3.7) using the ansatz (3.3.14) for $\mathbf{Q} = N$ [in units of $4\pi(\varepsilon/\lambda)$]

$$H[\theta] = \int_0^\infty dr \left[\theta'^2 \left(\frac{r^2}{2} + 2\sin^2\theta \right) + \sin^2\theta + \frac{\sin^4\theta}{r^2} \right]. \tag{3.3.17}$$

Let us pick out the points r_k, where the condition $\theta(r_k) = k\pi$, $k = \overline{0,N}$ holds, with $r_0 = \infty$, $r_N = 0$ and define the following functions

$$\theta_k(r) = \begin{cases} \theta(r) - (k-1)\pi, & r \in [r_k, r_{k-1}], \\ 0, & r > r_{k-1}, \\ \pi, & r \le r_k. \end{cases}$$

We rewrite the Hamiltonian (3.3.17) in the form

$$H = \sum_{k=1}^{N} H[\theta_k] \tag{3.3.18}$$

and note, that the introduced functions $\theta(r_k)$ and solutions of the equation of motion for the skyrmion with $\mathbf{Q} = N$ do satisfy the same boundary conditions, and because of that they belong to the first homotopy class. But there in the first homotopy class the absolute minimum of the energy is realized for the skyrmion configuration, and therefore we have $H[\theta_k] > E_1$. In such a case the inequality (3.3.16) follows at once from (3.3.18).

Numerical calculations support the validity of this inequality. In particular, for $N < 10$ the mass spectrum of the spherically-symmetric configurations of the "hedgehog" type is described with an accuracy $\sim 1\%$ by the following formula:[48]

$$E_N \approx \frac{1}{2} N(N+1) E_1. \tag{3.3.19}$$

For $N \gg 1$ the following asymptotic representation of the mass spectrum [in units of $4\pi(\varepsilon/\lambda)$] is valid[49]

$$E_N \approx [8{,}310 N(N + 0{,}8726)/2] + O(1). \tag{3.3.20}$$

It is easy to see that the inequality (3.3.16) follows from (3.3.19) and (3.3.20) with a good margin. In particular $E_1 \approx 3E_1$, and this leads to a strong repulsion between skyrmions at small distances and allows one to explain the effect of saturation of nuclear forces.

Let us write the basic equation for the chiral angle $\theta(r)$:

$$\theta''(r^2 + 4\sin^2\theta) = -2r\theta' + \sin 2\theta \cdot \left(1 - 2\theta'^2 + \frac{2\sin^2\theta}{r^2} \right). \tag{3.3.21}$$

Unfortunately, until now no one has succeeded in obtaining an exact solution of equation (3.3.21). The function $\theta(r)$ shows a linear behavior for $r \ll 1$: $\theta(r) \simeq N\pi - r/r_0$, and for $r \gg 1$ (in the linear regime) it falls off at the rate $\theta(r) = g/r^2$. The numerical value for the skyrmion energy is $E_1 \simeq 8.206749$.

A rather good approximation of the solution with $\mathbf{Q} = 1$ is provided by the Atiyah-Manton trial function:[50]

$$\theta = \pi \left[1 - \left(1 + \frac{\lambda^2}{r^2} \right)^{-1/2} \right], \quad \lambda^2 \approx 4{,}22. \tag{3.3.22}$$

By making use of it one obtains $E_1 \simeq 8.285$. If one uses a more complicated two-parametric trial function

$$\theta = 2\,\mathrm{arctg}[y(1 + by^2)^{1/2}], \quad y = r/a,$$

it results in $E_1 \simeq 8.239$ for $a = 1.495$, $b = 0.6984$. It is interesting to compare the above values for E_1 with the estimate (3.1.10)

$$E_1 = 6\pi^2\sqrt{2}\frac{\varepsilon}{\lambda} \cdot 1{,}231445.$$

Thus the excess over the estimate corresponds to the factor 1.23 for $N = 1$.

3.4. The direct minimization methods and the proof of the existence of the Skyrmion

To prove the existence of skyrmions with $\mathbf{Q} = N$ as described by the chiral angle $\theta(r)$, satisfying the equation (3.3.21) and the boundary conditions

$$\theta(0) = N\pi, \quad \theta(\infty) = 0, \tag{3.4.1}$$

it is possible by means of the direct method in the calculus of variations (see Refs. 45, 46 and for more details Ref. 12). Indeed, from the estimate (3.1.10) follows the existence of a lower bound for the Hamiltonian $H[\theta]$. In units of $4\pi(\varepsilon/\lambda)$ it takes the form

$$H = \int_0^\infty \mathscr{H} dr = \int_0^\infty \left[\frac{\theta'^2}{2}(r^2 + 4\sin^2\theta) + \sin^2\theta + \frac{\sin^4\theta}{r^2} \right] dr. \tag{3.4.2}$$

Therefore one may regard the existence of skyrmions as a consequence of the attainability of this lower bound of H on some set M, made up of functions $\theta(r)$ satisfying the boundary conditions (3.4.1). Recall that the direct method consists of the construction of a minimizing sequence $\theta_n(r) \in M$; of the proof of the convergence of this sequence to a limiting point $\theta_0 \in M$, and of the proof that

$$\inf H[\theta] = \varliminf_{n\to\infty} H[\theta_n] = H[\theta_0]. \tag{3.4.3}$$

Note, that (3.4.3) follows from the semicontinuity from below of the functional $H[\theta]$, i.e., from the following property

$$\varliminf_{n\to\infty} H[\theta_n] \ge H[\theta_0],$$

since in accordance with the definition of the lower bound $\inf H[\theta] \leqslant H[\theta_0]$. Therefore it is enough to convince oneself of the semicontinuity from below of $H[\theta]$.

To construct a minimizing sequence we establish some *a priori* estimates of the limiting function $\theta_0(r)$. First, we show the boundedness of its derivative at zero, i.e.,

$$|\theta_0'(0)| \le C < \infty. \tag{3.4.4}$$

Putting $N\pi - \theta(r) \equiv y(r)$ we assume the opposite, that as $r \to 0$, $|y'| \to \infty$, although the conditions (3.4.1) tell us that $y \to 0$. Then from the equation (3.3.21) we derive for $r \to 0$:

$$yy'' + y'^2 = y^2/r^2. \qquad (3.4.5)$$

The solution of this equation is $y = Cr$, which contradicts the previous assumption and proves (3.4.4).

Let us also establish some limitations on the behavior of $\theta(r)$ and $\theta'(r)$ for $r \to \infty$. It is clear that the functional $H[\theta]$ is bounded from above, since when constructing a minimizing sequence one can take as the first term $\theta_1(r) = N\pi\exp(-r)$, which gives $H[\theta_1] < \infty$. Therefore from (3.4.2) follows the limitation:

$$\int_0^\infty r^2\theta_0'^2\,dr \le C_0^2 < \infty,$$

whence, for $b > 0$ we have according to the Schwarz inequality

$$|\theta_0(b)| \le \int_b^\infty |\theta_0'|\,dr \le \left(\int_b^\infty \frac{dr}{r^2}\right)^{1/2}\left(\int_b^\infty r^2\theta_0'^2\,dr\right)^{1/2} < C_0 b^{-1/2}. \qquad (3.4.6)$$

Now we rewrite (3.3.21) as an integral equation (with variable limits a,b):

$$\left[r\theta_0'^2(r^2 + 4\sin^2\theta_0) - 2r\sin^2\theta_0\cdot\left(1 + \frac{\sin^2\theta_0}{r^2}\right)\right]_a^b = \int_a^b dr\left[-r^2\theta_0'^2 + 2\sin^2\theta_0\cdot\left(2\theta_0'^2 - 1 + \frac{\sin^2\theta_0}{r^2}\right)\right]. \qquad (3.4.7)$$

Then, setting in (3.4.7) $a \to 0$, $b \gg 1$ and taking into account (3.4.4) and (3.4.6) we find

$$|\theta_0'(b)| = O(b^{-3/2}). \qquad (3.4.8)$$

Finally, notice that the linearized equation (3.3.21), i.e., $r^2\theta_0'' = 2\theta_0 - 2r\theta_0'$, has the Green's function

$$G(r > s) = -\frac{s^4}{3r^2},$$

$$G(r < s) = -\frac{rs}{3}.$$

Therefore (3.3.21) is equivalent to the integral equation

$$\theta_0(r) = \int_0^\infty ds\, G(r,s)(s^2 + 4\sin^2\theta_0)^{-1} \times\left[8\sin^2\theta_0\cdot\left(\frac{\theta_0}{s}\right)' - 2\sin 2\theta_0 \times\left(\theta_0'^2 - \frac{\sin^2\theta_0}{s^2}\right) + \sin 2\theta_0 - 2\theta_0\right]. \qquad (3.4.9)$$

By making use of (3.4.8) we derive from (3.4.9) a more precise estimate for $b \to \infty$:

$$\theta_0(b) = O(b^{-2}), \quad |\theta_0'(b)| = O(b^{-3}). \qquad (3.4.10)$$

Now we consider the problem of minimization of a Hamiltonian auxiliary for (3.4.2):

$$H_{ab}[\theta] = \int_a^b \mathcal{H}\,dr,$$

specified on a finite interval $[a,b]$ in the class of smooth functions $\theta(r)$, satisfying the conditions $\theta(a) = N\pi, \theta(b) = 0$. Then in accordance with estimates (3.4.5) and (3.4.6) these functions belong to the Sobolev space $\mathbb{H}_1(a,b)$ with the norm

$$\|\theta\|_{H_1} = \left(\int_a^b dr\, r^2\theta'^2 + \theta^2(1)\right)^{1/2}, \quad a < 1 < b.$$

We choose the set M to be a sphere in $\mathbb{H}_1(a,b)$, which in accordance with the Banakha–Alaoglu theorem[51] has to be a weakly compact one, and therefore the sequence $\theta_n \in M$ is weakly convergent in this sphere to some limiting function $\theta_0(r)$.

Next we prove that the functional $H_{ab}[\theta]$ is weakly semicontinuous from below. To this end we use the fact that any Hilbert space, including $L_2(a,b)$ in particular, possesses a scalar product of positive functions, which is weakly semicontinuous from below.[52] Denoting by $\|\cdot\|$ the norm in $L_2(a,b)$, we represent $H_{ab}[\theta]$ as a sum of squared norms of some vectors from $L_2(a,b)$:

$$H_{ab}[\theta] = \sum_{k=1}^4 \|h_k\|^2,$$

where we have set

$$h_1 = r|\theta'|/\sqrt{2}, \quad h_2 = \sqrt{2}|\theta'\sin\theta|,$$

$$h_3 = |\sin\theta|, \quad h_4 = (\sin^2\theta)/r.$$

Now we take into consideration that the Sobolev space, i.e., $\mathbb{H}_1(a,b)$ is compactly embedded in the space of continuous functions,[53] $\mathbb{H}_1(a,b) \subsetneq \mathbb{C}(a,b)$. This means that the sequence $\theta n(r)$ is strongly convergent to an element $\theta_0(r)$ in $\mathbb{C}(a,b)$. Since the functions $h_3^{(n)}$ and $h_4^{(n)}$ are continuous with respect to θ, this means they converge to their limits. Furthermore, the sequence θ_n' is weakly convergent to θ_0' in $L_2(a,b)$, whence $h_1^{(n)}$ is weakly convergent in $L_2(a,b)$ to the limit vector $h_1(\theta_0')$. Finally, $h_2^{(n)}$ is the product of two sequences: $\sqrt{2}\theta_n'$, which is weakly convergent in $L_2(a,b)$, and $\sin\theta_n$, which in its turn is strongly convergent in $\mathbb{C}(a,b)$. Therefore $h_2^{(n)}$ would be weakly convergent in $L_2(a,b)$.

Thus we have found that all vectors $h_i^{(n)}$ are weakly convergent in $H_1(a,b) \subsetneq C(a,b)$ to the corresponding limits, and consequently the functional H_{ab} is weakly semicontinuous from below in $\mathbb{H}_1(a,b)$. Finally, by performing the limit transition to $a \to 0$, $b \to \infty$, we convince ourselves that

$$\lim_{\substack{a\to 0,\\ b\to\infty}} H_{ab} = H,$$

since in accord with (3.4.4) and (3.4.10)

$$\int_0^a \mathcal{H}\mathrm{d}r = O(a^3), \quad \int_b^\infty \mathcal{H}\mathrm{d}r = O(b^{-3}).$$

Thus, the weak attainability of the lower bound of the Hamiltonian H in the space $\mathrm{H}_1(0,\infty)$ for $\mathbf{Q} = N$ has been proven. It remains only to check the regularity of the limiting function $\theta_0(r)$. The latter follows directly from the presentation of (3.3.21) in the form

$$\theta_0'(r) = -\int_r^\infty \mathrm{d}r F(r, \theta_0, \theta_0'),$$

since from (3.4.4) and (3.4.10) it is straightforward to deduce that $F \in L_1(0,\infty)$.

3.5. The structure of topological solitons in higher homotopy classes

As was stated above, the "hedgehog" ansatz does not realize the minima of energy in higher homotopy classes. For this reason, we shall minimize the Hamiltonian in a less extended (compared with that in Sec. 3.3) phase space. If, for example, in (3.3.2) we vary only the direction of the vector $\mathbf{Z}$, then instead of (3.3.3) there would remain only two conditions:

$$(\nabla\theta\,\nabla\gamma) = (\nabla\beta\,\nabla\gamma) = 0. \tag{3.5.1}$$

The combination $(\nabla\gamma)^2$ will still appear in the Hamiltonian [see the expression (3.3.4)] and this will lead to the condition (3.3.10). Substituting (3.3.10) into (3.5.1) we arrive at the axially-symmetric configuration (3.2.16). These are the basic facts, giving rise to the conclusion, that the G_2-invariant fields do realize the minima of energy in higher homotopy classes.

This property is not an accidental one, and is supported by the statement of the following theorem:[54]

Theorem 3.5.1.

Assume that a G-invariant field ϕ_0, where $G = G_1$ (or G_2) realizes the minimum of a G-invariant functional $H[\phi]$ in the class of invariant fields. Then if the functional $H[\phi]$ is convex with respect to derivatives taken at the point ϕ_0, then the field ϕ_0 realizes the true minimum of $H[\phi]$, i.e., it is the minimum with respect to noninvariant perturbations as well.

Proof. Let us take the second variation of H at the point ϕ_0 and present it in the form of a scalar product in the Hilbert space L_2:

$$\delta^2 H = (y, \hat{K}(x)y), \quad y = \phi(x) - \phi^0(x),$$

where $\hat{K}$ is the Jacobi operator of the functional H. From the convexity of H with respect to derivatives it follows, that $\hat{K}$ is an elliptic operator and from the G-invariance of H we deduce that

$$\hat{K}(x) = \hat{T}_g \hat{K}(g^{-1}x)\hat{T}_g^{-1}, \tag{3.5.2}$$

where T_g is the operator of the group G representation. The relation (3.5.2) tells us that the operator $\hat{K}$ is a G-invariant *one and therefore can be expressed in terms of the Casimir* operators $\hat{\mathbf{C}}_\alpha$ of G.

Taking, for instance, $G = G_1, \hat{\mathbf{C}} = (\hat{\mathbf{J}} + \hat{\mathbf{T}})^2$, where $\hat{\mathbf{J}}$ are the angular momentum operators, $\hat{\mathbf{T}}$ are isospin operators. Finally, from the properties of the elliptic operator $\hat{K}$ follows the monotonic dependence of $\hat{K}$ on $\hat{\mathbf{C}}_\alpha$, which implies that the eigenvalues λ of the operator $\hat{K}$ increase with increasing eigenvalues α_k of the Casimir operator. In other words $d\lambda/d\alpha_k > 0$ when $\alpha_k > 0$ (in virtue of ellipticity one can restrict oneself to the positive branch of the spectrum). On the other hand, for an invariant eigenfunction y_0 of $\hat{K}$ we will have $\hat{\mathbf{C}}y_0 = 0$, and for noninvariant eigenfunctions y_k the eigenvalue equation is $\mathbf{C}y_k = \alpha_k y_k, \alpha_k > 0$. With the assumption that the spectrum of $\hat{K}$ is positive on invariant functions, i.e., $\lambda_0 > 0$, it follows that $\lambda > \lambda_0 > 0$.

The application of theorem 3.5.1 to the G_1-invariant solutions of the Skyrme model has been already illustrated in previous sections. As far as G_2-invariant configurations are concerned we have at our disposal only the numerical data on the dibaryons (Refs. 55, 56), i.e., the states with $\mathbf{Q} = 2$, when $k = 2$, $\theta(0) = \pi$ and $E_2 = 1.92E_1$. The available information on the states with $\mathbf{Q} \geqslant 3$ is contradictory, since it is obtained as a rule under assumption of discrete symmetry, when H is minimized in a certain sector of space, and configurations in other sectors are reconstructed by a continuation procedure. Nevertheless, in each of these sectors the axial symmetry of minimal-energy configurations is confirmed (Ref. 57, see also Ref. 111).

We note an interesting method of approximating soliton configurations with higher values of topological charge $\mathbf{Q} = k$, as suggested in Ref. 50. Consider Euclidean Yang–Mills fields

$$A_\mu = \frac{i}{2}\tau^a A^a{}_\mu,$$

which satisfy the self-duality equation

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3.5.3}$$

and possess a finite action. This means, that $F_{\mu\nu} \to 0$ as $|x| \to \infty$. For this reason at infinity these fields are just pure gauge fields:

$$A_\mu \underset{|x|\to\infty}{\to} U^{-1}\partial_\mu U = L_\mu.$$

The corresponding solutions are called instantons and are classified by means of topological charge $\mathbf{Q} = k$, which coincides numerically with the degree of mapping $\mathrm{S}^3 \to \mathrm{S}^3$ and can be calculated by the formula (3.1.9). The vacuum states ($F_{\mu\nu} = 0$), as is obvious from (3.5.3), are also pure gauge expressions and are characterized by the same topological charge $\mathbf{Q}$. Formally, the instanton solution links (in time) two vacuum states with charges $\mathbf{Q} = n$ and $\mathbf{Q}' = n + k$. One can readily see that, if one imposed the gauge condition $A_0 = 0$.

Let us suppose that a found instanton solution (A_t, A_i) does not satisfy this gauge condition, i.e., let its time component be nonzero ($A_t \neq 0$). Then we perform the gauge transformation

$$A_0 = V^{-1}A_t V + V^{-1}\partial_t V = 0,$$

wherefrom we obtain an "evolution" equation

$$\partial_t V = -A_t V,$$

with a formal solution (in the form of a holonomy along the time axis)

$$V(\mathbf{x}, t_1) = T\exp\left(-\int_{t_0}^{t_1} A_t(\mathbf{x}, t)\,dt\right)V(\mathbf{x}, t_0). \quad (3.5.4)$$

Putting $V(\mathbf{x} - \infty) = I$, we find $V(\mathbf{x}, +\infty) = \mathbf{U}(\mathbf{x})$. If A_t falls off fast enough for $|x| \to \infty$, then for its solution we have $\mathbf{U}(\mathbf{x}) \to I$ as $|\mathbf{x}| \to \infty$. The thus constructed field $\mathbf{U}(\mathbf{x})$ is the required chiral field with the topological charge $\mathbf{Q} = k$.

If we take, for example, the t'Hooft's formula for instantons:

$$A_t = \frac{i}{2\rho}(\tau\nabla)\rho, \quad \rho = 1 + \sum_{i=1}^{k}\lambda_i^2(x - X_i)^{-2}.$$

then from (3.5.4) we find the soliton configuration

$$U(\mathbf{x}) = T\exp\left(-\int_{-\infty}^{\infty} A_t(\mathbf{x}, t)\,dt\right). \quad (3.5.5)$$

In particular, for $k = 1$ and $X_1 = T_1 = 0$ we have

$$A_t = i(\mathbf{x}\tau)\left[(t^2 + r^2 + \lambda^2)^{-1} - (t^2 + r^2)^{-1}\right],$$

on substitution of which into (3.5.5) we obtain the trial function (3.3.22). Calculations for $k > 1$ are much more complicated, but use of the algorithm presented here allows one to obtain good enough approximations to soliton configurations in higher homotopy classes.

3.6. The rotating Skyrmion

The semiclassical quantization method, known also as the method of collective coordinates, forms the basis for attempts to describe the static properties of baryons in the framework of the Skyrme model. The method itself is based on the assumption, that the main contributions to the spectrum of excitations are those of rotational modes, given by Witten's ansatz:

$$U(t, \mathbf{r}) = A(t)U_0(\mathbf{r})A^{-1}(t), \quad (3.6.1)$$

where $A(t) \in SU(2)$ and $U_0(\mathbf{r})$ is the hedgehog ansatz (3.2.12). The substitution (3.6.1) describes the rigid-body-like rotation of the skyrmion in isospace, which is not consistent with the equation of motion and can be regarded as admissible only under assumption of slow rotations.[58] A more rigorous description of steady-state rotations, for instance, around the z axis might be undertaken using field functions with the following characteristic dependence on coordinates: $\alpha - \omega t$, where ω is the angular velocity. However, such an approach leads to equations, that do not possess soliton solutions, since one obtains typical radiative asymptotic behavior $\exp(i\omega r)/r$. Such a result can be predicted from physical considerations, as a rotating Skyrmion becomes deformed and starts to radiate pions, which are regarded as massless in the chiral limit. To improve such a shortcoming, let us include into Lagrangian the pion mass term

$$L_m = -\frac{m_\pi^2}{\lambda^2}(1 - \cos\theta), \quad (3.6.2)$$

which destroys chiral invariance.

In order to separate the time variable we perform the substitution $\partial_t = -\omega\partial_\alpha$, and, for the G_2-invariant ansatz (3.6.1), obtain the Lagrangian density:

$$\begin{aligned} L = &-\left[\frac{1}{2\lambda^2} + \varepsilon^2\sin^2\theta\cdot\sin^2\beta(r^{-2}\sin^{-2}\vartheta - \omega^2)\right.\\ &\times[(\partial_r\theta)^2 + r^{-2}(\partial_\vartheta\theta)^2 + \sin^2\beta\cdot((\partial_r\beta)^2 + r^{-2}(\partial_\vartheta\beta)^2)]\\ &- \frac{1}{2\lambda^2}\sin^2\theta\cdot\sin^2\beta(r^{-2}\sin^{-2}\vartheta - \omega^2) - \frac{m_\pi^2}{\lambda^2}(1 - \cos\theta)\\ &- \varepsilon^2 r^{-2}\sin^2\theta\cdot(\partial_r\theta\,\partial_\vartheta\beta - \partial_r\beta\,\partial_\vartheta\theta)^2. \end{aligned} \quad (3.6.3)$$

If we make a transition to dimensionless variables

$$r \to \varepsilon\lambda r, \quad \omega \to \omega/\varepsilon\lambda, \quad m_\pi \to m/\varepsilon\lambda,$$

then the typical combination $1 - \omega^2r^2\sin^2\vartheta$, which appears in (3.6.3), shows that for $\omega r \ll 1$ one can perform a decomposition in terms of ω^2, taking the latter as a small parameter and the "hedgehog" ansatz as a first approximation.

Then one can present the functions θ and β in the form

$$\theta \approx \theta_0(r) + \omega^2\theta_1, \quad \beta \approx \vartheta + \omega^2\beta_1. \quad (3.6.4)$$

The perturbations θ_1 and β_1 in (3.6.4) might be taken as expansions in terms of the Legendre polynomials:

$$\theta_1 = \sum_{n=0}^{\infty} a_n(r)P_{2n}(\vartheta), \quad \beta_1 = \sum_{n=1}^{\infty} b_n(r)P_{2n}^1(\vartheta). \quad (3.6.5)$$

Finally, notice that for $r\omega \gg 1$ the corresponding equations of motion are nearly linear in terms of field variables

$$X = \sin\theta\cdot\cos\beta, \quad Y = \sin\theta\cdot\sin\beta.$$

Therefore, if we restrict ourselves to the amplitudes a_0, a_1, b_1, then the solutions of equations for X and Y would be consistent with the expansion (3.6.5), if one chooses:

$$X = \cos\vartheta\cdot(a + b\cos^2\vartheta), \quad Y = \sin\vartheta\cdot(c + d\cos^2\vartheta), \quad (3.6.6)$$

where a,b,c,d, are radial functions with the asymptotic behavior given by

$$a, b \sim \frac{e^{-mr}}{r}, \quad c, d \sim \frac{e^{-m'r}}{r}, \quad m' = (m^2 - \omega^2)^{1/2}. \quad (3.6.7)$$

Thus, for $\omega \sim m$ the transverse component of the isovector field falls off slowly at large distances. This is a manifestation of the centrifugal effect.[59] As is clear from (3.6.7), the angular velocity $\omega = m$ (the pion mass) is the critical one. When the skyrmion attains this velocity it will be unstable and will start to radiate pions. Appropriate calculations show, that taking into account deformations of the rotating skyrmion allows one to improve the Skyrme model predictions.[60]

3.7. Toroidal and string-like vortex solutions. The relation between the Skyrme model and Faddeev's σ-model nonlinear on S^2

If one sets $\beta = \pi/2$ in the Skyrme model which corresponds to the condition of electrical neutrality of the "pion fluid",[34,35] the remaining angle variables θ and γ will determine the field with values on the S^2 sphere, or the so-called $\mathbf{n}$-field. The Hamiltonian takes the form

$$H=\int d^3x\left\{\frac{1}{2\lambda^2}((\nabla\theta)^2+\sin^2\theta\cdot(\nabla\gamma)^2)+\varepsilon^2\sin^2\theta\cdot[\nabla\theta\times\nabla\gamma]^2\right\}$$

(3.7.1)

and corresponds to Faddeev's $\mathbb{S}^2$ nonlinear σ-model.[61] Fields, which satisfy the boundary condition $\theta(\infty)=0$, split into homotopy classes, characterized by the value of the specific topological invariant, known as the Hopf index $\mathbf{Q}_H$. It turns out that the Hopf index reduces to the degree of mapping $\mathbb{S}^3\to\mathbb{S}^3$, if one introduces the auxiliary $\mathbb{S}^3$ manifold, parametrized by the angles A,B,C in such a way that $\mathbf{n}=\varphi^+\tau\varphi$, where φ is a two-component spinor, given by

$$\varphi^{\mathrm{T}}=(\cos A+i\sin A\cdot\cos B,\quad \sin A\cdot\sin Be^{iC}).$$

In this case $\mathbf{Q}_H$ is defined by the formula (3.3.5) with the following relabeling of angles $\theta,\beta,\gamma\to A,B,C$. A minimum estimate of the Hamiltonian (3.7.1) in terms of $|\mathbf{Q}_H|^{3/4}$ is:[61]

$$H>\mu|Q_{\mathrm{H}}|^{3/4},\quad \mu=\varepsilon\lambda\cdot 8\pi^2\sqrt{2}\cdot 3^{3/8}. \tag{3.7.2}$$

Minimizing the functional $H-\mu|\mathbf{Q}_H|^{3/4}$ in the extended phase space, as has been performed in Sec. 3.5, we obtain a G_2-invariant configuration in terms of A,B,C. Going back to variables θ,γ, we obtain the following structure:

$$\theta=\theta(r,\vartheta),\quad \gamma=k\alpha+\upsilon(r,\vartheta),\quad k\in\mathbf{Z}. \tag{3.7.3}$$

The Hopf index takes the form

$$Q_{\mathrm{H}}=\frac{k}{4\pi}\int_0^\infty dr\int_0^\pi d\vartheta(1-w)(\partial_r w\,\partial_\vartheta\upsilon-\partial_\vartheta w\,\partial_r\upsilon),$$

and the Hamiltonian is equal to

$$H=\int d^3x\left\{\frac{1}{2\lambda^2}\left[(\nabla\theta)^2+\sin^2\theta\cdot\left(\frac{k^2}{\rho^2}+(\nabla\upsilon)^2\right)\right]+\varepsilon^2\sin^2\theta\cdot\left(\frac{k^2(\nabla\theta)^2}{\rho^2}+[\nabla\theta\times\nabla\upsilon]^2\right)\right\}, \tag{3.7.4}$$

where $\rho=r\sin\vartheta$.

As one can see from (3.7.4), the corresponding regular solutions are to behave so that $\sin\theta\to0$ for $\rho\to0$. It means that they have to be of toroidal structure,[62] and one can imagine them as closed twisted strings (or vortices, smoke rings, etc.[63]). The additional twist of the n-field along the string is described by the function $\upsilon(r,\vartheta)\in[-n\pi/2,n\pi/2]$, $n\in\mathbb{Z}$, so that $\mathbf{Q}_H=kn$. Since the $\mathbf{n}$-field manifests a toroidal structure, one can make an attempt to approximate it by a string-like (vortex) solution, by taking a segment of the string and joining its endpoints. The string-like solutions for the Hamiltonian (3.7.4) have the form

$$\theta=\theta(\rho),\quad \upsilon=\varkappa z,\quad \theta(0)=m\pi, \tag{3.7.5}$$

where we have used the cylindrical coordinates ρ,z, with z being the coordinate along the string. The existence of these solutions can be proven by the direct variational method.[62] In Ref. 62 there are also given numerical data for the function $\theta(\rho)$ for $m=1$. The Hopf index for the closed segment of string of length l is determined by the formula

$$Q_{\mathrm{H}}=\frac{\varkappa l}{2\pi}=N.$$

The following estimate of the energy for such a configuration (in units of $4\pi\varepsilon/\lambda$) was obtained in Ref. 62:

$$E=23{,}65N. \tag{3.7.6}$$

Since the baryon charge of configurations in question is equal to zero, they should be interpreted as heavy mesons (torons). Let us also note, that in Ref. 64 a simple trial function was suggested to approximate the solution (3.7.3) with $k=1$:

$$\cos\theta=1-2\sin^2\Psi\cdot\sin^2\vartheta,\quad \upsilon=\mathrm{arctg}(\mathrm{tg}\,\Psi\cdot\cos\vartheta), \tag{3.7.7}$$

where $\Psi=\Psi(r)$ is an unknown radial function. Substituting (3.7.7) into (3.7.4), integrating over angle variables and performing the change of the radial variable $r\to\varepsilon\lambda r\sqrt{2}$, one finds

$$H[\Psi]=\left(32\pi\sqrt{2}\varepsilon/3\lambda\right)\int_0^\infty dr\left[\Psi'^2\left(\frac{r^2}{2}+\sin^2\Psi\right)+\sin^2\Psi+\frac{\sin^4\Psi}{r^2}\right].$$

Thus, the Skyrme Hamiltonian H_S (3.3.17) was obtained for a new chiral angle $\Psi(r)$ with the following relation $H=(8\sqrt{2}/3)H_S$. If one restricts oneself to the approximation (3.3.19) of the spectrum, then it implies the following value for the energy

$$E\approx 30{,}95N(N+1)/2. \tag{3.7.8}$$

One can see that the estimate (3.7.8) much exceeds (3.7.6), obtained in the string approximation. The latter works well for $N\gg1$, when the toroid has a big radius. At the same time for small values of N, the approximation (3.7.6) gives values that are definitely too low, and therefore one can indicate an approximate value for the toron mass with $\mathbf{Q}_H=1$:

$$2.88E_1\approx 23.65<E_{\mathrm{tor}}<30.95\approx 3.77E_1,$$

where $E_1=8.206749$ is the skyrmion mass.

4. THE SKYRME MODEL AND HADRON PHYSICS

Formulated in the early 1970s Quantum Chromodynamics (QCD), the SU(3)-gauge theory of quarks and gluons, is at present widely accepted as an indisputable candidate to become the strong interaction theory. However, using builders' terminology, one can say that until now there is only a general plan of a future building, and with respect to many subdivisions of QCD, one can say, that things are far from a completion of even drafts of working drawings. As a matter of fact, as of now we are provided neither with sufficient experimental data, nor with sufficiently reliable methods of calculation. At present distinct contours are seen only in the high energy QCD region, i.e., for short distances between quarks. Here QCD possesses the asymptotic freedom property and the description of quarks as of almost free particles appears to be a good approximation in this region, and the running coupling constant α_s appears to be a natural expansion parameter. Since methods of renormalization group analysis and of perturbation theory in powers of α_s provide a reliable calculation scheme for this region, this theory is called the perturbative QCD. The current study of phenomena in this high energy region encounters mostly technical difficulties. However, we are mostly interested in the nonperturbative QCD region which refers to the low energies.

Let us note one more problem of an independent character, and, possibly, the most difficult problem in QCD. This is the problem to study QCD dynamics in the region of intermediate energies. Here both quark-gluon and hadron degrees of freedom are essential, and this makes it rather difficult to develop effective methods for this region.

4.1. The low-energy sector of QCD, methods of investigation, and unsolved problems

The low-energy region of QCD is characterized by strong interactions between quarks, when the standard perturbation theory does not work. It is generally assumed that quarks are here in the confinement phase. However, in contrast to the asymptotic freedom property of quarks at short distances, which has been proved rigorously, the confinement phase of quarks is not a consequence of the QCD basic elements and is considered only as a likely hypothesis, which is not in contradiction with the QCD principles. In accord with this hypothesis, in the low-energy QCD there are significant only colorless (hadronic) degrees of freedom, i.e., QCD can be reduced to an effective theory of mesons and baryons. Along these lines, there arise two problems: 1) how to derive the Lagrangian of this effective theory from the QCD fundamental Lagrangian; 2) how to extract all the required information on the hadron properties out of the effective Lagrangian.

A possible way to solve the first problem has been suggested by t'Hooft[65] when he attempted to find an implicit expansion parameter for the low-energy QCD. It turned out that as an effective expansion parameter one can use the quantity $1/N_c$, where N_c is the number of color degrees of freedom. The only essential restriction here is the required existence in the theory of a continuous limit transition from large N_c values to the three-dimensional case. If such an expansion is an appropriate one, then for $N_c \to \infty$ the theory simplifies drastically, which allows one to obtain reasonable predictions for $N_c = 3$. We give a more detailed account of this method in the next section. Here we just note that the suggested scheme of the $1/N_c$-expansion has its rigorous grounds only in the two-dimensional QCD.[66] In Sec. 4.3 we shall consider some schemes of derivation of effective Lagrangians from QCD (such as the Andrianov–Novozhilov bosonization and so on), which are based on additional assumptions.

As is clear in advance, a Lagrangian of such an effective theory will be essentially nonlinear with a rather complicated structure.[67] Therefore different model concepts of hadron structure, which allow one to calculate the hadron characteristics and the hadron processes, appear to be rather topical. Among them let us mention the quark potentials model and various types of bag models. Starting from the middle 1980s, the Skyrme model joined this group. Great hopes are associated with the so-called hybrid models, which unite within themselves the attractive features of the bag models and of the soliton model of baryons—the Skyrme model (see Sec. 4.4). Some intermediate chiral quark models[112] are also being developed.

4.2. t'Hooft–Witten's $1/N$-expansion and renaissance of Skyrme's ideas

The idea of $1/N$-expansion appeared in physics in the early 1950s, almost simultaneously with Skyrme's model of "pion fluid."[68] It is well-known, that there are some problems in physics where a perturbation expansion in powers of a coupling constant is a meaningless one. Let us show this, taking as an example the hydrogen atom problem with the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{e^2}{r}. \tag{4.2.1}$$

Since e^2 is small enough, it seems that one can treat the potential energy term $-e^2/r$ as a perturbation, but after a rescaling $r \to r/me^2$; $p \to p \cdot me^2$ the expression (4.2.1) becomes

$$H = me^4\left(\frac{p^2}{2} - \frac{1}{r}\right), \tag{4.2.2}$$

where the overall factor me^4 can be used to determine an energy scale. Therefore we have to look for another small parameter. It was found, that in these situations one can use as a parameter the reciprocal dimension of an extended physical space. If we write down the Schrödinger equation for an s-state of an N dimensional hydrogen atom

$$\left[-\frac{1}{2m}\left(\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}\right) - \frac{e^2}{r}\right]\Psi = E\Psi, \tag{4.2.3}$$

and eliminate the term with the first derivative in (4.2.3) by means of redefinition of the wave function $\Psi \to \Psi r^{(N-1)/2}$, and perform the rescaling $r = (N-1)^2 R$ we obtain the equation

$$\frac{1}{(N-1)^2}\left[\frac{-1}{2m(N-1)^2}\frac{d^2}{dR^2} + \frac{N-3}{8m(N-1)R^2} - \frac{e^2}{R}\right]\Psi = E\Psi. \tag{4.2.4}$$

The equation (4.2.4) is the equation of motion for a particle with an effective mass $M_{\text{eff}} = m(N-1)^2$, moving in an effective potential

$$V_{\text{eff}}(R) = \frac{\gamma}{8mR^2} - \frac{e^2}{R}, \quad \gamma = \frac{N-3}{N-1}. \tag{4.2.5}$$

The problem simplifies drastically for large N, since the effective mass M_{eff} becomes very large and because of that it will not be a great mistake to assume that the particle simply sits at the bottom of the effective potential well (4.2.5). The ground state energy is assumed to be the absolute minimum of V_{eff} for $\gamma = 1$, and all the excited levels can be obtained by an $1/N$ expansion. Then, in accordance with (4.2.4), we will find

$$E_0 = -2me^4(N-1)^{-2}, \tag{4.2.6}$$

so that for $N = 3$ we get $E_0 = -me^4/2$, i.e., the exact binding energy value. Experience proves that the above method is a rather effective one.[69]

Let us note the common features of the above simple example and of the nonperturbative QCD, which in the limit of a large number of colors ($N_c \to \infty$) simplifies essentially (Refs. 65, 70). To clarify the reason for this simplification, recall, that the QCD fundamental Lagrangian for the N_c color case can be written as

$$L_{\rm QCD} = -\frac{1}{4}{\rm tr}(G_{\mu\nu}G^{\mu\nu}) + \bar{q}(iD_\mu\gamma^\mu - m)q, \tag{4.2.7}$$

where the quark fields q^α_a transform according to the fundamental representation of the color group SU(N_c) (correspondingly, the color label a takes on the values $a = 1,2,3,...N_c$) and of the flavor group U(N_f) (the flavor label is $\alpha = u,d,s,...N_f$). Gluon vector fields $A_\mu = A^c_\mu\lambda^c$ take on values in the Lie algebra of the SU(N_c) group. Here $c = 1,2,...N^2_c - 1, \lambda^c$ are the generators of SU(N_c); $G_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu + gf^{abc}A^b_\mu A^c_\nu)\lambda^a$ is the gluon field strength tensor, f_{abc} are the structure constants for the SU(N_c) group, $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative g is the color coupling constant (the color charge), m is the current quark mass.

Since the quark q^a and antiquark $\bar{q}^a$ fields (here for the sake of simplicity we omit the flavor index) both have N_c colored components, then it is convenient to describe the gluon fields by a traceless $N_c \times N_c$ matrix $(A_\mu)^a_b$ with $N^2_c - 1$ components, where $(A_\mu)^a_b = A^c_\mu(\lambda^c)^a_b$. Therefore even after a specification of the color quantum numbers of the initial and the final gluon states we have at least N_c possibilities of choosing the values of the quantum numbers of the intermediate gluon fields. In other words, this fact leads to the appearance of large combinatoric factors in Feynman diagrams when we sum up over all intermediate states. The values of these factors are determined by the number of closed gluon loops in a particular diagram. The technique of those calculations in terms of the graphic double-line notations is presented in Refs. 65, 70. Here we give as an illustration the diagram of the one-loop contribution of gluon fields to the gluon vacuum polarization (see Fig. 2), for brevity omitting vector indices in it.

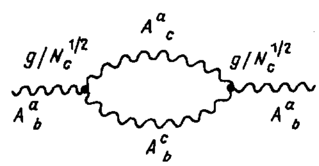

FIG. 2.

In this diagram there are two three-gluon vertices with the contribution $A_\mu{}^a_b A_\nu{}^b_c \partial^\mu A^{\nu c}{}_a$, where the index c corresponds to the closed gluon loop and runs over N_c possible values. The summation over all possible states gives the combinatoric factor of N_c. To provide the existence of a $N_c \to \infty$, it is convenient to redefine the charge $g \to g/\sqrt{N_c}$, then the factor of N_c would be canceled out. Then the resulting factor $N_c(g/\sqrt{N_c})^2 = g^2$ would not depend on N_c. After examination of all possible types of diagrams 't Hooft[65] came to the conclusion, that for $N_c \to \infty$ only some diagrams, which were called planar diagrams, would survive. They can be distinguished as those which can be drawn on the plane without line crossing or without an exit into the third dimension. The contributions of diagrams, which contain such an exit into the third dimension, fall off as N_c^{-2}, and of diagrams with internal quark loops fall off as N_c^{-1}.

A further simplification results from the hypothesis that the confinement of quarks continues to hold in the limit of large N_c, so that only colorless objects are observable. The detailed analysis of dominating diagrams, performed by Witten,[70] resulted in the following conclusions:

a. Mesons for large N_c might be regarded as stable and noninteracting particles, since their decay amplitudes are of order $N_c^{-1/2}$.

b. Meson-meson elastic scattering amplitudes are of order N_c^{-1} and are determined by contributions of tree diagrams only.

c. Baryonic states might arise in the effective theory of mesons with a small coupling constant $N^{-1}c$ as solitons with masses of the order of the reciprocal coupling constant, i.e., N_c.

As the main result of performed investigations one can regard the proof of the equivalence of QCD at the limit of $N_c \to \infty$ to the theory of meson fields (and glueballs) with an effective interaction of the order $1/N_c$. In this way at a new level Skyrme's idea was revived to consider a baryon as a soliton state, appearing as a result of collective excitations of meson fields. Of course, in such an approach there remain unclear questions concerning the meson masses and their coupling constants. Other problems are those of the structure of an effective meson Lagrangian and of fermion, the properties of baryons, regarded as chiral solitons. We intend to discuss all these questions and problems in the subsequent sections. Now let us explain how in the framework of QCD one can obtain a confirmation of Skyrme's idea that the topological charge can be identified with the baryon number.

The latter result has been obtained in the Balachandran group[71] and we reproduce it on the basis of the lecture notes of Ref. 72. Physically the idea consists of the study of the "polarization" of the Dirac "sea" of quarks in the presence of an external classical chiral field U. It turns out that the baryon current of quarks, averaged over the fermion states, has exactly the same form as the topological current in the Skyrme model. To start we present the baryon current $J^{(B)}_\mu$ in the form which is standard for QCD with N_c colors:

$$J^{(B)}_\mu = N_c^{-1}\bar{q}\gamma_\mu q, \tag{4.2.8}$$

where the factor $1/N_c$ reflects the fact that each quark has the baryon number $1/N_c$. Let us calculate the expectation value $\langle J^{(B)}_\mu\rangle = B_\mu(U)$ in the ground state, when the quark field q is coupled to the external field U. In the Lagrangian (4.2.7) we preserve only the quark part

$$L^{(q)}_{\rm QCD} = i\bar{q}\gamma^\mu\partial_\mu q - m(\bar{q}_{\rm L}Uq_{\rm R} + {\rm h.c.}), \tag{4.2.9}$$

where

$$q_{\rm L} = \frac{1}{2}(1+\gamma_5)q, \quad q_{\rm R} = \frac{1}{2}(1-\gamma_5)q. \tag{4.2.10}$$

If one assumes that the baryon current conservation law $\partial_\mu B^\mu = 0$ holds independently of the dynamics of the external field $U(x) \in {\rm SU}(N_f)$, then in general such a current B_μ, as defined on the U-manifold, should be of the form

$$B^\mu(U) = \beta J^\mu(U) + \varepsilon^{\mu\nu\sigma\tau}\partial_\nu\Psi_{\sigma\tau}, \tag{4.2.11}$$

where β is a constant, $\Psi_{\sigma\tau}$ is any smooth function of U and J^μ(U) is the conserved (topological) current. From general considerations it should be clear, that B^μ cannot be reduced to the last term in (4.2.11) only, since the corresponding charge density would have the form of a divergence

$$\varepsilon^{ijk}\partial_i\Psi_{jk}$$

and due to the boundary condition $\Psi_{jk}(r\to\infty)\to$const, it would not contribute to the baryon charge.

From the above considerations it follows, that the baryon number of the Dirac sea

$$\int B_0(U)\,\mathrm{d}^3x=\beta Q \qquad (4.2.12)$$

is completely determined by the proportionality constant β, which has to be fixed from the QCD fundamentals. In order to achieve this, let us change variables in (4.2.9) as follows: $u_{\mathrm{L}}=q_{\mathrm{L}}$, $u_{\mathrm{R}}=\mathrm{U}q_{\mathrm{R}}$:

$$L_{\mathrm{q}}=i\bar{u}_{\mathrm{L}}\gamma^{\mu}\partial_{\mu}u_{\mathrm{L}}+i\bar{u}_{\mathrm{R}}\gamma^{\mu}(\partial_{\mu}-A_{\mu})u_{\mathrm{R}}-m(\bar{u}_{\mathrm{L}}u_{\mathrm{R}}+\mathrm{h.c.}),$$

where $A_{\mu}=-U\partial_{\mu}\mathrm{U}^{\dagger}$. As a result of interaction, the right hand currents acquire the anomaly[113]

$$\partial_{\mu}(\bar{u}_{\mathrm{R}}\gamma^{\mu}u_{\mathrm{R}})-im(\bar{u}_{\mathrm{L}}u_{\mathrm{R}}-\bar{u}_{\mathrm{R}}u_{\mathrm{L}})$$
$$=\frac{N_{\mathrm{c}}}{8\pi^2}\varepsilon^{\mu\nu\alpha\beta}\partial_{\mu}\mathrm{tr}(A_{\nu}\partial_{\alpha}A_{\beta}+\tfrac{2}{3}A_{\nu}A_{\alpha}A_{\beta}), \qquad (4.2.13)$$

while for the left hand currents we have the standard relation

$$\partial_{\mu}(\bar{u}_{\mathrm{L}}\gamma^{\mu}u_{\mathrm{L}})-im(\bar{u}_{\mathrm{R}}u_{\mathrm{L}}-\bar{u}_{\mathrm{L}}u_{\mathrm{R}})=0, \qquad (4.2.14)$$

which follows from the equations of motion.

On adding (4.2.13) and (4.2.14), we find

$$\partial_{\mu}J^{\mu(\mathrm{B})}=\frac{1}{N_{\mathrm{c}}}\partial_{\mu}(\bar{q}\gamma^{\mu}q)$$
$$=\frac{1}{8\pi^2}\varepsilon^{\mu\nu\alpha\beta}\partial_{\mu}\mathrm{tr}(A_{\nu}\partial_{\alpha}A_{\beta}+\tfrac{2}{3}A_{\nu}A_{\alpha}A_{\beta}). \qquad (4.2.15)$$

The expectation value of the baryon current of the Dirac sea one obtains from (4.2.15) by averaging it over the fermion states:

$$\langle J^{\mu(\mathrm{B})}\rangle=\frac{1}{8\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{tr}(A_{\nu}\partial_{\alpha}A_{\beta}+\tfrac{2}{3}A_{\nu}A_{\alpha}A_{\beta}).$$

Taking into account the definitions for A_{μ} and for the topological current in the Skyrme model [see (2.4.5) and (3.1.9)] we obtain the expression

$$\langle J^{\mu(\mathrm{B})}\rangle=-\frac{1}{24\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{tr}(A_{\nu}A_{\alpha}A_{\beta})=J^{\mu},$$

which supports the Skyrme hypothesis on the interpretation of topological charge as the baryon number. Hence, in (4.2.12) one should set $\beta=1$ (Refs. 73–75).

4.3. Andrianov–Novozhilov bosonization and other approaches to derivation of σ-model Lagrangian from QCD

The problem of transition from a fundamental (microscopic) description to an effective (macroscopic) one is not new in theoretical physics. To this end let us recall the "low-energy" dynamics in solid state physics, where the fundamental objects are ions in the centers of a crystalline lattice and electrons, which interact via the Coulomb law. Nevertheless an adequate description of low-temperature phenomena is given in terms of electrons with an effective mass (polarons) and phonons. It is remarkable that in this case these effective degrees of freedom arise as a consequence of the spontaneous breaking of the translational invariance. Coming back to a description of the low-energy dynamics in QCD, it is easy to note some analogy with the above example. Here we also have to pass in the fundamental Lagrangian (4.2.7) from quark and gluon degrees of freedom, which prove to be essential only at distances of the order 0.3 fm to hadron degrees of freedom describing the physics of strong interactions at distances of 1–2 fm. In the process of formation of this sector (hypothetically) there arise such essentially nonperturbative phenomena as color confinement and spontaneous breaking of chiral symmetry. If the latter effect can be subject to some description, the situation with understanding confinement is in a much worse state. To date even a concrete statement of the problem is lacking. Moreover, there are some examples of calculations of hadron properties based on sum rules without any use of the confinement hypothesis (Refs. 76, 77). This hypothesis is also not used for the explanation of existence of pions, which appear as Goldstone bosons due to the spontaneous breaking of chiral symmetry. One can deduce the main pion characteristics from the partial conservation of axial current (PCAC).

We note that frequently there appear opposite views on the problem of an effective Lagrangian derivation from QCD. On the one hand, the problem can be formulated as that of finding an appropriate "change of variables" in the functional integral

$$\int \mathrm{d}A\,\mathrm{d}\bar{q}\,\mathrm{d}q\,\exp\left(\frac{i}{\hbar}\int \mathrm{d}x\,L_{\mathrm{QCD}}(\bar{q},q,A)\right)$$
$$\Rightarrow\int \mathrm{d}U\exp\left(\frac{i}{\hbar}\int \mathrm{d}x\,L_{\mathrm{eff}}(U)\right), \qquad (4.3.1)$$

where $\mathscr{L}_{\mathrm{QCD}}$ is given by the formula (4.2.7), U is the chiral field, parametrized by meson fields, and $\mathscr{L}_{\mathrm{eff}}$ is the effective Lagrangian being sought which depends only on meson fields.

Actually, in its full extent this problem is equivalent to that of the low-energy QCD solvability. Even if one were to be successful, it would result in a meson Lagrangian rather complicated for further study and special efforts would be necessary to extract information out of such a Lagrangian.[67] Such methods are being actively developed now, and they are being tested on those effective Lagrangians, which either can be derived under additional assumptions, or can be constructed in a plausible manner. The construction of reasonable approximations to the effective Lagrangian is regarded at present as an alternative approach, since as will be shown a direct derivation without any simplification is still impossible.

Besides purely meson effective Lagrangians, the so-called quark-meson, or hybrid models where the quark degrees of freedom are not regarded as suppressed, are under intensive study. Such an approach is justified by different phenomenological considerations, and in particular, by the lack in our understanding of the confinement problem. We shall return to these problems in Sec. 4.4, and here we will list some symmetry considerations, which are accepted by most researchers (see, for example, Ref. 12). We shall also pay attention to some approaches and approximations, which are used in attempts to solve the problem of (4.3.1).

A rather reliable approximation in the low-energy region of QCD turns out to be the chiral limit, when the current quarks are taken to be massless. This is based on the fact, that the masses of u- and d-quarks ($m_u \sim 4$ MeV, $m_d = 7$ MeV), and to a much lesser extent the mass of the s-quark $m_s \sim 130$ MeV are small compared to the energy scale $\Lambda_{\rm QCD} \simeq 300$ MeV characteristic of this region. Calculations show, that the baryon masses and almost all meson masses practically do not change under transition to this limit. The exceptions are the pions, which in virtue of the Gell Mann–Oaks–Renner relation $m_\pi^2 \simeq (m_u + m_d)\Lambda_{\rm QCD}$, become massless.

Substituting (4.2.10) into (4.2.7), it is easy to see that for $m \to 0$ the QCD Lagrangian possesses the global $U(N_f)_L \otimes U(N_f)_R$ symmetry with respect to the left and right rotations in the flavor space. More precisely, we are dealing with transformations from the group

$$U(3)_L \otimes U(3)_R = SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes U(1)_A, \quad (4.3.2)$$

where $U(1)_V$ and $U(1)_A$ are the one-parametric subgroups of vector and axial-vector transformations with generators $q \to \exp(i\alpha)q, q \to \exp(i\gamma_5\alpha)q$, respectively. The above considerations prove that it is unreasonable to extend the chiral symmetry to the case of $N_f > 3$, and therefore we restrict ourselves to $N_f = 3$ in (4.3.2). At the quantum level the $U(1)_A$-symmetry turns out to be explicitly broken in virtue of the Adler–Bell–Jackiw–Bardeen anomaly:[78]

$$\partial_\mu(\bar{q}\gamma^\mu\gamma_5 q) = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\lambda\rho}\mathrm{tr}(G_{\mu\nu}G_{\lambda\rho}). \quad (4.3.3)$$

The remaining symmetry group

$$SU(3)_L \otimes SU(3)_R \otimes U(1)_V$$

includes transformations of the type

$$q \to \exp\left(-i\gamma_5 \frac{\pi^a\lambda^a}{2}\right) q, \quad (4.3.4)$$

which mixes states with different parities. Here π^k are chiral phases, λ^a are the $SU(3)$ generators. For the left and the right quark fields (4.2.10) the transformation (4.3.4) takes the form

$$q \to U^{-1/2}q_L + U^{1/2}q_R, \quad U = \exp(i\pi^a\lambda^a). \quad (4.3.5)$$

The spontaneous breaking of the global $SU(3)_L \otimes SU(3)_R$ symmetry means that the relative chiral phases of the left and right handed quarks became locally fixed so that phase functions $\pi^k(x)$ might be related with the octet of pseudo-Goldstone bosons ($\pi^0, \pi^\pm, \eta, K^0, K^\pm, \bar{K}^0$). Since at spatial infinity the meson fields vanish, then for the principal chiral field $U(x)$ it is equivalent to the boundary condition

$$\lim_{|r|\to\infty} U(x) = I. \quad (4.3.6)$$

From the transformation laws of quark fields:

$$q_L \to Wq_L, \quad q_R \to Vq_R, \quad W, V \in SU(3),$$

follows the transformation law for the principal chiral field $U(x)$ under the group $SU(3)_L \otimes SU(3)_R$:

$$U(x) \to VU(x)W^{-1}. \quad (4.3.7)$$

Consequently, the vacuum state $\mathbf{U} = \mathbb{1}$ is invariant only under transformations from the subgroup of G, picked out by the condition $V = W$, i.e., under transformations from the vector subgroup

$$G_V = \mathrm{diag}(SU(3)_L \otimes SU(3)_R) \approx SU(3)_V. \quad (4.3.8)$$

One will come to the analogous conclusion under assumption that in the vacuum state quarks form a condensate $\langle \bar{q}_\alpha, q_\beta \rangle = \langle \bar{q}, q \rangle \delta_{\alpha\beta}/3$, which is not invariant with respect to chiral transformations (4.3.4). This leads to the spontaneous breaking of the chiral symmetry $G \to SU(3)_V$. A more detailed discussion of the scenario of spontaneous chiral symmetry breaking one can find in Ref. 79.

Thus, one can consider that the field $U(x)$ determines the orbit of the group G passing through the unit element I and therefore takes on the values in the homogeneous coset space $G/G_V \simeq SU(3)$, realizing a one-to-one correspondence between the weak (phase) vacuum excitations, which parametrize the field $\mathbf{U}(x)$ in virtue of (4.3.5), and elements of the $SU(3)$ group manifold. Now, if one takes into account the validity of the chiral limit with acceptable accuracy only for light u- and d-quarks, which corresponds to $SU(2)_L \otimes SU(2)_R$-symmetry, that breaks down according to the same scenario to $SU(2)_V$, then it is possible to regard the above consideration as one more manifestation of Skyrme's deep intuition. Actually, long before the appearance of the concepts of quarks and QCD, Skyrme suggested that the chiral field has to take on values on the $S^3 \simeq SU(2)$ manifold. On the other hand, since the above considerations are represented as being valid for any approach to the derivation of the effective Lagrangian then as a probable result one would obtain a nonlinear σ-model with spontaneous chiral symmetry breaking.

As a result of a more detailed QCD symmetry analysis, Witten[70] and the Syracuse group[71] came to the conclusion that the sought effective action of the σ-model must necessarily contain the kinetic term, (known as the Weinberg–Gürsey action) and the Wess–Zumino term[80] related to the Adler–Bell–Jackiw–Bardeen axial anomaly in QCD (4.3.3):

$$S = -\frac{F_\pi^2}{16}\int d^4x\, \mathrm{tr}(L_\mu L^\mu) - \frac{iN_c}{240\pi^2}\int_{D_5} d^5x\, \epsilon^{\mu\nu\lambda\rho\sigma}\mathrm{tr}(L_\mu L_\nu L_\lambda L_\rho L_\sigma), \quad (4.3.9)$$

where F_π is the pion decay constant and D_5 in 5-disc with the Minkowski spacetime as its boundary. This conclusion was rigorously confirmed by Karchev and Slavnov[81], where the chiral phase of quarks has been picked out on the basis of the Faddeev–Popov procedure. It was shown that after making the change of variables:

$$q^U = U^{-2}q_R + q_L, \quad \bar{q}^{U} = \bar{q}_R U^2 + \bar{q}_L \quad (4.3.10)$$

In the generating functional (4.3.1) with the massless variant of the Lagrangian (4.2.7), and after the calculation of *the Jacobian of this transformation the Wess–Zumino term* arises in the effective Lagrangian.

The next step in the procedure is the integration over the color variables. Without going into details of the path integration technique in gauge theories (cf. the well-known monographs of Refs. 20, 82), we note that to date this procedure cannot be carried through to the end. The limitation is mostly related to the need to sum up all planar gluon diagrams. The latter problem requires the extension of the technique of the $1/N$-expansion to matrix fields. This still remains an unsolved problem. Therefore one cannot uniquely reproduce the higher order terms (in powers of L_μ) in the effective Lagrangian, calculate the constant F_π in terms of the quark-gluon parameters, and so on.

At the present it is possible to answer the above questions only under some additional assumptions, using, for example, the Andrianov–Novozhilov bosonization (Refs. 83, 84). Spontaneous chiral symmetry breaking is admitted in this bosonization method, and due to the assumption of the presence of the quark $\langle \bar{q}, q\rangle$ and gluon $\langle \mathrm{Tr} G_{\mu\nu}^2\rangle$ condensate parameters of the low-energy sector are fixed in a self-consistent manner. Leaving aside any possible correlations between the flavor and color phases of quarks, one can deal with the following quark part of the Lagrangian (4.2.7):

$$L_q = \bar{q}\mathrm{D}q = i\bar{q}\gamma^\mu(\partial_\mu - V_\mu - \gamma_5 A_\mu)q - \bar{q}(S + \gamma_5 P)q,$$

where $\mathbf{D}$ is the Dirac operator and it is assumed that the system of quarks is in the following external fields: vector V_μ, pseudovector A_μ, scalar S and pseudoscalar P. If one ignores the quark masses, then the Dirac operator spectrum: $\mathbf{D}q_k = Kq_k$ is symmetrical and a gauge-invariant definition of the low energy region employs only the QCD scale parameter $\Lambda_{\rm QCD}$: $|K| < \Lambda_{\rm QCD}$. For massive current quarks, in order to limit the corresponding spectrum, it is necessary to introduce an additional parameter M, which accounts for the asymmetry of the Dirac operator spectrum. Then one can define the low energy region by the condition

$$|K - M| \le \Lambda_{\rm QCD}, \quad M \in [0, \Lambda_{\rm QCD}], \tag{4.3.11}$$

which leads to the quark condensate density

$$\langle \bar{q}q\rangle = -\frac{N_c}{2\pi^2}\left(\Lambda_{\rm QCD}^2 M - \frac{M^3}{3}\right). \tag{4.3.12}$$

Thus the Andrianov–Novozhilov bosonization method is based on the assumption that in the low-energy region of QCD the nonperturbative quark fluctuations dominate, violating the chiral symmetry and leading to the formation of the quark condensate.

The main difficulty of this approach consists in the derivation of the path integral in the left-hand side of (4.3.1), when integrating over the quark degrees of freedom, taking into account the restrictions on the Dirac operator spectrum. Utilizing the finite-mode regularization technique,[85] the authors of the presented method calculate the Adler–Bell–Jackiw–Bardeen anomaly and reproduce the Wess–Zumino–Witten action (4.3.9). Moreover, making use of a change of variables, analogous to (4.3.10), they succeeded in obtaining the explicit form of the fourth order terms in the chiral currents L_μ in the effective Lagrangian

$$L_{\rm eff}^{(4)}(U) = \frac{N_c}{384\pi^2}\mathrm{tr}\{[L_\mu, L_\nu][L^\mu, L^\nu] - 2(L_\mu L^\mu)^2 + 4\partial_\mu\partial^\mu U\partial_\mu\partial^\mu U^{-1}\}. \tag{4.3.13}$$

(The more complete expression also depends on introduced external fields and one can find it in Ref. 84). Finally, the above approach allows one to establish that the stability condition on the quark condensate with respect to fluctuations is the positive definiteness of the gluon condensate, given by the expression

$$\langle \mathrm{tr}(G_{\mu\nu}G^{\mu\nu})\rangle = \frac{6N_c}{g^2}(6\Lambda_{\rm QCD}^2 M^2 - \Lambda_{\rm QCD}^4 - M^4. \tag{4.3.14}$$

Note also, that in (4.3.13) besides the Skyrme term, which is necessary for a description at the classical level of a stable extended particle, there are tachion corrections, violating the positive definiteness of the energy functional. On the other hand, the developed approach leads to an effective potential allowing one to describe on an equal footing the low-energy region and the asymptotic freedom phase in QCD, as well. We discuss this topic a little further in the next section.

In conclusion it should be noted that there are some other methods of derivation of an effective meson Lagrangian in QCD.[77]

4.4. Hybrid models: quark bags and skyrmions

As already mentioned above, besides the effective meson Lagrangians to describe the low-energy QCD region various bag models are used (the Dubna bag, the MIT-bag and so on). Quarks are regarded as relativistic particles with spin 1/2, which are situated inside a bounded volume—a bag, with dimensions that have to correspond to the characteristic scale of confinement. Inside the bag quarks can be considered as free particles, in accordance with the asymptotic freedom property, and their confinement is provided by imposed boundary conditions. For instance, in the MIT-bag model[86] with the Lagrangian:

$$L_{\rm MIT} = \left(\frac{i}{2}\bar{q}\gamma^\mu\bar{\partial}_\mu q - m\bar{q}q - B\right)\theta_{\rm B}(x) - \frac{1}{2}\bar{q}q\delta_{\rm B}(x) \tag{4.4.1}$$

the confinement of quarks is provided by the step function $\theta_{\rm B}(x)$, which equals 1 inside the bag and vanishes outside the bag. The surface δ-function $\delta_{\rm B}(x)$ arises here due to the relation $\partial_\mu\theta_{\rm B} = n_\mu\delta_{\rm B}$, where n_μ is the outer normal to the bag surface. The bag stability is provided by the "vacuum" pressure B. Since quarks are free inside the bag, their behavior is governed by the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)q = 0, \tag{4.4.2}$$

with the boundary conditions on the surface

$$in_\mu\gamma^\mu q = q. \tag{4.4.3}$$

Though the idea of an artificial quark confinement inside the bag allows one to employ the usual methods of description of point-like particles, nevertheless it is clear that in a consistent theory the confinement itself must appear as a result of quark interactions. Therefore, in order to avoid the image of a bag, whose dimensions are taken from considerations external to the model, it was suggested in Ref. 87 to consider the bag as a defect in the sigma-model field configuration. As such a configuration, in particular, one can consider the "hedgehog" skyrmion configuration or a more complicated one, which includes fields of vector ρ- and ω-mesons.[88] For these models, which are known as chiral or hybrid bag models, in their $(1+1)$-dimensional variant,[89] a curious phenomenon was discovered. The results of calculations in the framework of such bag models show that the bag might be considered not as a physical substance, but rather as a way to divide the space into the "inside" and "outside" parts. In this case the boundary conditions (on the "bag" surface) are not inserted into the model, but instead are derived from the bosonization condition

$$\partial_\mu \phi = \sqrt{\pi}\bar{q}\gamma_\mu \gamma_5 q. \tag{4.4.4}$$

One can divide the space region, occupied by a hadron, into any number of pieces, so that in some of these pieces only the quark degrees of freedom would be essential, while in other pieces only meson degrees of freedom would be essential. It turns out that if one imposes appropriate boundary conditions on the boundaries of these pieces and takes into account the Casimir effect, then the physically significant results will not depend on the concrete method of division. This phenomenon acquired the name—the "Cheshire cat" principle, which, as people say, was able to disappear in such a way, that his smile remained after disappearance of the cat itself. Here we have a similar situation, there is no bag in essence, but one can still exploit the corresponding formalism.[90]

Strictly speaking, an extension of this scheme to the $(3+10)$-dimensional chiral bag model meets with the same difficulty that was encountered in the process of derivation of an effective Lagrangian from QCD: the lack of an exact bosonization scheme for fermions in the $(3+1)$-dimensional case. Nevertheless, as has been demonstrated by publications of the Stony Brook group,[91] in the $(3+1)$ hybrid model an approximate Cheshire cat principle proves to be valid, which leads to a reasonable value for a spherical bag of radius $R<0.5$ fm.

On the basis of hybrid models one succeeds in calculating static characteristics of hadrons and parameters of their interactions. What is even more important, in this framework one can obtain a unified description of phenomena both in the hadron sector of QCD and in the asymptotic freedom region. The latter is accomplished mostly thanks to the Cheshire cat principle, which allows one to perform an arbitrary division of space into pieces. consequently one can consider bags (defects) of an arbitrary radius R. Taking $R\to 0$ we should obtain the Skyrme model (or any other effective σ-model) and for large R we should obtain the standard bag model of the type (4.4.1).

We note that such an interpolation can be obtained beyond the scope of hybrid models as well, for instance, on the basis of some bosonization scheme. In this manner, using the Andrianov–Novozhilov bosonization one can get the $V(U,S)$ potential, which is a function of chiral U and scalar S fields, where S one can regard as a characteristic of quarkonium with the effective coupling constant g_{eff}. In the region of high energies the latter constant coincides with the running coupling constant α_s, and in the low-energy region $g_{\text{eff}}\simeq$const. These two regions correspond to two distinct minima of the $V(U,S)$ potential, corresponding to different values of S. Calculations based on this interpolating potential yield the reasonable value for the bag of radius $R=(0.42\pm 0.27)$ fm.

4.5. The Berry phase and skyrmion as a fermion

The effects mentioned in the heading are under intensive study during the last decade and, as it turned out, they have in fact the same origin. The Berry phase and the emergence of fermion states within the framework of a purely boson field theory were at first sight considered as exotic phenomena, but gradually their deep topological nature was elucidated together with the universality and the general character of these effects. Of the numerous articles on this subject we would like to mention here the reviews of Refs. 92–94 and the proceedings of recent conferences (Refs. 95,96) devoted to these problems. In accordance with the aims of this paper stated previously, we would like to focus our attention on the background ideas, with a special emphasis on the fermion properties of the skyrmion. Here is the right place to stress that in recent years the term "skyrmion" has acquired a more general meaning, denoting states in boson field theories [not necessarily $(3+1)$-dimensional ones and frequently without any relevance to the original Skyrme model], which are quantized as fermions.[94] Since the phenomena in question are in close connection with the topological properties of models, their most adequate description requires an intensive use of algebraic topology concepts and techniques. Lacking such a possibility in the framework of this review, we shall concentrate on the exposition of the principal results, leaving technical details aside.

As has been outlined in Sec. 2.3, the first hope to realize the idea of construction of fermions from bosons occurred to Skyrme after the discovery of the topological charge in the sine-Gordon model. We recall that this quantity represents the number of kinks minus antikinks. Let x_0^i denote the coordinates of localization of a kink (antikink) center. Then it is possible to write the following distribution function for the system of kinks and antikinks along the real axis:

$$\bar{\alpha}(x) = \frac{1}{2\pi}\left(\sum_i \theta^+(x-x_0^i) + \sum_j \theta^-(x-x_0^j)\right), \tag{4.5.1}$$

where $\theta^\pm(x)$ are 2π-kink solutions of the sine-Gordon equation (2.3.3), which are increasing or decreasing ones, respectively. In virtue of the expression (2.3.3), the topological charge $\mathbf{Q}$ can be presented in the form

$$Q = \bar{\alpha}(+\infty) - \bar{\alpha}(-\infty) = \sum_i n_i^+ - \sum_j n_j^-. \tag{4.5.2}$$

Skyrme interpreted the conserved quantity (4.5.2) as the number of fermions minus antifermions. In support of this interpretation he quantized the sine-Gordon model,[6] giving the explicit form of creation and annihilation operators, which obey anticommutation relations. On the basis of obtained results Skyrme concluded that quantum sine-Gordon solitons are equivalent to fermions, interacting via the four-fermionic type of interaction (such a model has been considered independently by Thirring). Later this result, on the basis of a different approach, was rigorously confirmed by Coleman[25] (see the book of Ref. 20 for details). This is the right place to note that in $(1+1)$-dimensional theories there is no difference in spins of bosons and fermions, and therefore the bosonization process, i.e., the process of transformation of a fermion theory into a boson theory has a clear scenario (see Sec. 2.3). Skyrme made an attempt to extend this result to a $(3+1)$-dimensional model,[36] but a strict justification of his hypothesis on the fermion properties of a skyrmion has been achieved only on the basis of a topological treatment of spin, due to Finkelstein (Refs. 27, 97).

The essence of this approach can be presented briefly as follows. In contrast to tensor fields, which transform according to single-valued representations of the Poincaré group, in particular the SO(3) group of spatial rotations, spinors are characterized by their double-valuedness under 2π rotations. In homotopy language this fact can be stated in the following way: the group SO(3) is a doubly-connected one, i.e., its fundamental group (the group of closed loops) $\pi_1(SO(3))$ is isomorphic to the abelian group of integers (modulo 2) $\mathbb{Z}_2$, which consists of two elements. In quantum physics this leads to two possible quantizations according either to the Fermi–Dirac or to the Bose–Einstein scheme.

Topological treatment of spin, according to Finkelstein, consists in construction of double-valued functionals on the set of classical fields Φ. These fields can be regarded (at any fixed moment of time t) as mappings $\phi(\mathbf{x}):\mathbb{R}^3\to\Phi$. The space of all such continuous maps Map $(\mathbb{R}^3;\Phi)$, denoted in short by M, is not, in general, a pathwise-connected one and its connected components can be thought of as elements of the zeroth homotopy group $\pi_0(\mathbf{M})$. On the other hand, the homotopy classes $[\mathbb{R}^3,\Phi]_i$ of maps $\phi(\mathbf{x})$ form the third homotopy group $\pi_3(\Phi)$ and it is clear that these two groups can be regarded as identical. Moreover, there is an isomorphism between the higher homotopy groups of the space M and the manifold Φ: $\pi_k(\mathbf{M}) = \pi_{k+3}(\Phi)$. The field $\phi = \phi(\mathbf{x},t)$, $t_1\leqslant t\leqslant t_2$ can be considered as a path p in $\mathbf{M}$ connecting the points $\phi_1(\mathbf{x}) = \phi(\mathbf{x},t_1)$ and $\phi_2(\mathbf{x}) = \phi(\mathbf{x},t_2)$. The amplitude A of the probability of a quantum transition from the state $\Psi[\phi_1(\mathbf{x})]$ to the state $\Psi[\phi_2(\mathbf{x})]$ can be written according to Feynman as a complex-valued functional, taking path integral of $\exp\{\frac{i}{\hbar}\int L[\phi]\mathrm{d}^4_x\}$ over the space $\mathbf{M}(\phi_1,\phi_2)$ of paths connecting the points $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$ in $\mathbf{M}$. Since in general $\mathbf{M}(\phi_1,\phi_2)$ is not pathwise-connected, then denoting its connected components by $\mathbf{M}_\alpha(\phi_1,\phi_2)$ one can present the transition amplitude as

$$A(\phi_1\to\phi_2) = \sum_\alpha \chi(\alpha)\int \exp\left\{\frac{i}{\hbar}\int L[\phi]\,\mathrm{d}^4x\right\}\mathrm{d}\mu[\phi], \quad (4.5.3)$$

where $L[\phi]$ is the Lagrangian density of classical theory; $\mathrm{d}\mu[\phi]$ is a quasimeasure in the functional space, and the coefficients $\chi(\alpha)$ determine a one-dimensional representation of $\pi_1(\mathbf{M})$ (Ref. 99). The sum in (4.5.3) runs over the set of connected components $\{M_\alpha(\phi_1,\phi_2)\}$ of the space $\mathbf{M}(\phi_1,\phi_2)$. It turns out that without any loss of generality the paths $\{\phi(\mathbf{x},t)\} = \{p\}$ might be considered as closed ones, i.e., $\phi_1 = \phi_2$. This allows one to identify the set of components $\{M_\alpha(\phi_1,\phi_2)\}$ with $\pi_1(\mathbf{M}) = \pi_{3+1}(\Phi)$. The coefficients $\chi(\alpha)$ in this case obey the composition law: $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$. The calculation of $\chi(\alpha)$ in the general case can be reduced also to that of the case $\phi_1 = \phi_2$, and therefore the result is of a general character.

Let us now give the definition of the double-valuedness of the functionals $A(\phi_1\to\phi_2)$ according to Finkelstein, which is just a translation into the homotopy language of the property of spinors to change their sign under a 2π rotation. Taking a point $\phi_1\in[\mathbb{R}_3,\Phi]_i$, then to the same homotopy class there should belong a closed path p, starting and ending at this point ϕ_1, such that the values of $\Psi[\phi(\mathbf{x},t)]$ along the path p vary continuously and do not attain the original value at the end point. After going twice around the same path p, i.e., after going around the path p^2 the functional $\Psi[\phi(\mathbf{x},t)]$ must be led to the initial value. In other words, the path p in the homotopy class $[\mathbb{R}_3,\Phi]_i$ should be a nontrivial one, while the path p^2 should be a trivial one.

The answer to the question when is such a situation possible can be obtained on the basis of calculation of coefficients $\chi(\alpha)$.[99] In the skyrmion case, we denote by $\lambda(\nu)$ the homotopy class of the composition of consecutive mappings: Ψ_ρ: $S^4\to S^3$ and φ_ν: $S^3\to\Phi$, i.e., $\lambda(\nu) = \nu\cdot\rho$, where $\nu\in\pi_3(\Phi)$, and ρ is a nontrivial element of the group $\pi_4(S^3) = \mathbb{Z}_2$. In accordance with the homotopical definition of a fermion state one can assert, that when the identity $\lambda(\nu)\equiv 0$ holds, all the above mappings would correspond to bosons, and to fermions in the opposite situation. More precisely, this statement can be formulated as follows: if a skyrmion is characterized by a quantum number $\nu\in\pi_3(\Phi)$ and $\chi[\lambda(\nu)] = 1$, then it obeys the Bose–Einstein statistics, and if $\chi[\lambda(\nu)] = -1$ it corresponds to Fermi–Dirac statistics (Ref. 100). In Ref. 99 calculations are performed for the cases, when the field manifold is $\Phi = G/H$, here G is a simple Lie group and H is its subgroup. Both can be replaced respectively by their maximal compact subgroups and this does not lead to a change of homotopy groups $\pi_n(\Phi)$. For the case of the Skyrme model we are interested in $\Phi = G = SU(2)$ and $\pi_3(SU(2)) = \mathbb{Z}$. If G is locally isomorphic to the group SO(n), where $n = 3$ or $n\geqslant 5$, then $\pi_4(G) = \mathbb{Z}_2$. In this case $\lambda(\nu) = 0$ for all even numbers $\nu\in\pi_3(G) = \mathbb{Z}$, and $\lambda(\nu) = 1$ for all odd numbers ν. If g is not locally isomorphic to SO(n), then $\pi_4(G) = 0$ and $\lambda(\nu) = 0$, i.e., in such theories only boson states are possible. Before turning to a description of a dynamical realization of this formalism, let us note that all the above statements completely agree with the results of papers of Refs. 97, 98, where necessary and sufficient conditions for the existence of "spinor structures" on field manifolds Φ have been formulated. In this context we consciously leave aside the problem related with the SU(3) generalization of the Skyrme model. It is known, that for the SU(3) group $\pi_4(SU(3)) = 0$, therefore we shall consider the statistics of skyrmions in this case within the framework of a dynamical realization of double-valued functionals, which is the next topic in our exposition.

A dynamical realization of Finkelstein's scheme has been suggested in a series of papers by Witten[114] and by Balachandran *et al.* (Refs. 71, 101–103), which revived interest in the Skyrme model and demonstrated its applicability to the QCD low-energy limit. In particular, Witten noted that there are a number of processes with pseudoscalar mesons, which are allowed by the QCD symmetries, but are prohibited in the framework of an effective low-energy theory of the Skyrme type. The reason for such a discrepancy is that the effective vertices in the description of processes, typical for this region, contain as a rule the Levi–Civita symbol $\varepsilon_{\mu\nu\lambda\rho}$. For example, a typical process $K^+K^-\to\pi^+\pi^0\pi^-$ is defined by the vertex of the form:

$$\varepsilon^{\mu\nu\lambda\rho}\pi^0\partial_\mu K^+\partial_\nu K^-\partial_\lambda\pi^+\partial_\rho\pi^-.$$

Calculations of this vertex in the framework of the chiral Skyrme model lead to combinations $\varepsilon^{\mu\nu\lambda\rho}L_\mu L_\nu L_\lambda L_\rho$, which are identically equal to zero by virtue of the invariance of the trace operation under cyclic permutations. Therefore the vertices of this type are prohibited in the framework of chiral models and one can understand this as a consequence of an extra conservation law.

Witten noted the fact, that the presence of an extra conservation law in the chiral models is related with an extra discrete symmetry of the effective Lagrangian, which is absent in the QCD fundamental Lagrangian. This extra symmetry can be easily discovered from the explicit form of the chiral field $\mathbf{U}(\mathbf{x},t)$, modeling the low-energy limit of QCD with three flavors:

$$U(\mathbf{x}, t) = \exp\left(\frac{i}{F_\pi}\lambda^a\pi^a(\mathbf{x}, t)\right), \quad a = \overline{1,8}, \tag{4.5.4}$$

where λ^a are the Gell-Mann matrices, and $\pi^a(\mathbf{x},t)$ are fields, describing the pseudoscalar octet of mesons. Noticing that under a space reflection

$$P:\ \pi^a(\mathbf{x}, t) \to -\pi^a(-\mathbf{x}, t), \tag{4.5.5}$$

we derive from (4.5.4) the transformation law for the chiral field $\mathbf{U}(\mathbf{x},t)$:

$$P:\ U(\mathbf{x}, t) \to U^+(-\mathbf{x}, t). \tag{4.5.6}$$

But the operation (4.5.6) can be represented as a product of two operations $P = P_0\cdot(-1)^{N_B}$, where

$$P_0:\ \mathbf{x} \to -\mathbf{x}, \quad t \to t; \quad (-1)^{N_B}:\ U \to U^+. \tag{4.5.7}$$

The notation $(-1)^{N_B}$ adopted here for the second operation is related with the fact that the operation $\mathbf{U}\to\mathbf{U}^+$ is equivalent to $\pi^a \to -\pi^a$ and thus counts modulo 2 the number of bosons, N_B. The indicated symmetry causes the special selection rule: all Green's functions must be invariant under the replacement $\pi^a(x) \to -\pi^a(x)$, and hence they should vanish for all combinations, which include an odd number of fields π^a. In other words, all reactions with an odd number of pseudoscalar participants must be suppressed.

Fortunately, the general recipe is known how to avoid this kind of trouble (Refs. 101, 102), which had occurred already, for instance in the description of the dynamics of "the charge-monopole" system. The equation of motion of an electric charge e on a unit sphere $S^2 = \{\mathbf{r}^2 = 1\}$ and in the Dirac monopole field with the magnetic charge μ, which is localized at the origin, has the form (in units $c = 1$):

$$m\ddot{x}_i = e\mu\varepsilon_{ijk}\dot{x}_j\frac{x_k}{r^3} - mx_i\dot{r}^2. \tag{4.5.8}$$

We note that the Lorentz force on the right-hand side of Eq. (4.5.8) excludes the extra discrete symmetries, namely: $t\to -t$; $\mathbf{r}\to -\mathbf{r}$, which are characteristic of the system in the absence of a magnetic field, leaving as an admissable symmetry only their product. But in this case the problem arises of how to reconstruct a Lagrangian of the system, which would correspond to the equation (4.5.8), since an introduction of a vector potential $\mathbf{A}$ such that

$$\mathrm{curl}\ \mathbf{A} = \frac{\mu\mathbf{r}}{r^3} \tag{4.5.9}$$

definitely forces $\mathbf{A}$ to be a singular function at some points along some line passing through the origin (a filament or a Dirac string), and consequently to a singular interaction term in the Lagrangian $e(\dot{\mathbf{r}}\cdot\mathbf{A})$.

Perhaps this is an appropriate place to recall the Berry phase.[104] The point is that M. Berry faced an analogous problem when investigating the applicability of the Born–Oppenheimer adiabatic approximation to quantum systems, where it is possible to separate the "fast" and the "slow" variables. Then, as is known, the solution of the problem can be divided into two stages. At first one has to consider the motion of the "fast" subsystem under the assumption that the "slow" coordinates are fixed. Next one has to take into account the motion of the "slow" subsystem. If one rejects the Born–Oppenheimer assumption, that coupling between the "fast" and "slow" variables in the adiabatic approximation can be neglected, then as a result of decoupling of the "fast" degrees of freedom in the space of the "slow" variables $\mathbf{R}$ a nontrivial structure—a peculiar "gauge potential" $\mathbf{A}(\mathbf{R})$ is formed.

Furthermore, if in the energy spectrum of the eigenvalue problem for the "fast" variables there is a degeneracy (the "slow" variables are considered as parameters in this case), then the gauge potential $\mathbf{A}(\mathbf{R})$ acquires a singularity. As has been shown by Berry,[104] if the electronic degrees of freedom are treated as "fast" variables (an electron in a magnetic field $\mathbf{B}(t)$), then in the space of the "slow" variables a singular potential of the monopole type will be induced. In its turn, the problem of calculation of the phase of the wave function (the Berry phase) is similar to the problem of reconstruction of an interaction Lagrangian for the "charge-monopole" system. On this basis Aitchison[105] emphasized the analogy between the Berry phase and the Wess–Zumino term. As it will be clear from what follows, the latter concept allows one to solve the problem of reconstruction of a regular Lagrangian for the "charge-monopole" system, solves the problem of elimination of extra symmetries in an effective chiral Lagrangian, and provides us with a treatment of a skyrmion as a fermion as well.

In order to eliminate singularities from the interaction Lagrangian $e(\dot{\mathbf{r}}\cdot\mathbf{A})$, one can note, for example, that the equation (4.5.9) is insensitive with respect to gradient transformation $\mathbf{A}' = \mathbf{A} + \nabla\psi$, which lead to the following change in the action:

$$S' - S = e\int_{t_1}^{t_2} dt(\dot{\mathbf{r}}\nabla\psi) = e\int_1^2 d\psi. \tag{4.5.10}$$

It is known that such an ambiguity of the action is quite undesirable in a quantum description. Let us estimate it, for example, for a closed path of an electron on the sphere $S^2 = \{\mathbf{r}^2 = 1\}$ setting $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ and calculating the contribution of the interaction term in two different gauges [see Eq. (4.5.10)]

$$e\int_{t_1}^{t_2} dt(\dot{\mathbf{r}}\mathbf{A}) = e\oint_\gamma(\mathbf{A}\, d\mathbf{r}). \tag{4.5.11}$$

Here γ is a closed contour, which separates two distinct regions D^+ and D^- on $S^2 = D^+ \cup D^-$ (Fig. 3).

It is possible to apply Stokes' theorem and to convert the line integral (4.5.10) into a surface integral if, and only if, the integrand is non-singular. Therefore when integrating over D^+ we relegate the singularities of $\mathbf{A}$ to the other disc D^- and vice versa. Then we will find

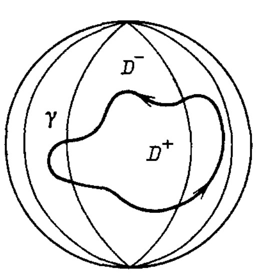

FIG. 3.

$$\oint_{\gamma}(\mathbf{A}_1\,d\mathbf{r}) = \int_{D^+}(\mathbf{B}_1\,d\mathbf{S}),$$

$$\oint_{\gamma}(\mathbf{A}_2\,d\mathbf{r}) = -\int_{D^-}(\mathbf{B}_2\,d\mathbf{S}), \tag{4.5.12}$$

where the minus sign is related to the choice of opposite orientation, and $\mathbf{B} =$ curl $\mathbf{A}$ is the magnetic induction of a monopole. In passing from one gauge to the other we obtain the action ambiguity as

$$\Delta S = e\int_{D^+\cup D^-}(\mathbf{B}\,d\mathbf{S}) = e\int_V d^3x \operatorname{div}\mathbf{B} = 4\pi e\mu, \tag{4.5.13}$$

where V is the volume bounded by the S^2 surface. Since the quantum transition amplitude

$$A(t_1\to t_2) = \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} d\mu[\mathbf{r}(t)]\exp\left\{\frac{i}{\hbar}S[\mathbf{r}(t)]\right\}$$

must not be gauge-dependent, we obtain the condition

$$\exp\left\{\frac{ie}{\hbar}\int_{D^+}(\mathbf{B}\,d\mathbf{S})\right\} = \exp\left\{-\frac{ie}{\hbar}\int_{D^-}(\mathbf{B}\,d\mathbf{S})\right\},$$

from where, and taking into account Eq. (4.5.13), we get the Dirac quantization condition

$$\exp\left(\frac{i}{\hbar}\Delta S\right) = 1, \quad \Delta S = 2\pi n\hbar, \quad e\mu = \frac{n}{2}\hbar. \tag{4.5.14}$$

We have met here with a rather widespread situation, when multivalued functionals appear, and an elimination of their ambiguity leads to quantization of physical parameters.[106]

Another way out, suggested by Balachandran *et al.* (Refs. 101, 102), provides the possibility to obtain a nonsingular action, and, as will be shown below, leads to some interesting generalizations. It was proposed to extend the configuration space Q to the space of paths PQ over Q, which is defined as follows. Let $\mathbf{x}_0$ be a fixed reference point in Q (an arbitrary one). Then as an element of PQ one can take a path p from $\mathbf{x}_0$ into $\mathbf{x}$, defined by a parameter σ:

$$p = \{\mathbf{p}(\sigma),\quad 0\le\sigma\le 1,\quad \mathbf{p}(0) = \mathbf{x}_0,\quad \mathbf{p}(1) = \mathbf{x}\}.$$

The time dependence $\mathbf{p}(\sigma,t)$ is introduced in the same manner. Then a singular-free action has the form:

$$S_{\rm int} = e\int_{t_1}^{t_2} dt\int_0^1 d\sigma\, F_{ij}[p]\frac{\partial p^i(\sigma,t)}{\partial\sigma}\frac{\partial p^j(\sigma,t)}{\partial t}, \tag{4.5.15}$$

where

$$F_{ij}[p] = \mu\varepsilon_{ijk}\frac{p^k(\sigma,t)}{|\mathbf{p}(\sigma,t)|}.$$

The integrand in (4.5.15) can be regarded as a closed 2-form

$$\omega = \frac{e}{2}F_{ij}[p]dp^i\wedge dp^j$$

with the closedness condition $d\omega = 0$, which coincides with the Bianchi identity

$$\partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} = 0.$$

The further development of this approach can be found in papers of Refs. 93, 107. Here we summarize briefly the main results. Since a wave function, which describes a (pure) state of a quantum system, is always determined up to the phase, then it might be regarded as not a function on Q, but rather as a function on the corresponding U(1) bundle over Q, denoted by $\hat{Q}$. This $\hat{Q}$ one can imagine as obtained by associating a circle S^1 to each point of Q. Thus obtained $\hat{Q}$ would be trivial or nontrivial, depending on the way these fibers S^1 are attached to points of Q. If $\hat{Q} = Q\otimes S^1$, then the bundle would be a trivial one, or, in other words, will not contain a twist. In the opposite case the bundle $\hat{Q}$ would be nontrivial and one can determine the Wess–Zumino action $S_{\rm WZ}$ on it.

A general method of constructing wave functions in the presence of the nontrivial Wess–Zumino term is described in Refs. 93 and 101. If the group of transformations G is defined on a configuration space so that the action S, including $S_{\rm WZ}$ is invariant under transformations from G, then on the corresponding bundle $\hat{Q}$ there would act a different, quantum group $\hat{G}$. The latter, in general, does not coincide with G, and at times even does not contain G as a subgroup. For example, in the above discussed "charge-monopole" system $G = SO(3)$, while for $e\mu = \hbar(n + 1/2), n\in\mathbb{Z}$ one can choose the SU(2) group to be the quantum group $\hat{G}$. It is because of this replacement of the group of classical symmetries G by the quantum group $\hat{G}$ that the fermion-boson transmutations predicted by Skyrme appear to be possible.

To eliminate the redundant symmetries in the σ model it is possible, as has been shown by Witten,[114] to add to the equations of motion (2.4.8) a term, analogous to the Lorentz force in (4.5.8), that is of the type $\varepsilon^{\mu\nu\lambda\rho}L_\mu L_\nu L_\lambda L_\rho$. But once again here arises the problem with reconstruction of the Lagrangian since the simplest possible term $\varepsilon^{\mu\nu\lambda\rho}\mathrm{Tr}(L_\mu L_\nu L_\lambda L_\rho)$ is a trivial one. Exploiting further the analogy with the "charge-monopole" system, instead of the path S^1 on S^2 we introduce the "path" S^4 on S^5, i.e., we extend the domain of definition of chiral fields and consider mappings of the form $U(x)\colon S^4\to SU(3) = G$ and the extension corresponding to the space of paths PQ, $U(x,\sigma)\colon S^5\to SU(3)$. It turns out, that such an extension can be performed only under the condition $\pi_4(G) = 0$, which fortunately is satisfied in our case. By virtue of the other well-known isomorphism $\pi_5(SU(3)) = \mathbb{Z}$ for mappings $U(x,\sigma)$ it is possible to construct a topological invariant:

$$Q_5 = \int_{S^5} d^5x\, J_5^0 = -i\frac{\varepsilon^{\mu\nu\lambda\rho\tau}}{240\pi^2}\int_{S^5} d^5x\,\mathrm{tr}(L_\mu L_\nu L_\lambda L_\rho L_\tau), \tag{4.5.16}$$

where $x^5 = \sigma$. It is not difficult to verify that the integrand in (4.5.16) is the 5-form

$$\omega_5 = \frac{-i}{240\pi^2}\operatorname{tr} l^5, \quad l = L_\mu\, dx^\mu,$$

which proves to be closed and inexact. Following the argumentation used in the description of the "charge-monopole" system, we obtain the explicit form of S_{WZ} for the SU(3) Skyrme model:

$$S_{WZ} = \frac{i\eta}{240\pi^2}\int_{D_5} d^5x\,\varepsilon^{\mu\nu\lambda\rho\tau}\operatorname{tr}(L_\mu L_\nu L_\lambda L_\rho L_\tau), \tag{4.5.17}$$

where $\eta = \text{const}$. In order to make the value of X_{WZ} independent with respect to a choice of discs D_5^+ of D_5^- as in (4.5.14), we impose the condition

$$\eta\Big(\int_{D_5^+}\omega_5 + \int_{D_5^-}\omega_5\Big) = \eta\int_{S^5}\omega_5 = 2\pi\eta n = 2\pi\hbar n',$$

where $n,n'\in\mathbb{Z}$ and the Gauss theorem generalized for S^5 has been used.[114] Once again we obtain quantization of the parameter η, and comparing (4.5.17) and the formula (4.3.9), derived from QCD taking axial anomaly into account, we obtain Witten's result: $\eta = \hbar N_c$.

Now it remains to adduce some arguments to prove that skyrmions might in this case behave as fermions. With this in mind we consider the vacuum-to-vacuum transition amplitude for a skyrmion at rest:

$$\langle \mathrm{Sk}(T)|\mathrm{Sk}(0)\rangle = \exp\Big(-\frac{i}{\hbar}HT\Big)(1 + O(h)),$$

where H is the Hamiltonian of the system, and T is a time interval. Now, if we rotate the skyrmion adiabatically through 2π around some axis, then according to quantum mechanics rules, the amplitude acquires the phase factor of the form $\exp(-i2\pi J/\hbar)$, where J is the total spin of the skyrmion:

$$\langle \mathrm{Sk}(T)|\mathrm{Sk}(0)\rangle_{2\pi} = \exp\Big(-\frac{i}{\hbar}HT\Big)\exp\Big(-\frac{i}{\hbar}2\pi J\Big)(1 + O(\hbar)). \tag{4.5.18}$$

To evaluate this factor we have to embed the static SU(2) skyrmion (3.2.12) into a SU(3)-valued chiral field, i.e., to deal with the classical chiral fields of the form

$$U_c(\mathbf{x}) = \begin{pmatrix} \exp(i\tau_r\theta(r)) & . & 0 \\ & \cdot & 0 \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot \\ 0 \quad\quad 0 & . & 1 \end{pmatrix}, \tag{4.5.19}$$

and to choose the rotated field $\mathbf{U}$ in the form $\mathbf{U}(\mathbf{x},t) = A(t)\cdot\mathbf{U}(\mathbf{x})\cdot A^+(t)$, where $A(t)$ is an SU(3) matrix of rotation around a spatial axis. We write the amplitude as a Feynman path integral

$$\langle \mathrm{Sk}(T)|\mathrm{Sk}(0)\rangle_{2\pi} = \int_{t=0(U_c)}^{t=T(A^+U_cA)} d\mu\,[U(\mathbf{x},t)]\exp\Big(\frac{i}{\hbar}S\Big), \tag{4.5.20}$$

where

$$S = -MT - \frac{i\hbar}{240\pi^2}\int_{D_5^+}\operatorname{tr} l^5, \tag{4.5.21}$$

M is the skyrmion mass, and the 1-form $l = L_\mu dx^\mu$ is calculated for fields $\mathbf{U}(\mathbf{x},t,\sigma)$, obtained as the extension of fields $\mathbf{U}(\mathbf{x},t)$ on D_5 by means of the replacement $A(t)\to A(t,\sigma)$. Here we have set $A(t,1) = A(t), \partial_t A(t,0) = 0, A(t+T,\sigma) = A(t,\sigma)$. Calculations in (4.5.20) and (4.5.21) yield the following result:

$$\frac{-i}{240\pi^2}\int_{D_5^+}\operatorname{tr} l^5 = \pi Q,$$

$$\langle \mathrm{Sk}(T)|\mathrm{Sk}(0)\rangle_{2\pi} = \exp\Big(-\frac{i}{\hbar}MT\Big)\exp(iN_c\pi Q)(1 + O(\hbar)), \tag{4.5.22}$$

where $\mathbf{Q}$ is the topological charge of the skyrmion.

From a comparison of the expected answer (4.5.18) with the obtained answer (4.5.22), we conclude, that for $\mathbf{Q} = 1$ the spin of the SU(3) skyrmion is equal to

$$J = \frac{\hbar}{2}N_c,$$

i.e., that for an even number of colors a skyrmion has an integer spin, and for an odd number—a half-integer spin.

The above considerations do not provide one with an answer for the SU(2) skyrmion spin, since the Wess–Zumino term is trivial in this case. The answer, which however can be obtained in the "Cheshire cat" spirit, is as follows. Indeed, the Wess–Zumino term vanishes in the transition from the case $N_c = 3$ to the $N_c = 2$ case, but its "smile" remains in the form of the discrete group $\mathbb{Z}_2$. It is often said that in the latter case the discrete version of the Wess–Zumino term (Refs. 72, 93) does work, and, in particular, it picks out the possible methods or types of quantization for the given system. The point is, that, when applying the collective coordinates method as a quantization scheme, there exist three possible types of quantization (see lecture notes of Ref. 72, and also Ref. 108): a purely boson, a purely fermion, and mixed (both boson and fermion) types. The standard procedure is the following: after embedding the Skyrme ansatz into the SU(3)-valued chiral field as in (4.5.19) and introducing the collective coordinates $A(t)$ we obtain the Lagrangian $L(A,\dot{A})$, which admits a gauge symmetry, i.e., the group of time-dependent transformations which changes $L(A,\dot{A})$ by a total derivative: $L\to L + d\chi/dt$. As usual, the availability of time-dependent symmetries does not lead to a conservation law, but instead imposes some restrictions on the phase space of the classical system as well as on admissible states in corresponding quantum theory. It is just these additional constraints that remove the freedom in the choice of the quantization scheme for the SU(3) skyrmion. The time-dependent transformations of the collective coordinates, which leave invariant the field $\mathbf{U}(\mathbf{x},t) = A^+(t)\mathbf{U}_c(\mathbf{x})A(t)$ and the corresponding equation, can be easily found from (4.5.19) assuming, for example,

$$A(t)\to A(t)\exp(iY\alpha(t)), \text{where } 3Y = \operatorname{diag}(1,1,-2). \tag{4.5.23}$$

Since Y commutes with $\mathbf{U}_c$, then $\mathbf{U}$ will be invariant with respect to transformations (4.5.23), which represent the one-parameter U(1) subgroup of hypercharge.

For the SU(2) Skyrme model the existence of alternative quantization schemes reflects the fact that physical states in this case might belong to any one of two possible irreducible representations of $\mathbb{Z}_2$. It means that state vectors can be either even or odd function of the collective variable A. One can regard the SU(2) model as obtained from the SU(3) model after imposing an appropriate constraint. Since

$$\exp(i3\pi Y) = \mathrm{diag}(-1, -1, 1),$$

then the reduction of the group of gauge invariance for the SU(3) model to the two-flavor case will contain $\mathbb{Z}_2$. The gauge condition in the SU(3) model leads to the unique value of $\exp(i3\pi Y)$ for physical states. In this respect, the ambiguity in the quantization of the two-flavor Skyrme model can be eliminated if one regards the latter as some contraction of the three-flavor model.

Finally, one more effect should be mentioned here. This effect was discovered recently and is relevant, in a sense, to the fermion properties of the skyrmion. We are referring to the results of the European Muon Collaboration (EMC) experiment[109] on the measurement of the spin-dependent structure function for protons, and to the explanation of this result in Ref. 110 based on the Skyrme model. As it turned out, the results of these calculations are confirmed with high accuracy by the experimental data. The attempts to obtain these results by lattice QCD calculations, using the most powerful of modern computers, are still unsuccessful. We are not going into the details of this effect any further, but just note that it proves once again the necessity to develop dynamical scenarios in order to extract definite answers from QCD. As is becoming apparent, the Skyrme model copes rather successfully with such functions in the low-energy sector. The explanation of the above effect within the framework of the Diakonov–Petrov chiral quark model (Refs. 112, 116) was given in the paper of Ref. 117.

4.6. Skyrmion interactions and internucleon forces

When considering hadron interactions in the framework of the Skyrme model, the following should be taken into account. This model, regarded as a low-energy approximation to QCD up to the leading order terms in a $1/N_c$-expansion, provides only an effective account of the exchange between hadrons of massive vector and scalar mesons. Therefore there could not be any claim of obtaining a complete picture of hadron interactions. Nevertheless, in spite of its relative simplicity, in many cases the Skyrme model produces a correct qualitative (and sometimes even quantitative) description of hadronic processes. In particular, this is the case when one deals with distant interactions, i.e., with low-energy phenomena, where this model works rather well.

First, we give a graphic topological description of the system of two interacting SU(2) skyrmions, i.e., the $\mathbf{Q} = 2$ configurations, in the adiabatic approximation. The latter means that we are going to minimize the energy of the configuration after fixing the geometrical centers of the skyrmions at some points, say $\mathbf{r}_1$ and $\mathbf{r}_2$, determined by the condition $\theta(\mathbf{r}_i) = \pi$, or

$$U(\mathbf{r}_1) = U(\mathbf{r}_2) = -I. \tag{4.6.1}$$

The condition (4.6.1) means that the field configuration $\mathbf{U}$ in the neighborhoods of points $\mathbf{r}_1$ and $\mathbf{r}_2$ coincides with the field $\mathbf{U}_0$ of a single skyrmion (3.2.12). Therefore the surfaces

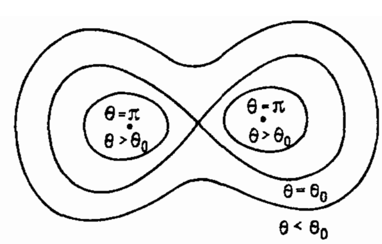

FIG. 4.

$\theta(\mathbf{r}_i) = \mathrm{const} \approx \pi$ are close to spherical surfaces (see Fig. 4). Further, there exists the surface $\theta = \theta_0$ homeomorphic to a connected sum of two spheres $\mathbb{S}^2$, which divides the space $\mathbb{R}^3$ into three parts: two internal ones ($\theta > \theta_0$), with points $\mathbf{r}_1$ and $\mathbf{r}_2$, and one outer region ($\theta < \theta_0$). Each surface $\theta = \mathrm{const}$ is mapped into some region $\Omega \subset \mathrm{SU}(2)$, which is homeomorphic to the sphere $\mathbb{S}^2$: if θ is fixed, then the unit vector $\mathbf{n} \in \mathbb{S}^2$ remains free in $\mathbf{U}$. From Fig. 4 it is clear that all the connected parts of the constant θ surfaces with $\theta > \theta_0$ are mapped into $\mathbb{S}^2$ with the degree 1. On the other hand all the constant θ surfaces with $\theta < \theta_0$ are mapped into the sphere $\mathbb{S}^2$ with degree 2. For this reason the topology of the field configurations in the region $\theta < \theta_0$ coincide with the topology of the G_2-invariant fields (3.2.16) with $k = 2$. As the skyrmions approach each other $\theta_0 \to \pi$. The field configuration obtained in this way is identical to an axisymmetric one (3.2.16), which, however, does not yet correspond to the closest possible approach of the skyrmions, since we have a toroidal distribution for the energy density of this state. A still closer approach would mean a draining of the "pion fluid" into the central region, and this process would lead to the formation of a region with $\theta > \pi$. Growth of this region would finally lead to the formation of a G_1-invariant field configuration with $\mathbf{Q} = 2$, and since the energy of such a configuration is three times greater than the energy E_1 of a single skyrmion, there would be a strong repulsion between the skyrmions (with an energy of the order of E_1).

The standard description of the skyrmion interaction is based on the product ansatz approximation

$$U_{12} = U_0^A(\mathbf{r} - \mathbf{r}_1)U_0^B(\mathbf{r} - \mathbf{r}_2), \tag{4.6.2}$$

where

$$U_0^A = AU_0A^+, \quad U_0^B = BU_0B^+; \quad A, B \in \mathrm{SU}(2).$$

If the matrices A and B are different, then (4.6.2) describes skyrmions with a relative angular displacement in isospace. In particular, from Fig. 4 it follows that the maximal attraction between skyrmions arises when the relative rotation is through an angle π about an axis perpendicular to the line of separation of the two skyrmions. Indeed, the spherical constant θ surfaces in the limit $\theta \to \theta_0$ must touch each other at points that are mapped to the same SU(2) element, so that precisely the rotation of the spheres through angle π is obtained (Refs. 47, 118, 119). From the above considerations it is clear that the configuration (4.6.2) faithfully reproduces the interactions between skyrmions at large distances and gives only a qualitatively correct description at small distances. At the same time, one cannot extract from that kind of description, for example, a formation of a G_2-invariant configuration at the intermediate distances (Refs. 55, 57).

To calculate the interaction one usually proceeds as follows. From the energy value computed for the configuration U_{12} is subtracted the energy of the two widely separated skyrmions, i.e., $2E_1$. The obtained result is declared to be the skyrmion-skyrmion potential $V(\mathbf{R}; A,B$, where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$. In a quantum description the matrices A and B are random and one has to average over them. It turns out that the result of averaging would be the same, if one were to average some products of the Pauli matrices σ_i and τ_i, $i,j = 1,2$ over the standard spin-isospin states of a nucleon $|N\rangle$, where σ_i and τ_i are spin and isospin operators of individual nucleons (Refs. 60, 120, 121). In particular, the following relation holds (cf. Ref. 60):

$$\langle N'|\mathrm{tr}(\tau_\alpha A^+\tau_\beta A)|N\rangle = -\frac{2}{3}\langle N'|\sigma_\alpha\tau_\beta|N\rangle.$$

Taking it into account leads to the potential for the NN-interaction

$$V^{NN} = V_c + (\tau_1\tau_2)[(\sigma_1\sigma_2)V_{\sigma\tau} + S_{12}V_{T\tau}], \tag{4.6.3}$$

where $S_{12} = 3(\mathbf{n}\cdot\sigma_2)(\mathbf{n}\cdot\sigma_2) - (\sigma_1\cdot\sigma_2)$ is the conventional tensor operator, with $\mathbf{n} = \mathbf{R}/R$. A comparison of (4.6.3) with the widely accepted "Paris" phenomenological potential of Ref. 122 shows a satisfactory agreement in the distant region ($R > 2$ fm) and a qualitative agreement at intermediate distances (1 fm $< R <$ 2 fm). However, the central potential V_c comes out to be purely repulsive, although the "Paris" potential does contain in the intermediate range an attraction, which is small (in comparison with the nucleon mass), but plays a crucial role in nuclear physics, forming the binding energy of nuclei. In the standard boson-exchange theory this attractive potential is explained by an isoscalar $\pi\pi$-exchange (an exchange of an effective σ-meson), and can be obtained within the framework of the Skyrme model, if one accounts for skyrmion perturbation in the course of interaction.[123] This problem in the intermediate-range attraction is, in our opinion, mostly related with the shortcomings of the product-ansatz approximation (4.6.2).

We note that in a relatively simple way one can extract from (4.6.3) the one-pion-exchange potential, which is related with the skyrmion structure in the asymptotic ($r\to\infty$) region

$$\theta(r) \approx \frac{3\lambda f_{\pi NN}}{8\pi m_\pi}(1 + m_\pi r)\exp(-m_\pi r)r^{-2}, \tag{4.6.4}$$

where $f_{\pi NN}$—a numerical constant, $\lambda = 2/F_\pi$, $F_\pi = 186$ MeV. By making use of (4.6.4) one obtains the following potential:

$$V_\pi^{NN} \approx \frac{m_\pi f_{\pi NN}^2}{12\pi}[Y_2(m_\pi r)S_{12} + Y_0(m_\pi r)(\sigma_1\sigma_2)](\tau_1\tau_2),$$

where the Yukawa functions were used, defined as $Y_0(x) = e^{-x}/x$; $Y_2(x) = [1 + (3/x) + (3/x^2)]Y_0(x)$. Thus, the one-pion-exchange interaction evidently contributes into the components $V_{\sigma\tau}$ and $V_{T\tau}$. One should say, that a satisfactory agreement with the "Paris" potential is not only due to the one-pion-exchange. For example, at $R < 1.5$ fm the contribution of V_π^{NN} is less than 30%. Therefore the Skyrme model qualitatively gives a correct account also of the multi-pion-exchange process.

But the phenomenological "Paris" potential, besides terms of the kind of (4.6.3), also contains other terms of considerable importance. Among them, for example, is the spin-orbit interaction, which has the form

$$[V_{LS} + (\tau_1\tau_2)V_{LS\tau}](\mathbf{L}\,\mathbf{S}), \tag{4.6.5}$$

where now $\mathbf{S} = (\sigma_1 + \sigma_2)/r$, $\mathbf{L} = -i(\mathbf{R}\times\nabla_R)$. To derive the spin-orbit interaction in the Skyrme model[124] one has to account for the kinetical non-steady-state part of the Hamiltonian, related with $\mathrm{Tr}\mathbf{L}_0^2$. However, the isospin-independent component V_{LS} in (4.6.5) is obtained with the incorrect sign. To improve the situation it is possible by the inclusion in the Skyrme Lagrangian of the term

$$L_6 = \gamma^2 J_\mu J^\mu, \tag{4.6.6}$$

where J_μ is the topological current density. The term (4.6.6) in the Lagrangian is quadratic in velocities and does not violate the skyrmion stability.

4.7. Meson-baryon processes

In the course of analyzing interaction processes between mesons and baryons there were revealed hitherto unknown phenomenlogically useful relations, which are in good agreement with experiment and first appeared due to attempts to describe πN-scattering in the framework of the Skyrme model. Later on it became clear, that these relations are to some extent model-independent ones and are relevant to the approximate conservation law of the K-spin, namely:

$$\mathbf{K} = \mathbf{L} + \mathbf{T}, \tag{4.7.1}$$

where $\mathbf{L}$ is the total angular momentum of mesons (here only 0^--multiplets are considered) and $\mathbf{T}$ is their total isospin. The conservation law for the K-spin manifests the availability of the so-called K-symmetry of baryons, which is relevant to the G_1-invariance of the skyrmion configuration. It is useful to stress here the analogy with the central potential scattering in quantum mechanics, when the orbital angular momentum is conserved.

We note that thus defined the K-symmetry holds only in the adiabatic approximation, at a relatively high energy of incident particles in the absence of resonances with low-lying excitations of skyrmions. Roughly speaking, the scattering time has to be small compared with the period of rotation. As one of the consequences of the K-symmetry there appear linear relations between partial amplitudes, the so-called Mattis–Peskin–Karliner relations.[125] So, if we consider the πN-scattering, including the N- and Δ-states, then the T-matrix can be brought tothe form (cf. Ref. 125):

$$\begin{aligned} T_{LTJ} &\equiv T(\{LsTJ\} \to \{L's'TJ\}) \\ &= (-1)^{s'-s}(2s+1)^{1/2}(2s'+1)^{1/2} \\ &\quad\times\sum_K (2K+1)\begin{Bmatrix} KTJ \\ s'L'1 \end{Bmatrix}\begin{Bmatrix} KTJ \\ sL1 \end{Bmatrix}\tau_{KL'L} \end{aligned} \tag{4.7.2}$$

where $\tau_{KL'L}$ is the reduced T-matrix, $\mathbf{L}$ and $\mathbf{L}'$ are the initial and final pion angular momenta, s and s' are the initial and final values of baryon spins (equal to isospins), $\mathbf{T}$ is the total isospin, and $\mathbf{J}$ is the total angular momentum. In (4.7.2) the K values range in accordance with the restrictions $|L-1| \leqslant K \leqslant L+1$, $|L'-1| \leqslant K \leqslant L'+1$. Equation (4.7.2) implies two linear relations:

$$(4L+2)T_{L3/2,L-1/2} - (L-1)T_{L1/2,L-1/2} = (3L+3)T_{L1/2,L+1/2}, \quad (4.7.3)$$

$$(4L+2)T_{L3/2,L+1/2} - 3LT_{L1/2,L-1/2} = (L+2)T_{L1/2,L+1/2}.$$

Comparison with experiments shows a fairly good fulfillment of the relations (4.7.3) in the F-channel,[60] and a worse agreement in the D-channel. This should have been expected, since the higher values of L correspond to higher energies.

Recently, it became clear that the Mattis–Peskin–Karliner relations can be generalized also to the SU(3) case.[126] Moreover, the K-symmetry conservation approach is successfully applied also in the analysis of baryon-baryon interaction process (Refs. 126, 127).

As far as calculations of the scattering amplitudes in the framework of the Skyrme model are concerned, usually this is performed by considering configurations of the form

$$U = \exp[i(\mathbf{n}\boldsymbol{\tau})\theta(r) + i\boldsymbol{\tau}\boldsymbol{\pi}(\mathbf{r})], \quad \mathbf{n} = \mathbf{r}/r,$$

where $\theta(r)$ describes the skyrmion and $\boldsymbol{\pi}(\mathbf{r})$ describes the meson field. Now in the Lagrangian there are left only terms quadratic in $\boldsymbol{\pi}$ and this leads to linear equations for the pion field with coefficients dependent on the "hedgehog" profile. Expanding the $\boldsymbol{\pi}$-field in spherical harmonics taking the K-symmetry into account, we obtain a set of equations for the radial field functions. The solutions of this set enable us to define the scattering matrix.[60]

4.8. The Skyrme model and nuclear matter

The first attempts to describe dense nuclear matter by means of effective mesonic fields were made by Skyrme.[33] Introducing, in accordance with Sec. 3.3, the vecotrs

$$\mathbf{X} = \nabla\theta, \quad \mathbf{Y} = \sin\theta \cdot \nabla\beta, \quad \mathbf{Z} = \sin\theta \cdot \sin\beta \cdot \nabla\gamma,$$

we can write for the energy density $u = E/V$ and for the baryon density $n = |N|/V = |\mathbf{Q}|/V$ the following expressions:

$$u = \frac{1}{2\lambda^2}(\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2) + \varepsilon^2([\mathbf{X}\,\mathbf{Y}]^2 + [\mathbf{Y}\,\mathbf{Z}]^2 + [\mathbf{Z}\,\mathbf{X}]^2), \quad (4.8.1)$$

$$n = \frac{1}{2\pi^2}|(\mathbf{X}[\mathbf{Y}\,\mathbf{Z}])|. \quad (4.8.2)$$

Recalling that a geometric average does not exceed an arithmetic average, one obtains the inequalities:

$$\frac{1}{3}(\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2) \geq (|\mathbf{X}| \cdot |\mathbf{Y}| \cdot |\mathbf{Z}|)^{2/3} \geq |(\mathbf{X}[\mathbf{Y}\,\mathbf{Z}])|^{2/3},$$

$$\frac{1}{3}([\mathbf{X}\,\mathbf{Y}]^2 + [\mathbf{Y}\,\mathbf{Z}]^2 + [\mathbf{Z}\,\mathbf{X}]^2) \geq |(\mathbf{X}[\mathbf{Y}\,\mathbf{Z}])|^{4/3}.$$

Using these inequalities we deduce from (4.8.1) and (4.8.2) an estimate of the energy density of nuclear matter:

$$u \geq \frac{1}{2\lambda^2}(2\pi^2 n)^{2/3} + \varepsilon^2(2\pi^2 n)^{4/3}. \quad (4.8.3)$$

In the dense matter limit the second term in (4.8.4) becomes the dominant one, whence:

$$u > \varepsilon^2(2\pi^2 n)^{4/3} \approx 160\varepsilon^2 n^{4/3}. \quad (4.8.4)$$

It is of interest to compare Skyrme's estimate (4.8.4) with that obtained in the bag approximation of the quark-gluon plasma theory,[128] namely

$$u = B + \frac{4}{9}\pi^{2/3}\left(1 + \frac{2\alpha_s}{3\pi}\right)n^{4/3}, \quad (4.8.5)$$

where B is a quark bag constant and α_s is the running QCD constant. It is evident that estimates (4.8.4) and (4.8.5) are in good agreement with each other. This fact tells us that the Skyrme model accounts quite well for high-energy hadron physics as well.

Numerical calculations of Ref. 129 show that if one considers a system of isolated skyrmions then as its density is increased they would form a structure similar to a face-centered cubic lattice with a spacing a and a symmetry of the form:

$$U(x + \frac{a}{2}, y + \frac{a}{2}, z) = \tau_3 U(x, y, z)\tau_3; \quad (4.8.6)$$

where dots stand for transformations obtained by cyclic permutations of (4.8.6). The symmetry (4.8.6) means that situated at the center of a face a skyrmion is turned in isospace by an angle π around the outward normal to the face. Under further condensation skyrmions expand in size losing their individuality. Therefore a phase transition to a condensed medium takes place. If we set $U = \sigma + i(\boldsymbol{\pi} \cdot \boldsymbol{\tau})$, then for this phase transition the parabolic approximation to the mean-field value is valid, i.e.,

$$\langle\sigma\rangle \sim (a - a_{cr})^2,$$

thus indicating that the above phase transition is of the second kind. The field configuration itself acquires an additional symmetry at the phase transition point:

$$\sigma(x + \frac{a}{2}, y, z) = -\sigma(x, y, z),$$

$$\pi_1(x + \frac{a}{2}, y, z) = -\pi_1(x, y, z),$$

$$\pi_2(x + \frac{a}{2}, y, z) = \pi_2(x, y, z),$$

$$\pi_3(x + \frac{a}{2}, y, z) = \pi_3(x, y, z);$$

and so on by cyclic permutations.

Finally, with a further decrease of the spacing $a < a_{cr}$, the energy per individual skyrmion continues to decrease, its minimal value at some $a = a_0$, where the field configuration is approximately described by the following functions:

$$\sigma = \cos kx \cdot \cos ky \cdot \cos kz, \quad k = 2\pi/a,$$

$$\pi_1 = \sin kx \cdot \left(1 - \frac{\sin^2 ky}{2} - \frac{\sin^2 kz}{2} + \frac{1}{3}\sin^2 ky \cdot \sin^2 kz\right)^{1/2};$$

and so on by cyclic permutations. Thus, the surfaces defined by $\sigma = 0$ are orthogonal planes with $[x,y,z$

$= (a/4) + m(a/2)]$, $m \in \mathbb{Z}$. We note this leads to a considerable diminution of the mass per skyrmion (with a factor $\approx (5/6)$) compared with that for a free one.

An interesting interpretation of this effect was suggested by N. Manton,[130] who considered the Skyrme model as defined not on the flat space $\mathbb{R}^3$, but instead on an $\mathbb{S}^3$ sphere of some radius R. Since the metric element on $\mathbb{S}^3$ in terms of spherical coordinates (μ,ϑ,α) is

$$dS^2 = R^2[d\mu^2 + \sin^2\mu(d\vartheta^2 + \sin^2\vartheta d\alpha^2)], \quad \mu \in [0,\pi],$$

the transition from $\mathbb{R}^3$ to $\mathbb{S}^3$-model is equivalent to the following replacement of variables in the Skyrme Hamiltonian: $r \to R \sin\mu$, $dr \to d\mu$. As the result, with the "hedgehog" ansatz the Hamiltonian takes the form

$$H = 2\sqrt{2}\pi \frac{\varepsilon}{\lambda} \int_0^\pi d\mu \sin^2\mu \cdot \left\{ L\left[\left(\frac{d\theta}{d\mu}\right)^2 + \frac{2\sin^2\theta}{\sin^2\mu} \right. \right.$$

$$\left. \left. + L^{-1}\left(\frac{\sin^2\theta}{\sin^2\mu}\right)\left[2\left(\frac{d\theta}{d\mu}\right)^2 + \frac{\sin^2\theta}{\sin^2\mu}\right]\right\}, \qquad (4.8.7)$$

where we have set $L = R/\lambda\varepsilon\sqrt{2}$. It is not difficult to see that minimization of (4.8.7) leads to the Euler–Lagrange equation, which admits the simple uniform solution

$$\theta = \pi - \mu \qquad (4.8.8)$$

with the associated energy

$$E = 3\pi^2\sqrt{2}\,\frac{\varepsilon}{\lambda}(L + L^{-1}).$$

For $L = 1$, the solution (4.8.8) realizes the absolute minimum of the energy and, moreover, it saturates the topological lower bound $E \simeq 6\sqrt{2}\pi^2(\varepsilon/\lambda)$, while this is impossible in $\mathbb{R}^3$ (see Sec. 3.1).

It is straightforward to see that the solution (4.8.8) corresponds to uniform distribution of matter on $\mathbb{S}^3$, that is to a condensed phase. As has been shown in Ref. 131 this solution turns out to be stable for $L < \sqrt{2}$, i.e., for $R = R_{cr} \equiv 2\lambda\varepsilon$ a phase transition should occur. Therefore in the Skyrme-Manton model the radius R of the spatial sphere $\mathbb{S}^3$ should depend on the baryon density: $R \sim n^{-1/3}$. If $L > \sqrt{2}$, the condensed phase becomes unstable and in the limit $L \to \infty$ isolated skyrmions should be formed. In the latter case it is possible to obtain L-dependent corrections to the skyrmion mass:[131]

$$E_1 \simeq 6\pi^2\sqrt{2}\frac{\varepsilon}{\lambda}\left(1{,}231445 - \frac{0{,}419}{L^2} + \frac{1}{4L^3}\right). \qquad (4.8.9)$$

The expression (4.8.9) implies the skyrmion mass diminution effect, correlated with the nuclear matter density.

5. CONCLUSION

One of the tasks, which we had set for ourselves when writing this review, was a graphic demonstration of the efficiency of topological methods, especially in the study of essentially nonlinear phenomena. The Skyrme model appears to be a clear illustration of such an approach in theoretical physics, which has evolved from the hydrodynamical Helmholz–Kelvin notions at the end of the last century to the modern models for particles and nuclear matter. Among other examples of the development of this approach, the theory of defects in solids, the liquid crystals theory, the theory of strong excitations in magnetic materials, superfluidity and superconductivity theories should be mentioned. All theories listed here are at present at different stages of their development, but nevertheless, it is already possible to notice their common features. This is the similar structure of model Hamiltonians, the deep topological origin of nonlinear phenomena and so on. Let us hope that the methods presented in this review and tested in applications to the Skyrme model will find successful applications in all the theories listed above as well, and that the methods developed in these theories will be useful in the physics of skyrmions. If this will come to pass, then the aim of this paper will have been achieved.

6. A BRIEF OUTLINE OF THE LIFE AND WORK OF T. H. R. SKYRME

Tony Hilton Royle Skyrme was born on 5 December 1922. After graduation from Trinity college, Cambridge, U.K. in 1943, he worked during three years at Los Alamos (U.S.A.) as a theorist. Then he returned to Trinity College and continue with his studies in theoretical nuclear physics. During the period 1950–1961 he worked in the Atomic Center in Harwell (U.K.), being the Head of the Nuclear Physics group, which specialized in the theory of nuclei. It is just during this period that his principal results appeared, which made him known worldwide.

From 1962 Skyrme devoted himself to teaching activity. For two years he worked in the Department of Mathematics, University of Malaya at Kuala Lumpur before returning in 1964 to Birmingham University, first as Professor and Head of the Department of Mathematical Physics, and latterly as Professor of Applied Mathematics in the Department of Mathematics.

In 1985 the Royal society awarded Tony Skyrme the Hughes medal in recognition of his contribution to theoretical particle and nuclear physics.

T. H. R. Skyrme died on 25 June 1987.

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