## On the linear theory of waves in media with periodic structures

P.S. Landa and V.F. Marchenko

M. V. Lomonosov Moscow State University (Submitted 11 April 1990, after revision 1 April 1991) Usp. Fiz. Nauk 161, 201–209 (September 1991)

The dispersion properties of the elastic vibrations of a one-dimensional periodicallyinhomogeneous chain are examined. It is shown that in both the acoustic and optical passbands both normal and anomalous dispersion branches are always generated in pairs. The amplitudes of the waves corresponding to these branches are mutually linked. Results are presented of experimental simulation in electric transmission lines with periodically varying parameters.

Both surveys and special articles in the literature have been devoted to the propagation of waves in media with periodic structure (see, for example, Refs. 1-7). Textbooks have also been written on this subject.<sup>8-10</sup> The specialized literature deals mainly with three areas of knowledge: solid state physics, where models of crystal lattices are examined,<sup>1,3,6</sup> electronics, where different types of delay systems are examined<sup>4,5,7</sup>, and optics, which examines multi-layer coatings.<sup>5</sup> We note that, as a rule, publications on electronics present true dispersion relations for linear waves propagating in delay systems. Solid state physics publications that examine one-dimensional discrete models of crystal lattices and sections of textbooks, even new textbooks,<sup>8,9</sup> on the theory of vibrations and waves do not provide a complete picture of these dispersion relations. As in earlier publications,<sup>1,2,11</sup> the range of wave numbers in the above publications is limited to the interval  $\mp \pi/2$ , which may lead to the erroneous conclusion that the so-called "acoustic" branches of the spectrum of elastic vibrations have only the normal dispersion, that is, the frequency of vibrations increases as the modulus of the wave number increases. Conversely, the "optical" branches have only an anomalous dispersion, that is, the frequency of vibrations falls as the modulus of the wave number increases. In fact, as will be shown below, in both the acoustic and optical ranges there should be branches with both a normal and an anomalous dispersion, and their amplitudes are unambiguously mutually linked.

The purpose of this article is to draw the attention of a wide range of physicists and instructors at institutes of higher learning to the fact presented above, which is important both from the point of view of fundamental research and also for the solution of practical problems. In fact, thousands of university students learn from the aforementioned textbooks.

First of all, let us examine the well-known<sup>1-3,5,6</sup> onedimensional model of a diatomic crystal lattice (for example, NaCl) which is an alternating sequence of heavy (with mass M) and light (with mass m) balls joined by springs with a force constant k (Fig. 1a). Let us number the balls in order, and to be definite we will assume that the heavier balls correspond to even numbers, and the lighter balls correspond to odd numbers. Then the equations for the vibrations of the balls can be written in the form

$$m\ddot{x}_{s} + k(2x_{s} - x_{s+1} - x_{s-1}) = 0$$
 for  $s = \pm 1, \pm 3, \pm 5, ...,$   
(1)  
 $M\ddot{x} + k(2x_{s} - x_{s+1} - x_{s-1}) = 0$  for  $s = \pm 2, \pm 4, \pm 6, ...,$ 

830 Sov. Phys. Usp. 34 (9), September 1991

where  $x_s$  is the displacement of the *s*-th ball from the equilibrium position.

The masses of the balls differ, and so the amplitudes of their vibrations will differ from one another. Thus, the solution of Eq. (1) will be sought in the form

$$\begin{aligned} \mathbf{x}_{s} &= A e^{i s \beta} \cos \omega t \quad \text{for} \quad s = \pm 1, \pm 3, \dots \\ &= B e^{i s \beta} \cos \omega t \quad \text{for} \quad s = \pm 2, \pm 4, \dots, \end{aligned}$$

where  $\beta$  is an unknown quantity which plays the role of the wave number. As follows from Eq. (2),  $\beta$  is the phase change in one cell.

Substituting Eq. (2) into Eq. (1), we obtain equations to determine the amplitudes of A and B, from which we find the dispersion equation

$$\omega^4 - 2(\omega_1^2 + \omega_2^2)\omega^2 + 4\omega_1^2\omega_2^2\sin^2\beta = 0, \qquad (3)$$

where  $\omega_1 = (k/M)^{1/2}$ ,  $\omega_2 = (k/m)^{1/2}$  are the eigenfrequencies of the uncoupled vibrations of the balls, as well as the ratio of amplitudes

$$\frac{B}{A} = \left[ \left( 1 - \frac{\omega^2}{2\omega_2^2} \right) \left( 1 - \frac{\omega^2}{2\omega_1^2} \right)^{-1} \right]^{1/2} \operatorname{sign} \left[ \left( 1 - \frac{\omega^2}{2\omega_2^2} \right) \cos \beta \right].$$
(4)

Then it is clear that at  $\omega \leqslant \sqrt{2}\omega_1$  the amplitude of vibrations of the heavy balls *B* exceeds the amplitude of vibrations of the light balls *A*, so *A* goes to zero as frequency  $\omega$  approaches the boundary value  $\sqrt{2}\omega_1$ . At  $\omega = \sqrt{2}\omega_2$ , the heavy balls are immobile (B = 0), and in the second region of frequencies  $\omega$ , when  $\sqrt{2}\omega_2 \leqslant \omega \leqslant [2(\omega_1^2 + \omega_2^2)]^{1/2}$ , it is always true that B < A.

Up to this point the results have completely coincided with known results.<sup>1-9</sup> The divergence begins with the following question: in which range should one examine values of  $\beta$ ? In a chain with identical balls the range of variation in  $\beta$  corresponds to the interval from  $-\pi$  to  $\pi$ , since all other values of  $\beta$  are indistinguishable from the indicated values. In the transition to a chain with periodically alternating



FIG. 1. One-dimensional mechanical model of a diatomic crystal lattice. a) infinite; b) bounded.

0038-5670/91/090830-05\$01.00

© 1992 American Institute of Physics 830

balls the following argument is adduced: since the period of the structure is doubled, the range of variation in  $\beta$  should be halved. This would be correct if the chain remained homogeneous as before. For an inhomogeneous lattice this statement, generally speaking, is untrue.

Let us show that for the case of a chain with alternating elements the wave number  $\beta$  should still be examined in the interval  $-\pi$  to  $\pi$ . Let us examine a finite chain consisting of *n* balls which are fastened at the ends (Fig. 1b). The boundary conditions for such a chain have the form

$$x_0 = x_{n+1} = 0. (5)$$

It is easy to obtain a particular solution of Eq. (1) with the boundary conditions in Eq. (5) in the form

$$\begin{aligned} x_{sq} &= \sin s\beta_q (A_{q1} \cos \omega_{q1} t + A_{q2} \cos \omega_{q2} t) \text{ for } s = \pm 1, \pm 3, ..., \\ &= \sin s\beta_q (B_{q1} \cos \omega_{q1} t + B_{q2} \cos \omega_{q2} t) \text{ for } s = \pm 2, \pm 4, ..., \end{aligned}$$
(6)

where the ratios  $B_{q1}/A_{q1}$  and  $B_{q2}/A_{q2}$  are defined by Eq. (4) for  $\omega = \omega_{q1}$  and  $\omega = \omega_{q2}$  respectively, and  $\omega_{q1}$  and  $\omega_{q2}$  are two roots of Eq. (3) for  $\beta = \beta_q$ . It follows from the boundary conditions in Eq. (5) that  $\beta_q = \pm q\pi/(n+1)$ , where q = 1, 2, ..., n. Then it is clear that the range of changes in  $\beta_q$  is enclosed within the interval  $-\pi$ ,  $\pi$ . For each value of  $\beta_q$ there are two fundamental frequencies  $\omega_{a1}$  and  $\omega_{a2}$ . All n fundamental frequencies of the lattice can be obtained by changing q from 1 to n/2, that is, in the range of variation in  $\beta$  from  $-\pi/2$  to  $\pi/2$ . Expansion of the range of variation in  $\beta$  to the interval  $[-\pi, \pi]$ , that is, for q = n/2 + 1, ..., n does not lead to the appearance of new fundamental frequencies. Thus, in theories where only the number of degrees of freedom is significant, for example, Born's theory of specific heat,<sup>1</sup> restricting the range of variation of  $\beta$  to the interval  $[-\pi/2, \pi/2]$  does not lead to errors.

If we are interested in the character of wave processes occurring in the chain, in particular, the phase and group velocity of waves, as well as nonlinear processes, then the limitation on the range of variation of  $\beta$  to the interval  $[-\pi/2, \pi/2]$  is not correct. In this interval we obtain from



FIG. 2. Dependence of frequency  $\omega$  on wave number  $\beta$ . a)  $\beta$  ranges from  $-\pi/2$  to  $\pi/2$ . b)  $\beta$  ranges from  $-\pi$  to  $\pi$ .

831 Sov. Phys. Usp. 34 (9), September 1991

Eq. (3) the known dispersion equations presented in Fig. 2a. Branches I and I' have been named by Born acoustic branches, and branches 2 and 2' optical branches of the spectrum of elastic vibrations in a diatomic lattice. It is clear from Fig. 2a that the acoustic branches correspond to a normal dispersion law in which the frequency of vibrations increases as the modulus of the wave number increases, that is, the phase velocity  $(\omega/\beta)$  and group velocity  $(d\omega/d\beta)$  are in one direction. In the optical branches the dispersion law is anomalous, that is, the frequency decreases as the modulus of the wave number increases, which leads to the fact that the phase and group velocities of the waves are in opposite directions. It is this very case which Mandelshtam<sup>10</sup> used as an example of a medium with these interesting properties.

In the range of variation of  $\beta$  from  $-\pi$  to  $\pi$  the shape of the dispersion relations is shown in Fig. 2b. It follows from this figure that the optical  $(\omega > \sqrt{2}\omega_2)$  and the acoustic  $(\omega < \sqrt{2}\omega_1)$  branches of the spectrum have segments with normal (1,1',I,I') and anomalous (2,2',II,II') dispersion laws. But what does actually occur? For an answer to this question we present the solution of Eq. (2) in a form which is universal for the description of vibrations both of heavy and light balls:

$$x_s = \frac{A+B}{2} \left( 1 + \mu^{\operatorname{sign}(B/A)} \cos \pi s \right) e^{is\beta} \cos \omega t, \tag{6'}$$

where

$$\mu = \left| \left[ \left( 1 - \frac{\omega^2}{2\omega_2^2} \right)^{1/2} - \left( 1 - \frac{\omega^2}{2\omega_1^2} \right)^{1/2} \right] \left[ \left( 1 - \frac{\omega^2}{2\omega_2^2} \right)^{1/2} + \left( 1 - \frac{\omega^2}{2\omega_1^2} \right)^{1/2} \right]^{-1} \right|$$

Taking into account that  $\exp[is(\beta + \pi)] = \exp[is(\beta - \pi)]$ , Eq. 6 can be written in a more convenient form

$$x_{s} = \frac{A+B}{2} \left( e^{is\beta} + \mu^{\operatorname{sign}(B/A)} e^{is(\beta-\pi)} \right) \cos \omega t.$$
(7)

Then it follows that at a given frequency  $\omega$  two waves are always generated with wave numbers  $\beta$  and  $\beta - \pi$ . These waves correspond to the two branches in Fig. 2b (I and II', II and I', 1 and 2', or 2 and 1'). Both waves have the same group velocities and differ in the value and the direction of the phase velocities. The amplitudes of the generated waves are not independent. The ratio of the amplitudes of the waves in branches II, II', 2 and 2' to the amplitudes of waves in branches I', I, 1', and 1, respectively, is  $\mu \leq 1$ . The dependence of  $\mu$  on  $\beta$  for frequency intervals  $\omega \leq \sqrt{2}\omega_1$  and  $\omega \geq \sqrt{2}\omega_2$ is shown in Fig. 3. It is clear from the figure that waves with a normal dispersion law in the examined lattice always (with the exception of critical points  $\omega = \sqrt{2}\omega_1$  and  $\omega = \sqrt{2}\omega_2$ ) have a larger amplitude than waves with an anomalous dispersion law.

As  $m \rightarrow M$  the amplitudes of waves with an anomalous dispersion law go to zero, and  $\omega_1$  goes to  $\omega_2$ . The derivative  $d\omega/d\beta$ , that is, the group velocity, as follows from Eq. (3), is equal to

$$\frac{d\omega}{d\beta} = \pm \frac{\omega_1^2 \omega_2^2 \sin 2\beta}{\omega} [(\omega_1^2 - \omega_2^2)^2 + 4\omega_1^2 \omega_2^2 \cos^2\beta]^{-1/2}$$

Then it is clear that the value of  $d\omega/d\beta$  as  $\beta \rightarrow \pm \pi/2$ ,

P. S. Landa and V. F. Marchenko 831

- Fright 6-

· · · · ·



FIG. 3. Dependence of coefficient  $\mu$  on wave number  $\beta$ . a) for  $\omega \leq \sqrt{2}\omega_1$ ; b) for  $\omega \geq \sqrt{2}\omega_2$ .

 $\omega_1 \rightarrow \omega_2$  depends on the order of transition to the limit. If  $\omega_1$ goes to  $\omega_2$  first, and then we set  $\beta = \pm \pi/2$ , then  $d\omega/d\beta$ acquires a finite value as for a chain with identical balls. If, conversely,  $\beta$  goes to  $\pm \pi/2$  first, and then  $\omega_1$  goes to  $\omega_2$ , then  $d\omega/d\beta$  will go to zero. However, the second derivative  $d^2\omega/d\beta^2$  will go to infinity. Thus, the character of the transition to the limit as  $m \rightarrow M$  becomes clear, and the difficulties which occur when one limits the range of variation of  $\beta$  to the range of values  $\pm \pi/2$  do not arise (see, for example, Refs. 2 and 8).

The results we have obtained can be confirmed with a simple experiment. It is known<sup>2,12</sup> that the equations for charges (and consequently, for currents) in an LC line, such as a low-frequency filter in which alternating inductances  $L_1$  and  $L_2$  are used (the capacitances C of the cells are the same), are identical to Eq. (1), that describe the vibrations of balls in the mechanical chain shown in Fig. 1. Thus, Eqs. (3) and (4) obtained above can be applied to the calculation of the dispersion relations  $\beta(\omega)$  and the parameter of the electric filter  $\mu(\omega) = |1 - |B/A||/(1 + |B/A|)$ , if one sets  $\omega_1 = 1/(L_1C)^{1/2}$ ,  $\omega_2 = 1/(L_2C)^{1/2}$ . We note that the analog of a mechanical chain with identical ball masses, but with alternating spring force constants  $k_1$  and  $k_2$ , is a low-frequency filter in which capacitors with different capacitances  $C_1$  and  $C_2$  are used and the inductances of the L cells are the same.

Experiments were conducted in the radio frequency range, and in addition to a low-frequency filter, more complex lines were studied with a periodic change in parameters. A high-ohm probe, selective voltmeter and phase meter were used to measure the amplitudes of the voltages in the cells of the line and the dependences of the phase change per cell on the frequency. The amplitudes of the currents were determined by calculation. The degree of agreement of theoretical and experimental results is determined by the care taken in manufacturing the cells in the line (in our case, the requirement of exact agreement was of no fundamental importance).

Figure 4 shows the theoretical values (solid curves) and experimental data for the relations  $\beta(\omega)$  and  $\mu(\omega)$  as functions of the normalized frequencies  $\omega/\omega_B$  where  $\omega_B = [2(L_1 + L_2)/CL_1L_2]^{1/2} (L_1/L_2 = 2)$ . When a harmonic voltage was fed to the input of the line, which consist-



FIG. 4. Dependence of the phase change  $\beta$  per line cell and of the coefficient  $\mu$  on frequency in a low-frequency filter with parameters  $L_1 = 40$   $\mu$ H,  $L_2 = 20 \mu$ H, C = 330 pF. The solid curves for  $\beta$  and the dashed curves for  $\mu$  correspond to theoretical data, and the symbols indicate experimental data.

ed of 24 cells, predominantly waves with a normal dispersion law were generated in both the first and the second passbands. In a real system anomalous branches are always generated with a "weight" less than unity. This is due to the fact the parameter  $\mu$  does not reach unity even at points corresponding to the boundary frequencies  $\sqrt{2}\omega_1$  and  $\sqrt{2}\omega_2$  due to losses not considered in the theory presented above.

For comparison we also studied a high frequency filter (with  $L_1/L_2 = 2$ ,  $C_1 = C_2 = C$ ) for which an anomalous dispersion of the generated waves is characteristic. Figure 5 presents the relations  $\beta(\omega)$  and  $\mu(\omega)$  for normalized frequencies  $\omega/\omega_{\rm H}$ , where  $\omega_{\rm H} = [(L_1 + L_2)/2CL_1L_2]^{1/2}$ . In this case predominantly waves with an anomalous dispersion are generated both in the first and second passbands, and the pattern of the dispersion relations in this case is, to a certain degree, inverted in frequency with respect to the corresponding relations for a low-frequency filter. Thus, one can conclude that the appearance of a cut-off band due to the difference in the parameters of neighboring cells in the line (or in a mechanical chain) does not lead to a change in the type of dispersion if the character of the dispersion (conventionally, "inductive" or "capacitative") of the cells them-



FIG. 5. Dependence of  $\beta$  and  $\mu$  on frequency in a high-frequency filter with parameters  $L_1 = 20 \,\mu$ H,  $L_2 = 40 \,\mu$ H,  $C = 330 \,\text{pF}$ .

832 Sov. Phys. Usp. 34 (9), September 1991

selves is preserved over the entire frequency interval.

A question arises: in what kind of line (or chain) will waves with a normal dispersion predominate in an "acoustic" band of frequencies, and in what kind of line or chain will waves with an anomalous dispersion predominate in an "optical" band of frequencies. To answer this question we studied a line consisting of two types of alternating resonators whose resonance frequencies  $\omega_1$  and  $\omega_2$  differ appreciably from each other. Such a separation of resonance frequencies is necessary to change the character of dispersion in the resonators with lower values of  $\omega_1$ . The schematic of this line is shown in Fig. 6. By a choice of suitable circuit parameters it is possible to arrange both a case of normal dispersion in both passbands (Fig. 6a) as well as a case where in the high-frequency band ("optical" band) waves with an anomalous dispersion will dominate (Fig. 6b). In the latter case the pattern of the dependence  $\mu(\omega)$  changes as well, because this parameter reaches its maximum value at  $\beta = +\pi/2$ .

Analogous results are obtained also for bulk media with a periodic structure, which also is not reflected in textbooks.<sup>8,9</sup> In Ref. 8, for example, only branches with a normal dispersion are presented, and there is no indication of the existence of branches with an anomalous dispersion.



FIG. 6. Dependence of  $\beta$  and  $\mu$  on frequency in lines with resonance cells having the following parameters: a)  $L_1 = 12 \,\mu$ H,  $L_2 = 24 \,\mu$ H,  $C_1 = 330$  pF,  $C_2 = 540$  pF,  $C_0 = 1000$  pF; b)  $L_1 = 12 \,\mu$ H,  $L_2 = 24 \,\mu$ H,  $C_1 = 33$  pF,  $C_2 = 160$  pF,  $C_0 = 410$  pF.

833 Sov. Phys. Usp. 34 (9), September 1991

Let us compare the results obtained above with results known in the theory of delay systems in the microwave range of electromagnetic waves.<sup>4</sup> In the majority of cases the delay system can be conveniently depicted as a continuous medium (regions of interaction with an electron beam) with discretely actuated elements (for example, resonators). Let us examine a two-stage system in which two types of resonators are actuated along the z axis at a distance l from each other. According to Ref. 4, the field in the system can be written in the form of sums of spatial harmonics

$$E(x, y, z) = \sum_{j=-\infty}^{\infty} e_j(x, y) \exp[i(\beta' + 2\pi j/d)z] \cos \omega t, \quad (8)$$
  
$$d = 2l.$$

The difference in the resonators in one period makes it possible to separate out two sets of spatial harmonics. To do this, let us split the series in Eq. (8) into two series with even and odd numbers j. Writing  $j_1 = j/2$  for even j, and  $j_2 = (j + 1)/2$  for odd j, and setting  $z = z_s = sl$ , where  $s = \pm 1, \pm 2,...$ , we get

 $E(x, y, z_s) = (\alpha_0 e^{is\beta} + \alpha_{-1} e^{is(\beta - \pi)}) \cos \omega t, \qquad (9)$ 

where

$$\beta = \beta' l, \quad \alpha_0 = \sum_{j_1} e_{j_1}, \quad \alpha_{-1} = \sum_{j_2} e_{j_2}$$

This expression coincides in form with Eq. (7), which describes waves in a diatomic chain. The ratio of the amplitudes of the minus first component to that of the zero component,  $R = \alpha_{-1} / \alpha_0$ , is defined now as the sums of spatial harmonics, which is a consequence of the transition from a discrete model to a discretely-distributed model. In contrast to the parameter  $\mu$  presented above, which has the physical sense of the ratio of the amplitude of a wave with a normal dispersion to the amplitude of a wave with an anomalous dispersion, the parameter R does not have this clear physical sense. When the interval of variation of  $\beta$  is limited to the values  $\pm \pi/2$ , the amplitude, for example, of the minus first component characterizes in the first passband a wave with an anomalous dispersion, and in the second passband, a wave with a normal dispersion. In these bands the values of R are respectively  $\mu$  and  $1/\mu$ . As noted above, in the case of resonance dependences of  $\alpha_0$  and  $\alpha_{-1}$  on frequency, the branch with an anomalous dispersion may predominate.

In conclusion, we note that the correct interpretation of dispersion relations is especially important in the examination of nonlinear processes in the described chains: the generation of a second harmonic, decay instability, the formation of solitons, etc. It can be shown for example, that in a certain range of mass ratios, namely

$$\frac{4}{3} \le \frac{M}{m} \le 3,$$

for branches with normal dispersion, at some frequency  $\omega - \omega^*$  exact synchronism is possible, that is,  $\beta(2\omega^*) = 2\beta(\omega^*)$ , where  $\omega^*$  is defined by the expression

$$\frac{\omega^{*}}{\sqrt{2}\omega_{0}} = \left(1 - \sqrt{3}\frac{\omega_{1}\omega_{2}}{\omega_{0}^{2}}\right)^{1/2};$$

 $\omega_0 = (\omega_1^2 + \omega_2^2)^{1/2}$ . Here  $\omega^* < \sqrt{2}\omega_1$ ,  $\sqrt{2}\omega_2 \le 2\omega^* \le \sqrt{2}\omega_0$ , that is, the frequency  $\omega^*$  is located on the acoustic branch,

P. S. Landa and V. F. Marchenko 833

. . .

and  $2\omega^*$  is located on the optical branch. Due to the presence of the condition of synchronism one should expect, due to the quadratic nonlinearity of the effective transformation of vibration energy in the acoustic range of frequencies into vibrations in the optical range of frequencies. However, inevitable losses existing in real chains, can substantially decrease the efficiency of this transformation.

- <sup>1</sup>M. Born and K. Huang, *Dynamical Theory of Crystal Lattices*, Oxford U. Pr., Oxford, 1954 and 1985 [Russ. transl. IL, M., 1958].
- <sup>2</sup> L. Brillouin and M. Parodi, *Wave Propagation in Periodic Structures*,
- Dover, N.Y., 1953. [Russ. transl., IL, M., 1959].
- <sup>3</sup> A. M. Kosevich, *Theory of Crystal Lattices* (in Russian), Izd-vo KhGU, Kharikov, 1988.
- <sup>4</sup> R. A. Silin and V. P. Sazonov, *Delay Systems* (in Russian), Sov. radio, M., 1966.

- <sup>5</sup>C. Elachi, Waves in Active and Passive Periodic Structures: A Review, Proc. IEEE 64, 1666 (1976). [Russ. transl., TIIER 64/12, 22 1976].
- <sup>6</sup>O. Madelung, Introduction to Solid State Theory, Springer-Verlag, N.Y., 1978 [Russ. transl., Nauka, M., 1985].
- <sup>7</sup> L. A. Vaĭnstheĭn, *Electromagnetic Waves* (in Russian), Radio i svyaz' M., 1988.
- <sup>8</sup>M. B. Vinogradova, O. V. Pudenko, and A. P. Sukhorukov, *Theory of Waves* (in Russian), Nauka, M., 1979, 1990.
- <sup>9</sup>M. I. Rabinovich and D. I. Trubetskov, *Introduction to the Theory of Vibrations and Waves* (in Russian), Nauka, M., 1984.
- <sup>10</sup> V. E. Sotin, Bragg Reflection in Periodic Structures (in Russian), Izd-vo Universiteta druzhby narodov, M., 1981.
- <sup>11</sup> L. I. Mandelshtam, Group Velocity in a Crystal Lattice in Collected Works, Vol. 2 (in Russian), Izd-vo AN SSSR, M., 1947.
- <sup>12</sup> S. P. Strelkov, Introduction to the Theory of Vibrations (in Russian), Nauka, M., 1964.

Translated by C. Gallant