# The self-avoiding walk problem 

V.I. Alkhimov

## N. K. Krupskaya Moscow Regional Education Institute

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Different approaches to the self-avoiding walk problem are reviewed. The problem first arose in the statistical physics of linear polymers in connection with the evaluation of the average size of a polymer. The probability distribution density $\boldsymbol{W}_{N}(\mathbf{R})$ for the vector $\mathbf{R}$ connecting the end-points of an $N$-step self-avoiding walk is the main quantity in this problem. The equation for $W_{N}(\mathbf{R})$ seems to be invariant under the scaling transformation group. This means that the renormalization group method can be used to determine the asymptotic form of $W_{N}(\mathbf{R})$ as $N \rightarrow \infty$.

## 1. INTRODUCTION

The self-avoiding walk (SAW) problem first arose in the statistical physics of linear polymers in connection with the evaluation of the average spatial dimensions of a polymer. Linear polymers are, of course, exceedingly long chain molecules (macromolecules) in which the number of links can reach values of the order of $10^{4}-10^{5}$. They include synthetic polymers and biopolymers. Each link of a linear polymer has a degree of orientational freedom relative to the positions of neighboring links, which is responsible for the flexibility of the polymer chain. The number of degrees of freedom in such macromolecules is very large, i.e., of the order of the number of monomers in the chain. This enables us to look upon a macromolecule as a macroscopic system, and to use statistical methods to determine the average values of variables such as, for example, the spatial dimensions of a macromolecule. There is abundant experimental evidence showing that the spatial distribution of monomers in a macromolecule relative to its center of mass has a significant influence on the macroscopic properties of polymer solutions such as viscosity, diffusion, light scattering, and so on. The most important measurable quantities that characterize the dimensions of a macromolecule are the mean square end-to-end distance and the mean square of its radius of inertia.

The so-called excluded volume effect has a major influence on the monomer distribution mentioned above. Its essence is that a given volume element cannot contain more than one monomer at any given time. The excluded volume problem for a polymer chain is concerned with the effect of this volume phenomenon on the monomer density distribution in a macromolecule and, ultimately, on its dimensions. The excluded volume effect is a long-range order phenomenon because it is due to the interaction between monomers with very different position numbers on the polymer chain. There is, however, a short-range order effect due to the interaction between monomers that are neighbors on the chain sequence. The spatial configuration of a macromolecule is nevertheless largely determined by the long-range order effect.

The interaction between monomers in a macromolecule can be taken into account by methods that are essentially similar to those used in the theory of real gases. The behavior of the gas molecules is, however, very different from the behavior of monomers in a macromolecule because of the chain structure of the latter. Actually, the probability of a
collision between two monomers depends not only on the separation between them along the chain, but also on their joint position on the chain. In other words, this probability depends on the monomer position numbers on the chain sequence. Thus, gas molecules are indistinguishable, but the monomers in a macromolecule are not. In addition, the structure of a linear polymer chain enables us to exploit the ideas and methods of the theory of random walks of Brownian particles. It is precisely the chain character of the path of a random particle and of the linear polymer that points to the analogy between these systems. The excluded volume effect must, however, be taken into account in the theory of polymers. In terms of random-walk theory, this effect signifies that the particle must not cross its own trajectory during a random walk. The excluded volume effect in linear polymers is therefore sometimes referred to as the problem of the self-avoiding walk (SAW). Under these circumstances, we have a non-Markov process because the particle must avoid portions of space that it has already visited, i.e., it must remember its entire past. This memory phenomenon attaches exclusive properties to the problem that have no analogs among physical problems. The main quantity in the SAW problem is usually taken to be the probability density $W_{N}(\mathbf{R})$ of the vector $\mathbf{R}$ joining the end-points of the $N$-step random walk $(N \gg 1)$. Once the function $W_{N}(\mathbf{R})$ has been defined, we can immediately deduce the mean square end-toend distance $\left\langle R^{2}\right\rangle_{N}$, which is an important quantity in the theory. The classical result for $\left\langle R^{2}\right\rangle_{N}$ was deduced by Flory ${ }^{1}$ from the thermodynamic theory of polymer solutions and takes the form

$$
\begin{equation*}
\left\langle R^{2}\right\rangle_{N} \sim \text { const } \cdot\left(v_{0} / l^{3}\right)^{2 / 5} N^{6 / 5} l^{2} \tag{1.1}
\end{equation*}
$$

where $v_{0}$ is the excluded volume of a monomer and $l$ is the length of the link joining two neighboring monomers. The generalization of the Flory formula (1.1) to arbitrary dimensionality $d$ was first made by Fisher ${ }^{2}$ in the form

$$
\begin{equation*}
\left\langle R^{2}\right\rangle_{N} \sim \text { const } \cdot\left(v_{0} / l^{d}\right)^{2 y / 3} N^{2 \nu} l^{2} \tag{1.2}
\end{equation*}
$$

where the critical index $v$ is given by

$$
\begin{align*}
v & =\frac{3}{d+2}, & & \text { if } \quad d<4, \\
& =\frac{1}{2}, & & \text { if } \quad d>4 . \tag{1.3}
\end{align*}
$$

It is clear from (1.3) that the dimensionality $d=4$ is special
in the sense that, when $d\rangle 4$, the mean square $\left\langle R^{2}\right\rangle_{N}$ depends asymptotically on $N$ in the same way as in the absence of the excluded volume, i.e.,

$$
\begin{equation*}
\left\langle R^{2}\right\rangle_{N} \sim O\left(N l^{2}\right) \tag{1.4}
\end{equation*}
$$

Both numerous experimental facts and the results of numerical computer experiments are in agreement with (1.2). A simple derivation of this formula is presented below.

## 2. PHENOMENOLOGICAL APPROACH

Consider a linear polymer chain in $d$-dimensional Euclidean space $\mathrm{R}^{d}$. The chain consists of $N+1$ monomers that are freely joined together by $N$ links of length $l$ each. We shall number these monomers and links in the order in which they appear in the chain, taking the zeroth monomer at the origin of the coordinate system in $\mathrm{R}^{d}$. It is convenient to take the links to be the vectors $\mathbf{l}_{i}\left(\left|\mathbf{l}_{i}\right|=l, i=1,2, \ldots, N\right)$, assuming that $l_{i}$ is drawn from the $i$-lth monomer to the $i$ th monomer. The interaction between any two monomers joined by more than one link will be described with the help of the shortrange repulsion potential $U(\mathbf{R})$ of range $r_{0}$, where $r_{0}<l$. The quantity

$$
v_{0}=\int\left(1-e^{-U(R) / k T}\right) \mathrm{d}^{d} R
$$

is then the excluded volume of the monomer. In the simplest case, the monomer can be looked upon as a perfectly hard sphere of diameter $r_{0}$.

Next, we take

$$
U_{N}=\sum_{1 \leq i<j \leq N} U\left(\mathrm{~L}_{i j}\right)
$$

to be the potential energy of the polymer chain, where the vector

$$
\mathrm{L}_{i j}=\sum_{i \leq m \leq j} \mathbf{1}_{m}
$$

joins the geometric centers of the $i-1$ th and $j$ th monomers. The mean square distance between the $i$-1th and $j$ th monomers is then given by

$$
\begin{equation*}
\left\langle L_{i j}^{2}\right\rangle=(j-i+1) l^{2}+2 l^{2} \sum_{i \leq m<n \leq j}\left\langle\cos \theta_{m n}\right\rangle \tag{2.1}
\end{equation*}
$$

where $\theta_{m n}$ is the angle between $\mathbf{I}_{m}$ and $\mathbf{I}_{n}$, defined by

$$
\begin{equation*}
\left\langle\cos \theta_{m n}\right\rangle=Q_{N}^{-1} \int\left(\mathrm{e}_{m} \mathrm{e}_{n}\right) e^{-U_{N} / k T} \prod_{i=1}^{N} \mathrm{~d} \Omega_{i}, \tag{2.2}
\end{equation*}
$$

in which $Q_{N}$ is the normalizing factor given by

$$
Q_{N}=\int e^{-U_{N} / k T} \prod_{i=1}^{N} \mathrm{~d} \Omega_{i}
$$

$\mathbf{e}_{m} \cdot \mathbf{e}_{n}$ is the scalar product of the vectors $\mathbf{e}_{m}=\mathbf{l}_{m} / l$ and $\mathbf{e}_{n}=\mathbf{l}_{n} / l, d \Omega=\omega^{-1} d \omega, d \omega$ is the area element on a unit sphere in $\mathrm{R}^{d}, \omega=2 \pi^{d / 2} / \Gamma(d / 2)$ is the total surface area of the sphere, and $\Gamma(x)$ is the Euler gamma function. The quantity $\left\langle\cos \theta_{m n}\right\rangle$ can be looked upon as a measure of the correlation between the directions of the vectors $I_{m}$ and $I_{n}$ due to the excluded volume effect. It is clear that, when $U(\mathbf{R}) \equiv 0$, so that $v_{0}=0$, we have $\left\langle\cos \theta_{m n}\right\rangle=0$. It is natural to expect that $\left\langle\cos \theta_{m n}\right\rangle>0$, so that $v_{0}>0$, i.e., volume effects produce an increase in the mean square distance be-
tween monomers in a polymer chain. For example, it is clear that, when $d=1$, we have $\left\langle\cos \theta_{m n}\right\rangle=1$ and, according to (2.1), $\left\langle L_{i j}^{2}\right\rangle=(j-i+1)^{2} l^{2}$, i.e., the polymer chain is fully extended. When $d \geqslant 2$, the polymer chain may be looked upon, on average, as a ball whose spatial dimensions are characterized by a single quantity, namely, the mean square end-to-end distance $\left\langle L_{1 N}^{2}\right\rangle$.

We shall now use (2.1) to construct the difference relation

$$
\begin{equation*}
\left\langle L_{i j}^{2}\right\rangle-\left\langle L_{i+1 j}^{2}\right\rangle-\left\langle L_{i j-1}^{2}\right\rangle+\left\langle L_{i+1 j-1}^{2}\right\rangle=2 l^{2}\left\langle\cos \theta_{i j}\right\rangle \tag{2.3}
\end{equation*}
$$

and examine it for values of $i$ and $j(1 \leqslant i \leqslant j \leqslant N)$ for which $t \equiv j-i+1$ is much greater than unity. However, the values of $\left\langle\cos \theta_{i j}\right\rangle$ should then be much less than unity, and should decrease with increasing $t$, since otherwise (2.1) would require that $\left\langle L_{1 N}^{2}\right\rangle \sim N^{2} l^{2}$, i.e., the polymer chain is fully extended. Hence, we conclude that the dimensionless quantity $\left\langle\cos \theta_{i j}\right\rangle$ should depend on some dimensionless small parameter that depends, at most, on $v_{0} / l^{d}$ and $t$. When $t=N$, the small parameter can be naturally taken to be the sum of the excluded volumes of all the $N$ monomers, divided by the mean volume occupied by the polymer chain, i.e.,

$$
\begin{equation*}
\varepsilon_{N}=v_{1} N / 2\left\langle L_{1 N}^{2}\right\rangle^{d / 2} \tag{2.4}
\end{equation*}
$$

where $v_{1}=c_{d} v_{0}$ and the coefficient $c_{d}=(d / 2 \pi)^{d / 2}$ that characterizes the spatial shape of the ball can be found from perturbation theory for $\left\langle R^{2}\right\rangle_{N} \equiv\left\langle L_{i N}^{2}\right\rangle$ :

$$
\left\langle R^{2}\right\rangle_{N} / N l^{2}=1+4\left(d / 2 \pi l^{2}\right)^{d / 2} v_{0} N^{(4-d) / 2} /(4-d)(6-d)+\ldots .
$$

When $t<N$, the correlation between the directions of the vectors $l_{i}$ and $l_{j}$ is due to not only 'internal' monomers (with numbers between $i-1$ and $j$ ) but also 'external' monomers (with numbers between 0 and $i-2$ and between $j+1$ and $N$ ). The influence of 'external' monomers on the correlation between vectors $l_{i}$ and $l_{j}$ is due to both the interaction with the 'internal' monomers and their interaction with one another. However, as $t \rightarrow N$, the number of 'external' monomers decreases and their effect on the correlation between $l_{i}$ and $l_{j}$ can be neglected altogether for values of $t$ close enough to $N$. We therefore conclude that, when $t \sim N$, we may assume that, to the first order in the small parameter

$$
\begin{equation*}
\varepsilon_{t}=\frac{v_{1} t}{2\left(L_{i j}^{2}\right)^{d / 2}} \tag{2.5}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left\langle\cos \theta_{i j}\right\rangle=\varepsilon_{t} \tag{2.6}
\end{equation*}
$$

If we now approximately replace the difference relation (2.3) with the corresponding differential equation, we obtain the following equation for $y(t) \approx\left\langle L_{i j}^{2}\right\rangle / l^{2}$ :

$$
\begin{equation*}
\frac{\mathrm{d}^{2} y}{\mathrm{~d} t^{2}}=\frac{v_{1}}{l^{d}} \frac{t}{y^{d / 2}} \tag{2.7}
\end{equation*}
$$

The asymptotic solution of this for $t \rightarrow \infty$ is unique. When $d<4$, this solution has the following form: ${ }^{3}$

$$
\begin{equation*}
y(t) \sim\left[\frac{(d+2)^{2} v_{1}}{6(4-d) l^{d}}\right]^{2 v / 3} t^{2 v} \tag{2.8}
\end{equation*}
$$

Substituting (2.8) in (2.5) and (2.6), we obtain
$\left\langle\cos \theta_{i j}\right\rangle \sim \frac{1}{2}\left[\frac{6(4-d) v_{1}}{(d+2)^{2} l^{d}}\right]^{\nu d / 3} t^{-2 v(d-1) / 3}$.
Finally when $d>4$, the asymptotic solution of (2.7) for $t \rightarrow \infty$ is $y(t) \sim O(t)$, so that
$\left\langle\cos \theta_{i j}\right\rangle \sim O\left(t^{-(d-2) / 2}\right)$.
The excluded volume effect is thus seen to lead to a power-type expression for the reduction in the correlation between the directions of two links $l_{i}$ and $l_{j}$ with increasing $t=j-i+1$, namely, $\left\langle\cos \theta_{i j}\right\rangle \sim O\left(t^{-\alpha}\right)$ where $\alpha<1$ for $d<4$ and $\alpha>1$ for $d>4$.

## 3. CONNECTION WITH THE THEORY OF CRITICAL PHENOMENA

The important advances in the theory of phase transitions achieved in the early 1970s led to significant advances in our understanding of the nature of critical phenomena, which in turn had an appreciable influence on the development of the statistical physics of polymers. ${ }^{4,9}$ Wilson ${ }^{6,7}$ developed the renormalization group method as a means of calculating the critical exponents that determine the degree of singularity of different thermodynamic variables at the phase transition point. The conceptual basis for the Wilson method was provided by the so-called universality hypothesis put forward by Kadanoff. ${ }^{8}$ According to this, critical exponents should be insensitive to the detailed behavior of the interaction potential, and should be determined mainly by large-scale properties such as the dimensionality of the system and the symmetry of its Hamiltonian (see also Ref. 9). The central point of Wilson's theory is the assumption of a gradual reduction in the correlation function of the system, e.g., of a magnet, with increasing separation between its elementary magnetic moments at the phase transition point (this is the similarity or scaling hypothesis). Wilson and Fisher ${ }^{10}$ put forward the so-called $\varepsilon$-expansion as a means of evaluating the critical exponents, where $\varepsilon=4-d$ is the deviation of the dimensionality $d$ of space from $d=4$. The dimensionality $d=4$ occupies a special position in this theory in the sense that perturbation theory can be used for the evaluation of the critical exponents when $d>4$. The $\varepsilon$-expansion procedure can be used to determine the critical exponents in the form of power series in $\varepsilon$. Calculations show that the coefficients in these series initially decrease, but then begin to increase rapidly, which seems to suggest that the $\varepsilon$ series are asymptotic in character. Thus, the renormalization group method is, so far, effective only for dimensionalities near $d=4$, although only values of $d$ that are integers are physically meaningful.

The Wilson method was first used by de Gennes ${ }^{11}$ in the SAW problem. Essentially, the de Gennes approach consists of establishing an analogy between the behavior of an $n$-component magnet near the phase transition point and an $N$-step SAW of a particle where $N$ tends to infinity. The main characteristic of the magnet, i.e., its correlation function, is then determined by the standard diagram technique based on the expansion of the Gibbs exponential into an infinite series in powers of the interaction Hamiltonian (usually in the form used in the Ginzburg-Landau model), followed by averaging of each term in the series over the Gibbs exponential with the unperturbed Hamiltonian. ${ }^{12}$ The analog of the correla-
tion function of a magnet in the SAW problem is the quantity obtained by applying the Laplace transformation in $N$ to the probability density $W_{N}(\mathbf{R})$. However, the diagrams representing the correlation function of the magnet include diagrams that contain closed loops coupled to the remaining portions of the diagram by interaction lines alone. Moreover, all the diagrams that appear in the SAW problem should be simply-connected along the main line simply because the path of the particle undergoing the random walk is simply-connected. To exclude closed loops from the diagrams for the $n$-component magnet, and thus achieve a formal analogy between the two problems, we have to put $n=0$, which is one of the important results obtained by de Gennes. Hence it follows that, for example, to determine the exponent in the asymptotic formula $\left\langle R^{2}\right\rangle_{N} \sim N^{2 v} l^{2}$ as $N \rightarrow \infty$ and $\varepsilon \ll 1$, we can use the expression for $v$ obtained by the Wilson method ${ }^{12}$ and then allow $n$ to tend to zero. The result is

$$
\begin{equation*}
2 v=1+c_{1} \varepsilon+c_{2} \varepsilon^{2}+c_{3} \varepsilon^{3}+\ldots \tag{3.1}
\end{equation*}
$$

where $c_{1}=1 / 8, c_{2}=15 / 256, c_{3}=-0.0445 \ldots, \ldots$.
The SAW problem can be treated by Wilson's renormalization group method by applying the renormalization transformation directly in the form of the so-called decimation procedure ${ }^{13}$ (see also Refs. 4 and 14) in which a polymer chain consisting of $N$ links is divided into $N_{1}=N / g$ identical subunits containing $g$ links joined consecutively. Each such subunit is looked upon as a new link with new length $l_{1}$ and new excluded volume parameter $v_{1}$. To determine $l_{1}$, we must take into account the interaction between all the monomers belonging to a given subunit, and to find $v_{1}$, we must include both the interaction between two subunits that are far enough from one another and all the interactions within each of them. The result is a polymer chain with the new parameters $N_{1}, l_{1}, v_{1}$ that depend on the original values of $N, l$, and $v$. This procedure of arranging the units into subunits is then repeated several times and the renormalized quantities $l_{k}$ and $v_{k}$ are determined for each $k$ th stage. The relationship between successive steps in this process is described by recurrence relations for $l_{k}$ and $v_{k}$ or for the dimensionless quantity $u_{k}=v_{k} / l_{k}^{d}$ where $u_{k}=f\left(u_{k-1}\right)$. As $k \rightarrow \infty$, the sequence of numbers $u_{k}$ converges to a limit $u^{*}$, which is a root of the equation $u^{*}=f\left(u^{*}\right)$. This approach is a reflection of the universality hypothesis. It has been implemented quantitatively only for $\varepsilon \ll 1$, and the result for $v$ in the first order in $\varepsilon$ agrees with the expression given by (3.1). The subject discussed in this Section is examined in greater detail in Refs. 4 and 14.

The above iteration process is usually related to Wilson's renormalization group that is actually a semigroup because the reverse operations cannot be defined for the above iterations. Polymer statistics can also be investigated directly by means of renormalizing transformations associated with a true group that is also called renormalization group. The latter was first discovered in quantum field theory by Stueckelberg and Peterman ${ }^{15}$ and later by Gell-Mann and Low. ${ }^{16}$ The renormalization group method was subsequently developed by Bogolyubov and Shirkov. ${ }^{17,18}$ To distinguish this (true) group of renormalizing transformations from the Wilson renormalization group, the former is usually referred to as the field renormalization group. It is impor-
tant to note that the field renormalization group is a continuous group and can therefore be described by the Lie differential equations that are exceedingly useful in practice. It was found subsequently that the field renormalization group method can also be incorporated in different schemes for the investigation of the SAW problem. The next Section presents a brief account of the continuum approach to the SAW problem.

## 4. CONTINUUM MODEL AND THE RENORMALIZATION GROUP

The excluded volume problem was first formulated in terms of the functional integration method in the wellknown paper by Edwards, ${ }^{19}$ which led to the intensive development of this approach to the problem, and gave rise to an extensive literature (see, for example, Refs. 20-35).

In the continuum model of a polymer chain, the number of links is allowed to tend to infinity and the effective length of each link $\Delta s_{i}(i=1,2, \ldots, N)$ is allowed to tend to zero in such a way that the total length $L$ of the chain remains finite, i.e.,

$$
L=\lim \sum_{i=1}^{N} \Delta s_{i}<\infty, \quad \max \Delta s_{i} \rightarrow 0, \quad N \rightarrow \infty .
$$

The distribution density for the vector $\mathbf{R}_{i}=\mathbf{r}_{i}-\mathbf{r}_{i-1}$ joining two neighboring monomers along the chain at points $\mathbf{r}_{i-1}$ and $\mathbf{r}_{i}$ is then given by the Gaussian function

$$
\begin{equation*}
\tau\left(\mathrm{R}_{i}\right)=\left(3 / 2 \pi l \Delta s_{i}\right)^{3 / 2} \exp \left(-3 R_{i}^{2} / 2 l \Delta s_{i}\right) \tag{4.1}
\end{equation*}
$$

where $l$ is the so-called Kuhn length defined so that the mean square end-to-end distance of the ideal polymer chain (i.e., in the absence of the excluded volume) is equal to $l L$. If we now use $U_{i j}\left(\mathbf{r}_{i}-\mathbf{r}_{j}\right)$ to represent the volume interaction potential between the $i$ th and $j$ th monomers, the distribution of the vector $\mathbf{R}$ joining the ends of the chain can be written in the form

$$
\begin{align*}
W_{N}(\mathbf{R})=Z_{N}^{-1} \int & \exp \left(-\frac{1}{2 k T} \sum_{i \neq j} \sum_{i j}\left(\mathbf{r}_{i}-\mathrm{r}_{j}\right)\right) \\
& \times \delta\left(\mathbf{R}-\mathbf{r}_{N}+\mathbf{r}_{0}\right) \mathrm{d}^{3} r_{0} \prod_{i=1}^{N} \tau\left(\mathbf{R}_{i}\right) \mathrm{d}^{3} r_{i} \tag{4.2}
\end{align*}
$$

where

$$
\begin{equation*}
z_{N}=\int \exp \left(-\frac{1}{2 k T} \sum_{i \neq j} \sum_{i j}\left(\mathrm{r}_{i}-\mathrm{r}_{j}\right)\right) \mathrm{d}^{3} r_{0} \prod_{i=1}^{N} \tau\left(\mathbf{R}_{i}\right) \mathrm{d}^{3} r_{i} \tag{4.3}
\end{equation*}
$$

The next step is to proceed to the continuum model of the linear polymer by the method indicated above. As a result, any admissible spatial configuration of the polymer can be represented by the equation of a continuous curve $\mathbf{r}=\mathbf{r}(s)$ in which $s$ is the coordinate of a point on the curve that ranges from zero to $L$. The finite sums in (4.2) and (4.3) then become integrals

$$
\begin{aligned}
& \sum_{i=1}^{N} \frac{3}{2 l}\left(\frac{\mathrm{r}_{i}-\mathrm{r}_{i-1}}{\Delta s_{i}}\right)^{2} \Delta s_{i} \rightarrow \frac{3}{2 l} \int_{0}^{L}\left(\frac{\mathrm{dr}(s)}{\mathrm{d} s}\right)^{2} \mathrm{~d} s \\
& \frac{1}{2 k T} \sum_{i \neq j} \sum_{i j} U_{i j}\left(\mathrm{r}_{i}-\mathrm{r}_{j}\right) \rightarrow \frac{1}{2 k T l^{2}} \int_{0}^{L} \mathrm{~d} s \int_{0}^{L} \mathrm{~d} s^{\prime} U\left(\mathrm{r}(s)-\mathrm{r}\left(s^{\prime}\right)\right)
\end{aligned}
$$

where $U_{i j} / l^{2}$ can be interpreted as the interaction energy per unit link length between the $i$ th and $j$ th monomers. The next step is to replace the potential $U$ by the pseudopotential in accordance with the scheme

$$
(k T)^{-1} U\left(\mathrm{r}(s)-\mathrm{r}\left(s^{\prime}\right)\right) \rightarrow v_{0} \delta\left(\mathrm{r}(s)-\mathrm{r}\left(s^{\prime}\right)\right),
$$

in which $v_{0}$ is again the excluded volume parameter of the monomer. The condition $\left|s-s^{\prime}\right|>\lambda$, where $\lambda$ is of the order of the link length $l$, is then imposed in order to avoid the singularity when the integrals with respect to $s$ and $s^{\prime}$ are evaluated.

Finally, if we consider the continuum model of a polymer in $d$-dimensional space, and assume for simplicity that $l=1$, the distribution density for the vector $\mathbf{R}$ assumes the form

$$
\begin{equation*}
W(\mathbf{R}, L)=Z^{-1}(L) G(\mathbf{R}, L) \tag{4.4}
\end{equation*}
$$

where

$$
\begin{align*}
& Z(L)=\int G(\mathbf{R}, L) \mathrm{d}^{d} R,  \tag{4.5}\\
& G(\mathbf{R}, L)=\left[\int_{\mathrm{r}(0)=0}^{\mathrm{r}(L)=\mathrm{R}} D\left[\mathrm{r}(s) \operatorname{lexp}\left(-H_{0}(\mathrm{r})\right)\right]^{-1}\right. \\
& \times \int_{\mathrm{r}(0)=0}^{\mathrm{r}(L)=\mathrm{R}} D[\mathrm{r}(s) \exp (-H(\mathrm{r})),  \tag{4.6}\\
& H_{0}(\mathrm{r})=\frac{d}{2} \int_{0}^{L}\left(\frac{\mathrm{dr}(s)}{\mathrm{d} s}\right)^{2} \mathrm{~d} s \\
& H(\mathrm{r})=H_{0}(\mathrm{r})+\frac{v_{0}}{2} \int_{\left|s-s^{\prime}\right|>\lambda}^{L} \mathrm{~d} s \int^{L} \mathrm{~d} s^{\prime} \delta\left(\mathrm{r}(s)-\mathrm{r}\left(s^{\prime}\right)\right),
\end{align*}
$$

and $D[\mathbf{r}(s)]$ represents the measure on the configuration manifold of a continuous curve whose end-points lie at $\mathbf{R}(0)=0$ and $\mathbf{r}(L)=\mathbf{R}$, respectively.

It is readily seen from the definition of the above model that it ceases to be meaningful for distances of the order of $\lambda$. This forces us to proceed to the macroscopic description, i.e., to distances much greater than $\lambda$. A further scaling length parameter $\Lambda$ is therefore introduced and is such that $\lambda / \Lambda \ll 1$. This new length scale, which is not in general equal to the chain length $\widetilde{L}$, is not equal to the macroscopic chain length $L$ and enables us to employ the universality hypothesis in this model when $L \rightarrow \infty$.

For example, if we double $L$, the result will of course be a doubling of $\widetilde{L}$, which means that there is a linear relationship between $L$ and $\widetilde{L}$

$$
\begin{equation*}
\tilde{L}=X L \tag{4.7}
\end{equation*}
$$

where the factor $X$ depends on the excluded volume parameter $v_{0}$ and the ratio $\lambda / \Lambda$. On the new length scale, the quantity describing the excluded volume effect must reflect the collective character of the volume interactions over distances of the order of $\Lambda$. It follows that the macroscopic excluded volume parameter is now the renormalized quantity $v$ that is, of course, a function of $v_{0}$ and $\lambda / \Lambda$. However,
when the critical exponents are calculated, it is more convenient to use the quantities

$$
u_{0}=v_{0} \Lambda^{\varepsilon / 2}, \quad u=v \Lambda^{\varepsilon / 2}
$$

so that, in view of the foregoing, we can write

$$
\begin{equation*}
u=u\left(u_{0}, \lambda / \Lambda\right) \tag{4.8}
\end{equation*}
$$

Finally, let $G_{0}\left(\mathbf{R}, L, v_{0} ; \lambda\right)$ and $G(\mathbf{R}, \widetilde{L}, u ; \Lambda)$ represent the unrenormalized distribution densities in the microscopic and macroscopic approaches to the SAW problem, respectively. However, after normalization, each of the functions $G_{0}$ and $G$ must yield the same distribution density for the vector ${ }^{\prime} R$, so that it is obvious that these two functions must be proportional to one another:

$$
\begin{equation*}
G(\mathbf{R}, \tilde{L}, u ; \Lambda)=Y^{-1}(u, \lambda / \Lambda) G_{0}\left(\mathbf{R}, L, v_{0} ; \lambda\right) \tag{4.9}
\end{equation*}
$$

The relations given by (4.7)-(4.9) provide us with the basis for the application of the renormalization group method in the continuum model of a linear polymer chain.

Actually, if we rewrite (4.9) in the form

$$
\begin{align*}
& G_{0}\left(\mathbf{R}, L, v_{0} ; \lambda\right)=Y(u, \lambda / \Lambda) \\
& \quad \times G\left(\mathbf{R}, L X(u, \lambda / \Lambda), u\left(u_{0}, \lambda / \Lambda\right) ; \Lambda\right) \tag{4.10}
\end{align*}
$$

and recall that the left-hand side of this equation is independent of the parameter $\Lambda$, we immediately obtain the differential equation for the function $G(\mathbf{R}, \widetilde{L}, u ; \Lambda)$ :

$$
\begin{align*}
& \left(\Lambda \frac{\partial}{\partial \Lambda}+\beta(u) \frac{\partial}{\partial u}+\Lambda \frac{\partial \ln Y}{\partial \Lambda}+\right. \\
& \left.\quad+\Lambda \frac{\partial \ln X}{\partial \Lambda} \tilde{L} \frac{\partial}{\partial \tilde{L}}\right)_{L, v_{0}, \lambda} G(\mathbf{R}, \tilde{L}, u ; \Lambda)=0 \tag{4.11}
\end{align*}
$$

in which

$$
\begin{equation*}
\beta(u)=\left.\Lambda \frac{\partial u}{\partial \Lambda}\right|_{L, v_{0}, \lambda} \tag{4.12}
\end{equation*}
$$

represents the Gell-Mann-Low function. Equation (4.11) is the so-called renormalization group equation that contains information on the dependence of the required function $G$ on the macroscopic parameters of the model. If $u$ ceases to depend on $\Lambda$ as $\Lambda$ increases, i.e., the system exhibits asymptotic scale invariance, then we have a basis for the implementation of the scaling idea. The limiting value $u^{*}$ of $u$, which is called the 'fixed point', is found from the condition

$$
\begin{equation*}
\left.\beta(u)\right|_{u=u^{u}}=0 . \tag{4.13}
\end{equation*}
$$

The calculation of critical exponents becomes significantly simpler if we use the last equation in conjunction with (4.11). Actually, dimensional analysis of the solution of the renormalization group equation under the scaling regime leads to the following asymptotic behavior:

$$
\begin{equation*}
G\left(\mathbf{R}, \tilde{L}, u^{*} ; \Lambda\right) \sim \tilde{L}^{-v d-1} g\left(\mathbf{R} / \tilde{L}^{v}\right) \tag{4.14}
\end{equation*}
$$

in which the critical exponents $v$ and $\gamma$ are found from the formulas

$$
\begin{align*}
& \frac{2 v-1}{2 v}=\left.\Lambda \frac{\partial \ln X}{\partial \Lambda}\right|_{u=u^{\circ}}  \tag{4.15}\\
& \frac{\gamma-1}{2 v}=\left.\Lambda \frac{\partial \ln Y}{\partial \Lambda}\right|_{u=u^{\prime \prime}} \tag{4.16}
\end{align*}
$$

The determination of the values of $v$ and $\gamma$ and of the scaling
function $g\left(\mathbf{R} / \widetilde{L}^{v}\right)$ in (4.14) requires, in turn, a determination of $X, Y$, and $u$ that, so far, has been possible within the framework of the $\varepsilon$-expansion only for $\varepsilon \ll 1$. The result obtained by this method for $2 v$ agrees to within $\varepsilon^{2}$ with the expression given by (3.1) from which it follows that $\gamma \approx 2 \nu$. If we formally substitute $\varepsilon=1$ in (3.1), the sum of the first three terms of the series in (3.1) gives $2 v \approx 1.184$, but the inclusion of the fourth term produces a much poorer result, i.e., $2 v \approx 1.139$. The $\varepsilon$-expansion method used in the above approaches to the problem does not therefore enable us to extend our results beyond the small neighborhood of dimensionality $d=4$. A more detailed description of the functional integration metiod as applied to the SAW problem can be found in Freed's monograph. ${ }^{36}$

## 5. BASIC EQUATION

We shall now describe the approach based on the renormalization group method for the solution of the exact equation for the probability density $W_{N}(\mathbf{R})$.

Consider a spherical particle of diameter $r_{0}$, which begins a random walk at the origin of coordiates in $R^{d}$ in such a way that each individual step $l_{i}$ of its geometric center has a constant length $\left|l_{i}\right|=l(l \geqslant 1)$ and a random direction. In addition, after each step, the particle avoids all regions of space it had visited after all previous steps. It follows that $r_{0}<1$. We now use the Markov method ${ }^{39}$ to express the required probability density for the end-to-end vector $\mathbf{R}$ of the $N$-step path in the form

$$
\begin{equation*}
w_{N}(\mathrm{R})=Q_{N}^{-1} w_{N}(\mathrm{R}) \tag{5.1}
\end{equation*}
$$

where

$$
\begin{align*}
& w_{N}(\mathrm{R})=\int P_{1 N} \delta\left(\mathbf{R}-\sum_{i=1}^{N} \mathbf{1}_{i}\right) \prod_{i=1}^{N} \mathrm{~d} \Omega_{i} ;  \tag{5.2}\\
& P_{1 N}=\prod_{1 \leq i<j \leq N}\left(1+f_{i j}\right), \\
& f_{i j}=f\left(\left|\sum_{k=i}^{\prime} 1_{k}\right|\right), \begin{array}{rll}
f(r) & =-1, & \text { if } \quad r<r_{0}, \\
=0, & \text { if } & r>r_{0} ;
\end{array}
\end{align*}
$$

and the normalization factor $Q_{N}$ is given by

$$
\begin{equation*}
Q_{N}=\int P_{1 N} \prod_{i=1}^{N} \mathrm{~d} \Omega_{i} \tag{5.3}
\end{equation*}
$$

Next, we define the function $a(z, \vec{\rho})$ by the expression

$$
\begin{equation*}
a(z, \vec{\rho})=\sum_{N \geq 0} z^{N} \int e^{\overrightarrow{\vec{p}} \mathbf{k}_{w_{N}}(\mathrm{R}) \mathrm{d}^{d} R, ~ . ~} \tag{5.4}
\end{equation*}
$$

for which we have the equation ${ }^{38}$

$$
\begin{equation*}
a^{-1}(z, \vec{\rho})=a_{0}^{-1}(z, \vec{\rho})-b(z, \vec{\rho}) \tag{5.5}
\end{equation*}
$$

in which

$$
\begin{align*}
& a_{0}^{-1}(z, \vec{\rho})=1-z \Lambda_{s}(\rho l)  \tag{5.6}\\
& \Lambda_{s}(x)=\Gamma(1+s)(2 / x)^{s} J_{s}(x), \tag{5.7}
\end{align*}
$$

$s=(d-2) / 2, J_{s}(x)$ is the Bessel function, and $b(z, \vec{\rho})$ is an infinite series whose first few terms are given by

$$
\begin{aligned}
& b(z, \vec{\rho})=-\int v(\vec{x}) a(\vec{\rho}-\vec{x}) d^{d} x /(2 \pi)^{d} \\
& \quad+\int v(\vec{x}) v\left(\overrightarrow{x^{\prime}}\right) a(\vec{\rho}-\vec{x}) a\left(\vec{\rho}-\vec{x}-\overrightarrow{x^{\prime}}\right) a\left(\vec{\rho}-\overrightarrow{x^{\prime}}\right) \mathrm{d}^{d} x \mathrm{~d}^{d} x^{\prime} /(2 \pi)^{2 d}
\end{aligned}
$$

$-\int V(\vec{x}) \varphi\left(\overrightarrow{x^{\prime}}\right) \cup\left(x^{\prime \prime}\right)\left(a(\vec{\rho}-\vec{x}) a\left(\vec{\rho}-\vec{x}-\overrightarrow{x^{\prime}}\right) a\left(\vec{\rho}-\vec{x}-\overrightarrow{x^{\prime}}-\overrightarrow{x^{\prime}}\right)\right.$
$\times\left(a\left(\vec{\rho}-\overrightarrow{x^{\prime}}-\overrightarrow{x^{\prime}}\right) a\left(\vec{\rho}-\overrightarrow{x^{\prime}}\right)+\ldots+a(\vec{\rho}-\vec{x}) a\left(\vec{\rho}-\vec{x}-\overrightarrow{x^{\prime}}\right)\right.$
$\left.\times a\left(\vec{\rho}-\overrightarrow{x^{\prime}}\right) a\left(\vec{\rho}-\overrightarrow{x^{\prime}}-x^{\prime \prime}\right) a\left(\vec{\rho}-\overrightarrow{x^{\prime}}\right)\right) \mathrm{d}^{d} x \mathrm{~d}^{d} x^{\prime} \mathrm{d}^{d} x_{x^{\prime \prime}} /(2 \pi)^{3 d}+\ldots$,
in which

$$
\begin{equation*}
v(\vec{X})=-\int e^{j \vec{x} \pi} f(r) d^{d} r=v_{0} \Lambda_{1+5}\left(r_{0^{\chi}}\right), \tag{5.9}
\end{equation*}
$$

and the excluded volume parameter is given by

$$
\begin{equation*}
v_{0}=\left(\pi r_{0}^{2}\right)^{1+s} / \Gamma(2+s) \tag{5.10}
\end{equation*}
$$

The $z$ dependence of the function $a(\vec{\rho}-\vec{x})$ is not shown in (5.8) for the sake of simplicity. Equation (5.5) is closed in the required quantity $a(z, \vec{\rho})$ and its form is analogous to the well-known Dyson equation. It plays an important part in our problem and is therefore referred to as the basic equation. The series in (5.8) is conveniently represented graphically. This is done by associating each term of this series with a diagram consisting of a certain number of vertices and a set of solid and wavy lines joining these vertices. In addition, each diagram contains two further external solid lines, one of which is ingoing and the other outgoing. Each line is assigned a particular vector, i.e., 'momentum', which must be conserved at each vertex on which two solid and one wavy line converge. Finally, the quantity $a(z, \vec{\rho}-\vec{\kappa})$ is assigned to each internal solid line and $-v(\vec{x}) /(2 \pi)^{d}$ is assigned to each wavy line. The integration is then carried out over all the wavy-line vectors. When these rules are applied to (5.8), the result is as shown in Fig. 1.

Analysis of the individual terms in the series given by (5.8) shows that the problem associated with the divergence of the integrals does not arise here. Actually, if in all the integrals we transform to the configurational variables $\{\mathbf{r}\}$, we readily see that integration with respect to $\{r\}$ has the lower limit $|r|=r_{0}$, which is of course a reflection of the excluded volume effect, whereas at the upper limit $r \rightarrow \infty$ and the integrands tend exponentially to zero because of the finite number of steps executed by the particle in its random walk. However, we still have the problem of the convergence of the series as a whole. It can be shown that the series given by (5.8) converges absolutely, at least for $|z|<1$, for all real values of $\rho$.

Let us suppose now that we know the solution of (5.5). We can then use the Fourier transformation

$$
\begin{equation*}
w(z, \mathbf{R})=\int e^{-i \mathbb{R} \vec{\rho}_{a}}(z, \vec{\rho}) \mathrm{d}^{d} \rho /(2 \pi)^{d} \tag{5.11}
\end{equation*}
$$

to find the generating function

$$
\begin{equation*}
w(z, \mathbf{R}) \equiv \sum_{N \geq 0} z^{N} w_{N}(\mathbf{R}) \tag{5.12}
\end{equation*}
$$

and then use the inversion formula

$$
\begin{equation*}
w_{N}(\mathbf{R})=\frac{1}{2 \pi i} \oint \frac{d z}{z^{N+1}} w(z, \mathbf{R}) \tag{5.13}
\end{equation*}
$$

to determine $W_{N}(\mathbf{R})$. If we also recall that

$$
\begin{equation*}
Q_{N}=\int w_{N}(\mathbf{R}) \mathrm{d}^{d} R \tag{5.14}
\end{equation*}
$$

then the definition given by (5.1) gives the required probability density $W_{N}(\mathbf{R})$. We now apply (5.13) and (5.11) in succession to (5.14), and evaluate the integral over $\mathbf{R}$. The result is

$$
\begin{equation*}
Q_{N}=\frac{1}{2 \pi i} \oint \frac{\mathrm{~d} z}{z^{N+1}} a(z, 0) \tag{5.15}
\end{equation*}
$$

from which it follows that

$$
\begin{equation*}
a(z, 0)=Q(z) \equiv \sum_{N \geq 0} z^{N} Q_{N} \tag{5.16}
\end{equation*}
$$

Combining the last equation with (5.5), we obtain

$$
\begin{equation*}
Q^{-1}(z)=1-z-b(z, 0) \tag{5.17}
\end{equation*}
$$

If $z_{0}$ represents the singular point of $Q(z)$ closest to the origin, we have by definition

$$
\begin{equation*}
1-z_{0}-b\left(z_{0}, 0\right) \equiv 0 \tag{5.18}
\end{equation*}
$$

Hence we readily see that $z_{0}=z_{0}\left(v_{0} / l^{d}\right)$ and $z_{0}(0)=1$. The inclusion of the excluded volume effect thus leads to a shift of the singular point $z_{0}$ from its 'undisturbed' position $z_{0}(0)=1$. We know from the theory of analytic functions that the boundary of the circle of convergence of a power series crosses its singular point closest to the origin. ${ }^{39}$ Moreover, the asymptotic behavior of a function represented by a power series as the variable $z$ approaches the boundary of its circle of convergence can be related to the asymptotic behavior of the coefficients of this series when their order numbers tend to infinity. ${ }^{40}$ The importance of the latter statement lies in the fact that the next problem that we have to face is the determination of the asymptotic form of the probability density $W_{N}(\mathbf{R})$ for $N \rightarrow \infty$ and $R \gg l$. It will be convenient to transform from $z$ to the new variable $\zeta=z_{0} / z$ and introduce the following notation:

$$
\begin{aligned}
& A(\zeta, \rho)=\zeta^{-1} a(z, \rho) \\
& B(\zeta, \rho l, v ; A)=\zeta b(z, \rho) \\
& v=\zeta^{2} v_{0} / l^{d}
\end{aligned}
$$

In view of (5.18), we can now write (5.5) in the form


FIG. 1.

$A^{-1}=\zeta-1+z_{0}\left(1-\Lambda_{s}(\rho l)\right)+B(1,0, v ; A)-B(\zeta, \rho l, v ; A)$.

We now put $\rho_{1} \equiv i \xi(\operatorname{Re} \xi>0)$ to represent the root of the equation $A^{-1}(\zeta, \rho)=0$ that is closest to the origin $\rho=0$. Hence, putting $\rho=\rho_{1}$ in (5.19), we obtain the identity
$\zeta-1+z_{0}\left(1-\Lambda_{s}(i \xi l)\right)+B(1,0, v ; A)-B(\zeta, i \xi l, v ; A) \equiv 0$,
which relates the variables $\zeta$ and $\xi$. Finally, using the identity (5.20) in (5.19), we obtain the basic equation in the form

$$
\begin{equation*}
A^{-1}=A_{0}^{-1}(\xi, \rho)+B(\zeta, i \xi l, v ; A)-B(\zeta, \rho l, v ; A) \tag{5.21}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{0}^{-1}(\xi, \rho)=z_{0}\left(\Lambda_{s}(i \xi l)-\Lambda_{s}(\rho l)\right) \tag{5.22}
\end{equation*}
$$

Equation (5.21) and the identity given by (5.20) together constitute the basis for the above method of investigating SAWs. The method can be generalized to the case where the direction of each individual step in the random walk depends on the direction of the preceeding step and, moreover, the length of an individual step is not fixed, but is described by some generally speaking arbitrary distribution. ${ }^{38}$

We note one further important property of (5.21), i.e., the invariance of this equation under multiplicative transformations:

$$
\begin{align*}
& A \rightarrow A^{\prime}=\alpha A, \quad A_{0} \rightarrow A_{0}^{\prime}=\alpha A_{0},  \tag{5.23}\\
& v \rightarrow v^{\prime}=\alpha^{-2} v,
\end{align*}
$$

where $\alpha$ is a smoothly-varying nonzero parameter. ${ }^{41}$ It is readily seen that the transformations given by (5.23) form a continuous group that is usually referred to as the renormalization group (RG). To elucidate the physical significance of RG in this approach, we write the expression for the probability density in the form

$$
\begin{equation*}
W_{N}(\mathbf{R})=\oint_{\Gamma} \mathrm{d} \zeta \zeta^{N} \widetilde{W}(\zeta, \mathbf{R}) / \oint_{\Gamma} \mathrm{d} \zeta \zeta^{N} A(\zeta, 0) \tag{5.24}
\end{equation*}
$$

in accordance with (5.1), (5.13), and (5.14), where

$$
\begin{equation*}
\widetilde{w}(\zeta, \mathbf{R})=\int e^{-i \mathbb{R}_{\vec{\rho}}} A(\zeta, \rho) \mathrm{d}^{d} \rho /(2 \pi)^{d} \tag{5.25}
\end{equation*}
$$

and the closed contour of integration $\Gamma$ is chosen so that the origin is $\zeta=0$ and all the singular points of the integrand lie inside this contour. It follows from (5.24) and (5.25) that, to determine the asymptotic behavior of $W_{N}(\mathbf{R})$ for $N \rightarrow \infty$ and $R \gg l$, we must know the behavior of the function $A(\zeta, \rho)$ in a small neighborhood of the points $\zeta=1$ and $\rho=0$. We shall assume henceforth that the values of $\zeta$ and $\rho$ belong to the neighborhoods of these points. However, we then find that

$$
\begin{equation*}
\Lambda_{s}(x)=1-x^{2} / 2 d+O\left(x^{4}\right), \quad x-0 \tag{5.26}
\end{equation*}
$$

and (5.20) show that the values of $\xi$ belong to the neighborhood of the point $\xi=0$. Next, we can readily show with the help of (5.24), (5.25) and (5.19) that a change in the number of steps by a factor of $\alpha$, i.e., $N \rightarrow N^{\prime}=\alpha N$, is equivalent to a change in the step length $/$ in accordance with the equa-
tion $l^{\prime}=l / \sqrt{ } \alpha$ and to a change in the excluded volume parameter $v_{0}$ in accordance with the rule $v^{\prime}=\alpha^{s-1} v$. However, we must emphasize that this statement refers only to the asymptotic case ( $N \rightarrow \infty, r \gg l$ ) and can be written symbolically in the form

$$
W_{N}\left(R / l, v_{0} / l^{d}\right)=W_{a N}\left(\alpha^{1 / 2} R / l, \alpha^{s-1} v_{0} / l^{d}\right)
$$

Hence, we have the following asymptotic expression for the mean square end-to-end distance of the random walk:

$$
\left\langle R^{2}\right\rangle_{N} \sim N l^{2} \chi\left(N^{1-s} v_{0} / l^{d}\right)
$$

where the universal function $\chi(x)$ is to be determined. It is clear from the last formula that the dimensionality $d=4(s=1)$ is special insofar as the excluded volume effect is weak for $d>4(s>1)$ and we can use perturbation theory when we evaluate $\left\langle R^{2}\right\rangle_{N}$. It is shown in Ref. 42 that, when $d \geqslant 5$, the asymptotic form of the probability density $W_{N}(\mathbf{R})$ is Gaussian for $N \rightarrow \infty$.

### 5.1. Renormalization group equations

We now turn to (5.20) and (5.21), the first of which establishes the relationship between $\zeta$ and $\xi$ and the second defines the type of the singular point $\rho_{1} \equiv i \xi$ of the function $A(\zeta, \rho)$. It is clear that the solution of this problem must begin with the determination of the character of the singularity of $A(\zeta, \rho)$ at $\rho=\rho_{1}$. With this in view, we write the required function $A(\zeta, \rho)$ in (5.21) in the form

$$
\begin{equation*}
A=A_{0} G \tag{5.27}
\end{equation*}
$$

For the new unknown quantity $G$ we have the equation

$$
\begin{equation*}
G^{-1}=1+F(G) \tag{5.28}
\end{equation*}
$$

in which
$F(G)=A_{0}(\xi, \rho)\left(B\left(\zeta, i \xi l, v ; A_{0} G\right)-B\left(\zeta, \rho l, v ; A_{0} G\right)\right)$,
and the function $A_{0}(\xi, \rho)$ displays the following behavior for the small values of $\xi$ and $\rho$ that we are considering:

$$
\begin{equation*}
A_{0}(\xi, \rho)=2 d / z_{0}\left(\xi^{2}+\rho^{2}\right) l^{2} \tag{5.30}
\end{equation*}
$$

If we substitute $\rho=0$ in (5.28) and introduce the notation $G_{0}=\left.G\right|_{\rho=0}$, we can use the resulting equation for $G_{0}$ to rewrite (5.20) in the form

$$
\begin{equation*}
E^{-1}=G_{0}^{-1}+A_{0}(\xi, 0)\left(B\left(\zeta, 0, v ; A_{0} G\right)-B\left(1,0, v ; A_{0} G\right)\right), \tag{5.31}
\end{equation*}
$$

where

$$
\begin{equation*}
E^{-1}=(\zeta-1) A_{0}(\xi, 0) \tag{5.32}
\end{equation*}
$$

is proportional to the ratio $(\xi-1) / \xi^{2} l^{2}$ for small values of $\xi$. It is readily seen that (5.31) can be used for the direct evaluation of the critical exponent $v$ once the function $G$ has been determined.

Since $G$ is a dimensionless quantity, it may be looked upon as a function of dimensionless variables

$$
G=G\left(\frac{\xi^{2}}{\lambda}, \frac{e^{2}}{\lambda}, v\right)
$$

where the magnitude of $\lambda$, i.e., the normalization point, will be chosen so that the following normalization condition is satisfied for $\rho^{2}=\lambda$ :

$$
G\left(\frac{\xi^{2}}{\lambda}, 1, v\right)=1 .
$$

The renormalization group property of (5.28), which is equivalent to the basic equation ( 5.21 ), is conveniently written as

$$
\begin{equation*}
\alpha G\left(\frac{\xi^{2}}{\lambda}, \frac{\rho^{2}}{\lambda}, v\right)=G\left(\frac{\xi^{2}}{\lambda^{2}}, \frac{\rho^{2}}{\lambda^{\prime}}, v^{\prime}\right), \tag{5.33}
\end{equation*}
$$

where $\lambda$ ' is a new normalization point for which we now have

$$
\begin{equation*}
G\left(\frac{\xi^{2}}{\lambda^{\prime}}, 1, v^{\prime}\right)=1 \tag{5.34}
\end{equation*}
$$

and $v^{\prime}=\alpha^{-2} v$. Equation (5.33) shows that multiplication of $G$ by a nonzero number $\alpha$ is equivalent to a change of the normalization point and a renormalization of the excluded volume. Substituting $\rho^{2}=\lambda^{\prime}$ in (5.33), and recalling (5.34), we obtain

$$
\alpha^{-1}=G\left(\frac{\xi^{2}}{\lambda}, \frac{\lambda^{\prime}}{\lambda}, v\right),
$$

and, in terms of the new variables

$$
x=\xi^{2} / \lambda, \quad y=\rho^{2} / \lambda, \quad t=\lambda^{\prime} / \lambda
$$

we obtain (5.33) in the form

$$
\begin{equation*}
G(x, y, v)=G(x, t, v) G\left(x / t, y / t, v G^{2}(x, t, v)\right) \tag{5.35}
\end{equation*}
$$

Squaring both sides of this equation and then multiplying the result by $v$, we finally obtain for the RG-invariant quantity

$$
\begin{equation*}
V(x, y, v)=v G^{2}(x, y, v) \tag{5.36}
\end{equation*}
$$

the functional equation

$$
\begin{equation*}
V(x, y, v)=V(x / t, y / t, V(x, t, v)) \tag{5.37}
\end{equation*}
$$

with the normalization condition

$$
\begin{equation*}
V(x, 1, v)=v . \tag{5.38}
\end{equation*}
$$

Since RG is a continuous group, it can be described by the corresponding Lie differential equations that are exceedingly useful for practical purposes. For example, to obtain the Lie equation for $V(x, y, v)$, we differentiate both sides of (5.37) with respect to $y$ and then put $t=y$. The result is

$$
\begin{equation*}
y \frac{\partial V(x, y, v)}{\partial y}=\beta\left(\frac{x}{y}, V(x, y, v)\right), \tag{5.39}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta(x, v)=\left.\frac{\partial V(x, y, v)}{\partial y}\right|_{y=1}, \tag{5.40}
\end{equation*}
$$

and the normalization condition (5.38) serves as the boundary condition for (5.39). On the other hand, if we differentiate (5.37) with respect to $t$ and then put $t=1$, we obtain another form of the Lie equation

$$
\left(x \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}-\beta(x, v) \frac{\partial}{\partial v}\right) V(x, y, v)=0 .
$$

The above RG-equations are particularly useful in investigating the asymptotic properties of the function $G(x, y, v)$. Hence, to find the asymptotic form of $G(x, y, v)$ when, for
example, $x \rightarrow 0, y \rightarrow 0$, but $|x / y| \ll 1$, we pass to the limit as $x \rightarrow 0$ in (5.39). Substituting

$$
\begin{equation*}
V(y, v)=\lim _{x \rightarrow 0} V(x, y, v), \quad \beta(v)=\left.\frac{\partial V(y, v)}{\partial y}\right|_{y=1}, \tag{5.41}
\end{equation*}
$$

we then obtain the following equation:

$$
\begin{equation*}
y-\frac{\partial V(y, v)}{\partial y}=\beta(V(y, v)) \tag{5.42}
\end{equation*}
$$

subject to the boundary condition

$$
\begin{equation*}
V(1, v)=v \tag{5.43}
\end{equation*}
$$

Equation (5.42) and the boundary condition (5.43) can also be written in the Gell-Mann-Low form

$$
\begin{equation*}
\int_{v}^{V(y, v)} \beta^{-1}(t) \mathrm{d} t=\ln y . \tag{5.44}
\end{equation*}
$$

Thus, according to the above RG-equations, the invariant excluded volume is the effective parameter characterizing the strength of the volume interaction in a small neighborhood of the points $\xi=0$ and $\rho=0$. Hence, to determine the asymptotic behavior of, for example, the function $G(y, v)$ as $y \rightarrow 0$, we must know the attendant behavior of $V(y, v)$. However, this behavior is determined by the properties of the function $\beta(v)$. Since perturbation theory is usually used to calculate $\beta(v)$, we can judge its behavior only in a small neighborhood of the point $v=0$ at which $\beta(0)=0$. Actually, if $\beta(v)$ is positive in this neighborhood, then according to (5.42) the invariant excluded volume $V(y, v)$ tends to zero as $y \rightarrow 0$. If on the other hand the function $\beta(v)$ is negative near the origin, then $V(y, v)$ increases as $y \rightarrow 0$, and this takes us outside the range of validity of perturbation theory.

As already noted, before we can determine the asymptotic behavior of the probability density $W_{N}(\mathbf{R})$ as $N \rightarrow \infty$ for $R \gg l$, we must know the behavior of $A(\zeta, \rho)$ in a small neighborhood of the points $\zeta=1$ and $\rho=0$ or, equivalently, the behavior of the function $G(x, y, v)$ near $x=0$ and $y=0$. The most significant ranges of integration in the terms of the series ( 5.8 ) will then be regions in which the arguments of all the integrands are numerically small, i.e., when $x l \leqslant 1$ for each integration variable $\varkappa$. For such values of $\varkappa$, we also have $x r_{0} \ll 1$, since $r_{0}<l$. According to (5.9) and (5.26) we then find for sufficiently small values of $\varkappa$ that the equation $v(\varkappa)=v_{0}$ is approximately valid, and this corresponds to the well-known equation $f(r)=-v_{0} \delta(r)$ of the theory of polymers.

On the other hand, as the quantity $x$ increases and assumes values such that $\varkappa>\sim^{-1}$ (and the associated $R$ decreases to $R<\sim l$ ), the volume accessible to the particle executing the SAW decreases because of the increasing role of the excluded volume effect. Hence, the probability density $W_{N}(\mathbf{R})$ for $R<\sim l$ and, consequently, the function $A(\zeta, \chi)$ for $x>\sim l^{-1}$, should assume their limiting low values. The range of integration $x-l^{-1}$ in all terms of the series (5.8) will therefore provide a negligible contribution, and we are entitled to use the approximation $v(\varkappa)=v_{0}$ in (5.8) for all $\varkappa$. The series (5.8) can then be written in the form

$$
\begin{aligned}
& B\left(\zeta, \rho l, v_{0} ; A\right)=-v_{0} \int A(\zeta, \kappa) \mathrm{d}^{d} \kappa /(2 \pi)^{d} \\
& \quad+v_{0}^{2} \int A(\zeta, \vec{\rho}-\vec{\kappa}) M(\zeta, \kappa) \mathrm{d}^{d} \kappa /(2 \pi)^{d}
\end{aligned}
$$




FIG. 2.

$$
\begin{equation*}
-2 v_{0}^{3} \int A(\zeta, \vec{\rho}-\vec{x}) M^{2}(\zeta, x) \mathrm{d}^{d} x /(2 \pi)^{d}+\ldots \tag{5.45}
\end{equation*}
$$

where $v_{0}=\zeta^{2} v_{0}$

$$
\begin{align*}
& \int A(\zeta, x) \mathrm{d}^{d} \mu /(2 \pi)^{d}=\tilde{w}(\zeta, 0),  \tag{5.46}\\
& \begin{aligned}
M(\zeta, \varkappa) & =\int A\left(\cdot x-\varkappa^{\prime}\right) A\left(\zeta, x^{\prime}\right) \mathrm{d}^{d} x^{\prime} /(2 \pi)^{d} \\
\quad= & \omega \int_{0}^{\infty} \Lambda_{s}(\varkappa R) \tilde{w}^{2}(\zeta, R) R^{2 s+1} \mathrm{~d} R
\end{aligned}
\end{align*}
$$

in accordance with (5.25). As a result of the above approximation, some of the terms of the series (5.8) are now described in the same way. This means that the expression in (5.45) that corresponds to such terms must be multiplied by a factor that represents their symmetry; this is done in the third term in (5.45). The diagram representation of (5.45) can be deduced from the corresponding diagram representation of (5.8) in Fig. 1 by shrinking the wavy line to a point, so that the latter can be assigned the factor $-v_{0} /(2 \pi)^{d}$ and the conservation of momentum is sat fied at this point as before. The final result is the series shown in Fig. 2 where the square in diagram ( $\Sigma$ ) represents the sei of diagrams of Fig. 3, usually called the vertex part. It is very important to remember in this procedure that a closed equation ${ }^{9}$ can be established for the vortex part which, together with the basic equation, forms a closed system.

We note in conclusion that by following the above scheme for the derivation of the RG-equations and interchanging the roles of $\xi^{2}$ and $\rho^{2}$, i.e., by assuming that $\xi^{2}$ is the leading variable, we finally obtain the RG-equations in a similar form.

### 5.2. Self-avoiding walks in a space of dimensionality $\boldsymbol{d} \leqslant 4$

Finally, we turn to the determination of the proability density $W_{N}(\mathbf{R})$. We must first find the explicit form of the


FIG. 3.

$$
\beta(v)-v^{3}, \quad v \sim 0
$$

Finally, substituting this expression for $\beta(v)$ in (5.44), and solving it, we obtain the following expression for the invariant excluded volume:

$$
V(y, v)=v\left(1-2 v^{2} \ln y\right)^{-1 / 2},
$$

Hence, we readily find that

$$
\begin{equation*}
A^{-1}(1, \rho)=(4 \pi)^{-1} \rho^{2}\left(4 v_{0}^{2} \ln \rho^{-1}\right)^{1 / 4} \tag{5.53}
\end{equation*}
$$

In the present case, the equation given by (5.99) can be replaced by the following approximate equation when $\zeta \sim 1$ :

$$
\begin{equation*}
A^{-1}(\zeta, \rho)=\zeta-1+A^{-1}(1, \rho) \tag{5.54}
\end{equation*}
$$

Using (5.53) on the right hand side of (5.54), and then substituting the result for $A(\zeta, \rho)$ in (5.25), we can readily determine the form of the function $\widetilde{w}(\zeta, R)$ for large $R(R \gg l)$. We now substitute the expression obtained for $\widetilde{w}(\zeta, R)$ into the inversion formula (5.24) and use the method of steepest descents to obtain an asymptotic estimate for the integral as $N \rightarrow \infty$. This gives us the following expression for the leading term in the asymptotic expansion of the probability density $W_{N}(\mathbf{R})$ (Ref. 43):

$$
\begin{equation*}
W_{N}(\mathbf{R}) \sim\left(\pi N_{1}\right)^{-2}\left(\ln \frac{N_{1}}{R}\right)^{-1 / 2} \exp \left\{-\frac{R^{2}}{N_{1}\left[\ln \left(N_{1} / R\right)\right]^{1 / 4}}\right\} \tag{5.55}
\end{equation*}
$$

where $N_{1}=\left(2 v_{0} / \pi^{2}\right)^{1 / 2} N$. Hence, we can readily find the corresponding asymptotic form of the mean square end-toend distance for the random walk:

$$
\left(R^{2}\right\rangle_{N}-N_{1}\left(8 \ln N_{1}\right)^{1 / 4}
$$

This result is in good agreement with numerical calculations of $\left\langle R^{2}\right\rangle_{N}$ when $d=4$ (Ref. 44).
5.2.2. The case $d<4$ ( $s<1$ ). We must first briefly summarize the traditional approach to the problem. The initial approximation to $A(\zeta, \rho)$ in (5.45)-(5.47) can be taken to be (5.49). If we proceed to the limit as $\zeta \rightarrow 1(\xi \rightarrow 0)$, and then evaluate the integral in (5.45), we obtain the following results:

$$
\begin{equation*}
\left.\left.G^{-1}\right|_{\xi=0}=1+\frac{1}{2 \varepsilon} v_{1}^{2} \rho l\right)^{-2 \varepsilon}-\frac{8}{3 \varepsilon^{2}} v_{1}^{3}(\rho l)^{-3 \varepsilon}+\ldots \tag{5.56}
\end{equation*}
$$

where $v_{1}=v_{0} d / \pi^{2} z_{0}^{2} l^{d}$. Hence, we readily find $\beta(v)$ in accordance with its definition in (5.41). In the above approximation,

$$
\begin{equation*}
\beta(v)=v^{3}\left(1-\frac{8}{\varepsilon} v\right) \tag{5.57}
\end{equation*}
$$

When $\beta(v)$ can assume negative values, it is convenient to use the following form of the Gell-Mann-Low equation:

$$
\begin{equation*}
\int_{v(y, v)}^{v}|\beta(t)|^{-1} \mathrm{~d} t=\ln y \tag{5.58}
\end{equation*}
$$

It follows from (5.57) that $\beta(v)$ vanishes linearly at the point

$$
v=v_{*} \equiv \frac{\varepsilon}{8}
$$

so that the numerical value of the integral in (5.58) increases without limit as $V(y, v) \rightarrow v_{*}$. Hence $v_{*}$ is the limiting value of the invariant excluded volume $V$ :

$$
V(0, v)=v_{*}
$$

To analyze the behavior of $V(y, v)$ as $y \rightarrow 0$ in (5.58), we use the approximation

$$
\beta(v) \approx \frac{\varepsilon^{2}}{64}\left(v_{*}-v\right) .
$$

This gives

$$
\begin{equation*}
V(y, v) \approx v_{*}+C\left(v-v_{*}\right) y^{\xi^{2} / 64} \tag{5.59}
\end{equation*}
$$

and it follows that the critical exponent $\eta$ characterizing the asymptotic function $G \sim(\rho l)^{\eta}$ (or $A \sim(\rho l)^{\eta-2}$ ) for $\rho \sim 0$ is $\eta=\varepsilon^{2} / 64$, and is small in comparison with unity.

We now turn to the evaluation of the other critical exponent $v$ that determines the asymptotic form $\xi l \sim(\xi-1)^{v}$ when $\xi \sim 1$. For this we turn to (5.31) and evaluate its right hand side to within terms of the second order in $v_{1}$. The result is

$$
\begin{equation*}
E^{-1}=1+\frac{2}{\varepsilon} v_{1}(\xi l)^{-\varepsilon}-\frac{6}{\varepsilon^{2}} v_{1}^{2}(\xi l)^{-2 \varepsilon}+\ldots \tag{5.60}
\end{equation*}
$$

The role of the invariant excluded volume is now played by the quantity

$$
\bar{v}(x, v)=v E^{-\varepsilon / 2}
$$

where $x=C(\zeta-1)$. If we raise both sides of (5.60) to the power $\varepsilon / 2$, and then multiply the result by $v_{1}$, we finally obtain

$$
\bar{v}(x, v)=v\left(1+x^{-\varepsilon / 2} \bar{v}(x, v)-\frac{4}{\varepsilon} x^{-\varepsilon} \bar{v}^{2}(x, v)+\ldots\right)
$$

Hence, using the definition

$$
\beta(v)=\left.\frac{\partial \bar{v}(x, v)}{\partial x}\right|_{x=1}
$$

we obtain

$$
\beta(v)=\frac{4 v^{2}\left(v-v_{*}\right)}{1+v\left(v-v_{*}\right) / v_{*}}
$$

and the behavior of this function near $v=v_{*}$ is

$$
\beta(v) \approx \frac{\varepsilon^{2}}{16}\left(v-v_{*}\right) .
$$

Using the last approximation in the Gell-Mann-Low equation, we obtain

$$
\bar{v}(x, v) \approx v_{*}+C\left(v-v_{*}\right) x^{-\varepsilon^{2} / 16}
$$

from which it follows that $E \sim x^{\varepsilon / 8}$ or $(\xi l)^{2} \sim(\xi-1)^{1+\varepsilon / 8}$ when $\zeta \sim 1$. We thus obtain the well-known result $2 v \approx 1+(\varepsilon / 8)$.

We have already noted that, when the SAW problem is tackled by methods based on the $\varepsilon$-expansion, the results are not in general valid in real space. This may be due to the fact that the initial approximation for $A(\zeta, \rho)$ is usually taken to be the 'unperturbed' function $A_{0}(\xi, \rho)$. Moreover, the fact that the critical exponent is not equal to zero shows that the
singular points of the function $A(\zeta, \rho)$ are not simple poles. To extend the range of validity of our results, we must therefore start by choosing for $A(\zeta, \rho)$ an approximation that is free from these nonpole singularities. A method that can be used to determine the critical exponents for $d<4$ is presented below.

We now turn to (5.21) and note that, when $\xi \sim 0$ and $\rho \sim 0$, the variables $\xi$ and $\rho$ appear in (5.21) only in form $\xi^{2}+\rho^{2}$, in accordance with (5.30). Since $A(\zeta, \rho)$ is even in $\rho$, and the singular points closest to the origin $\rho=0$ are $\rho_{1,2}= \pm i \xi$, it is natural to take the initial approximation for $G$ to be

$$
\begin{equation*}
\tilde{G}(\xi, \rho) \sim u^{2 \mu}\left(\xi^{2}+\rho^{2}\right)^{\mu} \tag{5.61}
\end{equation*}
$$

in which $\rho \ll L^{-1}$ and the parameters $u=u\left(v_{0}\right)$, $\mu=\mu(d)(0<\mu<1)$, and $L>\sim l$ are to be determined. The function $\widetilde{G}(\xi, \rho)$ can be regarded as negligible in this case for $\rho \gg L^{-1}$. In accordance with the approximation adopted for $G$, and the equation given by (5.27), we take the function

$$
\begin{equation*}
\tilde{A}(\xi, \rho)=C z_{0}^{-1} u^{2 \mu}\left(\frac{L^{2}}{\xi^{2}+\rho^{2}}\right)^{(1-\mu) / 2} K_{1-\mu}\left(L\left(\xi^{2}+\rho^{2}\right)^{1 / 2}\right) \tag{5.62}
\end{equation*}
$$

as the initial approximation for $A(\zeta, \rho)$. It is important to emphasize that the choice of an explicit form for this approximation is not unique, but at any rate its behavior for $\rho \ll L^{-1}$ and $\rho \gg L^{-1}$ must satisfy the above conditions. Actually, since $\zeta \sim 1(\xi \sim 0)$, the quantity $\widetilde{A}(\xi, \rho)$ in (5.62) behaves as

$$
O\left(u^{2 \mu}\left(\xi^{2}+\rho^{2} y^{\mu-1}\right)\right.
$$

for $\rho L \ll 1$, and tends exponentially to zero for $\rho L \gg 1$. We therefore conclude that the problem is now reduced to the evaluation of the parameters $u, \mu, L$ for which the trial function $\widetilde{A}(\xi, p)$ is in a certain definite sense a solution of (5.21).

Let us now suppose that $u, \mu, L$ have been determined. Substituting (5.62) in (5.25), we obtain instead of $A(\zeta, \rho)$ the expression
$\tilde{W}(\xi, R)=C z_{0}^{-1} u^{2 \mu}\left(\frac{\xi^{2}}{R^{2}+L^{2}}\right)^{(\mu+s) / 2} K_{\mu+s}\left(\xi\left(R^{2}+L^{2}\right)^{1 / 2}\right)$,
which we shall use later in the inversion formula

$$
\begin{equation*}
\tilde{w}_{N}(R)=\frac{i}{2 \pi z_{0}^{N}} \oint d \zeta e^{N \ln \zeta \widetilde{w}(\xi, R) .} \tag{5.64}
\end{equation*}
$$

To find the asymptotic estimate for the integral in (5.64) when $N$ and $R$ tend to infinity, but their ratio $R / N$ remains fixed and $R / N \ll 1$, we use (5.20) to transform to the new integration variable $\xi$. If we take the asymptotic properties of $K_{v+s}(t)$ as $t \rightarrow \infty$ into account in (5.63) and introduce the notation

$$
D(\xi, t)=B\left(\zeta, i \xi l, v_{0} ; A\right)-B\left(1,0, v_{0} ; A\right)-t \xi+O\left(\xi^{2}\right)
$$

we can reduce (5.64) to the form

$$
\begin{align*}
& \tilde{w}_{N}(R) \sim\left(C u^{\left.2 \mu / z_{0}^{N+1} R^{\mu+s+1 / 2}\right) \times}\right. \\
& \times \int \mathrm{d} \xi(\mathrm{~d} \xi / \mathrm{d} \xi) \xi^{\mu+s-1 / 2} \exp (N D(\xi, R / N)) \tag{5.65}
\end{align*}
$$

Finally, using the method of steepest descents for the asymptotic evaluation of the integral in (5.65) as $N \rightarrow \infty$, we obtain the required asymptotic form of the function $\widetilde{w}_{N}(R)$.

To determine $u, \mu, L$, we use the fact that they are independent of $\xi$ and $\rho$. We therefore set $\xi=0$ in (5.28) and, to keep the formulas as simple as possible, we do not explicitly indicate the dependence on $\xi=0$. The function $\widetilde{G}(\rho)=\tilde{A}(\rho) / A_{0}(\rho)$ then has the following form for $\rho=0$ :

$$
\begin{equation*}
\tilde{G}(\rho) \sim\left(u^{2} \rho^{2} y^{\mu}\right. \tag{5.66}
\end{equation*}
$$

and this can be taken as the asymptotic behavior of the solution of (5.28) for $\xi=0$ and $\rho \sim 0$ if

$$
\begin{align*}
F(\tilde{G}) & =A_{0}(\rho)\left(B\left(1,0, v_{0} ; \tilde{A}\right)-B\left(1, \rho l, v_{0} ; \tilde{A}\right)\right) \\
& \sim \sum_{m \geq 1}\left(-\mu \ln u^{2} \rho^{2}\right)^{m / m} \tag{5.67}
\end{align*}
$$

However, according to the RG-method, to determine the asymptotic form of $G(\rho)$ for $\rho \sim 0$ we need only know the coefficient of the linear logarithmic term in the expansion for $F(\widetilde{G})$

$$
\begin{align*}
F(\tilde{G})= & \left(2 d / z_{0} \rho^{2} l^{2}\right)\left[\omega v_{0}^{2} \int_{0}^{\infty} \mathrm{d} R R^{2 s+1}\left(1-\Lambda_{s}(\rho R)\right) \tilde{w}^{3}(R)\right. \\
& -2 \omega v_{0}^{3} \int_{0}^{\infty} \mathrm{d} R R^{2 s+1}\left(1-\Lambda_{s}(\rho R)\right) \tilde{w}(R) \\
& \left.\times \int \tilde{w}^{2}(R-r) \tilde{w}^{2}(r) \mathrm{d}^{d} r+\ldots\right] \tag{5.68}
\end{align*}
$$

in powers of $\ln \rho^{2} L^{2}$. The first term in (5.68), which determines the contribution of diagram (b) to $F(\widetilde{G})$, will be written in the form

$$
\begin{equation*}
F_{\mathrm{b}}(\rho)=A_{0}(\rho)\left(B_{\mathrm{b}}(0)-B_{\mathrm{b}}(\rho)\right), \tag{5.69}
\end{equation*}
$$

where

$$
\begin{align*}
B_{b}(\rho) & =\omega v_{0}^{2} \int_{0}^{\infty} \Lambda_{s}(\rho R) \tilde{w}^{3}(R) R^{2 s+1} \mathrm{~d} R \\
& =C v_{0}^{2} z_{0}^{-3} u^{6 \mu}(\rho / L)^{3 \mu+2 s-1} K_{3 \mu+2 s-1}(\rho L) \tag{5.70}
\end{align*}
$$

represents the contribution of this diagram to $B\left(1, \rho l, v_{0} ; \widetilde{A}\right)$. If we now substitute ( 5.70 ) into the right-hand side of (5.69), and recall that

$$
t K_{1}(t)=1+\left(t^{2} / 2\right) \ln t+O\left(t^{2}\right), \quad t \sim 0
$$

we can readily see that the logarithmic behavior of the function $F_{\sigma}(\rho)$ for $\rho \sim 0$ can be assured by putting $3 \mu+2 s-2=0$, i.e.,

$$
\begin{equation*}
\mu=(4-d) / 3 \tag{5.71}
\end{equation*}
$$

Actually, it follows from (5.69)-(5.71) that

$$
\begin{equation*}
F_{6}(\rho) \sim-C\left(v_{0} / z_{0}^{2}\right)^{2} u^{6 u} \ln \rho^{2} L^{2} \tag{5.72}
\end{equation*}
$$

Comparison of this with the first term in the series in (5.67) suggests that the quantities $u$ and $L$ can be defined by

$$
\begin{equation*}
\left(v_{0} / z_{0}^{2}\right)^{2} u^{6 \mu}=C \mu, \quad L=C u \tag{5.73}
\end{equation*}
$$

so that

$$
\begin{equation*}
F_{\mathrm{b}}(\rho) \sim-C \mu \ln u^{2} \rho^{2} . \tag{5.74}
\end{equation*}
$$

We now turn to the second term in (5.68) which defines the contribution of diagram (c) in Fig. 2 to $F(\widetilde{G})$. The difference between this diagram and ( $b$ ) is that it contains a further vertex and two lines that form a loop subdiagram. As a result of the addition of this loop to the above term in (5.68), we have the additional factor

$$
\begin{equation*}
-C \mu^{1 / 2}(\mu L)^{\mu / 2} K_{\mu / 2}(\mu L) \tag{5.75}
\end{equation*}
$$

under the integral sign. For sufficiently small values of $\varkappa$, this factor is independent of $x$, and is negligible for $x \gg L^{-1}$. The expression given by (5.75) can also serve as an estimate for the contribution to $F(\widetilde{G})$ due to a transition from an arbitrarily chosen diagram to the diagram containing the next number of vertices, since any such transition is always accompanied by the addition of one vertex and two lines.

We now turn to the diagram series of Fig. 3 which represents the set of all possible diagrams with four outgoing lines. This set of quadrilateral diagrams, sometimes called a complete four-pole, constitutes the vertex part $Y$ in diagram ( $\Sigma$ ) of Fig. 2. As already noted, a closed equation can be established for the complete four-pole, which can be solved in the so-called parquet approximation for $\mu \ll 1$. The solution obtained in this way for $Y$ can be used to estimate $F(\widetilde{G})$. The final result takes the form of a series in powers of $-\mu \ln \rho^{2} L^{2}$.

Next, the constants in (5.73) can be chosen so as to ensure that the terms in (5.67) and (5.68) that are linear in $\ln u^{2} \rho^{2}$ are identical. The resulting expansion for $F(\widetilde{G})$ can now be used in the RG-equation to show that $G(\rho) \sim\left(u^{2} \rho^{2}\right)^{\mu}$, i.e., $G(\rho)=\widetilde{G}(\rho)$ when $\rho \sim 0$. It is precisely this result that justifies the choice $\widetilde{G}=\widetilde{A} / A_{0}$ as the asymptotic solution of (5.28).

Since the values of the parameters $u, \mu$ and $L$ have been determined, i.e., we have determined the functions in (5.62) and (5.63), we are able to determine the function $D(\xi, R / N)$ in (5.65). The final result is

$$
\begin{equation*}
D(\xi, R / N)=C v_{1}^{1 / 3} \xi^{1 / \nu_{F}}-\xi R / N+O\left(\xi^{2}\right) \tag{5.76}
\end{equation*}
$$

where

$$
\begin{align*}
& v_{1}=z_{0} v_{0} /(1-s)^{2},  \tag{5.77}\\
& v_{F}=3 /(d+2) . \tag{5.78}
\end{align*}
$$

Since the integrand in (5.65) has been determined, we can use the method of steepest descent to estimate the asymptotic form of the integral in (5.65) as $N \rightarrow \infty$ and $R \gg L$, and hence obtain the required asymptotic form of the probability density: ${ }^{45}$

$$
\begin{align*}
W_{N}(R) & \sim R_{F}^{-1} \delta(t-1), \quad d=1,  \tag{5.79}\\
& \sim C R_{F}^{-d} t^{p} \exp \left(-t^{q}\right), 1<d<4,
\end{align*}
$$

where

$$
t=R / R_{F}, \quad R_{F}=C\left(v_{1}^{1 / 3} N\right)^{\nu_{F}}
$$

$$
\begin{equation*}
p=\frac{(4-d)(d+2)}{6(d-1)}, \quad q=\frac{d+2}{d-1} . \tag{5.80}
\end{equation*}
$$

The mean square end-to-end distance for the random walk is therefore given by

$$
\left\langle R^{2}\right\rangle_{N} \sim C R_{F}^{2}
$$

which is the well-known generalization of the Flory formula to the $d$-dimensional case $(d<4)$. When $\varepsilon=4-d$ is small enough, we find from (5.78) that the critical exponent $v_{F}$ is given by

$$
\nu_{F}=\frac{1}{2}+\frac{\varepsilon}{12}+O\left(\varepsilon^{2}\right)
$$

To the first order in $\varepsilon$, this is identical with the corresponding result obtained by Wilson for the standard model of scalar field theory. The above method is thus fundamentally different from the de Gennes theory and leads to a new $\varepsilon$ expansion near the critical dimensionality $d=4$, which is now found to be convergent.

Next, according to (5.79) the asymptotic form of $W_{N}(\mathbf{R})$ has a valley for $R<R_{F}$ due to the excluded volume effect. The valley is convex downward ( $p=4 / c$ ) and convex upward ( $p=5 / 12$ ) for $d=2$ and 3 , respectively. When the density $W_{N}(\mathbf{R})$ is evaluated for $R<R_{F}$ it is common to replace $p$ with a different critical exponent $\gamma$ in calculations of the probability that the particle undergoing the random walk will return to the initial point. The relationship between $\gamma$ and $p$ is given by the Cloizeaux formula ${ }^{46}$ $\gamma=1+p \nu$. If we replace $v$ in this formula with the expression for $v_{F}$ given by (5.78), and use the value of $p$ given by (5.80), we obtain $\gamma=2$ for $d=2$ and $\gamma=5 / 4$ for $d=3$. We note, for comparison, that the values of $\gamma$ reported in the literature (see, for example, Refs. 47-51, where they are obtained by both analytic and computational methods) are found to be be somewhat lower than the above results. However, we must remember that (5.79) is an asymptotic formula obtained for $R>l$. In terms of the notation commonly employed in the theory of critical phenomena, we may write $A(1, \rho) \sim O\left(\rho^{\eta-2}\right)$ when $\rho \sim 0$ and $A(\zeta, 0) \sim O(\xi-1)^{-\gamma}$ when $\zeta \sim 1$. If we now adopt ( 5.62 ) as the asymptotic solution of the basic equation with $\zeta \sim 1$ and $\rho \sim 0$, we find that $\eta=2 \mu$ and the critical exponents $\gamma, \eta, v_{F}$ are related by the well-known formula $\gamma=(2-\eta) v_{F}$. The other critical exponent, $q$, is given by the generally accepted formula in (5.80).

## 6. CONCLUSION

We have presented an outline of the different schemes for investigating the SAW problem. This problem differs from other well-known physical problems by the presence of a 'memory' in the system, which is indicated by the integral form of the basic equation. The important property of this equation is its invariance under the continuous group of scaling transformations, and this has enabled us to use the renormalization group method to determine the asymptotic form of the probability density $W_{N}(R)$. Despite the fact that, to some extent, the asymptotic character of these results makes it difficult to compare them with experimental data, the agreement between theoretical and experimental results es-
tablished, for example, for critical exponents may be regarded as completely satisfactory.

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