METHODOLOGICAL NOTES

Conceptual problems in quantum mechanics

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This review is devoted to a discussion of the interpretation of quantum mechanics. The heuristic role and limitations of the principle of observability and of operationalism are discussed. It is shown that the probabilistic approach to quantum mechanics is essential as a way of reconciling the conflicting concepts of particle and wave. The reason why the reduction of the wave packet is not a physical process, but a logical act is explained. The discussion of the paradoxes of quantum mechanics covers many well known examples and includes the Aharonov–Bohm effect and interference between two independent laser beams. It is suggested that the causality principle does not reduce to determinism, but has certain other manifestations too. This is illustrated by the fact that Newtonian mechanics was at one time considered as abstract and impenetrable, in contrast to the 'natural', but eventually fruitless mechanics of Descartes. It is shown that a classical foundation cannot be provided for quantum mechanics, i.e., it is impossible to introduce hidden variables into quantum mechanics. Mathematical manipulation is reduced to the essential minimum, and many examples are provided to illustrate the discussion. Outstanding contributors to physics are extensively quoted. The review is intended for readers with higher education in both the natural sciences and the humanities, who are interested in conceptual problems in modern science.

> It may be that these electrons
> Are worlds with five continents,
> Arts, knowledge, wars, kingdoms,
> And the memory of forty centuries!
>
> It may also be that each atom
> Is a universe with a hundred planets
> Containing all we have in compressed volume
> And perhaps some things that we do not have.
>
> Their measures are small, but their infinity
> Is nevertheless no smaller than ours.
> (V. Ya. Bryusov, *The Electron World*, a
> fairly literal translation).
>
> '....If quantum theory does not disturb
> on first acquaintance, it could not have been properly
> understood.' (Niels Bohr[1a])

The fundamental ideas of quantum theory have altered the picture of the world that we have inherited from the nineteenth century. They have caused a conceptual revolution and thus touch the lives of many people. However, the literature devoted to the interpretation of quantum mechanics suffers from a significant gap. Whilst there are extensive discussions of the paradoxes of quantum mechanics, the unexpected nature of its conclusions, and the contradictions between quantum mechanics and our intuition, there are relatively few books that try to develop the reader's intuition so that the new facts become more readily understood and accepted.

Our aim in this review is to examine the connection between quantum mechanics, on the one hand, and intuition and common sense on the other. We make extensive use of quotations while being fully aware that many authors employ quotations as a means of dissimulation or of covering up their poverty of thought. On the other hand, an article on conceptual problems in natural science that is devoid of quotations is just as clumsy as mathematical paper without formulas. At the same time, eminent scientists often express their thoughts epigrammatically, which is why the re-telling of such quotations often causes a loss of the 'feeling of direct contact with beauty.'[3] Such retelling makes the source more remote and tends to reduce nuances and color. In our own presentation, citations serve not merely as illustrations, but are treated as an integral part of the text.

Any serious publication on conceptual questions in

physics must by now contain a certain amount of mathematics. This notwithstanding, we have tried to reduce the number of mathematical formulas to a minimum. Even so, some of the sections presented below may be difficult for readers whose education is mostly in the humanities. These sections may be omitted on first reading.

We have taken every opportunity to enliven our discussion with examples. '....Examples provide better explanations...than abstract general discussions' (Ref. 5, p. 247). At the same time, we would not wish to emulate the textbook writer[6] who, having listed the units of power, recalls the quaint fact that the German and British units of horse power are different (75 and $76 \, \text{kg ms}^{-1}$, respectively). Indeed, we confine our attention to the more substantial interpretations of quantum mechanics.

Our review is intended for readers with higher education in the sciences and the humanities, who are interested in conceptual problems in modern science. In order to help the nonspecialist reader, we have given considerable thought to the presentation of the more important questions. Some readers may therefore find that this paper is a bit like *Gulliver's Travels* (with apologies to Jonathan Swift), which is seen by children as a mere tale and by adults as political satire. On the other hand, anyone familiar with the history of physics will recall Leibnitz's proposition about the relativity of length: nothing will change if all objects are reduced in length by a factor of 12. Since our paper is intended for a wide range of readers, it contains fragments that are rather elementary. Some of them may seem unnecessary to specialists, but they will be useful to others. Topics requiring greater mathematical sophistication are collected together in Sections 8 and 9. More detailed discussions of particular topics may be found in the references scattered throughout the text. Similar accounts are presented in Refs. 17, 50, and 52. Our review differs from other presentations by its examination of the contributions of Ernst Mach as the ideologue of scientific revolutions, by its analysis of the process of measurement, and by the proof that it gives of the impossibility of hidden parameters in quantum mechanics. (An annotated bibliography of interpretations of quantum mechanics may be found in Ref. 7).

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INTRODUCTION

The history of physics is an expression of the victory of reason over ignorance. Every year, an increasing number of physical phenomena is explained in terms of material causes rather than supernatural forces. Newton's laws of motion and of universal gravitation led to an explanation of the motion of celestial bodies, and to predictions of solar and lunar eclipses that had been 'explained' by divine intervention. Newton's laws explained all the known celestial phenomena. When he came across Laplace's *On the System of the World*, Newton told him 'I found no mention of God in your book.' To which Laplace replied 'I had no need for this hypothesis' (Ref. 8, p. 217).

The application of the laws of physics to terrestrial phenomena has resulted in a veritable avalanche of inventions that would have been regarded as miraculous in the past. Examples include the steam engine and the gramophone, electricity and the aeroplane, cinematography and radio, and so on. It seemed that the victory of materialism was irreversible. True, materialism was paralleled by idealism: Bishop Berkeley wrote 'It is indeed an opinion strangely prevailing amongst men, that houses, mountains, rivers, and in a word all sensible things, have an existence, natural or real, distinct from their being perceived by the understanding... . But, say you, though the ideas themselves do not exist without the mind, yet there may be things like them, whereof they are copies or resemblances; which things exist without the mind, in an unthinking substance. I answer, an idea can be like nothing but an idea; a color or figure can be like nothing but another color or figure... . The table I write on I say exists, that is I see and feel it... . To exist is to be perceived.'[9] This argument is fallacious: a portrait is a collection of colors, but it can resemble a person.'

Physicists are natural materialists because it is their job to study the laws of nature, irrespective of human feelings which are the province of physiology and psychology. The arguments of Bishop Berkeley and his followers seemed so unreasonable to physicists that they felt it was below their dignity even to criticize subjective idealism. Thus "the gradual accumulation of information about atomic and small-scale behavior during the first quarter of this century, which gave some indications about how small things do behave, produced an increasing confusion... ." (Ref. 2, vol. 3, p. 199). There arose a "mysticism that was contrary to the spirit of science" (Ref. 10, p. 204).

It was common to find in papers and books statements to the effect that physics was concerned exclusively with the ordering of our sensory perceptions and not with the discovery of objective laws independent of the observer. Jordan wrote: "When it is characterized as the framework for mathematical formulas, the atom is an auxiliary device for ordering experimental facts, much like the geographical grid of the Earth."

Let us now consider how materialists interpreted quantum mechanics. Here is V. A. Fock: "What are...these features of quantum mechanics that prevent us from treating it in a classical spirit and see the wave function as a field[1] distributed in space and time, in many ways similar to the classical field... . For a complex system consisting of a large number of particles, the wave function depends on all the degrees of freedom of the system and not just on three coordinates. It is a function in multidimensional configuration space and not in the real physical space... . The wave function doe sot always exist and is not always described by the Schrödinger equation; under certain well-known conditions, it is simply deleted and replaced by another (this is the so-called reduction of a wave packet). It is clear that this type of instantaneous change is not consistent with the concept of a field" (Ref. 11, p. 461). These ideas are correct and far-reaching, but they are presented in brief form and may therefore be misunderstood.

If the wave function is not a "field distributed in space," if it is defined not in real space, but in a multidimensional abstract space, and if varies not according to the Schrödinger equation, but is simply deleted, does it not follow that quantum mechanics describes our sensory perceptions or our knowledge? Fock's aim was to demonstrate that quantum

mechanics could not be reduced to classical mechanics. He therefore contrasted classical and quantum concepts. Our aim here, on the other hand, is to understand quantum mechanics and to reconcile it to common sense. The paradoxical properties of the wave function mentioned by Fock are due to the probabilistic character of quantum mechanics. The same properties are encountered in the classical theory of random processes (see Secs. 3.1 and 3.2), but no-one regards them as paradoxical. At this point we merely mention that the real physical space is not always three-dimensional, even in classical physics, when randomness is absent. For example, a table tennis player must take into account not only the three coordinates of the ball, but also the three components of translational and of angular velocities. Hence the "real physical space" involved in the game of table tennis is a nine-dimensional configuration space and not the familiar three-dimensional space.

As far as the term "abstract" (as applied to space) is concerned, it by no means denotes "unreal," i.e., existing only inside a human head. There are many mathematics books devoted to abstract multidimensional spaces. These books imply many concrete physics applications of abstract spaces, but their authors restrict themselves only to their general properties, leaving aside the specific properties of each particular space. Therefore the expression "the wave function is defined in a multidimensional abstract space" means only that the concept of the space of a wave function is a special case of a more general mathematical concept.

The 'strangeness' of quantum mechanics does not lie exclusively in the existence of specific quantum effects that cannot be explained by classical physics. Many effects, e.g., discreteness, randomness, and uncertainty relations, that are usually referred to as quantal are actually found in classical physics, too. However, in quantum mechanics, these classical effects combine in a totally 'senseless' way. To achieve a better understanding of such quantum effects, we begin by discussing them in terms of the usual classical language (Section 2), emphasizing those aspects of the phenomena that are usually left on the sidelines, but assume particular prominence in quantum theory.

1. THE THRESHOLD OF THE 20TH CENTURY SCIENTIFIC REVOLUTION

> It is brutal necessity and not mere speculation or the desire for novelty that forces us to change the old classical views. (*Einstein and Infeld* [11])

1.1. The Mach observability principle

How did it happen that so many physicists engaged in studying nature, which exists independently of observers, have come to regard the positivist Mach as their idol? The fact is that the description of Mach as a positivist is correct, but too crude. In fact, Mach was a very inconsistent personality. In some questions he was a dialectician; in others he was a major materialist (e.g., in his proposition about the materiality of space). He drew attention to the fact that the history of physics is an alternation of evolution (i.e., gradual gathering of knowledge) and revolution (i.e., a radical change in our picture of the physical world). [12] At the time, the last revolution had been engineered by Newton. In an earlier epoch, Aristotle considered that uniform motion of a body required a constant force for its continuance. Newton,

on the other hand, considered that a body continued in uniform motion if there were no external forces acting upon it.

Strictly speaking, there were two further revolutions after Newton: the electromagnetic field was given the status of objective reality and the idea of randomness was introduced into physics (in statistical physics), but these did not result in the repudiation of the fundamentals of classical mechanics. "Following the discovery of new phenomena in electricity and magnetism, electrical and magnetic forces were likened to gravitational forces and their effect on the motion of bodies could again be described in terms of the axioms of Newtonian mechanics. By the end of the nineteenth century, even the theory of heat was reduced to mechanics by introducing the idea that heat was in reality a complicated statistical motion of minute particles of matter" (Ref. 57a, p. 53). Both these revolutions could therefore be more properly regarded as "palace revolts."

Mach was one of the very few who foresaw the radical changes in physics that were to come. He therefore understood that Newtonian mechanics could not be regarded as absolute truth. He wrote: [14] "If we now assume that the facts established in mechanics are so much better understood than other facts that they can be used as the foundation for all other physical facts, then this must be an illusion. The explanation is that the history of mechanics is so much older and richer than the history of physics that we tend to take the facts of mechanics as primary." [13] Mach's remarks stimulated the revision of classical mechanics. Einstein had a high regard for Mach's critique of mechanics: "I see Mach's true greatness in his incorruptible skepticism and independence' (Ref. 15, vol. 4, p. 266). Einstein continues: "In his historio-critical publications, in which he followed with great care the evolution of science and explored the internal laboratory of individual researchers who have laid new pathways in their own branches of science, Mach had an enormous influence on the scientists of our generation" (Ref. 16, p. 113). Indeed, mach developed a program for a new revolution in physics. In particular, if we were to abandon all our acquired knowledge, we would regress to the level of the ape. We must therefore retain something of the prevailing theory, but the question is what? The answer to this question is supplied by Mach's principle of observability: the only true phenomena are those that can be observed directly (Ref. 17, p. 70).

Feynman discusses this in greater detail: 'We just have to take what we see, and then formulate all the rest of our ideas in terms of our actual experience' (Ref. 2, vol. 1, p. 47).

We note that when Newton developed classical mechanics, he also relied on the principle of observability: '...hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whatever metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction.' [18] Mach's observability principle may turn out to be a return to Berkley's subjective idealism: "to exist is to be perceived." In actual fact, it is only the unrestricted application of this principle that leads to idealism. On the other hand, a reasonable use of the observability principle is a powerful tool for constructing new theories. This principle enables us

to extract from the ruins of the old theory the components that will remain in the new.

It is precisely with the help of the observability principle that Einstein was able to create the theory of relativity, and Bohr and Heisenberg were led to the conclusion that the electron in an atom does not have a definite position or a definite momentum. At the same time, the energy of an electron in the atom is defined precisely: '...as far as the periodic orbit of the electron is concerned, it may be that it does not exist at all. The only directly observable entities are the energies of the discrete stationary state, the spectral-line intensities, and, possibly, the corresponding amplitudes and phases, but not the electron orbits.' (Ref. 19, p. 82).

To be fair, we note that, the two other founding fathers of quantum mechanics, namely, de Broglie and Schrödinger, took the route of classical physics and treated quantum effects in terms of the flow of some subquantal fluid.

However, having opposed metaphysical materialism, Mach proceeded to argue against materialism generally: "the majority of scientists, acting as philosophers, adhere to a materialism that is now 150 years old and has long been regarded as inadequate not only by philosophers but also by people who are more or less familiar with philosophical thinking... . My aim has been not so much to introduce a new philosophy into natural science as to remove from it the old philosophy that has outlived its purpose." (Ref. 5, p. 12; Ref. 4).

1.2. Limitation of the principle of observability

Theory cannot, however, be confined to the description of observations. It must necessarily include generalization. "...It would be quite wrong to try to construct a theory entirely on the basis of observable quantities. Indeed, the reverse is the case. It is only theory that determines what can be observed."[20] If we were to follow the principle of observability to the letter, science would be just as unpredictable as the result of a horse race. "Theory is not a listing of individual observations, but an account of general regularity" (Ref. 21, p. 294). "A scientific law is not only the expression of a particular number of experimental facts; it reflects the thinking of the scientist: the selection of facts, comparison, *fantasy*, and the spark of genius" (Ref. 21a, p. 349).

Before Newton, people could see that apples fell on the ground and that the Moon orbited the Earth. However, only Newton saw the common law underlying both the fall of the apple and the motion of the Moon, and used it to predict a multitude of effects that had not been previously observed.

No theory can be verified precisely in a finite number of experiments (Ref. 21, p. 288). For example, when Heisenberg constructed quantum theory, he did not confine himself to the principle of observability, but also assumed that Newton's equations of motion were also valid in quantum theory if the position coordinate and the momentum were assumed to be matrices rather than numbers.[22]

Mach's observability principle is essential at the first stage of an investigation, but it must be abandoned once a formulation of a physical law has been found. Mach's himself wrote: "...naturally, it is only an infinite number of observations, performed by excluding all interfering factors, that can yield a law" (Ref. 5, p. 241). Strictly speaking, the principle of observability is not satisfied even in classical mechanics: "although one can see throughout that Newton

was trying to present his system as the necessary outcome of experiment, and to introduce the smallest number of concepts that were not related directly to experiment, he nevertheless introduced the concepts of absolute space and absolute time" (Ref. 15, vol. 4, p. 85). In relativistic quantum mechanics, the background of electrons with negative energy is unobservable (Ref. 23, p. 62). Einstein related Mach's observability principle not to positivism or contemporary philosophy, but to passive realism: "From the philosophical point of view, this picture of the world is closely related to naive realism because the supporters of the latter consider that objects in the outside world are presented to us directly by sensory perception. However, the introduction of immutable material points signified a step toward more refined realism, since it is clear from the very outset that the introduction of such atomistic elements is not based on direct observation" (Ref. 15, vol. 4, p. 317). Mach assigned absolute significance to the principle of observability and refused to acknowledge the existence of atoms (Ref. 14, p. 55).

Einstein regarded Mach as an inspiring innovator and almost the co-author of the theory of relativity.[23] However, this theory is, strictly speaking, in conflict with Mach's philosophical ideas. In particular the theory of relativity rejects the allegedly absolute principle of observability. Thus, the theory of relativity involves not only our observations, but also the properties of matter that exist independently of people. After some fruitless attempts to convince Mach, Einstein bitterly concluded that 'Mach was as good at mechanics as he was wretched at philosophy,'[24] thus unwittingly repeating Lenin's words.

1.3. Objective processes and subjective sensations

The philosophical interpretation of quantum mechanics was greatly influenced by Mach's assertion about their inseperability of objective processes and subjective sensations: 'Everything physical that I find I can resolve into elements, but entities that remain unresolved at present are colors, pure tones, pressure, heat, odor, space, time, and so on. Depending on circumstances, these elements lie[2] outside or inside U. Because, and only because, these elements depend on conditions prevailing inside and outside U, we also call them sensations' (Ref. 5, p. 17). This interpretation makes physical events the consequences of their observation instead of considering that events are observed because they have actually occurred (Ref. 21, p. 292).

Mach's assertion that sensations cannot be separated into subjective and objective parts is not consistent with reality. For example, we perceive five minutes spent in a dentist's chair as being longer than half an hour spent in the company of a beautiful lady. Here the 'unresolvable element' is actually readily resolved into 'conditions inside I and those not inside U.' Thus Mach himself, when he passes on to specific examples, in fact resolves the unresolvable elements and rejects all conditions that lie both inside our body and inside our mind: 'a hot body A (an incandescent iron ball) will heat a cooler body B (thermometer) by radiation even if the two are not in contact' (Ref. 5, p. 196). Where is U and where is I in this example? Here we have only non-I, i.e., objective reality, which is independent of the observer's sensations. Mach's view of the world is clearly illustrated by an episode in his life. He was interested in ballistics and was often present at shooting practice. Once he turned to a col-

league and said 'I am constantly bothered by the question whether the bullet exists in the interval between the firing of the gun and the striking of the target. We can't see what is happening and cannot perceive it.' 'You are mad' replied the colleague 'How can you doubt the existence of the bullet? And this quite apart from the fact that you yourself have done calculations on bullet trajectories, and your calculations agree with experiment. Doesn't this prove the existence of the bullet?' 'This proves nothing' replied Mach. 'It may be that the trajectory is merely an auxiliary mathematical concept that serves only for the prediction of further observations. It may be that the bullet does not travel on the trajectory at all. It is possible that the bullet vanishes at the point of firing and reappears just before it hits the target.' The colleague just shrugged his shoulders, but Mach remained dubious. He constructed an instrument that could be used to photograph the bullet in flight and saw on his photograph some lines emerging from the bullet. They are now known as Mach lines.[14]

It was thus his doubts about the existence of a flying bullet that led Mach to lay the foundations of supersonic gas dynamics. The ratio of the speed of a flying object to the speed of sound is now called the Mach number in his honor.

1.4. Operationalism

The principle of observability has led to the development of operationalism. The founding father of operationalism was Bridgman who defined it in the following words: 'We understand by any concept no more than a sequence of operations. The concept is synonymous with a known sequence of operation (Ref. 25, p. 5). Operationalism is discussed in greater detail in Refs. 21, 26, and 27.

We shall elucidate the concept of operationalism in terms of an example. Suppose we ask: what is the time? "Webster defines 'a time' as 'a period,' and the latter as 'a time,'" (Ref. 2, vol. 1, p. 86). This is more of a vicious circle than a definition. To give 'time' a meaning we must specify how it is to be measured. In other words time is defined by the operation of its measurement. The extension of this to all other physical entities is operationalism. Without the operational approach there would be no theory of relativity and no quantum mechanics (Ref. 28, p. 2).

Returning to the concept of time we note that it is measured with a clock, and the terrestrial globe is a natural clock. Indeed the unit of time, the second, is the time taken by the globe to complete 1/86400th part of its revolution around its axis. In particular, it follows from this definition that there is no point in asking whether the Earth rotates uniformly because time is defined in terms of its rotation. 'Recently' writes Feynman 'we have been gaining experience with some natural oscillators which we now believe would provide a more constant time reference than the Earth, and which are also based on a natural phenomenon available to everyone. These are called *atomic clocks*. Their basic internal point period is that of an atomic vibration which is very insensitive to temperature or any other external effects. These clocks keep time to an accuracy of one part in 10^9 or better' (Ref. 2, vol. 1, p. 93).

Atomic clocks have been used to measure the extent to which Earth's rotation is nonuniform. '...the Earth's rotation on its axis is slightly slowing down. It is due to tidal friction' (Ref. 30, p. 98). We see that the question whether the Earth rotates uniformly is not as meaningless as suggested by operationalists. Time does not reduce to some specific measuring device, not even such an accurate instrument as the rotating Earth. Time is a much deeper concept; it is meaningless unless we specify the method used to measure it, but it does not reduce entirely to the method of measurement. 'The operational point of view, taken as the only criterion, always presupposes a structure at a lower level. However, the most striking theoretical achievements involve abstractions at a very high level' (Ref. 31, p. 184).

Literal adherence to operationalism in the theory of elementary particles has also been unsuccessful: 'Heisenberg taught that theory must operate exclusively with experimental facts and insisted that this principle be applied in elementary-particle physics by removing from it any mention of the time dependence of the state vector between preparation and measurement. This more radical form of quantum mechanics was called by him the S-matrix theory, and he presented it as a competitor to quantum field theory... . It did not turn out to be a successful theory of elementary particles: quantum field theory was totally victorious in this area (Ref. 21, p. 294).

Bridgman himself subsequently acknowledged that purely operational definitions of different concepts were incomplete: 'If I were to write all this again I would try to emphasize the importance of both mental and pencil-and-paper operations. One of the most important mental operations is the verbal operation. It plays a much greater part that I suggested previously...' (Ref. 32, p. 184). We note in this connection that all physical quantities such as momentum, energy, and so on have a precise meaning only within the framework of a particular theory (Newtonian mechanics, theory of relativity, quantum mechanics), but some theories are more universal than others. This is why the concepts of momentum and energy survived (with modifications) when Newtonian mechanics was superceded by relativity and quantum mechanics.

2. Particles and waves

Three blind men encounter an elephant for the first time.
'He is like a wall'—says one.
'No, he is like a column'—says another.
'You are both wrong'—says the third—'he is like a dragon.'

2.1. Particles and waves in classical physics

There are two forms of physical reality in classical physics: substance and field. Substance consists of individual particles of infinitesimal size, namely, electrons, protons, and neutrons. Field, on the hand, is distributed in all space. Excited states of the field propagate in space in the form of waves. Waves on a corn field, driven by wind, are a clear example of this. Although the waves all travel in the same direction, the corn ears themselves do not take part in net translational motion because they are attached to the ground.

Waves play a major part in physics. For example, sound waves propagate in air. Electromagnetic waves are important in nature and in technology. Electromagnetic waves with wavelengths between a meter and a kilometer are

known as radio waves, whereas visible light has wavelengths of the order of 10^{-4} cm. We note that fields exist even when waves are absent. A particular physical field can be specified by specifying it at all points in space. For example, sound is the excited state of a pressure field.

The fundamental conflict between the concepts of *particle* and *wave* disappears in quantum mechanics. To comprehend this, we must first examine these concepts within the framework of classical physics in which they cannot be combined. Consider particles first. Particles typically occupy a negligible volume, i.e, they are practically point objects (Fig. 1). However, their more important property is that they are indivisible. A liter of water can be readily divided into two parts with identical properties. However, a molecule of water cannot be divided into two parts simply by tilting a glass: a much more powerful means of division is necessary to achieve this end, e.g., electrolysis. The main point is that the division of a molecule of water does not result in two half molecules of water, but in the atoms of two new materials, namely, oxygen and hydrogen. A further important property of classical particles is their individual identity. We can always label each particle and follow its individual fate.

Waves constitute the exact opposite of all this. The ideal wave has the sinusoidal shape shown in Fig. 2 and is called a harmonic wave. We note that measurement on a low-intensity wave always involves a measure of distortion. For example, when we tune to a particular radio station, we use resonance to amplify a particular frequency and suppress all other frequencies. The result is a highly distorted wave, but its intensity is high enough for the purpose.

2.2. Interference of waves

The characteristic feature of waves is that they can not only amplify, but also 'extinguish' one another. This mutual amplification and extinguishing of waves is called interference. When the crests and troughs of two waves coincide (Fig. 3), they add constructively, but when the crests of one fall on the troughs of the other (Fig. 4), they interfere destructively. We note that the phenomenon of interference—especially destructive interference—is inconceivable in the case of particles. We can illustrate this by considering a machine gun and a target. An armored plate, with a vertical slot
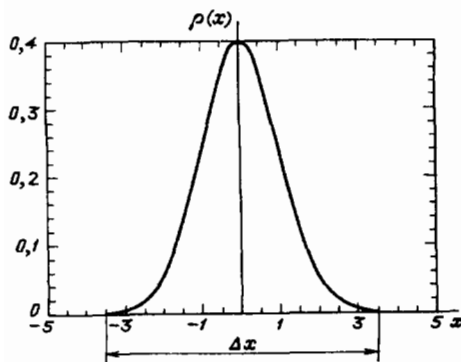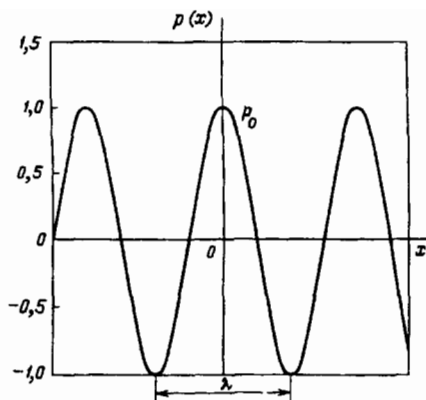

FIG. 2. Acoustic wave profile. Excess pressure above normal, plotted against position x (p_0 is the wave amplitude and λ is the wavelength which determine the frequency).

cut into it, is placed between the machine gun and the target, and bullets can pass freely through the slot. Most of the bullets hit the target center A which lies directly opposite the center of the slot. If we take the x axis to be horizontal and parallel to the plate, we can define a function $N(x)$ to represent the number N of hits at x. This function has the bell shape shown in Fig. 5. Now suppose that two closely spaced slots are cut in the plate (Fig. 6). If we cover slot 2, the number of hits will be described by the function $N_1(x)$ shown in Fig. 7. On the other hand, if we cover slot 1, the number of hits will be described by a similar, but shifted, curve $N_2(x)$. When both slots are open, the number of hits $N_{12}(x)$ is obviously equal to the sum of N_1 and N_2:

$$N_{12} = N_1 + N_2. \qquad (2.1)$$

We note that the resultant curve $N_{12}(x)$ lies above each of the individual curves $N_1(x)$ and $N_2(x)$. In other words, by opening a further slit we can only increase the number of hits at each point on the target, and there is no way of reducing this number.

FIG. 1. Distribution of the density of matter in a classical particle [$\rho(x)$—density of matter, Δx—particle size]. For the sake of simplicity, the particle is assumed to be one-dimensional.
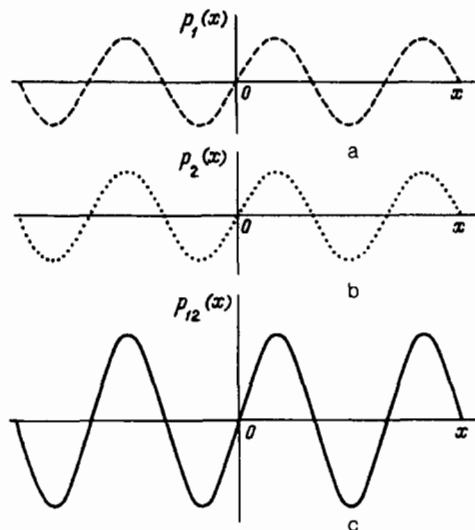

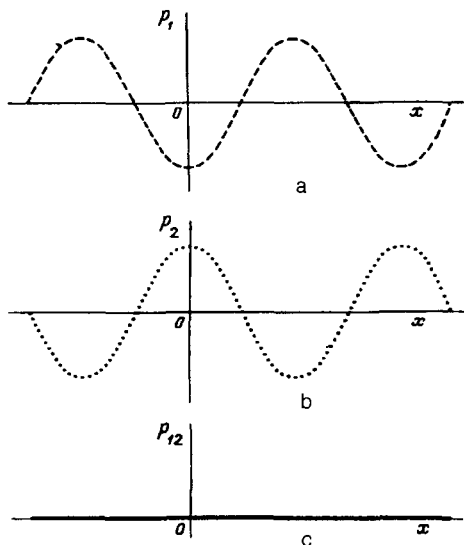FIG. 3. Interference when crests and troughs coincide: a—first wave, b—second wave, c—resultant wave.

FIG. 4. Interference when crests of one wave fall on the troughs of the other: a—first wave, b—second wave, c—resultant wave.

FIG. 6. Firing through two slits.

A totally different situation occurs in the case of waves. Consider an experiment that is the analog of the experiment described above, but uses water instead of bullets. The target is now replaced by a vertical wall and the plane of the drawing is the surface of water in its undisturbed state. The machine gun is replaced by a source of water (an oscillating object). The vertical displacement of a point on the surface of water, measured from the undisturbed state, will be denoted by $p(x)$. Consider the case where only slot 1 is open. In contrast to the number of bullets, $N_1(x)$, the displacement $p_1(x)$ can be either positive (crest) or negative (trough). When only the second slot is open, the displacement is $p_2(x)$. When both slots are open, the resultant displacement $p_{12}(x)$ is the algebraic sum of the two displacements:

$$p_{12}(x) = p_1(x) + p_2(x). \qquad (2.2)$$

Since the quantities $p_1(x)$ and $p_2(x)$ can be either positive or negative, we have the possibility of mutual cancellation of displacements, i.e., destructive interference (cf. Fig. 4).

The energy W of the wave per unit volume is proportional to the square of displacement. Omitting, for the sake of simplicity, the proportionality coefficient, we can write

$$W_1(x) = p_1^2(x), \qquad (2.3)$$

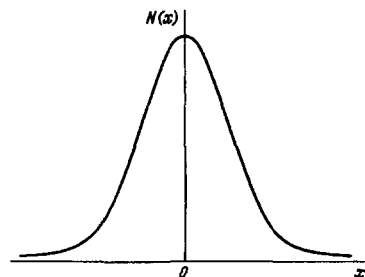$$W_2(x) = p_2^2(x), \qquad (2.4)$$
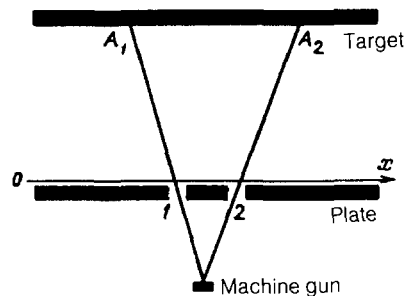
and the energy of the resultant wave is

$$W_{12}(x) = (p_1(x) + p_2(x))^2. \qquad (2.5)$$

We see that

$$W_{12}(x) = W_1(x) + W_2(x) + 2p_1 p_2, \qquad (2.6)$$

i.e.,

$$W_{12}(x) \neq W_1(x) + W_2(x). \qquad (2.7)$$

Thus, in the case of waves, we add not the energies, but the amplitudes. This gives rise to an apparent violation of the law of conservation of energy (2.7). However, in reality, this is not so. Indeed, what we are dealing with is the outflow of wave energy from the volume under consideration (which can be either positive or negative), so that the energy remaining in the chosen volume is not constant. The law of conservation of energy actually demands that the rate of loss of energy from a given volume must be equal to the rate at which energy flows out of the volume (Ref. 33, p. 358).

At a point M_a, for which the difference between its distances to the two slits is equal to an integral multiple of the wavelength λ (Fig. 8), i.e.,

$$A_1 M_a - A_2 M_a = n\lambda \qquad (2.8)$$

where n is an arbitrary integer and we are assuming, for the sake of simplicity, that the waves are one-dimensional, we have the condition

$$p_1(x) = p_2(x), \qquad (2.9)$$

and the displacement of a floating detector when both slots are open is twice as large as it was when only one slot was open (cf. Fig. 3), so that the energy of the resultant wave is greater by a factor of four. On the other hand, at a point M_b at which

FIG. 5. Number of bullets N striking the target as a function of distance from center of target.
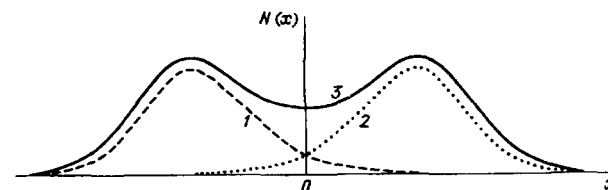

FIG. 7. Number of bullets reaching the screen after two slits: 1—number of hits N_1 when slit 1 is open, 2—number of hits N_2 when slit 2 is open, 3—number of hits N_{12} when both slits are open.
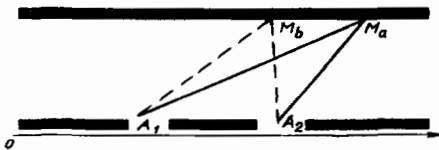
FIG. 8. Conditions for destructive and constructive interference.

$$A_1M_b - A_2M_b = n\lambda + (\lambda/2), \tag{2.10}$$

the crests of the wave from one slot fall on the troughs of the wave from the other, and we have

$$p_1(x) = -p_2(x). \tag{2.11}$$

This produces destructive interference (cf. Fig. 4), which is inconceivable in the case of particles. Thus, the addition of salt to seawater can never convert it into fresh water. The opening of the second slot produces an increase in the wave amplitude at some points and a reduction at other points. When the number n in (2.8) and (2.10) runs through all integral values between $-\infty$ and $+\infty$, the corresponding points 'move' along the wall. Regions of high energy of oscillations alternate with regions of low oscillation energy (cf. Fig. 9 which plots the wave energy, proportional to the square of its amplitude).

2.3. Coherence

Let us consider in greater detail the interference of the two waves

$$p_1(x) = P_1\sin(kx + \varphi_1), \tag{2.12}$$
$$p_2(x) = P_2\sin(kx + \varphi_2); \tag{2.13}$$

where P_1 and P_2 are constants (the respective amplitudes of the two waves), $k = 2\pi/\lambda$ is the wave number, and φ_1 and φ_2 are the phases. The energy of the first wave is

$$W_1 = (p_1(x))^2 = P_1^2\sin^2(kx + \varphi_1). \tag{2.14}$$

Usually, the wavelength λ is small in comparison with the typical linear dimensions of the apparatus, so that $\sin^2(kx + \varphi_1)$ is a rapidly oscillating function. In a measurement, we always average x over a certain interval Δx that is small in comparison with macroscopic dimensions, but is large in comparison with the wavelength λ. The average of the energy over Δx has a physical meaning:

$$\langle W_1 \rangle = P_1^2 \langle \sin^2(kx + \varphi_1) \rangle. \tag{2.15}$$
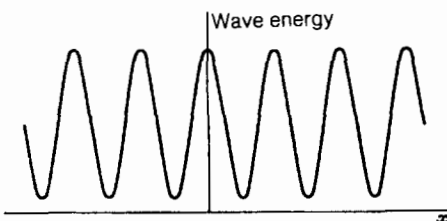
We know from trigonometry that

FIG. 9. Interference of two waves.

$$\sin^2(kx + \varphi_1) = \frac{1}{2} - \frac{1}{2}\cos[2(kx + \varphi_1)] \tag{2.16}$$

and if we note that

$$\langle \cos[2(kx + \varphi_1)] \rangle = 0,$$

we obtain

$$\langle W_1 \rangle = \frac{1}{2}P_1^2. \tag{2.17}$$

Similarly,

$$\langle W_2 \rangle = \frac{1}{2}P_2^2. \tag{2.18}$$

Combining these results, we obtain

$$\langle W_{12} \rangle = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + 2P_1P_2 \langle \sin(kx + \varphi_1)\cdot\sin(kx + \varphi_2) \rangle. \tag{2.19}$$

Since

$$\sin(kx + \varphi_1)\sin(kx + \varphi_2)$$
$$= \frac{1}{2}\cos(\varphi_1 - \varphi_2) - \frac{1}{2}\cos(2kx + \varphi_1 + \varphi_2), \tag{2.20}$$

we have

$$\langle \sin(kx + \varphi_1)\cdot\sin(kx + \varphi_2) \rangle = \frac{1}{2}\cos(\varphi_1 - \varphi_2). \tag{2.21}$$

Consequently,

$$\langle W_{12} \rangle = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + P_1P_2\cos(\varphi_1 - \varphi_2). \tag{2.22}$$

The term containing $\cos(\varphi_1 - \varphi_2)$ describes interference. In particular, when $P_1 = P_2$ and $\cos(\varphi_1 - \varphi_2) = -1$, the two waves cancel one another out.

Since light is an electromagnetic wave, illumination by two electric lamps can produce either an increase or a reduction in intensity at certain points in space. In practice, this is not observed because each atom in the filaments of the lamps emits a photon within a very short interval of time, so that the phase difference $\varphi_1 - \varphi_2$ is a rapidly-varying random function of time. We thus observe the average value of $\cos(\varphi_1 - \varphi_2)$ which is zero, i.e., the interference term disappears from (2.19) and there is no interference.

Waves for which there is a strict relation between φ_1 and φ_2 are called coherent. We thus see that interference is observed only for coherent waves. In this sense, incoherent waves behave like particles.

2.4. Uncertainty relation in classical physics

A further difference between a wave and a particle is that a harmonic wave extends to infinity, whereas a particle is localized within an infinitesimal portion of space Δx. However, this difference is unimportant because it is shown in the theory of Fourier integrals that any function that vanishes outside a finite interval Δx can be represented by a superposition (sum) of an infinite number of sinusoids with different wavelength λ and different amplitude. The wave amplitudes in this sum usually decrease rapidly with increasing difference between λ and some average wavelength λ_0. It can be said that the superposition of waves results in the wavelength λ being confined to the neighborhood of λ_0 defined by

$$\lambda_0 - (\Delta\lambda/2) < \lambda_0 < \lambda_0 + (\Delta\lambda/2),$$

where $\Delta\lambda$ maybe looked upon as the uncertainty in the wavelength.

The resolution of sunlight into harmonic waves is performed in practice by, for example, a glass prism. White sunlight is thus resolved into a rainbow of colors, namely, red, orange, yellow, green, blue, blue-violet, and violet. 'There were numerous discussions in the nineteenth century about whether the monochromatic components of white light were there in the first place, i.e., in the incident beam, or whether the components were produced by the prism. The question did not receive a satisfactory answer. In the final analysis, the most cautious position was: the monochromatic components are present in the incident light in a virtual, i.e., a potential state' (Ref. 34, p. 171).

Let us now return to the harmonic wave and introduce the wave number k, defined by

$$k = 2\pi/\lambda. \tag{2.23}$$

The uncertainty $\Delta\lambda$ then corresponds to the following uncertainty in the wave vector:

$$\Delta k = 2\pi\Delta\lambda/\lambda^2. $$

It will be shown in Sec. 9.1 that the two uncertainties Δx and Δk are linked by the uncertainty relation

$$\Delta x \cdot \Delta k \sim 1. \tag{2.24}$$

In our discussion above, we considered a wave at different points x in space at a given time t. We can also consider a wave at a fixed point x a different times t (Fig. 10). The wavelength λ then replaces the oscillation period T, whereas the wave vector k is replaced by the frequency ω:

$$\omega = 2\pi/T. \tag{2.25}$$

In terms of these new quantities, the uncertainty relation becomes

$$\Delta\omega \cdot \Delta t \sim 1. \tag{2.26}$$

We emphasize that the uncertainty relations given by (2.24) and (2.26) have nothing to do with quantum mechanics. They apply to wave processes in classical physics (Ref. 29, pp. 54, 35, 191, 36, and 37). The uncertainty relation given by (2.26) is encountered in connection with television. For example, one can ask why in many towns there are television aerials, but no radio aerials? The answer is that
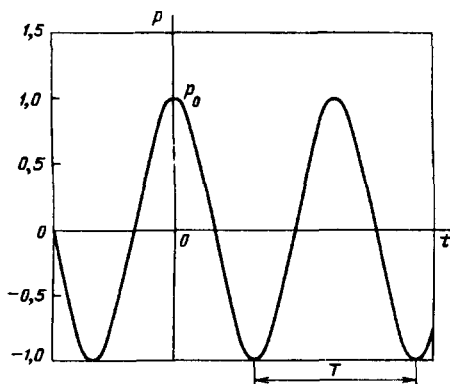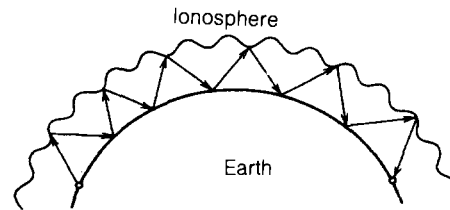

FIG. 11. Radio-wave propagation.

radio transmission can be received from stations thousands of kilometers away whereas television transmissions come from neighborhood TV centers. This is so because radio waves have wavelengths ranging from dozens of meters to several kilometers. They are reflected from the ionosphere and can therefore propagate to any point on the Earth's surface (Fig. 11).

TV transmissions, on the other hand, employ ultrashort waves, whose wavelengths are of the order of a meter. Such short waves pass freely through the ionosphere (Fig. 12), so that TV sets receive only transmissions from TV stations in their direct line of sight (Fig. 13). We then ask again: why is it that TV transmissions cannot be made at longer wavelengths? The answer is that, when compared with the acoustic information carried by radio transmission, the rate of transmission of information for a TV set is enormous. The screen has a very large number of points, so that to ensure that the successive frames are received not as blips on the screen, but as a moving image, the entire picture must be changed completely at the rate of 24 per second. The duration Δt of each signal is therefore very short and it is clear from the uncertainty relation given by (2.26) that $\Delta\omega$ is then very large.

On the other hand, the TV receiver can cope with extremely weak signals. This is possible only because the natural frequency of ω_0 of the television circuit is equal to the frequency ω of the transmitting station (resonance):

$$\omega_0 = \omega. \tag{2.27}$$

This is possible if $\Delta\omega \ll \omega$, which means that, since $\Delta\omega$ must be large, ω must be large. Since the electromagnetic wavelength λ is inversely proportional to frequency, i.e.,

$$\lambda \sim 1/\omega \tag{2.28}$$

the wavelength transmitted by the TV station must be sufficiently short.

The uncertainty relation given by (2.24) also makes its appearance in the case of the brass band. Anyone who has seen a marching band will have noticed the social inequality of these people: the flautist carries a small instrument, but the tuba player has a large one. Why is it that the bass tuba
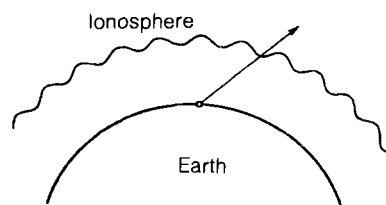

FIG. 10. Harmonic wave at a fixed point x at different times t. $p = p_0 \sin(\omega t + \pi/2)$—excess of pressure above normal, p_0—amplitude, T—period.
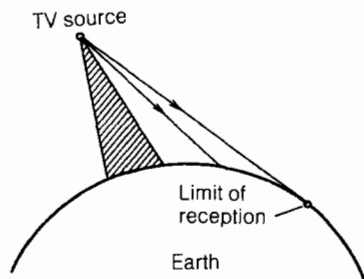

FIG. 12. Propagation of TV waves.

FIG. 13. Region of TV reception.

cannot be made smaller? Low frequency (bass) sound correspond to large λ, i.e., small k and Δk. The uncertainty principle (2.24) then shows that the corresponding Δx (the size of the tube) must be sufficiently large.

The fundamental difference between classical waves and particles is that, in classical physics, waves are indefinitely divisible, i.e., there are no wave 'atoms.' Any classical wave, however small its amplitude, can be divided into two waves of even smaller amplitude. In contrast to particles, classical waves are indistinguishable. For example, suppose that at the initial time t_0 the amplitude at a point A in water is 1 cm, whereas at a point B it is 3 cm. If at some subsequent time t_1 the wave amplitude at A becomes 3 cm, we can say that the wave has traveled from B to A. Equally so, we are entitled to say that the waves have remained in place, but the amplitude A has increased.

2.5. Particles and waves in quantum mechanics

Quantum mechanics was born when Planck discovered minute particles of light, i.e., he found that the energy of a light wave was not indefinitely divisible, but consisted of indivisible packets (quanta) given by

$$\mathcal{E} = \hbar\omega ; \tag{2.29}$$

where $\hbar$ is Planck's constant ($\hbar = 1.055 \times 10^{-27}$ erg·s) and ω is the wave frequency. (We note that, in early work, it was common to use the quantities $h = 2\pi\hbar$ and $\nu = \omega/2\pi$.) The formula given by (2.29) then took the form

$$\mathcal{E} = h\nu . \tag{2.30}$$

'...The energy of a beam of light emerging from a particular point is not distributed continuously in the entire expanding volume, but consists of a finite number...of indivisible quanta of energy that are absorbed or emitted only as complete quanta' (Ref. 38). Moreover the quanta of light are emitted by molecules in random directions (Ref. 17, p. 30). The law expressed by (2.29) can be generalized to any wave process. 'Wherever it occurs in nature, the energy of a sinusoidal oscillatory process of frequency ν always assumes values that are integral multiples of $h\nu$. Intermediate values of the energy of sinusoidal oscillatory processes are not found in nature' (Ref. 15, vol. 4, p. 58). Einstein and de Broglie use (2.29) to derive the relation between the momentum p and the wave number k:

$$p = \hbar k . \tag{2.31}$$

The Planck relation (2.29) and the Einstein-de Broglie relation given by (2.31) show that each particle of energy $\mathcal{E}$ and momentum p is also a wave of frequency

$$\omega = \mathcal{E}/\hbar , \tag{2.32}$$

and wavelength

$$\lambda = 2\pi\hbar/p . \tag{2.33}$$

On the other hand, we have seen that particles and waves are mutually exclusive concepts. The question is: how can they be unified? 'There is an entirely new idea involved [here] to which one must get accustomed and in terms of which one must proceed to build up an exact mathematical theory, without having any detailed classical picture' (Ref. 61, p. 29). We shall illustrate this idea by the example of polarized light.

We know from classical electrodynamics that light is an electromagnetic wave. Actually, it is a transverse wave: when light propagates in, say, the direction of the z axis, the electric field vector $\mathbf{E}$ lies in the perpendicular x, y plane and, in the case of linear polarization, the direction of $\mathbf{E}$ does not vary in time (or only its sign varies). The energy $\mathcal{E}$ of the beam of light is proportional to the square of the vector $\mathbf{E}$:

$$\mathcal{E} = a\mathbf{E}^2 \quad (a = \text{const}) .$$

If we now fix the x and y axes, we can resolve an arbitrarily polarized light into two beams, one of which is polarized along the x axis and the other along the y axis. The energy of the original beam, $\mathcal{E}$, can then be written as the sum of the energies of these two beams:

$$\mathcal{E} = \mathcal{E}_x + \mathcal{E}_y ,$$

where

$$\mathcal{E}_x = aE_x^2, \quad \mathcal{E}_y = aE_y^2 .$$

When the vector $\mathbf{E}$ is at an angle α to the x axis, we have

$$E_x = |\mathbf{E}|\cos\alpha .$$

and hence

$$\mathcal{E}_x = \mathcal{E}\cos^2\alpha .$$

A beam of light polarized along the x axis can be separated out by means of a tourmaline crystal with the optical axis lying along the y direction. A beam polarized in the x direction will pass through the crystal unimpeded. However when it is polarized at an angle α to the x axis, a fraction $\cos^2\alpha$ will pass through the crystal. In particular, when $\alpha = 45°$, the energy $\mathcal{E}$ of the transmitted beam is equal to one-half of the energy of the original beam, i.e.,

$$\mathcal{E}_x = \mathcal{E}/2 . \tag{2.33a}$$

In quantum physics, light consists of indivisible particles, namely, photons. A beam of light that is linearly polarized in a particular direction must be looked upon as consisting of photons, each of which is linearly polarized in that direction. This presents no difficulty when the incident beam is polarized along the x or y axis. When this is so, we need only assume that each photon polarized along the x axis passes through the crystal, whereas every photon polarized at right angles to the x is absorbed. A difficulty arises when a photon is polarized at an angle of, say 45° to the x axis. According to (2.33a), the energy of the transmitted photon should then be equal to half the energy of the incident photon, which implies a division of the incident photon into two

halves. However this is impossible because the photon is a particle and there are no 'half photons.' We thus see that the concept of a particle is in conflict with the concept of a wave. In quantum mechanics, on the other hand, the two opposites merge together, but this unification is achieved at considerable cost. In particular, in quantum mechanics, we have to abandon the determinism of classical physics, which had been elevated to a philosophical principle. Returning to the photon polarized at 45° to the x axis, we can say that, according to quantum mechanics, the photon has two possibilities: it can either pass through the crystal or be absorbed by it. Sometimes a complete photon polarized along the x axis is found after the tourmaline crystal, and sometimes no photon is detected. Half a photon is never observed. If we repeat the experiment a large number of times, we find that in one half of all cases a photon crosses the tourmaline crystal, i.e., the probability that a photon will cross the crystal is 1/2. On the other hand if the polarization vector of the incident is at an angle α to the x axis, the probability that a photon will get through is $\cos^2 \alpha$. This result for the probability leads to the correct classical result for an incident beam containing a large number of photons (Ref. 61, p. 21).

Thus, the probabilistic character of quantum mechanics, i.e., the violation of classical determinism, is not due to external causes (experimental factors), but has an internal reason, namely, the need to combine the two opposites, i.e., waves and particles.

2.6. Heisenberg uncertainty relations

It follows from the Planck and Einstein–de Broglie relations [(2.29) and (2.31)] that the universal classical concepts of time, position coordinate, energy and momentum have only limited utility in quantum mechanics. In particular, their simultaneous use is restricted by the Heisenberg uncertainty relations. We can deduce the latter[3] by multiplying the uncertainty relations (2.24) and (2.26) by $\hbar$, and then, using (2.29) and (2.31), obtain[36]

$$\Delta p \cdot \Delta x \sim \hbar , \qquad (2.34)$$
$$\Delta \mathscr{E} \cdot \Delta t \sim \hbar . \qquad (2.35)$$

We shall not pause to consider the difficulties associated with the interpretation of the energy-time uncertainty relation (Ref. 39 and Ref. 40, p. 103), and confine ourselves to two limiting cases of (2.34):

(1) $\Delta p = 0$, in which case $\Delta x = \infty$; this is a wave. It has a definite momentum, but occupies all space. We note that this case corresponds to an electron with definite velocity $v = p/m$ and therefore specific energy

$$\mathscr{E} = [(mc^2)^2 + (pc^2)]^{1/2} . \qquad (2.36)$$

(2) $\Delta x = 0$, in which case $\Delta p = \infty$; this is a particle. The particle lies at a particular point in space, but its momentum is completely undetermined. This means that, according to quantum mechanics, the particle cannot be at rest.

We now turn to a more realistic situation in which δx is different from zero, but is negligible. For example, consider an electron in an atom. We then have

$$\Delta x \sim 10^{-8} \text{ cm.} \qquad (2.37)$$

and

$$\Delta p \sim \hbar / \Delta x . \qquad (2.38)$$

The momentum uncertainty Δp corresponds to the following uncertainty in the kinetic energy:

$$\Delta \mathscr{E}_{kin} = (\Delta p)^2 / 2m .$$

Substituting $\hbar \sim 10^{-27}$ erg·s and $\sim 10^{-27}$ g, we find that the uncertainty in the kinetic energy of an electron in an atom is

$$\Delta \mathscr{E}_{kin} \sim 5 \cdot 10^{-12} \text{ erg.}$$

In atomic physics, energy is usually measured in electron volts (eV), defined by

$$1 \text{ eV} = 1.6 \cdot 10^{-12} \text{ erg.}$$

In order to keep the electron in the atom, its binding energy must not be less than $\Delta \mathscr{E}_{kin}$, i.e., it must be of the order of an electron volt.

The uncertainty in velocity is more illustrative in this case:

$$\Delta v \sim \Delta p / m . \qquad (2.39)$$

For an electron in an atom

$$\Delta v \sim 1000 \text{ km/s.} \qquad (2.40)$$

We thus see that the velocity of an atomic electron is a random quantity ranging between 0 and 1000 km/s (values of v much greater than 1000 km/s therefore have low probability). Thus, if the electron is to fit into the volume of the atom, its velocity must be random and its maximum value must exceed the velocity of a bullet by a factor of at least a thousand!

The following thought experiment provides a very clear illustration of the Heisenberg uncertainty relations given by (2.34) and (2.34). To determine the position of an electron under a 'microscope', we have to illuminate it. Since light is a wave, the uncertainty Δx in the position of the electron is of the order of the wavelength of light λ:

$$\Delta x \sim \lambda . \qquad (2.41)$$

If we reduce λ indefinitely, we increase without limit the precision with which the position of the electron is determined. However, the quantum of light—the photon—is also a particle with momentum given by (2.31). When an electron collides with a photon, it receives the additional momentum

$$\Delta p \sim \hbar / \lambda . \qquad (2.42)$$

By comparing (2.41) with (2.42), we obtain the Heisenberg relation (2.34). In other words, the more accurately we measure position, the more we disturb the original momentum. To put it another way, position and momentum cannot be measured simultaneously with absolute precision.

The above thought experiment serves as an illustration, but is hardly a proof (Ref. 41, p. 21). Paraphrasing Spinoza, we may say that 'inability to measure is not proof.' The above discussion does not therefore entirely remove the basic possibility that the position and momentum of an electron could be measured accurately by some other method. Moreover, many physical quantities have been obtained not by direct measurement, but by numerical calculation. For example, the temperature at the center of the Sun was determined not

with a thermometer or a bolometer, but by computer calculation.

There are many examples in the history of physics in which a radical improvement in measurement technique resulted in the observation of 'fundamentally unobservable objects.' For example, prior to he advent of X-ray microstructure analysis it was considered that an individual atom could not be observed. Here are a few lines from a letter written by E. S. Fedorov, the father of crystallography, to N. A. Morozov in 1912: 'Dear Nikolaĭ Aleksandrovich: You end your letter by saying that no man will ever see an atom. But you wrote this more or less at a time when man had already seen the atom with his own eyes; if not the atoms themselves, then photographic images of them, certainly...' (Ref. 42, p. 59). We can now see that atoms in crystals as the regularly distributed spots on an X-ray diffraction pattern.

The essence of the uncertainty relationships is not so much that we cannot simultaneously measure position and momentum, but that these concepts are often inadequately defined. The Heisenberg uncertainty relations is not a consequence of the fundamental imperfection of measuring devices, but a mathematical theorem (Ref. 43, p. 67). 'It is usually said that the uncertainty relation arises from the interaction between the measurer and the object being measured... . The relation actually arises at the very beginning, well before there is any question of measurement' (Ref. 29, p. 358).

The uncertainty relation for position and momentum is 'a consequence of the formalism of quantum mechanics' (Ref. 44, p. 13). The uncertainty described by the Heisenberg relation arises because we are attempting to measure something that has no definite meaning. 'If you ask a silly question, you get a silly answer' (Ref. 44a). For example, according to the Einstein–de Broglie relation, in a state with definite momentum p, the electron has a precisely defined value k, i.e., it is a harmonic wave and occupies all space. Its coordinates can then have arbitrary values. Contrariwise, in a state with definite position coordinate r_0, the momentum of the electron does not have a definite value. We thus see that a quantum object is a single entity that in one limiting case ($\Delta x = 0$) behaves like a particle and in the other limiting case ($\Delta k = 0$) behaves like a wave. However, in general ($\Delta x \neq 0$, $\Delta k \neq 0$), the quantum object has the properties of both particles and waves. The quantum-mechanical unification of waves and particles is often exploited in classical physics too, e.g., in the analysis of wave interactions (Ref. 45, p. 540 and Ref. 46). Since quantum theory becomes identical with classical theory in the limit as $\hbar \to 0$, waves are regarded quantum mechanically as particles. The interaction between particles is mathematically simple to describe than the interaction between waves. In the final formulas, Planck's constant cancels out in this case as $\hbar \to 0$.

3. MEASUREMENT AND RANDOMNESS

> **Passenger**: 'What chaos! There are three pairs of clocks in this station and they all show different time.'
> **Stationmaster**: 'What would be the sense of having three pairs of clocks in the same station if they all showed the same time?'

3.1. Randomness in classical physics

The unification of the two opposites—waves and particles—is possible because quantum mechanics describes not the established, but the potential, state of micro-objects. In other words, quantum mechanics contains the elements of randomness (i.e., it is statistical in character). Random processes are described by the theory of probability, but before we consider such processes in quantum mechanics, we must examine the more usual question of randomness in classical physics. We shall place particular emphasis on questions that are of special relevance to quantum mechanics, but are usually inadequately explored. The possibility of a random event A is characterized by the probability $p(A)$ defined in the following way. When the number N of trials is sufficiently large (more precisely, when $N \to \infty$), the ratio of the number of trials M in which A occurs to the total number of trials is given by

$$p(A) = M/N. \tag{3.1}$$

For example, suppose a factory has produced 10 000 radio components ($N = 10\ 000$) and 300 of them are rejected as faulty ($M = 300$). The probability of a faulty component is then

$$p = 300/10000 = 0,03. \tag{3.2}$$

It may be expected that a batch of 20 000 components will then contain 600 faulty ones.

If in a certain problem all events can be represented by a combination of equally possible events, then the probability can be caculated theoretically. The probability of an event A will then be

$$p(A) \leq m/n, \tag{3.3}$$

where n is the total number of equally possible events and m is the number of equally possible events in which A occurs. For example, let us determine the probability that by throwing dice we obtain at least a 5. We then have $n = 6$ (the dice has six faces) and $n = 2$ (the acceptable outcomes are 5 or 6). Hence

$$p = 2/6 = 1/3. \tag{3.4}$$

Similarly, the probability of getting a head in a coin tossing session is

$$n = 2, m = 1, p = 1/2.$$

We note two special cases of (3.3). The first is the impossible event $m = 0$ in which case $p = 0$; the second is $m = n$ (certainty) for which $p = 1$. In general,

$$0 \leq m \leq n, \tag{3.5}$$

i.e.,

$$0 \leq p \leq 1. \tag{3.6}$$

Thus, the probability of any event is nonnegative and does not exceed unity. This necessary condition is not satisfied by Wigner's hidden variables model (cf. Section 7.6) in which certain values of hidden parameters have negative probability. We have already discussed the probability of different events, but there were only two possibilities: the event either took place or it did not.

We now turn to the probability of different values of a continuous variable, i.e., a quantity that can assume an infinite number of values. For example, consider the coordinates x of a point reached by an electron. We can define the probability density $f(x)$ such that $f(x)dx$ is the probability that the electron will fall into the interval $[x, x + dx]$. We note that the probability density can exceed unity, but cannot be negative. Similarly, we can introduce a probability density in three-dimensional space: $f(\mathbf{r})d\mathbf{r}$ is the probability that a particle found near the point $\mathbf{r}$ will be in an infinitesimal volume $d\mathbf{r}$. We emphasize that the probability density $f(\mathbf{r})$ is an objective characteristic of the classical particle, but is not a field. It describes the potential possibility that the particle will be found in a particular part of space, but it is not a form of matter. The probability density contains t as a parameter. The time rate of change of the probability density is given by the transport equation

$$\frac{\partial f(\mathbf{r},t)}{\partial t} = \hat{K} f(\mathbf{r},t) , \tag{3.7}$$

where $\hat{K}$ is an operator. For example, if $\hat{K}$ is the Laplace operator, the transport equation takes the form

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} . \tag{3.8}$$

We emphasize that the transport equation (3.7) describes an objective process that is independent of the state of our knowledge.

We note that the transport equation is a determinate equation, i.e., nonrandom, although it describes the evolution of a random process. This is so because randomness is the absence of regularity. On the other hand, mathematics is concerned with regularities and can operate with random variables only symbolically. For example, $Y = 2X$. To obtain a result that could be compared with experiment, we must translate randomness into a deterministic language. A random event is then described by a determinate number, i.e., its probability. A random quantity, on the other hand, is described by a determinate function, namely, the probability density. The evolution of a random variable is then described by a determinate equation, i.e., the transport equation given by (3.7).

We now turn to the description of two particles. We begin with determinate particles, i.e., with the case where the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ are known precisely. When the particles do not interact, each of them travels independently of the other in the same three-dimensional space. On the other hand, if the particles do interact, then knowledge of the three-dimensional vector $\mathbf{r}_1$ will not be enough to enable us to determine the motion particle 1: we must also know the position $\mathbf{r}_2$ of the other particle. Hence the state of two interacting particles is described by the vector $(x_1, y_1, z_1, x_2, y_2, z_2)$ in six-dimensional space. This space is just a real as the familiar three-dimensional space.

Next, consider two particles with random coordinates. If the particles are mutually independent, the state of each of them is described by the probability density $f(\mathbf{r}) = f(x,y,z)$. On the other hand, if the probability of finding one particle in a certain volume depends on the position of the other particle, the density depends not on three but on six coordinates, i.e, $x_1, y_1, z_1, x_2, y_2, z_2$. This is the 'multidimensional

configuration space' which Fock associated with 'real physical space' (cf. **Introduction**).

In the discussion given below, we shall examine complex events. Let us suppose that, in an event C, at least one of two events A and B takes place. This complex event is called the sum of the two simple events and is denoted by

$$C = A + B . \tag{3.9}$$

It is shown in probability theory that incompatible[4] events satisfy the following law of composition of probabilities:

$$p(A + B) = p(A) + p(B) . \tag{3.10}$$

Let us now consider two types of randomness in classical physics. The fundamental laws of classical physics are determinate in character and randomness occurs for two reasons.

The first reason is: *uncontrollable interaction.* If we toss a coin, we get heads in some cases and tails in other cases. Here randomness arises because we give the coin different initial translational and angular velocities in each case.

The second reason is: *hidden parameters.* There is a certain probability that a particular person is color blind. There is nothing random about this: the retina simply has a congenital defect; the randomness is merely apparent. However, the defect is hidden from us and the randomness is actually a hidden regularity.

The question is: how can we distinguish between uncontrollable interaction and hidden parameters? The answer will depend on the outcome of repeated trials. If we toss a certain number of coins and select those that show tails, then a second tossing of the chosen coins will produce a similarity random result, i.e., we again obtain heads or tails. However, if we select people who are color blind and perform the selection again on the chosen set of people, we will again find that they are all blind.

There are two approaches to probability: objective and subjective. The former was discussed above: the probability is the fraction of events in which we are interested among the total number of events. However, the subjective approach is quite common and is concerned with 'our degree of confidence.' If this approach were to be correct, probability would only be used in logic, but not in physics in which we deal with objective processes that do not depend on whether the observer is confident about them or not.

3.2. Conditional probability

The concept of conditional probability plays an important part in our understanding of quantum mechanics. We shall illustrate this by considering again the example of a component in a radio set. Suppose that the reject probability for components of a new design is not 0.03, but 0.01. It is then clear that we have to distinguish between two probabilities, namely, unconditional probability ($p = 0.03$) and conditional probability ($p = 0.01$). The conditional probability is relevant under certain conditions: in this example, the condition is that a new design is used.

The concept of conditional probability is often used as the basis for the subjective approach to probability, regarded as a measure of our confidence. If we do not know the design of the radio component, then the supporters of the subjective approach would say that the reject probability is 0.03. On the other hand, if we do not know that we are dealing with a

component of new design, the reject probability falls to 0.01. However, our knowledge is of little significance. Different probabilities were obtained not because we knew or did not know, but because we considered different sets of radio components (Ref. 47, p. 10). In the above example, in the first case the batch consisted of components of different design, both old and new, whereas in the second case we had components of new design alone.

Despite the fact that its probability is an objective characteristic of an event, its dependence on the prevailing conditions introduces a subjective element into this concept, i.e., the selection of events satisfying particular conditions depend on the person making the selection. 'It was once decided to determine the average size of a family by asking people how many children their parents had. It is clear that this could not yield a true average because childless families were automatically excluded.' (Ref. 29, p. 355).

It is essential in each particular case to analyze the conditions under which the probabilities are obtained. Probabilities can depend on the prevailing conditions, on position, and on time. In the example of radio components, the reject probability can depend not only on the design, but also on other and often unexpected conditions. It may be found that the reject probability for components manufactured in Moscow and Khar'kov is different. It may also be different for components manufactured at the end and at the beginning of a quarter.

We now turn to the delicate question of reduction of probability, which is important for the understanding of quantum mechanics. Let us consider coin tossing again. The experiment is carried out in three different stages:

(1) the coin has not been tossed; the probability of getting tails is 1/2

(2) the coin has been tossed; the result is tails, but we have not looked at the coin and therefore believe that the probability of tails is 1/2, as before

(3) we have seen that the result is tails; we can now usefully exploit this information to improve our knowledge of the state of the coin, since we are now sure that we have tails and therefore the probability of this event has become equal to unity.

The transition from the second to the third stages, i.e., the transition from a definite, but unknown, state to a known state can be referred to by analogy with quantum mechanics as a reduction of probability. The reduction of probability does not correspond to any objective process: it is a purely logical operation whereby we cross out probability and replace it with certainty. Because of the reduction of probability we can say that 'there are in the physical worlds events that cannot be regarded as occurring in space and time' (Ref. 48, p. 276). This also happens in the case of the reduction of a wave packet in quantum mechanics (cf. Sec. 3.7). However, because quantum-mechanical concepts are complex and unfamiliar, the process is sometimes treated in a subjective-mystical spirit.

3.3. Probabilistic interpretation of quantum mechanics

Even if an atomic object is under fixed external conditions, the result of its interaction with an instrument is not in general unambiguous. Only the probability of the result is definite. The most complete expression of the results of a series of measurements is not the accurate value of the mea-

sured quantity, but the probability distribution obtained for it' (Ref. 11, p. 467). Randomness does occur in classical physics, but it has a totally different status in quantum mechanics: "Whilst all the great classical minds from Laplace to Poincaré have always proclaimed that natural phenomena are always determinate and that probability, when it is introduced into scientific theories, is a consequence of our lack of knowledge or our inability to understand the entire complexity of determinate phenomena, the situation in the currently accepted interpretation of quantum physics is that we are dealing with 'pure probability,' that does not appear to be a consequence of hidden determinism. In classical theories such as the kinetic theory of gases, probabilistic laws have regarded as a consequence of our lack of knowledge of the completely determinate, but disordered and complicated, motions of countless molecules of a gas; if we knew the positions and velocities of all the molecules then, in principle we could predict precisely the evolution of a gas. However, in practice, we do not know these hidden parameters and have to introduce probabilities. The pure probabilistic interpretation of wave mechanics rejects this interpretation of probabilistic laws' (Ref. 49, p. 25).

The probabilistic laws of quantum mechanics are not due to our ignorance about some hidden parameter: there are in fact no such parameters (see Section 7). Randomness in quantum mechanics is one of its postulates. '...the concept of probability is a primary concept in quantum physics in which it plays a fundamental role. The quantum-mechanical concept of the state of an object is closely related to it,' (Ref. 11, p. 468).

The state of a quantum object is characterized by its wave function $\psi(\mathbf{r})$ which is not a determinate field, but a probability field. The probability dw of finding a particle near a point x, y, z in an infinitesimal parallelepiped with edges $dx \, dy \, dz$ is proportional not to the function $\psi(x, y, z)$ but to the square of its modulus

$$dw = |\psi(x,y,z)|^2 \, dx \, dy \, dz . \qquad (3.11)$$

In this discussion, we are treating the micro-object as a particle. To emphasize that the micro-object has wave properties as well, the function ψ is often referred to as the wave function. '...The wave function of a particle describes the possibility of a subsequent observation' (Ref. 17, p. 45). For example, when an electron is in a state with a particular momentum $\mathbf{p}$, it is described by the following function (in the coordinate representation):

$$\psi = \exp(i\mathbf{p}\mathbf{r}/\hbar) . \qquad (3.12)$$

The momentum of the electron is precisely determined and is equal to the vector $\mathbf{p}$. On the other hand, the coordinates are completely indeterminate and can have any value with equal probability. The wave function (3.12) describes an infinite wave that has the same intensity at all points in space. On the other hand, in a state with particular position vector $\mathbf{r}_0$, the electron is described by the wave function

$$\psi = \delta(\mathbf{r} - \mathbf{r}_0) . \qquad (3.13)$$

The position of the electron is then determined precisely and is given by the vector $\mathbf{r}_0$, but its momentum is then totally undetermined and can assume any value with equal probability.

We now turn to the unperturbed electron in the hydrogen atom. Its state is described by a wave function, which in terms of the polar coordinates r, ϑ, φ is

$$\psi(r,\vartheta,\varphi) = \frac{1}{(\pi a^3)^{1/2}} \exp\left(-\frac{r}{a}\right), \tag{3.14}$$

where a is the Bohr radius, given by

$$a = \hbar^2/me^2. \tag{3.15}$$

The wave function (3.14) is independent of ϑ and φ, which means that we are dealing with an isotropic situation. In this state, the coordinates of the electron are indeterminate. The probability dw that the electron is in a cell $(r, r + dr)$, $(\vartheta, \vartheta + d\vartheta)$, $(\varphi, \varphi + d\varphi)$ is

$$dw = |\psi|^2 r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi = \frac{1}{\pi a^3} e^{-2r/a} r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi. \tag{3.16}$$

The magnitude of the momentum in the state described by (3.14) is also indeterminate, but we shall not reproduce the expression for the probability of the different values of the momentum. As far as the energy $\mathscr{E}$ of the electron is concerned, it is given by the following expression in the state described by the wave function (3.14):

$$\mathscr{E} = -me^4/2\hbar^2 \tag{3.17}$$

where the negative sign signifies that the electron is in a bound state.

The evolution of the wave function in time is described by a determinate equation similar to the transport equation given by (3.7):

$$\frac{\partial\psi}{\partial t} = \frac{i}{\hbar} \hat{H}\psi ; \tag{3.18}$$

where $\hat{H}$ is the Hamilton operator obtained from the expression for the energy in which momentum is replaced with the differentiation operator

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial r}. \tag{3.19}$$

The relation given (3.18) is called the Schrödinger equation.

3.4. Catastrophe in the micro-world

When Mark Twain heard the words 'It is surprising how Columbus found America!,' he is said to have responded: 'It would be even more surprising if he didn't find it, since it has always been there.' These words are relevant to measurements in quantum mechanics. The fact that the momentum and the position of an electron cannot be measured simultaneously with great precision is often said to be surprising. However, it is even more surprising that we can measure the position and momentum of an individual electron with coarse macroscopic devices whose mass exceeds the mass of the electron by a factor of 10^{26}. "...The macroscopic measuring device should be an *unstable system* (or more precisely an *almost* unstable system). It is only then that a micro-particle can change its state, and it is this change that is a macroscopic phenomenon. A micro-particle cannot affect an instrument in the form of a stable macroscopic system. It cannot 'displace' its 'pointer' from its zero position!" (Ref. 50, p. 120). In other words, a measurement performed on a micro-object by a macroscopic device is a 'catastrophe in the micro-world. It is precisely such catastrophes that enable us to perform measurements on individual micro-objects. For example, let us consider how a Geiger counter records the position of an electron. The counter is a capacitor in which the space between the electrodes is filled with air. The voltage between the electrodes is low enough to avoid breakdown, but high enough to accelerate an electron to an energy that enables it to ionize the atoms of air by collision. This releases a number of electrons that are in turn accelerated by the electric field in the capacitor and thus produce further ionization. The result is a growing avalanche of free electrons which finally causes electrical breakdown, which is readily recorded.

Another example is the detection of an electron by a photographic plate. The photographic emulsion contains silver bromide molecules. The state of the AgBr molecule is shown schematically in Fig. 14 in which, for the sake of illustration, we have replaced the chemical bonding force by the more familiar gravitational force. The state of the molecule is represented by a ball rolling on the smooth surface. Gravity pulls the ball down and its state becomes stable when it reaches the bottom of the well. The potential in which AgBr molecule finds itself consists of a very shallow well and, next to it, a very deep well that corresponds to the slit in of the molecule into the individual atoms of silver and bromine. Silver bromide is therefore stable, but it can be split by supplying a relatively small amount of energy to it. The energy released in this process is received by neighboring molecules, and the result is a chain reaction that continues until all the molecules in the emulsion grain have split into the individual atoms. The resulting black grain can then be readily observed by the unaided eye. "In the case of a photographic plate or a counter, we are dealing with an amplifying device in which avalanche-type processes develop' (Ref. 44, p. 6).

"Measurements" in a nuclear reactor have a somewhat different character. The fission of the uranium-235 nucleus results in the release of a few neutrons which results in a chain reaction. However, some of the neutrons are observed by other nuclei or leave the reactor. The relative number of neutrons captured by uranium-235 nuclei can be increased by exploiting the phenomenon of resonance. When it captures a neutron, the uranium-235 nucleus is raised to an excited state (before fission takes place) with low excitation energy $\mathscr{E}_0$. According to Planck's formula (2.29), the frequency corresponding to this energy is $\omega_0 = \mathscr{E}_0/\hbar$. Neutrons released in fission have high energy $\mathscr{E} \gg \mathscr{E}_0$ which corresponds to a high frequency $\omega = \mathscr{E}/\hbar \gg \omega_0$. Resonance takes place when $\omega \approx \omega_0$ and the neutrons are rapidly cap-
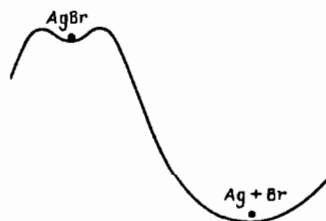

FIG. 14. Schematic representation of a weakly-stable state of the silver bromide molecule.

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tured by the uranium-235 nuclei, which in turn leads to the fission of these nuclei. This means that a 'moderator' must be introduced into the reactor for a chain reaction to take place. This is usually graphite which does not absorb neutrons, but does slow them down. Neutrons colliding with graphite nuclei lose some of their energy $\mathscr{E}$, so that the frequency ω must increase. We note that, in this particular "measurement," the 'macro-instrument' is a micro-object, namely, the uranium-235 nucleus. The quantity that is being measured is the neutron energy. Since the neutrons are located at large distances from the uranium-235 nucleus, their potential energy is zero, and their total energy is equal to their kinetic energy

$$\mathscr{E} = p^2/2M \qquad (3.20)$$

where M is the neutron mass. The magnitude of the momentum p of the neutron is thus accurately determined, but its position remains undetermined. It is precisely this uncertainty in position coordinates that enables the uranium-235 nucleus to interact with a large number of neutrons at once.

In classical physics, measurement or observation do not usually affect the state of the object being examined. On the contrary, in quantum mechanics, measurement or observation of a micro-object is accompanied by the destruction of its previous state. For example, a Nicol prism is used to determine the polarization of light by allowing light to pass through it. Only those photons emerge from the prism for which the polarization vector lies along a particular direction. Photons whose polarization vector is perpendicular to this direction are destroyed.

A more 'humane' method of observation is the determination of the position of an electron with a Geiger counter. In this measurement, an initial electron with accurately known momentum, but unknown position, undergoes a transition to a different state, namely, a state in which its coordinates x, y at right angles to the direction of its motion have small uncertainties Δx, Δy of the order of the transverse dimensions of the counter. Heisenberg's uncertainty relations then show that the uncertainties in the transverse components of the momentum are Δp_x, Δp_y. 'By suitably choosing a particular method of observation, we actually decide which properties of nature will be determined and which will be erased in the course of our observation. This distinguishes the smallest particles of matter from the range in which our sensory perception operates' (Ref. 51, p. 68).

3.5. Superposition of states

Quantum mechanics is "...a new system of exact laws of nature. One of the most fundamental and radical among them is the principle of superposition of states" (Ref. 61, p. 19). (This principle leads, among other things, to the linearity of the equation describing the evolution of the wave function, i.e., the Schrödinger equation.) We shall illustrate the superposition principle by an example. Suppose that a beam of electrons is incident on a screen containing two slits.[5] The state of the electron behind the screen will be described by the wave function ψ. We now cover slit 2 so that only slit 1 is open. The electrons are then transmitted by slit 1 alone. Let the state of an electron in this case be denoted by ψ_1 and the state when slit 2 is open, but slit 1 is closed, by ψ_2. The above principle then states that the wave function ψ is a linear combination of ψ_1 and ψ_2:

$$\psi = c_1\psi_1 + c_2\psi_2 \qquad (3.21)$$

where c_1 and c_2 are constants. This linear combination is called a superposition or a wave packet (to distinguish superposition from a mixture, we also refer to it as a pure state). The quantity

$$A(C_1) = c_1\psi_1(\mathbf{r}) \qquad (3.22)$$

is the probability amplitude that an electron which has crossed the first slit will reach a given point $\mathbf{r}$ on the photographic plate placed beyond the screen (event C_1). The corresponding probability will be denoted by $P(C_1)$.

One of the postulates of quantum mechanics is that probability is measured by the square of the modulus of the amplitude:

$$P(C_1) = |A(C_1)|^2 . \qquad (3.23)$$

Similarly, for the second slit

$$A(C_2) = c_2\psi_2(\mathbf{r}) , \qquad (3.24)$$

and

$$P(C_2) = |A(C_2)|^2 . \qquad (3.25)$$

Let us now consider the composite event $C_1 + C_2$ in which an electron reaches a given point on the photographic plate when both slits are open. The probability amplitude for this event will be denoted by $A(C_1 + C_2)$. It is clear that the amplitude $A(C_1 + C_2)$ is equal to the wave function

$$A(C_1 + C_2) = \psi . \qquad (3.26)$$

From (3.21), (3.22), and (3.24) we then find that

$$A(C_1 + C_2) = A(C_1) + A(C_2) . \qquad (3.27)$$

This means that the amplitude for the sum of the events is equal to the sum of their amplitudes. The probability of the event $C_1 + C_2$ will be denoted by $P(C_1 + C_2)$. According to (3.23) and (3.27), we have

$$P(C_1 + C_2) = |A(C_1 + C_2)|^2 = |A(C_1) + A(C_2)|^2$$
$$= P(C_1) + P(C_2) + A(C_1)A^*(C_2) + A^*(C_1)A(C_2) . \qquad (3.28)$$

It is clear that interference takes place, which means that the superposition of two events results in the addition of the probability amplitudes, but not of the probabilities themselves.

We note that the representation of the wave function ψ by the superposition (3.21) is natural, but not unique. For example, instead of the functions ψ_1 and ψ_2 in (3.21) we can take their linear combinations

$$\psi'_1 = \frac{\psi_1 + \psi_2}{\sqrt{2}}, \quad \psi'_2 = \frac{\psi_2 - \psi_1}{\sqrt{2}} . \qquad (3.29)$$

The formula given by (3.21) then takes the form

$$\psi = c'_1\psi'_1 + c'_2\psi'_2 , \qquad (3.30)$$

where

$$c'_1 = (c_1 + c_2)/\sqrt{2} , \quad c'_2 = (c_2 - c_1)/\sqrt{2} . \qquad (3.31)$$

We have already noted that quantum-mechanical ran-

domness arises when we try to find something that does not exist. In the above example, randomness arises because we try to determine which particular slit was traversed by the electron whereas the state ψ in (3.21) describes the passage of an electron through *both* slits. On the other hand, if we look for something that does exit, we find there is no randomness. In particular, if we measure the momentum of the electron in state (3.12), we obtain a perfectly definite value for p. It can be shown that any superposition corresponds to a precise value of a particular physical quantity. This proposition will be illustrated by an example in Section 9.4. Randomness arises only when we measure a physical quantity that does not have a particular value in a given state.

3.6. Mixed states

To obtain a mixture of states, we place a Geiger counter in front of each of the slits, so that we can detect electrons from each slit separately. As noted above, the operation of measurement or detection is not as innocent in quantum mechanics as it is in classical physics. Thus, in classical physics, we can record an event without affecting it appreciably. In quantum mechanics, on the other hand, the situation is totally different because the process of measurement is accompanied by a significant change in the state of the micro-system. When an electron interacts with a Geiger counter, the unconditional probability described by the wave function ψ is replaced by the conditional probability. This is described mathematically by saying that the original wave function ψ no longer characterizes the state of the electron and is replaced is by two new wave functions ψ_1 and ψ_2 defined in the last Section. The probability that the state of an electron is described by ψ_1 is

$$p_1 = |c_1|^2. \tag{3.32}$$

and, similarly, the probability of the state ψ_2 is

$$p_2 = |c_2|^2. \tag{3.33}$$

The state of the electron is now no longer described by the single wave function, but requires two wave functions ψ_1 and ψ_2 and their probabilities p_1 and p_2. This type of state is called a mixed state (mixture of ψ_1 and ψ_2).

We note that in, contrast to the superposition of states, the decomposition of a wave function ψ into the two wave functions ψ_1 and ψ_2 in the case of a mixed stated is unique i.e., the basis functions ψ_1 and ψ_2 are the eigenfunctions of the operator $\hat{A}$ (see Section 9.4) that corresponds to the measured quantity a. For a mixed state, the law of composition of probabilties is

$$P(C_1 + C_2) = P(C_1) + P(C_2), \tag{3.34}$$

i.e., there is no interference.

We note that the concepts of *superposition* and *mixed state* are not specifically quantum mechanical. They are also encountered in classical theory. In Sec. 2.2, we considered the wave passing through two slits as a *superposition* of two waves. On the other hand, a stream of bullets crossing two slits is a *mixture* of the two currents emerging from the slits.

3.7. Reduction of a wave packet

When a single electron is incident on the two slits, it is recorded by only one Geiger counter. As noted above, we

cannot say in advance which slit will be crossed by the electron. All we can do is to specify the probabilities p_1 and p_2 corresponding to the passage of the electron through slit 1 and slit 2, respectively. When the observer recognizes that the counter in front of slit 1 has recorded an electron, he knows that there is no point in describing the state of the electron by a mixture. Using the new information, he then replaces the unconditional probability with the conditional probability. To describe this state of the electron, the observer therefore replaces the mixture of ψ_1 and ψ_2 with the single wave function ψ_1. This process is called a reduction of the wave packet.

We note that the reduction of the wave packet is not a physical process that occurs in space and requires a certain interval of time for its completion. Wave-packet reduction is a change in the method of description—a purely logical process. This is often used as a basis for the conclusion that quantum mechanics describes only the information we have about micro-objects and not the objective reality that is independent of our perception. This is incorrect. Quantum mechanics employs a probabilistic and not a determinate description. It does, however, provide a description of objective processes that occur independently of the observer. Wave-packet reduction is a transition from conditional probability, as in the case of coin tossing. This transition, i.e., the recognition of the results of measurement, is 'familiar even in classical theory' (Ref. 58, p. 50).

3.8. Objective and subjective components of the process of measurement

Quantum mechanics describes three different processes, namely: (1) the evolution of a micro-system on its own, in the absence of macroscopic instruments, (2) the interaction of a micro-object with a macroscopic instrument described by classical mechanics, and (3) improved description of the state of the micro-object once information about the result of its interaction with the macroscopic instrument becomes available.

3.8.1. State of microsystem described by the superposition of wavefunctions, and the evolution of a micro-system described by the Schrödinger equation

The operation whereby a wave function ψ is represented by the superposition of basis functions ψ_1, ψ_2, ... is a thought operation that does not correspond to any particular physical process. It is a mathematical device whose aim is to calculate the probabilities of different results of measurement.

Similarly, an arbitrary vector $\mathbf{r}$ on a plane can be represented by the superposition of two unit vectors $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{r} = c_1\mathbf{i} + c_2\mathbf{j}$$

as illustrated in Fig. 15. The same vector can be represented by the superposition of two other unit vectors $\mathbf{i}'$ and $\mathbf{j}'$, obtained by rotating the coordinate frame through, say 45°:

$$\mathbf{i}' = (\mathbf{i} + \mathbf{j})/\sqrt{2}, \quad \mathbf{j}' = (\mathbf{j} - \mathbf{i})/\sqrt{2}.$$

In the new coordinate frame

$$\mathbf{r} = c'_1\mathbf{i}' + c'_2\mathbf{j}',$$

where

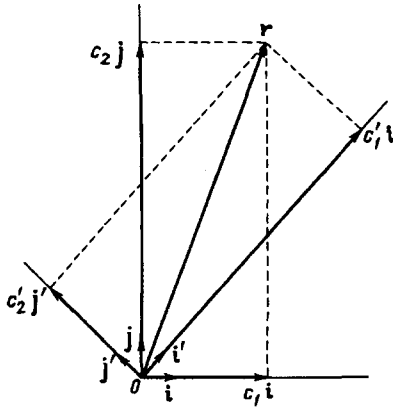$$c'_1 = (c_1 + c_2)/\sqrt{2}, \quad c'_2 = (c_2 - c_1)/\sqrt{2}.$$

FIG. 15. Transformation to other basic vectors.

The choice of any particular coordinate frame is a subjective operation. In principle, any coordinate frame can be employed, but the most appropriate frame is the one that takes into account the shape of the body under investigation. A successful choice of the basis function ensures the least laborious calculations and the simplest final formula. If we now return to the quantum-mechanical superposition described by (3.21), we note that the simplest expressions are obtained by taking the basis functions ψ_1, ψ_2, ... to be the eigenfunctions (see Sec. 9.4) of the operator A corresponding to the measured quantity a. We emphasize that the representation of a wave function by a superposition of eigenfunctions is still not a measurement, but a choice of an appropriate coordinate frame.

3.8.2. Interaction of a micro-object with a macroscopic device

"The first step in measurement is to subject the system to an external, physically real, perturbation that alters the course of events... . The perturbation produces a transition of the system to a 'mixture' of states" (Ref. 41, p. 50). The detection of the micro-object is accompanied by a catastrophe in the micro-world. Although in the course of this, the wave function also changes in a determinate way, this change is so tortuous that it is actually random. It seems to us that we are dealing here with one further manifestation of dynamic chaos[52] (see also Ref. 53 and 54). The question of measurement in quantum mechanics is also discussed in Refs. 21, 37, 55, and 56.

Thus, the transformation of a superposition into a mixture is an objective process that is independent of the observer, but depends on the measuring instrument.

3.8.3. Recording the result of measurement

This process is described by the reduction of a wave packet. The reduction is not objective: it is simply an improved description of the micro-object in the light of additional information. "The second act of measurement is to choose from an infinite number of states in the mixture a state that is completely determinate because it is actually realized. This second step is a process that does not itself influence the course of events, but simply alters our knowledge of the true relationships' (Ref. 41, p. 50). "Of course"—Heisenberg continues—"the introduction of an observer should not be interpreted incorrectly: we must avoid introducing subjective features into the description of nature. The observer merely performs the functions of recording devices, i.e, the recording of processes occurring in space and time, and it is totally irrelevant whether the observer is an instrument or a person. The process of recording, i.e., the transition from the possible to the real, is here an essential feature, and cannot be excluded from the interpretation of quantum theory' (Ref. 57, p. 36).

The relation between the objective and subjective aspects of the theory was very aptly described by Weizsäcker: "...nature existed before mankind, but mankind was there before science" (Ref. 57a, p. 26).

3.9. Example: the measurement of spin

The spin wave function is the simplest. We shall often use it to illustrate different aspects of quantum mechanics. A graphic, but very crude, measure of spin is the angular velocity of a particle around its axis. In reality, spin is not a vector, but an operator (see Section 4.1) that represents angular momentum (Ref. 43, p. 108).

In classical physics, the angular momentum 1 of a particle is defined as the produce of its moment of inertia I and its angular velocity $\vec{\omega}$:

$$1 = I\vec{\omega}.$$

The moment of inertia I is a constant typical of the given particle. Spin can therefore be imagined as angular velocity.

In quantum mechanics, spin is usually measured in units of Planck's constant $\hbar$. It is then a dimensionless quantity represented by the symbol S. In classical physics, the magnitude of angular velocity of a particle can assume any real value between zero and infinity. Accordingly, the component of the angular velocity vector along an arbitrary axis can assume any value between $-\infty$ and $+\infty$. In quantum mechanics, on the other hand, spin can assume only integral of half-integral values. Each particle (or micro-system in a particular state) can only have one value of spin. For example, the spin of the electron is $S = 1/2$. The helium atom has $S = 0$ in the singlet state and $S = 1$ in the triplet state. The spin of ^7Li is $S = 3/2$. The projection of spin onto a given axis, say, the z axis, can assume $2S + 1$ possible values (the values are numbered in descending order):

$$S_z^{(1)} = S, \quad S_z^{(2)} = (S-1), ..., S_z^{(2S+1)} = -S. \quad (3.35)$$

In particular, the z-component of the spin of the electron is either $1/2$ or $-1/2$.

The spatial state of a particle was discussed in some detail on the previous pages and was described by the wave function $\psi(r)$. The probability of finding a particle near a point r in an infinitesimal volume dr was taken to be

$$|\psi(r)|^2 dr. \quad (3.36)$$

The spin state is described in a somewhat different way. Thus, the spin state of an electron is characterized by the two quantities ψ_1 and ψ_2, written in the form of a column ψ, given by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (3.37)$$

This column can be treated as the wave function ψ whose argument is not the position vector r, but the spin index j that can assume two values, namely, 1 or 2.

The wave function ψ does not predict the value of S_z obtained by measurement. The above column merely gives the probability of different values of S_z, namely, the probability p_1 that measurement will yield $S_z = 1/2$ is

$$p_1 = |\psi_1|^2 , \qquad (3.38)$$

and the probability p_2 that measurement will yield $S_z = -1/2$ is

$$p_2 = |\psi_2|^2 . \qquad (3.39)$$

Similarly, the state of a particle with spin S is characterized by a column consisting of $2S + 1$ components:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \cdot \\ \psi_{2S+1} \end{pmatrix} . \qquad (3.40)$$

The probability p_1 that the measurement of S_z will yield S is

$$p_1 = |\psi_1|^2 ; \qquad (3.41)$$

and the probability p_2 that the result will be $S_z = S - 1$ is

$$p_2 = |\psi_2|^2 , \qquad (3.42)$$

and so on. Since the particle must be in one of the possible spin states defined by (3.35), the probabilities $|\psi_j|^2$ must satisfy the normalization condition

$$\sum_{j=1}^{2S+1} |\psi_j|^2 = 1 . \qquad (3.43)$$

4. THE 'PARADOXES' OF QUANTUM MECHANICS

There is no motion, said the bearded sage.
Whereupon another sage became quiet and started walking in front of him.
That was the strongest gesture of objection he could make.
Everyone praised the subtle response,
But this entertaining case brings to mind another:
The Sun rises every day and yet the stubborn Galileo was right.

(A.S. Pushkin, *Motion*)

4.1. Imaginary numbers and operators

Quantum mechanics is a logically consistent theory. This means that, strictly speaking, it does not involve any paradoxes. However, our intuition is formed in early childhood and is based on macroscopic experience founded on classical mechanics. When quantum mechanics is subsequently encountered, people subconsciously tend to replace quantum concepts with classical concepts, which results in apparent paradoxes. We shall discuss the best known of these. Some authors see a paradox in the fact that the basic equation of quantum mechanics, i.e., the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi .$$

contains the imaginary quantity i. However, the imaginary unit is not actually a symbol of anything really unusual. A complex number that consists of an imaginary and a real part is simply a compact way of writing two real equations (see. Sec. 7.3).

By writing two real equations in the form of a single complex Schrödinger equation we ensure not only that the result is compact, but also that the Schrödinger equation is linear whereas the two real equations (7.4) and (7.5) are nonlinear. The advantage of having a linear equation is that there is a well established mathematical formalism for solving such equations. This relationship between linear and nonlinear equations is used for the precise analytic solution of nonlinear problems. In particular, the real nonlinear equation is treated as a component of a quantum-mechanical particle scattering problem. The latter is a linear problem whose solution can be derived in an explicit form. The solution found in this way is then translated into the language of nonlinear problems, thus obtaining a solution of the original nonlinear equation.[59] The measurement of any physical quantity always produces a real number. In the above mathematical formalism, this is guaranteed by the fact that physical quantities are always represented by Hermitian operators (see Sec. 9.2) whose eigenvalues are always real.

We now turn to operators. "An essential feature of the new theory is that physical quantities or, in Dirac's terminology, *observables* (momentum, particle energy, field components, and so on) are represented not by variables, but by symbols with a noncommutative multiplication law or, to be specific, operators" (Ref. 36, p. 105). For example, the momentum p of a particle is expressed not by a number, but the differentiation operator

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x} \qquad (4.1)$$

where, for simplicity, we have confined our attention to the one-dimensional case. Quantum-mechanical operators are related in this approach by the same expressions as in classical mechanics. In particular, in the absence of an external field, the energy of a classical particle is given by

$$\mathscr{E} = p^2/2m . \qquad (4.2)$$

According to (4.1), energy must be represented in quantum mechanics by the operator

$$\hat{\mathscr{E}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} . \qquad (4.3)$$

It is important to note that, strictly speaking, even in classical mechanics, physical quantities are expressed not by numbers, but by operators. For example, when we say that a car is traveling with a speed of 60 km/h, the number 60 does not mean 'sixty pieces,' but signifies an operator producing an expansion by a factor of 60 when applied to the speed of 1 km/h. In precisely the same way, a temperature of 100° does not mean 100 temperatures of 1° (Ref. 58, p. 33). The difference between classical and quantum mechanics is that, in classical mechanics, we use only simple operators such as operators producing the expansion of physical quantities, whereas in quantum mechanics we use more complicated operators. Expansion operators always commute with one another, whereas quantum-mechanical operators often do not.

4.2. The particle identity paradox

The indistinguishability of waves leads in quantum mechanics to the indistinguishability, or more precisely, the *identity* of particles.[6] For example, all the electrons in an atom are absolutely identical. Even if at the initial time $t = 0$ we label all the electrons, we have no way of telling which is which at a subsequent time $t > 0$ because the concept of a particle trajectory is meaningless for an electron. For example, there is no change in any physical phenomenon when two electrons are interchanged. This property seems 'strange' in classical physics. For example, the planets in the solar system are all different which means, for example, that if we were to interchange the Earth and Mercury, we would soon notice a difference. When she buys a classical object, for example, a blouse, a lady always feels the quality of the material. However, when she buys a gold ring, she is interested only in its quantitative content, i.e., its weight and the number of carats. She's not normally interested in the quality of the gold. Gold atoms are quantum objects and are therefore identical; in contrast to a fabric, there is no such thing an 'inferior' gold.

The atoms of chemical elements are so small that they exhibit quantum-mechanical effects such as the indistinguishability of particles of the same kind. For atoms other than gold, this property is masked by the fact that different atoms enter different chemical compounds and thus become distinguishable. Gold, however, is a noble metal and does not readily enter chemical reactions, so that gold atoms cannot be distinguished from one another. The existence of the same, immutable gold is a quantum effect that cannot be explained in terms of classical physics. We have become accustomed to the fact that gold remains unaltered even after it has been exposed to a huge number of external factors. However, the property of identity is so 'strange' to us that even the founding fathers of quantum mechanics have been known to be wrong. For example, Dirac writes, '...the wave function gives information about the probability of *one* photon being in a particular place and not the probable number of photons in that place. The importance of the distinction can be made clear in the following way. Suppose we have a beam of light consisting of a large number of photons split up into two components of equal intensity. On the assumption that the intensity of a beam is connected with the probable number of photons in it, we should have half the total number of photons going into each component. If the two components are now made to interfere, we should require a photon in one component to be able to interfere with one in the other. Sometimes these two photons would have to annihilate and other times they would have to produce four photons. This would contradict the conservation of energy. The new theory, which connects wave function with probabilities for one photon, gets over the difficulty by making each photon go partly into each of the two components. Each photon then interferes only with itself. Interference between two different photons never occurs' (Ref. 61, p. 25). The fact is that two photons of the same frequency cannot be different. Two photons from different components of the original beam do not differ in any way from one photon that belongs 'partly to each of the two components.' The reference to the conservation of energy is also inconsistent because the violation of the law is only apparent and merely a manifestation of interference (see Sec. 2.2).

Recent experiments have shown that photons from two different lasers do interfere with each other.[62] In contrast to thermal sources of photons, an individual emission event in a laser takes a relatively long interval of time, so that interference between photons from statistically independent lasers *can* be observed.[63] It is irrelevant whether there are two photons from one source or from different sources. Because of the identity of photons, all sources throughout the universe must be looked upon as a single source. An observed photon can be related to a particular source only if the probability of arrival of a photon from all other sources at a given point is negligible.[62]

In the case of electrons, one of the manifestations of the principle of indistinguishability is Pauli's exclusion principle: no two electrons can be found in the same state. This principle is a revival at a higher level of the ancient principle of impenetrability of matter: two different bodies cannot occupy simultaneously the same position. The principle of impenetrability of matter has already been violated in classical physics: radio waves pass freely through a wall.

4.3. Schrödinger's cat

In one of his papers on quantum mechanics. Schrödinger produces an example of a paradoxical situation. Suppose that a chamber contains a speck of radium, a Geiger counter, a glass vial containing prussic acid, and a cat. The decay of a radium nucleus causes the emission of an alpha particle which crosses the Geiger counter. The counter produces a pulse which is used to initiate a mechanical device that breaks the vial and releases the prussic acid, which kills the cat. Since the decay of radium is random, we have a superposition of two quantum states, namely, the live cat and the dead cat, and the two states can interfere. This interference means that the cat does not occupy one particular state (dead or alive), but is half dead and half alive, which is absurd. The true situation is different. The discussion that we have just given does not take into account the fact that the operation of the counter is a catastrophe in the micro-world which converts a superposition into a mixture. The state of the cat is therefore described not by the single wave function

$$\psi_M = c_1\psi_1 + c_2\psi_2 ,$$

but by two wave functions, namely, ψ_1 with probability $|c_1|^2$ and ψ_2 with probability $|c_2|^2$, and interference between the two states is not possible. This situation is consistent with the classical theory of probability. For example, 48% of all people are men. This means that a person selected at random has a probability of 0.48 of being a man. This is not a paradox; we understand the result. However, the same fact can be formulated in a mystical,paradoxical form: "each person is 48% man and 52% woman."

4.4. The Einstein–Podolsksy–Rosen paradox

Einstein, Podolsky, and Rosen devised an example of a physical situation which, in their opinion, demonstrated the incompleteness of quantum mechanics. The incompleteness was understood in the sense that there were some hidden parameters which, when discovered, would show that quantum mechanics was in fact a determinate theory. Einstein, Podolsky, and Rosen maintained that the denial of the existence of such parameters leads to a paradox, i.e., a logical inconsistency. Einstein, Podolsky and Rosen considered the

measurement of the position and momentum of an electron. We shall discuss a simple modification of this thought experiment, due to Bohm, which involves the measurement not of the position and momentum, but of a component of the spin of the electron.

Consider two electrons with zero resultant spin. For example, this can be the atomic shell of an atom of helium in the singlet state. When a neutron knocks out the nucleus from the helium atom, the two electrons fly apart because of the Coulomb repulsion between them. The projection of the spin of one of them on an arbitrary axis, say, the x axis, is a random quantity equal to $1/2$ or $-1/2$. The projection of the spin is of the other electron on the same axis is also random and equal to $\pm 1/2$. Since momentum has to be conserved, the resultant spin of the two electrons must be zero. Let us now determine the projection of the spin of one of the electrons along the x axis. Suppose that the resultant of this measurement is $S_x^{(1)} = +1/2$. The projection of the spin of the other electron onto the x axis must then be $S_x^{(2)} = -1/2$. We thus see that the state of the second electron has changed instantaneously: if prior to the measurement on the first electron, $S_x^{(2)}$ could be $+1/2$ or $-1/2$ with equal probability, then after the measurement $S_x^{(2)} = -1/2$. However, the electrons can be at an arbitrarily large distance. For example, one electron could be in Paris and the other electron in Peking. This means that the measurement of the projection of the spin off the Paris electron could not possibly affect the Peking electron. This instantaneous reaction between electrons separated by an enormous distance is the Einstein–Podolsky–Rosen paradox. Einstein considered that the paradox could be regarded as evidence for the incompleteness of quantum mechanics.

In reality, the Einstein–Podolsky–Rosen paradox does not contain a logical inconsistency. Prior to measurement, the two electrons were not localized, and each of them was potentially both in Paris and in Peking.[64] Hence, during the measurement of the spin of the Paris electron there is an instantaneous change not in the state of the Peking electron, but in the probability of its state. Such an instantaneous change in probability is not specific to quantum mechanics: it is also encountered in classical physics (Ref. 17, p. 96). For example, consider two rooms, in one of which there is a princess and in the other a tiger. The two rooms are a great distance apart. A slave can, at his wish, open the doors of one of the two rooms whereupon he either marries the princess or is torn to pieces by the tiger. Thus, by opening the doors of one of the rooms we know immediately who is in the other. This does not involve a paradox, but it does involve a hidden parameter ξ. For example, $\xi = 1$ if a given room contains the princess and $\xi = 0$ if it contains the tiger. We shall see in Sec. 7 that hidden parameters are impossible in quantum mechanics. The Einstein–Podolsky–Rosen effect is therefore in conflict with common sense. This has given rise to doubts about the validity of quantum mechanics in the Einstein–Podolsky–Rosen situation. However, experiments[65-67] have revealed no evidence for a deviation from the predictions of quantum theory. Nevertheless, it seems appropriate to recall Mach's words: 'The history of science teaches us that experiments with a negative result must never be regarded as conclusive. Hooke did not succeed with his balance to demonstrate the effect of distance from the Earth on the weight of a body, but this presents no particular difficulty to the more

sensitive modern balances' (Ref. 5, p. 219).

A detailed discussion of the Einstein–Podolsky–Rosen paradox is given in Refs. 67 and 68.

4.5. Aharonov–Bohm paradox

We shall illustrate this paradox by an example. Consider a particle carrying an electric charge e and traveling in a region with constant potential φ. The total energy of the particle is

$$H = (p^2/2m) + e\varphi . \qquad (4.4)$$

We know that the potential φ has no direct physical meaning. The physical situation does not change when we add an arbitrary constant C to φ:

$$\varphi \to \varphi + C .$$

It is only the electric field

$$E_{el} = -\partial\varphi/\partial x \qquad (4.5)$$

that has a direct physical meaning (for the sake of simplicity, we confine our attention to the one-dimensional case).

In classical physics, the fact that the physical picture is independent of C means that C does not appear in Hamilton's equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} , \qquad (4.6)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -e\frac{\partial\varphi}{\partial x} . \qquad (4.7)$$

In quantum mechanics, the Schrödinger equation has the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + e\varphi\psi . \qquad (4.8)$$

The solution of this equation for a particle with momentum p is

$$\psi = A \exp\left\{\frac{i}{\hbar}\left[px - \left(\frac{p^2}{2m} + e\varphi\right)t\right]\right\} \qquad (4.9)$$

where A is a constant of integration.

In contrast to classical mechanics, the state ψ of a particle in quantum mechanics depends directly on the potential φ. This can be demonstrated experimentally by means of interference. For example, a particle beam can be divided into two parts by two slits in a screen. One part is sent through a region with potential φ and the other through a region with zero potential. When the two parts of the beam are recombined, and interference pattern is observed because a path difference has been introduced between them. The paradox is that we can experimentally detect[68a] the potential φ that contains the arbitrary term C. In reality, there is no paradox.[68b] We cannot say in quantum theory that one particle crosses the region of potential φ and the other the region with zero potential. Each electron can be present in both regions. Hence the interference pattern reflects not the potential φ itself, but the difference between φ and 0, which means that the arbitrary constant C is eliminated.

A more detailed discussion of the Aharonov + Bohm effect may be found in Ref. 64.

5. THE PRINCIPLE OF CAUSALITY

> *Napoleon*: "Why have I not been greeted with a gun salute?"
>
> *Fortress commandant*: "There are a dozen reasons. First, we have no gunpowder..."
>
> *Napoleon*: "That'll do."

5.1. Determinism

We shall understand the principle of causality as requiring the following physical properties:

1. Determinism: cause uniquely determines effect.

2. Materiality of cause: cause must be material.

3. Time asymmetry: cause always preceeds effect.

4. No action at a distance: cause has a direct effect only on objects in close proximity. Interactions propagate with finite velocity.

5. Matter is inexhaustible: the reason for any physical law is a more fundamental physical law.

Kant saw causality as determinism: "This rule, namely, the unique combination of cause and effect must... be assumed at the beginning if we wish to objectivize perception... Hence it necessarily follows that all science must imply the law of causality; that natural science exists only to the extent that the law of causality exits" (cited in Ref. 57a, p. 240). Determinism prevails in classical mechanics: '...The laws governing the external world were considered complete in the following sense: if the state of objects at a given time is known completely, then their state at some subsequent time is completely determined by the laws of nature. It is this that we have in mind when we speak of *causality* (Ref. 15, vol. 4, p. 317).

The situation is quite different in quantum mechanics: '...When an observation is made on an atomic system that is in a given state, in general the result will not be determinate, i.e., if the experiment is repeated several times under identical conditions several different results may be obtained' (Ref. 61, p. 30). For example, when an electron having a definite momentum passes through an aperture in a screen, it can reach any point on a photographic plate placed beyond the screen. Quantum mechanics provides us with only the probability of finding the electron at different points on the photographic plate. However, there is a fear of admitting the absence of causality in quantum mechanics because this is somehow identified with philosophical idealism. It is then said that the principle of causality is valid in quantum mechanics, but is probabilistic in character. In other words, the absence of causality is renamed quantum causality. In our view, there is little point in reversing the meaning of causality, especially since this word has a history stretching over millenia.

There is also a fear of parting company with the concept of 'determinism' because in philosophical idealism there is the concept of 'indeterminism.' For example, an editor's note added to Gliozzi's book (Ref. 71, p. 411) states: '... Instead of emphasizing the new form of determinism in quantum mechanics due to its probabilistic character, the author introduces the more fashionable *indeterminism*, which leads to some very confused philosophical conclusions.' This is why the absence of determinism from quantum mechanics is often renamed probabilistic determinism. Of course, certain elements of determinism remain even in quantum mechanics. The variation of the wave function

with time is determinate. However, the wave function gives only probabilistic predictions for the behavior of a micro-object. For example, when a large number experiments is performed, the fraction of electrons reaching a particular point on the photographic plate tends to a determinate quantity, i.e., a probability. However, when a single experiment is performed, the electron can reach different points on the plate. It is precisely in this sense that we speak of the absence of determinism in quantum mechanics. "...Despite the successful application of quantum mechanics to many practical problems, there are still serious doubts (and not only in philosophy!) about the final significance and self-consistency of the quantum mechanical formalism. These doubts are serious enough for some physicists to consider that, eventually, a new and intuitively more acceptable picture of the world will replace quantum theory which will come to be seen as a set of recipes capable of yielding the correct answer under the experimental conditions attainable in the twentieth century" (Ref. 72, p. 671).

Einstein considered quantum mechanics to be an incomplete and temporary theory because the basic laws of quantum mechanics included randomness. This is particularly surprising because Einstein himself introduced randomness into quantum mechanics in his paper on the quantum theory of radiation. 'The most important point in this paper by Einstein is the introduction of probability into the description of a micro-object. In addition to the probabilities of spontaneous and stimulated emission, it is necessary to assume a random direction of emission of a photon by the molecule, i.e., the direction of emission cannot be predicted' (Ref. 17, p. 30). '...My scientific instinct drives me against this type of departure from strict causality' (Ref. 15, vol. 4, p. 108).

We note that the elevation of determinism to the status of an absolute principle is in conflict with the principle of causality even in classical physics. It is referred to as Laplace determinism: 'All phenomenon, even those that because of their relative insignificance do not appear to depend on the major laws of nature, are in fact consequences of these laws, just as unavoidable as the periodicity of the Sun... . all phenomena are related to the past by the obvious principle whereby no phenomenon can arise without its generating cause... . We must therefore consider the present state of the universe as the effect of the preceeding state and as the cause of the next state... . A mind possessing the knowledge of all the forces existing in nature at a given time, and the relative motion of all its components, would, if it were powerful enough to subject these data to analysis, be able to combine in a single formula the motion of all the major bodies of the universe as well as the motion of the smallest atoms: Nothing would be beyond the reach of this formula, and neither the future or the past would escape it' (Ref. 73, p. 10–11).

The Laplace determinism is often looked upon as a triumph of materialism or, more precisely, a triumph of the principle of causality. Actually, it is a rejection of the principle of causality because the concept of a cause includes the possibility of the absence of a cause. If all causes are inevitable, they cease to be causes. The entire scenario of the world is then subject to predestination. We then have neither cause nor effect, but merely a rigid sequence of events, one after another. The analog of this in the cinema is the sequence of frames on film which do not cause one another, but merely

constitute a series of takes that are photographed strictly in accordance with the script, and are independent. The death of a hero can be photographed before his birth. The Laplace determinism was criticized by Engels: 'According to this view, nature is governed entirely by simple direct necessity. The fact that a given pod contains five peaks and not four or six, the fact that the tail of a given dog is five inches long and neither shorter nor longer..., the fact that a given clover has been pollinated by a bee and another has not (and that it was pollinated by a particular bee at a particular time), the fact that a particular dandelion seed carried by the wind eventually produced a plant but another did not, the fact that I was bitten by a flea at four o'clock in the morning and not at three or five o'clock, and that I was bitten on the arm and not on the leg—all this is the effect of an unbreakable chain of inevitable causes, such that even the gas cloud from which our solar system originally evolved was constructed in such a way that all these events occurred in a particular way and not in some other way; in science, it is almost irrelevant whether we call this...divine intervention or else...a necessity' (Ref. 74, p. 173).

We note that determinism has assumed the status of an all-pervading philosophical principle because of the phenomenal success of Newtonian mechanics. Many philosophers rejected the principle of determinism before Newton, and considered it not only false, but actually amoral because it could be used as a justification for practically anything, including crime.

5.2. Materiality of cause and action at a distance

Newtonian mechanics does not require the principle of materiality of cause and allows action at a distance. Actually, the law of universal gravitation states that any motion of a body is transmitted by vacuum and that it instantaneously affects other bodies however distant. In other words, the gravitational interaction propagates through vacuum and does so with infinite speed, which actually constitutes a violation of the principle of causality. This was well understood by Newton himself: "Tis unconceivable that inanimate brute matter should (without ye mediation of something else which is not material) operate upon & affect other matter without mutual contact... . That gravity should...act...at a distance through a vacuum without the mediation of anything else... is to me so great an absurdity that I believe no man who has in philosophical matters any competant faculty of thinking can ever fall into" (Ref. 75, p. 45).

It is difficult to understand how such a considerable force can be transmitted through a vacuum. If the Earth were to be held in its orbit not by the force of attraction to the Sun but by a steel cable, the diameter of the latter would have to be greater than the diameter of the Earth. It would appear that the conflict with the principle that there is no action at a distance can be avoided by saying that gravitation propagates with finite but very high velocity, which only *seems* to us to be infinite.

Newton maintains that his "...law must be regarded not as a final explanation, but as a rule deduced from experiment." (Ref. 15, vol. 3, p. 86). However, in reality, the infinite speed of propagation of interactions adopted in Newton's mechanics is not so much a generalization from observations as a philosophical principle. Thus, suppose that a body 1 is a point A and is at rest, whereas a body 2 travels
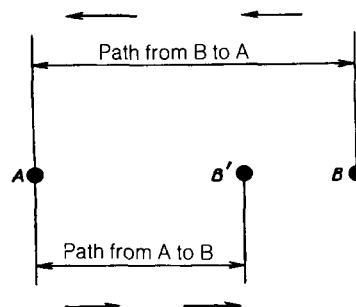
FIG. 16. Interaction of two bodies.

from a point B in the direction of A (Fig. 16). During the time that the gravity wave leaving the point B at the initial time takes to reach the stationary body 1, body 2 reaches the point B'. Since the distance AB is greater than AB', the gravity wave propagating from A to B' will arrive before the wave emitted at B arrives at A. This means that there will be an interval of time during which body 1 already acts on body 2, but body 2 does not act on 1. This is in conflict with Newton's third law which demands that the force with which body 1 acts on body 2 must be equal and opposite to that with which body 2 acts of body 1. On the other hand, it can be shown that the third law implies the conservation of momentum. Any violation of this law would be a violation of the principle that motion cannot be created or destroyed. Hence an infinite speed of propagation of interaction is, in Newtonian mechanics, a consequence of the philosophical principle of conservation of motion. This means that the speed of propagation of interaction in Newtonian mechanics must in principle be infinite. We note that "when we use the phrase 'in principle' we have in mind a particular theory and its principles that allow some things and forbid others" (Ref. 50, p. 140).

We recall that, in relativity theory, gravity propagates with finite speed, equal to the speed of light. The principle of causality is therefore preserved in relativistic mechanics in the sense that there is no interaction at a distance. However, we have just shown that the speed of propagation of interaction must in principle be infinite. Does this mean that our demonstration contains an error? This is indeed the case. The error lies in the fact that we have implicitly assumed that only particles can have momentum. In actual fact, the gravitational field also has momentum, so that the finite speed of propagation of gravitation that appears in relativity is not in conflict with the principle of conservation of momentum. It is therefore clear that the theory of relativity is in better agreement with the principle of causality than Newtonian mechanics.

5.3. Asymmetry of time

"It is obvious to everybody that the phenomena of the world are evidently irreversible. ...You drop a cup and it breaks, and you can sit there a long time waiting for the pieces to come together and jump back into your hand. ...The demonstration of this in lectures is usually made by having a section of moving picture in which you take a number of phenomena, and run the film backwards, and then wait for all the laughter. The laughter just means this would not happen in the real world. But actually that is a rather

weak way to put something which is as obvious and as deep as the difference between the past and the future;... we feel that we can do something to affect the future but none of us or very few of us believe there is anything we can do to affect the past... the most obvious interpretation of this evident distinction between past and future and this irreversibility of all phenomena would be that some laws, some of the motion laws of the atoms are going one way—that the atom laws are not such that they can go either way. There should be something in the works some kind of a principle that uxles only make wuxles, and never vice versa... but we have not found this principle yet. That is, in all the laws of physics we have found so far there does not seem to be any distinction between the past and the future.' (Ref. 30, p. 96–97).

The asymmetry of time does not appear in the equation of either classical or quantum mechanics. Indeed, Newton's equations of motion are unaffected by time reversal. On the other hand, time reversal makes cause and effect change places. Classical mechanics enables us to predict not only solar eclipses but also to determine the time and place of previous eclipses (for example, it is possible to deduce a more accurate date and place for the solar eclipse described in *The Lay of Prince Igor*). The requirement that cause must always precede effect is used as a boundary condition for the differential equations of classical mechanics.

Time asymmetry has the same status in quantum mechanics.[7] We note at this junction that the role of time asymmetry is quite different from the role of the above manifestations of the principle of causality. Determinism, materiality of cause, and action at a distance are either already incorporated in the postulates of the theory or are in conflict with the theory (we have shut our eyes to this difficulty). As far as time asymmetry is concerned, this must be formulated explicitly when the corresponding differential equations are solved. This is the reason why physicists usually understand the principle of causality as only time reversal: "we must also satisfy the principle of causality which demands that any event that has occurred in the system can influence the evolution of the system only in the future, but cannot effects its behavior in the past' (Ref. 76, p. 192). It may be said that, according to the laws of modern physics, the asymmetry of time is not a 'legal' principle but an actual law (Ref. 76a).

5.4. Inexhaustability of matter

We now turn to the principle of inexhaustability of matter which states that any physical law must have its own cause, i.e., a deeper law. All theories are based on postulates, i.e., propositions that cannot be explained by the theory. In this sense, all theories violate the principle of causality. Newton's contemporaries rejected his mechanics because it was based on 'strange' postulates that had no explanation and involved the causeless motion of an isolated body traveling with constant velocity, and the causeless mutual attraction of all bodies: "why bodies continue moving once they start, and what is the origin of the law of gravitation—all this was unknown" (Ref. 2, vol. 1, p. 40).

The principle of inexhaustability of matter means that the subdivision of physical theories into fundamental (microscopic) and phenomenological (macroscopic, i.e., consequences of fundamental) is temporary in character. A theory can be fundamental only at a given level of development of science. When a more fundamental theory subsequently appears, the earlier fundamental theory becomes phenomenological. However, the new fundamental theory always contains postulates that are its phenomenological elements. For example, Planck's constant is introduced phenomenologically into quantum mechanics, i.e., without any explanation. The main point is, however, that the basis of quantum mechanics—its probabilities character—is postulated; in other words, the random character of its laws has no cause (Ref. 77, p. 46–47). The postulates of quantum mechanics should have a cause: "we cannot be sure that the equal charges of electrons are the result of pure coincidence: this fact should be fundamental in the natural scheme and should have a cause" (Ref. 48, p. 278). "Our picture of physical reality"—wrote Einstein—"can never be final. We must always be ready to change it, i.e., to change the axiomatic basis of physics..." (Ref. 15, vol. 4, p. 136). This applies to all physical theories, including both the theory of relativity and quantum mechanics, which are often treated as absolutely true: "it was perfectly sensible for the classical physicists to go happily along and suppose that the concept of position—which obviously means something for a baseball—meant something also for an electron. ...Today we know that the law of relativity is supposed to be true at all energies, but someday somebody may come along and say how stupid we were. We do not know where we are 'stupid' until 'we outgrow ourselves' " (Ref. 2, vol. 3, p. 234).

Let us now consider an alternative point of view in which the number of laws of nature is finite. It is based on the analogy with geography. The middle ages produced great geographical discoveries: new continents, seas, and even oceans were found for the first time. However, by now, the surface of the Earth is fully explored, except for a few particularly inaccessible areas. A new continent will never be discovered again. The entire body of modern physics can now be reduced to four types of interaction, namely, electromagnetic, strong, weak, and gravitational. A unified field theory is being created at present and will combine all four interactions. Opponents of the principle of inexhaustibility of matter say that the development of the unified theory will spell the end of physical science. Physics will then develop, they say, exclusively by expanding its applications. Fundamentally new theories will no longer appear.

Opponents of the principle of inexhaustability of matter say that the world must be similar to a *matryoshka*—one of those multiple Russian dolls, nestled inside the other. However, this is not an argument but mere sophistry. One could just as well object to a spherical Earth on the grounds that it should not be similar to a watermelon. We are suggesting that matter is similar to a *matryoshka* not in the geometric, but in the causal sense. We are not saying that all particles consist of smaller particles and the latter in turn consist of still smaller particles, and so on *ad infinitum*. We merely say that any physical law is a consequence of some more fundamental law.

Berkeley rejected Newton's mechanics through the following questions: 'Can conclusions be scientific when principles are not evident? And can principles be evident if they cannot be understood?' (Ref. 78, p. 172). Another, this time contemporary, Bishop has expressed himself more bluntly: "Imagine that we are standing near a railway track and the windows of the tenth carriage pass in front of us. What is

responsible for the motion of the tenth carriage? The answer: the ninth carriage. And what moves the ninth carriage? Answer: eighth carriage... . We thus see that the materialist approach suggests that the train has no locomotive." Indeed, materialists maintain that there is no supernatural force (locomotive) that controls physical processes at different depths (carriages). Materialism thus leads to "an infinite number of carriages," i.e., an infinite chain of successively more fundamental theories. Returning now to Berkeley's argument, we note that principles can be understood only on the basis of more fundamental principles that in turn rely on still more fundamental principles and so on *ad infinitum*.

The absence of determinism from quantum mechanics is not a rejection of materialism because quantum mechanics is not a fundamental theory.[8] It seems to us that a more fundamental theory, in which determinism will prevail, will be created at some point in the future. We shall show in Secs. 7 and 8 that it is impossible to construct a determinate theory that leads to the same observational results as quantum mechanics. A deeper, more, more determinate theory should therefore be able to predict other experimental results. At the same time, for the parameter values accessible to modern science, the more fundamental theory should pass over to quantum mechanics. This more fundamental future theory will violate other philosophical principles because it will not be fundamental either. The more fundamental theory will be even 'stranger' than quantum mechanics because it will be even more remote from our everyday experience. Niels Bohr was once asked how he saw a hypothesis that was claiming to lead to a fundamental theory. He replied: 'It is not lunatic enough for the purpose.'

The theory of relativity has now invalidated Newton's action at a distance which is in conflict with the principle of causality. However, in Newton's time, attempts to reconcile action at a distance with causality were just as hopeless as any attempts today to reconcile quantum mechanics with determinism. Quantum mechanics is 'the only thing that provides a satisfactory logical explanation of the dual (corpuscular and wave) properties of matter.' (Ref. 15, vol. 3, p. 295–296). The predictions of quantum mechanics have been confirmed experimentally for a huge number of physical systems ranging from nuclear reactors to biological molecules (Ref. 72, p. 671).

We conclude this Section with a historical analogy. It was believed in antiquity that the Earth was flat and rested on three elephants. It was subsequently discovered that neither proposition was true: they relied on two falsities. The proposition that the Earth is flat, although false, remains relatively true. The radius of the Earth is much greater than the linear dimensions of the human body. This means that, provided we confine our attention to relatively short distances (say, no more than a hundred kilometers), we can neglect the curvature of the Earth's surface. However, the second proposition, i.e., that the Earth is supported by three elephants is absolutely false and contains not even a grain of truth. Where do these elephants come from? Has anyone seen a proboscis or some other indicator of an elephant? And why *three* elephants and not four or five? Three elephants were probably the first thing that came to someone's mind as a means of establishing a causality principle and of explaining why the Earth did not fall whereas all other bodies did.

The formalism of quantum mechanics is in a sense analogous to the proposition that the Earth is flat. If we try to guess the form of a more fundamental theory that would include quantum mechanics as a special case, we may well find the result to be as prophetic as the three elephants mentioned above!

6. METHODOLOGICAL QUESTIONS IN QUANTUM MECHANICS

'You have written that man derives from a species of monkeys such as marmosets, orangutans, and similar creatures. Please forgive an old man, but I cannot agree with you about this important point and, indeed, must ask a question in return. If man, the dominant figure in the world, the most intelligent of all mammals, has descended from a stupid and ignorant ape, then he should have a tail and an uncultivated voice. You have written and published in your erudite essay...that even the largest star, the Sun, has black spots upon it. This can not be, because it can never happen... for what is the purpose of such spots if there is no need for them?'

(A. P. Chekhov, *'A letter to a learned neighbor'*.)

6.1. Against the ignorant criticism of modern physical theories

This was the title of a paper by V. A. Fock published in March 1953 in *Voprosy Filosofii* (Problems in Philosophy). He wrote '...Some of our philosophers, who have not bothered to study physics, often display total ignorance and reduce their problem to general accusation that modern physics is prone to idealism' (Ref. 80, p. 169).

Here are just two examples. "...The question is: should we retain and develop the philosophical foundation of Marxism–Leninism, or abandon them, as suggested by Professor M. A. Markov and his supporters on the late editorial board of *Voprosy Filosofii*? ...According to M. A. Markov, we cannot consider the space-time properties or the velocity and energy of an individual electron as essentially macroscopic' (A. A. Maksimov, Ref. 81, pp. 222 and 223). "The idealism inherent in the ideas of Bohr, Heisenberg, Schroödinger and other bourgeois scientists was not merely an appendage that could readily be jettisoned, but something that was cleverly and penetratingly waved into the very fabric of theoretical structures" (L. I. Storchak, Ref. 82, p. 202).

Criticism of a generally accepted theory is not as yet a criminal act. A scientific mistake is not a transgression either. But is it not acceptable that the ideas put forward by the authors cited above should be put forward as "the only correct deductions" from Marxism, i.e., from dialectical materialism. Still less acceptable are the conclusions drawn by these authors. This type of criticism of quantum mechanics brings to mind the criticism of the Copernican system in the sixteenth century. The Jesuit Clavius wrote: "We are unsure which system is to be preferred—the Ptolemaic or the Copernican. Both are in agreement with observed phenomena. However, the principles of Copernicus contain too many absurd propositions" (Ref. 83). We note that Clavius had more reasons to reject the Copernicus system that the igno-

rant twentieth century critics of quantum mechanics. The 'natural' Ptolemaic system did indeed lead to the same observational results as the 'absurd' system of Copernicus, whereas classical physics is unable to explain quantum effects.

The postulates of quantum mechanics are not in conflict with dialectical materialism, but do contradict those of classical mechanics, raised to the status of dogma. The scientific revolution has removed the 'properties of matter that were thought to be absolute, invariable, primary... and which are now seen as relative and inherent only in certain states of matter... . The acceptance of these invariable elements, immutable essences of matter, and so on, is not materialism, but a metaphysical, i.e., antidialectical, materialism' (Ref. 84, p. 225). Ignorant critics of quantum mechanics did not, as they claimed, defend Marxism, but natural philosophy (Ref. 85, pp. 552–553), in which physics was expected to fit a philosophical scheme of things.

6.2. The micro-world and its macroscopic description

"Bohr's suggestion that the quantum-mechanical description of the properties of an atomic object must be combined with the classical description of the means of observation (i.e., the apparatus) has played an essential role... in the interpretation of quantum mechanics. In his papers devoted to fundamental questions in quantum mechanics, Bohr insisted that it was essential to consider the experiment as a whole, and to extend its description to include instrumental readings" (Ref. 11, p. 463). Thus, in his discussion of the Einstein–Podolsky–Rosen paradox, Bohr considered the passage of an electron through an aperture and wrote: "...We start by assuming that our screen with a slit cut in it, the second screen with several slits parallel to the first, and the photographic plate are initially rigidly coupled to a heavy base... . But we could have used a different apparatus, in which the first screen was not rigidly coupled to the rest of the apparatus' (Ref. 10, p. 183). If the screen is rigidly coupled to the base, the transverse coordinate of the electron passing through its slit is determined, but the transverse component of its momentum is not. On the other hand, if the screen is "not coupled to the rest of apparatus" the transverse momentum component of the electron crossing it is zero because its transverse coordinate is undetermined. In the first case we have an electron with a particular position coordinate, but unknown momentum, whereas in the second we have an electron with a known momentum, but unknown position.

"The recording of observations," wrote Bohr, "reduces in the final analysis to the creation of stable marks on measuring devices, e.g., spots on a photographic plate, produced by an incident photon or electron" (Ref. 110, p. 603). This feature of quantum mechanics is often made absolute. Quantum mechanics is then treated as a science limited to the study of the interaction between a micro-object and a macroscopic device, and no attempt is made to examine the micro-object itself. "The 'output' of any instrument always presents a microscopic phenomenon: the rotation of a pointer, the formation of droplets in the Wilson cloud chamber, the blacking of a photographic emulsion, and so on. ...It is therefore correct to say that quantum mechanics investigates the micro-world in so far as it relates to the macro-world. Mac-

roscopic (classical) instruments present us with reference systems in which the state of micro-systems is defined in quantum theory' (Ref. 50, p. 84).

An analogous point of view is encountered in mathematics. It is said that "we cannot understand infinity because our brain is finite." However, if we were to argue this way, we would conclude that were are unable to recognize sheep because we are not sheep ourselves.

The proposition that a micro-object must always be treated in relation to a particular measuring instrument is confined to philosophical publications which Bohr himself referred to as 'pseudorealistic' (Ref. 10, p. 414). Descriptions of real experiments that involve the micro-world often employ expressions such as 'a 100 MeV proton' or 'a hydrogen atom in the S-state.' The associated macroscopic apparatus is very rarely described, i.e., little reference is made to the way in which these states are produced and measured. An electron with momentum p is described by the wave function

$$\psi = \exp(ipr/\hbar),$$

and not in terms of the readings of macroscopic instruments. Indeed, this formula is valid for *any* method of observation.

We are forced, say the supporters of the absolute interpretation of the role of macroscopic instruments, to describe quantum-mechanical objects in the language of classical physics, which is the language of our instruments and also the language in which we think. This again is incorrect. Is not the language of means observation the language of ψ functions? It is the language of milliamperes, the number of counts, the blackening of photographic plates, and so on, whereas we think in the nonclassical language of quantum states, Pauli's principle, and so on. It is only in the above philosophical paper by Bohr (Ref. 10, p. 414) that he describes the 'first screen' and the method in which it is supported when the state of the electron is measured. The recording of experimental results in the form of instrumental readings or spots on a photographic plate is the function of a laboratory assistant and not the research scientist. The latter is more concerned with the interpretation of the experiment, i.e, with drawing certain conclusions about the properties of the microobject 'as such.' No journal will accept for publication a report confined to the description of 'spots on a photographic plate.' Potential authors are warned about this, for example, in the editorial note in Physical Review Letters. Of course, the readings of macroscopic instrument constitute the final stage of any experiment, but this does not mean that physics can be reduced to instrument readings. Indeed, every experimenter has not only a pair of eyes but a head too!

Similarly, algebraic derivations end in formulas that relate Latin letters such as a, b, c, and so on, but this doe sot mean that algebra is written in Latin. "The properties of atomic objects such as charge, mass, and spin, the form of the energy operator, and the law describing the interaction of particles with an external field are, on the one hand, completely objective and can be treated separately from the means of observation, and, on the other hand, they require new quantum-mechanical concepts for their formulation. This applies in particular to the formulation of the many-body problem" [V. A. Fock (Ref. 11, p. 463)].

Feynman was also against attaching absolute significance to macroscopic instruments: "It is not true that

science can be constructed by using only concepts that are directly related to experiment. Indeed, even quantum mechanics operates with both the amplitude of the wave function and the potential, as well as other mental constructs that cannot be measured directly" (Ref. 2, vol. 3, p. 233). "...Every observation," writes Heisenberg, "leads to a certain discontinuous change in the mathematical quantities characterizing the atomic process and, consequently, to an abrupt change in the physical phenomenon itself... . For heavy bodies, for example, planets revolving around the Sun, the pressure due to sunlight which is reflected from their surface and which is necessary for observation plays no part in this; however, for the very small particles of matter, each observation does affect their physical behavior because their mass is so small" (Ref. 85a, p. 27–28). Here we encounter two different entities, namely, 'observations" and 'physical process,' which is in conflict with the alleged absolute role of macroscopic description. These two different entities correspond to two different mathematical formalisms. Thus, the Schrödinger equation describes the evolution of micro-system as such, and without interaction with an observer or a macro-object. On the other hand, the reduction of the wave packet describes the process of measurement.

6.3. Does the wave function represent information in the observer's possession?

It is sometimes said that the wave function carries information about a micro-object. Of course, any record constitutes information. More than that, any additional information about a micro-object forces us to modify the wave function (Secs. 3.7 and 3.8). However, the supporters of the 'informational interpretation of the wave function' would make us believe something more: they maintain that it is meaningless to speak of the state of a micro-object and that we can only say something about information. The question is: information about what? If this is information about the micro-object then the micro-object must objectively exist because "...information is a reflection of the objective laws of nature as represented by modern science' (Ref. 87). On the other hand, if this is abstract information, then it is totally unrelated to quantum mechanics. Abstract information can describe practically anything, from the result of a football match to the evolution of the universe. 'Of course we can examine the observer's knowledge of physics (but not the physics itself), which is not our purpose here. For example, the observer's knowledge of a particular system may radically change when he is hit on the head and loses his memory, or when he receives new information. Subjectivists tend to ignore the former and emphasize the latter' (Ref. 88, p. 371–372).

The wave function is not only information: it has an objective meaning too. It describes the motion of micro-particles in an external field (Ref. 17, p. 129). A common objection to the objectivity of the wave function ψ is that it is not uniquely defined. The physically meaningful quantity is $|\psi|^2$ and not ψ itself. The same observational results are obtained when we multiply all wave functions by i or -1. However, this is not a real objection because most mathematical objects are not uniquely defined. For example, nothing changes if we write 1/2 instead of 3/6 or 5/10.

6.4. Quantum mechanics and philosophical principles

"The interpretation of Bohr's ideas in the spirit of positivism, performed by some of his successors, naturally gave rise to a reaction that resulted in the rejection of the new ideas in the name of materialism (de Broglie, Bohm, Vigier, and so on). The principal factor that led these scientists to reject the usual probabilistic interpretation of quantum mechanics is the erroneous belief that the probabilistic interpretation constitutes a rejection of the objectivity of the micro-world and its laws, i.e., a rejection off the basic proposition of materialism. The followers of de Broglie's school believe that it is only classical-type determinism that is compatible with materialism. They therefore describe their point of view as deterministic (Ref. 11, p. 464). Einstein did not accept quantum mechanics although he was himself responsible for supplying its foundation stone (the theory of the photoelectric effect): "Quantum mechanics is the last, highly successful creation of theoretical physics... . The quantities that appear in its laws do not claim to represent physical reality itself; they provide only the corresponding probabilities... . I am nevertheless inclined to think that physicists will not for long be limited to such an indirect description of reality" (Ref. 89, p. 243–247).

Einstein was unwilling to accept quantum mechanics because he believed that the fundamental laws of nature should be deterministic and not random. However, we noted in Sec. 5.4, that the existence of fundamental laws of nature would be in conflict with the principle of causality because a fundamental law is a law without cause. On the other hand, a fundamental law that can be explained ceases to be fundamental. We can illustrate this by the law governing the free fall of a body on the Earth. Aristotle explained this by saying that a body raised above ground tends to return to its natural position, i.e., to the Earth's surface. However, this does not explain anything because it is not clear why a body raised above ground is in an unnatural position whereas a body resting on the ground is in a natural state. Such pseudo-explanations were very popular among scholastics. They were ridiculed by Molière: in one of his plays, a scientist says: 'Opium makes you sleepy because it has a soporific effect.'

Actually, the free fall of bodies is a consequence of Newton's law of universal gravitational. It can be shown that, according to this law, not all bodies fall to the Earth: the Moon does not, nor do satellites. When its velocity is high enough (much higher than was possible in Aristotle's time), a body raised above ground may actually leave the Earth. This means that, having explained the law of free fall for all bodies, we have shown that the law is generally invalid because it turns out that not all bodies will undergo free fall. "Dialectical materialism insists on the approximate and relative character of all scientific propositions about the structure and properties of matter" (Ref. 90, p. 275). This means that no individual physical theory can be absolutely true. All such theories have a limited range of validity. "I suggest that, strictly speaking, and with the exception of mathematics, there are no inviolable principles" (Ref. 36).

All theories involve a degree of distortion of reality: "We cannot represent, express, measure, or image motion without interrupting the continuous, without simplifying, without approximating, separating, or kill in a live specimen" (Ref. 91, p. 233). Every physical theory is in conflict with some experiment (possibly one that has not yet been

performed). Moreover, a physical theory that disrupts existing connections contradicts certain philosophical principles.[92] This so because, "all principles known to us are mutually incompatible, so that some things have to be rejected" (Ref. 30, p. 147).

Quantum mechanics is in conflict with determinism: single events are not determined. Newtonian mechanics was in conflict with the asymmetry of time, the materiality of cause, and the contact interaction.

"...The test of all knowledge is experiment" (Ref. 2, vol. 1, p. 47). However, if every physical theory can contradict philosophical principles, is there some limitation on a physical theory that follows from the world picture? "The only property of matter acknowledged by philosophical materialism is that it must be an objective reality, i.e., it must exist outside our perception" (Ref. 90, p. 275). This principle, i.e., the principle of objectivity of matter, must be obeyed by all physical theories. The principle is violated by the proposition that the quantum-mechanical wave function does not have an objective meaning and expresses only the information we have about the state of a micro-object. In fact, the wave function determines probability, i.e, an objective measure of possibility.

6.5. A clear theory and a deep theory

In the beginning of the history of experimental observation, ...or any other kind of observation on scientific things, it is intuition, which is really based on simple experience with everyday objects, that suggests reasonable explanations for things. But as we try to widen and make more consistent or description of what we see, as it gets wider and wider and we see a greater range of phenomena, the explanations become what we call laws instead of simple explanations. One odd characteristic is that they often seem to become more and more unreasonable and more and more intuitively far from obvious... .

...There is no reason why we should expect things to be otherwise, because the things of everyday experience ...involve ...conditions that are special and represent in fact a limited experience with nature. It is a small section only of natural phenomena that one gets from direct experience. It is only through refined measurements and careful experimentation that we can have a wider vision. And then we see unexpected things: we see things that are far from what we would guess—far from what we could have imagined. Our imagination is stretched to the utmost, not, as in fiction, to imagine things that are not really there, but just to comprehend those things that are there" (Ref. 30, p. 115–116).

All explanations are based on some initial assumptions that are assumed to be correct, i.e., they are based on postulates. The conditions that such postulates must meet are radically different for laymen and for scientists. For a layman, a postulate must be obvious and readily visualized. Moreover, the layman is not bothered by the fact that different postulates are required to explain different effects. On the contrary, in science, we should be able to explain all known effects in terms of a small number of postulates. It is, however, practically impossible to ensure that the postulates are obvious or readily visualized. The further we are from our everyday experience, the stranger the postulates. "...the main object of physical science is not the provision of pictures, but is the formulation of laws governing phenomena and the application of these laws to the discovery of new phenomena. If a picture exists, so much the better; but whether a picture exists or not is a matter of only secondary importance" (Ref. 61, p. 26). In the seventeenth and eighteenth centuries, two theories were put forward to explain the motion of celestial bodies, namely the theory of Descartes and the theory of Newton (Ref. 8, pp. 93–95). Descartes considered that the necessary postulates had to be visualized and obvious. Similar demands were introduced at the beginning of the Renaissance: the starting point should be reason and not religious dogma.

The requirement that the postulates had to be obvious and readily visualized was a step forward as compared with blind faith. However, it had to be regarded as idealistic because thought and not experiment was regarded as primary. Moreover, the principal instrument of cognition was considered to be intuition. Descartes derived the properties of nature by reasoning. For example, nature's abhorrence of vacuum was justified as follows. Matter is extension, i.e., space. Vacuum is impossible because one cannot imagine a place in the universe in which there is no space with length, depth, and width. Descartes considered that matter was inert and passive. Motion arose only as a result of an impulse. The followers of Descartes rejected inertial motion and attraction at a distance because they considered them to be causeless. Descartes though that natural motion was not rectilinear (which could not be directly observed) but circular because it could be seen on the celestial sphere. According to him, all planets are brought into motion by vortices in the aether. Descartes confined himself to a qualitative explanation of the motion of celestial bodies and made no attempt to explain quantitative relationships, such as, for example, Kepler's laws of motion.

In contrast to Descartes, Newton considered that experiment was the exclusive source of our knowledge. Theory arises only as a generalization of individual uncoordinated facts. According to Newton, the rotation of a planet around the Sun is due to the laws of inertia and the force of universal gravition. This force acts instantaneously and always points in the radial direction, i.e., at right angles to the orbit, which roughly speaking, can be regarded as circular. Descartes' postulates seemed natural because they involved interaction by contact. Uniform motion required the constant application of a force pointing along the trajectory. On the other hand, Newton's law of inertia seemed strange because motion with constant velocity required no cause.

Next, Newton's second law made use of a previously unknown and strange idea, namely, that of acceleration, i.e., the derivative of velocity or, in other words, the second derivative of position. To formulate his second law, Newton had to create the differential calculus—a new branch of mathematics. Integral calculus also had to be created to solve the equations of motion that arose in this way. The differential calculus and the integral calculus seemed to Newton's contemporaries to be so complicated that they were combined under the respectful title of 'higher mathematics.' "Contemporary and subsequent criticism of Newtonian mechanics (including criticism by Huygens and Leibnitz) was largely concerned with this abstract construction, just as Maxwell's electrodynamics, Einstein's theory of relativity, and, especially, quantum mechanics were subse-

quently criticized for their high degree of abstraction, and were regarded as difficult to visualize. ...In his researches, Newton always employed his new mathematics, but when he presented his results he often used the old synthetic method of presentation in order to avoid placing technical complexities in the way of an appreciation of his results" (Ref. 96, pp. 55 and 61).

Many of Newton's contemporaries rejected instanteous attraction at a distance. "The advance of science," wrote Mach, "would undoubtedly be accelerated if we were to abandon the assumption of action at a distance because we have no true or even apparent explanation of it (Ref. 5, p. 245). The advantage of Newtonian mechanics was that it led directly to the three laws of Kepler. However, the force acting along the radius produced circular motion. This conclusion was obtained by long abstract calculations and was therefore unconvincing. The fact that all planets revolve in the same sense was obvious in Descartes theory, but required the application of an abstract theory in Newton's case. Newtonian mechanics seems to us understandable and natural because we first encounter it in our childhood, when we are prone to instinctive imitation and our critical faculties are not fully developed. Newton's contemporaries, on the other hand, gave his theory a hostile reception. Today, the remnant of the lively disputes between Newton's and Descartes' supporter is the unit of length, i.e., the meter which was defined as 1/400000000th part of the length of the Paris meridian. However, this unit of length was very inconvenient in practical applications and as soon replaced by the distance between two marks on a platinum-irridium rod used as standard. The length of the Paris meridian is then no longer equal to exactly 4×10^7 m. The question is: why was it not clear at the very outset that the length of the Paris meridian would be very inconvenient as a standard?

The answer is that it *was* clear. So why was the meter defined in this way? The answer is that Laplace, who was the chairman of the metric commission, wished to determine whether Newtonian or Cartesian mechanics was valid. According to Newton, the terrestial globe is slightly flattened at the poles by centripetal forces, whereas Descartes proposed that it is slightly compressed along the equator by the action of the aether. Two very closely expeditions, one to the equator and the other to the polar region, were proposed as a means of verifying who was right. To help these expeditions, Laplace proposed that the unnatural unit of length commonly used at the time, namely, the length of Charlemagne's foot, should be replaced with a natural standard, namely, the length of the circumference of the Earth. Expeditions were dispatched to Brazil and Finland and confirmed that the Earth was flatter at the poles, thus verifying the validity of Newton's theory.

However, while Newton tried to understand the motion of celestial bodies, assuming it as given, Descartes speculated how the universe evolved to its contemporary form and structure, having originally arisen in accordance with natural laws. Since Newtonian mechanics has now been replaced by the more rigorous relativistic theory, in which interactions propagate with finite speed and the universe is nonstationary, we can say that Descartes and not Newton was right. However, this is like saying that a nonworking clock shows the right time twice a day. Descartes' theory was obvious and readily visualized, but qualitative in character. It

contained no quantitative laws that could be used in an experimental verification. Descartes incorrectly guessed the laws of motion, so that his theory is fruitless. Newton's strange and abstract theory, on the other hand, provided a highway for the development of science, by which humanity has arrived at its present stage of civilization.

Our common sense has now reached a stage where newtonian mechanics seems obvious and is readily visualized. Quantum mechanics, on the other hand, is still natural only for well-prepared scientists.

7. THE IMPOSSIBILITY OF HIDDEN-PARAMETER MODELS

> ...When the mayor's clerk, report in hand, entered the mayor's office in the morning, he faced a curious sight: the mayoral body, dressed in uniform, was seated behind a desk, and on a pile of records of unpaid taxes there lay, like a foppish paperweight, the totally empty mayoral head... . The town's leading physician was summoned, and three questions were put to him: (1) could the mayoral head have separated from the mayoral trunk without any blood being spilled? (2) Could it possibly have happened that the mayor removed and emptied his head himself? And (3) could it be assumed that the mayoral head having been removed could be reinstated later by some as yet unknown process? The medic thought long and hard, and murmured something about a 'mayoral substance' that allegedly issued from the mayoral body....'
>
> (M.E. Saltykov-Shchedrin, *History of a Town*).

7.1. The problem of hidden parameters

Quantum mechanics rejects the determinism of Newtonian mechanics. Many physicists consider this unacceptable. Their point of view has been articulated by David Bohm: "The usual interpretation of quantum theory, which is internally closed, nevertheless includes the assumption that the most complete description of the state of an individual system is achieved by using the wave function that determines only the probable results of actual measurement processes. The only way of verifying the validity of this proposition is to try to find some other interpretation of quantum theory in terms of what are still 'hidden parameters,' but which in principle would determine exactly the behavior of the individual systems; measurements that are practicable at present constitute averages over these parameters" (Ref. 97, p. 34). In other words, supporters of the hypothesis of hidden parameters consider that nondeterministic quantum mechanics is only the visible part of the complete edifice. This visible part rests on an invisible foundation, i.e., some deeper deterministic theory, created in the spirit of classical physics. Supporters of hidden parameters assume that the situation in quantum mechanics is the same as in classical kinetic theory. For example, the blue color of the sky is a consequence of the scattering of sunlight by random pulsations in the density of air, which have a certain particular size. Random-

ness is then only apparent because the air molecules follow determined motion and randomness arises because we do not know the positions and velocities of the individual molecules.

Many hidden-parameter models have been proposed. To illustrate the situation, we shall consider three of them: the model of subquantal particles (Sec. 7.2), the model of the subquantal liquid (Sec. 7.3), and the model of the subquantal wave function (Sec. 7.4). These models explain only some of the quantum effects, and are in conflict with others. The assumptions on which hidden-parameter models rely are therefore inconsistent.

The founding father of quantum mechanics knew that reasonable hidden-parameter models were impossible in principle. However, a rigorous proof of this proposition was lacking, and the speculations remained unpublished. The first rigorous mathematical proof that hidden parameters could not be introduced without radical change to quantum mechanics was provided by von Neumann (Ref. 98, pp. 234–244). The proof is based on certain postulates, one of which is that the equations of the subquantum (i.e., more fundamental) theory should be linear in the same way the Schrödinger's equation is linear. Supporters of hidden parameters objected to this and noted that classical mechanics is nonlinear, so that the linearity postulate was inadmissible.

The first proof that it is impossible to construct a hidden-parameter model, without assuming the linearity of the equations of this model, was given by Bell (Ref. 64, pp. 66 and 69). A simple proof was given soon after by Kochen and Specker[100] (see Sec. 7.5). These proofs are based on the idea, that, in quantum mechanics, randomness combines with necessity in such a way that it is impossible to reduce randomness to a set of hidden parameters. A very short and simple proof of the impossibility of hidden-parameter models has been given by Turner,[101] but his proof relies on familiarity with quantum logic. These questions are discussed in Sec. 8 and a detailed discussion of the problem of hidden parameters is given in Ref. 65–67, 88, 102 & 103.

We note that Maxwell's electromagnetic field theory was initially regarded as unsatisfactory because it described the behavior of the abstract vectors E and H and not the motion of matter. "Many models were proposed to overcome this difficulty. They were based on the behavior of a fictitious continuous medium, called the 'aether,' which was capable of transmitting action from point to point. Unfortunately, calculations and experiments showed that the existence of the aether could not be proved for the electromagnetic field, and even a description of it could not be provided" (Ref. 66, p. 147).

7.2. Model of subquantal particles

The decay of the radium-226 nucleus is accompanied by the emission of α-particles with an energy of $\mathscr{E} = 4.8$ MeV.[9] We know that the α-particle is attracted to the nucleus by nuclear forces and is repelled by the electrostatic interaction (the α-particle and the nucleus are both positively charged). Nuclear forces are much stronger than electrical forces, but their range is much shorter, i.e., of the order of 10^{-13} cm. For distances $r \lesssim 10^{-13}$ cm, the α-particle is attracted to the nucleus, whereas for $r \gg 10^{-13}$ cm it is repelled by it. The potential energy of the α-particle as a function of
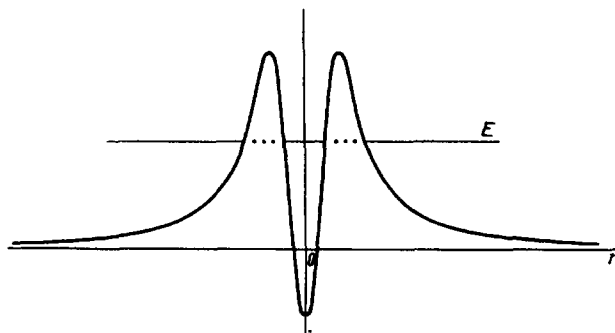

FIG. 17. Potential energy of an α-particle in the field of a nucleus.

the distance r from the nucleus is therefore of the form shown in Fig. 17. The height of the potential barrier is of the order of 30 MeV. According to classical mechanics, a 4.8—MeV particle cannot cross a potential barrier of this height, since the kinetic energy inside the barrier would then be negative, which is impossible. The fact that the α-particle nevertheless does cross the potential barrier, and the quantum-mechanical randomness of the process, could be explained classically by assuming that the α-particle collided with some as yet unknown small subquantal particles ('zerons'[104]) or vacuum fluctuations.[105–107] This was suggested by the fact that equation (7.5), which is a consequence of Schrödinger's equation, has the same structure as the diffusion equation,[108,109] namely, it contains the derivative with respect to time and the second derivative with respect to the coordinates. However, if this were so, the α-particles crossing the potential barrier would have random energies, ranging from zero to the height of the barrier, whereas all the α-particles leaving the nucleus are known to have an energy of exactly 4.8 MeV. The subquantal particle model is therefore inconsistent.

7.3. The subquantal liquid model

Let us now separate the complex Schrödinger equation into two real equations. This can be done by writing the wave function ψ in the form of

$$\psi = Re^{iS/\hbar}, \tag{7.1}$$

where R and S are real. Denoting

$$\rho = R^2, \tag{7.2}$$

and

$$v = \frac{1}{m}\nabla S, \tag{7.3}$$

we obtain the real continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\rho v = 0 \tag{7.4}$$

and the real equation of motion[97,110]

$$m\frac{\partial v}{\partial t} + m(v\nabla)v = -\nabla V + \frac{\hbar^2}{2m}\nabla\left(\frac{\Delta\rho^{1/2}}{\rho^{1/2}}\right). \tag{7.5}$$

Equation (7.4) may be looked upon as the equation describing the continuity of a (subquantal) liquid, whereas (7.5) is the equation of motion of a particle that experiences both the classical potential V and the further 'quantum potential'[97,110]

$$-\frac{\hbar^2}{2m}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}.\tag{7.6}$$

The 'quantum potential' keeps the electron on a quantized orbit around the nucleus and not at an arbitrary distance, which would be the case if there were an analogy with the planets in the solar system. The velocity $\mathbf{v}$ of a particle in the subquantal liquid model is interpreted as a hidden parameter, and it is assumed that, after measurement, the particle momentum $\mathbf{p}$ is different from the "true momentum" $m\mathbf{v}$—the value prior to measurement. On the other hand, the measured position is the same as the true position. Bohm has examined a number of simple measurement processes that could lead to agreement with the predictions of quantum theory. However, the paradox of quantum mechanics lies not so much in the fact that it predicts some special effects that cannot be explained in terms of classical theory as in the existence of classical effects that are in conflict with one another. In particular, the laws of quantum mechanics are invariant under the transformation of the variables.

The subquantal liquid model does not meet this requirement and cannot therefore explain the results of more complicated experiments. We shall not pause to examine these experiments because we shall show in Sec. 7.5 that hidden parameters cannot, in principle, be introduced into quantum mechanics.

7.4. Subquantal wave function model

In the model proposed by Wiener and Siegel,[111] the state of a micro-system is described by two wave functions, namely, the usual quantum-mechanical wave function ψ and the 'hidden' wave function ξ. The latter is introduced to ensure that we can accurately predict which of the eigenvalues of the observed variable is obtained by measurement. We shall illustrate the Wiener–Siegel model by considering measurement of the projection of electron spin. Wiener and Siegel assumed that, in addition to the explicit electron wave vector

$$\psi = \begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}\tag{7.7}$$

there was a further 'hidden' vector

$$\xi = \begin{pmatrix}\xi_1\\\xi_2\end{pmatrix},\tag{7.8}$$

which predetermines the result of any measurement of S_z. In particular, when

$$\frac{|\psi_1|}{|\xi_1|} > \frac{|\psi_2|}{|\xi_2|},\tag{7.9}$$

we have $S_z = 1/2$ whereas if

$$\frac{|\psi_1|}{|\xi_1|} < \frac{|\psi_2|}{|\xi_2|},\tag{7.10}$$

we have $S_z = -1/2$. In contrast to ψ, the vector ξ is not normalized: $|\xi_1|^2 + |\xi_2|^2$ is not necessarily equal to unity. To ensure that the hidden-parameter model provides the foundation for quantum mechanics, it must lead to the quantum-mechanical postulates. In particular, the Wiener–Siegel model must lead to (3.38) and (3.39). It can be shown that these probabilities are obtained if it is assumed that the 'hid-

den' vector ξ is random and that the quantities $|\xi_1|$ and $|\xi_2|$ are independent and lie between zero and infinity with the distribution

$$f(|\xi_j|) = |\xi_j|\exp(-|\xi_j|^2/2)\quad(j = 1,2).\tag{7.11}$$

This model can be directly generalized to the case of arbitrary spin.

The Wiener–Siegel model explains the single measurement of spin projection, but cannot explain more complicated sets of several measurements (see Sec. 7.5).

7.5. The Kochen–Specker proof

We have already noted that quantum mechanics is nonclassical not so much because it involves randomness as because the randomness combines in a strange way with necessity. The random results of simple measurements could be explained by hidden parameters, but the random result of more complicated measurement are subject to essential constraints (correlations) which, as we shall show, exclude the possibility of hidden parameters.

To prove that hidden parameters cannot be introduced into quantum mechanics, we need only find one example that cannot be explained by the existence of such parameters. We shall take this example to be the measurement of the square of the spin projection for a particle with unit spin

$$S = 1.\tag{7.12}$$

It follows from Sec. 3.9 that the square of the spin projection on an arbitrary axis l can assume two values, namely,

$$S_l^2 = 0 \text{ or } S_l^2 = 1.\tag{7.13}$$

Suppose that the l, m, n, ..., axes emerge from the same point O, and let us construct sphere of arbitrary radius centered on O. The sphere cuts the l, m, n, ..., axes at points L, M, N, ... We now lay out the sphere on a plane that coincides with the plane of the drawing. Each point L, M, N on the plane is then associated with its own directions of the l, m, n axes leaving the point O. Whenever the directions of l and m are orthogonal (i.e., perpendicular), the corresponding points and M are joined by a line. On the other hand, whenever the two directions are not orthogonal, the corresponding points are not joined by a line. For example, in Fig. 18, we show the eight directions A, B, C, D, E, F, G, H. Directions A and B are orthogonal, whereas A and D are not. The question is: what is the difference between the orthogonality and nonorthogonality of axes? We shall show in Sec. 9.5 that, when l and m are orthogonal, the values of S_l^2 and S_m^2 are compatible. This means that, when

$$\mathbf{l}\cdot\mathbf{m} = 0\tag{7.14}$$

there is a quantum state in which the quantities S_l^2 and S_m^2 are simultaneously determined. When the three axes l, m, n are orthogonal in pairs, then according to Sec. 9.5, the values of S_l^2, S_m^2, S_n^2 are compatible and two of them are equal to unity where the third is zero. In other words, for the three directions that are orthogonal in pairs, the only possibility is the combination of zeros and units illustrated in Fig. 19. Hence it follows that, for two mutually orthogonal directions, the only possible combinations of zeros and units are those shown in Fig. 20, whereas the combination shown in Fig. 21 is not possible. Consequently, the two points marked
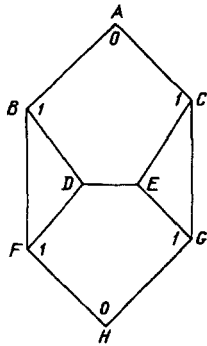
FIG. 18. Hidden-parameter models cannot be constructed. Orthogonal directions correspond to points joined directly by straight lines.

0 in Fig. 18 cannot be joined by a line, i.e., the corresponding directions be orthogonal.

Now consider the case where the directions of l and m are not orthogonal:

$$l \cdot m \neq 0. \tag{7.15}$$

We show in Sec. 9.5 that, when this is so, the values of S_l^2 and S_m^2 are incompatible. This means that, for any particular value of S_l^2, the quantity S_m^2 is always random and can assume two values, 0 or 1. For example, when

$$S_a^2 = 0$$

(see Fig. 18), S_h^2 can be 0 or 1. If hidden parameters were to exist, S_h^2 would be equal to 0 for some and to 1 for other such parameters.

We shall now show that the experimental situation illustrated in Fig. 18 is inconsistent with any hidden-parameter model. We can prove this by *reductio ad absurdum*. If hidden parameters were to exist, then quantities S_a^2 and S_h^2 would be simultaneously equal to 0 for any such parameters. It is clear from Fig. 20 that $S_a^2 = 0$ leads to

$$S_b^2 = 1.$$

Similarly

$$S_c^2 = 1.$$

In precisely the same way,

$$S_h^2 = 0$$

leads to

$$S_f^2 = 1, \quad S_g^2 = 1.$$

It follows from the triangle *BFD* of Fig. 19 that

$$S_d^2 = 0.$$
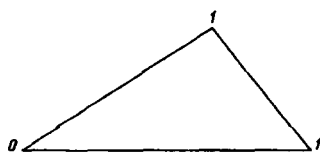
Similarly, it follows from the triangle *CEG* that

FIG. 19. A possible combination of S_l^2, S_m^2, S_n^2 for three orthogonal directions.

FIG. 20. Possible combinations of S_l^2 and S_m^2 for two orthogonal directions.

$$S_e^2 = 0.$$

We thus have obtained zero values for the square of the projection of the spin for the two mutually orthogonal directions **d** and **e**, i.e., we have the impossible situation illustrated in Fig. 21. Hence the assumption that hidden parameters exist leads to a contradiction. In other words, hidden parameters cannot be introduced into quantum mechanics.

It is tacitly assumed in the above proof that the configuration shown in Fig. 18 is realistic. This follows from the following example:

$$a = i - j + k, \quad b = j + k, \quad c = i + j, \quad d = i,$$
$$e = k, \quad f = j - k, \quad g = i - j, \quad h = i + j + k, \tag{7.16}$$

where **i**, **j**, **k** are three mutually orthogonal unit vectors. The above proof of the impossibility of hidden parameters in quantum mechanics is due to Kochen and Specker,[100] modified in Ref. 65.

7.6. Negative probabilities

In the proof given in the last Section, we confined our attention to 'reasonable' models. However, if we now turn to 'strange' models, we find that hidden parameters are possible. A 'strange' model of this kind was proposed by Wigner[112] (see also Refs. 113–121) who introduced the following compatible probability density for position coordinate x and momentum p:

$$f(x,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^*(x + \frac{h\tau}{2}) e^{ip\tau} \psi(x - \frac{h\tau}{2}) d\tau, \tag{7.17}$$

where $\psi(x)$ is the wave function and $h = 2\pi\hbar$. This function $f(x, p)$ can be used to obtain the probability $dw_x(x)$ of finding a particle in the range $(x, x + dx)$, which is given by the following expression:

$$dw_x(x) = dx \int f(x, p) dp. \tag{7.18}$$

Similarly, the probability that the particle momentum lies in the range $(p, p + dp)$ is given by

$$dw_p(p) = dp \int f(x, p) dx. \tag{7.19}$$

However, the expression given by (7.17) does not actually signify that the particle has simultaneously determined position and momentum because the function $f(x, p)$ can assume negative values, which is inadmissible (see Sec. 3.1). It is important to note that the utility of the Wigner distribution (7.17) lies not only in introducing hidden parameters
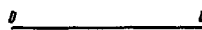

FIG. 21. Impossible combinations of S_l^2 and S_m^2 for two orthogonal directions.

into quantum mechanics, but also in that it is convenient for the evaluation of different quantum effects.[122]

At this point, it is appropriate to recall a simple fact of life: a working man is rarely a conman. Our usual understanding of probability as a nonnegative quantity may mean that our interpretation of the concept of probability is too narrow.[123,124] Here is a historical analogy. In the sixteenth century, Cardano derived the formula

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \quad (7.20)$$

for the solution of the cubic

$$x^3 + px + q = 0. \quad (7.21)$$

This solution often involves a negative expression under the square root, i.e., it is a complex number, which was regarded as inadmissible at the time. However, if we formally treat imaginary numbers in the same way as real numbers, the imaginary parts eventually disappear and we obtain real roots. Thus, for

$$x^3 - 3x + \sqrt{2} = 0$$

we have

$$x = \sqrt[3]{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}} + \sqrt[3]{-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}.$$

If we write this expression in trigonometric form, i.e.,

$$x = \sqrt[3]{\cos 135° + i\sin 135°} + \sqrt[3]{\cos 135° - i\sin 135°}$$

and use the de Moivre formula

$$(\cos \alpha + i\sin \alpha)^n = \cos n\alpha + i\sin n\alpha$$

with $n = 1/3$, we obtain

$$x = 2\cos 45° = \sqrt{2}.$$

8. DIFFERENCE BETWEEN LOGICAL STRUCTURES IN CLASSICAL PHYSICS AND QUANTUM MECHANICS

A mathematician could not find his glasses. After a long and fruitless search he called logic to his aid: "I had spectacles, i.e., I had poor vision. But since I can see that they are nowhere to be found, this must mean that I am wearing them!" And then by touching the bridge of his noise he verified that he had not taken his glasses off.

8.1. Classical logic

The above proof that hidden-parameter models are impossible, i.e., that quantum mechanics cannot be reduced to classical physics, has all the appearances of artificiality. There is a very simple and natural proof[101] that is based on the incompatiblity of the logical structures of quantum and classical physics. However, to understand this proof we have to be familiar with classical and quantum logic. 'Classical logic' and 'quantum logic' are generally accepted, but some-

what infelicitous, phrases. Logic is the science of general laws of thinking. The laws of thinking in quantum mechanics are no different from the laws of thinking in classical physics, in the same way that high-temperature plasma logic is not different from low-temperature plasma logic. *Quantum logic* is the phrase usually applied to mathematical logic augmented by the postulate of superposition (see Sec. 3.5). In contrast, mathematical logic without any additional postulates is called classical logic. In other words, classical logic is simply the calculus of propositions.

The word 'calculus' means that logical operations are denoted by mathematical symbols of composition and multiplication, and the operations upon them constitute a special algebra. To each proposition A there corresponds a certain set Ω_A of points in phase space in which the proposition is true. The set Ω_A is called the support of the proposition A. The example, the support Ω_A of the proposition $x^2 + p^2 < 1$ is a circle of unit radius on the x, p plane with center at the origin. If we have two propositions A and B, and proposition C is that at least one of the two propositions A and B is valid, we say that proposition C is the sum of A and B, and we write this in the form of the equation

$$C = A + B. \quad (8.1)$$

If proposition C is that both A and B are true, then we say that proposition C is the product of these two propositions and we write this in the form of the equation

$$C = A \cdot B. \quad (8.2)$$

Colloquially, logical composition corresponds to the union 'or' and logical multiplication corresponds to the union 'and.' The operations of composition and multiplication constitute, in classical logic, the set-theoretical composition and multiplication of the supports of the corresponding propositions:

$$\Omega_{A+B} = \Omega_A + \Omega_B, \quad \Omega_{AB} = \Omega_A \Omega_B. \quad (8.3)$$

Between *certain* pairs of propositions $A, B, C, ...$ we can establish a cause and effect relation

$$A \to B, \quad (8.4)$$

which means that if prediction A is true then prediction B is also true. In other words, prediction B is a consequence of A. The relation $A \to B$ means that the support Ω_A is a subset of the support Ω_B of proposition B:

$$\Omega_A \subset \Omega_B. \quad (8.5)$$

For example, if proposition B is $p > 0$ and is a consequence of proposition C which states that $p > 1$, we have

$$C \to B. \quad (8.6)$$

Figure 22 shows the domain Ω_B by the oblique shading whereas the domain Ω_C is shown by the cross hatching. It is clear that

FIG. 22. Cause-and-effect relation in classical physics.

$$\Omega_C \subset \Omega_B. \tag{8.7}$$

We note that the cause-and-effect relation does not apply to every pair of propositions. For example, it cannot be valid for $p > 0$ and $p < 0$ because neither is a consequence of the other.

The above notation provides us with a compact way of writing down complicated logical structures. For example, consider the proposition "if anyone is late for prayers or if there is any suggestion that the cadets have taken part in a prank, or if a lady teacher has been seen out late at night with an officer, then he would become very agitated, and would keep on repeating that he hoped that nothing untoward would come of it." This can be written in the compact form

$$A + B + C \rightarrow D \cdot E, \tag{8.8}$$

where A represents 'anyone is late for prayers,' B stands for 'there is any suggestion that the cadets have taken part in a prank,' C is "a lady teacher has been seen out late at night with an officer," D is 'he would become very agitated,' and E stands for "would keep on repeating that he hoped that nothing toward would come of it."

The cause-and-effect relation clearly involves the operations of composition and multiplication of propositions

$$A \rightarrow A + B, \quad A \cdot B \rightarrow A, \tag{8.9}$$

called laws of implication, and also the laws

$$\text{if } A \rightarrow B, \text{ then } A + B = B \text{ and } A \cdot B = A, \tag{8.10}$$

called the laws of absorption.

We shall illustrate these properties of propositions by an example. Suppose that μ is the magnetic moment of an atom. Proposition A is that this magnetic moment points along the x axis, whereas proposition B is that it points along the y axis; proposition C states that it lies in the x, y plane. The supports of propositions A, B, C are: Ω_A is the x axis, Ω_B is the y axis, and Ω_C is the x, y plane. Next, proposition $A + B$ is that the vector μ lies either along the x axis or along the y axis. The support of this proposition Ω_{A+B} is the set of two straight lines x and y. Proposition $A \cdot B$ is that the vector μ points along the x axis *and* along the y axis, which is impossible. This is an absurd proposition and does not therefore have any support. In mathematical language, the support of this proposition is an empty set. Obviously

$$\Omega_A \subset \Omega_C. \tag{8.11}$$

so that

$$A \rightarrow C. \tag{8.12}$$

8.2. Quantum logic

The structure of phase space in quantum mechanics is quite different from that in classical physics. In classical physics, the support Ω_A of proposition A can be any region of phase space. On the other hand, in quantum mechanics, because of the superposition principle, the state of a system described by a wave function ψ is also described by the wave function

$$\Psi = C\psi \quad (C = \text{const}). \tag{8.13}$$

We are assuming in this Section that the wave functions are not normalized. Proposition (8.13) signifies that the support of proposition A, namely, "the state of the system is described by wave function ψ" is not a point in the phase space of ψ, but the straight line L_A described by (8.13), where the constant C is arbitrary. Next, if the system is in both states ψ_1 and ψ_2, it can also be in any state

$$\Psi = c_1\psi_1 + c_2\psi_2. \tag{8.14}$$

In other words, the support of this proposition is the plane (8.14) spanning the vectors ψ_1 and ψ_2.

In quantum logic, as in classical logic (see Sec. 8.1), we can construct a calculus of propositions based on the operations of composition, multiplication, and the cause-and-effect relation.[127-129] The operation of multiplication and the cause-and-effect relation induce the same relations between the supports of propositions, just as in classical logic:

$$L_{AB} = L_A \cdot L_B; \tag{8.15}$$

$$\text{if } A \rightarrow B, \text{ then } L_A \subset L_B. \tag{8.16}$$

As far as the operation of composition is concerned, this corresponds not to the set-theoretic sum $L_A + L_B$ of the supports of the individual terms L_A and L_B, but the set of all the possible sums of vectors $x + y$, where $x \in L_A$ and $y \in L_B$. This set of vector sums is called the direct sum of supports L_A and L_B, and is written as follows:

$$L_{A+B} = L_A \oplus L_B \neq L_A + L_B. \tag{8.17}$$

Suppose, for example, that proposition A is that the vector μ points along the x axis and proposition B is that it points along the y axis, whereas proposition C is that it lies in the x, y plane. Proposition $A + B$ is then that the vector μ has the form

$$\vec{\mu} = c_1\vec{\mu_1} + c_2\vec{\mu_2}, \tag{8.18}$$

where μ_1 and μ_2 point along the x and y axes, respectively, and c_1 and c_2 are constants. In other words, proposition $A + B = C$ is that the vector μ lies in the x, y plane. (We recall that, in classical logic, proposition $A + B$ is that the vector μ points either along the x or along the y axis.)

8.3. Impossibility of introducing hidden parameters

We shall now show that it is impossible to introduce hidden parameters into quantum mechanics.[101] The proof is based on the fact that the cause-and-effect relation $A \rightarrow B$ may be violated as we pass from quantum logic to classical logic. This is not possible if we demand that some new classical theory should be the foundation of quantum mechanics. We shall now prove the impossibility of hidden parameters by *reductio ad absurdum*. Let us suppose that quantum mechanics has some classical foundation. For any proposition A in quantum theory there will then be several propositions $A(\xi)$ in classical theory, where ξ is the value of some hidden parameter that uniquely determines the results of arbitrary measurements. In quantum phase space L, proposition A corresponds to a set of vectors (L_A is the support of the proposition). In classical phase space Ω, a set of vectors corresponds to the same proposition [$\Omega_{A(\xi)}$ is the support of proposition $A(\xi)$]. The hidden parameter ξ then runs through all values compatible with the quantum proposition A.

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The cause-and-effect relation $A \to B$ means in quantum theory that the support L_A of proposition A is the subset of the support L_B of proposition B:

$$L_A \subset L_B. \tag{8.19}$$

Similarly, in the proposed classical theory with hidden parameter ξ, which is the foundation of quantum mechanics, the cause-and-effect relation $A(\xi) \to B(\xi)$ means that the support $\Omega_{A(\xi)}$ of the proposition $A(\xi)$ is a subset of the support $\Omega_{B(\xi)}$ of proposition $B(\xi)$:

$$\Omega_{A(\xi)} \subset \Omega_{B(\xi)}. \tag{8.20}$$

The cause-and-effect relation $A \to B$ then means that (8.19) and (8.20) are equivalent.

To verify the hidden-parameter hypothesis, it is sufficient to provide at least one example in which (8.19) and (8.20) are not equivalent. Consider the three vectors μ_1, μ_2, μ_3, which lie the same plane. Suppose that propositions A, B, C are that the system is in state μ_1, μ_2, μ_3, respectively. Proceeding as in Sec. 8.2, we then find that

$$L_C \subset L_A \oplus L_B \equiv L_{A+B}. \tag{8.21}$$

Hence

$$C \to A + B. \tag{8.22}$$

On the other hand, according to Sec. 8.1, the corresponding formula is

$$\Omega_C \subset \Omega_{A+B}, \tag{8.23}$$

and both it and also relation (8.22) are not valid.

We see that cause-and-effect relations can be violated, so that there is no classical theory that could serve as the foundation for quantum theory, i.e., there are no hidden parameters.

9. MATHEMATICAL ADDENDA

Erudite Göttingen mathematicians keep talking about Hermitian matrices, and I don't even know what a matrix is. (*W. Heisenberg*[130])[11)]

9.1. Proof of uncertainty relations

The relation

$$\Delta k \cdot \Delta x \sim 1 \tag{9.1}$$

can be obtained with the help of the Fourier transformation. We confine our attention to a simple case in which the deviation of a field (for example, pressure) from a constant value is given by

$$u(x) = \mathrm{Re}\, U(x), \tag{9.2}$$

$$U(x) = A e^{ikx} e^{-x^2/(\Delta x)^2}; \tag{9.3}$$

where A is a constant complex number. We consider the field at a fixed instant of time, which we take to be $t = 0$.

The wave intensity at a point x is $|U(x)|^2$. If the exponential factor $\exp[-x^2/(\Delta x)^2]$ were absent from (9.3), then the wave intensity would be the same at all points:

$$|U(x)|^2 = |A|^2 = \mathrm{const}. \tag{9.4}$$

The uncertainty in the position coordinate would then be infinite:

$$\Delta x = \infty.$$

The factor $\exp[-x^2/(\Delta x)^2]$ shows that the wave intensity decreases with x^2 and becomes infinitesimal when $x^2 \gg (\Delta x)^2$. The wave described by (9.2) is therefore localized at the point $x = 0$, and the uncertainty in the position coordinate is Δx. On the other hand, the wave can be written as a superposition of plane waves $e^{ik'x}$ with different wave numbers k':

$$U(x) = \int_{-\infty}^{\infty} e^{ik'x} a(k') \mathrm{d}k', \tag{9.5}$$

where $a(k')$ is the amplitude corresponding to wave vector k'. To find $a(k')$, we invert the Fourier transform:

$$A e^{ikx} e^{-x^2/(\Delta x)^2} = \int_{-\infty}^{\infty} e^{ik'x} a(k') \mathrm{d}k'. \tag{9.6}$$

We know that this inversion yields

$$a(k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} A e^{-i(k'-k)x} e^{-x^2/(\Delta x)^2} \mathrm{d}x. \tag{9.7}$$

which is readily evaluated:

$$a(k') = \frac{1}{2} A \frac{|\Delta x|}{\sqrt{\pi}} \exp\left[-\frac{(\Delta x)^2 (\Delta k)^2}{4}\right], \tag{9.8}$$

where

$$\Delta k = k' - k.$$

We thus see that the wave intensity decreases with increasing Δk and becomes infinitesimal when $(\Delta k)^2 \gg 1/(\Delta x)^2$. Hence the uncertainties Δk and Δx are related by

$$|\Delta k| \cdot |\Delta x| \sim 1,$$

which was proved.

9.2. Eigenvalues and eigenvectors

The mathematical formalism of quantum mechanics makes use of the concept of Hilbert space. This space is abstract, which gives rise to difficulties in understanding quantum mechanics. It is therefore useful to begin by illustrating this mathematical scheme by a simple model involving electrical conductivity in two-dimensional space. We know that an electric field **E** applied to a medium produces a current density **j**. These two quantities are related by Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}, \tag{9.9}$$

where σ is the electrical conductivity of the medium. The conductivity σ is a number in the simple case where the properties of the medium are the same in all directions (isotropic medium). However, in crystals, the conductivity is a function of direction (anisotropy). The electric field E_1 pointing along the x_1 axis then produces not only a current j_1 along this axis, but also a current j_2 along another axis x_2 perpendicular to x_1. Similarly, a field E_2 pointing along the x_2 axis

produces current components j_1 and j_2.

For small values of $\mathbf{E} = (E_1, E_2)$ the dependence of $\mathbf{j} = (j_1, j_2)$ on $\mathbf{E}$ is linear, i.e.,

$$j_1 = \sigma_{11}E_1 + \sigma_{12}E_2, \tag{9.10}$$

$$j_2 = \sigma_{21}E_1 + \sigma_{22}E_2. \tag{9.11}$$

These two equations can be written in the compact form

$$\mathbf{j} = \hat{\sigma}\mathbf{E}, \tag{9.12}$$

where $\hat{\sigma}$ is the conductivity operator acting on the vector $\mathbf{E}$ and is actually a matrix:

$$\hat{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \tag{9.13}$$

The two vectors $\mathbf{j}$ and $\mathbf{E}$ in (9.12) are column vectors:

$$\mathbf{j} = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{9.14}$$

There are directions in a crystal in which the current is parallel to the electric field:

$$\mathbf{j} = \alpha\mathbf{E}, \tag{9.15}$$

where α is a number and not a matrix. The vector $\mathbf{E}$ corresponding to this direction is called an eigenvector and the number α is an eigenvalue of the operator $\hat{\sigma}$. The eigenvector and eigenvalue of the operator $\hat{\sigma}$ can be found from (9.10), (9.11), and (9.15):

$$(\sigma_{11} - \alpha)E_1 + \sigma_{12}E_2 = 0,$$
$$\sigma_{21}E_1 + (\sigma_{22} - \alpha)E_2 = 0. \tag{9.16}$$

This set of two linear homogeneous equations always has a zero solution:

$$E_1 = E_2 = 0.$$

This solution corresponds to the trivial proposition that, when there is no electric field, there is also no current. On the other hand, the electrical conductivity describes the nontrivial solution for which there is a nonzero current. The condition for a nontrivial solution is that the determinant of (9.16) must be zero:

$$\begin{vmatrix} \sigma_{11} - \alpha & \sigma_{12} \\ \sigma_{21} & \sigma_{22} - \alpha \end{vmatrix} = 0. \tag{9.17}$$

This second-order algebraic equation defines two eigenvalues α_1 and α_2. When $\alpha = \alpha_1$ or $\alpha = \alpha_2$, one of the solutions in (9.16) is a consequence of the other. For example, if we take the first equation and put $\alpha = \alpha_1$, we obtain the eigenvector $\mathbf{E}^{(1)} = (E_1^{(1)}, E_1^{(2)})$:

$$E_1^{(1)} = \sigma_{12}, \quad E_2^{(1)} = \alpha_1 - \sigma_{11}. \tag{9.18}$$

The eigenvector $\mathbf{E}^{(1)}$ is defined to within an arbitrary factor C: if $\mathbf{E}^{(1)}$ is an eigenvector then $C\,\mathbf{E}^{(1)}$ is also an eigenvector. Similarly, we can define a second eigenvector $\mathbf{E}^{(2)}$ corresponding to the eigenvalue α_2.

In the above discussion, we have tacitly assumed that all the quantities were real. However, we often have to deal with an electric field that varies in accordance with the harmonic law

$$\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}. \tag{9.19}$$

in which case $\mathbf{E}$, $\mathbf{j}$ and $\hat{\sigma}$ are all complex.

We note that, according to the well known Onsager relations,[131] we have

$$\sigma_{12} = \sigma_{21}^*, \quad \sigma_{11} = \sigma_{11}^*, \quad \sigma_{22} = \sigma_{22}^*, \tag{9.20}$$

where the asterisk represents the complex conjugate. This type of matrix is called Hermitian (or self-adjoint).

Vectors usually have three or more components. For example, the position of two particles with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is characterized by the six-dimensional vector $\mathbf{l} = (x_1, y_1, z_1, x_2, y_2, z_2)$. The set of such vectors is called a six-dimensional space. More complicated physical systems are described in a space of a larger number of dimensions.

The relations given by (9.10), (9.11), (9.15), (9.16), and (9.17) can be directly generalized to the case of n-dimensional space:

$$m_i = \sigma_{ij}l_j, \tag{9.21}$$

$$\mathbf{m} = \alpha\mathbf{l}, \tag{9.22}$$

$$(\hat{\sigma} - \alpha\hat{I})\mathbf{l} = 0, \tag{9.23}$$

$$\det(\hat{\sigma} - \alpha\hat{I}) = 0; \tag{9.24}$$

where $\mathbf{m} = (m_1, m_2, ..., m_n)$, $\hat{\sigma}$ is an n-dimensional matrix with elements σ_{ij} and $\hat{I}$ is a unit matrix (a matrix with units along the diagonal and all other elements equal to zero); repeated indices indicate summation between 1 and n.

9.3. Hilbert space

This is an infinite-dimensional space in which we can define the scalar product of two vectors.[12] In finite-dimensional space, the scalar product of two vectors $\mathbf{l}$ and $\mathbf{m}$ is defined by

$$(\mathbf{l}, \mathbf{m}) = l_i^* m_i \tag{9.25}$$

(repeated indices indicate summation). This formula is a direct generalization of the scalar product in ordinary Euclidean space. The scalar square of a vector defines its norm (length):

$$\| \mathbf{l} \|^2 = l_i^* l_i = \sum_{i=1}^{n} |l_i|^2. \tag{9.26}$$

We note that the norm of any nonzero vector is positive. This is the reason for the complex conjugation symbol introduced in (9.25). A vector of unit length is said to be normalized. To normalize an arbitrary vector $\mathbf{l}$ we divide it by its norm. This means that the vector $\mathbf{l}/\|\mathbf{l}\|$ has unit length. The orthogonality (i.e., perpendicularity) of two vectors is another important concept that can be expressed in terms of the scalar product. In particular, two vectors $\mathbf{l}$ and $\mathbf{m}$ are called orthogonal if their scalar product is zero

$$(\mathbf{l}, \mathbf{m}) = 0. \tag{9.27}$$

The Hermitian property can be directly generalized to the case of n-dimensional space:

$$\sigma_{ij} = \sigma_{ji}^*. \tag{9.28}$$

It can be shown that all eigenvalues of hermitian matrix are real, and different eigenvectors are orthogonal in pairs. The

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latter property enables us to write any vector $\mathbf{q}$ in the form of a superposition (sum) of eigenvectors $\mathbf{l}^{(i)}$ of an arbitrary Hermitian operator $\hat{\sigma}$:

$$\mathbf{q} = c_i \mathbf{l}^{(i)}. \tag{9.29}$$

The eigenvectors $\mathbf{l}^{(i)}$ can be considered to be normalized. The orthogonality relations ensure that the coefficients c_i have a very simple form:[132]

$$c_i = (\mathbf{l}^{(i)}, \mathbf{q}). \tag{9.30}$$

In quantum mechanics, we consider more complicated objects such as the wave functions $\psi(\mathbf{r})$ where $\mathbf{r}$ is a vector in ordinary three-dimensional space. This function is an infinite-dimensional vector because its description involves an infinite number of values (components of the vector) $\psi_1(\mathbf{r})$, $\psi_2(\mathbf{r})$, ... On the other hand, the component l_i of a vector $\mathbf{l}$ can be regarded as a function of the index i. This is why there is no difference between vectors and functions in the theory of Hilbert space. In an infinite-dimensional space, a scalar product such as (9.25) contains an infinite number of terms. Since an integral is defined by

$$\int \varphi(t)dt = \lim_{\Delta t_i \to 0} \sum_i \varphi(t_i)\Delta t_i, \tag{9.31}$$

it is natural to define a scalar product of two functions in the form

$$(\varphi, \psi) = \int \varphi^*(\mathbf{r})\psi(\mathbf{r})d\mathbf{r}. \tag{9.32}$$

Formulas (9.26) and (9.27) can be directly generalized to the case of Hilbert, i.e., infinite-dimensional, space.

9.4. Possible results of measurements

If we had to express the entire content of quantum theory by a single formula, we would write

$$P(a_i) = |(\psi^{(i)}, \psi)|^2, \tag{9.33}$$

where $P(a_i)$ is the probability that a measurement of a quantity a will result in a value a_i if the quantum object is in a state of ψ. In this Section, we examine (9.33) in some detail. In quantum mechanics, we can associate an operator $\hat{A}$ with every physical quantity a. When a is measured, the result can only be an eigenvalue of the operator $\hat{A}$.

We shall now illustrate a measurement scheme in quantum mechanics by considering the measurement of electron spin. The components of electron spin along the x, y, z axes have associated with them following operators:[133]

$$\hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{9.34}$$

$$\hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{9.35}$$

$$\hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.36}$$

Using (9.17) and (9.36), we then obtain

$$\begin{vmatrix} \frac{1}{2} - S_z & 0 \\ 0 & -\frac{1}{2} - S_z \end{vmatrix} = 0. \tag{9.37}$$

Hence the eigenvalues of the operator $\hat{S}_z$ are

$$S_z = \pm\frac{1}{2}. \tag{9.38}$$

Equations similar to (9.16) then show that the eigenfunction corresponding to $S_z^{(1)} = 1/2$ is

$$\psi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{9.39}$$

whereas

$$\psi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.40}$$

corresponds to the eigenvalue $S_z^{(2)} = -1/2$. The arbitrary constants in (9.39) and (9.40) were chosen to ensure that the vector eigenfunctions were normalized:

$$\| \psi^{(1)} \| = 1, \quad \| \psi^{(2)} \| = 1.$$

The physical meaning of the eigenfunction $\psi^{(1)}$ is that the electron spin component S_z in this state is determinate and equal to $+1/2$. Similarly, the magnitude of S_z in the state $\psi^{(2)}$ is $-1/2$.

Quantum mechanical randomness arises only in those cases where we measure a quantity that does not have a definite value. For example, suppose we measure the projection S_z of the spin of the electron in the state $\psi^{(\alpha)}$ in which the projection S_α on the α axis has a particular value. It can be shown that the operator $\hat{S}_\alpha$ takes the form

$$\hat{S}_\alpha = \alpha_x \hat{S}_x + \alpha_y \hat{S}_y + \alpha_z \hat{S}_z; \tag{9.41}$$

where α_x, α_y, α_z are the components of the unit vector α on the coordinate axes. We now choose the coordinate frame in such a way that the vector α lies in the x, z plane and makes an angle φ to the z axis. We then have $\alpha_x = \sin\varphi$, $\alpha_y = 0$, $\alpha_z = \cos\varphi$, and

$$\hat{S}_\alpha = \frac{1}{2}\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}. \tag{9.42}$$

One of the eigenvalues is $\hat{S}_\alpha = 1/2$. The corresponding eigenvector is

$$\psi^{(\alpha)} = \begin{pmatrix} \cos\dfrac{\varphi}{2} \\ \sin\dfrac{\varphi}{2} \end{pmatrix}. \tag{9.43}$$

When we measure S_z in the state $\psi^{(\alpha)}$, the only possible outcomes are $S_z = 1/2$ and $S_z = -1/2$. These values are random. The only determined quantity is the probability of these values. To determine this probability, we write $\psi^{(1)}$ and $\psi^{(2)}$:

$$\psi^{(\alpha)} = c_1 \psi^{(1)} + c_2 \psi^{(2)}. \tag{9.44}$$

The probability of the result $S_z = 1/2$ in state $\psi^{(\alpha)}$ is $|c_1|^2$ and the probability of $S_z = -1/2$ is $|c_2|^2$. Equations (9.39), (9.40), and (9.43) then yield

$$\psi^{(\alpha)} = \psi^{(1)}\cos\frac{\varphi}{2} + \psi^{(2)}\sin\frac{\varphi}{2}. \tag{9.45}$$

Hence the probability of obtaining $S_z = 1/2$ in the state $\psi^{(\alpha)}$ is $\cos^2(\varphi/2)$ and the probability of $S_z = -1/2$ is $\sin^2(\varphi/2)$.

In the special case $\varphi = \pi/2$, i.e., when the vector α points along the x axis, we have

$$\psi^{(\alpha)} = \frac{1}{\sqrt{2}} \psi^{(1)} + \frac{1}{\sqrt{2}} \psi^{(2)}. \qquad (9.46)$$

When the spin projection along the z axis is measured, we obtain $1/2$ with 50% probability. If, on the other hand, we measure the spin projection along the x axis, we find there is no randomness and he result is always

$$S_x = 1/2.$$

Let us now consider the general case. Suppose we measure a quantity a in the state ψ. We associate with a a particular operator $\hat{A}$. This operator has eigenvectors $\psi^{(1)}, \psi^{(2)}, ...$ and the corresponding eigenvalues are $a_1, a_2, ...$. Measurement of a can result only in one of the eigenvalues a_i of the operator $\hat{A}$.

The eigenvalues of an operator are frequently discrete. This occurs, for example, in the case of the energy operator of an electron in an atom. The electron energy, which in classical physics can assume any of a continuous series of values, then takes on only certain definite discrete values in quantum mechanics.

We note that this type of discreteness is encountered in classical physics as well. For example, the frequency of a string cannot be arbitrary, but must assume one of the discrete values in the sequence $\omega_0, 2\omega_0, 3\omega_0, ...$. The difference between classical and quantum mechanics lies in the fact that, in the former, the oscillation energy is unrelated to the oscillation frequency and can assume any value. In quantum mechanics, on the other hand, the energy of the oscillations is related to their frequency by Planck's formula, given by (2.29).

We now turn to the discrete eigenvalues a_i of the quantum-mechanical operator $\hat{A}$. To find the probability of a value a_i, we write the state vector ψ in the form of the superposition $\psi^{(1)}, \psi^{(2)}, ...$:

$$\psi = c_1 \psi^{(1)} + c_2 \psi^{(2)} + ... \qquad (9.47)$$

The probability that a_i will be obtained is $|c_i|^2$. If we now use (9.30), we obtain (9.33).

9.5. Missing points in the Kochen–Specker proof

The proof that hidden parameters cannot be introduced into quantum mechanics, given in Sections 7.5, makes use of the following propositions:

1. The squares of the projections of the unit spin ($S = 1$) on the l and m axes, i.e., S_l^2 and S_m^2 are compatible if and only if the two axes are orthogonal.

2. If the three axes l, m, n are orthogonal in pairs, then two of the values S_l^2, S_m^2, S_n^2 are equal to unity and the third is zero.

We shall now prove the first of these two propositions. The compatibility condition for S_l^2 and S_m^2 is

$$\left[\hat{S}_l^2, \hat{S}_m^2 \right] \equiv \hat{S}_l^2 \hat{S}_m^2 - \hat{S}_m^2 \hat{S}_l^2 = 0; \qquad (9.48)$$

where

$$\hat{S}_l = l_x \hat{S}_x + l_y \hat{S}_y + l_z \hat{S}_z, \qquad (9.49)$$
$$\hat{S}_m = m_x \hat{S}_x + m_y \hat{S}_y + m_z \hat{S}_z, \qquad (9.50)$$

and the three operators are given by the matrices[133]

$$\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{S}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (9.51)$$

Since the z axis is not special in any way, we may suppose that it lies along the direction of $\mathbf{m} = (0,0,1)$. From (9.49), (9.50), and (9.51) we find that

$$\left[\hat{S}_l^2, \hat{S}_m^2 \right] = \begin{pmatrix} 0 & -\frac{l_z}{\sqrt{2}}(l_x - il_y) & 0 \\ \frac{l_z}{\sqrt{2}}(l_x + il_y) & 0 & -\frac{l_z}{\sqrt{2}}(l_x - il_y) \\ 0 & \frac{l_z}{\sqrt{2}}(l_x + il_y) & 0 \end{pmatrix}. \qquad (9.52)$$

For nonparallel l and m, i.e., when l_x and l_y are not simultaneously equal to zero, the matrix given by (9.52) is equal to zero if and only if $l_z = 0$. Hence the quantities S_l^2 nd S_m^2 are compatible if an only if the directions of l and m are orthogonal.

We now turn to the proof of the second proposition. To do this, we must evaluate the operator representing the square of the spin:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2. \qquad (9.53)$$

It follows from (9.51) that

$$\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = 2\hat{I}, \qquad (9.54)$$

where $\hat{I}$ is the unit matrix. Since the values of $S_x^2, S_y^2,$ and S_z^2 are compatible, they must be related by

$$S_x^2 + S_y^2 + S_z^2 = 2. \qquad (9.55)$$

Since S_x^2, S_y^2, S_z^2 can only be equal to unity or zero, we find that two of them are equal to unity and the third to zero, which was to be proved. We note, by the way, that (9.55) is a further manifestation of the quantum nature of spin. The classical unit vector would be subject to the three dimensional Pythagoras theorem

$$S_x^2 + S_y^2 + S_z^2 = 1. \qquad (9.56)$$

[1] A field is a form of matter that is different from particles. The field is not concentrated at particular discrete points, but is distributed throughout all space. For example, in the atmosphere, each point has a particular pressure associated with it. The set of these pressure values constitutes the pressure field. A more detailed discussion of this point is given in Sec. 2.1.

[2] In this context, it seems to us that the difference between U (our body) and the philosophical concept I adopted in the literature (our perception) seems unimportant.

[3] Formulas (2.34) and (2.35) differ from the analogous classical expressions (2.24) and (2.26) not only by the factor $\hbar$ but also by the meaning of the different quantities that appear in them. In classical physics, harmonic components with different k and ω do actually exist. On the contrary, in quantum mechanics, and as we shall see in the next Section, the values of p and $\mathscr{E}$ are random and the corresponding uncertainties

Δp and $\Delta\mathscr{E}$ are defined as the square roots of the corresponding variances.

[4]Two events are incompatible if occurrence of one excludes the occurrence of the other.

[5]The distance between the slits does not exceed in order of magnitude the wavelength of the electron λ given by (2.33).

[6]We note that, within a sufficiently short interval of time, any two particles behave as if they were identical.[60]

[7]In quantum mechanics, the replacement of t with $-t$ means that the wave function ψ must be replaced with the complex conjugate ψ^*, but this has no observable effect.

[8]A theory is fundamental if it postulates are primary, i.e., they do not emerge from some other more fundamental theory. In economics, the expression 'fundamental science' is used in another sense: it is said to be a science for which direct economic effects cannot be forseen.

[9]MeV is the abbreviation for million electron volt (1.6×10^{-13} J) and is commonly used in nuclear physics.

[10]We present only the essentials of the proof that hidden variables cannot be introduced into quantum mechanics. A more detailed discussion can be found in Refs. 125 and 126.

[11]Soon after stating this, Heisenberg created the matrix form of quantum mechanics.

[12]We omit some of the mathematical details (Ref. 98, Chap. 2).

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