Groups and probabilities at the foundations of quantum mechanics

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The current state of the foundations of quantum mechanics is discussed. The analysis takes as its starting point the theory of probability amplitudes, which is intimately related to the group-theoretic approach. A detailed examination is presented of the relationships of classical and quantum theories, the transition to the classical limit, the different forms of uncertainty relations, and the properties of quantum structures determined by the Clebsch–Gordan coefficients. Possible future generalizations are examined, including those involving quantum algebras.

1. INTRODUCTION

Analysis of the foundations of quantum theory has a long history. It reached its peak when quantum mechanics was being created, and is closely associated with names such as de Broglie, Schrödinger, Heisenberg, Bohr, and Dirac. The foundations of quantum mechanics were essentially laid in 1932 with the publication of von Neumann's book[1] in which he gave a logically consistent and mathematically rigorous presentation based on the theory of Hilbert space.

These problems then gradually receded from the front line of research, although the more important concepts continued to evolve. For example, nonrelativistic quantum mechanics became part of general quantum theory, group-theoretic and probability aspects were recognized as being of fundamental importance, the use of coherent states (CS) has altered our views on the inter-relation between classical and quantum theories, there has been a change in, and a gradual complication of, the formalism that has entered physics from branches of mathematics and has seemingly become more and more abstract, and significant changes have taken place in recent years in what seemed to be pretty basic ideas.

All this has forced us to return once again to the examination of the foundations of quantum theory.

Our approach to this problem is closely linked to the theory of Clebsch–Gordon coefficients (C–G coefficients).

In 1972, the senior authors of the present review published an article in the present journal[2] under the title "Clebsch–Gordan coefficients, viewed from different sides." By that time, the theory of the Clebsch–Gordan coefficients of the group SU(2), i.e., the theory of angular momenta, had become an integral part of several branches of physics, namely, quantum mechanics and field theory, the theory of atomic collisions and atomic spectra, molecular physics and the physics of elementary particles, and relativistic equations and coherent phenomena. It was shown that the Clebsch–Gordan coefficients constituted a new calculus that was closely related to algebra, multi-dimensional geometry, topology, projective geometry, theory of analytic functions, special functions, differential equations, combinatorial analysis, and finite differences.

In the last twenty years, the theory of Clebsch–Gordan coefficients has undergone radical changes. We now have continuous as well as discrete coefficients, and a unified theory has evolved that includes not only SU(2), but also other groups. It was then suggested that we should examine the situation in a review entitled "Clebsch–Gordan coefficients, viewed from different sides (twenty years later)". However, something unexpected has happened. From the physical point of view, the theory of Clebsch–Gordan coefficients is the science of the structure of quantum-mechanical systems, of the connection between a system and subsystems, and of the transitions between different structures, but the last twenty years have witnessed changes in things that seemed totally unshakeable and routine. The result is that the problem of the Clebsch–Gordan coefficients has been subsumed by the general problem of the foundations of quantum theory.

The key point of this theory is that of probability amplitudes. These amplitudes (wave functions ψ) were considered as auxiliary quantities used in quantum mechanics to calculate physical quantities. In actual fact, the probability amplitudes are independent probabilistic objects for which a systematic theory can be constructed. Dirac[3] noted that of the two fundamental properties of quantum theory, namely, noncommutativity of observables and the concept of probability amplitude, the latter is the more important and crucial to the subsequent development of the theory and to the resolution of existing difficulties. Dirac's prediction has come true. One of the fundamental steps in this direction was the introduction of the path integral by Feynman.[4] The theory of probability amplitudes discussed in Ref. 5 runs parallel to the usual theory of probability and has its own distribution functions, limit theorems, and Markov processes. There are thus two equivalent languages in quantum theory, namely, the language of operators in Hilbert space and the language of probability amplitudes.

Probability amplitudes are intimately related to the group-theoretic approach which has a universal character in quantum theory. Each process has associated with it its own

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group, in much the same way that, in statistics, each particular phenomenon has its own probability distribution. In other words, there is no preferred group and we can speak of a definite 'group democracy'. At the same time, different group and probability approaches are becoming increasingly intertwined, forming a unified theory. Even the Clebsch–Gordan coefficients that emerged from pure group-theoretic considerations are actually probability amplitudes.

Studies of CS as a basis of irreducible representations (IR) of groups have resulted in a version of quantum theory that is wave-particle symmetric, and have led to a reformulation of the question of its relation to classical theory. A group-theoretic treatment of the uncertainty relations, a rigorous transition to the classical limit, and the construction of states that are as close as desired to classical states have all become possible.

Still greater generality is emerging from the formalism of overlap formulas for basis vectors of the space of states in which the well-known Dirac notation is employed (many of Dirac's ideas are presented in Ref. 6). This formalism appears to combine the two languages, i.e., the language of the theory of operators in Hilbert space and the language of the theory of probability amplitudes. Its central relation, namely, the formula for the decomposition of unity, can be treated either as a consequence of the completeness of the basis or as a consequence of the completeness of the field of events under consideration. The overlap formalism involves the C–G coefficients and a broad class of special functions, which results in a radical simplification of notation and calculations. We know from the history of science that the introduction of efficient and simple notations has far-reaching consequences.

Advances in the development of a general quantum theory, containing the usual quantum mechanics as a special case, have been invigorated by new ideas that rely, for example, on the use of infinite-dimensional generator algebras, pseudo-differential operators, and quantum algebras. Here we can see the importance of a systems approach and the identification of fundamental principles.

The aim of this review is not only to bring together the different strands of the subject, but also to identify new ideas in order to demonstrate the internal unity of the three basic approaches (group-theoretic, probability, and operator) and to provide an outline of a general quantum theory. Particular attention will be devoted to the simplicity of presentation and to the use of particular formulas as working tools.

Our review presents a systematic account of the group-theoretic prerequisites for the analysis of the foundations of quantum theory; the calculus of finite differences which includes the vigorously developing quantum algebras; coherent states and the associated question of uncertainty relations and the transition to the classical limit; the C–G coefficients for continuous and discrete bases; and the theory of probability amplitudes and its connection with the group-theoretic approach and the general structure of quantum theory. Relatively complex formulas that are widely used in group quantum theory are relegated to the Appendix.

Because of the wide range of topics covered in this review, we have had to confine ourselves to a minimum number of references. With exception of the formulas given in Sec. 4.1, which refer to the uncertainty relations, we use the system of units in which $\hbar = c = 1$.

2. PREREQUISITES FOR THE GROUP-THEORETIC APPROACH

2.1. Lie algebras, representations, and bases

Group-theoretic methods have become an inseparable part of quantum theory, whose basic concepts can be expressed in group-theoretic language and whose extensive spectrum of problems is amenable to group-theoretic analysis. Different representations of quantum mechanics can be placed in correspondence with particular bases for irreducible representations. Until recently, attempts were being made to find some fundamental all-embracing group for which there were many candidates, right up to exclusive groups. However, the reality is that different groups apply to different problems, and each group describes a particular aspect of reality.

In this Section we examine a systems approach based on an equivalent set of simple groups whose Lie algebras contain a triple of operators, namely, raising, lowering, and diagonal operators. This is the foundation of the group-theoretic analysis. These groups can be used as a framework for a systematic examination of basic quantum concepts (coordinate/momentum, angular momentum/angle, phase/number of particles, energy/mass equivalence, and the components of the angular momentum vector). We note that all the basic consequences can be fully examined within the framework of simple (three-parameter) groups, and a systematic and complete group-theoretic description can be given (including generator algebras, representations, bases, C–G series and coefficients, coherent states, operator symbols, and transition to the classical limit).

The theory of representations of more complicated Lie groups is still incomplete, so that it is best to use the constructive principle for them, based on results obtained for fundamental groups. We note that groups such as the Poincare group M(3,1) or the Lorentz group SL(2C)-SO(3,1) are really topics for separate monographs.

Advances in the group-theoretic approach have been closely related to applications in physics. The successes of quantum field theory are well known from this point of view. There is also optics in which an enormous amount of research has been carried out and which typically involves an intertwining of group-theoretic and statistical problems. The development of the group-theoretic formalism and the wide range of group-theoretical research in optics have become prerequisites for a re-examination of the foundations of quantum theory.

The material presented in this Section can be regarded as a generalization of the theory of angular momenta to the entire set of simple groups lying at the foundation of the group-theoretic approach. They include the five three-parameter non-Abelian Lie groups, namely, the semisimple compact group SU(2), the noncompact group SU(1,1), the nilpotent group, the Heisenberg group (or Heisenberg-Weyl group) (W1), the group of motions of a plane (or the Fourier-Bessel group) M(2), and the group of motions of a pseudo-Euclidean plane M(1,1) [sometimes denoted by MH(2)]. Their Lie algebras contain three operators each (the raising operator $\hat{E}_+$, the lowering operator $\hat{E}_-$, and the diagonal operator $\hat{H}$) and they have similar structure:

$$[\hat{H}, \hat{E}_\pm] = \pm\hat{E}_\pm, \quad [\hat{E}_+, \hat{E}_-] = f(\hat{H}). \tag{2.1}$$

The difference lies in the form of the function $f(\hat{H})$. We shall use a specific operator designation for each group:

$$\begin{array}{lccc} & \mathrm{SU}(2), \mathrm{SU}(1,1) & \mathrm{M}(2), \mathrm{M}(1,1) & \mathrm{W}(1) \\ f(\hat{H}): & 2\hat{H} & 0 & \hat{I} \\ \hat{E}_{\pm}: & \hat{J}_{\pm} & \hat{\Phi}_{\pm} & \hat{a}_{\pm}\ ; \\ \hat{H}: & \hat{J}_z & \hat{J}_z, \hat{J}_{xt} & \hat{n} \end{array} \tag{2.2}$$

where $\hat{n} = \hat{a}_+ \hat{a}_-$ is the particle number operator and $\hat{I}$ is an operator that is a multiple of the unit operator. For unitary irreducible representations of the group W(1), $\hat{a}_-^+ = a_+$, $\hat{I}^+ = \hat{I}$, instead of $\hat{a}_+$ and $\hat{a}_-$ we usually employ the notation $\hat{a}^+$ and $\hat{a}$ (these are the creation and annihilation operators).

The Lie algebras in (2.1) correspond to different transformations of the groups $\exp \hat{L}$, defined by three parameters:

$$\begin{aligned} &\hat{L} = \alpha\hat{E}_+ - \bar{\alpha}\hat{E}_- + i\alpha_0\hat{J}_z, && \hat{E}_{\pm}^+ = \hat{E}_{\mp},\ \hat{J}_z^+ = J_z\ (\mathrm{SU}(2), \mathrm{M}(2)), \\ &\hat{L} = \alpha\hat{E}_+ + \bar{\alpha}\hat{E}_- + i\alpha_0\hat{J}_z, && \hat{E}_{\pm}^+ = -\hat{E}_{\mp},\ \hat{J}_z^+ = \hat{J}_z\ (\mathrm{SU}(1,1)), \\ &\hat{L} = i\alpha_1\hat{E}_+ + i\alpha_2\hat{E}_- + \alpha_0 J_{xt}, && \hat{E}_{\pm}^+ = E_{\pm},\ \hat{J}_{xt}^+ = -\hat{J}_{xt}\ (\mathrm{M}(1,1)), \\ &\hat{L} = \alpha\hat{a}_+ - \bar{\alpha}\hat{a}_- + i\alpha_0\hat{I}, && \hat{a}_{\pm}^+ = \hat{a}_{\mp},\ \hat{I}^+ = I\ (\mathrm{W}(1)), \end{aligned} \tag{2.3}$$

where α is a complex number and all the α_i are real. The Hermitian conditions in (2.3) correspond to unitary operators $\exp \hat{L}$. For arbitrary (nonunitary) representations, these conditions are not in general satisfied. We note that all the generators of (2.3) are usually chosen to be Hermitian in the literature of physics. The factors i and -1 are then found to appear in the commutation relations given by (2.1).

The last decade saw the development of deformed algebras (that are no longer Lie algebras) in which $f(\hat{H})$ is an arbitrary function. It has been found that, for some deformed algebras (the so-called quantum algebras), the basic structure of the theory of representations is retained in a transformed form (see Sec. 3).

The three-parameter groups that we are considering are groups of motions of two-dimensional manifolds (i.e., transformations of these manifolds into themselves that preserve separations between points and orientations) of constant curvature K, namely, the Euclidean and pseudo-Euclidean planes [$K=0$, $M(2)$ and $M(1,1)$], the sphere [$K>0$, SU(2)], and the Lobachevskiĭ plane or the hyperboloid [$K<0$, SU(1,1)]; W(1) is the group of motions of a phase plane.

The irreducible representations of the corresponding groups are constructed with the help of the Lie algebras (2.1). The discrete basis of irreducible representations is introduced as a basis of the eigenfunctions of the diagonal operator $\hat{H}$ whose eigenvalues are then called the weights of the irreducible representations. The raising and lowering operators define transitions between basis vectors and couple them into one irreducible representation of a Lie algebra.

An irreducible representation is itself defined by a number (its signature), i.e., an eigenvalue of an operator that commutes with all the group generators. The basis constructed with the help of the operators of the algebras (2.1) can be looked upon as a basis of occupation numbers. It is conveniently written in the form $|n\rangle$, $|n_1 n_2\rangle$. We shall also use the following two types of notation for the irreducible representations: $T_{(\lambda)}(g)$ will denote a finite transformation operator where g is a group element, λ is the signature of the representation, and $D(\lambda)$ is the same operator, but for those cases where the absence of the argument g does not give rise to a misunderstanding.

We now turn directly to the construction of the unitary IR of the Heisenberg algebra W(1). We introduce the basis of $|\omega n\rangle$ as the basis consisting of the eigenfunctions of two commuting operators, namely, the operator $\hat{I}$ that is a multiple of the unit operator and the particle-number operator $\hat{n} = \hat{a}_+ \hat{a}_-$:

$$\hat{I}|\omega n\rangle = \omega|\omega n\rangle,\ \hat{a}_+\hat{a}_-|n\rangle = \omega n|\omega n\rangle. \tag{2.4}$$

From the first pair of commutation relations in (2.1) we have

$$\begin{aligned} \hat{a}_+|\omega n\rangle &= c_1(\omega, n)|\omega, n+1\rangle,\ \hat{a}_-|\omega n\rangle \\ &= c_2(\omega, n)|\omega, n-1\rangle. \end{aligned}$$

The Hermitian conditions (2.3) are satisfied for unitary IR, so that ω and n are real. Moreover, $\hat{a}_-^+ = \hat{a}_+$, from which $c_1^2 = \omega(n+1)$, $c_2^2 = \omega n$, $\bar{c}_k(n) = c_k(n)$ and

$$\begin{aligned} \hat{a}_+|\omega n\rangle &= [\omega(n+1)]^{1/2}|\omega, n+1\rangle,\ \hat{a}_-|\omega n\rangle \\ &= (\omega n)^{1/2}|\omega, n-1\rangle, \end{aligned} \tag{2.5}$$

$$\omega n \geq 1,\ \omega(n+1) \geq 0. \tag{2.6}$$

The formulas given by (2.4)–(2.6) define the weights of the unitary infinite-dimensional IR specified by the real $\omega \neq 0$ (when $\omega = 0$, the unitary IR are one-dimensional). Since $\hat{I}$ commutes with all the operators in the group, the generators specify transitions only between states with the same ω. When $\omega > 0$, (2.6) shows that the IR have the lowest weight $n = 0$, whereas for $\omega < 0$, the highest weight is $n = -1$ (Fig. 1). The representations $D(\omega)$ and $D(-\omega)$ are conjugate.

From the physical point of view, n specifies the oscillator level number and ω is the level separation (oscillator frequency). We note that much of the physics literature discusses only one IR with fixed $\omega = 1$.

The generators $\hat{\Phi}_{\pm}$ of the group of motions of a plane M(2) which consists of rotations and translations can be written in the form

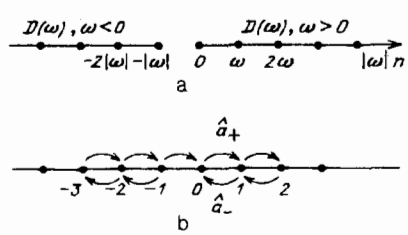

FIG. 1. Representations of the Heisenberg group: a—weight composition of unitary IR for fixed $|\omega|$, b—reducible but nodecomposable representations.

$$\hat{\Phi}_{\pm} = \hat{p}_x \pm i\hat{p}_y, \ \hat{\Phi}_+\hat{\Phi}_- = \hat{p}_x^2 + \hat{p}_y^2, \tag{2.7}$$

where $\hat{p}_x$ and $\hat{p}_y$ are infinitesimal operators corresponding to translations (momentum operators). The discrete basis of the IR of the group of motions of a plane M(2) is constructed as a basis consisting of the eigenfunctions of commuting angular momentum operators $\hat{J}_z$ and the square of momentum operator $\hat{p}^2 = \hat{\Phi}_+\hat{\Phi}_-$:

$$\hat{\Phi}_+\hat{\Phi}_-|pm\rangle = p^2|pm\rangle, \ \hat{J}_z|pm\rangle = m|pm\rangle. \tag{2.8}$$

From (2.1) we obtain

$$\hat{\Phi}_+|pm\rangle = pc^m|p, m+1\rangle, \ \hat{\Phi}_-|pm\rangle = pc^{-m}|p, m-1\rangle.$$

For unitary irreducible representations, (2.3) shows that m and p are real and

$$\hat{\Phi}_{\pm}|pm\rangle = p|p, m \pm 1\rangle. \tag{2.9}$$

Infinite-dimensional unitary IR are specified by real $p \neq 0$ (for $p = 0$ the unitary IR are one-dimensional; the corresponding states are $|0,m\rangle$); the irreducible representations $D(p)$ and $D(-p)$ are equivalent; and m runs through both positive and negative values, i.e., the IR do not have either the highest or the lowest weight.

In the irreducible representation $D(p)$, i.e., for a fixed square of momentum p^2, the generators can be written in the form

$$\hat{J}_z = -i\frac{\partial}{\partial\varphi}, \ \Phi_{\pm} = p\exp(\pm i\varphi), \ \hat{p}_x = p\cos\varphi, \ \hat{p}_y = p\sin\varphi. \tag{2.10}$$

This representation is the foundation for the description of angular momentum and angle variables in quantum mechanics.[8,9]

The generators of the group of motions of a pseudo-Euclidean plane M(1,1) [the separation between the points is given by $\rho^2 = (x_1 - x_2)^2 - (t_1 - t_2)^2$] are infinitestimal operators corresponding to translations and hyperbolic rotations:

$$\hat{\mathscr{E}} = i\hat{p}_t = i\frac{\partial}{\partial t}, \ \hat{p}_x = -i\frac{\partial}{\partial x}, \ \hat{J}_{xt} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$
$$\hat{\Phi}_{\pm} = \hat{p}_x + \hat{\mathscr{E}}, \ \hat{\Phi}_+\hat{\Phi}_- = \hat{p}_x^2 - \hat{\mathscr{E}}^2. \tag{2.11}$$

We can now introduce a basis consisting of eigenfunctions of the Hermitian operators $\hat{\mathscr{E}}$ and $\hat{p}$: $\hat{\mathscr{E}}|\mathscr{E}, p\rangle = \mathscr{E}|\mathscr{E}p\rangle, \hat{p}|\mathscr{E}, p\rangle = p|\mathscr{E}, p\rangle$ where $\mathscr{E}$ and p real, and

$$\hat{\Phi}_+\hat{\Phi}_-|\mathscr{E}, p\rangle = (p^2 - \mathscr{E}^2)|\mathscr{E}, p\rangle. \tag{2.12}$$

The unitary representations are characterized by the real number $m_0^2 = \mathscr{E}^2 - p^2$ where m_0 is the particle mass. However, the theory of representations of M(1,1) (Ref. 7) is more complicated than in the case of M(2); in this sense, M(2) and M(1,1) are in the same relationship as SU(2) and SU(1,1).

2.2. The groups U(2) and SU(1,1) as one system

The standard theory of angular momentum examines states with integer and half-integer angular momenta $j \geqslant 0$. The basis of irreducible representations $D(j)$ contains $2j + 1$ vectors $|jm\rangle$, $-j \leqslant m \leqslant j$. The unified theory of representations of SU(2) and SU(1,1) can be usefully interpreted as a generalization to arbitrary real and complex angular momenta.

When the irreducible representations of SU(2) and SU(1,1) are constructed, it is more convenient to start not with (2.1) but with the commutation relations for the generators of the groups U(2) and U(1,1):

$$[\hat{n}_1, \hat{n}_2] = 0, \qquad [\hat{n}_1, \hat{J}_{\pm}] = \pm\hat{J}_{\pm},$$
$$[\hat{J}_+, \hat{J}_-] = \hat{n}_1 - \hat{n}_2, \ [\hat{n}_2, \hat{J}_{\pm}] = \mp\hat{J}_{\pm}. \tag{2.13}$$

This algebra splits into two, namely, U(1) ($\hat{n}_1 - \hat{n}_2$ commutes with all the others) and SU(2) [SU(1,1)] ($\hat{J}_{\pm}, \hat{n}_1 - \hat{n}_2$). The Casimir operator of the latter system is

$$\mathbf{J}^2 = \frac{1}{4}(2\hat{J}_+\hat{J}_- + 2\hat{J}_-\hat{J}_+ + (n_1 - n_2)^2). \tag{2.14}$$

The basis of the representations is specified as a basis consisting of the functions $\hat{n}_1$ and $\hat{n}_2$:

$$\hat{n}_1|n_1n_2\rangle = n_1|n_1n_2\rangle, \ \hat{n}_2|n_1n_2\rangle = n_2|n_1n_2\rangle. \tag{2.15}$$

In the usual notation $n_1 + n_2 = 2j$, $n_1 - n_2 = 2m$. When the unitary IR are constructed, we can use either directly the commutation relations (2.13) (Refs. 10 and 11) or the representations of the generators in terms of the operators $\hat{a}_+$ and $\hat{a}_-$ of the Heisenberg group for $\omega = 1$

$$\hat{J}_{\pm} = \hat{a}_{1\pm}\hat{a}_{2\mp}, \ \hat{n}_1 = \hat{a}_{1+}\hat{a}_{1-}, \ \hat{n}_2 = \hat{a}_{2+}\hat{a}_{2-} \tag{2.16}$$

and the formula given by (2.5):

$$\hat{J}_+|n_1n_2\rangle = [n_1(n_2+1)]^{1/2}|n_1 - 1, n_2 + 1\rangle,$$
$$\hat{J}_-|n_1n_2\rangle = [n_2(n_1+1)]^{1/2}|n_1 + 1, n_2 - 1\rangle. \tag{2.17}$$

For unitary IR of SU(2) and SU(1,1), we have $\hat{J}_z^+ = \hat{J}_z \Rightarrow n_1 - n_2$ real, i.e., Im n_1 = Im n_2, and

$$\text{SU (2)}: \hat{J}_-^+ = \hat{J}_+ \Rightarrow n_1(n_2+1) \geq 0, \ n_2(n_1+1) \geq 0, \tag{2.18}$$
$$\text{SU(1,1)}: \hat{J}_-^+ = -\hat{J}_+ \Rightarrow n_1(n_2+1) \leq 0, \ n_2(n_1+1) \leq 0. \tag{2.19}$$

Figure 2a shows the plane of all the admissible real parts of n_1 and n_2; the unitary regions for SU(2) (I) and SU(1,1) (II) are defined by (2.18) and (2.19). Since it follows from (2.17) that for the irreducible representations $n_1 + n_2 = 2j$ is conserved, their weights must lie on the straight lines $n_1 + n_2 =$ const parallel to the horizontal axis. The necessary condition for unitary IR is that the weights on the straight line (separated by two units of length) do not enter the nonunitary region. The operators $\hat{J}_{\pm}$ can be used to reach an arbitrary weight of a given representation by starting from any given weight. Whenever the numerical factor in (2.17) is zero, the representation cuts off and we reach the highest (lowest) weight. We note that the operators $\hat{J}_{\pm}$ do not act on the imaginary part of n_1 and n_2, i.e., the weight diagram of the representation on the Im n_1, Im n_2 plane is a point.

From (2.17) we obtain the following classification and weight composition of representations of the algebra (2.13):

1. n_1 and n_2 are integers. The representation of $D(j)$ is reducible; it splits into three: $D^-(j)$ with the highest weight, the finite-dimensional $D^0(j)$, and dim $D^0(j)$

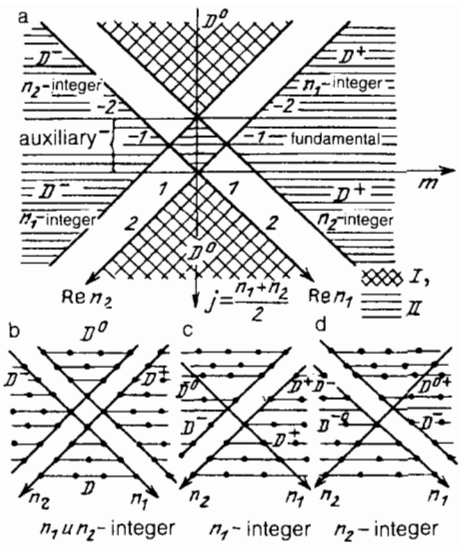

FIG. 2. Weight diagrams for the irreducible representations of SU(2) and SU(1,1).

$= 2j + 1$, $D^+(j)$ with the lowest weight (Fig. 2b).

2. n_1 (or n_2) is an integer and n_2 (n_1) is not an integer (Figs. 2c and d, respectively). We have two types of representation with the highest (lowest) weight, namely, $D^-(j)$ and $D^{0-}(j)$ [$D^+(j)$ and $D^{0+}(j)$] that differ by the weight composition.

3. n_1 and n_2 are not integers. The representation $D(j)$ is irreducible and has neither the highest nor the lowest weight.

Let us consider the region in which SU(2) is unitary.

1. Suppose n_1 and n_2 are real numbers. To leave the unitary region (see Fig. 2a), we must ensure that both the highest and the lowest weights are integers. The representations are finite-dimensional. There are two series, arranged symmetrically around the straight line $n_1 + n_2 = 2j = -1$. The eigenvalues of the operator $\hat{\mathbf{J}}^2 = j(j+1)$ are then equal for both of them.

2. Suppose that n_1 and n_2 have an imaginary part: $\operatorname{Im} n_1 = \operatorname{Im} n_2 \neq 0$. We then find that (2.18) are not satisfied. These representations are nonunitary.

Consider the unitary representations of SU(1,1).

1. Suppose that n_1 and n_2 are real numbers. It is readily seen that the weights of all the representations $D^+(j)$ and $D^-(j)$, $-\infty < j < \infty$ are then in the unitary region. This is the so-called discrete series of representations.

For $-2 < 2j < 0$, we have the discrete series and also an additional series due to the fact that we can 'step over' the nonunitary region. The representations of the additional series require for their characterization a further number E that specifies the value of $2m = n_1 - n_2$ that is closest to $m = 0$; $-1 + |2j+1| \leqslant E \leqslant 1 - |2j+1|$. The equal sign in the last relation corresponds to the representations $D^{0+}(j)$ and $D^{-0}(j)$ with the lowest (highest) weight, and the inequality corresponds to the irreducible $D(j)$.

2. Suppose that n_1 and n_2 have an imaginary part. From (2.19) we obtain

$$\operatorname{Re} n_1 + \operatorname{Re} n_2 = -1, \quad \operatorname{Im} n_1 = \operatorname{Im} n_2.$$

These are the representations of the so called fundamental series

$$D\left(-\frac{1}{2} + \frac{i\rho}{2}\right):$$

When ρ is real and $-1 \leqslant E \leqslant 1$, the representations

$$D\left(-\frac{1}{2} + \frac{i\rho}{2}\right), \quad D\left(-\frac{1}{2} - \frac{i\rho}{2}\right)$$

are equivalent and $j^2 = (\rho^2 - 1)/4$.

When j is real, the representations $D(j)$ that lie symmetrically relative to the line $2j = -1$ are equivalent.

Thus, the plane with axes j, m (see Fig. 2) gives a very clear picture of the unitary IR of the groups SU(2) and SU(1,1). Different symmetries are clearly present, including the mirror symmetry ($j \to -j-1$), and also a set of weights of representations. If for a compact group SU(2), the IR have a finite number of basis vectors (weights), including highest and lowest weights, then for a noncompact SU(1,1) there are three types (series) of unitary IR that contain an infinite number of basis vectors that are either bounded on one side by the highest (lowest) weight (discrete series) or are unbounded on both sides (fundamental and auxilary series).

The abstract bases constructed with the help of a Lie algebra can have specific realizations in function space. From many points of view it is convenient to use the symmetric (polynomial) basis that is closely related to finite transformations (see Sec. 4) and group invariants.[11,12]

The transformations $T_{2j}(g)$ of the group SU(2) are realized in the space of polynomials of degree $2j$ of z_1, z_2:

$$\hat{T}_{2j}(g) f_{2j}(z_1, z_2) = f_{2j}(z'_1, z'_2), \quad z'_k = z_k(g) = g^i_k z_i \tag{2.20}$$

where z_1, z_2 form the basis of the two-dimensional fundamental irreducible representation $D(1/2)$, $|z_1|^2 + |z_2|^2 = 1$ and $\|g^i_k\|$ is a unitary 2×2 matrix. For the symmetric basis of the irreducible representation $D(j)$ of dimension $2j+1$ we have

$$|n_1 n_2\rangle \sim \psi_{jm}(z) = \left[\frac{(n_1+n_2)!}{n_1! n_2!}\right]^{1/2} u_1^{n_1} u_2^{n_2}, \quad n_1 = j+m, \quad n_2 = j-m \tag{2.21}$$

where n_1 and n_2 are nonnegative integers. For a discrete positive SU(1,1) series, the irreducible representation $D^+(j)$ ($j < -1/2$) is finite-dimensional and

$$|n_1 n_2\rangle \sim \psi_{jm}(z) = \left(\frac{\Gamma(-n_2)}{n_1! \Gamma(-n_1 - n_2)}\right)^{1/2} u_1^{n_1} u_2^{n_2} \tag{2.22}$$

where $n_1 \geqslant 0$ is an integer, $n_2 < -1$; for noninteger n_2 the representation is multivalued. The effect of finite transformations is analogous to (2.21), but now $|u_1|^2 - |u_2|^2 = 1$, and $\|g^i_k\|$ is a pseudounitary matrix. The bases of the conjugate IR are obtained by complex conjugation of the bases for (2.21), (2.22). The generators of SU(2) and SU(1,1) are

$$\hat{J}_+ = z_1 d/dz_2, \quad \hat{J}_- = z_2 d/dz_1,$$
$$\hat{J}_z = z_1 d/dz_1 - z_2 d/dz_2. \tag{2.23}$$

For fixed j, we can also have a realization of the irreducible representations $D(j)$ and $D^+(j)$ in projective space

$z = z_1/z_2$ (Refs. 7 and 13–15).

For fixed $\omega \neq 0$, the operators $\hat{a}_+$ and $\hat{a}_-$ of the W(1) algebra can be realized in the space of analytic functions of the complex variable z or real x

$$\hat{a}_- = \frac{d}{dz},\ \hat{a}_+ = \omega z,\ \hat{a}_- = \frac{1}{\sqrt{2}}(\omega x + \frac{d}{dx}),$$

$$\hat{a}_+ = \frac{1}{\sqrt{2}}(\omega x - \frac{d}{dx}). \tag{2.24}$$

The effect of unitary operators of finite transformations $T_\omega(g)$ is specified by the exponential in (2.3)

$$\hat{T}_\omega(g)f(x) = \exp\left[i\omega(\alpha_0 + \alpha_2 x) + \alpha_1 \frac{d}{dx}\right] f(x)$$
$$= \exp[i\omega(\alpha_0 + \alpha_1\alpha_2 + \alpha_2 x)]\, f(x + \alpha_1),$$
$$\hat{T}_\omega(g)f(z) = \exp(i\alpha_0\omega)\hat{D}(\alpha)f(z)$$
$$= \exp[\omega(i\alpha_0 - |\alpha|^2 - \alpha z)]\, f(z - \alpha),$$
$$\alpha_1 = \sqrt{2}\,\mathrm{Re}\,\alpha,\ \alpha_2 = \sqrt{2}\,\mathrm{Im}\,\alpha. \tag{2.25}$$

The operator $D(\alpha) = \exp(\alpha\hat{a}^+ - \bar{\alpha}\hat{a})$ is called the shift operator. Discrete bases are specified in the space of functions

$$\psi_{\omega n}(z) = (\omega^{1/2}z)^n/(n!)^{1/2},$$
$$\psi_{\omega n}(x) = H_n(\omega^{1/2}x)\exp(-\omega x^2/2)(\omega/\sqrt{\pi}2^n n!)^{1/2} \tag{2.26}$$

where H_n are the Hermite polynomials.

For the groups W(1), M(2), and M(1,1) there are also finite-dimensional nonunitary representations by triangular matrices.[7,14] The commutation relations can be realized not only by differential operators, but also by 2×2 or 3×3 matrices. We note that the theory of nonunitary representations is relatively complicated. For example, the set of vectors $(\omega^{1/2}z)^n$, where n is an integer, forms a reducible but not a decomposable representation of the W(1) algebra: while we cannot reach $n = -1$ from the state with $n = 0$ (application of $\hat{a}_-$ produces zero), the reverse transition is possible see Fig. 1b). This situation is also typical for other algebras.[17–20]

2.3. Clebsch–Gordon series and coefficients

The Clebsch–Gordon series specify the decomposition of the direct product of two irreducible representations into irreducible representations. For quantum-mechanical systems, the Clebsch–Gordan series provide us with information about the possible state of the system that arises when subsystems are combined. We begin with two simple examples. For the group SU(2)

$$D(j_1) \otimes D(j_2) = \sum_{\alpha=0}^{\min\{j_1,j_2\}} D(j_1 + j_2 - \alpha), \tag{2.27}$$

i.e., if the subsystems are characterized by angular momenta j_1 and j_2, the system as a whole can have angular momentum j that runs through values ranging from $|j_1 - j_2|$ to $j_1 + j_2$. For M(2) (Ref. 7)

$$D(p_1) \otimes D(p_2) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} D(p)d\varphi,$$
$$p^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\varphi. \tag{2.28}$$

Correspondingly, the absolute magnitude of the momentum p varies from $|p_1 - p_2|$ (momenta pointing in opposite directions) to $p_1 + p_2$ (momenta pointing in the same direction).

While the Clebsch–Gordan series show which states can arise when subsystems are combined, the Clebsch–Gordan coefficients define the probability amplitudes for finding the system in a given state for known states of the subsystems. The most extensively investigated example is that involving the Clebsch–Gordan coefficients $\langle j_1 m_1 | j_2 m_2 \| jm\rangle$ of the group SU(2). They are the coefficients of the redecomposition of the bases $D(j_1) \otimes D(j_2) \to D(j)$

$$|jm\rangle = \sum_{m_1,m_2} \langle j_1 m_1 | j_2 m_2 \| jm\rangle |j_1 m_1\rangle |j_2 m_2\rangle,$$
$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{jm} \langle jm \| j_1 m_1 | j_2 m_2\rangle |jm\rangle. \tag{2.29}$$

A detailed description of this can be found in many reviews and monographs, for example, those in Refs. 2 and 21–25. The Clebsch–Gordan coefficients are closely related to the Wigner coefficient $\langle j_1 m_1 | j_2 m_2 | j_3 m_3 \| 00\rangle$ which specifies the decomposition of an invariant over the product of the bases for three irreducible representations.

The Clebsch–Gordan coefficients of the group M(2) are defined by

$$|p_1 m_1\rangle |p_2 m_2\rangle = \int_0^{2\pi} \langle pm \| p_2 m_2 | p_1 m_1\rangle |pm'\rangle d\varphi,$$
$$|pm\rangle = \sum_{m_1,m_2} \langle p_1 m_1 | p_2 m_2 \| pm\rangle |p_1 m_1\rangle |p_2 m\rangle. \tag{2.30}$$

Their explicit form can be found in Ref. 7 where the relations given by (2.30) are looked upon as formulas defining multiplication and composition of Bessel functions; $\langle p_1 m_1 | p_2 m_2 \| pm\rangle = \exp[i(m_1\varphi - m\beta)]/2\pi$, $\exp(i\beta) = [p_1 + p_2\exp(i\varphi)]/p$ for $m = m_1 + m_2$, and are equal to zero for $m \neq m_1 + m_2$. The integral with respect to φ can be replaced with an integral with respect to p between $|p_1 - p_2|$ and $p_1 + p_2$ with $d\varphi$ replaced by $2pdp[4p_1^2p_2^2 - (p^2 - p_1^2 - p_2^2)^2]^{-1/2}$ in accordance with (2.28). The factor in the denominator is equal to four times the area of the triangle with sides p_1, p_2, p.

When the IR of groups were constructed above, we used Lie algebras of these groups, i.e., the infinitesimal approach. To construct the Clebsch–Gordan series and coefficients, it is convenient to use group invariance, namely, the so-called method of generating invariants. This method has been applied by van der Waerden to SU(2) (see also Ref. 25) and was subsequently developed in Refs. 11, 12, and 26 for semisimple (both compact and noncompact) Lie groups.

The method of generating invariants is based on the decomposition of functions of group invariants over polyno-

mial bases of irreducible representations such as (2.21), (2.22), and (2.26). We begin with SU(2) and SU(1,1) whose invariants are the convolution and the determinant

$$(u,\bar{v}) \equiv u_1\bar{v}_1 \pm u_2\bar{v}_2,\ (u,v) = u_1v_2 - u_2v_1. \tag{2.31}$$

Decomposing the SU(2) invariant $(u,v)^{2j}$ over the basis (2.21)

$$\rho(u,v)^{2j} = \sum_m (-1)^m \psi_{jm}\psi_{jm},$$

we obtain the Clebsch–Gordan coefficient $\langle jm|jm\|00\rangle = (-1)^m\rho$, where the normalizing factor is $\rho = [\dim D(j)]^{-1/2} = (2j+1)^{-1/2}$. The generating invariant $\langle j_1u|j_2v\|jw\rangle$ for the product of two irreducible representations of SU(2) [SU(1,1)] is given by

$$\rho(M_1, M_2, \alpha)(uv)^{\alpha}(u\bar{w})^{M_1-\alpha}(v\bar{w})^{M_2-\alpha} \tag{2.32}$$

where $M_1 = 2j_1, M_2 = 2j_2, \alpha = 2j_1 + 2j_2 - 2j$. Decomposition of (2.32) yields the following expression (to within an arbitrary coefficient):

$$u_1^{n_1+\alpha'}u_2^{M_1-\alpha_1-n_1+\alpha''}v_1^{n_1'+\alpha''}v_2^{M_2-\alpha-n_1'+\alpha'}\bar{w}_1^{M_1+M_2-2\alpha-n_1-n_1'}\bar{w}_2^{n_1+n_1'}, \tag{2.33}$$

$(\alpha' + \alpha'' = \alpha)$, i.e., a polynomial in u, v, and $\bar{w}$ of degree M_1, M_2, $M_1 + M_2 - 2\alpha$, respectively. The transition $D(M_1/2) \otimes D(M_2/2) \to D[(M_1 + M_2 - 2\alpha)/2]$ thus corresponds to the generating invariant (2.32).

For compact groups, the basis vectors contain the variables u, v, w with only nonnegative integral powers, i.e., only integral positive powers of the invariants are possible. This leads to the series (2.27) for the finite-dimensional irreducible representations $D(j)$ of SU(2). In the case of unitary IR of noncompact groups, the powers can in general be arbitrary complex numbers; in particular, for a discrete SU(1,1) series, one of the powers is negative, and we have to use negative nonintegral powers of the invariants.[10,11]

For the product $D^+(M_1/2) \otimes D^+(M_2/2)$, if we compare the product of the bases $(u_1^{n_1}u_2^{M_1-/n_1})(v_1^{n_1'}v_2^{M-n_1'})$ with the expression given by (2.33) we find that α should be a nonnegative integer since otherwise u_1 or v_1 will appear in (2.33) with a negative power. Hence we obtain the following expression for the Clebsch–Gordan series:[11,14]

$$D^+(j_1) \otimes D^+(j_2) = \sum_{\alpha=0}^{\infty} D^+(j_1 + j_2 - \alpha). \tag{2.34}$$

Similarly, for $j_1 + j_2 < -1/2$

$$D^+(j_1) \otimes D^0(j_2) = \sum_{\alpha=0}^{2j_2} D^+(j_1 + j_2 - \alpha). \tag{2.35}$$

The product $D^+(j_1) \otimes D^-(j_2)$ can be decomposed into the direct integral of irreducible representations because α need not be an integer.

The C–G coefficients are obtained as the coefficients of the normalized product of basis functions $\psi_{j_1m_1}\psi_{j_2m_2}\bar{\psi}_{jm}$ in the decomposition of the generating invariant (2.32). For a discrete positive series D^+ of SU(1,1), they are found to be equal to the SU(2) coefficients, apart from the change in the parameters:[27,28]

$$\langle j_1^+m_1|j_2^+m_2\|j^+m\rangle_{SU(1,1)} = (-1)^{j_2+m_2}$$
$$\times\left\langle \begin{matrix}(m-j_1+j_2-1)/2\\(m_1-m_2-j_1-j_2-1)/2\end{matrix}\middle|\begin{matrix}(m+j_1-j_2-1)/2\\(m_2-m_1-j_1-j_2-1)/2\end{matrix}\middle|\begin{matrix}-j-1\\-j_1-j_2-1\end{matrix}\right\rangle;$$
$$j<0,\ m=-j,\ -j+1,\ -j+2\ldots \tag{2.36}$$

Since α in (2.34) runs through an infinite number of values, the finite sum in one of the normalization conditions must be replaced by an infinite sum [cf. (2.29)]:

$$\sum_{\alpha=0}^{\infty} |\langle j_1m_1|j_2m_2\|j_1 + j_2 - \alpha, m\rangle|^2 = 1,$$
$$\sum_{m_1+m_2=m} |\langle j_1m_1|j_2m_2\|jm\rangle|^2 = 1. \tag{2.37}$$

We note that for integral and half-integral j_1, j_2, j the parameters of the SU(2) Clebsch–Gordan coefficients in (2.36) are also integers or half-integers.

The product $D(\omega_1) \otimes D(\omega_2)$ of irreducible representations of the Heisenberg group is a multiple of the irreducible representation $D(\omega_1 + \omega_2)$ (with infinite multiplicity) for $\omega_1 + \omega_2 \neq 0$; for $\omega_1 + \omega_2 = 0$, we obtain the representation that splits into one-dimensional irreducible representations.[14]

Strange though it may be, the theory of C–G coefficients of the group W(1) (i.e., the quantum theory of composition of oscillators) is still in its infancy, whereas the analogous SU(2) theory is discussed not only in textbooks of quantum mechanics, but has even had specialized monographs devoted to it. To fill this gap, at least partly, we turn to the examination of the invariant of the W(1) group. This invariant can be found as the overlap of coherent states (see Sec. 4 for further details):

$$\langle \omega u|\omega v\rangle = \exp(-\omega J),\ J = -\bar{u}v + (|u|^2/2) + (|v|^2/2). \tag{2.37'}$$

The real part of the invariant is the separation between points on the complex (phase) plane, $2\,\mathrm{Re}\,J = |u - v|^2$ [this invariant is analogous to the invariant of the group of motions of a plane M(2)], while the imaginary part is twice the area S_Δ of a triangle with vertices at the origin and at the points v and u, i.e., $\mathrm{Im}\,J = 2S_\Delta$ (Fig. 3).

The generating invariant $\langle \omega_1u|\omega_2v\|\omega_1 + \omega_2w\rangle$ takes the form

$$\exp\{\omega_1[u\bar{w} - (|u|^2/2) - (|w|^2/2)]\}$$
$$\times \exp\{\omega_2[v\bar{w} - (|v|^2/2) - (|w|^2/2)]\}. \tag{2.38}$$

Expanding the exponentials into series in powers of u, v, and $\bar{w}$, and separating out the bases (2.26), we obtain the C–G coefficient of the special form

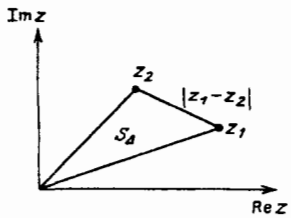

FIG. 3. Invariants of the Heisenberg group: distance $|z_1 - z_2|$, area of triangle S_Δ.

$$\langle \omega n_1 | \omega n_2 \| \omega_1 + \omega_2, n_1 + n_2 \rangle = \left[\frac{\omega_1^{n_1} \omega_2^{n_2}}{(\omega_1 + \omega_2)^n} \frac{(n_1 + n_2)!}{n_1! n_2!} \right]^{1/2}, \tag{2.39}$$

$$\sum_{n_1+n_2=n} |\langle \omega_1 n_1 | \omega_2 n_2 \| \omega_1 + \omega_2, n \rangle|^2 = 1,$$

$$|\omega_1 + \omega_2, n \rangle = \sum_{n_1+n_2=n} \langle \omega_1 n_1 | \omega_2 n_2 \| \omega_1 + \omega_2, n \rangle | \omega_1 n_1 \rangle | \omega_2 n_2 \rangle. \tag{2.40}$$

Replacing $|\omega n\rangle$ with the bases $\psi_{\omega n}(x)$ of (2.26), we obtain the following series for the Hermite polynomials in terms of products of these polynomials:

$$\frac{H_n((\omega_1 + \omega_2)x)}{(n!)^{1/2}} = \sum_{n_1+n_2=n} \left[\frac{\omega_1^{n_1} \omega_2^{n_2} n!}{(\omega_1 + \omega_2)^n n_1! n_2!} \right]^{1/2} \frac{H_{n_1}(\omega_1 x)}{(n_1!)^{1/2}} \frac{H_{n_2}(\omega_2 x)}{(n_2!)^{1/2}}. \tag{2.41}$$

The general C–G coefficients specify the recoupling of the bases of the irreducible representations $D(\omega)$ $|\omega_1 x_1 n_1\rangle$, $|\omega_2 x_2 n_2\rangle$, $|\omega_1 + \omega_2 x_3 n_3\rangle$, where $|\omega x_i n\rangle$ corresponds to the state $|\omega n\rangle$ shifted to the point x_i away from the origin.

The C–G coefficients of the above groups specify the physical characteristics that arise when subsystems are combined. They can also be used in scattering and decay problems. The theory of representations of these groups is characterized by many symmetries that find applications in different branches of physics.

We start by noting the symmetries that preserve commutation relations. For the Heisenberg group W(1) there are the canonical transformations

$$\hat{a}' = u\hat{a} + v\hat{a}^+, \quad (\hat{a}^+)' = \bar{v}a + \bar{u}\hat{a}^+, \quad |u|^2 - |v|^2 = 1. \tag{2.42}$$

These transformations belong to the group SU(1,1), i.e., the group of automorphisms of the W(1) algebra. For the semisimple groups SU(2) and SU(1,1), the transformations that preserve the commutation relations belong to SU(2) and SU(1,1), respectively.

There is considerable interest in the symmetry between operators and functions. In the Heisenberg group, this is the symmetry between functions of $\hat{a} = \mathrm{d}/\mathrm{d}z$ and $\hat{a}^+ = z$. According to (2.42), the latter transform in accordance with the representation $D(1/2)$ of the group SU(1,1). The operators $\hat{a}_1^+ = z_1$ and $\hat{a}_2^+ = z_2$ may be looked upon as the basis of the irreducible representation $D(1/2)$ of the group SU(2). The operators $\hat{a}_1 = \mathrm{d}/\mathrm{d}z_1$ and $\hat{a}_2 = \mathrm{d}/\mathrm{d}z_2$ then form the basis of the conjugate IR. In the same way that z_1 and z_2 are used to construct the symmetric basis (2.21), we can construct an operator basis for any representation $D(j)$ of SU(2) or SU(1,1). In particular, the generators (2.23) that transform in accordance with the associated irreducible representations $D(1)$ are constructed from z and $\mathrm{d}/\mathrm{d}z$ with the help of the C–G coefficients corresponding to $D(1/2) \otimes D(1/2) \to (1)$. We note that the set of operators $\hat{a}_1^+, \hat{a}_2^+, \hat{a}_3^+$ can be looked upon as the basis of the irreducible representation $D(10)$ of SU(3), and $\hat{a}_1, \hat{a}_2, \hat{a}_3$ as the basis of $D(01)$; on the other hand, the SU(3) generator may be regarded as the analog of (2.23) if we use the C–G coefficients corresponding to $D(10) \otimes D(01) \to D(11)$.

In the Dicke model,[29,30] a set of two-level systems is described by the representation $[D(1/2)]^n$ that decomposes into the irreducible representation $D(j)$ where j is the cooperative number representing the coherent properties of the set (for example, a set of radiating molecules). A similar construction can be developed for the operators $\hat{a}$ and $\hat{a}^+$ or $\hat{a}_1$ and $\hat{a}_2$ because they may be looked upon as the components of a spinor.

An arbitrary operator of a group can be expressed in terms of the sums of products of generators and, in principle, we can consider any functions of them, i.e., pseudodifferential operators. There is a symmetry between pseudodifferential and ordinary functions. It is reflected in the topology of the concepts of a chain and a cochain.[31]

Problems connected with the symmetry of C–G coefficients, including the Regge symmetry, their interpretation and applications are discussed in Secs. 5 and 6.

2.4. Optics as a topic in group theory

The development of the theory of fundamental groups has been closely coupled to problems in physics generally and in optics in particular. Initially, the starting point was the spherical symmetry of atoms. The IR of the group SU(2) and the theory of angular momenta were widely used in the analysis of atomic spectra.[32–34] It eventually transpired that the group properties arise not just from the symmetry of the radiating object, but are intrinsic to optics. Problems of coherence, interference, diffraction, polarization, image formation, radiation transport in a medium, and the interaction between light beams were gradually translated into the group language. The basis for this development of the group-theoretic approach was provided by algebraic or more particularly matrix methods which, after Ref. 35, became part and parcel of most branches of modern optics.[36–38]

The propagation of radiation is a typical example. The equation for a paraxial ray takes the form[38,39]

$$\begin{vmatrix} y_2 \\ V_2 \end{vmatrix} = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{vmatrix} y_1 \\ V_1 \end{vmatrix}, \tag{2.43}$$

where y is the height above the optical axis, $V = n \sin \vartheta$, n is the refractive index, and ϑ is the angle between the axis and a ray. For small angles, $V = n\vartheta$. The determinant of the ma-

trix is $AD - BC = 1$. With each element of an optical system we can associate its own unimodular matrix, and the system as a whole can be described by the product of the matrices representing elementary translations, refractions, and reflections. The group SL(2,R) which is isomorphous with SU(1,1) corresponds to the transformations defined by (2.43) with a unimodular matrix. In the absence of axial symmetry, in which case the position of the ray is specified by two coordinates and two angles, we have to consider the methaplectic group Sp(4,R) (Ref. 40).

The wave properties of light in paraxial optics are described in terms of the theory of Gaussian beams. A Gaussian beam describes coherent radiation with diffraction spreading. Its energy is concentrated in the axial region and falls off rapidly toward the periphery in accordance with the Gaussian function.

Because of diffraction, a Gaussian beam propagating in free space expands in such a way that the spot radius w and the radius of curvature R of the surface of constant phase undergo a slow variation. These two parameters can be combined in a complex curvature parameter defined by

$$q = \frac{1}{R} + \frac{i\lambda}{\pi w^2}$$

where λ is the wavelength. This new parameter satisfies the relation

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{2.44}$$

which is the analog of the relation for the real curvature parameter, namely, $1/R = y/V$ that follows from (2.43). The bilinear transformation defined by (2.44) specifies the parameters of the Gaussian beam after it passes through the optical system. It is often referred to as the $ABCD$ rule.[41] It is used in laser physics to calculate the properties of resonators. We note that a beam that has been multiply reflected between mirrors eventually become Gaussian.

A general analysis of group structures in paraxial optics, including methaplectic groups and their connection with the Poincare group is given in Ref. 42. Nonparaxial beams are also examined and a group treatment of abberations is given in Ref. 43. The representations of SU(1,1) and of its generalizations describe image transfer and scene recognition processes.

The second typical example is provided by the polarization properties of light, which are described by the coherence matrix[36]

$$G(\tau) = \left\| \begin{matrix} \langle E_x^*(t)E_x(t+\tau)\rangle & \langle E_x^*(t)E_y(t+\tau)\rangle \\ \langle E_y^*(t)E_x(t+\tau)\rangle & \langle E_y^*(t)E_y(t+\tau)\rangle \end{matrix} \right\|. \tag{2.45}$$

Any measured polarization variable can be expressed in terms of this matrix in accordance with relation $F = \mathrm{Sp}(\hat{F}G)$. The spectral matrix with elements

$$\mathscr{R}_{ik}(\nu) = \sum_{-\infty}^{\infty} G_{ik}(\tau)\exp(i2\pi\nu\tau)\mathrm{d}\tau \tag{2.46}$$

can be expanded in terms of the Pauli matrices

$$\mathscr{R} = \frac{1}{2}\sum_{i=0}^{3} s_i\sigma_i,$$

where the Stokes parameters

$$s_i = \mathrm{Sp}(\sigma_i\mathscr{R}) \tag{2.47}$$

specify the intensity s_0 of the wave and the spectral polarization states (s_1, s_2, s_3). They satisfy the conditions

$$4\det\mathscr{R} = s_0^2 - s_1^2 - s_2^2 - s_3^2 \geq 0,$$
$$\mathrm{Sp}\,\mathscr{R} = s_0 > 0.$$

The transformations of the Stokes parameters (2.47) belong to the group SL(2,C) and, when $s_0 = \mathrm{const}$, to the group SU(2). The description of the polarization properties of quasimonochromatic, but not plane, waves involves the group SU(3) (Refs. 44 and 45), and the superposition of n light beams involves the group SU(n) (Ref. 46).

A variation in the polarization states at constant wave intensity s_0 can be treated as the motion of a mapping point on a Poincaré sphere of radius $s_1^2 + s_2^2 + s_3^2$. The statistical properties of a light beam were investigated in Ref. 47 by the method of random walks on a sphere, and a calculation was made of the distribution of trains over polarization states. Similarly to the random walk on an invariant manifold of the group SU(2), i.e., the Poincaré sphere, fluctuational changes in the parameters A, B, C, D during the propagation of radiation can be treated as a random walk on a hyperboloid.

The third example is that of the effects of correlation between radiators in problems involving the generation of radiation. In the Dicke model, cooperative, collective states of a system of two-level molecules are described by representations of SU(2). The cooperative number (signature of representation) remains unaltered during radiative transitions. Specific collective effects are discussed in Refs. 30 and 48 within the framework of the Dicke model. A system of nonequidistant n-level molecules is an object in SU(n) (Refs. 49 and 50).

The quantum description of a field is closely related to the Heisenberg group. Quantum electrodynamics is founded on the representation of the free electromagnetic field by an infinite number of harmonic oscillators. The electric and magnetic fields are then looked upon as operators in the space of states that describe the field, and are expressed in terms of the operators $\hat{a}$ and $\hat{a}^+$. States $|n\rangle$ with a precisely defined number of photons are the eigenvectors of the operator representing the number of photons $n = \hat{a}^+\hat{a}$, and form an orthonormal system for which the following completeness conditions are satisfied:

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1}.$$

From the quantum point of view, all the information about the statistics of the systems is contained in the density matrix ρ which can be expanded over these states:

$$\hat{\rho} = \sum_{n,m} \rho(n,m)|n\rangle\langle m|, \tag{2.48}$$

where the pure states are described by the matrix $\rho = |n\rangle\langle n|$. Since $|n\rangle$ transform in accordance with the representations of the group W(1), the matrix ρ transforms in accordance the representations of the group $W(1)\times W(1)$. Different statistical, probability distributions arise under different physical conditions for $\rho(n)$ (Refs. 36, 51, and 52).

For chaotic (Gaussian) radiation, characterized by maximum entropy $S = -\mathrm{Sp}(\hat{\rho}\ln\hat{\rho})$, we have the Bose–Einstein distribution

$$\rho(n) = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{1+n}}. \tag{2.49}$$

The only nonzero elements of the density matrix are then the diagonal elements.

For ideal coherent radiation, the distribution of recorded photon counts is given by the Poisson distribution

$$\rho(n) = \frac{\langle n\rangle^n}{n!}\exp(-\langle n\rangle). \tag{2.50}$$

Here group-theoretic problems are intertwined with statistical problems.

New possibilities have arisen in the investigation of the statistical properties of radiation following the advent of photon counting techniques and measurements of correlation functions, which began with the well-known work of Hanbury–Brown and Twiss.[53] Studies were made of the distribution function for photon counts under different conditions.[36,54]

The concept of the occupancy (of a state with a particular number of photons $|n\rangle$) provides no information about the phase of the field, but this knowledge is unnecessary when coherence is analyzed. Glauber[55] has formulated a quantum theory of optical coherence in a form close to the classical description. It is based on the representation of a coherent state in which the basis vectors $|z\rangle$ correspond to a superposition of $|n\rangle$ and depend on the complex variable z that is related to the field amplitude and phase:

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n}{(n!)^{1/2}}|n\rangle. \tag{2.51}$$

The field in the state $|z\rangle$ satisfies the condition of complete coherence. From the group-theoretic point of view, (2.51) is a transition to a new basis, i.e., quantum optics can be formulated either in the $|n\rangle$ representation or in the representation of coherent states.

The experimental confirmation of a new phenomenon, namely, that of squeezed states of light[56–58] is closely connected with a development of the theory of coherent states.

Studies of coherent states began in optics and have acted as a powerful stimulus to the development of group-theoretic investigations, including the theory of C–G coefficients and the exploration of the very close connection between probability and group-theoretic approaches. There is, however, another important point. There have been rapid advances in data processing methods,[59,60] and this has occurred in parallel with advances in group methods. Optical systems are now capable of performing, quite simply and naturally, the evaluation of Fourier transforms, convolutions, and Hilbert, Laplace, Mellin, and other transforms. The number of such operations is continually increasing and there has been a tendency to implement group-theoretic operations by optical methods or, in pictorial language, to create an experimental basis for group theory.

It is thus clear that current attempts to create a unified group-theoretic approach that would include simple Lie groups, the translation into the group language of an extensive range of physical problems, and the interweaving of probability and group-theoretic aspects are leading us to the re-examination of the fundamentals of quantum theory.

3. DISCRETE GENERALIZATIONS (FINITE DIFFERENCES AND QUANTUM ALGEBRAS)

3.1. Discrete calculus and finite differences

A natural generalization of the Lie algebras (2.1), in which $f(\hat{H}) = 2\hat{H}$, 0, $\hat{I}$, is provided by deformed algebras with arbitrary $f(\hat{H})$. In general, they are not Lie algebras, but they contain many of the specific features of the algebras (2.1). The problem will, however, will be treated below from the wider standpoint of discrete generalizations of the theory.

In a previous paper,[2] two of us examined the fundamentals of the relationship between C–G coefficients and the calculus of finite differences that arose from the fact that the objects treated in the theory of angular momenta (C–G coefficients, Racah coefficients, and jm and j symbols) are functions of discrete variables and can be subjected, for example, to the operations of finite-difference differentiation and integration. Further advances have taken place in this area during the last twenty years.

In addition, a new subject has emerged, namely, quantum algebras, which since the mid-1980s have attracted considerable attention. They originally appeared in publications on inverse-scattering problems in statistical physics and in quantum field theory.[61–65] The operators of the quantum algebra $su_q(2)$ satisfy the relations

$$[\hat{H},\hat{E}_\pm] = \pm\hat{E}_\pm,$$
$$[\hat{E}_+,\hat{E}_-] = \mathrm{sh}(h\hat{H})/\mathrm{sh}(h/2) = (q^{\hat{H}} - q^{-\hat{H}})/(q^{1/2} - q^{-1/2}). \tag{3.1}$$

According to the first pair of commutation relations, $\hat{E}_\pm$ are again the raising and lowering operators in the basis of the eigenfunctions of $\hat{H}$. If $q = \exp h$, then for $h\to 0$ we obtain the usual commutation relations of a Lie algebra and the SU(2) group. This property has indeed led to the designations "quantum algebras" and "quantum groups." Of course, this was initially no more than a new jargon. However, similarly to the concepts of strangeness and charm, which have nothing in common with the usual meaning of these words, the new terminology became accepted in the literature. In fact, we are dealing here with a special case of deformed algebras [with arbitrary $f(\hat{H})$].

It eventually became clear that quantum algebras are related to the so-called q-calculus which is a calculus of finite differences on an exponential (and nonuniform) net. Correspondingly, if the elements of the Lie algebras (2.1) are differential operators, then the elements of the quantum algebra (3.1) are finite-difference differential operators. In the

q-calculus, we see the surprising coexistence of century and a half old results[66–68] with some recent results.[69] The renewed interest in the q-calculus has also led to systematic studies of nonuniform nets for which there are specific concepts of number, difference function, and differentiation.

We begin by considering the relationships that apply to any finite-difference net. Three finite-difference differentiation operators are defined on such nets:

$$\hat{D}_x^+ f(x) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}, \quad \hat{D}_x^0 f(x) = \frac{f(x_{n+1/2}) - f(x_{n-1/2})}{x_{n+1/2} - x_{n-1/2}},$$
$$\hat{D}_x^- f(x) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{3.2}$$

For a step $h_n = x_{n+1} - x_n \to 0$ we have $\hat{D}_x^{(k)} \to \mathrm{d}/\mathrm{d}x$. The finite-difference integral is defined by

$$\int_{x_k}^{x_l} f(x)\,\mathrm{d}_{\mathrm{f.d.}}x = \sum_{n=k}^{l} f(x_n). \tag{3.3}$$

Analogs of the Leibnitz shift formulas (involving a shift in the argument) can be established for the derivatives, and the formulas for the differentiation of a complicated function take the form

$$\hat{D}_x^0(f_1(x)f_2(x)) = f_1(x_{n+1/2})\hat{D}_x^0 f_2(x) + f_2(x_{n-1/2})\hat{D}_x^0 f(x)$$
$$= f_1(x_{n-1/2})\hat{D}_x^0 f_2(x) + f_2(x_{n+1/2})\hat{D}_x^0 f_1(x),$$
$$\hat{D}_x^{(k)} f(\varphi(x)) = \hat{\tilde{D}}_\varphi^{(k)} f(\varphi)\hat{D}_x^{(k)} f(x), \tag{3.4}$$

where $\tilde{D}_\varphi$ is the derivative on the net $\varphi_k = \varphi(x_k)$. In the limit as $h \to 0$, we have the usual formulas that are familiar from continuous analysis.

Finite-difference functions are then defined on the net, namely, the right (shifted factorial) and left (simple difference, generalized, or symbolic) finite differences $(x)_n$ and $x^{(n)}$:

$$(x_l)_n = \prod_{k=0}^{n-1} x_{k+l}, \quad x_l^{(n)} = \prod_{k=-1}^{-n} x_{k+l}, \quad \lim_{h\to 0}(x)_n = \lim_{h\to 0} x^{(n)} = x^n,$$
$$(a + x)^{(n)} = a^n(1 + xa^{-1})^{(n)}. \tag{3.5}$$

For a uniform net, $x_k = a + kh$, $x^{(n)} = (x - nh)_n$, and

$$(x)_n = \prod_{k=0}^{n-1}(x + kh) = h^n\Gamma\left(\frac{x}{h} + n\right)/\Gamma\left(\frac{x}{h}\right),$$
$$x^{(n)} = \prod_{k=1}^{n}(x - kh) = h^n\Gamma\left(\frac{x}{h} + 1\right)/\Gamma\left(\frac{x}{h} - n + 1\right). \tag{3.6}$$

For integral x and $h = 1$, we have $(x)_n = (x + n - 1)!/(n - 1)!$, $x^{(n)} = x!/(x - n)!$. For an exponential net, $x_k = aq^k$, $h_k = aq^k(q - 1) = x_k(q - 1)$, it is usual to consider functions of the form

$$(x, q)_n \equiv (1 - x)_n = \prod_{k=0}^{n-1}(1 - xq^k),$$
$$(1 - x)^{(n)} = \prod_{k=-1}^{-n}(1 - xq^k) = (q^{-n}x, q)_n. \tag{3.7}$$

The sums Σx_n in the above cases represent sums of an arithmetic progression with a step h or a geometric progression with ratio q.

We now turn to the finite-difference calculus on a uniform net.[70–73] Here we encounter ordinary numbers x, the power function a^x, and the finite-difference power $x^{(n)}$:

$$(a + x)^{(n)} = \sum_k \frac{n!}{k!(n-k)!} x^{(k)} a^{(n-k)}, \quad a^{(m+k)}$$
$$= a^{(m)}(a - m)^{(k)} \tag{3.8}$$

where $x^{(n)}$ is specified by the Bernoulli polynomial for natural n and the coefficients of x^m in the polynomial are called Stirling numbers of the first kind whereas for the inverse expansion they are Stirling numbers of the second kind. The necessary expressions for these numbers are given in Ref. 73. The finite-difference derivative

$$\hat{D}_x^+ = \frac{f(x + h) - f(x)}{h} \tag{3.9}$$

is often used with simplified operator Δ_x for $h = 1$, the sth power of which is given by

$$\Delta_x^s f(x) = \sum_{k=0}^{s} (-1)^{s+k} \frac{s!}{k!(s-k)!} f(x + s). \tag{3.10}$$

Typical examples of finite-difference differentiation and integration are as follows:

$$\Delta_x x^{(n)} = nx^{(n-1)}, \qquad \Delta_x^s x^{(n)} = n^{(s)} x^{(n-s)},$$
$$\int_0^x t^{(n)} \mathrm{d}_h t = \frac{x^{(n+1)}}{n+1}, \qquad \Delta_x^s a^x = (a - 1)^s a^x.$$

There is a considerable degree of correspondence between continuous and discrete analysis. It is often sufficient to make the formal replacements

$$x^k \leftrightarrow x^{(k)}, \quad (\mathrm{d}/\mathrm{d}x)^s \leftrightarrow \Delta_x^s.$$

However the isomorphism is incomplete: for example, the formula for the differentiation of a product is different, and the multiplication rule for finite-difference powers has a different structure.

The solution of the finite-difference second-order hypergeometric equation is given by the function[74,75]

$${}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\middle| h, x\right) = {}_3F_2\left(\begin{matrix}\alpha, \beta, x \\ \gamma, \delta\end{matrix}; 1\right) = \sum \frac{(\alpha)_n(\beta)_n(x)_n}{(\gamma)_n(\delta)_n n!}, \tag{3.11}$$

which is a series in powers of $x^{(n)}$ where

$$\delta = \frac{1}{h} + \frac{\alpha + \beta + 1}{2}.$$

As $h \to 0$, this equation becomes the ordinary hypergeometric equation, and

$$ {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\middle| h, \frac{x}{h}\right) \xrightarrow[h\to 0]{} {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; x\right). \tag{3.12}$$

The relation given by (3.11) explains the appearance of hypergeometric functions ${}_mF_n$ of unit argument in discrete analysis. The generalization of a power can be used in a convenient notation for the generalized-geometric functions

$$ {}_pF_q\left(\begin{matrix}(a_i)_p \\ (b_i)_q\end{matrix}; x\right) = \sum_{h=0}^{\infty} \prod_{i=1}^{p} (a_i - 1 + n)^{(n)} \prod_{j=1}^{q} \frac{1}{(b_j + 1 + n)^{(n)}} \frac{z^n}{n!}.$$

These functions with arguments $z = \pm 1$ constitute the standard object studied in the calculus of finite differences. Their fundamental importance for finite-difference differentiation is demonstrated in Ref. 73. The operator Δ_x^s can be applied to the generalized power and to the power function, and this has resulted in a large set of relations of the form

$$\begin{aligned}
&\Delta_x^s(a - x)^{(\alpha)} = (-1)^s(a - x)^{(\alpha)}{}_2F_1\left(\begin{matrix}x + \alpha - a, s \\ x - a\end{matrix}; 1\right), \\
&\Delta_x^s[(a + x)^{(\alpha)}]^{-1} = (-1)^s[(a + x)^{(\alpha)}]^{-1}{}_2F_1\left(\begin{matrix}a + x - \alpha, s \\ a + x + 1\end{matrix}; 1\right), \\
&\Delta_x^s(b + x)^{(a+x)} = (-1)^s(b + x)^{(a+x)}{}_2F_0(b + x + 1, -s; 1), \\
&\Delta_x^s[(a + x)^{(b+2x)}] = (-1)^s(a + x)^{(b+2x)}{}_3F_0(b - x \\
&\qquad - a, a + x + 1, -s; -1).
\end{aligned} \tag{3.13}$$

In all these relations, the application of the finite-difference differentiation operator Δ_x^s to the corresponding function produces multiplication of the function by a generalized hypergeometric function with the argument $x = \pm 1$. The differentiation rules for more complicated functions are also given in Ref. 73. When ${}_pF_q$ is determined from formulas such as (3.13), the result is not unique. For example, in the case of the function ${}_3F_2$, there are three equivalent expressions, namely,

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix}a + x + 1, x - b + \beta, -s \\ x - b, a + x - \alpha + 1\end{matrix}; 1\right) \\
&\quad = [(a - x)^{(\alpha)}(b - x)^{(\beta)}]^{-1}(-1)^s\Delta_x^s[(a + x)^{(\alpha)}(b - x)^{(\beta)}], \\
&{}_3F_2\left(\begin{matrix}x - b, a + x - \alpha + 1, -s \\ x - b - \beta, a + x + 1\end{matrix}; 1\right) \\
&\quad = (a + x)^{(\alpha)}(b - x)^{(\beta)}(-1)^s\Delta_x^s[(a + x)^{(\alpha)}(b - x)^{(\beta)}], \\
&{}_3F_2\left(\begin{matrix}a + x + 1, b + x - \beta + 1, -s \\ a + x + 1 - \alpha, b + x + 1\end{matrix}; 1\right) \\
&\quad = (-1)^s\frac{(b + x)^{(\beta)}}{(a + x)^{(\alpha)}}\Delta_x^s\frac{(a + x)^{(\alpha)}}{(b + x)^{\beta}}.
\end{aligned} \tag{3.14}$$

Transitions between these expressions are performed by means of formulas such as (3.8). Several basic quantities in the theory of angular momenta are expressed in terms of ${}_mF_n$ with argument $x = 1$. For example, the C–G coefficient expressed in terms of ${}_3F_2$ is then identical with (3.14) apart from a normalizing factor, whereas the Racah coefficient is expressed in terms of ${}_4F_3$. A specific expression for the normalizing factor for the C–G coefficient $\langle j_1 m_1 | j_2 m_2 \| jm\rangle$ in the form given by (3.14) is given in Refs. 2 and 76. It is readily shown that in a composition of C–G coefficients, the factors in front of the finite-difference differentiation operator cancel out and all the operations performed in the theory of angular momenta are expressed in terms of Δ_x^s and the product of generalized powers.

The fact that the C–G coefficients and their compositions can be regarded as special objects in the calculus of finite differences is only one aspect of the interrelation between the two. Another involves the development of a systematic theory of angular momenta in discrete time. Many aspects of this problem are examined in Ref. 77 where finite-difference equations are constructed for discrete spherical functions and also analogs of the Schröndinger, Klein–Gordon, and Dirac equations. Apart from the evaluation of finite differences, there are also other possibilities for constructing discrete topologies, in particular, the dyadic[77,78] and the p-adic[79] coordinate systems. Specifically, the use of group methods and the development of the theory of angular momenta within the framework of a nonstandard calculus is illustrated below by the q-calculus.

3.2. The q-calculus

In the last few years, different aspects of the theory of quantum algebras (q-calculus) have attracted a rapidly growing literature (see, for example, Refs. 80–84). A modern presentation of the theory of the algebras $su_q(2)$ and $su_q(1,1)$ can be found in Ref. 85. The salient points of the q-calculus are briefly summarized below insofar as they are related to the q-analogs of number, power, exponent, and also finite-difference differentiation and expansion of functions.

According to (3.2), finite-difference operators on an exponential net take the form

$$\begin{aligned}
&\hat{D}_x^+ f(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad \hat{D}_x^- f(x) = \frac{f(x) - f(xq^{-1})}{(1 - q^{-1})x}, \\
&\hat{D}_x^0 f(x) = \frac{f(q^{1/2}x) - f(q^{-1/2}x)}{(q^{1/2} - q^{-1/2})x},
\end{aligned} \tag{3.15}$$

or more simply

$$\hat{D}_x^+ f(x) = \frac{f(x + \alpha x) - f(x)}{\alpha x}, \quad \alpha = q - 1. \tag{3.16}$$

It is clear that there is a similarity with the usual finite-difference derivative (3.8). However, in the latter case, the addition to x was independent, whereas here it is proportional to x. Under the deformation $\alpha \to 0$ ($q \to 1$) we have, as for $h \to 0$ in (3.9), $\hat{D}_x^{(k)} \to \mathrm{d}/\mathrm{d}x$ ($k = +, 0, -$). The integration operator in q-calculus takes the form of (3.3) where $x_n = aq^n$. In particular,

$$\begin{aligned}
&\int_0^a f(x)\mathrm{d}_q x = \sum_{n=0}^{\infty}(x_n - x_{n-1})f(x_n), \quad 0 < q < 1, \\
&\int_a^{\infty} f(x)\mathrm{d}_q x = \sum_{n=0}^{\infty}(x_{n+1} - x_n)f(x_n), \quad q > 1.
\end{aligned} \tag{3.17}$$

Let us apply the finite-difference q-differential operators to a power function:

$$\hat{D}_x^+ x^n = [n]_+ x^{n-1},\ \hat{D}_x^- x^n = [n]_- x^{n-1},\ \hat{D}_x^0 x^n = [n] x^{n-1}. \tag{3.18}$$

The quantity $[n]$ that appears in this expression is the analog of a number in q-calculus:

$$[n] = q^{-(n-1)/2}[n]_+ = q^{(n-1)/2}[n]_-$$
$$= \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \sum_{k=-(n-1)}^{n-1} q^{k/2}. \tag{3.19}$$

The quantity $[n]$ is invariant under the replacement $q \to q^{-1}$. In the limit as $q \to 1$, the q-numbers become ordinary numbers. We note the following properties of the numbers $[m]$:

$$[m \pm n] = q^{\pm\frac{n}{2}}[m] \pm q^{-\frac{m}{2}}[n] = q^{\mp\frac{n}{2}}[m] \pm q^{\frac{m}{2}}[n],$$
$$[n][n' + n''] - [n + n''][n'] = [n - n'][n''],\ [0]=[1] = 1. \tag{3.20}$$

The law of composition of q-numbers [polynomials such as (3.19)] follows from the requirement that their sum must produce polynomials of the same type; bearing in mind the fact that (3.20) is symmetric under the replacement $n \rightleftarrows m$, it be can rewritten in the form ($q = \exp h$)

$$[m + n] = (\mathrm{ch}\,\frac{nh}{2} \cdot \mathrm{sh}\,\frac{mh}{2} + \mathrm{ch}\,\frac{mh}{2} \cdot \mathrm{sh}\,\frac{nh}{2})/\mathrm{sh}\,\frac{h}{2}.$$

We note that this formula corresponds to the composition of 4-velocities in the special theory of relativity and that the 4-velocity itself is measured in terms of the "elementary" $\sinh(h/2)$, i.e., we have a particular kind of quantization of the Lobachevskiĭ space. At the same time, all the relationships of the q-algebra can be realized on orispheres of the Lobachevskiĭ space, which constitute a parabolic section of a hyperboloid, parallel to the light cone.

The factorial of a q-number is defined as follows:

$$[m]! = [1][2][3]\ldots[m],$$
$$[n]_+! = (q,q)_n(1-q)^{-n} = [n]!q^{\frac{n(n-1)}{2}} = [n]_-!q^{n(n-1)}. \tag{3.21}$$

and the q-analogs of exponential functions are defined by

$$e_q^{(k)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{(k)}!},\ \hat{D}_x^{(k)} e_q^{(k)}(ax) = a e_q^{(k)}(ax),\ k = +,0,-. \tag{3.22}$$

As $q \to 1$, $e_q^{(k)}(x) \to e^x$; the transition between the $e_q^{(k)}(x)$ is given by formulas of the form

$$\hat{D}^- e_q^0(x) = \sum_{n=0}^{\infty} \frac{(xq^{-1/2})^n}{q^{n^2/2}[n]!} = e_q^+(xq^{-1})$$

and the q-analog of the Taylor series[85] is

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^{(n)}}{[n]_+!} \hat{D}_x^{+n} f(x)\Big|_{x=a},$$

where $\hat{D}_x^{+n}$ is the nth finite-difference derivative and $(x-a)^{(n)}$ is the generalized difference power (3.7). The q-binomial theorem plays an important role in applications:

$$\sum_{n=0}^{\infty} \frac{(a,q)_n}{(q,q)_n} x^n = \frac{(ax,q)_\infty}{(x,q)_\infty}, \tag{3.23}$$

which has been quoted as due to a number of scientists, but seems to have been first mentioned by Rote in 1811. If we apply it to (3.22), we obtain $e_q^+[x/(1-q)] = 1/(x,q)_\infty$.

The expansion of a power of a quasibinomial is obtained with the help of (3.6) and (3.7), and is found to contain the q-analog of the binomial coefficient:

$$(b-x)^{(n)} = b^n(1 - \frac{x}{b})^{(n)} = \sum_{n_1=0}^{n} \frac{(q^{-n},q)_{n_1}}{(q,q)_{n_1}} x^{n_1} b^{n-n_1}$$
$$= \sum_{n_1=0}^{k} \frac{[n]!}{[n_1]![n-n_1]!}(-1)^{n_1} q^{-n_1(n_1+1)/2} x^{n_1} b^{n-n_1}. \tag{3.24}$$

There are also other far-reaching analogies with ordinary theory. Differences arise because many of the relationships, including (3.24), contain additional factors that depend on the deformation q.

The following q-analog of confluent hypergeometric functions plays a fundamental part:

$${}_n\Phi_m\left(\begin{matrix} a_1 \ldots a_n \\ b_1 \ldots b_m \end{matrix}\middle| q, x\right) = \sum_{r=0}^{\infty} \frac{(a_1,q)_r \ldots (a_n,q)_r}{(b_1,q)_r \ldots (b_m,q)_r} \frac{x^r}{(q,q)_r}. \tag{3.25}$$

It contains q-calculus fractorials instead of ordinary fractorials. The function ${}_2\Phi_1$ was introduced by Heine in 1847. It satisfies a second-order finite-difference equation.[66] By analogy with the case of a uniform net (3.12), we have

$${}_2\Phi_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}\middle| q, (q-1)x\right) \xrightarrow[q \to 1]{} {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; x\right).$$

Most of the formulas of the theory of ordinary hypergeometric functions remain valid for the q-analogs, e.g., the integral representations

$${}_2\Phi_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| q, x\right)$$
$$= \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)} \int_0^1 u^{a-1}(1-qu)^{c-a-1}(1-q^b xu)^{(-b)} d_q u,$$

where $\Gamma_q(n) = [n-1]_+!$ is the analog of the Γ-function. Similar functions on the exponential net, including q-polynomials, are discussed in Sec. 7.

The q-calculus as a whole has a closed and logical structure. We note that the equation

$$(a, q)_n = (a, q)_\infty/(aq^n, q)_\infty$$

or the analog of the integral representation of the Γ-function, can be used to extend the theory to arbitrary complex n.

3.3. Theory of angular momenta in the q-calculus

The q-calculus can be used as the starting point for a systematic development of q-deformed theory of angular momenta. (We note in passing that the phrase "quantum theory of angular momentum" could lead to confusion because this refers to the usual theory.)

We begin with the creation and annihilation q-operators $\hat{a}_q = \hat{D}^0_z$, $\hat{a}_q^+ = z$ that act in the space of the functions $f(z)$ of the complex variable z:

$$\begin{aligned} &[\hat{a}_q, \hat{a}_q^+] = \hat{D}^0_z z - z\hat{D}^0_z = \hat{T}_z, \\ &[\hat{a}_q^+, \hat{N}] = \hat{a}_q^+, \ [\hat{a}_q, \hat{N}] = -\hat{a}_q, \end{aligned} \tag{3.26}$$

where $\hat{N} = z\mathrm{d}/\mathrm{d}z$ is the particle number operator. The fundamentally new point is the emergence of the shift operator $\hat{T}_z$, $\hat{T}_z f(z) = f(q^{1/2}z)$. This is connected with the finite-difference character of $\hat{a}_q$, namely, the shift of the argument in the formula for the differentiation of a product (3.4):

$$\begin{aligned} \hat{D}^0_z(f_1(z)f_2(z)) &= f_1(q^{1/2}z)\hat{D}^0_z f_2(z) + f_2(q^{-1/2}z)\hat{D}^0_z f_1(z) \\ &= f_1(q^{-1/2}z)\hat{D}^0_z f_2(z) + f_2(q^{1/2}z)\hat{D}^0_z f_1(z). \end{aligned} \tag{3.27}$$

If we choose

$$|n\rangle_q = z^n/[n]!, \tag{3.28}$$

as our basis in the space of functions [cf. (2.26)], we obtain

$$\begin{aligned} &\hat{a}_q^+|n\rangle_q = [n+1]^{1/2}|n+1\rangle_q, \\ &\hat{N}|n\rangle_q = n|n\rangle_q, \ \hat{T}_z|z\rangle = q^{n/2}|n\rangle, \\ &\hat{a}_q|n\rangle_q = [n]^{1/2}|n-1\rangle_q, \ \hat{a}_q\hat{a}_q^+|n\rangle = z\hat{D}^0_z|n\rangle_q = [n]|n\rangle_q. \end{aligned} \tag{3.29}$$

The q-exponential (3.22) is the analog of coherent states:

$$|u\rangle_q = e^0_q(uz), \ \hat{D}^0_z|u\rangle_q = u|u\rangle_q, \ \hat{T}|u\rangle_q = |q^{1/2}u\rangle_q. \tag{3.30}$$

In the limit as $q \to 1$, we have $\hat{a}_q^+ \to \hat{a}^+$, $\hat{a}_q \to \hat{a}$, $\hat{T}_z \to 1$, and all the formulas in (3.28)–(3.30) become identical with the usual formulas for the group W(1).

To examine $su_q(2)$, we take the operators and the IR basis $D(j)$ in a form analogous to (2.23) and (2.21):

$$\hat{E}_q^+ = z_1\hat{D}^0_{z_2}, \ \hat{E}_q^- = z_2\hat{D}^0_{z_1}, \ 2\hat{H} = z_1\frac{\mathrm{d}}{\mathrm{d}z_1} - z_2\frac{\mathrm{d}}{\mathrm{d}z_2}, \tag{3.31}$$

$$|jm\rangle_q = \left(\frac{[2j]!}{[j+m]![j-m]!}\right)^{1/2} z_1^{j+m}z_2^{j-m}. \tag{3.32}$$

In this realization, the space of the representation $D(j)$ is the space of polynomials of z_1, z_2 of degree $2j$:

$$\begin{aligned} &\hat{E}_q^\pm|jm\rangle_q = ([j\mp m][j\pm m+1])^{1/2}|j, m\pm 1\rangle_q, \\ &\hat{H}|jm\rangle_q = m|jm\rangle_q. \end{aligned} \tag{3.33}$$

It is readily verified that the commutation relations given by (3.1) are satisfied. As $q \to 1$, we have the usual theory of angular momentum. The Casimir operator $su_q(2)$ is[86]

$$\begin{aligned} &\hat{C}_2 = \hat{E}_q^-\hat{E}_q^+ + \left[\hat{H} + \frac{1}{2}\right]^2 = \hat{E}_q^+\hat{E}_q^- + \left[\hat{H} - \frac{1}{2}\right]^2, \\ &\hat{C}_2|jm\rangle = \left[j + \frac{1}{2}\right]^2, \ \hat{C}_2 \xrightarrow[q\to 1]{} j(j+1) + \frac{1}{4} \end{aligned} \tag{3.34}$$

and the q-analog of the C–G coefficient is defined in the standard fashion:

$$|j_1j_2jm\rangle_q = \sum_{m_1,m_2}\langle j_1m_1|j_2m_2\|jm\rangle_q|j_1m_1\rangle_q|j_2m_2\rangle_q. \tag{3.35}$$

However, the problem of composition of angular momenta for $su_q(2)$ has certain specific features. We have to establish the rule for the effect of $\hat{E}_q^\pm$ in the basis $|j_1m_1\rangle \otimes |j_2m_2\rangle$. As in the case of the formula for the differentiation of a product, given by (3.27), the finite-difference character of the operators $\hat{E}_q^\pm$ leads to a complication of the picture: instead of the usual tensorial product of irreducible representations, we have the coproduct[80,85]

$$\begin{aligned} &\hat{E}^\pm_{(1,2)} = \hat{E}^\pm_{(1)} \otimes q^{\frac{\hat{H}}{2}(2)} + q^{-\frac{\hat{H}}{2}(1)} \otimes \hat{E}^\pm_{(2)}, \\ &\hat{H}_{(1,2)} = \hat{H}_{(1)} \otimes 1 + 1 \otimes \hat{H}_{(2)}. \end{aligned} \tag{3.36}$$

In the usual theory, $\hat{E}^\pm_{(12)} = \hat{E}^\pm_{(1)} \otimes 1 + 1 \otimes \hat{E}^\pm_{(2)}$, whereas for $su_q(2)$ this operator differs from (3.36) in that it does not satisfy the commutation relations in (3.1). The formulas given by (3.36) can be obtained by applying the finite-difference differentiation operators (3.31) to the product $f_1(z) \otimes f_2(z) = f_1(q^{-1/4}\cdot z) \times f_2(q^{1/4}\cdot z)$.

If we apply $\hat{E}^\pm_{(12)}$ to (3.35), we obtain the recurrence relations

$$\begin{aligned} &([j\mp m][j\pm m+1])^{1/2}\langle j_1m_1|j_2m_2\|jm\rangle_q \\ &\quad = q^{\frac{m_2}{2}}([j_1\mp m_1][j_1\pm m_1+1])^{1/2} \\ &\quad \times \langle j_1m_1\pm 1|j_2m_2\|jm\pm 1\rangle_q \\ &\quad + q^{-\frac{m_1}{2}}([j_2\mp m_2][j_2\pm m_2+1])^{1/2} \\ &\quad \times \langle j_1m_1|j_2m_2\pm 1\|jm\pm 1\rangle_q, \end{aligned} \tag{3.37}$$

that can be used to find the C–G coefficients. In particular,

$$\langle jm|j-m\|00\rangle_q = (-1)^{j-m}q^{\frac{m}{2}}/[2j+1]^{1/2}. \tag{3.38}$$

We have examined the theory of representations of the algebra $su_q(2)$. The group-theoretic aspect is still not entirely clear at present. The C–G coefficients have been used to establish the q-analog of the matrices of finite transformations in $su_q(2)$, which have very unusual properties.[80,85] We

shall now use the explicit form of the coefficients (3.38) and the states $|jm\rangle$ given by (3.32) to find the invariants of $su_q(2)$. According to (3.35),

$$\left\langle \frac{1}{2}\frac{1}{2}\left|\frac{1}{2}-\frac{1}{2}\right\|00\right\rangle\left|\frac{1}{2}\frac{1}{2}\right\rangle\left|\frac{1}{2}-\frac{1}{2}\right\rangle + \left\langle \frac{1}{2}-\frac{1}{2}\left|\frac{1}{2}\frac{1}{2}\right\|00\right\rangle\left|\frac{1}{2}-\frac{1}{2}\right\rangle\left|\frac{1}{2}\frac{1}{2}\right\rangle = |00\rangle = \text{inv},$$

and, if we denote the basis functions of the irreducible representations $D_{(1)}(1/2)$ and $D_{(2)}(1/2)$ by z_1, z_2 and u_1, u_2, we obtain the q-deformed analog of the determinant (2.31):

$$z_1 u_2 q^{1/4} - u_1 z_2 q^{-1/4} = \text{inv}, \tag{3.39}$$

which together with the convolution $z_1\bar{u}_1 + z_2\bar{u}_2$ is an invariant of $su_q(2)$. We note that the symmetry $u \leftrightarrow z$ is combinatorial in character: we must also replace q with q^{-1}.

The C–G series and coefficients can be obtained not only with the help of recurrence relations, but also by the method of generating invariants. This involves the use of generalized (finite difference) powers and formulas of the form given by (3.24). The analog of the CS overlap (see Sec. 4) is the generalized power of the invariant (convolution), whereas the above basis functions (3.28) and (3.32) are the overlaps of the CS and states in a discrete basis: $\langle z|n\rangle$, $\langle jz_1z_2|jm\rangle$.

There is an extensive literature[80,82,85,87] on the construction of specific q-analogs. Such analogs have been obtained for the C–G coefficients, the Racah coefficients, the 6j-symbols, and many other objects in the theory of angular momenta, and their properties such as symmetry, orthogonality, recurrence relations, and asymptotic behavior have been investigated. The C–G coefficient in the van der Waerden form is given by[88]

$$\begin{aligned}\langle j_1m_1|j_2m_2\|jm\rangle_q &= \Delta(j_1,j_2,j)q^A([j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]!\\ &\times [j+m]![j-m]![2j+1])^{1/2}\\ &\times \sum_\alpha (-1)^\alpha q^{B(\alpha)}([\alpha]![j_1-j_2-j-\alpha]!\\ &\times [j_1-m_1-\alpha]![j_2+m_2-\alpha]!\\ &\times [j_1-j_2+m_1-\alpha]![j-j_1-m_2+\alpha]!)^{-1},\end{aligned} \tag{3.40}$$

where the sum over α is evaluated so that none of the factorials have a negative argument and

$$\begin{aligned}\Delta(abc) &= ([-a+b+c]!\\ &\quad\times [a-b+c]![a+b-c]!/[a+b+c+1]!)^{1/2},\\ A &= \{j_1(j_1+1)+j_2(j_2+1)-j(j+1)\\ &\quad + 2(j_1j_2+j_1m_2-j_2m_1)\}/4,\\ B(\alpha) &= -\alpha(j_1+j_2+j+1)/2.\end{aligned}$$

Apart from the fact that ordinary numbers are replaced with q-numbers $[a]$, and additional factors of the form q^A are found to appear, the resulting formulas are identical with the standard SU(2) formulas. This is also typical for the theory of angular momenta in the q-calculus as a whole. Complete agreement with ordinary theory (subject to the replacement $a \to [a]$) occurs for the orthogonality formulas (5.11) and the 6j-symbols:

$$\begin{aligned}\begin{Bmatrix} abc \\ def \end{Bmatrix} &= \Delta(abe)\Delta(acf)\Delta(cde)\Delta(dbf)\sum_\alpha (-1)^\alpha[\alpha+1]!\\ &\times([\alpha-a-b-e]![\alpha-a-c-f]![\alpha-b-d-f]![\alpha-d-c-e]!\\ &\times[a+b+c+d-\alpha]![a+d+e+f-\alpha]![b+c+e+f-\alpha]!)^{-1}.\end{aligned} \tag{3.41}$$

Like many other quantities in the theory of angular momenta, the 6j-symbol of the q-calculus is invariant under $q \to q^{-1}$ [cf. (3.1)]. For the C–G coefficient, the symmetry under this replacement is combinatorial in character:

$$\langle j_1m_1|j_2m_2\|jm\rangle_q = (-1)^{j_1+j_2-j}\langle j_2m_2|j_1m_1\|jm\rangle_{q^{-1}}. \tag{3.42}$$

All the usual 144 symmetries, including the Regge symmetry, hold for the q-analogs of the 6j-symbols. However, factors of the form q^A appear in compositions of 6j-symbols, and invariance under $q \leftrightarrow q^{-1}$ is lost. For example, this applies to the 9j-symbol written in the form of a sum of products of three 6j-symbols.[85]

The q-theory of angular momenta preserves the formula relating the C–G coefficients and the 6j-symbols on the one hand and the q-analogs of confluent hypergeometric functions on the other. The C–G coefficient is given by[85]

$$\begin{aligned}\langle j_1m_1|j_2m_2\|jm\rangle_q &= (-1)^{j_1-m_1}q^B\frac{\Delta(j_1j_2j)[j+j_2-m_1]!}{[j_1-j_2+j]![j+j_2-j_1]!}\\ &\times\left(\frac{[j_1+m_1]![j_2-m_2]![j+m]![2j+1]}{[j_1-m_1]![j_2+m_2]![j-m]![j_2-j+m_1]!}\right)^{1/2}\\ &\times {}_3\Phi_2\left(\begin{matrix} m_1-j_1, j_1+m_1+1, m-j \\ m_1-j+j_2+1, m_1-j-j_2\end{matrix}\middle| q, q\right),\end{aligned} \tag{3.43}$$

where

$$B = \frac{1}{4}[j_2(j_2+1)-j_1(j_1+1)-j(j+1)+2m_1(m+1)].$$

The representation of the q-analog of the 6j-symbol in terms of ${}_4\Phi_3$ is identical with the usual form [cf. (5.24) in Ref. 2] if we put $a \to [a]$. The structure of the theory of angular momenta is thus largely preserved in q-calculus.

The new point now is the appearance of the gauge transformation (q-gauge). We have already shown that there are two equivalent variants of the formulas for the composition of q-numbers and the differentiation of a product, coproduct, and invariant, given by (3.20), (3.27) (3.36), and (3.39), respectively, and obtained by introducing the replacement $q \leftrightarrow q^{-1}$. The two variants can be used to form a linear combination. For example, the formula for the composition of q-numbers, given by (3.20), can be written in the form

$$[m+n] = (\beta q^{\frac{n}{2}} + (1-\beta)q^{-\frac{n}{2}})[m] + (\beta q^{-\frac{m}{2}} + (1-\beta)q^{\frac{m}{2}})[n],\ 0 \le \beta \le 1.$$

The gauge transformation is defined by the parameter β. The operations of differentiation, composition of q-numbers, and the Raccah coefficients remain unaltered under the transformations. However, the C–G coefficient does depend on the gauge. It is determined by the choice of the factor in the coproduct (3.36), and all the calculations for it must be performed for a single chosen β.

Physical applications of the q-calculus involve topics such quantum field theory,[64] the Yang–Baxter[65] equation, conformal field theory,[89] exactly-solvable problems in statistical physics,[61,62] and phenomenological models of nuclear and molecular rotational terms.[90] An increasing number of methodological applications present q-calculus generalizations of everything within reach, up to exclusive q-algebras.[91] The classification of the irreducible representations of q-deformed algebras is not very different from the usual classification. For example, for $su_q(1,1)$, there are also discrete, fundamental, and auxillary series, but there is also a further "strange" series of unitary irreducible representations.[85] There is considerable interest in the connection between q-functions on the one hand and elliptic theta-functions that do not satisfy the usual differential equations. This connection was noted as far back as 1847 by Heine.[66] The relevant formulas are reproduced in Chapters 4 and 11 of Ref. 92. We note finally the connection between the q-calculus and the theory of partitions.[93]

The quantum algebra (3.1) is a special case of deformed algebras with $f(\hat{H})$ in (2.1) specified by a power series. For example, it is possible to put $f(\hat{H}) = \sin(h\hat{H})/\sin(h/2)$, i.e., consider that h in (3.1) is purely imaginary. Apart from quantum algebras, deformed algebras also include the so-called W-algebras and Casimir algebras.[94–97] Possible applications of these algebras in quantum optics are discussed in Ref. 94. Casimir algebras include, for example, the algebra $su^{(n)}(1,1)$:

$$\hat{E}_+ = (\hat{a}^+)^n = z^n,\ \hat{E}_- = (\hat{a})^n = \frac{d^n}{dz^n},\ \hat{H} = \frac{\hat{a}^+\hat{a}}{n} = \frac{z}{n}\frac{d}{dz},$$
$$[\hat{H}, \hat{E}_\pm] = \pm\hat{E}_\pm,\ [\hat{E}_-, \hat{E}_+] = (n(\hat{H}+1))^{(n+1)} - (n\hat{H})^{(n)}, \tag{3.44}$$

since its Casimir operator $\hat{C}_2 = (n\hat{H})^{(n)} - \hat{E}_+\hat{E}_-$ is a deformation of the SU(1,1) Casimir operator [here (n) is the generalized finite-difference power defined by (3.8)]. The operators of this algebra are functions of the $W(1)$ generators, namely, mth order differential operators. The finite-difference derivatives on uniform and exponential nets are defined by

$$\begin{aligned}
&\hat{D}_z^+ f(z) = \frac{\exp(hd/dz) - 1}{h} f(z) = \frac{f(z+h) - f(z)}{h},\\
&\hat{D}_z^0 = \frac{\operatorname{sh}[(h/2)d/dz]}{h/2},\\
&\hat{D}_z^+ f(z) = \frac{\exp(hzd/dz) - 1}{(\exp h - 1)z} f(z) = \frac{f(qz) - f(z)}{(q-1)z},\\
&\hat{D}_z^0 = \frac{\operatorname{sh}[(hz/2)d/dz]}{2\operatorname{sh}(h/2)},
\end{aligned} \tag{3.45}$$

where $q = e^h$, and are also functions of d/dz, i.e., they are pseudodifferential operators.

Thus, in contrast to the theory of Lie algebras, in which generators can be represented by differential operators of order no higher than 1, in the theory of deformed algebras we deal with pseudodifferential operators that can be represented by integral operators; the kernel is then called the operator symbol. In addition to the power and exponential functions of d/dx, we can also consider other functions, for example, $\delta(d/dx)$. We can then use the usual representation of the δ-function as an integral of $\exp(ikx)$:

$$\int_{-\infty}^{\infty} \delta\left(\frac{d}{dx}\right) f(x)dx = \frac{1}{\sqrt{2\pi}}\iint f(z)d^2z. \tag{3.46}$$

Examination of physical systems with Hamiltonians consisting of operators belonging to q-algebras is equivalent to the examination of systems described by differential equations of infinite order. It is precisely equations of this type and the theory of representatons of Kac–Moody algebras that can be used in relation to certain problems in quantum field theory. In this sense, the results presented above can serve as a bridge to a generalization of existing theory (see also Sec. 6).

4. COHERENT STATES AND TRANSITION TO THE CLASSICAL LIMIT

4.1. Coherent states and uncertainty relations

A notable event in the development of the group-theoretic approach in physics was the introduction of the continuous bases for irreducible representations in addition to discrete bases. The point of departure for this was provided by studies in optics that involved the development of coherent sources (lasers) and a new method of detection of radiation.[53,98,99] This placed on the current agenda the transition from the representation of the electromagnetic field by a particular set of quantum states with precisely determined number of photons to the consideration of states with a given *phase*, i.e., an uncertain number of photons. In 1963, Glauber[55] showed that coherent phenomena could be described with the help of states introduced by Schrödinger[100] and describing nonspreading wave packets for quantum operators. He examined such states in detail, and called them coherent states. In the 1960s, coherent states became the subject of careful analysis and were used in many physical applications (cf. the books by Klauder and Sudarshan,[51] Malkin and Man'ko,[101] and the collection of papers in Ref. 102).

Among the many remarkable properties of coherent states, we single out two particularly important ones. First, the coherent states $|z\rangle$ are functions of a continuous variable, i.e., a complex number z, and not a set of eigenvectors of diagonal operators of a Lie algebra. They constitute a complete though not orthogonal system over which any vector can be expanded. The system becomes orthogonal only in the classical limit. Second, coherent states have minimum uncertainty and can be treated as quantum states that are maximally close to classical states. Coherent states can therefore be used in the transition to the classical limit, in which quantum formulas assume their classical form.

The coherent states introduced by Schrödinger and Glauber were found to be closely related to the Heisenberg group. Radcliff[103] has constructed the coherent states for the group SU(2). In 1972, Perelomov[104] put forward a de-

finition of generalized coherent states as states that are arise when the representation operator is applied to some fixed vector $|z_0\rangle$ in the space of the representation. The usual coherent states correspond to the choice of the vacuum vector $|0\rangle$ as the vector $|z_0\rangle$. For an arbitrary group G, the generalized coherent states CS$|z_g$ of the representation $T(g)$ $(g\in G)$ are defined by

$$|z_g\rangle = T(g)|z_0\rangle. \tag{4.1}$$

In other words, the system of generalized coherent states is an orbit of the element $|z_0\rangle$ of Hilbert space in which the representation $T(g)$ acts. For each irreducible representation we have a set of such orbits and the properties of the system of generalized coherent states depend on the choice of z_0 (Ref. 13). The definition given by (4.1) signifies that the generalized coherent states $|z\rangle$ are transformed under finite transformations to $|z'\rangle$, and not to a superposition of them, which is in contrast to the states of the orthogonal basis. Thus, for SU(2), the state with $j = m = 1/2$ transforms under rotation to the superposition of states with $m = \pm 1/2$. This is one of the main differences between the CS basis generated by finite transformations of a group and the infinitesimal (discrete) basis produced by the generators of a Lie algebra.

For a given state $|z_0\rangle$ we can find a stationary subgroup $H = \{h\}$, i.e., a set of elements of a group G whose application changes only the phase or a numerical factor. For a stationary compact subgroup,

$$T(h)|z_0\rangle = \exp(i\alpha(h))|z_0\rangle, \tag{4.2}$$

while the factor $\exp[\alpha(h)]$ arises in the case of a noncompact subgroup. The generalized CS associated with $|z_0\rangle$ are parametrized by points of the factor-space G/H whose dimensionality is equal to the difference between the numbers of parameters of G and H. They can be looked upon as points on a surface defined by group invariants.

To construct the CS as quantum states that are maximally close to classical states, we must select from the different systems of generalized CS (orbits) the states with minimum uncertainty. Initially, the criterion for this relied on taking $|z_0\rangle$ as the IR of highest weight or the selection of states with a maximum stationary subalgebra.[104,105] However, these approaches were unsuccessful. The former is unsuitable even in the case of representations of the principal and auxillary SU(1,1) series which, in general, are not limited by the highest weight; when the latter is applied to the group W(1), it produces not only the coherent states, but also other systems of generalized coherent states (squeezed states). The direct use of uncertainty relations would therefore appear to be the best way to procede. There are two types of such relations between the root mean square deviations ΔT_a, $(\Delta T_a)^2 = \langle \hat{T}_a^2\rangle - \langle \hat{T}_a\rangle^2$, where $\hat{T}_a$ are the generators of the group G.

The first type (discussed in detail in the review paper of Ref. 106) includes relations containing products of uncertainties in terms of different variables, for example,

$$\mathrm{W}(1):\ \Delta x\cdot\Delta p \geq \hbar/2, \tag{4.3}$$

$$\mathrm{SU}(2):\ \Delta J_x\cdot\Delta J_y \geq |\langle \hat{J}_z\rangle|\hbar/2. \tag{4.4}$$

However, these relations are unsuitable for the determination of the coherent states of the corresponding groups because states that minimize them can include states with arbitrarily large uncertainties. For example, in (4.3), the equality can be achieved for $\Delta x\to\infty$ and $\Delta p\to 0$, whereas in (4.4) we obtain an identity in all states with a definite component J_x of the angular momentum, irrespective of the dependence on ΔJ_y and ΔJ_z. Moreover, like the analogous relations for other semisimple groups, (4.4) is invariant under the finite transformations $T(g)$ of the group. On the other hand, according to (4.1), the CS defined as a special case of a set of generalized CS are invariant under finite transformations $T(g)$.

A consistent definition of coherent states can be based on relations containing the sum of squares of uncertainties

$$\mathrm{W}(1):\ \left(\frac{\Delta x}{x_0}\right)^2 + \left(\frac{\Delta p}{p_0}\right)^2 \geq 1,\ x_0^2 = \frac{\hbar}{m\omega},\ x_0p_0 = \hbar, \tag{4.5}$$

$$\mathrm{SU}(2):\ (\Delta \mathrm{J})^2 = (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j\hbar^2. \tag{4.6}$$

The development of this approach is due to Delbourgo,[107,108] who derived relations such as (4.6) for compact semisimple groups, and to Gitman and Schelepin[15,109] who considered (4.5) for the Heisenberg group and the analogous relations for SU$(N,1)$ and SU(N). We note that (4.5) can be written in the form

$$\langle \hat{a}^+ a\rangle - \langle \hat{a}^+\rangle\langle a\rangle \geq 0. \tag{4.7}$$

The equality is achieved if and only if the states over which the average is evaluated are Glauber coherent states (eigenstates of $\hat{a}$).

The uncertainty relations given by (4.5) and (4.6) involve the invariant variance that is unaffected by finite transformations in the group to states over which the average is evaluated. The relations themselves are also invariant. The measure of uncertainty such as (4.6) can be written for compact semisimple groups in the following general form:[15,108]

$$\Delta C_2 = g^{ab}(\langle \hat{T}_a\hat{T}_b\rangle - \langle \hat{T}_a\rangle\langle \hat{T}_b\rangle) \geq 0, \tag{4.8}$$

where $C_2 = g_{ab}\hat{T}_a\hat{T}_b$ is the quadratic Casimir operator, $\hat{T}_a$ are the generators, and g^{ab} is the Cartan–Killing metric tensor. For noncompact groups, the ΔC_2 given by (4.8) is also a measure of uncertainty. However, in this case, since ΔC_2 can be negative, we must start with $|\Delta C_2| = \min$ when we determine the coherent states. Thus, for the irreducible representations of the SU(1,1) discrete series,

$$\Delta C_2 = (\Delta \mathrm{J})^2 = \langle \mathrm{J}^2\rangle - \langle \mathrm{J}\rangle^2 = j(j+1) - m^2 \leq j < -1/2.$$

It is readily verified that ΔC_2 is an invariant variance. For example, if we transform from the Cartan–Weyl basis of the infinitestimal algebra $\hat{H}_i, \hat{E}_\alpha, \hat{E}_{-\alpha}$ to a new set of Hermitian and anti-Hermitian operators $\hat{T}_a$ consisting of $\hat{H}_i$, $\hat{E}_\alpha + i\hat{E}_{-\alpha}$, $\hat{E}_\alpha - i\hat{E}_{-\alpha}$ with determined coefficients, and if they are properly chosen, we have $\Delta C_2 = (\Delta T)^2$ and

$$(\Delta \mathrm{T})^2 = \sum_{a=1}^{R}(\Delta T_a)^2 = \sum_{a=1}^{R}(\langle \hat{T}_a^2\rangle - \langle \hat{T}_a\rangle^2), \tag{4.9}$$

where R is the dimensionality of the algebra of the Lie group G.

Thus, in all the cases that we have considered (compact

and noncompact semisimple groups and the nilpotent Heisenberg group), the invariant relation that determines the coherent states can be written in the form

$$|(\Delta T)^2| = \min \tag{4.10}$$

with variance $(\Delta T)^2$ defined by (4.9). The completeness of the corresponding systems of coherent states follows from the invariance of (4.10) under finite transformations. It is clear that this is a general approach to the determination of coherent states and that the use of finite transformations and of the invariant formulation of uncertainty relations creates the necessary prerequisites for the creation of a particular CS basis and for the analysis of the transition from quantum theory to the classical limit.

4.2. Coherent states as an IR basis

A rigorous and consistent formulation of quantum mechanics was first put forward by von Neumann.[1] It relied on the representation of observable and dynamic variables by operators in Hilbert space. Concrete analyses of coherent states as quantum states, and examinations of fundamental types of IR bases (including infinitestimal, symmetric, and CS) from a single point of view, are closely related to the properties of Hilbert space. We now turn to a brief examination of this question.

It is convenient to adopt the Dirac notation $|\psi\rangle$ (and $\langle\psi|$) for the vectors and the conjugate vectors in Hilbert space. A complex number then corresponds to the product

$$\langle\varphi|\psi\rangle = \overline{\langle\psi|\varphi\rangle} \tag{4.11}$$

In each (separable) Hilbert space we can then introduce a complete orthonormal discrete basis, namely, a set of vectors $|n\rangle$ for which

$$\langle n|m\rangle = \delta_{mn} \tag{4.12}$$

and there is a formula for the decomposition of the unit operator over the one-dimensional projection operators

$$\hat{1} = \sum_n |n\rangle\langle n|. \tag{4.13}$$

The latter can be used to expand any vector over the set $\{|n\rangle\}$

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle. \tag{4.14}$$

The linear operator $\hat{A}$ has a matrix associated with it, whose elements are given by

$$A_{mn} = \langle m|\hat{A}|n\rangle, \tag{4.15}$$

the matrix being interpreted as the mth component of the vector $\hat{A}|n\rangle$.

The general scheme changes when we consider coherent states $|z\rangle$ $(z = \{z_k\})$. Instead of δ_{mn} in (4.12) or the δ-function [for example $\delta(x - x')$], we find that the following CS overlap arises for the basis $|x\rangle$ of the irreducible representation $D(1)$ of W(1):

$$\langle z|z'\rangle = f(\bar{z}, z') = \overline{f(\bar{z}', z)},\ f(\bar{z}, z) = 1, \tag{4.16}$$

where $|f(\bar{z},z')|$ has a maximum at $z = z'$. The CS overlap function is an invariant of the group. Actually, for unitary irreducible representations $\hat{T}^{-1}(g) = \hat{T}^{+}(g)$ and, according to (4.11),

$$f(\bar{z}_g, u_g) = \langle z_g|u_g\rangle = \langle z|\hat{T}^{+}(g)\hat{T}(g)|u\rangle = \langle z|u\rangle = f(\bar{z}, u). \tag{4.17}$$

The system of coherent states is overfull and the expansion of the unity operator takes the form

$$\hat{1} = \int |z\rangle\langle z|\,d\mu(z), \tag{4.18}$$

where $d\mu(z) = d\mu(z_g)$ is the invariant (under the transformations of the group) measure and

$$|\psi\rangle = \int |z\rangle\langle z|\psi\rangle d\mu(z). \tag{4.19}$$

Hence it follows that the coherent state of an overlap function $f(\bar{z},u) = \langle z|u\rangle$ is a generating kernel in the space of the functions $\psi(\bar{z}) = \langle z|\psi\rangle$:[13,110]

$$\psi(\bar{u}) = \int f(\bar{u}, z)\psi(\bar{z})d\mu(z).$$

The operator A can be associated with the following function (covariant symbol)

$$A(\bar{z}, z) = \langle z|\hat{A}|z\rangle.$$

Discussions of particular systems of coherent states are closely related to the overlap formalism that enables us to standardize the theory. We begin with the Heisenberg group W(1) which is the starting point of the general theory.[13,51,102] The minimum of the invarant variance (4.5) is achieved according to (4.7) if, and only if, the states over which the average is evaluated are the eigenstates of the annihilation operator $\hat{a}$,$\hat{a}|\omega z\rangle = \omega z|\omega z\rangle$

$$|\omega z\rangle = \exp\left(-\frac{1}{2}\omega|z|^2\right)\sum_{n=0}^{\infty}\frac{\omega^{n/2}z^n}{(n!)^{1/2}}|\omega n\rangle. \tag{4.20}$$

The conjugate basis $\langle\omega z|$ coincides with (4.20) subject to the replacement of $|\omega n\rangle$ with $\langle\omega n|$ and z with $\bar{z}$. Comparison of (4.20) with

$$|\omega z\rangle = \sum_n \langle\omega n|\omega z\rangle|\omega n\rangle,$$

yields

$$\langle\omega n|\omega z\rangle = \frac{(\omega^{1/2}z)^n}{(n!)^{1/2}}\exp\left(-\frac{\omega}{2}|z|^2\right), \tag{4.21}$$

i.e., none other but the symmetric basis of the irreducible representation (2.26).

The CS overlap has the form

$$\langle\omega z|\omega n\rangle = \sum_n \langle\omega z|\omega n\rangle\langle\omega n|\omega u\rangle$$
$$= \exp\{[\bar{z}u - (|u|^2/2) - (|z|^2/2)]\omega\},$$
$$|\langle\omega z|\omega u\rangle|^2 = \exp(-\omega|z - u|^2). \tag{4.22}$$

No two states can be mutually orthogonal, since for any z and u we have $0 < |\langle \omega z|\omega u\rangle| \leqslant 1$. Moreover, the states $|\omega z\rangle$ and $|\omega z'\rangle$ are approximately the same for neighboring points of the complex plane, $\langle \omega z|\omega z\rangle = 1$. In coherent states

$$\langle \hat{x}\rangle = \sqrt{2}\,\mathrm{Re}\,z,\ \langle \hat{p}\rangle = \sqrt{2}\omega\,\mathrm{Im}\,z,$$
$$(\Delta x)^2 = 1/2\omega,\ (\Delta p)^2 = \omega/2. \tag{4.23}$$

The invariant measure is $d\mu_\omega(z) = \omega d^2z/\pi$ and the integral in (4.18) is evaluated over the entire complex plane.

Other bases for generalized CS employ the semicoherent states[15]

$$|\omega n, z\rangle = \hat{D}(z)|\omega n\rangle.$$

When $n = 0$, we have the usual CS with $\hat{D}(z)$ being the shift operator (2.25).

Squeezed and correlated states have recently attracted considerable attention.[57-59] These states minimize the Schrödinger uncertainty relations [like the expressions of Sec. (4.1), the expressions given by (4.24)–(4.26) involve the Planck constant $\hbar$, whereas in all other cases we use the system of units in which $\hbar = c = 1$]:

$$(\Delta x\cdot\Delta p)^2 - \sigma_{px}^2 \geq \hbar^2/4,\ \sigma_{px} = (\langle \widehat{px} + \widehat{xp}\rangle/2) - \langle \hat{p}\rangle\langle \hat{x}\rangle, \tag{4.24}$$

and are characterized by squeezing and correlation coefficients, respectively defined by

$$k = \left(\frac{\Delta x/x_0}{\Delta p/p_0}\right)^2,$$
$$r = \frac{\sigma_{px}}{\Delta x\cdot\Delta p}. \tag{4.25}$$

These coefficients can be used to rewrite (4.24) in the form $\Delta x\cdot\Delta p \geqslant \hbar/[2(1-r^2)^{1/2}]$ and the invariant variance[15,110] in the form

$$(\Delta x/x_0)^2 + (\Delta p/p_0)^2 \geq (k^{1/2} + k^{-1/2})\cdot\frac{1}{2(1-r^2)^{1/2}}. \tag{4.26}$$

When $r = 0$, the squeezed states minimize the Heisenberg relation $\Delta x\cdot\Delta p \geqslant \hbar/2$. However, in contrast to coherent states, they are in general far from classical because Δx and Δp can be as large as desired for $k\to\infty$ and $k\to 0$, respectively. These limiting cases correspond to states with definite momentum $|\omega p\rangle$ or definite coordinate $|\omega x\rangle$ and also the basis-forming irreducible representations $D(\omega)$, $\langle \omega x|\omega x'\rangle = \delta[\omega(x-x')]$. The squeezed and correlated states are the eigenfunctions of the annihilation operators $\hat{a}'$ obtained by means of the canonical transformations (2.42) in the group SU(1,1); $\hat{a}' = u\hat{a} + v\hat{a}^+$,

$$\hat{a}'|\omega, uv, z\rangle = \omega z|\omega, uv, z\rangle,$$

where the coefficients k and r depend only on the ratio $s = v/u$:

$$r = -\mathrm{Im}\,s\{[(1+|s|^2)^2/4] - (\mathrm{Re}\,s)^2\}^{-1/2},\ -1 \leq r \leq 1,$$
$$k = (1 + |s|^2 - 2\,\mathrm{Re}\,s)/(1 + |s|^2 + 2\,\mathrm{Re}\,s),\ 0 < k < \infty. \tag{4.27}$$

This is not a fortuitous result. Actually, squeezed and correlated states $|\omega, uv, z\rangle$ depend on the single complex parameter $s = v/u$, $|s| < 1$ because u and v are related by $|u|^2 - |v|^2 = 1$ and, moreover, multiplication of u and v by $e^{i\alpha}$ leads to a change in z such that $z' = ze^{i\alpha}$. Thus, without loss of generality, we can put $u = \cosh\theta\cdot e^{i\varphi/2}$, $v = \sinh\theta\cdot e^{-i\varphi/2}$, i.e., squeezed and correlated states are determined by the point (θ,φ) in the homogeneous space SU(1,1)/U(1), i.e., the upper cavity of the hyperboloid of two sheets. When θ is large, $k\to\tan^2\varphi/2$, $r\to -1\,(0<\varphi<\pi)\,r\to 1\,(\pi<\varphi<2\pi)$, $r = 0\,(\varphi = 0, \varphi = \pi)$. The invariant variance (4.26) is then equal to $1 + 2|v|^2$.

The CS overlap $\langle z|\psi\rangle$ defines the decomposition over the coherent-state basis. For semicoherent, squeezed, and correlated states[15] and also states with the definite coordinate and momentum, we have

$$\langle \omega z|\omega n, w\rangle = \frac{1}{(n!)^{1/2}}\omega^{n/2}(\bar{z} - \bar{w})^n \exp(\omega\bar{z}w)\exp(-\omega|z|^2/2), \tag{4.28}$$

$$\langle \omega z|\omega, uv, w\rangle = u^{-1/2}\exp\left[\omega\left(\frac{w\bar{z}}{u} - \frac{v\bar{z}^2 - \bar{v}w^2}{2u} - \frac{|w|^2 + |z|^2}{2}\right)\right],$$

$$\langle \omega z|\omega x\rangle = (\pi/\omega^2)^{-1/4}\exp\{\omega[-(x^2/2) + \bar{z}\sqrt{2}x - (\bar{z}^2/2) - (|z|^2/2)]\}, \tag{4.29}$$

$$\langle \omega z|\omega p\rangle = (\pi\omega^2)^{-1/4}\exp[-(p^2/2\omega) + i\bar{z}\sqrt{2}p + (\omega\bar{z}^2/2) - (\omega|z|^2/2)]. \tag{4.30}$$

We now turn to the consideration of coherent states for the IR of the groups SU(2) and SU(1,1). The SU(2) coherent states introduced in Ref. 103 were subsequently used in Refs. 111–113 where they were referred to as the atomic, spin, and angular momentum coherent states. The SU(1,1) coherent states were investigated in Refs. 13, 104, and 110. We begin by considering the general scheme for constructing the IR in function space. Suppose that the irreducible representations $T(g)$ of the group G act in the space of functions $\psi(z)$, $z = \{z_k\}$ in accordance with the formula

$$\hat{T}(g)\psi(z) = \psi(gz). \tag{4.31}$$

Different irreducible representations correspond to different types of function. The variable z is interpreted as an element of the group G, and the tranformation (4.31) as an internal group automorphism. However, z can also be interpreted as the column $\|z_k\|$ with the matrix of fundamental irreducible representation g acting upon it. We shall use this below.

Realizations of the irreducible representations in terms of function space has been performed mostly by mathematicians[14,114,115] whereas realizations in terms of a discrete basis were tackled mostly by physicists (for the connection between these methods see Ref. 17). From the physical point of view, the work of these mathematicians was essentially concerned with the construction of generalized coherent states (orbits) in function spaces.

For the irreducible representations $D(j)$ of SU(2) and $D^+(j)$, $D^-(j)$ of SU(1,1), the space of irreducible representations is the space of polynomials of the form $z_1^{n_1}z_2^{n_2}$ of degree $n_1+n_2=2j$ [cf. (2.20)–(2.23)] where $|z_2|^2 \pm |z_1|^2$ is an invariant of SU(2) [SU(1,1)]. Next, let us put $|z_2|^2-|z_1|^2=1$ [if we assume that $|z_2|^2-|z_1|^2=0$ and follow steps similar to those employed below, we arrive at an irreducible representation of the fundamental series of SU(1,1)].

If, in accordance with (2.20), we act on the highest weight $f_{2j}(z_1,z_2)=\psi_{jj}(z_1,z_2)=(z_1)^{2j}$, we obtain the coherent states $\psi_{ju}(z_1,z_2)=(z_{1g})^{2j}=(\bar{u}_2z_2\pm\bar{u}_1z_1)^{2j}$ where $\bar{u}_1$ and $\bar{u}_2$ are the elements of the first row of the matrix $\|g_k^i\|$, $|u_1|^2\pm|u_2|^2=1$. Expanding the binomial, we can readily verify that the symmetric bases (2.21) and (2.22) are none other than the overlaps $\langle jz|jm\rangle$. Hence for $D(j)$ and $D^+(j)$ we have

$$|j,z_1z_2\rangle=\sum_{m=-j}^{j}\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}z_1^{j+m}z_2^{j-m}|jm\rangle, \tag{4.32}$$

$$|j,z_1z_2\rangle=\sum_{m=-j}^{\infty}\left[\frac{\Gamma(-(j-m))}{\Gamma(-2j)(j+m)!}\right]^{1/2}z_1^{j+m}z_2^{j-m}|jm\rangle. \tag{4.33}$$

For $D^-(j)$, we have $m\leqslant j<-1/2$. The formula for the coherent states is obtained by putting $m\to -m$, and the sum is evaluated between $m=-\infty$ and $m=j$.

Decomposition of the CS over a discrete basis

$$\langle ju|jz\rangle=\sum_m\langle ju|jm\rangle\langle jm|jz\rangle, \tag{4.34}$$

readily yields the overlaps for $D(j)$ and $D^+(j)$:

$$\langle ju|j'z\rangle=(\bar{u}_1z_1+\bar{u}_2z_2)^{2j}\delta_{jj'},\quad 2j\geq 0, \tag{4.35}$$

$$\langle ju|j'z\rangle=(\bar{u}_2z_2-\bar{u}_1z_1)^{2j}\delta_{jj'},\quad 2j<-1. \tag{4.36}$$

It is clear that, in contrast to the CS of the group W(1), there are always CS that are orthogonal to a given CS. The formula given by (4.34) can be used directly to construct the CS as in Ref. 110 (see also Refs. 15 and 16) if we know the discrete functional basis of the IR. The invariant measure takes the form[15]

$$D(j):\ d\mu_j(z)=\frac{2j+1}{(2\pi)^2}\delta(|z_1|^2+|z_2|^2-1)d^2z_1d^2z_2, \tag{4.37}$$

$$D^+(j):\ d\mu_j(z)=\frac{-2j-1}{(2\pi)^2}\delta(|z_2|^2-|z_1|^2-1)d^2z_1d^2z_2. \tag{4.38}$$

We can now transform from the variables $z-1$ and z_2 to the angular variables θ, φ:

$$|j\theta\varphi\rangle=|jz_1z_2\rangle e^{-i\omega j}.$$

We then have

$$\begin{aligned}
\mathrm{SU}(2):\ & z_1=\cos\frac{\theta}{2}\cdot e^{i(-\varphi/2+\omega/2)},\\
& z_2=\sin\frac{\theta}{2}\cdot e^{i[(\varphi/2)+(\omega/2)]},\\
& d\mu_j(\theta,\varphi)=\frac{2j+1}{4\pi}\sin\theta d\theta d\varphi,\\
& 0\leq\theta\leq\pi,\ 0\leq\varphi<2\pi,
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\mathrm{SU}(1,1):\ & z_1=\mathrm{ch}\frac{\theta}{2}\cdot e^{i(-\varphi/2+\omega/2)},\\
& z_2=\mathrm{sh}\frac{\theta}{2}\cdot e^{i[(\varphi/2)+(\omega/2)]},\\
& d\mu_j(\theta,\varphi)=\frac{-2j-1}{4\pi}\mathrm{sh}\,\theta d\theta d\varphi,\\
& 2j<-1,\ 0\leq\theta<\infty,\ 0\leq\varphi\leq 2\pi
\end{aligned} \tag{4.40}$$

where in contrast to (4.47) and (4.38) there is no integration with respect to the phase ωj [this actually signifies a transition from $U(2)$ to SU(2)]. The mean evaluated over the generators (covariant symbols) $J_k(\theta,\varphi)=\langle\hat{J}_k\rangle$ for $D(j)$ and $D^{(\pm)}(j)$, respectively,

$$\begin{aligned}
J_x&=j_x=j\sin\theta\cdot\cos\varphi, & J_x&=ij_x=\mp ij\,\mathrm{sh}\,\theta\cdot\cos\varphi,\\
J_y&=j_y=j\sin\theta\cdot\sin\varphi, & J_y&=ij_y=\mp ij\,\mathrm{sh}\,\theta\cdot\sin\varphi,\\
J_z&=j_z=j\cos\theta, & J_z&=j_z=\mp j\,\mathrm{ch}\,\theta,\\
j_x^2+j_y^2+j_z^2&=j^2, & -j_x^2-j_y^2+j_z^2&=j^2.
\end{aligned} \tag{4.41}$$

A very graphic picture is obtained for different types of IR bases by considering the space of the averages with axes j_x, j_y, j_z. It follows from (4.41) that the CS of the group SU(2) are defined by a point on a sphere (Fig. 4a), whereas the CS of representations of discrete series of SU(1,1) are determined by a point on the upper $[D^+(j)]$ or lower $[D^-(j)]$ plane of a hyperboloid of two sheets (Fig. 4b). For unitary IR of the auxiliary and fundamental series that are not limited by the highest (lowest) weight, the CS are specified by points of a single-sheet hyperboloid or cone. The points on the j_z axis in the figure correspond to states of the discrete basis $|jm\rangle$, and the dotted line shows surfaces (sphere and hyperboloid, respectively) corresponding to the generalized CS associated not with the highest weight, but with the states $|jj-1\rangle$ of the discrete basis.

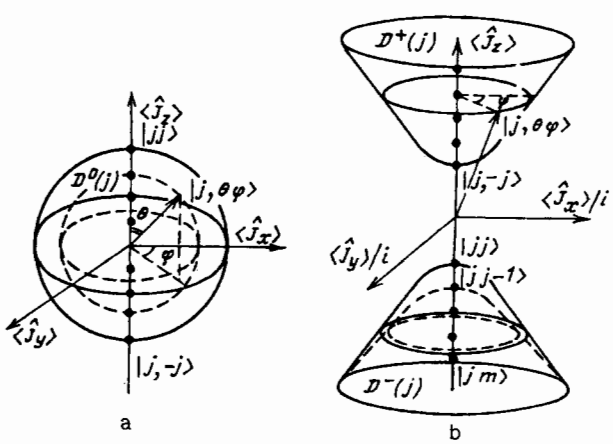

FIG. 4. CS of SU(2) (a) and SU(1,1), (b)—discrete series.

Let us consider in greater detail the CS of the group SU(2), i.e., the angular momentum CS. The covariant symbols (4.41) of operators representing the projections of the angular momentum are identical with the expressions for the component of the classical angular momentum vector $\mathbf{j} = j\mathbf{n}$ (whose direction is defined by θ, φ in a spherical frame) where $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Substitution of the coefficient of the covariant generator symbols into the Poisson bracket

$$\{f_1, f_2\} = \frac{1}{j\sin\theta}\left(\frac{\partial f_1}{\partial\theta}\frac{\partial f_2}{\partial\varphi} - \frac{\partial f_1}{\partial\varphi}\frac{\partial f_2}{\partial\theta}\right) \tag{4.42}$$

gives relations analogous to the commutation relations for the operators themselves: $\{j_x(\theta,\varphi), j_y(\theta,\varphi)\} = j_z(\theta,\varphi)$ and so on; for the CS variances we have $(\Delta J_k)^2 \leqslant j/2$; the modulus of the CS overlap can be found from (4.35):

$$|\langle j\theta_1\varphi_1|j\theta_2\varphi_2\rangle| = \left(\cos\frac{\theta'}{2}\right)^{2j}, \tag{4.43}$$

where θ' is the angle between the vectors $\mathbf{j}_1$ and $\mathbf{j}_2$ (see Fig. 4a). The different CS become orthogonal only in the limit as $j\to\infty$; for arbitrary j only pairs of states characterized by opposite directions of $\mathbf{j}_1$ and $\mathbf{j}_2$ are orthogonal.

The generalized CS $|j, |m|; \theta\varphi\rangle$ obtained by operating on $|jm\rangle$ with finite transformations can also be found from

$$(\mathbf{nJ})|j, |m|, \theta\varphi\rangle = |m||j, |m|, \theta\varphi\rangle, \tag{4.44}$$

i.e., as states with a particular projection of the angular momentum onto the direction of $\mathbf{n}$ (see Ref. 116 about the analog of this relation for semisimple groups). The wave functions of the states $|j, |m|, \theta\varphi\rangle$ are the spherical harmonics $Y_{jm}(\theta'',\varphi'')$ where θ'' and φ'' are functions of the state variables θ', φ' and the spherical coordinates θ, φ. In the special case of CS

$$Y_{jj}(\theta,\varphi) = (-1)^j\left[\frac{(2j+1)!}{2\pi(2j)!!}\right]^{1/2}(\sin\theta)^j e^{i\varphi j}.$$

It is clear that, as $j\to\infty$, the CS wave function is localized near $\theta = \pi/2$ i.e., in the plane perpendicular to the angular momentum vector $\mathbf{j}$, which corresponds to the classical rotation of this plane. On the other hand, in the case of the generalized CS $|j, |m|, \theta\varphi$, if the condition $|m/j|\to 1$ is not satisfied, this localization does not occur even for $j\to\infty$ and the states do not become the classical states. The overlaps $\langle j\theta_1\varphi_1|j,|m|,\theta_2\varphi_2\rangle$ are given in Ref. 15.

We note that the spherical functions are overlaps of states with a definite angular momentum j and its projection m onto the axis and states with a definite angle (i.e., uncertain j):

$$\begin{aligned}
&\langle jm|\theta\varphi\rangle = Y_{jm}(\theta,\varphi),\\
&\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\langle\theta\varphi|jm\rangle\langle jm|\theta'\varphi'\rangle\\
&\qquad = \langle\theta\varphi|\theta'\varphi'\rangle = \delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta'),\\
&\hat{1} = \iint|\theta\varphi\rangle\langle\theta\varphi|\sin\theta\,d\varphi\,d\theta,\ 0\le\varphi<2\pi,\ 0\le\theta\le\pi.
\end{aligned} \tag{4.45}$$

For the IR $D(p)$ of the group of motions of a plane, M(2), it is natural to consider the two bases $|pm\rangle$ and $|p\alpha\rangle$. The states $|pm\rangle$ are the eigenstates with eigenvalues m and p^2 for operators representing the projection of the angular momentum $\hat{J}_z$ and the square of the momentum (Laplace operator) $\hat{p}_x^2 + \hat{p}_y^2 = -(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ [cf. (2.7) and (2.8)], and $\hat{p}_x|p\alpha\rangle = p\cos\alpha\cdot|p\alpha\rangle$, $\hat{p}_y|p\alpha\rangle = p\sin\alpha\cdot|p\alpha\rangle$. For fixed p,

$$\int_0^{2\pi}|p\alpha\rangle\langle p\alpha|\,d\alpha = \hat{1},\ \sum_{m=-\infty}^{\infty}|pm\rangle\langle pm| = \hat{1}. \tag{4.46}$$

The wave functions, i.e., the overlaps with states $|r\varphi\rangle$, where r, φ are the polar coordinates of a point on a plane, can be obtained by solving the differential equations[7] (2.8) and (2.9):

$$\begin{aligned}
&\langle r\varphi|pm\rangle = e^{im\varphi}i^m J_m(pr),\\
&\sum_{m=-\infty}^{\infty}\langle r\varphi|pm\rangle\langle pm|r\varphi\rangle = \sum_{m=-\infty}^{\infty}J_m^2(pr) = 1,
\end{aligned} \tag{4.47}$$

$$\begin{aligned}
&\langle r\varphi|p\alpha\rangle = e^{ipr\cos(\varphi-\alpha)},\\
&\langle p\alpha|pm\rangle = e^{im\alpha}/\sqrt{2\pi},\\
&\langle p\alpha|p\beta\rangle = \sum_{m=-\infty}^{\infty}\langle p\alpha|pm\rangle\langle pm|p\beta\rangle\\
&\qquad = \frac{1}{2\pi}\sum_m e^{im(\alpha-\beta)} = \delta(\alpha-\beta),
\end{aligned} \tag{4.48}$$

where $J_m(pr)$ is the Bessel function of order m. To construct the generalized CS, we apply the finite transformation operator to $|p0\rangle$ (the parameters in front of $\hat{J}_z$ will be set to zero because $\{\exp(i\alpha_0\hat{J}_z)\}$ form a stationary subgroup of the state $|pm\rangle$)

$$\begin{aligned}
&\exp[(z\hat{\Phi}_+ - \bar{z}\hat{\Phi}_-)/2]J_0(pr)\\
&\qquad = \sum_{n_1,n_2=0}^{\infty}\frac{1}{n_1!}\frac{1}{n_2!}J_{n_1-n_2}(pr)\left(\frac{pz}{2}\right)^{n_1}\left(-\frac{p\bar{z}}{2}\right)^{n_2}\\
&\qquad = J_0[p(r^2+\rho^2-2\rho r\cos(\beta-\varphi))^{1/2}]\\
&\qquad = \langle r\varphi|pz\rangle,\ z = \rho e^{i\beta}.
\end{aligned} \tag{4.49}$$

Here we have used the summation formulas given in Ref. 117. The argument of J_0 is the product of p and the distance between points with coordinates (r,φ) and (ρ,β). Hence, $|pm, z\rangle$ is the state $|pm\rangle$ "shifted" from the origin of coordinates to the point $(\rho\beta)$.

For the CS overlap we obtain

$$\langle p, u|p, z\rangle = J_0(p|u-z|). \tag{4.50}$$

In accordance with the general theory [cf. (4.17)], the CS overlap is a function of the invariant of the group M(2), i.e., the separation between two points on a plane.

For large p, $p\gg m$, the wave functions $\langle r\varphi|pm,z\rangle$ are localized at (ρ,β). The states $|p,z\rangle = |p0,z\rangle$ are special

among the $|pm,z\rangle$ because the spacial uncertainty $\Delta x^2 + \Delta y^2$ is a minimum for them.

Some questions associated with the transition to the classical limit and the generalized CS for angular momentum/angle variables are discussed in Refs. 9 and 16 which constructs states that are the eigenstates of the "rotated" operator

$$\hat{J}'_z = \hat{J}_z - \bar{z}\hat{\Phi}_+ + z\hat{\Phi}_-$$

which is a particular analog of squeezed states of the group W(1) that are the eigenstates of the annihilation operator $\hat{a}'$.

From an arbitrary Lie group G, the analogs of squeezed states can be constructed with the help of linear automorphisms (canonical transformations) G_A of the Lie algebra of the group G. For this we have to "turn" the generators $\hat{T}_i$, $\hat{T}'_i = \hat{T}_i(g_A)$, $g_A \in G_A$ and construct the usual states for the new operators $\hat{T}'_i$, which are in fact squeezed states. They are parametrized by points of the homogenous space G_A/G_1 where G_1 is a subgroup of the group G consisting of elements that are common to $\{\exp \Sigma\alpha_i\hat{T}'_i\}$ and $\{\exp \Sigma\beta_i\hat{T}_i\}$, $\hat{T}'_i = \hat{T}_i(g_A)$, and α and β are parameters of the transformations in the group G. This construction gives a substantial theory in the case of resolvable groups; for the semisimple group SU(2), where G_1, G_A, and G coincide, it does not lead to new states because $\exp(\Sigma\alpha_i\hat{T}'_i)$ are finite SU(2) transformations.

The overlap formulas provide us with a very convenient formalism for calculations. In addition to the usual overlaps, we can use those that include the differential operators $\langle jw|jd/du\rangle$, $\langle jd/dw|jd/du\rangle$ (cf. Refs. 11 and 118). For the group SU(2),

$$\langle jw|jd/du\rangle = (\bar{w}_1 d/du_1 + \bar{w}_2 d/du_2)^{2j}. \tag{4.51}$$

It is also interesting to consider overlaps in the discrete cases presented in Sec. 3, e.g., $\langle jz|j\hat{D}_z^+\rangle_q$. The structural features of these overlaps are readily seen in the case of SU(2). The expression given by (4.35) leads directly to relations of the form

$$\langle u|w\rangle^{2j} = \langle ju|jw\rangle, \ \langle j_1u|j_2w\rangle\langle j_1u|j_2w\rangle$$
$$= \langle j_1 + j_2u|j_1 + j_2w\rangle.$$

The composition and decomposition of overlaps is realized on the basis of the formulas for the decomposition of unity, given by (4.13) and (4.18). Essentially, the overlaps constitute a new universal language that enables us to perform very complicated operations with functions at a standard, elementary level. The theory of C–G coefficients and certain portions of the theory of special functions may be looked upon as studies of generalized overlaps.

We note that we have used the traditional realization of the space of representations of W(1), which is valid for fixed ω. In general, if a state does not have a particular frequency, i.e., it does not transform in accordance with some particular irreducible representation $D(\omega)$ of the group W(1), we must consider the space of functions $\psi(x,\tau)$,

$$\hat{p} = -i\partial/\partial x, \ \hat{x}_\omega = -ix\partial/\partial\tau, \ \hat{I} = -i\partial/\partial\tau,$$
$$[\hat{x}_\omega, \hat{p}] = i\hat{I}, \ [\hat{x}_\omega, \hat{I}] = [\hat{p}, \hat{I}] = 0. \tag{4.52}$$

For example, $\psi(x,\tau) = \exp(i\omega\tau)\psi(\omega,x)$, $\hat{p} = -i\partial/\partial x$, $x_\omega = \omega x$, $\hat{I} = \omega$ [see (2.25)]. If, in general, $[\hat{a},\hat{a}_\omega^+] = -i\partial/\partial\tau$, then for states with fixed frequency $[\hat{a},\hat{a}_\omega^+] = \omega$ and we can employ the usual operator $\hat{a}^+ = \hat{a}_\omega^+/\omega$, $[\hat{a},\hat{a}^+] = 1$. Such states are examined in the next Section where instead of $|\omega n\rangle$ and $|\omega z\rangle$ we use the abbreviated notations $|n\rangle$ and $|z\rangle$ and the usual operators $\hat{a}^+$ and $\hat{a}$.

4.3. CS-representation of quantum mechanics and the transition to the classical limit

The CS provide a certain classical representation of quantum mechanics. They were defined above as states with minimum uncertainties, using the relations for the invariant (under finite transformations of the symmetry group) variance. This invariance of the uncertainty relations ensures that the CS form a complete IR basis of the group G. They are parametrized by points in the phase space of the classical system.

The wave function ψ in the Schrödinger equation is essentially none other but an overlap that is different for different representations of quantum mechanics: $\langle x|\psi\rangle$ for coordinate, $\langle p|\psi\rangle$ for momentum, $\langle n|\psi\rangle$ for energy, and $\langle z|\psi\rangle$ for the CS representations (these are the Bargmann–Fock representations).[13,51,119]

In the last case, the Schrödinger equation takes the form

$$i\frac{\partial}{\partial t}\psi(z) = -\frac{\omega}{4}\left(\frac{d}{dz} - z\right)^2\psi(z) + u\left(\frac{d/dz + z}{(2m\omega)^{1/2}}\right)\psi(z), \tag{4.53}$$

where $\psi(z) = \langle\bar{z}|\psi\rangle\exp(|z|^2/2)$ is an analytic function of z and

$$\langle\psi_1|\psi_2\rangle = \int\overline{\psi_1(z)}\psi_2(z)\exp(-|z|^2)d^2z/\pi,$$

$$\hat{a}=d/dz, \ \hat{a}^+=z, \ \hat{x} = \frac{d/dz + z}{(2m\omega)^{1/2}}.$$

An oscillator is described by the first-order equation

$$i\frac{\partial}{\partial t}\psi(z) = \omega\left(\frac{1}{2} + z\frac{d}{dz}\right)\psi(z). \tag{4.54}$$

States with determined energy $E = \omega[(1/2) + n)]$ have $\psi_n(z) = z^n \times \exp(-iEt)/(n!)^{1/2}$, and coherent states have

$$\psi_n(z) = \exp[u(t)z - (i\omega t/2) - (|u|^2/2)], \ u(t) = u_0e^{-i\omega t}.$$

The evolution of a state thus reduces to the variation of the CS parameter $u(t)$. The Bargmann-Fock representation has been used in a variety of quantum-mechanical problems.[13,51]

Let us now consider generally the transition to the quasiclassical equations, using the calculus of symbols. Suppose that the Hamiltonian of the physical system and a certain operator $\hat{L}$ that is not an explicit function of time are operator functions of the group generators. The Heisenberg form of the equation of motion is then

$$\frac{\partial\hat{L}_H}{\partial t} = i[\hat{H}_H, \hat{L}_H]. \tag{4.55}$$

The quasiclassical equations of motion are obtained by formally replacing the operators in the Heisenberg equations

with the corresponding dynamic variables and the commutator with the Poisson bracket (the word "quasiclassical" means that the resulting equations retain the spin and color variables that do not appear in the classical form).

The question is: when can this type of transition to c-number equations take place or, at least, provide a good approximation? The answer to this question can be obtained by averaging the Heisenberg equations over the coherent states. The quasiclassical equations are then obtained (in general, under certain additional conditions that define the range of their validity) as equations for the evolution of the CS that are parameterized by points in the phase space of the system.

If the coherent states of the corresponding group do not spread out, i.e., they remain coherent in time ($\hat{u}|z(0)\rangle = |z(t)\rangle$ where $\hat{u}$ is the evolution operator), we can rewrite (4.55) in the form

$$\frac{\partial}{\partial t}\langle z|\hat{L}|z\rangle = i\langle z|[\hat{H},\hat{L}]|z\rangle, \tag{4.56}$$

where $z = z(t)$ and the operators are taken in the Schrödinger representation, i.e., (4.55) converges to the equations for $z(d)$ (this is the case of the so called "exact quasiclassical behavior"). To ensure that the CS do not spread, it is sufficient for $\hat{H}$ to be linear in the group generators because the evolution operator is then an operator for finite transformations in the group (the nonspreading of coherent states is discussed in Ref. 120). The left-hand side of (4.56) contains the derivative of the covariant symbol of the operator $\hat{L}$ whereas the right-hand side contains the commutator symbol. Thus, in symbol language, the second correspondence principle signifies that the commutator symbol becomes the Poisson symbol bracket (cf. 110 and 122).

We now turn to the coherrent states of the group W(1). We know that they do not spread for a Hamiltonian of the form

$$\hat{H} = (\omega\hat{a}^+\hat{a} + F\hat{a} + \hat{F}\hat{a}^+ + \beta),\ \omega = \bar{\omega},\ \beta = \bar{\beta}.$$

If we put $\hat{L} = \hat{a}$ in (4.56), we obtain

$$\frac{\partial z}{\partial t} = i\{H, z\} = -i\frac{\partial H}{\partial \bar{z}} = -i\omega z + \bar{F}, \tag{4.57}$$

where $H = H(\bar{z},z)$ is the covariant symbol for the Hamiltonian. The solution of this equation is

$$z = |z|\exp[-i(\omega t + \varphi)] + \int \bar{F}(t)\mathrm{d}t.$$

The quantity $|z|$ is interpreted as the amplitude of the oscillations, φ is the initial phase, and

$$\langle\hat{x}\rangle = \sqrt{2}x_0|z|\cos(\omega t + \varphi),$$
$$\langle\hat{p}\rangle = \sqrt{2}p_0|z|\sin(\omega t + \varphi). \tag{4.58}$$

It is clear that the average values vary in the coherent states in accordance with classical laws, but the classical picture of motion is a good approximation only in the case of large amplitudes of the oscillations $\langle\hat{p}\rangle$ and $\langle\hat{x}\rangle$ in (4.58), $\sqrt{2}|z|x_0 \gg \Delta x = x_0/\sqrt{2}$, $\sqrt{2}|z|p_0 \gg \Delta p = p_0/\sqrt{2}$, and the classical limit is reached for $|z| \to \infty$.

For Hamiltonians that are linear in the generators of SU(N) and SU($N-1$,1) (see Appendix), we find similarly for

$$\hat{H} = c_k^l\hat{T}_l^k,\ \bar{c}_k^l = c_l^k(\pm 1)^{\delta_{kN}+\delta_{lN}}$$

[the lower sign corresponds to SU(N-1,1)] that for symmetric irreducible representations[15]

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = -ic_k^l z_l. \tag{4.59}$$

In particular, for the SU(2) group, we have $\hat{H} = i(a_x\hat{J}_x + a_y\hat{J}_y + a_z\hat{J}_z)$ and, using the CS parametrization $|j\theta\varphi\rangle$ and (4.39)–(4.41), we obtain the equation describing the motion of a point on a sphere:

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = [\mathbf{a}(t), \mathbf{n}]. \tag{4.60}$$

For SU(1,1) ~ SO(2,1), the equations for the evolution of the coherent states describe the motion of a point on the upper $[D^+(j)]$ or lower $[D^-(j)]$ sheets of the hyperboloid [cf. (4.41) and Fig. 4].

If the Hamiltonian is nonlinear in the group generators, then if we neglect the possible spreading of coherent states, we can obtain from (4.56) an approximate equation for z_k which, in contrast to the quasiclassical (4.57) and (4.60), contains the signature of the irreducible representation. The quasiclassical equations given by (4.57) and (4.59) are obtained for certain values of the parameters of state ($|z| \to \infty$ for W(1) and $j \to \infty$ for SU(2); the case of SU(N) was investigated in Ref. 15 in connection with the derivation of the quasiclassical evolution equations for color variables, i.e., the Wong equations).

We have already noted that if the CS do not spread (the evolution operator $\hat{u}(t,t_0)$ reduces to finite transformations in the group), then $\psi(z,t) = \langle z|\hat{u}(t,t_0)|\psi_0\rangle = \langle z(t)|\psi_0\rangle = \psi_0[z(t)]$. For the irreducible representations $D(j)$ and $D^+(j)$ of SU(2) and SU(1,1), $\psi = \psi[(z_1(t_1), z_2(t_2)]$, $z_1(t)$, and $z_2(t)$ transform as the components of a spinor under the fundamental irreducible representation $D(1/2)$, $|z_2|^2 \pm |z_1|^2 = 1$, i.e., there are two equations for $z_1(t)$ and $z_2(t)$. If, on the other hand, the CS do spread, then the equations obtained by averaging (4.56) over the coherent states are only approximate; the exact equations are either the system for $\psi_m(t) = \langle jm|\psi\rangle$ [for SU(2) we have dim $D(j) = 2j + 1$ equations, and for unitary representations of SU(1,1) there is an infinite number] or a partial differential equation for $\psi(z,t) = \langle jz|\psi\rangle$.

A concrete and consistent analysis starts with the symbol calculus based on the mutually single valued mapping of operator functions of generators and vectors of the group representation space onto c-number functions of complex variables z_k and $\bar{z}_k$. This calculus was developed by Berezin[110,121] and plays an important part in his concept of quantization[110,122] and the evaluation of the functional integral.[121,123] Certain general propositions including those relating to W(N), SU(N), and SU(N,1) can be found in Refs. 13, 15, 16, and 110. By using the CS-representations that rest on the calculus of symbols, we are able to follow the transition to the classical description, and also simplify the calculations for particular systems.

Let us briefly examine the basic concepts in the theory of symbols. The symbol of a vector $|\psi\rangle$ is the function $\psi(z) = \langle z|\psi\rangle$, i.e., the wave function in the CS representation. It is uniquely determined by the symbol that specifies the decomposition (4.19) over the CS basis. The function $\langle z|n\rangle$ that specifies the symmetric basis can be looked upon

as the symbol of vectors in an abstract basis of occupation numbers. The overlaps (4.21), (4.22), (4.30), (4.35), and (4.36) determine the symbols of vectors in different IR bases.

Operators in the Hilbert space of irreducible representations can also be assigned symbols that define them completely. The covariant and contravariant symbols of an operator $\hat{A}$ are defined as the functions $Q_A(\bar{z},z)$ and $P_A(\bar{z},z)$

$Q_A(\bar{z},z)$ and $P_A(\bar{z},z)$

$$Q_A(\bar{z},z) = \langle z|\hat{A}|z\rangle, \tag{4.61}$$

$$\hat{A} = \int P_A(\bar{z},z)|z\rangle\langle z|\,d\mu(z), \tag{4.62}$$

$$Q_A(\bar{z},z) = \int P_A(\bar{u},u)|\langle z|u\rangle|^2 d\mu(u). \tag{4.63}$$

The representation of an operator by a P-symbol is very convenient in problems involving the density matrix. We note that the possibility of such definitions relies on the overcompletion of the basis $|z\rangle$, which ensures that $\langle z|\hat{A}|u\rangle$ is completely determined by its diagonal elements $\langle z|\hat{A}|z\rangle$, where $Q_A(\bar{z},u) = \langle z|A|u\rangle/\langle z|u\rangle$ is the analytic continuation of the symbol $Q_A(\bar{z},z)$.

For the group W(1), the covariant symbol Q_A corresponds to the normal way of writing down operators in which all the $\hat{a}_+$ are written on the left of $\hat{a}_-$ (followed by the replacement $\hat{a}_+ \to \bar{z}$, $\hat{a}_- \to z$) and the contravariant symbol P_A corresponds to the antinormal method of writing operators in the reverse way. Just as important as Q_A and P_A is the Weyl symbol that corresponds to the symmetrized way of writing $\hat{a}_+$ and $\hat{a}_-$. The explicit form of the P and Q symbols for the groups SU(2) and SU(1,1) can be found in Ref. 13; the corresponding results for SU(N) and SU(1,1) are given in Refs. 15 and 16 together with a simple method for their evaluation, based on the representation of generators in terms of $\hat{a}_+$ and $\hat{a}_-$, analogous to (2.16).

The decomposition of unity given by (4.18) readily yields the rules

$$\hat{A}\psi(\bar{z}) = \int Q_A(\bar{z},u)\langle z|u\rangle\psi(\bar{u})\,d\mu(u), \tag{4.64}$$

$$\hat{A}\psi(\bar{z}) = \int P_A(\bar{u},u)\langle z|u\rangle\psi(\bar{u})\,d\mu(u) \tag{4.65}$$

and the formulas for *-multiplication (multiplication of symbols)

$$Q_{A_1} * Q_{A_2} = Q_{A_1A_2}(\bar{z},z) = \int Q_{A_1}(\bar{z},u)Q_{A_2}(\bar{u},z)|\langle z|u\rangle|^2 d\mu(u), \tag{4.66}$$

$$Q_{A_1A_2}(\bar{z},z) = \iint P_1(\bar{u},u)P_2(\bar{v},v)\langle z|u\rangle\langle u|v\rangle\langle v|z\rangle\,d\mu(v)\,d\mu(u). \tag{4.67}$$

For the group W(1), the expressions given by (4.63) and (4.66) can be written in the differential form[121]

$$Q_A(\bar{z},z) = \exp(\Delta(\bar{z},z))P_A(\bar{z},z),\quad \Delta(\bar{z},z) = \partial^2/\partial z\,\partial\bar{z}, \tag{4.68}$$

$$Q_{A_1A_2}(\bar{z},z) = (\exp\Delta(\bar{v},v))Q_{A_1}(\bar{z},v)Q_{A_2}(\bar{v},z)\big|_{v=z}. \tag{4.69}$$

The formulas for the *-multiplication of symbols, given by (4.66), and (4.69), are important in the analysis of the transition to the classical limit, i.e., for deciding which states can be described classically. Such states must have minimum uncertainty and be parametrized by points in phase space. Moreover, the expectation values of operators in such states must be much greater than the mean square deviations.

All the above equations are satisfied by the CS for certain values of the parameters. For an oscillator, these are the CS $|z\rangle$ with large $|z|$ (oscillation amplitude) and the angular momentum CS $|j\theta\varphi\rangle$ become classical for high angular momenta j. The CS become orthogonal in the classical limit [cf. (4.22) and (4.43)], the product of the overlap $|\langle u|z\rangle|^2$ by the normalizing factor in the measure $d\mu$ (4.66) becomes a δ-function, and for operators with nonsinglular symbols we have the first correspondence principle up to terms of order $1/|z|^2$ [W(1)] or $1/j$ [SU(2)], where the operator product symbol is equal to the product of symbols: $Q_{A_1}*Q_{A_2} = Q_{A_1}\cdot Q_{A_2}$. The following are the corresponding formulas for the group W(1):

$$\begin{aligned}|\langle u|z\rangle|^2 d\mu(u) &= e^{-|u-z|^2} d^2u/\pi \\ &= e^{-\mathrm{Re}^2(u-z)-\mathrm{Im}^2(u-z)} d^2(u-z)/\pi \\ &= \frac{1}{\sqrt{\pi}h} e^{-(\mathrm{Re}\,\alpha)^2/h^2}\,d\,\mathrm{Re}\,\alpha \\ &\times \frac{1}{\sqrt{\pi}h} e^{-(\mathrm{Im}\,\alpha)^2/h^2}\,d\,\mathrm{Im}\,\alpha \xrightarrow[n\to 0]{} \delta(\mathrm{Re}\,\alpha)\,d\,\mathrm{Re}\,\alpha\,\delta(\mathrm{Im}\,\alpha)\,d\,\mathrm{Im}\,\alpha,\end{aligned} \tag{4.70}$$

where $\alpha = (u-z)/|z|$, $h = 1/|z|$. The second correspondence principle can be obtained with the help of (4.66) by including terms of order $1/|z|^2(1/j)$ (Ref. 16).

We note that, in the classical limit, the P and Q operator symbols are identical because both (4.63) and (4.66) contain the square of the modulus of the CS overlap.

It follows from the foregoing discussion that the classical description is possible for high values of certain quantum numbers, i.e., the parameters of state (and sufficiently "good" Hamiltonians that do not, in the course of time, leave the region of states close to classical states). The cases of SU(2) and SU(1,1) were investigated in Refs. 110 and 112, whereas semisimple compact groups were discussed by Simon[124] who showed that the classical limit corresponds to large values of the IR signature parameters; the discrete series of SU(N,1) are discussed in Refs. 15 and 16. The classical picture is then a good approximation for high values of the IR signatures and for large CS parameters $|z_i|$. A more general formulation of the problem of the classical limit as the limit corresponding to "large N" (in particular a large number of degrees of freedom) is given in Ref. 125. We shall return to this question in Sec. 6 where we shall consider the law of large numbers and the limit theorems of the theory of probability amplitudes.

4.4. Phase space and the relative variance as a measure of the uncertainty of states

Let us now examine in greater detail which states are maximally close to classical states. In classical theory, a state is represented by a point in phase space. In the quantum case, it is convenient to consider the $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$ plane (or, in the case of an arbitrary Lie group, the space of their mean generators). For each space there is a certain region with center at the point $x = \langle\hat{x}\rangle$, $p = \langle\hat{p}\rangle$ (for the CS, $x = \sqrt{2}\,\mathrm{Re}\,z$, $p = \sqrt{2}\,\mathrm{Im}\,z$ and dimensions of the order of Δx and Δp, i.e., a certain vector (x,p) with variance $[(\Delta x)^2,(\Delta p)^2]$. The uncertainty in this vector (and, hence, of the state) is naturally

characterized by the ratio of the invariant variance of the vector (4.9) to the square of its length:

$$D_{\rm rel} = (\Delta \mathrm{T})^2/\langle \hat{\mathrm{T}}^2 \rangle, \quad 0 \leqslant D_{\rm rel} \leqslant 1. \tag{4.71}$$

The uncertainty is a maximum for $D_{\rm rel} = 1$. The classical limit corresponds to $D_{\rm rel} \to 0$, i.e, the uncertainties are negligible in comparison with the means. The condition $D_{\rm rel} \to 0$ is essentially the correspondence principle for the states.

In the case of a flat phase space [group W(1)], the relative variance is

$$\frac{(\Delta \mathrm{T})^2}{\langle \hat{\mathrm{T}}^2 \rangle} = \frac{\left(\frac{\Delta x}{x_0}\right)^2 + \left(\frac{\Delta p}{p_0}\right)^2}{\langle (\hat{x}/x_0)^2 \rangle + \langle (\hat{p}/p_0)^2 \rangle}$$

$$= \frac{2(\langle \hat{a}^+\hat{a} \rangle - \langle \hat{a}^+ \rangle \langle \hat{a} \rangle) + 1}{2\langle \hat{a}^+\hat{a} \rangle + 1}. \tag{4.72}$$

For states with definite coordinate $|x\rangle$ and momentum $|p\rangle$, and also states corresponding to the nth level of the oscillator $|n\rangle$ and coherent $|z\rangle$, semicoherent $|nz\rangle$, and squeezed $|uv, z\rangle$ states, we have

States:	$\lvert x\rangle$	$\lvert p\rangle$	$\lvert n\rangle$	$\lvert z\rangle$	$\lvert nz\rangle$	$\lvert uv, z\rangle$.
$D_{\rm rel}$:	1	1	1	$\frac{1}{1+2\lvert z\rvert^2}$	$\frac{1}{1+\frac{2\lvert z\rvert^2}{1+2n}}$	$\frac{1}{1+\frac{2\lvert u\bar{z}-\bar{v}z\rvert^2}{1+2\lvert v\rvert^2}}$.
Classical limit	—	—	—	$\lvert z\rvert\to\infty$	$\lvert z\rvert\to\infty$; $n/\lvert z\rvert^2\to 0$	$\lvert z\rvert\to\infty$.

In the case of squeezed and correlated states,

$$D_{\rm rel} \xrightarrow[|z|\to\infty]{} 0$$

for fixed $|v/u| \neq 1$. In the limit as $|v/u| \to 1$, this will not in general be the case. In particular, for real z and $|v/u| \to 1$, we find that $D_{\rm rel}$ becomes equal to unity.

The relative variance of the angular momentum is given by

$$D_{\rm rel} = \frac{(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2}{\langle \mathrm{J}^2 \rangle}. \tag{4.73}$$

For the CS $|j\theta\varphi\rangle$ of the states $|jm\rangle$ of generalized CS $|j, |m|, \theta\varphi\rangle$, we have

States :	$\lvert j\theta\varphi\rangle$	$\lvert jm\rangle$, $\lvert j\lvert m\rvert, \theta\varphi\rangle$	$\lvert j0\rangle$.
$D_{\rm rel}$:	$\frac{1}{j+1}$	$1-\frac{m^2}{j(j+1)}$	1.
Classical limit	$j\to\infty$	$j\to\infty$, $\lvert m/j\rvert\to 1$	—

Pictorially speaking, the relative variance defined by (4.71) enables us to determine the "degree of quantization" of a state in percent. We note, that in the classical limit, the variances $(\Delta T)^2$ are small only in comparison with the mean; they can be constants [CS$|z\rangle$ of the group W(1)] or they can increase [CS$|j\theta\varphi\rangle$ of the group SU(2)], which is completely consistent with the uncertainty relations.

For states with $D_{\rm rel} \to 0$, the quantum-mechanical formulas assume the classical form. In particular, the C–G coefficients relating the angular momentum CS $|j\theta\varphi\rangle$ lead to the classical formula for the composition of angular momenta as $j \to \infty$.

To conclude this Section, we consider the common procedure of passing to the classical limit by allowing $\hbar$ to tend to zero. It is well known that, in the formal limit as $\hbar \to 0$, both the commutators (i.e., uncertainties) and the operators $\hat{J}_z = -i\hbar\partial/\partial\varphi$, $\hat{p} = -i\hbar\partial/\partial x$ (and, hence, the corresponding physical quantities) must vanish. It is therefore usually demanded (see for example Refs. 58 and 125) that physical quantities such as the energy $\hbar\omega[|z|^2 + (1/2)]$ or $\hbar\omega[n + (1/2)]$, the angular momentum, $j\hbar$, and so on, must remain finite. This gives $\omega|z|^2 \to \infty$ or $\omega n \to \infty, j \to \infty$, i.e., we are back to the consideration of the limit of high quantum numbers (parameters of state). For $\hbar \to 0$, the uncertainties vanish only for states for which the expectation values (which must be finite) are much greater than the corresponding uncertainties, i.e., for realitive variance $D_{\rm rel} \to 0$.

5. THEORY OF CLEBSCH–GORDAN COEFFICIENTS AND THEIR COMPOSITIONS

5.1. Unified theory of C–G coefficients of the group SU(2)

The structure of coupling between a system and its subsystems is determined in quantum mechanics with the help of the C–G coefficients. The square of the modulus of a C–G coefficient specifies the probability that the system is in a given state for known states of the subsystems and, from the group-theoretic point of view, the C–G coefficients specify the transition from the basis of the direct product of two IR to an IR basis, and constitute a unitary matrix.

The C–G coefficients are unique objects on which the problems of quantum theory are focused. They arise under very different guises: either as objects in the group-theoretic approach or as probability amplitudes that may be looked upon as sums over paths or as descriptors of the structure of quantum objects or as powerful computational tools. We thus have an intersection of very different branches of physics and mathematics.[2] The theory of SU(2) Clebsch–Gordan coefficients, i.e., the theory of angular momenta, can be found in one form or another in most textbooks and monographs on quantum mechanics, and a summary of the formulas deduced by the mid-1970s is given in Ref. 21.

Until recently, the theory was confined almost entirely to the discrete basis. In 1974, Bellisard and Holtz[126] gave an expression for the C–G coefficient of SU(2) in CS spaces, but systematic studies began relatively recently.[5,15,127] In contrast to the standard theory of angular momenta[21,24] in which the C–G coefficients and their compositions take the form of awkward sums of products of factorials, in the CS basis they are functions of continuous variables and the generators for the usual C–G coefficients.

The familiar theory of C–G coefficients has undergone a radical change (see also Ref. 5). It is now formulated in a unified manner for all the bases as a special generalization of overlap formulas. Because of the convenient language in which the various quantities that appear in the theory are written, it is now possible to have both continuous and discrete coefficients at the same time, and the development of increasingly complicated structures of particular C–G coefficients and their compositions is something akin to the

child-like assembly of toy building blocks. We begin with definitions. For the sake of convenience, we write down in parallel the formulas for SU(2) that refer to the C–G coefficients $\langle j_1 u | j_2 v \| jw \rangle$ and the Wigner coefficients $\langle j_1 u | j_2 v | jw \| 00 \rangle$ in the CS basis ($u = \{u_1, u_2\}$) [first rows in (5.1)–(5.3)], and the C–G coefficients $\langle j_1 m_1 | j_2 m_2 \| jm \rangle$ and Wigner coefficients $\langle j_1 m_1 | j_2 m_2 | jm \| 00 \rangle$ in a discrete basis [second row in (5.1)–(5.3)]:

$$|jw\rangle = \int \langle j_1 u | j_2 v \| jw \rangle |j_1 u\rangle |j_2 v\rangle \mathrm{d}\mu_{j_1}(u) \mathrm{d}\mu_{j_2}(v),$$

$$|jm\rangle = \sum_{m_1, m_2} \langle j_1 m_1 | j_2 m_2 \| jm \rangle |j_1 m_1\rangle |j_2 m_2\rangle, \tag{5.1}$$

$$|j_1 u\rangle |j_2 v\rangle = \sum_j \int \langle jw \| j_1 u | j_2 v \rangle |jw\rangle \mathrm{d}\mu_j(w),$$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{j,m} \langle jm \| j_1 m_1 | j_2 m_2 \rangle |jm\rangle, \tag{5.2}$$

$$|00\rangle = \int \langle j_1 u | j_2 v | jw \| 00 \rangle |j_1 u\rangle |j_2 v\rangle |jw\rangle \mathrm{d}\mu_{j_1}(u) \mathrm{d}\mu_{j_2}(v) \mathrm{d}\mu_j(w),$$

$$|00\rangle = \sum_{m_1, m_2, m_3} \langle j_1 m_1 | j_2 m_2 | j_3 m_3 \| 00 \rangle |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle. \tag{5.3}$$

The integrals in these expressions are evaluated over the measure (4.37). It is also possible to use the angular variables (4.39) in terms of which which a state is represented by $|j\theta\varphi\rangle$ and the integration is performed over the measure $\mathrm{d}\mu_j(\theta, \varphi)$.

The formulas given by (5.1)–(5.3) are written for abstract bases. In functional (Hilbert) space, it is best to use overlaps that enable us to give the theory in standard unified form. The resulting relationships often involve the following overlaps:

$$\langle ju | jw \rangle = (\bar{u}_1 w_1 + \bar{u}_2 w_2)^{2j}, \quad \langle jm | jm' \rangle = \delta_{mm'}, \tag{5.4}$$

$$\langle ju | jw \| 00 \rangle = \frac{1}{(2j+1)^{1/2}} (\bar{u}_1 \bar{w}_2 - \bar{u}_2 \bar{w}_1)^{2j},$$

$$\langle jm | jm' \| 00 \rangle = \frac{(-1)^{j-m}}{(2j+1)^{1/2}} \delta_{m,-m'}, \tag{5.5}$$

$$\langle ju | jm \rangle = \left[\frac{(2j)!}{(j-m)!(j+m)!} \right]^{1/2} \bar{u}_1^{j-m} \bar{u}_2^{j+m}, \tag{5.6}$$

which may be looked upon as a special case of C–G coefficients and as the definition (5.6) of the symmetric basis. The decomposition of unity given by formulas such as

$$\int \langle ju | jw \rangle \langle jw | jv \rangle \mathrm{d}\mu_j(w) = \langle ju | jv \rangle,$$

$$\sum_m \langle ju | jm \rangle \langle jm | jw \rangle = \langle ju | jw \rangle. \tag{5.7}$$

plays an important role.

We note that the C–G coefficients themselves are essentially overlaps, i.e., of the basis of the direct product of two IR with an IR basis. Hence in addition to the C–G coefficients that relate bases of the same type, we can consider the mixed quantities $\langle j_1 u | j_2 v \| jm \rangle$ or $\langle j_1 m_1 | j_2 m_2 \| j\theta\varphi \rangle$. These unusual objects can be subjected to systematic analysis that has much in common with the analysis of the C–G coefficients in the CS basis.

For each property and each relationship between the C–G coefficients in the discrete basis there is an analog in the continuous basis, and transfer formulas between the two are available. The nature of the resulting relationships is clearly seen if we consider the orthonormalization formulas:

$$\int \langle j_1 u | j_2 v \| jw \rangle \langle jw' \| j_1 u | j_2 v \rangle \mathrm{d}\mu_{j_1}(u) \mathrm{d}\mu_{j_2}(v) = \langle jw' | jw \rangle, \tag{5.8}$$

$$\sum_j \int \langle j_1 u | j_2 v \| jw \rangle \langle jw \| j_1 u' | j_2 v' \rangle \mathrm{d}\mu_j(w) = \langle j_1 u | j_1 u' \rangle \langle j_2 v | j_2 v' \rangle, \tag{5.9}$$

$$\int |\langle j_1 u | j_2 v | jw \rangle|^2 \mathrm{d}\mu_{j_1}(u) \mathrm{d}\mu_{j_2}(v) \mathrm{d}\mu_j(w) = \dim D(j),$$

$$\sum_{m_1, m_2} \langle j_1 m_1 | j_2 m_2 \| jm \rangle \langle jm' \| j_1 m_1 | j_2 m_2 \rangle = \delta_{mm'}, \tag{5.10}$$

$$\sum_{j,m} \langle j_1 m_1 | j_2 m_2 \| jm \rangle \langle jm \| j_1 m'_1 | j_2 m'_2 \rangle = \delta_{m_1 m'_1} \delta_{m_2 m'_2}, \tag{5.11}$$

$$\sum_{m_1, m_2, m_3} |\langle j_1 m_1 | j_2 m_2 \| jm \rangle|^2 = \dim D(j).$$

The interrelation between the C–G coefficients in continuous and discrete bases is given by the following formulas:

$$\langle j_1 m_1 | j_2 m_2 | j_3 m_3 \| 00 \rangle = \int \langle j_1 u | j_2 v | j_3 w \| 00 \rangle \langle j_1 m_1 | j_1 u \rangle \langle j_2 m_2 | j_2 v \rangle \times \langle j_3 m_3 | j_3 w \rangle \mathrm{d}\mu_{j_1}(u) \mathrm{d}\mu_{j_2}(v) \mathrm{d}\mu_j(w), \tag{5.12}$$

$$\langle j_1 u | j_2 v | j_3 w \| 00 \rangle = \sum_{m_1, m_2, m_3} \langle j_1 m_1 | j_2 m_2 | j_3 m_3 \| 00 \rangle \langle j_1 u | j_1 m_1 \rangle \langle j_2 v | j_2 m_2 \rangle \langle j_3 w | j_3 m_3 \rangle, \tag{5.13}$$

$$\langle j_1 u | j_2 v \| j_3 w \rangle = \sum_{m_1, m_2, m_3} \langle j_1 m_1 | j_2 m_2 \| j_3 m_3 \rangle \langle j_3 m_3 | j_3 w \rangle \langle j_2 v | j_2 m_2 \rangle \langle j_1 u | j_1 m_1 \rangle. \tag{5.14}$$

The C–G and Wigner coefficients are related by

$$\langle j_1 u | j_2 v | jw \| 00 \rangle = \int \langle j_1 u | j_2 v \| jw' \rangle \langle jw' | jw \| 00 \rangle d\mu_j(w'),$$

$$\langle j_1 m_1 | j_2 m_2 | jm' \| 00 \rangle = \sum_m \langle j_1 m_1 | j_2 m_2 \| jm \rangle \langle jm | jm' \| 00 \rangle. \tag{5.15}$$

All the properties of the standard discrete theory of angular momenta can be systematically transferred to the C–G coefficients in the CS basis and become part and parcel of the unified theory. The new point is the appearance of an efficient computational formalism due to the integration over the measure μ. This is a very simple operation that enables us to perform calculations in the CS basis without recourse to the usual theory. For example, consider $\int f_1(z) \overline{f_2(z)} \mathrm{d}\mu(z)$ where f_1 and f_2 are polynomials in z. When the powers of any z_k and $\bar{z}_k$ under the integral sign are different, then integration of $\exp(in\varphi_k)$ with respect to the phase φ_k over one period gives zero. When the powers are equal, the integral over φ_k reduces to a multiplication by 2π. We thus obtain integrals that contain only $|z_k|^2 = x$:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = B(\alpha,\beta),\ \alpha,\beta>0. \tag{5.16}$$

The C–G coefficients in the CS basis are the invariants of a concrete group and constitute the normalized product of elementary generating invariants such as determinants and convolutions. They can always be written down in factorized form. For example,

$$\langle j_1u|j_2v\|j_3w\rangle = \rho\langle R_1u|R_1w\rangle\langle R_2v|R_2w\rangle\langle R_3u|R_3v\|00\rangle$$

where

$$R_1 = -j_1+j_2+j_3,\ R_2 = j_1-j_2+j_3,\ R_3 = j_1+j_2-j_3$$

are the powers to which the elementary generating invariants are raised:

$$\langle j_1u|j_2v\|j_3w\rangle = \rho(\bar{u}_1w_1+\bar{u}_2w_2)^{R_1}(\bar{v}_1w_1 + \bar{v}_2w_2)^{R_2}(\bar{u}_1\bar{v}_2-\bar{u}_2\bar{v}_1)^{R_3},$$

$$\rho = \left[\frac{(2j_1)!(2j_2)!(2j+1)!}{R_1!R_2!R_3!(R_1+R_2+R_3+1)!}\right]^{1/2}. \tag{5.17}$$

This is a typical example of C–G coefficients in the CS basis. They are written down directly in the form of the product of group invariants. The normalizing factor can be calculated with the help of (5.16). The symmetries of the C–G coefficients (invariants) correspond to the symmetries in discrete bases. The analogs of the discrete recurrence relations can be obtained with the help of overlaps such as (4.51), which contain differential operators (see also Refs. 128 and 129). The general scheme used to construct compositions of C–G coefficients also remains in the CS basis. In the simplest case,

$$\int\langle j_1u|j_2v\|j_{12}w\rangle\langle j_{12}w|j_3s\|jz\rangle\mathrm{d}\mu_{j_{12}}(w),$$

$$\sum_{m'}\langle j_1m_1|j_2m_2\|j'm'\rangle\langle j'm'|j_3m_3\|jm\rangle. \tag{5.18}$$

This formulation of the theory of C–G coefficients is quite general and is valid not only for SU(2), but also for other groups.

5.2. Transformation matrices

The relation between the different structures in quantum mechanics is determined by the transformation matrices. From the group-theoretic point of view, they specify the transition between different bases for direct products of irreducible representations. For SU(2), this corresponds to a transition between different compositions of n angular momenta ($n\geqslant 3$). The simplest case of the $n=3$ matrix

$$((j_1j_2)j_{12}j_3j|j_1(j_2j_3)j_{23}j),$$

was investigated by Racah.[32] The transformation matrices are usually written in the form of compositions of C–G coefficients summed over all the projections of the angular momenta. In their symmetric form, they are referred to as the j-symbols.[21,24,130] The transformation matrices are unitary in their construction. This leads to formulas of the form

$$\sum_{j_{23}}((j_1j_2)j_{12}j_3j|j_1(j_2j_3)j_{23}j)\,(j_1(j_2j_3)j_{23}j|(j_1j_2)j'_{12}j_3j) = \delta_{j_{12}j'_{12}}.$$

A detailed listing of the relationships containing the transformation matrices and including discrete variables is given in Ref. 21.

In the CS basis, the transformation matrices specify the recoupling of the angular momenta in terms of continuous variables:

$$\int\langle j_1w_1|j_2w_2\|j_{12}w_{12}\rangle\langle j_{12}w_{12}|j_3w_3\|jw\rangle\mathrm{d}\mu_{j_{12}}(w_{12}) = \sum_{j_{23}}((j_1j_2)j_{12}j_3j|j_1(j_2j_3)j_{23}j) \times\int\langle j_1w_1|j_{23}w_{23}\|jw\rangle\langle j_2w_2|j_3w_3\|j_{23}w_{23}\rangle\mathrm{d}\mu_{j_{23}}(w_{23}). \tag{5.19}$$

Naturally, the transformation matrices that depend only on the angular momenta are the same in any SU(2) basis. They can also be expressed in terms of an integral. Thus, for the transformation matrix in (5.19), which corresponds to the Racah coefficient, we have

$$\begin{aligned}((j_1j_2)j_{12}j_3j|j_1(j_2j_3)j_{23}j) &= \sum_{m_i}\langle j_1m_1|j_2m_2\|j_{12}m_{12}\rangle\langle j_{12}m_{12}|j_3m_3\|jm\rangle\\ &\times\langle jm\|j_{23}m_{23}|j_1m_1\rangle\langle j_{23}m_{23}\|j_2m_2|j_3m_3\rangle\\ &= \int\langle j_1w_1|j_2w_2\|j_{12}w_{12}\rangle\langle j_{12}w_{12}|j_3w_3\|jw\rangle\\ &\times\langle jw\|j_{23}w_{23}|j_1w_1\rangle\langle j_{23}w_{23}\|j_2w_2|j_3w_3\rangle\\ &\times\mathrm{d}\mu_{j_{12}}(w_{12})\mathrm{d}\mu_{j_2}(w_2)\mathrm{d}\mu_{j_3}(w_3)\mathrm{d}\mu_{j_1}(w_1)\mathrm{d}\mu_{j_{23}}(w_{23})\mathrm{d}\mu_j(w).\end{aligned} \tag{5.20}$$

Thus, the original expression for the transformation matrices can be written in two equivalent forms in terms of discrete and continuous variables. The transformation matrix itself (which depends only on the signature of the representation) is an invariant of SU(2). It can also be looked upon as a SU(3) tensor. This duality is illustrated by the following simple example of the connection between the SU(2) and SU(3) overlaps:

$$\begin{aligned}\begin{vmatrix} s_1 & s_2 & s_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}^R &= \frac{1}{(R+1)^{1/2}}\,{}_3\langle Ru|Rv|Rs\|00\rangle\\ &= \sum_{R_1+R_2+R_3=R}\frac{R!}{R_1!R_2!R_3!}s_1^{R_1}s_2^{R_2}s_3^{R_3}\\ &\times\left\langle\tfrac{1}{2}(R_2+R_3)u_1v_1\middle|\tfrac{1}{2}(R_1+R_3)u_2v_2\middle|\tfrac{1}{2}(R_1+R_2)u_3v_3\|00\right\rangle.\end{aligned} \tag{5.21}$$

The subscript 3 on the left shows that the overlap refers to SU(3). The loss of simplicity that is apparent here is not fortuitous: the quantities that appear in (5.21) refer to different groups. This aspect of the theory is discussed in Refs. 2 and 12 which employ the following decomposition of the 3×3 determinant $|u_{ik}|$:

$$|u_{ik}|^R = [(R!)^3(R+1)]^{1/2} \sum_{\sum_k R_{ik}=1} \|R_{ik}\| \cdot \prod_{i,k=1}^{3} \frac{u_{ik}^{R_{ik}}}{(R_{ik}!)^{1/2}}, \tag{5.22}$$

$$\|R_{ik}\| = \left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| = \left\| \begin{matrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{matrix} \right\| = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{5.23}$$

and the R-symbols given by (5.23) and their unions are used to construct a theory of the jm- and j-symbols in a form that is covariant under SU(3).

The transformation matrix (5.20), regarded as a sum of the products of the four symbols $\|R_{ik}\|$, corresponds to a SU(3) tensor containing four triples of indices. In an expression for an arbitrary j-symbol, such as (5.20), we have a summation over the projections of the angular momenta in the second and third rows of $\|R_{ik}\|$, and a $3\times n$-matrix can be constructed from the first-row elements.

This matrix defines the j-symbol in the R-representation. The triads of elements R^{l}_{1k} corresponding to the R_{1k} of the lth symbol of $\|R_{ik}\|$ form a column. Each triad is linked to three others and the links lie in a row. The essence of the unified R-form is clear from the following two examples of $6j$- and $9j$-symbols:

$$\left\| \begin{matrix} R_{11} & R_{21} & R_{31} & R_{41} \\ R_{12} & R_{22} & R_{32} & R_{42} \\ R_{13} & R_{23} & R_{33} & R_{43} \end{matrix} \right\| = \left\| \begin{matrix} -j_1+j_2+j_{12} & -j_1+j_{23}+j & j_2-j_3+j_{23} & j_{12}-j_3+j \\ j_1-j_2+j_{12} & j_1+j_{23}-j & -j_2+j_3+j_{23} & j_{12}+j_3-j \\ j_1+j_2-j_{12} & j_1-j_{23}+j & j_2+j_3-j_{23} & -j_{12}+j_3+j \end{matrix} \right\|, \tag{5.24}$$

$$\left\| \begin{matrix} R_{11} & R_{21} & R_{31} & R_{41} & R_{51} & R_{61} \\ R_{12} & R_{22} & R_{32} & R_{42} & R_{52} & R_{62} \\ R_{13} & R_{23} & R_{33} & R_{43} & R_{53} & R_{63} \end{matrix} \right\|. \tag{5.25}$$

This way of writing these expressions takes into account the particular character of the transformation matrices because it indicates all the links. We note that if each triad in the R-symbols (5.24) and (5.25) is represented by a point, and we retain the link lines, then the linking scheme in the R-symbols goes back to the usual graphs of the theory of angular momenta. The general form of the graphs for the symbols given by (5.24) and (5.25) is shown in Figs. 5a, b. The total number of symmetries in the $3\times n$-symbols is equal to the product of the number of permutations of rows and the number of symmetries of the corresponding geometric figure (Fig. 5). For (5.24), we have $3!\cdot 24$ symmetries, whereas for

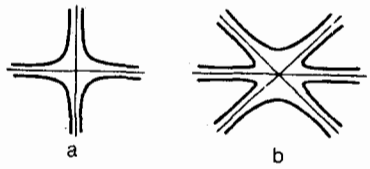

FIG. 5. Graphs for j-symbols: a—$6j$-symbol, b—$9j$-symbol.

(5.25) there are $3!\cdot 72$ symmetries. Figure 5 clearly shows the additional 3! symmetries as linked to the change in the order of lines approaching each vertex. This method takes into account all the symmetries, including the Regge symmetry.[131] The transformation matrices in the R-representation can also be discussed in the language of overlaps. The j-symbol given by (5.24) corresponds to an overlap of the form

$${}_3\langle R_1R_{1i}|R_2R_{2i}|R_3R_{3i}|R_4R_{4i}\|00\rangle,$$

in which, in addition, we have to indicate the linking scheme. This overlap is a SU(3) tensor that can be transformed into an invariant by using four vectors of the form

$$\left(\frac{R!}{R_{i1}!R_{i2}!R_{i3}!}\right)^{1/2} u_{i1}^{R_{i1}} u_{i2}^{R_{i2}} u_{i3}^{R_{i3}},$$

and can be written as the overlap

$${}_3\langle R_1u_1|R_2u_2|R_3u_3|R_4u_4\|00\rangle.$$

The relation between these overlaps can be written in the standard form

$$\begin{aligned} {}_3\langle R_1u_1|R_2u_2|R_3u_3|R_4u_4\|00\rangle &= \sum_{R_{1i},R_{2i},R_{3i},R_{4i}} {}_3\langle R_1u|R_1R_{1i}\rangle\, {}_3\langle R_2u_2|R_2R_{2i}\rangle \\ &\times {}_3\langle R_3u|R_3R_{3i}\rangle\, {}_3\langle R_4u_4|R_4R_{4i}\rangle \\ &\times {}_3\langle R_1R_{1i}|R_2R_{2i}|R_3R_{3i}|R_4R_{4i}\|00\rangle. \end{aligned} \tag{5.26}$$

We thus have a unified theory for the SU(2) transformation matrices and for the C–G coefficients, which include both discrete and continuous bases.

This theory emerges within the framework of the group SU(2), but eventually becomes part of the SU(3) overlap theory.

5.3. C–G coefficients of other groups and the classical limit

Even a cursory glance at the theory of the C–G coefficients of the group SU(2) reveals certain characteristic features of quantum structures. They include the absence of the impenetrable boundary, which is typical of the classical situation, between system and subsystem, [the C–G coefficients contain the symmetry not only between the subsystems (j_1m_1) and (j_2m_2), but also between them and the system (jm); see also Sec. 6], the possibility of describing different partitions into subsystems in terms of group characteristics, and the irreducibility of quantum structures to structures described by one group. Particular systems such as atoms, nuclei, and molecules are also characterized by the existence

of a whole series of different groups. Even when we systematically construct objects, starting with a single group, the appearance of other groups becomes inevitable (see also Ref. 132).

As already noted, the fundamentals of the theory of C–G coefficients in the CS basis, which were discussed above for SU(2), continue to be valid for an arbitrary group. Particular calculations are essentially based, as in the case of SU(2), on the decomposition of unity (M is the IR signature)

$$\int |Mz\rangle\langle Mz|\,d\mu_M(z) = \hat{1}, \tag{5.27}$$

from which we have, for example, relations of the form

$$\sum_M \int |\langle M_1 u|M_2 v\|Mw\rangle|^2 d\mu_M(w) = 1,$$
$$\int |\langle M_1 u|M_2 v\|Mw\rangle|^2 d\mu_{M_1}(u)d\mu_{M_2}(v) = 1. \tag{5.28}$$

The probabilistic meaning of the C–G coefficients is the same as for SU(2). The values of the square of the modulus in (5.28) are set by the probability density over the measure $d\mu_M$ that the system is in the state $|Mw\rangle$ if the subsystems are in states $|M_1 u\rangle$ and $|M_2 u\rangle$. For discrete series $D^+(M)$ of the group SU(1,1), the C–G coefficient in the CS basis has much in common with the C–G coefficient of the SU(2) group [cf. (5.17)]:

$$\langle M_1 u|M_2 v\|Mw\rangle = \rho'(M_1,M_2,M)\,(\bar{u}_1 w_1 - \bar{u}_2 w_2)^{M_1-\alpha} \times (\bar{v}_1 w_1 - \bar{v}_2 w_2)^{M_2-\alpha}(\bar{u}_1 v_2 - \bar{u}_2 \bar{v}_1)^{\alpha},$$

where

$$M = 2j,\ 2\alpha = M - M_1 - M_2 \geq 0,\ M_1, M_2, M$$

are negative,

$$\rho'(M_1,M_2,M) = \left(\frac{\Gamma(-M_1+\alpha)\Gamma(-M_2+\alpha)\Gamma(-M-1+\alpha)}{\Gamma(-M_1)\Gamma(-M_2)\Gamma(-M-1)\alpha!}\right)^{1/2}. \tag{5.29}$$

In this case, we can effectively use the formalism of integration over the measure μ. The basic relation is

$$\int_0^\infty x^{\alpha-1}(x+1)^{-\beta}dx = B(\alpha,\beta-\alpha),\ 0<\alpha<\beta, \tag{5.30}$$

which is the analog of (5.16). When we construct the C–G coefficients and their compositions in the CS basis and in the discrete basis, we use the same overlap technique as for SU(2).

For the groups SU(N) and SU($N-1$,1), the C–G coefficients in the CS basis were obtained in Refs. 5, 11, and 15 (see also Appendix). The analogs of (5.16) and (5.30) are given in Ref. 5. Their right-hand sides contain the multidimensional B-function. The construction of the C–G coefficients is based on the analogy between (5.4) and (5.5) which have the following form for SU(3):

$${}_3\langle Mv|M\xi\rangle = (\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3)^M,$$

$${}_3\langle Mu|Mv|Mw\|00\rangle = \frac{1}{[(M+1)^2(M+2)/2]^{1/2}}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}^M. \tag{5.31}$$

The C–G coefficients in the CS basis have a classical limit for certain parameter values. Consider the transition to the classical limit by considering the example of the group SU(2) (Ref. 15). In terms of angle variables (4.39), we have for the CS

$$|j\theta\varphi\rangle = \sum_{m=-j}^{j}\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2} \times \left(\cos\frac{\theta}{2}\right)^{j+m}\left(\sin\frac{\theta}{2}\right)^{j-m}\exp(-im\varphi)\,|jm\rangle. \tag{5.32}$$

The overlap is

$$|\langle j_1\theta_1\varphi_1|j_2\theta_2\varphi_2\rangle|\left(\cos\frac{\theta_{12}}{2}\right)^2,$$

where θ_{12} is the angle between $\mathbf{j}_1$ and $\mathbf{j}_2$, and the C–G coefficient is given by[15]

$$|\langle j_1\theta_1\varphi_1|j_2\theta_2\varphi_2\|j\theta\varphi\rangle| = \rho(j_1 j_2 j_3)\left(\sin\frac{\theta_{12}}{2}\right)^{j_1+j_2-j} \times \left(\cos\frac{\theta_{23}}{2}\right)^{-j_1+j_2+j}\left(\cos\frac{\theta_{13}}{2}\right)^{j_1-j_2+j},$$

where $\theta_{12}, \theta_{23}, \theta_{13}$ are the angles between $\mathbf{j}_1$ and $\mathbf{j}_2$, $\mathbf{j}_2$ and $\mathbf{j}_3$, and $\mathbf{j}_1$ and $\mathbf{j}_3$.

In the basis for the direct product $|j_1\theta_1\varphi_1\rangle \otimes |j_2\theta_2\varphi_2\rangle$,

$$\langle\hat{\mathbf{J}}^2\rangle = j_1(j_1+1) + j_2(j_2+1) + 2j_1 j_2\cos\theta_{12} = j_{\rm cl}^2 + j_1 + j_2; \tag{5.33}$$

where $j_{\rm cl}$ is the classical angular momentum, given by

$$j_{\rm cl}^2 = j_1^2 + j_2^2 + 2j_1 j_2\cos\theta_{12}. \tag{5.34}$$

For large values of j_1, j_2, the mean square deviation $\Delta\mathbf{J} = (j_1+j_2)^{1/2}$ is much smaller than either of the combining angular momenta, $\langle\hat{\mathbf{J}}^2\rangle \approx j_{\rm cl}^2$, and we obtain the classical formula for the addition of the angular momenta, given by (5.34).

For large j_1, j_2, the coherent C–G coefficients are significantly different from zero only near values of j, θ, φ given by the classical formula for the composition of angular momenta. The usual C–G coefficients $\langle j_1 m_1|j_2 m_2\|jm\rangle$ oscillate rapidly as $j\to\infty$ (Refs. 21 and 133), i.e., they have no classical limit, which is not unexpected because they couple states that are highly nonclassical. In the basis of the direct

product $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$, the variance $(\Delta \mathbf{J})^2 = (j_1^2 - m_1^2 + j_2^2 - m_2^2 + j_1 + j_2)$, i.e., it is generally of the same order as $\langle \hat{\mathbf{J}}^2 \rangle$ for any j_1 and j_2, however large.

6. THEORY OF PROBABILITY AMPLITUDES

6.1. IR bases, probability distributions, and limit theorems

In quantum mechanics, the square of the modulus of the wave function $|\psi_n|^2$ gives the probability of finding the system in the nth state, and the function ψ_n itself is referred to as the probability amplitude. The wave function has often been referred to as an auxiliary concept used to calculate different physical quantities. However, similarly to the Lobachevskiĭ geometry which, on the one hand, was a generalization of ordinary geometry and, on the other, a mapping or analog of it, the theory of probability amplitudes may be looked upon as generalization of probability theory, but also as a certain parallel theory that has analogies with ordinary probability theory for many of its objects. This is the starting point of Ref. 5 which presents the theory of probability amplitudes; its results will be extensively used in this Section.

We note that there is a limited number of publications that are directly concerned with the theory of probability amplitudes. We begin with Feynman's papers in which, as part of his development path integrals, he gave (probably for the first time) a systematic treatment of probability amplitudes as independent quantities (see Ref. 4). Maslov[134,135] considered complex Markov chains and on this basis gave a derivation of the discrete analog of the Feynman integral and the Schrödinger equation. Dicke (see Ref. 19) proposed a simple two-state scheme for the probability amplitudes. Various aspects of the relation between probabilistic and group-theoretic characteristics are discussed in Refs. 5, 11, 136, and 137.

The theory of probability amplitudes considers events in which a given system is in a given (pure) state. This set of events serves as the foundation for the systematic development of the theory, including the derivation of the basic equation. In principle, it is possible to take into account more complicated events (transmissions by a slit or measurement), but these are of secondary importance and the structure of the sequence of events then becomes much more complicated. According to Ref. 5, several qualititively new points arise in the theory of probability amplitudes, which distinguish it from the usual theory of probability. They are as follows.

1. Whereas, in probability theory, events form a σ-algebra,[138] in the theory of probability amplitudes they form a Hilbert space. A set of elementary events is then associated with the basis elements of the Hilbert space. The conditions of completeness of the set of elementary events $|n\rangle$ or $|y\rangle$ are written in the form

$$\sum_n |n\rangle\langle n| = \hat{1}, \quad \int |y\rangle\langle y| d\mu(y) = \hat{1}. \tag{6.1}$$

2. In probability theory, each elementary event A_n has associated with it a real nonnegative number p_n on the interval [0,1] with the normalization condition $\Sigma p_n = 1$; in the theory of probability amplitudes, on the other hand, the latter is a hypercomplex number $\psi_n = \langle n|\psi\rangle$ (real, complex, or quaternion), subject to the normalization condition

$$\sum_n \psi_n \bar{\psi}_n = 1. \tag{6.2}$$

Once $\{\psi_n\}$ is given, this implies that $|\psi\rangle$ is also given. For a continuous set of elementary states, each state can be assigned a function $\psi(y) = \langle y|\psi\rangle$, i.e., the probability amplitude density over the measure $d\mu(y)$.

3. In the usual probability theory, we can choose p_n so that any random quantity will be precisely defined; for example, $p_1 = 1$ means that the variance is

$$D(x) = \sum_n x_n^2 p_n (1 - p_n) = 0.$$

In the theory of probability amplitudes, we have an uncertainty principle: it is impossible to have an event, i.e., it is *impossible to choose ψ_n, for which all random quantities* (observables) will be precisely determined.

4. In probability theory, the transition matrices (operators) that specify the complete probability picture of possible changes between time t and t' form a subgroup, whereas in the theory of probability amplitudes they form a group. This means that it is possible to use the fully developed formalism of group theory to describe the extensive set of different structures that arise.

5. The usual theory of Markov processes employs the formalism based on the Kolmogorov-Chapman equation (the generalized Markov equation for the stochastic evolution operator), whereas the theory of probability amplitudes relies on the equation for the unitary evolution operator. Under certain additional conditions (no discontinuities), the transition probabilities satisfy the Fokker-Planck-Kolmogorov equations, whereas the transition amplitudes satisfy the nonrelativistic Schrödinger equation.

In a systematic presentation of the theory of probability amplitudes, it is useful to consider the parallel with the powerful formalism of the usual probability theory—above all the specific distribution functions—and also the limit theorems. This reveals a qualitatively new aspect, namely, the resulting distributions have group characteristics. In addition to the distributions and limit theorems of the theory of probability amplitudes, we present below the probabilistic treatment of C–G coefficients and their compositions, and also the fundamentals of the theory of Markov processes for the probability amplitudes. This enables us to develop the general principles of quantum theory in a consistent way.

We now turn to the consideration of specific distributions.

If the complex amplitudes ψ_i satisfying (6.2) form the basis for the fundamental irreducible representation $D(10...0)$ of the group SU(N), the complex polynomial distributions

$$\Psi_M^{n_i} = \left(\frac{M!}{n_1!...n_N!}\right)^{1/2} \psi_1^{n_1}...\psi_N^{n_N}, \quad M = \sum_{i=1}^{N} n_i, \tag{6.3}$$

form the basis for the irreducible representation $D(M0...0)$ for fixed M. If we multiply it by the complex conjugate that transforms under the irreducible representation $D(0...0M)$, we obtain the polynomial distribution for the usual probabilities $p_i = |\psi_i|^2$, $\Sigma p_i = 1$

$$\mathscr{P}_M^{n_i} = \Psi_M^{n_i}\bar{\Psi}_M^{n_i} = \frac{M!}{n_1!\ldots n_N!}p_1^{n_1}\ldots p_N^{n_N}. \tag{6.4}$$

Thus, the symmetric basis (6.3) for the irreducible representation $D(M0\ldots0)$ of the groups SU(N) is a polynomial distribution for the complex amplitudes from the standpoint of probability theory.

The negative binomial distribution

$$\mathscr{P}_n^{n_1n_2} = \frac{(-n+n_1-1)!}{(-n-1)!n_1!}p^{-n}q^{n_1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n_2+1)\,n_1!}\left(\frac{1}{p}\right)^{n_2}\left(-\frac{q}{p}\right)^{n_1} \tag{6.5}$$

for integral $n<0$ gives the probability that, before success with the number $|n|$ is achieved, there are n_1 failures (probability of success p; probability of failure $q=1-p$). For fixed $n=n_1+n_2$, we have

$$\sum_{n_1=0}^{\infty}\mathscr{P}_n^{n_1n_2} = \left(\frac{1}{p}-\frac{q}{p}\right)^n = 1.$$

The negative binomial distribution for complex amplitudes has the form

$$\Psi_n^{n_1n_2} = \left(\frac{\Gamma(-n_2)}{n_1!\Gamma(-n)}\right)^{1/2}\psi_1^{-n}\psi_2^{n_1}$$

$$= \left[\frac{\Gamma(-n_2)(-1)^{n_1}}{n_1!\Gamma(-n)}\right]^{1/2}u_1^{n_1}u_2^{n_2}, \tag{6.6}$$

$$\mathscr{P}_n^{n_1n_2} = \left|\Psi_n^{n_i}\right|^2,\ u_1 = -i\psi_2/\psi_1,\ u_2 = 1/\psi_1,$$
$$p = |\psi_1|^2 = |1/u_2|^2,\ q = |\psi_2|^2 = |u_1/u_2|^2.$$

Since $p+q=1$, we have $-u_1\bar{u}_1+u_2\bar{u}_2=1$ which is an invariant of SU(1,1); u_1 and u_2 form a basis for the fundamental nonunitary irreducible representation $D(1/2)$. Under transformations from SU(1,1), the amplitudes ψ_1 and ψ_2 transform under a bilinear transformation, whereas $\Psi_n^{n_1n_2}$ and $\bar{\Psi}_n^{n_1n_2}$ with fixed n form a basis for unitary irreducible representations in the positive and negative series, $D^+(j)$ and $D^-(j)$, respectively. In these expressions, $j=n/2$, $m=(n_1-n_2)/2$, and $m=-j,\ -j+1,\ldots$ for $D^+(j)$ and $m=j,\ j-1,\ j-2,\ldots$ for $D^-(j)$. According to (6.5), for fixed $n=n_1+n_2$, we have

$$\sum_{n_1=0}^{\infty}\left|\Psi_n^{n_1n_2}\right|^2 = 1.$$

The Poisson distribution for the complex amplitudes is

$$\Psi_k = \frac{u^k}{(k!)^{1/2}}e^{-|u|^2/2}. \tag{6.7}$$

If we multiply this by $\bar{\Psi}_k$ we obtain the usual Poisson distribution with the parameter $|u|^2$:

$$\mathscr{P}_k = \left|\Psi_k\right|^2 = \frac{|u|^{2k}}{k!}e^{-|u|^2},\ \sum_{n=0}^{\infty}|\Psi_k|^2 = 1.$$

In this expression, the Ψ_k form a basis for the unitary infinite-dimensional irreducible representation of the Heisenberg group. By analogy with the usual probabilities, the Poisson distribution for the probability amplitudes can be obtained from the binomial distribution for the complex amplitudes. Thus, for the probability p_k we have for $n\to\infty$, fixed $a=np$, and finite $k\ll n$,

$$\lim_{n\to\infty}\mathscr{P}_k^{k,n-k} = \lim_{n\to\infty}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} = a^ke^{-a}/k!,$$

and we find that for the amplitudes $\psi_k = |\psi_k|e^{i\varphi_k}$ with fixed $|u|^2 = |\psi_1|^2n$ and large $n\ll k$

$$\Psi_n^{k,n-k} = \frac{n!}{k!(n-k)!}\psi_1^k\psi_2^{n-k}\longrightarrow\frac{u^k}{\sqrt{k!}}e^{-|u|^2/2}e^{i\varphi_2n},$$
$$u = |\psi_1|n^{1/2}e^{i(\varphi_1-\varphi_2)}.$$

The normal distribution that is the limiting case of many discrete distributions plays a particular part among the distributions of probability theory. The density of the normal distribution

$$\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\rho^2(x,x_0)/2\sigma^2} \tag{6.8}$$

decreases monotonically and rapidly with increasing distance ρ from the position x_0 of the maximum. The reduction in the square of the modulus of the probability density in proportion to $\exp(-\rho^2)$ is indeed a general property of analogs of the normal distribution of probability amplitudes. In the theory of representations of groups, they correspond to coherent states. The CS overlap functions $\langle u|z\rangle$ are the least "smeared out" in homogeneous space, and we shall write them in the form

$$|\langle u|z\rangle|^2 = \exp(-\rho^2(u,z)), \tag{6.9}$$

where $\rho(u,z)$ is a symmetric invariant under group transformations;[16] $\rho(u,z)\geqslant0$, $\rho(u,z)=\rho(z,u)$ and $\rho(u,z)=0$ only for $u=z$ (the symmetric differs from the metric, i.e., distance, by the fact that the triangle rule does not necessarily apply to it).

The CS overlap function $\langle u|z\rangle$ is the analog of the normal distribution of the theory of probability amplitudes. For the Heisenberg group, the square of the CS overlap function is given by

$$|\langle u|z\rangle|^2 = \exp|z-u|^2 = \exp[-(x-x_1)^2-(y-y_1)^2],$$
$$u = x+iy,\ z = x_1+iy_1, \tag{6.10}$$

and corresponds to the usual two-dimensional normal distribution with

$$\int|\langle z|u\rangle|^2d^2u/\pi = 1.$$

For squeezed states, (4.29) shows that the square of the overlap function for the generalized CS corresponds to the two-dimensional normal distribution with different variance along different directions in homogeneous space.

The square of the CS overlap integral for SU(2) is, according to (4.43),

$$|\langle j\theta_1\varphi_1|j\theta_2\varphi_2\rangle|^2 = \exp(-\rho^2(\theta_1,\varphi_1;\theta_2,\varphi_2)) = [\cos(\theta_{12}/2)]^{2j},$$

where θ_{12} is the angle between the vectors determined by the states $|j\theta_1\varphi_1\rangle$ and $|j\theta_2\varphi_2\rangle$. For large j, the square of the modulus of the overlap integral [with allowance for $d\mu_j$ in (4.39)] approaches the normal distribution. Typically, the above discrete and continuous amplitude distributions are the IR bases for Lie groups.

As in the usual probability theory, the law of large numbers and the limit theorems play an important part in the theory of probability amplitudes. We shall now consider an analog of the law of large numbers for a set of observables (random quantities), i.e., for the state of the system as a whole. The coherent states parametrized by points in phase space become orthogonal to one another in the classical limit. This is clear from the CS overlap integrals (4.22), (4.36), and (4.43). In the limit of large quantum numbers ($j\to\infty$ or $|z|\to\infty$ in the above examples), the mean square deviations of random quantities in coherent states becomes negligible in comparison with their mathematical expectations.

The law of large numbers is also valid for distributions describing the characteristics of a system when subsystems are combined, i.e., for the C–G coefficients. The usual C–G coefficients $\langle j_1 m_1|j_2 m_2\|jm\rangle$ of the group SU(2) oscillate rapidly for $j\to\infty$ because they couple states that are very different from classical states. For large j, the C–G coefficients in the CS basis are significantly nonzero only near the values of j, θ, φ given by the classical formula for the addition of angular momenta, which is also obtained in the limit as j_1, $j_2\to\infty$. The transition to the classical limit, discussed in Secs. 4 and 5, now has a clear probabilistic interpretation.

The distributions associated with coherent states have their own limit theorems. In previous papers[15,16] we examined coherent states $|Mz_1...z_{N+1}\rangle$ that were the symmetric irreducible representations of SU($N+1$) and SU(N,1), and used the transformation $\alpha^i = z_i/z_N$ to perform a transition to the N-dimensional projective space $P_N(\mathbf{C})$ or the open sphere $D_N(\mathbf{C})$, respectively; we showed that, for large M (curvature of space $\sim 1/M$), the CS overlap integral was nonzero only for small $\Delta\alpha^i$, and could be written in the form of the multidimensional normal distribution

$$\langle M\alpha|M\alpha'\rangle = \exp(-\rho^2) \approx \exp(-g_{ik}\Delta\alpha^i\Delta\bar{\alpha}^k),\ \Delta\alpha^i = \alpha' - \alpha. \tag{6.11}$$

The expressions given by (6.11) have a simple geometric interpretation: if the distribution is nonzero only in a small neighborhood of a point α (and space can be looked upon as locally flat), then the corresponding CS become the CS of flat phase space (CS belonging to the Heisenberg group).

The first correspondence principle follows from the law of large numbers for the CS, which corresponds to the fact that, apart from a normalizing factor, in the measure $d\mu$, $\langle M\alpha|M\beta\rangle$ tends to $\delta(\alpha-\beta)$ as $|M|\to\infty$. Using the formula for the *-multiplication of symbols (4.66), we obtain the following expression for the mathematical expectation $\mathscr{M}$ up to terms of order $1/M$:

$$\mathscr{M}[AB] = \mathscr{M}[BA] = \mathscr{M}[A]\mathscr{M}[B], \tag{6.12}$$

where A and B are random quantities that correspond to the group generators. Since the overlap integrals can be written in the form of (6.11), i.e., if we use the limit theorems for the CS, we find that, for large $|M|$, the commutator transforms to a Poisson bracket (this is the second correspondence principle):[16]

$$\mathscr{M}[AB-BA] = g^{ik}\left(\frac{\partial A(\alpha,\bar{\alpha})}{\partial\bar{\alpha}^k}\frac{\partial B(\alpha,\bar{\alpha})}{\partial\alpha^i} - \frac{\partial A(\alpha,\bar{\alpha})}{\partial\alpha^i}\frac{\partial A(\alpha,\bar{\alpha})}{\partial\bar{\alpha}^k}\right) \tag{6.13}$$

where $\|g^{ik}\| = \|g_{ik}\|^{-1}$, $A(\alpha,\bar{\alpha}) = \mathscr{M}(A)$, $B(\alpha,\bar{\alpha}) = \mathscr{M}(B)$ are functions on homogeneous space.

Thus, in quantum mechanics, the first and second correspondence principles are associated with the law of large numbers and the above limit theorems. It is clear that the development of a general theory of representations for probability amplitudes is a very pressing problem.

6.2. C–G coefficients as analogs of the hypergeometric distribution

In group theory, the C–G coefficients relate different bases and define the redecomposition of the probability distributions corresponding to the bases. The hypergeometric distribution of the usual probability theory

$$\begin{Bmatrix} n_1^1 & n_2^1 \\ n_1^2 & n_2^2 \end{Bmatrix} = \frac{C^{n_1^1 n_2^1}C^{n_1^2 n_2^2}}{C^{n_1 n_2}} = \frac{n^1!n^2!n_1!n_2!}{n!n_1^1!n_1^2!n_2^1!n_2^2!}$$

$$n_i = n_i^1 + n_i^2,\ n^i = n_1^i + n_2^i,\ n = \sum_{i,k} n_k^i \tag{6.14}$$

specifies the redecomposition of binomial distributions

$$\mathscr{P}_{n^1}^{n_1^1 n_2^1}\mathscr{P}_{n^2}^{n_1^2 n_2^2} = \begin{Bmatrix} n_1^1 & n_2^1 \\ n_1^2 & n_2^2 \end{Bmatrix}\mathscr{P}_n^{n_1 n_2} \tag{6.15}$$

and is the analog of the SU(2) coefficient of the theory of complex amplitudes. Analysis of the connection with the hypergeometric distribution and its generalizations is part of the problem of the relationship between classical and quantum properties, and enables us to understand the probabilistic meaning of the symmetries of the C–G coefficients.

The hypergeometric distribution

$$\begin{Bmatrix} n_1^1 & n_2^1 \\ n_1^2 & n_2^2 \end{Bmatrix}$$

is defined as the probability that if we make a selection from $n^1 = n_1^1 + n_2^1$ objects in a total set of

$$n = \sum_{i,k} n_i^k$$

objects containing $n_1 = n_1^1 + n_1^2$ objects with properties A, and $n_2 = n_2^1 + n_2^2$ objects with properties $\overline{A}$, the number of objects with property A will be n_1^1. The numerical value of the hypergeometric distribution is unaffected if we interchange columns and rows or if we perform the operation of transposition.

The distribution given by (6.14) has a simple generalization to the case where the total number of properties is I and the number of sets of objects into which the original set can be partitioned is K. The probability of finding n_i^k objects in the k th set with properties $i(i = \overline{1,I},\ k = \overline{1,K})$ is given by

$$\{n_i^k\} = \begin{Bmatrix} n_1^1 \ldots n_I^1 \\ \ldots\ldots\ldots \\ n_1^K \ldots n_I^K \end{Bmatrix} = \frac{\prod_i n_i! \prod_k n^k!}{n! \prod_{i,k} n_i^k!} = \{n_k^i\}. \tag{6.16}$$

The numerical value of the multidimensional hypergeometric distribution is unaffected by the interchange of rows and columns and by the operation of transposition. The latter means that the property of belonging to a given sample is no different, from the probabilistic standpoint, from the properties of the objects themselves. The normalization of (6.16) takes the form

$$\sum_{n_k^i} \{n_k^i\} = 1,\ 0 < \{n_i^k\} \le 1. \tag{6.17}$$

where the sum is evaluated for fixed sums of rows and columns of the matrix $\|n_k^i\|$, $n_i = \Sigma_k n_i^k$, $n^k = \Sigma_i n_i^k$.

We note the transformation of the $I \times K$ hypergeometric distribution to the $I \times (K-1)$ distribution

$$\sum_{\alpha^i} \{n_1^i \ldots n_{K-1}^i - \alpha^i, \alpha^i\} = \{n_1^i \ldots n_{K-1}^i\}, \tag{6.18}$$

which plays an important role in the theory of hypergeometric functions of many variables (see Sec. 7). This can be proved for nonintegral n_k^i as well. The sum is evaluated for fixed $\alpha = \Sigma \alpha_i$. From the probabilistic point of view, (6.18) corresponds to the composition of the elements of the K th and the $(K-1)$-th sets into one.

In the same way that the hypergeometric distribution arose from the redecomposition of the binomial distributions, the C–G coefficients define the reconnection of the IR bases, i.e., complex binomial distributions. The new point is the appearance of the parameter $\alpha = j_1 + j_2 - j$ which corresponds to the convolution of the basis functions $D(j_1)$ and $D(j_2)$ into the invariant $(\psi_1\psi_2' - \psi_1'\psi_2)^\alpha = 1$.[5,137] Symmetrization ($\alpha = 0$) corresponds to the usual hypergeometric distribution of degree 1/2.

In contrast to the sampling problem (the problem of partition into subsystems), the problem of finding the characteristics of a system under the composition of subsystems has a trivial solution in probability theory: the properties of the system as a whole are uniquely determined by the properties of the subsystems of which it is composed. In the theory of probability amplitudes, the problems of sampling and of composition of subsystems ("composition of angular momenta") are equivalent and their solution is given by the corresponding C–G coefficients. In the simplest case of two properties A and $\overline{A}$, these are the C–G coefficients of the group SU(2). We shall write them in the form of the Regge symbol

$$\begin{Vmatrix} 2j^2 - \alpha & 2j^1 - \alpha & \alpha \\ n_1^1 & n_2^1 & 2j_2 - \alpha \\ n_1^2 & n_2^2 & 2j_1 - \alpha \end{Vmatrix}$$

$$= \langle j_1 m_1 | j_2 m_2 \| jm \rangle \frac{(-1)^{-j_1+j_2-\alpha-m}}{(2j+1)^{1/2}}, \tag{6.19}$$

where $j = j_1 + j_2 - 2\alpha$, $m = m_1 + m_2$, $n_1^i + n_2^i = 2j_i$, $n_i^1 + n_i^2 = 2j^i$, $n_1^i - n_2^i = 2m_i$.

There is a profound analogy between the symmetries of the C–G coefficients and of the hypergeometric distribution. Permutation of the representations $D(j_1)$ and $D(j_2)$ in products corresponds to the permutation of two particular partitions of the original set of objects. A change in the sign of the projections of the angular momenta corresponds to the permutation of the numeration of the internal properties of a set. Thus, transposition of the Regge symbol corresponds to the transposition of the matrix of the hypergeometric distribution (this symmetry is due to the fact that the partition of a set of objects in accordance with their internal properties is no different from the probabilistic point of view than partition by sampling). The symmetry of the C–G coefficients that corresponds to the permutation of the resulting representation with one of the factors (i.e., symmetry due to the equivalence of the sampling and partition problems) has no analog. When this symmetry is added to the symmetries of the hypergeometric distribution by the successive application of transposition and permutation, we obtain the 72 Regge symmetries of the C–G coefficients of the group SU(2). It is then clear from (6.19) that the matrix of the hypergeometric distribution can itself be looked upon as part of a Regge symbol.

The negative hypergeometric distribution is defined as a recoupling of negative binomial distributions. The corresponding formula is identical with (6.15) except that n_2^i are negative, $|n_2^i| > n_1^i$, and n_1^i are positive integers. The negative hypergeometric distribution can be defined as a conditional probability. In particular, if we continue to sample from $|n_2|$ elements until the number of elements with property A reaches a given number $|n^1|$, the probability that the number of elements with property $\overline{A}$ in a sample will be n_1^1, will be given by

$$\begin{Bmatrix} n_1^1 & n_2^1 \\ n_1^2 & n_2^2 \end{Bmatrix} = \frac{(-n^1 + n_1^1 - 1)!(-n^2 + n_1^2 - 1)!(-n-1)!n_1!}{(-n^1 - 1)!n_1^1!(-n^2 - 1)!n_1^2!(-n + n_1 - 1)!}$$
$$= \begin{Bmatrix} n_1^1 & -n^1 - 1 \\ n_1^2 & -n^2 - 1 \end{Bmatrix} \frac{-n-1}{-n_2 - 1}. \tag{6.20}$$

The C–G coefficients $\langle j_1^+ m_1 | j_2^+ m_2 \| j^+ m \rangle$ of the group SU(1,1) define the recoupling of the negative binomial distribution for complex amplitudes, i.e., they are the analogs of

the negative hypergeometric distribution. The C–G coefficients for the discrete series of SU(1,1) can be expressed in terms of the C–G coefficients of SU(2) in accordance with (2.36), and can be written in the form of the Regge symbol[5]

$$\langle j_1 m_1 | j_2 m_2 \| jm \rangle = \left\| \begin{matrix} -n-\alpha-2 & n_1-\alpha & \alpha \\ n_1^1 & -n^1-\alpha-1 & -n_2^2-1 \\ n_1^2 & -n^2-\alpha-1 & -n_2^1-1 \end{matrix} \right\| \times (-n-1+2\alpha)^{1/2},$$

$$j_l = \frac{n_1^l + n_2^l}{2}, \quad m_l = \frac{n_1^l - n_2^l}{2}, \quad j = j_1 + j_2 - \alpha. \tag{6.21}$$

6.3. Markov processes and the Schrödinger equation

The theory of Markov processes provides the foundation for the description of both stochastic and quantum-mechanical process. A Markov process for the probability amplitudes is defined by analogy with ordinary probabilities as a process without aftereffect: the state of a system at a particular instant of time depends only on which state it was in at an immediately preceeding time t, but is independent of its states at previous times $\tau < t$.

We shall follow Ref. 5 and consider stochastic and quantum Markov processes in parallel. In the case of a finite or denumberable number of states

$$p_k(t) = p_k^i(t, t_0) p_i(t_0), \quad p_k^i(t, t) = \delta_k^i, \tag{6.22}$$

$$\psi_k(t) = u_k^i(t, t_0)\psi_i(t_0), \quad u_k^i(t, t) = \delta_k^i, \tag{6.23}$$

where $p_k^i(t,t_0)$ $[u_k^i(t,t_0)]$ are the conditional probabilities (amplitudes) for a transition from state i to state k, $t > t_0$. From the normalization conditions $\Sigma p_i = 1$, $p_i \geqslant 0$ and $\Sigma |\psi_i|^2 = 1$ we then have

$$\sum_k p_k^i = 1, \quad p_k^i \geq 0, \quad \sum_k \bar{u}_i^k u_l^k = \delta_{il},$$

i.e., $\|p_k^i\|$ is a stochastic matrix (the sum of elements in a row is equal to unity) and $U = \|u_k^i\|$ is a unitary matrix with $U^+ = U^{-1}$. The product of stochastic matrices is again a stochastic matrix and a product of unitary matrices is a unitary matrix. However, while the inverse matrix $U^{-1}(t,t_0) = U^+(t,t_0)$ always exists, this cannot be said about the inverse matrix $\|p_k^i\|^{-1}$. The matrices $\|u_k^i\|$ form the groups $U(N,\mathbf{K})$ ($\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ for real, complex, and quaternion amplitudes, respectively) and the stochastic matrices $\|p_k^i\|$ form a semigroup. The amplitudes ψ_i form the basis for the fundamental irreducible representation $D(10...0)$ of the group $U(N,\mathbf{K})$, and the matrices $\|u_k^i\|$ are the matrices of the finite transformations of these irreducible representations.

Since the inverse operator for the amplitudes exists, it follows that if we know the evolution operator and the state $\psi(t)$ of the system at the present time, we can reconstruct the past, i.e., the Markov process for the amplitudes is symmetric between the past and the future. For stochastic processes, we can determine the future if know the transformation operator and the state of the system, but we cannot reconstruct the past. This becomes particularly clear if we consider the example of the transformation matrix $\|p_k^i\|$, $p_1^i = 1$ with the remaining $p_k^i = 0$. After one step, we obtain $p_1 = 1$, $p_i = 0$ for $i \neq 1$, whatever the preceeding state, i.e., all information about the past is lost.

In the general case, (6.22) and (6.23) can be written in the operator form

$$p(t) = \hat{P}(t, t_0) p(t_0), \tag{6.24}$$

$$\psi(t) = \hat{U}(t,t_0)\psi(t_0). \tag{6.25}$$

From the normalization conditions for p and ψ it follows that $\hat{P}(t,t_0)$ is a stochastic operator and $\hat{U}(t,t_0)$ is a unitary operator. Since, for Markov processes, the future is independent of the past for a known present, the transition operators must satisfy the equations

$$\hat{P}(t, t_0) = \hat{P}(t, \tau)\hat{P}(\tau, t_0), \tag{6.26}$$

$$\hat{U}(t, t_0) = \hat{U}(t, \tau)\hat{U}(\tau, t_0). \tag{6.27}$$

These generalizations of the Markov equation have been designated in different ways in different cases (discrete and continuous processes; probability and amplitude chains): they are referred to as the Markov, Smoluchowski, and Kolmogorov-Chapman equations, the equations for probability amplitudes, and so on.

Expanding $\hat{P}(t + \Delta t, t)$ and $\hat{U}(t + \Delta t, t)$ into a series in Δt, and retaining only the linear terms $\hat{\Lambda}\Delta t$ and $\hat{H}\Delta t$, we obtain in the limit as $\Delta t \to 0$

$$\frac{\partial}{\partial t}\hat{P}(t, t_0) = \hat{\Lambda}(t)\hat{P}(t, t_0), \tag{6.28}$$

$$\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}(t)\hat{U}(t, t_0), \quad \hat{H}^+ = -\hat{H}. \tag{6.29}$$

The equations for $p(t)$ and $\psi(t)$ take the analogous form

$$\frac{\partial}{\partial t}p(t) = \hat{\Lambda}p(t), \tag{6.30}$$

$$\frac{\partial}{\partial t}\psi(t) = \hat{H}\psi(t). \tag{6.31}$$

We shall now consider $\hat{P}(t,t_0)$ and $\hat{U}(t,t_0)$ that are independent of $p(t_0)$ and $\psi(t_0)$. Nonlinear complex chains with nonadditive amplitudes are discussed in Ref. 135. For a finite and denumerable number of states, (6.30) and (6.31) can be written in matrix form (i.e., the right-hand side is a sum):

$$\frac{\partial}{\partial t}p_k = \lambda_k^i p_i, \quad \Lambda = \|\lambda_k^i\|, \quad \sum_k \lambda_k^i = 0, \quad \lambda_k^i \geq 0 \ (i \neq k),$$

$$\frac{\partial}{\partial t}\psi_k = h_k^i \psi_i, \quad H = \|h_k^i\|, \quad H^+ = -H.$$

In the case of continuous processes, (6.26)–(6.31) can be written in integral form. For a one-dimensional continuous Markov process, we have

$$\psi(x, t) = \int u(x,t|x',t_0)\psi(x',t_0)\mathrm{d}x', \tag{6.32}$$

and

$$u(x,\tau|x_0,t_0) = \int u(x,\tau|x',t)u(x',t|x_0,t_0)\mathrm{d}x', \tag{6.33}$$

which is analogous to the equations of the theory of probability, subject to the replacement $\psi(x,t) \to p(x,t)$, $u(x,\tau|x_0,t) \to p(x,\tau|x_0,t)$ where $u(x,\tau|x_0,t)$ is the conditional density (transition amplitude density).

We shall use (6.32) and (6.33) to find the explicit form of $\hat{\Lambda}$ and $\hat{H}$. We shall put $\tau = t + \Delta t$ in (6.33) and will take the transition densities $p(x|x')$ and $u(x|x')$ as the Fourier integrals of the conditional characteristic function

$$p(\Omega|x') = \int_{-\infty}^{\infty} e^{i\Omega(x-x')} p(x|x')\mathrm{d}x,$$

$$u(\tilde{p}|x') = \frac{\langle \tilde{p}|\hat{U}|x'\rangle}{\langle \tilde{p}|x'\rangle} = \int_{-\infty}^{\infty} \frac{\langle \tilde{p}|x\rangle\langle x|\hat{U}|x'\rangle}{\langle \tilde{p}|x'\rangle}\mathrm{d}x = \int_{-\infty}^{\infty} e^{i\tilde{p}(x-x')} u(x|x')\mathrm{d}x. \tag{6.34}$$

The last formula shows that the conditional characteristic function $u(\tilde{p}|x')$ is none other but the pq (or px) symbol of the operator $\hat{U}$. Expanding the characteristic function into a Taylor series in powers of Ω, or the momentum $\tilde{p}$, and allowing Δt to tend to 0, we obtain the following expressions for the operators $\hat{\Lambda}$ (Refs. 139 and 140) and $\hat{H}$ (Ref. 5):

$$\frac{\partial\psi(x,t)}{\partial t} = \hat{H}\psi(x,t),\ \hat{H} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n} A_n(x,t), \tag{6.35}$$

$$\frac{\partial p(x,t)}{\partial t} = \hat{\Lambda}p(x,t),\ \hat{\Lambda} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n} K_n(x,t), \tag{6.36}$$

$$A_n = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int (x-x')^n u(x,t+\Delta t|x',t)\mathrm{d}x,\ n \geq 1\ ,$$

$$A_0 = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int [u(x,t+\Delta t|x',t) - u(x,t|x',t)]\mathrm{d}x, \tag{6.37}$$

$$K_n = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int (x-x')^n p(x,t+\Delta t|x',t)\mathrm{d}x. \tag{6.38}$$

When all the coefficients with $n > 2$ are zero, we obtain the Fokker-Planck and Schrödinger equations

$$\hat{\Lambda}^{(2)} = \frac{\partial}{\partial x}a(x,t) - \frac{1}{2}\frac{\partial^2}{\partial x^2}b(x,t), \tag{6.39}$$

where $b(x,t) = K_2(x,t) \geqslant 0$ is the diffusion coefficient, $a(x,t) = K_1(x,t)$ is the drift coefficient,

$$\hat{H}^{(2)} = \frac{1}{2}\frac{i}{m}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}A_1(x,t) + A_0(x,t), \tag{6.40}$$

where $m = i/A_2$ is the mass of a particle, and the field potentials are described by $A_1(x,t)$ and $A_0(x,t)$. For any continuous process, we can show that either only the first two coefficients K_n are nonzero or there is an infinite number of such coefficients (in particular, all the even coefficients[139]). We thus have an alternative: either the equation is of order no higher than 2 (these are the so-called diffusion processes) or the order is infinite (a few examples of such processes can be found in Refs. 140 and 141). In the theory of probability amplitudes, the analogous proposition has not been proved, but is likely to be valid.

Consideration of processes described by second-order equations is equivalent to the consideration of processes satisfying the following conditions for the probabilities[142] and amplitudes,[5] respectively:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-x|\geq\delta} p(y,t|x,t-\Delta t)\mathrm{d}y = 0, \tag{6.41}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-x|\geq\delta} u(y,t|x,t-\Delta t)\mathrm{d}y = 0. \tag{6.42}$$

These conditions ensure that large shifts (jumps) have low probability. We shall refer to such processes as processes with small increments or processes without jumps. The condition for small increments leads to diffusion processes or nonrelativistic quantum mechanics. We note, in passing, that, in mathematical literature, (6.41) is also referred to as the condition for enhanced continuity[143] or the Lindeberg condition[142] because of the similarity with the condition for the validity of the central limit theorem.

Equation (6.35) refers to processes with jumps, i.e., processes involving transitions between different stationary states (emission and absorption). The conditions that must be satisfied by the coefficients A_n can be established on the basis of (6.37). A general analysis of (6.35) is quite complicated. It is interesting to examine the possibility of finding its exact solutions when $\hat{H}$ is written in terms of the operators of deformed (including quantum) algebras, e.g., shift operators (3.45).

For finite or denumerable states, the analog of the condition for small increments

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_{k,|i-k|>1} p_k^i(t|t-\Delta t) = 0,$$
$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_{k,|i-k|>1} u_k^i(t|t-\Delta t) = 0 \tag{6.43}$$

signifies that a transition within a small Δt is possible only to neighboring states, i.e., the problem of a one-dimensional random walk.

It was noted in Ref. 142 that "the theory of subgroups leads to a unified theory of Markov processes, which cannot be attained by other methods." The transition function (transition probability) of a Markov process generates a subgroup of operators. If we know the infinitesimal operator of the subgroup (or, what amounts to the same thing, of the process), we can find the significant characteristics of the process; moreover, questions relating to the classification of Markov processes reduce to the description of the corresponding infinitesimal operators.[144,145]

The classification of Markov processes for probability amplitudes is related to the classification of groups or of the corresponding Lie algebras. In the case of processes with denumerable (finite) number of states, the transition amplitudes for $\Delta t \to 0$ are

$$u_j^i = \delta_j^i + h_j^i\Delta t,\ h_j^i = -\bar{h}_i^j,$$

or, in matrix form, $U(t+\Delta t,t) = E + H\Delta t$, $H^+ = -H$, $H = \|h_j^i\|$. The classification of processes thus reduces to the classification of the matrix unitary irreducible representations of Lie groups.

Let us now consider one-dimensional processes for the complex amplitudes that satisfy (6.43). They correspond to multilevel systems with transitions between neighboring levels: the infinitesimal operator for the process (the Hamilto-

nian) contains only the transition amplitudes h^i_k with $|i-k|\leqslant 1$, i.e., the matrix $\|h^i_k\|$ is a three-diagonal matrix. The algebra of infinitesimal operators (Lie algebra) consists of three operators, namely, the raising operator $\hat{E}_+$, the lowering operator $\hat{E}_-$, and the level number (particle number) operator $\hat{H}$ that corresponds to a change in the phase of the state satifying (2.1). A random walk corresponds to a transformation of a finite-dimensional irreducible representation $D(j)$ of the group SU(2) on a segment, $D^{\pm}(j)$ of SU(1,1) or $D(\omega)$ of W(1) on a half-line, and the irreducible representation of the principal series of SU(1,1) or $D(p)$ of the group M(2) on a line (see Ref. 5 for further details).

The transition densities for discrete processes satisfying (6.43) are shown graphically in Fig. 6. If we consider a schematic limiting transition to a continuous process, we find that the diffusion coefficient becomes $K_2 \sim \lambda+\mu-(\lambda-\mu)^2$, the shift coefficient becomes $K_1 \sim \lambda-\mu$ (Ref. 140), the mass becomes inversely proportional to the amplitude for the transition to the neighboring state, $m \sim 1/\mathrm{Im}\,\alpha$, and the potential becomes $A_0 \sim \partial\varphi/\partial t$ (Ref. 5).

If the process is not a process with small increments then, in addition to the transition operators $\hat{E}_\pm$, we have to consider the operators $\hat{E}_{2\pm}$, $\hat{E}_{3\pm}$, and so on, i.e., instead of the Lie algebras we have to consider the Kac–Moody algebras or deformed algebras.

6.4. Postulates of quantum theory

The theory of probability amplitudes can be used as a basis for formulating the postulates of quantum theory.[5] This formulation is closely related to that proposed by von Neumann (see Refs. 1 and 121). It contains the nonrelativistic quantum mechanics as a special case and consists of the following four proposition.

I. A change in the state of a quantum system is a Markov process.

II. The states of a quantum system are in one-to-one correspondence with sets of nonzero vectors in Hilbert space that differ only by a factor.

III. Normalized state vectors $|\psi\rangle$ are in mutually one-to-one correspondence with probability amplitudes ψ. If the decomposition of unity $\int|y\rangle\langle y|\mathrm{d}\mu(y)=\hat{1}$ exists, then $\psi(y)=\langle y|\psi\rangle$.

IV. Observables are in mutually one-to-one correspondence with random quantities with real mathematical expectation.

We can also specify additional conditions that are satisfied for a wide class of processes, but are not essential. Let us consider the following two propositions.

1. The transition amplitude is independent of ψ, in which case evolution is described by linear equations.

2. The transition amplitude is independent of ψ and the process of variation of state is a process with small increments for complex amplitudes. Evolution is then described by the nonrelativistic Schrödinger equation.

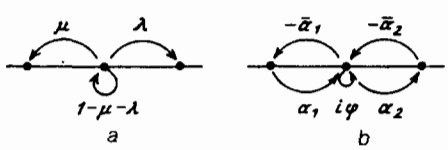

FIG. 6. Transition probabilities and amplitudes in the random walk problem.

In the general case, postulates I–III show that the evolution of a state can be described in two possible languages, namely, the language of operators in abstract Hilbert space or the language of transition amplitudes:

$$\hat{U}(t_2,t_1)|\psi(t_1)\rangle=|\psi(t_2)\rangle,\ \hat{U}(t,t)=\hat{1},$$
$$\int u(x,t_2|y,t_1)\psi(y,t_1)\mathrm{d}\mu(y)=\psi(x,t_2),\ u(x,t|y,t)=\delta(x-y). \tag{6.44}$$

For the sake of brevity, we shall use y to represent $y=\{y_i\}$. In the case of flat space

$$\mathrm{d}\mu(y)=\prod_i \mathrm{d}y_i.$$

Postulate I leads to the equation for the evolution operator or the generalized Markov equation for the amplitudes:

$$\hat{U}(t_3,t_1)=\hat{U}(t_3,t_2)\hat{U}(t_2,t_1),$$
$$\int u(z,t_3|y,t_2)u(y,t_2|x,t_1)\mathrm{d}\mu(y)=u(z,t_3|x,t_1). \tag{6.45}$$

Since $\psi(x,t)$ is the probability amplitude, the norm is $\int\psi(x,t)\bar{\psi}(x,t)\mathrm{d}\mu(x)$, i.e., $\langle\psi(t)|\psi(t)\rangle$ should remain constant in time, so that $\hat{U}^+=\hat{U}^{-1}$, i.e., $\hat{U}$ is a unitary operator.

Next, using (6.44) and (6.45), and the expansions of $\hat{U}(t,t-\Delta t)$ and $u(y,t|x,t-\Delta t)$ for small Δt, we can proceed to the Schrödinger equation (6.31) or the path integral.

Postulate IV enables us to define the classical limit as the case where the mathematical expectations of random quantities are much greater than their variances. The law of large numbers and the limit theorems of the theory of probability amplitudes, which are related to the first and second correspondence principles, specify the conditions for the transition to the classical limit. In the language of the theory of operators in Hilbert space, postulate IV demands that there should be a correspondence between linear self-adjoint operators and observables.

According to postulates II and III, a state has associated with it a set of amplitudes that differ only by a phase factor; in the case of complex amplitudes, this factor is $\exp(i\varphi)$ whereas in the case of quaternion amplitudes it is $\exp(i\varphi_1+j\varphi_2+k\varphi_3)$. By choosing the amplitude types, we are essentially defining a gauge group. For complex amplitudes, this gauge group is $U(1)$ of the electromagnetic interaction, whereas for quaternion amplitudes (corresponding to spin 1/2 particles) this group is $U(1,\mathrm{H})=\mathrm{SU}(2)$.

The requirement that the normalization of ψ must be preserved leads to the classification of states under the unitary irreducible representations of the symmetry groups. For noncompact groups, the unitary irreducible representations are infinite-dimensional. We know that spin S particles correspond, on the one hand, to unitary infinite-dimensional irreducible representations $D(SM)$ of the Lorentz group[146,147] whereas, on the other hand, they correspond to

nonunitary finite-dimensional irreducible representations (under which ψ transforms in relativistic wave equations), i.e., there are two different descriptions. It may well be that the relativistic equations for nonunitary irreducible representations are the CS evolution equations associated with the Lorentz group $SO(3,1)\sim SL(2,C)$ and the Poincaré group $M(3,1)$ [by analogy with the first-order equation for the spinor $\{z_1,z_2\}$, $|z_2|^2-|z_1|^2=1$ of $SU(1,1)$, which arises as the equation for the CS evolution of discrete unitary series of infinite-dimensional irreducible representations of $SU(1,1)$ (see Sec. 4.3)]. It follows that the development of the theory of coherent states and the operator symbols for the groups $SL(2,C)$ and $M(3,1)$ is a pressing problem.

In relativistic theory, one way of retaining the probabilistic intepretation is to introduce the proper time τ and to consider the first-order equations in $\partial/\partial\tau$; however, in contrast to the Dirac equation with proper time[148] (which, by the way, is very convenient in the transition to the classical limit[15,101]), ψ must transform under infinite-dimensional (and not finite-dimensional) unitary irreducible representations of the group $M(3,1)$.

The basic equation of the theory is (6.45) which expresses the Markov property of the process. When specific conditions are imposed on the transition amplitudes or the measure $d\mu$, the result is a variety of differential equations. The equations of quantum mechanics in curved space are obtained by considering processes with small increments on real and complex manifolds with metric tensor g_{ik} and $g_{i\bar{k}}$, respectively:

$$d\mu(x)=(\det g_{ik})^{1/2}\prod_n \wedge dx^n,\ ds^2=g_{ik}dx^idx^k,$$

$$d\mu(z)=\det g_{i\bar{k}}\prod_n dz^n\wedge d\bar{z}^n,\ ds^2=g_{i\bar{k}}dz^id\bar{z}^k.$$

If we do not demand that the transition amplitudes must be independent of ψ, i.e., the transition amplitude is regarded as a function of ψ, $u(x,t\,|x_0,t_0,\psi[x_0,t_0)]$, then for processes with small increments we obtain the equations of nonlinear quantum mechanics such as the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2}+\beta|\psi|^2\psi.$$

The derivation of the equations is conveniently based on the formalism of conditional characteristic functions, i.e., the pq; qp; and Weyl operator symbols.

7. QUANTUM THEORY, GROUPS, AND SPECIAL FUNCTIONS

7.1. Groups and special functions

Special functions usually arise in the course of the solution of problems in quantum mechanics. In one sense, they constitute the foundation for the entire formalism. It is therefore important to establish any correspondence with group and probability interpretations of quantum mechanics, i.e., a kind of translation from the language of special functions to the group and probability languages.

From the group point of view, the theory of special functions is a mixture of problems associated with respresentations, C–G coefficients, Lie algebras, coherent states, and the finite-difference calculus. However, for a long time, the theory of special functions developed independently of group methods and a huge variety of special formulas[92,117,149] has emerged without which modern physics cannot even be imagined. The connection between special functions and group representations was discovered by Cartan. A systematic presentation of the theory of special functions from the group-theoretic point of view was first given by Vilenkin[7] (see also Refs. 150 and 151). There are three qualitative points that can be made about this problem.

First, special functions appear in the theory as the elements $t^l_{mn}(g)$ of finite transformation matrices (l is the IR signature). Their discrete indices correspond to m and n, their continuous variables correspond to the parameters g of the group, e.g., the matrix elements of $SU(2)$ and $SU(1,1)$ are expressed in terms of Legendre polynomials and functions P^l_{mn} whereas those of $M(2)$ are expressed in terms of the Bessel functions J_{m-n} (Ref. 7). The composition theorems for special functions are a direct consequence of the relation

$$T_l(g_1g_2)=T_l(g_1)T_l(g_2). \tag{7.1}$$

The recurrence relations for $t^l_{mn}(g)$ in which the indices differ by unity constitute the infinitesimal variant of the composition theorem. The formulas take the form

$$\hat{A}t^l_{mn}=c_1(l,m,n)t^l_{m,n+1},\ \hat{A}'t^l_{m,n+1}=c_2(l,m,n)t^l_{mn},$$

where $\hat{A}$, $\hat{A}'$ are first-order differential operators (group generators). They can be used to establish the second-order differential equations for the special functions: $\hat{A}'\hat{A}t^l_{mn}=c_2(l,m,n+1)\times c_1(l,m,n)t^l_{mn}$. This approach was employed in Ref. 7.

Second, special functions appear as the overlaps of different bases of a given irreducible representation of a group. For example, the Hermite polynomials can be written in the form of the overlap $\langle n|x\rangle$ of the group $W(1)$:

$$\langle n|x\rangle=\pi^{-1/4}(2^nn!)^{-1/2}H_n(x)\exp(-x^2/2).$$

If we use the decomposition of unity, we obtain the generating function for the Hermite polynomials:

$$\langle z|x\rangle=\sum_n\langle z|n\rangle\langle n|x\rangle,$$

$$\exp[(-z^2/2)+z\cdot\sqrt{2}x]=\sum_n(z/\sqrt{2})^nH_n(x)/n!. \tag{7.2}$$

Similary, if we use (4.21) and (4.22), we obtain the integral representation and the orthogonality relation

$$\langle n|x\rangle=\frac{1}{\pi}\int\langle n|z\rangle\langle z|x\rangle d^2z,$$

$$H_n(x)=\frac{z^{n/2}}{\pi}\int\exp\left[-\frac{z^2}{2}+\sqrt{2}zx-|z|^2\right]\bar{z}^nd^2z,$$

$$\langle n|n'\rangle=\int\langle n|x\rangle\langle x|n'\rangle dx,\ \delta_{nn'}$$

$$=\int H_n(x)H_m(x)\exp(-x^2)dx. \tag{7.3}$$

The Bessel functions J_m can be written as the overlap $\langle r\varphi | pm \rangle$ of the bases of the irreducible representations for the group M(2). From (4.46)–(4.49) we obtain the generating function and the integral representation for J_m:

$$\langle r\varphi | p\alpha \rangle = \sum_m \langle r\varphi | pm \rangle \langle pm | p\alpha \rangle,$$

$$\exp[ipr\cos(\varphi - \alpha)] = \sum_{m=-\infty}^{\infty} \exp[im(\varphi - \alpha)] i^m J_m(pr),$$

$$\langle r\varphi | pm \rangle = \int_0^{2\pi} \langle r\varphi | p\alpha \rangle \langle p\alpha | pm \rangle d\alpha,$$

$$i^m J_m(pr) = \frac{1}{2\pi} \int_0^{2\pi} \exp[ipr\cos(\varphi - \alpha) + im\alpha] d\alpha. \tag{7.4}$$

Relations of the form $\langle x|\hat{a}|n\rangle = n^{1/2}\langle x|n-1\rangle$, $\langle r\varphi|\hat{\Phi}_{\pm}|pm\rangle = \langle r\varphi|p,m \pm 1\rangle$ readily yield recurrence relations and second-order differential equations. The multiplication (composition) formulas for the special functions define the decomposition of the product of special functions into a sum of functions of the same type (correspondingly, the decomposition of special functions into a sum of products). Within the framework of this approach, these operations correspond to the redecomposition of the bases of the direct product of irreducible representations and the reduced basis, performed with the help of the C–G coefficients [see (2.30), (2.29), and (2.41) for Bessel functions, spherical functions, and Hermite polynomials, respectively]. This approach enables us to exploit standard devices, and to elucidate the group significance of many of the relationships.

Thirdly, special functions appear as the basis recoupling coefficients. If we consider discrete bases, then all the parameters of these functions are also discrete, and studies in this area are closely related to finite-difference calculations on nets (see Sec. 3). The analysis of different finite-difference analogs of special functions is based on a generalization of the standard approach for a continuous variable. Following Refs. 75 and 152, we shall consider this in the case of orthogonal polynomials. The starting point is the differential equation for the classical polynomials

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \tag{7.5}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of degree not higher than 2 and 1, respectively. If the solution of this equation, $y(x)$, is a polynomial of degree n, then we have the Rodriguez formula for it, which defines a recurrence formula

$$y_n = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n}(\sigma^n(x)\rho(x)); \tag{7.6}$$

where B_n is a normalizing constant and $\rho(x)$ is a weight function defined by $(\sigma\rho)' = \tau\rho$, which appears, for example, in the orthogonality formula

$$\int_a^b y_m(x) y_n(x) \rho(x) dx = \delta_{mn}. \tag{7.7}$$

The function $\rho(x)$ can be reduced to three canonical forms corresponding to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (and their special cases, namely the Legendre, Chebyshev, and Gegenbauer polynomials), and the Laguerre and Hermite polynomials $L_n^{\alpha}(x)$ and $H_n(x)$. The quantities that appear in (7.6) and the corresponding orthogonality intervals (a,b) are listed in Table I (Ref. 152). All the polynomials are special cases of the hypergeometric function $F(\alpha,\beta;\gamma|z)$ and the equation $\alpha = -n$ is common to them.

The above standard scheme, originally developed for classical orthogonal polynomials, can be extended to the discrete case.[75,152] The relations given by (7.7)–(7.9) remain in force if we replace the usual variables with their finite-difference analogs [derivatives with finite-difference derivatives, powers with generalized powers, and the integral (7.7) with summation]. For the finite-difference analog of (7.5) on a uniform net, there are four types of finite-difference polynomials, namely, the Hahn, Meixner, Kravchuk, and Charlier polynomials $h_n^{(\mu,\nu)}(x,N)$, $m_n^{(\gamma,\mu)}(x)$, $K_n^p(x,N)$, and $C_n^{\mu}(x)$, respectively. The finite-difference analogs of the quantities appearing in (7.6) are listed in Table II (Ref. 152).

The Meixner, Kravchuk, and Charlier polynomials can be expressed in terms of the usual hypergeometric function ${}_2F_1$; the Hahn polynomials can be written in the form [cf (3.11)]

$$h_n^{(\alpha,\beta)}(x,N) = \frac{(-1)^n (N-1)!(\beta+1)^{(n)}}{n!(N-n-1)!} {}_3F_2\left(\begin{matrix}-n, \alpha+\beta+n+1, x \\ \beta+1, 1-N\end{matrix}; 1\right), \tag{7.8}$$

which, apart from normalization, is identical with the C–G coefficients of the group SU(2)

$$(-1)^{j_1-m_1}\langle j_1 m_1 | j_2 m_2 \| jm \rangle = \frac{(\rho(x))^{1/2}}{d_n} h_n^{(\alpha,\beta)}(x,N), \tag{7.9}$$

where $\rho(x)$ and d_n are, respectively, the weight and the norm of the Hahn polynomials and $n = j - m$, $x = j_2 - m_2$,

TABLE I.

$y_n(x)$	$P_n^{(\alpha,\beta)}(x)$; $\alpha,\beta > -1$	$L_n^{\alpha}(x)$; $\alpha > -1$	$H_n(x)$
$\rho(x)$	$(1-x)^{\alpha}(1+x)^{\beta}$	$x^{\alpha}\exp(-x)$	$\exp(-x^2)$
$\sigma(x)$	$1-x^2$	x	1
B_n	$(-1)^n/2^n n!$	$1/n!$	$(-1)^n$
(a,b)	$(-1,1)$	$(0,\infty)$	$(-\infty,\infty)$

TABLE II.

$y_n(x)$	$h_n^{(\mu,\nu)}(x,N);\mu,\nu>-1$	$m_n^{(\gamma,\mu)}(x);\gamma>0,\ 0<\mu<1$	$K_n^p(x,N);\ p,q>0,\ p+q>1$	$C_n^\mu(x);\mu>0$
$\rho(x)$	$\{\Gamma(x+1)\Gamma(x+\mu+1)\times$ $\times\Gamma(N+\nu-x)\Gamma(N-x)\}^{-1}$	$\frac{\mu^x\Gamma(\gamma+x)}{\Gamma(1+x)\Gamma(\gamma)}$	$\frac{N!p^xq^{N-x}}{\Gamma(1+x)\Gamma(N+1-x)}$	$\frac{e^{-x}\mu^x}{\Gamma(1+x)}$
$\sigma(x)$	$x(\mu+x)$	x	x	x
B_n	$1/n!$	$1/\mu^n$	$(-1)^nq^n/n!$	$1/\mu^n$
(a,b)	$(0,N)$	$(0,\infty)$	$(0,N+1)$	$(0,\infty)$

$N=j_1+j_2-m+1,\qquad \alpha=m-m',\qquad \beta=m+m',$ $m'=j_1-j_2$.

Representations of the group SU(2), or the Wigner D-functions

$$D^j_{mm'}(\alpha,\beta,\gamma)=\exp(-im\alpha)d^j_{mm'}(\beta)\exp(im'\gamma),$$

are directly related to the Jacobi and Kravchuk polynomials:

$$d^j_{mm'}(\beta)=\frac{(-1)^{m-m'}}{2^m}\left[\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}\right]^{1/2}\times(1-s)^{\frac{m-m'}{2}}(1+s)^{\frac{m+m'}{2}}P_{j-m}^{(m-m',m+m')},\quad(7.10)$$

$s=\cos\beta$

$$(-1)^{m-m'}d^j_{mm'}(\beta)=\frac{(\rho(x))^{1/2}}{d_n}K_n^p(x,N),\quad(7.11)$$

where $\rho(x)$ and d_n are the weight and the norm of the Kravchuk polynomials and $n=j-m$, $x=j-m'$, $N=2j$, $p=\sin^2(\beta/2)$. We note that, since the Hahn polynomials $h_n^{(\alpha,\beta)}(x,N)$ are the finite-difference analogs of the Jacobi polynomials $P_n^{(\alpha,\beta)}(s)$, it follows from (7.9) and (7.10) that the C–G coefficients are the finite-difference analogs of the function $d^j_{mm'}(\beta)$ for $s=\cos\beta$. Thus, the theory of the C–G coefficients can be translated into the language of special functions (see also Refs. 153). All its relationships are simultaneously replaced by relationships between finite-difference special functions.

A particular set of discrete polynomials can be considered by standard theory, but on a nonuniform net. Here again there are finite-difference analogs of the Rodriguez formula and the orthogonality formulas (7.6) and (7.7). For the square net $x_k=kh(kh+1)$, the differentiation operator is

$$D_x^+f(x)=\frac{f(x+2h(k+1))-f(x)}{2h(k+1)}.\quad(7.12)$$

For sufficiently large k $(hk+1)/hk\approx1$, it too corresponds approximately to the finite-difference operator (3.9) on a uniform net. Two types of polynomial were examined in Ref. 152 on a square net, namely, the Racah polynomials $u_n^{(\alpha,\beta)}(x,a,b)$ and the Hahn dual polynomials $W_n^c(x,a,b)$, which are orthogonal on (a,b). The existence of the Racah polynomials was established in Refs. 73, 154, and 155 by studying the Racah coefficients in their representation in terms of ${}_4F_3$ with argument $x=1$.

The formula connecting the $6j$-symbol and the Racah polynomials is

$$(-1)^{j_1+j+j_{23}}(2j_{12}+1)^{1/2}\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{Bmatrix}=\frac{(\rho(s))^{1/2}}{d_n}u_n^{(\alpha,\beta)}(x,a,b),\quad(7.13)$$

where $\rho(s)$ and d_n are the weight and the norm of the polynomials, respectively, and $n=j_{12}-j_1+j_2$, $x=s(s+1)$, $s=j_{23}$, $a=j_3-j_2$, $b=j_2+j_3+j$, $\alpha=j_1-j_2-j_3+j$, $\beta=j_1-j_2+j_3-j$. It is clear that by using (7.13) we can formulate a theory of transformation matrices in polynomial language.

We note that, as in the case of large k, we have in (7.12) a transition from the finite-difference differentiation on a square net to the usual finite difference one and then, for $h\to0$, to differentiation with respect to a continuous variable, so that we can write [cf. Refs. 21 and 152] the asymptotic formulas for the transition from the Racah polynomials to the Hahn polynomials and then to the Jacobi polynomials (correspondingly, Racah coefficients→C–G coefficients →d-functions).

In the case of the exponential net $x_k=\exp(2\gamma k)=q^k$, the differentiation rule is defined by (3.15) which involves the q-polynomials.[152,156,157] Their classification is analogous to the classification on a uniform net with which they become identical for $q\to1$. For example,

$$\lim_{q\to1}h_n^{(\alpha,\beta)}(q^k,N,q)=h_n^{(\alpha,\beta)}(k,N).\quad(7.14)$$

The C–G coefficients $su_q(2)$ given by (3.43) can be expressed in terms of the Hahn q-polynomials. For the net $x_k=\cosh(2\gamma k)$ we have, in particular, the q-analogs of the Racah polynomials and the Hahn dual polynomials. The case of the nets $\sinh(2\gamma k)$ and $\cos(2\gamma k)$ is briefly discussed in Ref. 152.

An extensive class of polynomials can be obtained from the above complexification of the argument. They are the so-called discrete polynomials of an imaginary argument. They are obtained by replacing summation with integration in the orthogonality formula. The C–G coefficients of the continuous series of noncompact groups, in particular, SO(3,1) (Ref. 152) can be expressed in terms of discrete polynomials of an imaginary argument. The $9j$-symbols of the group SU(2) correspond to discrete polynomials in two variables.[158] Functions of several variables are also encountered in the theory of representations of higher groups.

7.2. Probability distributions and hypergeometric functions

The theory of special functions is closely linked to probability distributions. Hypergeometric functions occupy a

special place in this theory because most of the functions encountered in mathematical physics can be expressed in terms of them. The last decade saw a very intensive development of the theory and an expansion of the range of applications of hypergeometric functions of many variables as well as of the confluent hypergeometric functions.[92,159-161] Their properties have much in common with the properties of ordinary hypergeometric functions ${}_2F_1$. Clearly, they are determined by the structure of the coefficients in the series in powers of x_i. These coefficients can be expressed in terms of $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$, e.g., for the hypergeometric functions of two variables, i.e., the Appell (F_1) and Horn (G_2) variables

$$F_1 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} x^m y^n,$$
$$G_2 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}}{m!n!} x^m y^n \tag{7.15}$$

or the confluent hypergeometric function

$${}_pF_q = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{x^n}{n!}. \tag{7.16}$$

The resulting complex functions must be analyzed by the constructive approach,[5] i.e., we must specify the basis elements, the method of composition used for them, and the construction rules. Here we have a direct analogy with the theory of angular momenta in which all the variables are constructed as compositions of C–G coefficients and the construction rules are specified by the graphical technique.[23] We note that, in addition to the method given in Ref. 23, in which the C–G coefficients are presented as "three-legged diagrams" it is possible to have a modification in which the C–G coefficient is represented by triangles and compositions of two coefficients are represented by two triangles with a common side.[2]

In the general theory of hypergeometric functions, the basic element (the analog of the C–G coefficient) is the hypergeometric distribution discussed in Sec. 6. The coefficients of continuous variables in hypergeometric functions of many variables and the confluent hypergeometric functions can be constructed as the compositions of these distributions, i.e., their theory has its own probability aspects.

We now turn to the consideration of methods for the determination of the basis elements.

The Gauss hypergeometric function is defined by the series

$$F(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!}. \tag{7.17}$$

This is the generating function for the hypergeometric distribution (See Sec. 6 for notation)

$$\begin{aligned}\widetilde{F}(x) &= \frac{n^2!n_2!}{n!(n-n_1-n^1)!} F(-n_1, -n^1, n-n_1-n^1+1; x) \\ &= \frac{n^1!n_1!n^2!n_2!}{n!} \sum_j \frac{x^j}{(n^1-j)!(n_1-j)!(n-n_1-n^1+j)!j!} \\ &= \sum_{n_1^1} \begin{Bmatrix} n_1^1 & n_2^1 \\ n_1^2 & n_2^2 \end{Bmatrix} x^{n_1^1}.\end{aligned} \tag{7.18}$$

If n_i and n^i are integers, then the series terminates for $j = \min\{n_1, n^1\}$ and we obtain polynomials. The summation in (7.18) continues for fixed sums of rows and columns n_i and n^i. The sum of probabilities of all the possible outcomes in (6.17) is $\widetilde{F}(1) = 1$.

For the 2×3 hypergeometric distribution, we can write two different generating functions, i.e., the hypergeometric functions of the two arguments F_1 and G_2:

$$\begin{aligned}&\widetilde{F}_1(x,y) \\ &= \frac{n^2!n_3!}{N!(N-n^1-n_1-n_2)!} F_1(-n^1, -n_1, -n_2; N-n^1-n_1-n_2+1; x, y) \\ &= \sum_{j,m} \begin{Bmatrix} j & m & n^1-j-m \\ n_1-j & n_2-m & N-n^1-n_1-n_2+j+m \end{Bmatrix} x^j y^m,\end{aligned} \tag{7.19}$$

$$\begin{aligned}&\widetilde{G}_2(x,y) \\ &= \frac{n^1!(N-n^1)!n_2!}{N!(N-n^1-n_1)!(n^1-n_2)!} G_2(-n_1, -n_2, \\ &\quad -N+n^1-n_1, -n_1+n_2; x, y) \\ &= \sum_{j,m} \begin{Bmatrix} j & n_2-m & n^1-n_2-j+m \\ n_1-j & m & N-n^1-n_1+j-m \end{Bmatrix} x^j y^m.\end{aligned} \tag{7.20}$$

If we replace the factorials in (7.18)–(7.20) with Γ-functions, we obtain the series for the hypergeometric functions for arbitrary values of the parameters, which take the form of sums of rows and columns of the distribution matrix.

The polyhypergeometric distribution as a function of a $M\times N$-matrix can be used effectively to construct and examine hypergeometric functions of many variables. In particular, if we start with (7.19) and (7.20), and the transformation properties of the polyhypergeometric distribution (6.18), we readily obtain the required reduction formulas, some of which are given in Ref. 92:

$$\begin{aligned}&\widetilde{G}_2(\beta,\beta', -\gamma+\beta'+1, \alpha-\beta'; x, 1) = \widetilde{F}_1(\alpha,\beta,\beta',\gamma; x, 1) \\ &\quad = \widetilde{F}(\alpha,\beta;\gamma-\beta', x), \\ &\widetilde{F}_1(\alpha,\beta,\beta',\gamma; x, x) \\ &\quad = \sum_{j,m} \begin{Bmatrix} j & m & -\alpha-j-m \\ -\beta-\gamma & -\beta-m & \gamma+j+m-1 \end{Bmatrix} x^{j+m} \\ &\quad = \sum_{j} \begin{Bmatrix} j' & -\alpha-j' \\ -\beta-\beta'-j' & \gamma+j'-1 \end{Bmatrix} x^{j'} = \widetilde{F}(\alpha,\beta+\beta',\gamma; x).\end{aligned} \tag{7.21}$$

For integral $\beta' < 0$, we can replace the summation parameter m in (7.19) with $k - n_2 - m$ and obtain[5]

$$\widetilde{F}_1(\alpha, \beta, \beta', \gamma; x, y) = y^{-\beta'} \widetilde{G}_2(\beta, \beta', -\gamma + \beta' + 1, \alpha - \beta'; x, \frac{1}{y}). \tag{7.22}$$

Analysis of the confluent hypergeometric functions is also related to the consideration of compositions of hypergeometric distributions. As already noted, the C–G and Racah coefficients can be expressed in terms of the functions ${}_3F_2$ and ${}_4F_3$, apart from normalizing factors. Using the expression for the C–G coefficient in terms of the hypergeometric distribution,[137] we obtain

$$\begin{aligned} &{}_3F_2(2\alpha - 2n^2, n_1^1 + 1, n_1; n^1 - n_1^2 + 1, -n^1 - n_2^2; x) \\ &= \rho \sum_{\alpha_1 + \alpha_2 = \alpha} (-1)^{\alpha_2} \begin{Bmatrix} n_1^1 - \alpha_1 & n_2^1 - \alpha_2 \\ n_1^2 - \alpha_2 & n_2^2 - \alpha_1 \end{Bmatrix}^{1/2} \\ &\times \begin{Bmatrix} n_1^1 - \alpha_1 & n_2^1 - \alpha_2 \\ \alpha_1 & \alpha_2 \end{Bmatrix}^{1/2} \begin{Bmatrix} n_1^2 - \alpha_2 & n_2^2 - \alpha_1 \\ \alpha_2 & \alpha_1 \end{Bmatrix}^{1/2} x^{\alpha_1}. \end{aligned} \tag{7.23}$$

The function ${}_3F_2$ is the generating function, apart from the normalizing factor ρ, for the composition (product) of three hypergeometric distributions, and its value for argument 1 gives the C–G coefficient of the group SU(2). Similarly, the function ${}_4F_3$ is the generating function for the composition of four C–G coefficients and its value for argument 1 corresponds to the Racah coefficient.

It is clear that the hypergeometric distributions serve as the building bricks for the construction, in accordance with a particular algorithm, of confluent hypergeometric functions and hypergeometric functions of many variables. The construction rules can be laid down with the help of a simple graphical technique. Hypergeometric functions of many variables are constructed as the generating functions for polyhypergeometric distributions.

In Fig. 7a, a parameter of the polyhypergeometric distribution specified in matrix form corresponds to each site on the graph. The diagrams are constructed in accordance with the following algorithm. The product of the factorials of numbers corresponding to the sites is written in the denominator and the product of factorials of sums over edges is written in the numerator. The factorial of the sum over the entire graph is placed in the denominator. A similar algorithm is used to construct the distribution for the graph of Figs. 7b–d. In the graphs of Figs. 7a, we associate with point x_i the hypergeometric functions of the form

$$F(x_1, x_2) = \sum_{n_1, n_2 = 0}^{\infty} \begin{Bmatrix} n_1 \cdots\cdots \\ \cdots n_2 \cdots \end{Bmatrix} x_1^{n_1} x_2^{n_2}.$$

We note that we then have $F(x,1) = F(x)$ and $F(1,1) = 1$. The sum is evaluated over matrices with fixed sums of columns and rows. These sums are the parameters of the functions.

The usual hypergeometric functions ${}_2F_1$, i.e., the Appell and Horn functions F_1 and G_2, correspond to the ordinary hypergeometric functions. The graph of Fig. 7b corresponds to the confluent hypergeometric function ${}_3F_2$. The sum is again evaluated with fixed sums along the graph edges. The functions that are obtained from one another by analytic continuation, e.g., F_1 and G_2 [see (7.22) and Fig. 7a] or F_2, F_3, H_2 (Ref. 92; see also Fig. 7c) have the same graphs.

We note that among the Lauricella functions (N-di-

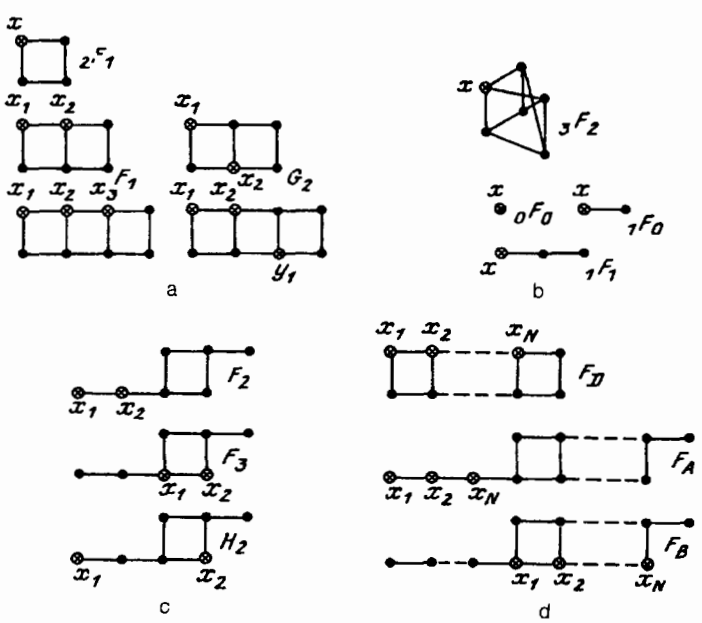

FIG. 7. Graphs of hypergeometric functions.

mensional analogs of ${}_2F_1$) F_A, F_B, F_C, and F_D (Fig. 7d), it is the last of these and its analytic continuations

$$\tilde{F}_D\left(\begin{matrix}\alpha_1,\ldots,\alpha_k,\alpha\\ \beta_1,\ldots,\beta_l,\beta\end{matrix}\middle|\begin{matrix}x_1,\ldots,x_k\\ y_1,\ldots,y_l\end{matrix}\right)$$

$$=\sum\left\{\begin{matrix} n_1,\ldots,n_k & \beta-m_1,\ldots,\beta_l-m_l & \beta+\sum n_i+\sum m_i\\ \alpha_1-n_1,\ldots,\alpha_k-n_k & m_1,\ldots,m_l & \alpha+\sum n_i-\sum m_i\end{matrix}\right\}$$

$$\times x^{n_1}\ldots x^{n_k}y_1^{m_1}\ldots y_l^{m_l},$$

that are connected to the $2\times(N+1)$ hypergeometric distribution and are the closest in their properties to those of ${}_2F_1$; they are particularly interesting from the group point of view.[150]

The above graphical technique satisfactorily represents the properties of the confluent hypergeometric function and the compositions of C–G coefficients. In the simplified graphical method[23] adopted for the C–G coefficients [the function ${}_3F_2(1)$], many of the properties, including the symmetry relations, are lost. In that case, instead of the figure twisted like the Möbius strip (see Fig. 7b) we have a projection, i.e., a triangle or its equivalent "three-legged diagram."

8. CONCLUSION

Analysis of the fundamentals of quantum theory shows that a particular synthesis of our concepts is occurring at the present time. Quantum theory now has two languages, namely, the language of operators in Hilbert space and the language of probability amplitudes. The latter is intimately connected with the group-theoretic description. Particular formulas of quantum theory often admit of a three-fold interpretation (operator, probability, and group). On the other hand, we are dealing with a single truth that remains the same however one views it. The universal aspect of the situation is then the overlap formalism that is distinguished by simplicity and convenience. It serves as a basis for the formalization and unification of the theory, including the C–G coefficients that specify the structure of quantum-mechanical objects and a variety of special functions. It also reveals the possibility of a smaller number of essential formulas and a smaller volume of reference literature.

The relationship between classical and quantum theories has been formalized to a considerable extent. The classical theory involves probabilities and semigroups whereas quantum theory deals with probability amplitudes and groups. The transition to the classical limit relies on the limit theorems of the theory of probability amplitudes.

Much new light has been thrown on the foundations of general quantum theory. The usual quantum mechanics is based on postulates that demand the existence of probability amplitudes for Markov processes (first-order equation in $\partial/\partial t$) and small increments (no discontinuities). The last postulate is responsible for the dominance of second-order equations (both in quantum theory and in classical theory). Any departure from this postulate and the admission that discontinuities are possible immediately leads to infinite-order equations and, as a consequence, groups with an infinite number of parameters and pseudodifferential operators. The approach that we have described provides a reasonably clear outline of the direction that general quantum theory is taking and of possible future developments.

Interestingly, many of the early papers on quantum theory remain topical in relation to the fundamentals of the theory and, as noted above, even publications dating to the middle of the nineteenth century are still relevant for the q-calculus. Here, we clearly see the link between the past, the present, and the future.

At this point, we would like to emphasize the outstanding contribution to this subject of the late Feliks Aleksandrovich Berezin and Naum Yakovlevich Vilenkin whose names are so familiar to every specialist in this field.

We are greatly indebted to D. M. Gitman, Yu. F. Smirnov, and V. N. Tolstoĭ for discussions of a number of problems.

9. APPENDIX

The analysis presented in this review was largely based on the theory of simple non-Abelian groups containing three generators, namely, SU(2), SU(1,1), W(1), M(2), and M(1,1). These groups constitute only a small part of the overall range of Lie groups that have found such wide applications in very different branchess of physics. Their theory has not been finally reduced to a single system, and one often finds physics journals reporting rediscoveries of the same formulas in different forms. It therefore seems useful to put together a short reference appendix on Lie groups as a whole, with particular emphasis on data that may be useful in practice.

9.1. Classification of Lie groups

Lie groups can be divided into solvable and semisimple. The Lie algebras L of solvable groups have their own commutative ideal $N([L,N]\subset N)$ or, in other words, the sequence of ideals $L^{(0)}=L$, $L^{(1)}=[L^{(0)},L^{(0)}]\ldots$, $L^{(n+1)}=[L^{(n)},L^{(n)}]=0$ terminates. Solvable groups include W(1), M(2), M(1,1), and also the Poincaré group. W(1) is a nilpotent group-*a* special case of solvable groups characterized by the sequence $L_{(0)}=L$, $L_{(1)}=[L_{(0)},L\,]\ldots$, $L_{(n+1)}=[L_{(n)},L\,]=0$.

Semisimple Lie groups do not contain a commutative ideal. They include groups of linear transformations of n variables (real **R**, complex **C**, and quaternion **H**) $GL(n,\mathbf{K})$, $\mathbf{K}=\mathbf{R},\mathbf{C},\mathrm{H}$ and their subgroups which are specified by unimodular, (pseudo)unitary, symplectic, and orthogonal matrices. All these groups are called classical. Semisimple groups also involve exclusive groups associated with the octave algebra. We note that the unitary quaternion group is sometimes denoted by USp(n) or SU(n,H); Sp(p,q) by Sp($2p,2q$); and SL(n,H) and SO(n,H) by SU*($2n$) and SO*($2n$).

Many groups of small dimensionality have between them equivalence relations (isomorphism and homeomorphism) such as SL(2,R)$\sim$SU(1,1)$\sim$SO(2,1), SL(2,C)$\sim$SO(3,C)$\sim$SO(3,1)$\sim$SL(1,C$\times$**H**). A complete listing of these relations is given in Ref. 28.

The basic characteristics of the group G include the matrices of infinitesimal operators L, the matrices of finite transformations in G, and the invariants J. The correspondence between the matrices of finite and infinitestimal transformations $g=\exp L$ is as follows:

TABLE III. Classification of classical groups.

	Unimodular	(Pseudo)unitary	Symplectic	Orthogonal
R:	SL(n,R)	SO(p,q)	Sp(n,R)	SO(n,R)
C:	SL(n,C)	SU(p,g)	Sp(n,C)	SO(n,C)
H:	SL(n,H)	Sp(p,q)	Sp(n,H)	SO(n,H)

$$g \in GL(n, \mathrm{K}): \quad \det g = 1 \quad I_{p,q}^{-1} g^{+} I_{p,q} = g^{-1} \quad F^{-1} g^{T} F = g^{-1} \quad g^{T} = g^{-1}$$

unimodular (pseudounitary) (symplectic) orthogonal

$$L \in gl(n, \mathrm{K}): \quad \mathrm{Tr}\, L = 0 \quad I_{p,q}^{-1} L^{+} I_{p,q} = -L \quad F^{-1} L^{T} F = L \quad L^{+} = -L$$

traceless (pseudoantihermitian) antisymmetric

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$, I_p is the unit matrix, $p \times p, p + q = n$,

$$F = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},$$

$2r = n$, and instead of F we can consider the matrix γ: $\gamma_{2l,2l-1} = 1, \gamma_{2l-1,2l} = -1, l = 1,2,\ldots,n/2$, and the remaining matrices $\gamma_{ik} = 0$.

The common invariant of classical groups is the determinant constructed from the basis vectors of fundamental irreducible representations. The quadratic invariants of pseudounitary, symplectic, and orthogonal subgroups SL(N,**K**), **K** = **R**, **C**, **H** have the form $X' I_{p,q} X^{+}$, $X' \gamma X^{T}$, $X' X^{T}$, where $X = \{x_i\}$ and $X' = \{x'_i\}$ are the rows of basis vectors of fundamental irreducible representations. For the quaternion bases, X^{T} is a column of transposed quaternions $q^{T} = a_1 + ia_2 - ja_3 + ka_4$. The invariants of classical groups are listed in Refs. 11 and 166 and those of exclusive groups in Refs. 166 and 167.

9.2. Lie algebras, C–G sequences, and reduction formulas

The commutation relations for group generators are written in the standard form

$$[\hat{H}_i, \hat{E}_\alpha] = \alpha(\hat{H}_i)\hat{E}_\alpha, \; [\hat{E}_\alpha, \hat{E}_{-\alpha}] = \hat{H}_\alpha, \; [\hat{E}_\alpha, \hat{E}_\beta] = \hat{N}_{\alpha,\beta}\hat{E}_{\alpha+\beta}.$$

In contrast to (2.1), they contain a whole set of raising, lowering, and diagonal operators. The constants $\alpha(\hat{H}_i)$ and $N_{\alpha,\beta}$ are expressed in terms of radical vectors that are characteristic for each group (the Cartan–Weyl basis). The algorithm for them and the structure of the set of radical vectors are specified by the Dynkin schemes. A detailed presentation of these questions is given in Refs. 28, 115, and 164. Figure 8 shows a set of Dynkin schemes for groups of relatively low rank. They give the dimensions of fundamental representations and the notation used below in the table of C–G series. The weight diagrams for the irreducible representations of the groups of rank 3 SU(4), SO(7), SP(3), and SU(3,1) are given in Ref. 11. An enormous range of factual material on C–G series and coefficients can be found in the literature. It is largely unsystematic. We confine our attention here to a table of series that are most frequently encountered in applications:

SU(3): $D(P_1Q_1) \otimes D(P_2Q_2)$

$$= \sum_{A=0}^{C_1} \sum_{S_1=0}^{\min\{P_1,P_2\}} \sum_{S_2=0}^{\min\{Q_1,Q_2\}} D(PQ), \; P \geq 0, \; Q \geq 0;$$

SU(2,1): $D^{+}(P_1Q_1) \otimes D^{+}(P_2Q_2)$

$$= \sum_{A=0}^{C_2} \sum_{S_1=0}^{\infty} \sum_{S_2=0}^{\min\{Q_1,Q_2\}} D^{+}(PQ), \; P < -2, \; Q \geq 0,$$

$P = P_1 + P_2 - 2S_1 + S_2 - A$, $Q = Q_1 + Q_2 - 2S_2 + S_1 - A$,

$C_1 = \min\{P_2 - S_1, Q_1 - S_2\}$

$+ \min\{P_1 - S_1, Q_2 - S_2\}$, $C_2 = Q_1 + Q_2 - 2S_2$;

SU(N,1): $D^{+}(P_1 0\ldots0) \otimes D^{+}(P_2 0\ldots0)$

$$= \sum_{S=0}^{\infty} D^{+}(P_1 + P_2 - 2S\, S0\ldots0);$$

SU(N): $D(P_1 0\ldots0) \otimes D(P_2 0\ldots0) = \sum_{S=0}^{\min\{P_1,P_2\}} D(P_1 + P_2 - 2S\, S0\ldots0)$,

$$D(P_1 0\ldots0 Q_1) \otimes D(P_2 0\ldots0 Q_2)$$

$$= \sum_{\alpha',\alpha'',S_P S_Q} D(P_1 + P_2 - \alpha' - \alpha'' - 2S_P\, S_P\, 0\ldots0\, S_Q\, Q_1 + Q_2 - \alpha' - \alpha'' - 2S_Q),$$

$$P_1 - S_P - \alpha' \geq 0, \; P_2 - S_P - \alpha'' \geq 0, \; Q_1 - S_Q - \alpha'' \geq 0,$$

$$Q_2 - S_Q - \alpha' \geq 0 \; (N \geq 4);$$

SO(N), Sp(N): $D(P_1 0\ldots0) \otimes D(P_2 0\ldots0)$

$$= \sum_{\alpha=0}^{\min\{P_1-S,\, P_2-S\}} \sum_{S=0}^{\min\{P_1,P_2\}} D(P_1 + P_2 - 2S - 2\alpha\, \alpha\, 0\ldots0),$$

$D(0\ldots0P_1) \otimes D(0\ldots0P_2)$

$$= \sum_{S_1,\ldots,S_N} \left\{ \begin{array}{l} D(2S_2\, 2S_3 \ldots 2S_N\, P_1 + P_2 - \sum_{i=1}^{N} 2S_i), \; \mathrm{Sp}(N) \\ D(S_1\, S_2 \ldots S_N\, P_1 + P_2 - \sum_{i=1}^{N} 2S_i), \; SO(2N+1) \end{array} \right\},$$

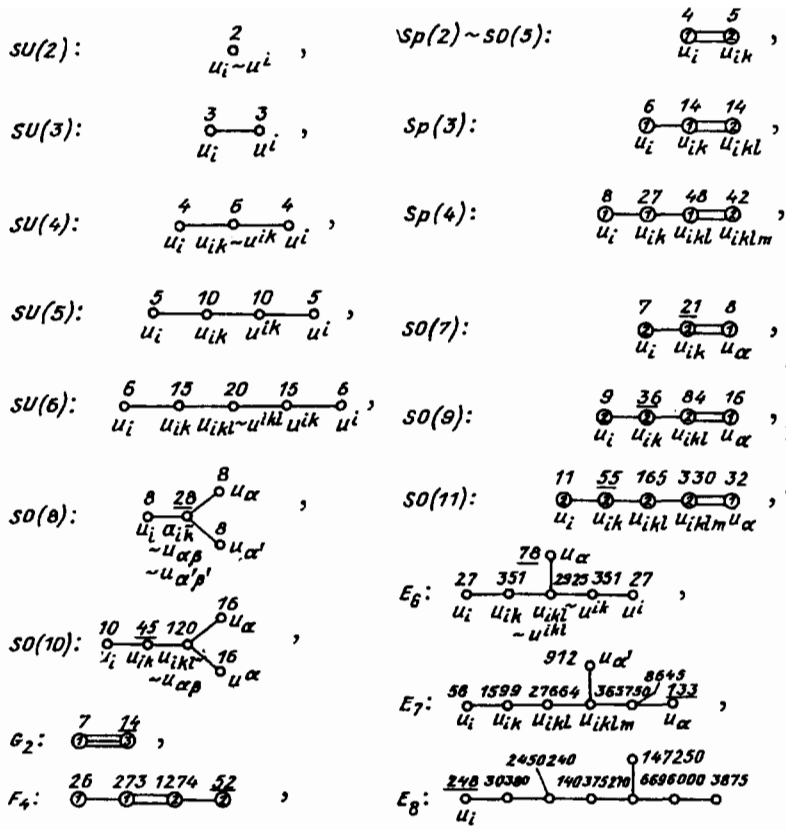

FIG. 8. Dynkin's schemes. The dimension of the fundamental IR is shown above each vertex. Numbers at the vertices are the ratios of the squares of the moduli of radical vectors [for SU(N), SO($2N$), E_6, E_7, E_8 they are equal to unity]. The dimensions of the associated representations are shown for the orthogonal and exclusive ones.

$$P_k - \sum_{i=1}^{N} S_i \geq 0;$$

$$\mathrm{Sp}(2) \sim \mathrm{SO}(5)\colon D(P_1Q_1) \otimes D(P_2Q_2)$$
$$= \sum_{\alpha_P,\alpha_Q,S_P,S_Q} D(P_1 + P_2 - 2S_P - 2\alpha_P + 2S_Q\ Q_1 + Q_2 - 2S_Q - \alpha_Q + S_P),$$

$$P_1 - \alpha_1 - S_P - 2\alpha'_1 - \alpha''_1 \geq 0,\ Q_1 - \alpha_2 - S_Q - \alpha'_2 - \alpha''_2 \geq 0,$$
$$\alpha_P = \alpha_1 + \alpha'_1 + \alpha'_2,$$
$$P_2 - \alpha_1 - S_P - 2\alpha'_2 - \alpha''_2 \geq 0,\ Q_2 - \alpha_2 - S_Q - \alpha'_1 - \alpha''_1 \geq 0;$$
$$\alpha_Q = 2\alpha_2 + \alpha''_1 + \alpha''_2.$$

A large number of concrete expressions for the C–G series of classical groups is given in Ref. 11 and for exclusive groups in Refs. 162, 163, and 165. The formulas for a reduction to a subgroup reflect the hierarchy of physical structures. The branching rules corresponding to the reduction SU($N+1$)$\to$SU(N), SO($N+1$)$\to$SO(N), SU($2N$)$\to$Sp(N), Sp(N)$\to$Sp($N-1$), SU(N)$\to$SO(N) for $N \leqslant 7$ are given in Ref. 12.

9.3. Bases of irreducible representations

The IR bases (both finite and infinite-dimensional) can be divided into two broad classes, namely, discrete (infinitesimal and tensor) and continuous (generalized CS bases and bases of the eigenfunctions of non-self-adjoint operators).

The infinitesimal bases are constructed as the IR bases of the corresponding Lie algebra (by operating on a chosen element of infinitesimal operators of the groups we obtain the complete IR bases). These bases constitute a set of eigenfunctions of self-adjoint operators; different choices of the latter lead to different bases.

The tensor (symmetric) basis is constructed as a polynomial of the basis functions of fundamental irreducible representations. It is closely related to the permutation group. In the case of unitary irreducible representations of noncompact groups, the tensors can have negative or noninteger rank. This basis is usually redundant (with the exception of simple cases).

The generalized CS basis is constructed by acting on a chosen element from the space of irreducible representations of finite transformations of a group; it is overcomplete. The basis of the eigenfunctions of raising or lowering operators can be constructed for infinite-dimensional unitary irreducible representations of noncompact groups, but not for finite-dimensional irreducible representations of compact groups.

9.4. The groups SU(N) and SU(N–1,1)

The fundamental point here is that it is possible to construct a theory of representations of higher groups on a constructive basis. In this approach, the formulas for the overlaps and for the effect of generators and of the invariant measure play a dominant role. Apart from the above simple groups, there is a detailed theory that includes coherent states, the symbol calculus, and a transition to the classical

limit, which applies to the unitary symmetric irreducible representations of SU(N) and SU($N-1,1$) (Refs. 15 and 16). The quantity

$$(u, z) = \bar{u}_N z_N \pm \sum_{i=1}^{N-1} \bar{u}_i z_i .$$

is an invariant of SU(N) and SU($N-1,1$) groups.

For symmetric finite-dimensional irreducible representations $D(M0...0)$ of SU(N), $M \geqslant 0$ is an integer; for the infinite-dimensional irreducible representations $D^+(M0...0)$ of SU(M 1,1), $M < -N+1$. The CS overlap is expressed in terms of the invariant $\langle Mu|Mz\rangle = (u,z)^M$ where $(u,u) = (z,z) = 1$. The relative variance (4.71) of the CS is $\Delta C_2/C_2 = N/(N+M)$. The invariant measure is

$$d\mu_M(z) = \frac{(M+N-1)!}{M!} \frac{1}{(2\pi)^N} \delta\left(\sum_{i=1}^{N} \rho_i^2 - 1\right) \prod_{i=1}^{N} d\rho_i^2 d\varphi_i,$$

$$d\mu_M(z) = \frac{\Gamma(-M)}{\Gamma(-M-N+1)} \frac{1}{(2\pi)^N} \delta\left(\rho_N^2 - \sum_{i=1}^{N-1} \rho_i^2 - 1\right) \prod_{i=1}^{N} d\rho_i^2 d\varphi_i,$$

$$z_i = \rho_i e^{i\varphi_i}.$$

The integration is performed over the N-dimensional complex sphere or hyperboloid $(z,z) = 1$, using formulas similar to (5.16) (Ref. 5). The transformation $z_i' = z_i/z_N$ is used to pass to the projective space $P_{N-1}(\mathrm{C})$ or the open sphere $D_{N-1}(\mathrm{C})$. The operator symbols can be found from the form of the generators $\hat{T}_j^i$ of the groups U(N) and U($N-1,1$) in terms of $\hat{a}_+$ and $\hat{a}_-$: $\hat{T}_j^i = \hat{a}_{i+}\hat{a}_{j-}$.

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