Similarity relations for low-temperature nonisothermal discharges

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(Submitted 22 January 1991; after revision 12 May 1991)
Usp. Fiz. Nauk **161**, 195–199 (September 1991)

Similarity relations are discussed for low-temperature gas discharges with a low degree of ionization. The relations are obtained by requiring that the Boltzmann equations (B-similarity) and the field equations (F-similarity) be invariant. Conditions are determined for these two relations to be mutually compatible and for complete similarity to exist in the discharge.

1. Many publications are devoted to studies of low-temperature nonisothermal gas discharges with a low degree of ionization. Such discharges are widely used in various fields of science and technology. Studies of these discharges and the use of such discharges frequently results in the need to use similarity relations for differently scaled discharges based on dimensional transformations. Despite the fact that there are a number of articles specially devoted to similarity relations in discharges, and that they are discussed in the most authoritative monographs,[1–4] one can even now encounter contradictory conclusions about discharge similarities. This situation prompted us to write this article, in which we attempt to eliminate the contradictions and clarify this issue.

2. Let us examine a steady-state low-temperature discharge with a constant chemical composition. We will proceed from the most general definition of the similarity of physical phenomena:[5] discharges are called similar if the physical quantities $G(\mathbf{r}, t)$ at similar spatial and temporal points linked by the linear transformation

$$\mathbf{r}' = s\mathbf{r}, \quad t' = st, \tag{1}$$

are also linked by the linear transformation

$$G'(\mathbf{r}', t') = s^{\alpha[G]}G(\mathbf{r}, t), \tag{2}$$

where $\alpha[G]$ is called the similarity index.

In the most general form, a gas discharge is described by a system of kinetic Boltzmann equations and field equations. Thus, to determine the transformation laws for physical quantities that characterize the discharge, one should proceed from the requirement that the Boltzmann equations and the field equations be invariant.

We note that it follows from Eqs. (1) and (2) that $\alpha[\mathbf{v}] = 0$, that is, when the transformations in Eq. (1) are made at similar spatial and temporal points of the discharge the velocities of the particles are identical. This, in turn, leads to the conclusion that in similar discharges, the distribution functions normalized to unity for all sorts of particles, $f_k(\mathbf{v},\mathbf{r},t)$ ($k = \mathrm{e, i, n}$) are invariant. Therefore, instead of the definition of discharge similarity presented above, one can use the following definition: two discharges are called similar if under the transformation specified by Eq. (1), the velocity distribution functions normalized to unity for all sorts of particles are invariant. This is a more physical definition. However, the definition presented above is more general, because it contains not only the requirement of invariance for the velocity distribution functions of particles of similar discharges, but also an indication of the form of the transformation of other physical quantities, for example, the densities of various components of the discharge plasma. We note that the definition for the similarity of discharges having invariant distribution functions was proposed by Margenau[6] in 1948. But his definition was limited only by the requirement that the distribution function of electrons, and not of all particles, be invariant.

Finally, we note that for complete similarity of discharges it is insufficient to speak of the similarity of quantities in bulk; the boundary conditions for the distribution functions of particles and the components of the electromagnetic field should also be similar (and in a noncontradictory manner).

3. Let us define the quantities $\alpha[G]$ proceeding from the invariance of kinetic Boltzmann equations for various sorts of particles

$$\frac{\partial f_k}{\partial t} + \mathbf{v}\frac{\partial f_k}{\partial \mathbf{r}} + \frac{e_k}{m_k}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right\}\frac{\partial f_k}{\partial \mathbf{v}} = I_k(\mathbf{r}, t). \tag{3}$$

All notations are the generally accepted ones. Since the operators $\partial/\partial t$, $\partial/\partial\mathbf{r}$, and $\partial/\partial\mathbf{v}$ have the following indices in the transformations specified by Eq. (1)

$$\alpha\left[\frac{\partial}{\partial t}\right] = \alpha\left[\frac{\partial}{\partial \mathbf{r}}\right] = -1, \quad \alpha\left[\frac{\partial}{\partial \mathbf{v}}\right] = 0, \tag{4}$$

then, for the left side of Eq. (3) to be invariant, the similarity indices for the fields $\mathbf{E}$ and $\mathbf{B}$ must be equal to

$$\alpha[\mathbf{E}] = \alpha[\mathbf{B}] = -1. \tag{5}$$

The requirement of invariance of the right side of Eq. (3) leads to the determination of the similarity indices for particle densities. We shall assume that in a discharge it is sufficient to limit ourselves to only the pair collisions of particles; thus,

$$I_k(\mathbf{r}, t) = \sum_{k'} I_{kk'}(n_k, n_{k'}), \tag{6}$$

and $I_{kk'}(n_k, n_{k'}) \sim n_k n_{k'}$. Then, for the right side of Eq. (3) to be invariant, the following equations must be satisfied

$$\alpha[n_n] = -1, \quad \alpha[n_e] = \alpha[n_i] = -1. \tag{7}$$

As a result of the invariance of $\mathbf{v}$, the current density $\mathbf{j} = \Sigma_k e_k n_k \mathbf{v}_k$ and the charge density $\rho = \Sigma_k e_k n_k$ have the similarity indices

 0038-5670/91/090827-03$01.00

$$\alpha[\mathbf{j}] = -1, \quad \alpha[\rho] = -1. \tag{8}$$

All that has been said up to this point imposes no limitations on the degree of ionization of the gas in the discharge. Hereafter we shall focus attention on a discharge with a low degree of gas ionization, which makes it possible to discard several of the limitations above that proceed from the requirement of invariance of the kinetic Boltzmann equations for the particles.

4. First of all, however, let us discuss the conclusions reached when one requires the invariance under the transformations in Eq. (1) of the field equations

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{div} \mathbf{B} = 0, \tag{9}$$

$$\operatorname{curl} \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \tag{10}$$

$$\operatorname{div} \mathbf{E} = 4\pi\rho. \tag{11}$$

Equation (9) shows that fields $\mathbf{E}$ and $\mathbf{B}$ should transform in the same way. Their similarity index is not defined by Eq. (9); thus, we are justified in assuming that they transform according to Eq. (5). It follows from the requirement of invariance of Eq. (10) that

$$\alpha[\mathbf{j}] = -2. \tag{12}$$

Finally, the requirement of invariance of the last equation (the Poisson equation) leads to the following equations for the charge density ρ, the electron density n_e, and the ion density n_i:

$$\alpha[\rho] = -2, \quad \alpha[n_e] = \alpha[n_i] = -2. \tag{13}$$

The similarity index of the density of the neutral component n_n is not determined by the requirement of invariance of the field equations; thus, we are justified in assuming that Eq. (7) is satisfied for this density.

We note that the similarity equations (12) and (13) have also been obtained without any limitations on the degree of ionization of the gas in the discharge.

5. By comparing Eqs. (7) and (8), and (12) and (13), we can see that in the general case it is impossible for the Boltzmann equations and the field equations to be simultaneously invariant under the transformations in Eq. (1). Naturally, the issue arises of determining the conditions for these equations not to be mutually contradictory and under which one can speak of complete similarity in a discharge. It turns out that this is possible in two limiting cases.

a) Let the gas be so weakly ionized that one can ignore the collisions of charged particles with each other, compared to their collisions with neutral particles, that is, in the Boltzmann equations for electrons and ions, $I_{ee}, I_{ei}, I_{ii} \ll I_{en}, I_{in}$. The requirement that these equations be invariant imposes no limitations on the expression for the transformation of n_e and n_i, and, consequently, $\mathbf{j}$ and ρ as well. Thus, we are justified in assuming the expressions in Eqs. (12) and (13) for them.

Strictly speaking, however, in order to discard the transformations in Eqs. (7) and (8) for n_e and n_i (and thus, for $\mathbf{j}$ and ρ), it is insufficient to analyze only the Boltzmann equations for electrons and ions. It is also necessary for the equation for the neutral particles to be invariant. For the transformations of n_e and n_i to be indefinite one must require that the collisions of neutral particles with charged particles be ignored, and this means ignoring heating of the neutral component in the discharge by the charged component. This, in turn, is possible if the specific heat of the neutral component of the plasma is much greater than the specific heat of the charged component, $p = n_n T_n \gg (n_e T_e + n_i T_i)$. Here, T_n is defined as the ambient temperature.

Thus, we have come to the classical theory of similarity.[1-4,7] Since the equations which proceed from the requirements of invariance of the field equations are the determining ones, the similarity of discharges under these conditions will be called F-similarity. The following quantities are simultaneously invariant in a discharge in the case of F-similarity: E/n_n, n_e/n_n^2, n_i/n_n^2, j/n_n^2.

b) Let us now discuss the other limiting case of a discharge with a relatively high density of charged particles and a high degree of ionization. The collision integrals of the charged particles with other charged particles are not considered small in comparison with those for the collisions of charged particles with neutral particles, that is, $I_{ee}, I_{ei}, I_{ii} \sim I_{en}, I_{in}$. Moreover, heating of the gas in the discharge may be significant. We also assume that the charge density and current density in the discharge are small. Thus, in the field equations (10) and (11) we can set

$$\mathbf{j} = 0, \quad \rho = 0. \tag{14}$$

This approximation is satisfied in a discharge if the magnetic pressure of the current field is negligibly small compared to the gas kinetic pressure of the charged component of the plasma, $B^2 \sim (jr_0)^2 \ll n_e T_e + n_i T_i$, where r_0 is the transverse size of the discharge, and the size of the discharge is, in turn, much larger then the Debye radius of the plasma:

$$r_0^2 \gg \frac{T_e}{4\pi e^2 n_e} + \frac{T_i}{4\pi e^2 n_i}$$

(or, the electrostatic pressure of the charge separation field is negligibly small compared to the pressure of the charged component of the plasma $E^2 \sim (n_e - n_i)^2 e^2 r_0^2 \ll (n_e T_e + n_i T_i)$.

Under these limitations, the invariance of the field equations does not impose the limitations of Eqs. (12) and (13), and we can consider Eqs. (7) and (8) to be satisfied in the discharge. These equations are obtained from the requirement that the Boltzmann equations be invariant. This type of similarity of dense discharges is called B-similarity. It was examined for the first time in Ref. 5, and was independently repeated in Refs. 8 and 9. In this case, the following quantities are invariant in the discharge: E/n_n, n_e/n_n, n_i/n_n, and j/n_n.

6. Let us illustrate the similarity equations formulated above using the example of a glow discharge burning in a cylindrical tube of radius r_0. The pressure of the neutral gas $p = n_n T_n$ is homogeneous in this discharge, and the gas temperature T_n is invariant according both to B-similarity and F-similarity. Then $\alpha[p] = -1$. We find the general similarity invariants

$$pr_0, \quad \frac{E}{p}, \quad U, \tag{15}$$

where U is the combustion potential of the discharge.

The invariance of j/p^2 and J (total current) is satisfied in F-similarity, and of j/p and JE is satisfied in B-similarity. The question arises of which of these occurs in a glow discharge. This depends on the degree of ionization of the gas. For low degrees of ionization, when there is no heating of the neutral component, and in accordance with the above, F-similarity is applicable to the whole discharge, and this is actually confirmed by experiment. In the case of a substantial degree of gas ionization and of heating of the neutral component, B-similarity will apply, not for the entire glow discharge, but only for its quasi-neutral column, excluding the layers of space charge at the electrode. In this case the role of the layers is small, and experiment confirms B-similarity of such discharges.

It is no accident that we have presented here the example of a glow discharge. A large number of publications has experimentally confirmed that the similarity equations hold true when applied to a glow discharge. However, these equations can be very confidently applied to a dark discharge and to low-pressure arcs as well, although there are substantially fewer experimental studies of these phenomena. Nonetheless, the similarity invariants are very useful in experiments and in calculations. Usually, instead of calculating all the quantities that characterize a discharge, only such invariants as E/p, pr_0, $\mathbf{j}/p$ or $\mathbf{j}/p^2$ are calculated (or measured), and this makes investigations much easier.

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Translated by C. Gallant