# Classical nonlinear dynamics and chaos of rays in problems of wave propagation in inhomogeneous media 


#### Abstract

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Institute of Space Research, Academy of Sciences of the USSR, Moscow (Submitted 24 September 1990; revision submitted 16 May 1991) Usp. Fiz. Nauk 161, 1-43 (August 1991) We discuss the geometrical theory of wave propagation in regularly inhomogeneous waveguide media from the point of view of nonlinear Hamiltonian dynamics. We consider ray dynamics in waveguides with periodic longitudinal inhomogeneities, including the phenomenon of spatial nonlinear resonance of rays, which leads to the formation of an effective waveguide channel in the neighborhood of the ray in resonance with the periodic inhomogeneities. We consider different properties of spatially resonant rays: the optical path length and propagation velocity of a signal along rays trapped in a separate nonlinear resonance; the fractal properties of rays, such as the "devil's staircase" form of the dependence of the spatial oscillation frequency of the ray and the propagation time of a signal along the rays. The trajectory of sound rays in a model of the ocean with transverse flow is considered using the adiabatic invariant method and the transverse drift of a ray with respect to the main propagation direction of sound is described. We consider the conditions for dynamical chaos of rays in a waveguide with longitudinal periodic inhomogeneities. We examine the conditions for internal spatial nonlinear resonance and chaos of rays in waveguides with an irregular cross section and their effect on the propagation velocity of a signal. We study the connection between the structure of the wave front and the dynamics of rays in waveguide channels with regular inhomogeneities. Finally, we discuss the applicability of geometrical optics in waveguides under the conditions of nonlinear resonance and chaos of rays, and the relation between this problem and quantum chaos.


## 1. INTRODUCTION

Geometrical optics is the oldest application of the laws of classical mechanics to phenomena far from the realm of mechanics. Modern methods adequately describe the conditions under which geometrical optics represents an adequate physical picture. There exists an obvious parallel between the quasiclassical approximation in quantum mechanics and the approximation of geometrical optics (or ray dynamics, in the more general case) to wave dynamics. The quantum condition for a particle

$$
\begin{equation*}
\frac{\hbar}{I} \ll 1 \tag{1.1}
\end{equation*}
$$

where $I$ is the action of the particle and $\hbar$ is Planck's constant, corresponds to the condition

$$
\begin{equation*}
k l \ll 1 \tag{1.2}
\end{equation*}
$$

where $l$ is the characteristic dimension of the wave propagation problem. In both (1.1) and (1.2) the dispersion of a wave packet is weak and hence one can use an approximation in which the dynamics of the wave packet is replaced by the dynamics of a much simpler object: a particle or a ray. ${ }^{1)}$

Significant progress in classical nonlinear dynamics has led to an understanding of a new phenomenon called dynamical chaos, or simply chaos. ${ }^{1-4}$ The incorporation of the new ideas in problems of ray dynamics is only in its infancy and the present review is the first ${ }^{2}$ ) in which the propagation of rays in weakly inhomogeneous media is studied in the approximation of geometrical optics using the methods of the modern nonlinear theory, in terms of which various types of nonlinear resonance and chaos can be treated.

Long-range and ultra-long-range wave propagation is possible when the phase velocity of the wave in the medium depends nonmonotonically on the coordinates transverse to the direction of propagation (for example, depth in the ocean or height in the atmosphere). Some examples of waveguide propagation are the propagation of low-frequency sound waves in the ocean and atmosphere, short radio waves in the ionosphere, seismic waves in the earth's crust, and optical radiation in optical waveguides. ${ }^{5-12}$

There are many factors which limit the distance of propagation of a wave in an inhomogeneous medium. An example is a regular inhomogeneity of the medium along the propagation direction of the wave. Examples of such inhomogeneities in waveguide media are periodic and quasiperiodic longitudinal inhomogeneities in the ionosphere, ${ }^{13,14}$ and internal waves in the ocean. ${ }^{7}$ The dimensions $l$ of the inhomogeneities parallel and perpendicular to the propagation direction of the wave are such that the ray approximation can be used (see Refs. 5, 15, 16, for example).

Therefore the problem of wave propagation in a waveguide medium can be reduced to a corresponding problem in ray dynamics. Ray dynamics in an inhomogeneous medium can be described in the Hamiltonian approximation. ${ }^{11,16}$

The study of nonlinear ray dynamics in regularly inhomogeneous waveguides leads to a number of new and interesting effects due to the interaction of the rays with the regular inhomogeneities of the medium.

The first effect, called nonlinear resonance of rays, ${ }^{17}$ is the formation of an effective waveguide channel in which a group of rays is trapped in the neighborhood of a ray in resonance with the periodic inhomogeneities (Sec. 3). In
particular, this effect leads to fractal localization and other properties of the rays ${ }^{18}$ (Sec. 4). A similar effect is observed in waveguides with complicated two-dimensional cross sections. ${ }^{19}$ Nonlinear resonance of rays is the analog of nonlinear resonance in classical mechanics. ${ }^{1-4}$

The second effect is the chaotic instability of rays in a regularly inhomogeneous (nonrandom!) waveguide and is the analog of stochastic instability in nonlinear mechanics. This phenomenon leads to effects such as the formation of a stochastic layer of rays near the separatrix, from which there is an effective radiation loss, a random distribution of propagation times of a signal along the channel, and an irreversible distortion of the wave front. ${ }^{20}$

By using the methods of nonlinear dynamics to study ray dynamics in inhomogeneous media, a number of problems can be formulated in a much more convenient form and unexpected results can often be obtained by a very short and physically transparent method. For example, the standard methods of studying ray dynamics use the idea of separation of the spatial variables in some form, which restricts the applicability of the theory. The techniques of modern nonlinear analysis make it possible to formulate and solve problems without assuming separation of variables and thereby avoid one of the most troublesome problems of the old theory.

With the help of another formal technique, the equations of motion of rays in a moving medium can be written in the form of Hamiltonian equations for a charged particle moving in an electromagnetic field. The velocity of the medium is then analogous to the vector potential in the dynamical problem.

The present review includes basic results obtained by the authors in collaboration over the last few years. The first example of stochastic instability of rays in a regularly inhomogeneous waveguide was described in Ref. 21. Nonlinear resonance of rays and the formation of a stochastic layer in a regular waveguide with periodic longitudinal inhomogeneities and with an irregular cross section were considered by the present authors in Refs. 17 and 19. The effect of nonlinearity of the ray oscillations on the parametric resonance of rays in a parabolic waveguide was studied in Ref. 24. The formation of a stochastic layer in a regular waveguide was also treated in Ref. 22 for the example of a wave channel in the ionosphere. The stochastic instability of rays in a regular horizontally inhomogeneous ocean was analyzed numerically in Refs. 23, 25, and 26. The fractal properties of rays in a waveguide with periodic corrugations on one of its walls were established in Ref. 18. The connection between dynamical chaos of rays and the structure of the wave front in waveguides was studied in Ref. 20. In the present review some new results obtained recently by the authors are also included.

At the end of the review we discuss a new aspect of ray dynamics: the analogy between ray propagation and the formation of images on the one hand, and so-called quantum chaos ${ }^{2}$ on the other.

## 2. FORMULATION OF THE BASIC EQUATIONS FOR RAYS

### 2.1. Hamiltonian equations for rays

We consider the propagation of a scalar monochromatic wave with frequency $v: u(\mathbf{R}, t)=u(\mathbf{R}) \exp (-i v t)$ in a medium with a spatially inhomogeneous phase velocity $c(\mathbf{R})(\mathbf{R}=(\mathbf{r}, z), \mathbf{r}=x, y)$. When the length $l$ over which the
wave velocity $c(\mathbf{r}, \boldsymbol{z})$ varies significantly in space is much larger than the wavelength of the radiation $\lambda / l \ll 1$, then the wave field can be represented in the form

$$
\begin{equation*}
u(\mathbf{R})=A(\mathbf{R}) \exp (i k S(\mathbf{R})) \tag{2.1}
\end{equation*}
$$

where the slowly varying amplitude $A(r, z)$ and phase function $S(r, z)$ satisfy the equations

$$
\begin{align*}
& (\nabla S)^{2}=n^{2}(r, z), \quad n(r, z)=\frac{c_{0}}{c(r, z)}, \\
& \nabla\left(A^{2} \nabla S\right)=0, \tag{2.2}
\end{align*}
$$

where $n(r, z)$ is the index of refraction of the medium and $c(r, z)$ is the phase velocity of the wave. In (2.1) and (2.2) $k=2 \pi / \lambda$ and $c_{0}$ are the wave number and the wave velocity in a homogeneous medium, respectively.

The first equation in (2.2) is called the eikonal equation. It is a first-order nonlinear differential equation of the Hamilton-Jacobi type. ${ }^{27}$ It can be solved by the method of characteristic equations. The most convenient form of the characteristic equation is the Hamiltonian representation. Introducing the generalized momentum $p=\nabla S$, the eikonal equation (2.2) can be written in the form

$$
\begin{equation*}
H=H(\mathbf{R}, \mathbf{p})=0 \tag{2.3}
\end{equation*}
$$

where $H(\mathbf{R}, \mathbf{p})$ is a function of the coordinates $\mathbf{R}$ and the generalized momenta $\mathbf{p}$.

The form of the function $H$ is determined by the eikonal equation and depends on the choice of the independent variable $\tau$ specifying the characteristic curves $\mathbf{R}=\mathbf{R}(\tau)$, $\mathbf{p}=\mathbf{p}(\tau)$ in the six-dimensional phase space ( $\mathbf{R}, \mathbf{p}$ ). The auxiliary variable $\tau$ determines the ray equations in parametric form. They are found with the help of the Hamiltonian canonical equations ${ }^{16,35}$

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{R}}{\mathrm{~d} \tau}=\frac{\partial H}{\partial \mathbf{p}}, \quad \frac{\mathrm{dp}}{\mathrm{~d} \tau}=-\frac{\partial H}{\partial \mathbf{R}}, \quad \frac{\mathrm{~d} S}{\mathrm{~d} \tau}=\frac{\mathbf{p} \partial S}{\partial \mathbf{p}} \tag{2.4}
\end{equation*}
$$

with $H(\mathbf{R}, \mathbf{p})$ as the Hamiltonian function. The independent variable $\tau$ is related to the element of arc length of the ray $\mathrm{d} \sigma=\left(\mathrm{dR}^{2}\right)^{1 / 2}$ :

$$
\mathrm{d} \sigma=\left|\frac{\partial H}{\partial \mathbf{p}}\right| \mathrm{d} x
$$

which follows from the first equation of (2.4). The projection of the characteristic curves $\mathbf{R}(\tau), \mathbf{p}(\tau)$ onto the threedimensional configuration space is called a ray.

We consider some forms of the Hamiltonian function $H$. Suppose $H(\mathbf{R}, \mathrm{p})$ has the form

$$
\begin{equation*}
H=\frac{1}{2}\left(n^{2}-\mathrm{p}^{2}\right)=0 . \tag{2.5}
\end{equation*}
$$

Hamilton's equations in this case are

$$
\begin{equation*}
\frac{d R}{d \tau}=p, \quad \frac{d p}{d \tau}=n(\mathbf{R}) \nabla n(\mathbf{R}) . \tag{2.6}
\end{equation*}
$$

In this representation the arc length of the ray is $d \sigma=n d \tau$. The tangent vector $d \mathbf{R} / d \tau$ to the ray points along the normal p to the surfaces of constant phase $S(\mathbf{R})=$ const.

In many problems of waveguide propagation it is convenient to use the representation where the independent variable is the coordinate along the axis of the waveguide. Let $z$ be this coordinate. Then the ray coordinates $\mathbf{r}=\mathbf{r}(z)$, $\mathbf{p}_{\perp}=\mathbf{p}_{\perp}(z)$ are determined by Hamilton's equations

$$
\begin{align*}
& \frac{\mathrm{dr}}{\mathrm{~d} z}=\frac{\partial H}{\partial \mathrm{p}_{\perp}}, \quad \frac{\mathrm{d} \mathrm{p}_{\perp}}{\mathrm{d} z}=-\frac{\partial H}{\partial \mathrm{r}} \\
& H \equiv-p_{\mathrm{I}}=H\left(\mathrm{r}, \mathrm{p}_{\perp}, z\right)=-\left[n^{2}(\mathrm{r}, z)-\mathrm{p}_{\perp}^{2}\right]^{1 / 2}  \tag{2.7}\\
& \mathrm{p}_{\perp}=\left(p_{x}, p_{y}\right)
\end{align*}
$$

It is not difficult to obtain a relation between the components of the normal vector $p$ to the surfaces of constant phase $S=$ const in the transverse plane ( $x, y$ ) on the one hand, and the angles $\theta$ and $\psi$ between the vector $p$ and the $z$ axis and between $p_{\perp}$ and the $x$ axis, respectively on the other:

$$
\begin{align*}
& p_{x}=n(\mathrm{r}, \mathrm{z}) \sin \theta \cdot \cos \psi, \quad p_{y}=n(\mathrm{r}, \mathrm{z}) \sin \theta \cdot \sin \psi \\
& H=-n(\mathrm{r}, \mathrm{z}) \cos \theta . \tag{2.8}
\end{align*}
$$

In the case of wave propagation in a weakly inhomogeneous medium the typical linear dimension $l$ of an inhomogeneity is much larger than the wavelength $\lambda$ (see the inequality (1.2)) and the waves are scattered by small angles $\theta$. Then the Hamiltonian function (2.8) can be simplified using the condition $\theta \ll 1$. Assuming also that the inhomogeneous part of the index of refraction $n(r, z)$ is small:

$$
\bar{\varepsilon}(\mathrm{r}, z)=n^{2}(\mathrm{r}, z)-1 \ll 1,
$$

we easily obtain the following approximate expression for $H$ :

$$
\begin{equation*}
H=-1+\frac{1}{2} \mathrm{p}_{\perp}^{2}-\frac{1}{2} \bar{\varepsilon}(\mathrm{r}, z) \tag{2.9}
\end{equation*}
$$

The corresponding approximation for rays is called the paraxial approximation.

### 2.2. Rays in an inhomogeneous moving medium

The study of sound propagation in an inhomogeneous moving medium is of great interest in connection with the acoustics of media in nature. Examples are the atmosphere in the presence of wind velocity inhomogeneities with height, and the ocean in the presence of inhomogeneous flow. The approximation of geometrical acoustics is convenient in studying sound propagation in these media. The eikonal equation in a moving inhomogeneous medium was obtained in Ref. 29 from the hydrodynamical equations. The convenient Hamiltonian formalism for the solution of the eikonal equation was developed in Refs. 31-33. A review of the basic assumptions and the latest results of the ray theory of sound propagation in an inhomogeneous moving medium was given in Ref. 28. Below we consider some aspects of the ray equations in moving inhomogeneous media using the results of Ref. 30, where an analogy was drawn between the equations of motion of a ray and the equations of motion of a charged particle in an electromagnetic field.

The eikonal equation in an inhomogeneous moving medium has the form ${ }^{29}$

$$
\begin{equation*}
(\nabla S)^{2}=\frac{\left[c_{0}-(v \nabla S)\right]^{2}}{c^{2}(\mathbf{R})} \tag{2.10}
\end{equation*}
$$

where $\mathbf{v}=\mathbf{v}(\mathbf{R})$ is the velocity field of the medium.
We introduce the generalized momentum $p=\nabla S$. The Hamiltonian function reducing to (2.5) in the limit $\mathbf{v} \rightarrow 0$ has the form

$$
\begin{equation*}
H=H(\mathbf{R}, \mathbf{P})=\frac{1}{2} \mathbf{p}^{2}-\frac{1}{2} n^{2}(\mathbf{R})(1-\mathbf{P A})^{2}=0 \tag{2.11}
\end{equation*}
$$

where $n(\mathbf{R})=c_{0} / c(\mathbf{R})$ and $\mathbf{A}=\mathbf{v}(\mathbf{R}) / c_{0}$ is the normalized velocity vector of the medium. It determines the ray equations

$$
\frac{\mathrm{dR}}{\mathrm{~d} \tau}=\frac{\partial H}{\partial \mathrm{P}}, \frac{\mathrm{dP}}{\mathrm{~d} \tau}=-\frac{\partial H}{\partial \mathrm{R}} .
$$

The ray equations take on an interesting and simple form in the case of a weakly inhomogeneous and slowly moving medium, as in the atmosphere and the oceans, i.e. $\bar{\varepsilon} \ll 1$ and $|v| / c_{0} \ll 1$. Then in (2.11) we can neglect higher-order terms such as $\bar{\varepsilon}^{2}, v^{2} / c_{0}^{2}, \bar{\varepsilon} v / c_{0}$. We then obtain

$$
\begin{equation*}
H=-1+\frac{1}{2} \mathbf{p}^{2}+\mathbf{P A}-\frac{1}{2} \bar{\varepsilon}(\mathbf{R})=0 \tag{2.12}
\end{equation*}
$$

The ray equations take the form

$$
\frac{d \mathbf{R}}{\mathrm{~d} \tau}=\mathbf{P}+\mathbf{A}, \quad \frac{\mathrm{dP}}{\mathrm{~d} \tau}=-\frac{1}{2} \nabla(\bar{\varepsilon}(\mathbf{R})+\mathbf{P A})
$$

Introducing the new generalized momentum $\mathbf{p}=\mathbf{P}+\mathbf{A}$, the ray equations can be written in the form

$$
\begin{equation*}
\frac{d \mathrm{R}}{\mathrm{~d} \tau}=\mathrm{p}, \quad \frac{\mathrm{~d} p}{\mathrm{~d} \tau}=\vec{G}+\frac{1}{c_{0}}[\mathrm{p}, \mathscr{H}] \tag{2.13}
\end{equation*}
$$

where

$$
\vec{E}=\frac{1}{2} \nabla \bar{\varepsilon}(R), \quad \overrightarrow{\mathcal{H}}=-\operatorname{rot} \mathbf{v}(\mathbf{R}) .
$$

The quantity $\mathscr{H}$ is the negative of the vortex vector of the moving medium.

In contrast to a nonmoving medium, the tangent vector to the ray $\mathbf{p}=d \mathbf{r} / d \tau$ does not lie along the normal vector $\mathbf{p}=\nabla S$ to the surfaces of constant phase $S(\mathbf{R})=$ const, i.e. $\mathbf{P} \neq \mathbf{p}=\mathbf{P}+\mathbf{A}$.

We consider the paraxial approximation. Suppose that the rays propagate at small angles to the main direction of wave propagation along the $z$ axis. Then in a weakly inhomogeneous and slowly moving medium the rays are determined by the Hamiltonian ${ }^{30}$

$$
\begin{align*}
H & =H(\mathrm{r}, \mathrm{P}, \mathrm{z}) \\
& =-1+\frac{1}{2} \mathrm{p}_{\perp}^{2}+\mathrm{P}_{\perp} \mathrm{A}_{\perp}-\frac{1}{2}\left(\bar{\varepsilon}(\mathrm{r}, z)-\frac{2 v_{z}}{c_{0}}(\mathrm{r}, \mathrm{z})\right) \tag{2.14}
\end{align*}
$$

where $\mathbf{P}_{\perp}=\partial S / \partial \mathbf{r} ; v_{z}$ is the longitudinal component of the velocity vector of the medium; $\mathbf{A}_{1}=\mathbf{v}_{1} / c_{0}$ is the transverse normalized velocity vector of the medium. The ray equations in a moving medium in terms of the canonical variables ( $\mathbf{r}, \mathbf{P}_{1}$ ) are

$$
\begin{equation*}
\frac{\mathrm{dr}}{\mathrm{dz}}=\frac{\partial H}{\partial \mathbf{P}_{\perp}}, \quad \frac{\mathrm{d} \mathbf{P}_{\perp}}{\mathrm{dz}}=-\frac{\partial H}{\partial \mathbf{r}} . \tag{2.15}
\end{equation*}
$$

The application of these equations to particular problems of sound propagation in an inhomogeneous moving medium will be considered in Sec. 6 .

### 2.3. Optical-mechanical analogy

Hamilton first suggested an analogy between classical mechanics and optics. ${ }^{34}$ A particle moving in a potential field behaves just like light in a medium, where the potential energy $U$ is related to the index of refraction $n$ by the equation

$$
\begin{equation*}
n=c[2 m(E-U)]^{1 / 2} / E \tag{2.16}
\end{equation*}
$$

where $m$ and $E$ are the mass and energy of the particle and $c$ is the speed of light in a vacuum. Indeed, substituting (2.16) into (2.3) and defining the momentum of the particle as $\mathbf{p}_{m}=\mathbf{p} E / c$, we obtain the classical expression for a particle in a potential field

$$
\begin{equation*}
E=\mathrm{p}_{m}^{2} / 2 m+U(\mathrm{r}) \tag{2.17}
\end{equation*}
$$

In addition to this direct analogy, there is also a formal analogy between the equations of classical mechanics and geometrical optics (see Ref. 11, for example). We see from (2.9) that the Hamiltonian in the paraxial approximation has the same form as the classical expression for the energy of a particle, where the momentum corresponds to $\mathbf{p}_{1}$, the potential energy corresponds to the inhomogeneous part of the index of refraction $\bar{\varepsilon}(\mathbf{r}, z)$ divided by two, and the spatial coordinates correspond to the transverse coordinates $r=(x, y)$ of the ray. The role of time is played by the $z$ coordinate defining the main direction of propagation of the wave.

There is also an interesting analogy between the ray dynamics in an inhomogeneous moving medium (described by (2.3)-(2.15)) and the dynamics of a charged particle in nonuniform electric and magnetic fields. It is evident from (2.13) and (2.14) that the scalar potential of the electromagnetic field corresponds to the inhomogeneous part of the index of refraction $\bar{\varepsilon}(\mathbf{R})$, while the vector potential corresponds to the negative of the velocity field of the medium $\mathbf{v}(\mathbf{R})$. Hence the electric field corresponds to the gradient of $\bar{\varepsilon}(\mathbf{R})$ and the magnetic field corresponds to the vortex vector of the medium ( $-\operatorname{curl} \mathbf{v}(\mathbf{R})$ ). Because of this analogy, the methods of particle dynamics can be used in the ray theory of wave propagation in inhomogeneous media.

### 2.4. Ray equations in an inhomogeneous medlum

The index of refraction of a waveguide medium perturbed by regular inhomogeneities is written in the form

$$
\begin{equation*}
n^{2}(r, z)=n_{0}^{2}(r)+\varepsilon n_{1}(r, z), \tag{2.18}
\end{equation*}
$$

where $n_{0}(\mathbf{r})$ corresponds to the unperturbed waveguide medium, which is homogeneous in the $z$ direction, and $\varepsilon n_{1}(\mathbf{r}, z)$ describes the perturbation of the medium in the $z$ direction. The quantity $\varepsilon \ll 1$ is the small dimensionless perturbation parameter. Because $\varepsilon$ is small, the Hamiltonian function (2.7) can be written in the form

$$
\begin{equation*}
H=H_{0}\left(\mathbf{r}, \mathrm{p}_{\perp}\right)+\varepsilon V\left(\mathrm{r}, \mathrm{p}_{\perp}, \mathrm{z}\right) \tag{2.19}
\end{equation*}
$$

where
$H_{0}=-\left[n_{0}^{2}(r)-p_{\perp}^{2}\right]^{1 / 2}, \quad \varepsilon V\left(r, p_{\perp}, z\right)=\frac{\varepsilon n_{1}(r, z)}{2 H_{0}}$.
Rays in the unperturbed waveguide are determined by the Hamiltonian $H_{0}$. Hence a ray propagating in an inhomogeneous waveguide medium is equivalent to a particle described by the unperturbed Hamiltonian $H_{0}$ and acted upon by an unsteady perturbation or by the interaction of the different degrees of freedom of the particle when it performs finite motion.

In the case of a plane waveguide $n$ is independent of $y$ and (2.7) takes the form

$$
\begin{aligned}
& \frac{\mathrm{d} x}{\mathrm{~d} z}=\frac{\partial H}{\partial p}, \quad \frac{\mathrm{~d} p}{\mathrm{~d} z}=-\frac{\partial H}{\partial x}, \quad p=p_{x}, \\
& H=H_{0}(x, p)+\varepsilon V(x, p, z), \quad H_{0}=-\left(n_{0}^{2}(x)-p^{2}\right)^{1 / 2}
\end{aligned}
$$

### 2.5. Action-angle variables

The trajectories of rays trapped by the waveguide channel are bounded in the transverse directions ( $x, y$ ) and are periodic along the $z$ direction. Therefore it is convenient to introduce action-angle variables ( $I, \vartheta$ ) (Ref. 35). In the case of a plane waveguide they are defined by
$I=(2 \pi)^{-1} \oint p \mathrm{~d} x, \quad \vartheta=\frac{\partial}{\partial I} \int^{x} p \mathrm{~d} x, \quad p=\left(n_{0}^{2}(x)-H^{2}\right)^{1 / 2}$.

In terms of these variables the equations of motion of rays in the perturbed waveguide take the form

$$
\begin{align*}
& \frac{\mathrm{d} I}{\mathrm{dz}}=-\frac{\varepsilon \partial V}{\partial \vartheta}, \quad \frac{\mathrm{~d} \vartheta}{\mathrm{~d} z}=\omega(I)+\varepsilon \frac{\partial V}{\partial I} \\
& H=H_{0}(I)+\varepsilon V(I, \vartheta, z) \tag{2.23}
\end{align*}
$$

where

$$
\omega(I)=\frac{\mathrm{d} H_{0}(I)}{\mathrm{d} I}
$$

is the nonlinear spatial oscillation frequency of a ray along the $z$ axis of the unperturbed waveguide. The quantity

$$
L(I)=\frac{2 \pi}{\omega(I)}
$$

is the spatial period of the ray.
The ray coordinates $x(z), p(z)$ are periodic functions of the angle $\vartheta$ in the unperturbed waveguide. Therefore they can be expanded in Fourier series
$x(\vartheta)=\sum_{m} x_{m}\left(\eta \exp (i m \vartheta), \quad p(\vartheta)=\sum_{m} p_{m}(I) \exp (i m \vartheta)\right.$.
We next consider a waveguide with a two-dimensional cross section and the index of refraction $n(\mathrm{r})=n(x, y)$. Suppose that in the unperturbed waveguide besides the integral of the motion $E=H(\mathbf{r}, \mathbf{p})$, where $\mathbf{p}=\left(p_{x}, p_{y}\right)$, there exists a second independent integral of the motion $W(r, p)$. Then there exists an invariant two-dimensional torus and the trajectory of a ray performing finite oscillations (i.e. a waveguide ray) coils around the surface of the torus. The actionangle variables $(\mathbf{I}, \vec{\vartheta}) \equiv\left(I_{1}, I_{2}, \vartheta_{1}, \vartheta_{2}\right)$ in this case are defined as $^{36}$

$$
\begin{equation*}
I_{k}=(2 \pi)^{-1} \oint_{C_{k}} \mathrm{pdr}, \quad \vartheta_{k}=\frac{\partial}{\partial I_{k}} \int_{\mathrm{r}_{0}}^{\mathrm{r}} p(\mathrm{I}, \mathrm{r}) \mathrm{d} r \quad(k=1,2) \tag{2.25}
\end{equation*}
$$

where the $C_{k}$ are the contours defining the two-dimensional torus.

In terms of these variables the unperturbed Hamiltonian has the form $H=H\left(I_{1}, I_{2}\right)$. The ray coordinates ( $\mathrm{r}, \mathrm{p}$ ) are periodic functions of the variables $\vartheta_{k}$ with period $2 \pi$ and with the oscillation frequency along the $z$ axis

$$
\omega_{k}\left(I_{1}, I_{2}\right)=\frac{\partial H\left(I_{1}, I_{2}\right)}{\partial I_{k}}
$$

The second integral $W(r, p)$ no longer exists for a waveguide with a general cross-section profile, and in this case the invariant torus does not exist and hence the contours $C_{k}$ do not exist. This case corresponds to a multidimensional dynamical system in which the integrals of the motion no longer exist and dynamical chaos develops in the system; ${ }^{1-4}$ it will be considered in Sec. 7.

Action-angle variables were successfully applied for the first time in Ref. 37 to wave propagation in waveguides in
a study of ray statistics in statistically irregular light guides.
The Hamiltonian formulation of the ray equations in terms of action-angle variables is also convenient in studying the propagation of rays in continuously regular waveguides using perturbation theory. ${ }^{39}$ This method is much simpler and more general than the asymptotic methods used to study continuously regular waveguides (see the literature cited in Ref. 39).

### 2.6. Wave-number spectrum

We point out an important feature of action-angle variables ( $I_{k}, \vartheta_{k}$ ) in problems of wave propagation in waveguides. We consider the integrable case, when there exist two integrals of the motion $H$ and $W$. Then the action variables ( $I_{1}, I_{2}$ ) are also integrals of the motion and the spectrum of wave numbers $k_{m}$ of the wave field is determined by the boundary-value problem

$$
\begin{equation*}
\left[\Delta+k^{2} n^{2}(r)\right] u_{m}(\mathrm{r})=k_{m}^{2} u_{m}(\mathrm{r}), \quad u_{m}(\mathrm{r}) \rightarrow 0 \quad(|\mathrm{r}| \rightarrow \infty) \tag{2.26}
\end{equation*}
$$

In the short-wave approximation the $k_{m}$ are determined by the dependence of the Hamiltonian (2.7) on the action variables $I_{1}$ and $I_{2}$ (see Refs. 37 and 38, for example):

$$
\begin{equation*}
k_{m}=-k H\left(I_{1}, I_{2}\right), \quad k=\frac{v}{c}, \tag{2.27}
\end{equation*}
$$

where $v$ is the wave frequency and the action variables $I_{1}, I_{2}$ are proportional to the mode numbers $m=\left(m_{1}, m_{2}\right)$ :

$$
\begin{equation*}
k I_{i}=m_{i} \quad(i=1,2) \tag{2.28}
\end{equation*}
$$

From (2.27) it is simple to determine the wave-number spectrum of the waveguide field $k_{m}$ and the group velocity of the wave $v_{m}$ with the help of the relation

$$
\begin{equation*}
\frac{c_{0}}{v_{m}\left(I_{1}, I_{2}\right)}=\frac{\mathrm{d} k_{m}}{\mathrm{~d} \nu}=I_{1} \omega_{1}+I_{2} \omega_{2}-H\left(I_{1}, I_{2}\right) \tag{2.29}
\end{equation*}
$$

without having to solve the boundary-value problem (2.26) and using only the classical ray equations.

## 3. RAY DYNAMICS IN A WAVEGUIDE WITH PERIODIC LONGITUDINAL INHOMOGENEITIES

For simplicity we consider a plane waveguide channel. For a periodic perturbation of the medium along the axis of the waveguide channel (the $z$ axis) the perturbation of the Hamiltonian $V(I, \vartheta, z)$ in (2.23) can be written as a double Fourier series:

$$
\begin{equation*}
\varepsilon V(I, \vartheta, z)=\frac{\varepsilon}{2} \sum_{m, s} V_{m s}(I) \exp (i m \vartheta+i s \Omega z)+\text { c.c. } \tag{3.1}
\end{equation*}
$$

where $\Omega$ is the spatial frequency of the perturbation of the medium along the $z$ axis ( $2 \pi / \Omega$ is the spatial period of the perturbation) and c.c. denotes the complex conjugate of the preceding term.

### 3.1. Spatial nonlinear resonance of rays

It often occurs in nature and in practical applications that the index of refraction $n(x)$ depends on the transverse coordinate $x$ in such a way (Fig. 1) that the spatial oscillations of the ray along the $z$ axis are nonlinear in general and their expansion in a Fourier series contains a large number of harmonics.


FIG. 1. Index of refraction $n(x)$ of a plane-layer waveguide.

When rays propagate in a medium perturbed by the periodic inhomogeneity (3.1), the perturbation has the strongest effect on the rays in the case of the resonance condition

$$
\begin{equation*}
m \omega(I)+s \Omega=0 \tag{3.2}
\end{equation*}
$$

where $m$ and $s$ are integers.
Let $I_{0}$ be the resonance value of the action $I$ satisfying (3.2) for certain values of $m$ and $s$. We consider the trajectory of a ray in the neighborhood of a single isolated resonance. We assume the condition of moderate nonlinearity

$$
\begin{equation*}
\varepsilon \ll \alpha \ll \varepsilon^{-1}, \quad \alpha=\frac{I}{\omega}\left|\frac{d \omega}{\mathrm{~d} I}\right|, \tag{3.3}
\end{equation*}
$$

where $\alpha$ is the dimensionless nonlinearity parameter. Then the nonresonant terms can be neglected in (2.23) with the perturbation (3.1). Introducing the new canonical variables

$$
\Delta I=I-I_{0}, \quad \Psi=m \vartheta+s \Omega z
$$

the ray equation in a small neighborhood of the resonance action $I_{0}$ can be written in the Hamiltonian form

$$
\begin{equation*}
\frac{\mathrm{d} \Delta I}{\mathrm{~d} z}=-\frac{\partial H_{\mathrm{u}}}{\partial \Psi}, \quad \frac{\mathrm{~d} \Psi}{\mathrm{~d} z}=\frac{\partial H_{\mathrm{u}}}{\partial \Delta I}, \tag{3.4}
\end{equation*}
$$

with the universal Hamiltonian

$$
\begin{align*}
& H_{\mathrm{u}}=\frac{1}{2} m \omega^{\prime}\left(I_{0}\right)\left(\Delta I^{2}+\varepsilon m\left|V_{m s}\right| \cos \Psi\right.  \tag{3.5}\\
& \omega^{\prime}\left(I_{0}\right)=\frac{\mathrm{d} \omega\left(I_{0}\right)}{\mathrm{d} I}
\end{align*}
$$

From the system of equations (3.4) we obtain an equation for the phase oscillations

$$
\begin{equation*}
\frac{d^{2} \Psi}{d z^{2}}-\bar{\Omega}^{2} \sin \Psi=0 \tag{3.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{\Omega}=m\left|\varepsilon V_{m s} \omega^{\prime}\left(I_{0}\right)\right|^{12} \tag{3.7}
\end{equation*}
$$

is the frequency of the modulated oscillation. Note that (3.6) is equivalent to the equation of motion of a pendulum, where $\bar{\Omega}$ is the frequency of small oscillations.

According to (3.5), when

$$
H_{\mathrm{u}}<\varepsilon m\left|V_{m s}\right|
$$

the rays perform phase oscillations bounded with respect to $\Psi$, i.e. the spatial oscillations of the ray in the $z$ direction are synchronized with the periodic inhomogeneities of the medium. In other words, the rays are trapped in nonlinear resonance. The widths of the region of trapped rays in terms of the action $I$ and the frequency $\omega$ are found with the help of (3.5):

$$
\begin{align*}
& \Delta I=2 \max \left|I-I_{0}\right|=4\left|\frac{\varepsilon V_{m s}}{\omega^{\prime}\left(I_{0}\right)}\right|^{1 / 2}, \\
& \Delta \omega=\Delta I\left|\frac{\mathrm{~d} \omega}{\mathrm{~d} I}\right|=4\left|\varepsilon V_{m s} \omega^{\prime}\left(I_{0}\right)\right|^{1 / 2} . \tag{3.8}
\end{align*}
$$

This phenomenon can be called spatial nonlinear resonance of rays. It is the analog of nonlinear resonance in classical mechanics, ${ }^{1-4}$ but occurs in space instead of time. It has been studied in Ref. 17.

The physical interpretation of nonlinear resonance of rays is as follows. In the absence of the perturbation the trajectory of a ray is an oscillating line along the $z$ axis with spatial period $L=2 \pi / \omega$. In the presence of a periodic perturbation in the $z$ direction a group of rays in the neighborhood of the unperturbed ray corresponding to the resonance action $I_{0}$ is trapped into resonance and the rays perform additional oscillations with the modulation frequency $\bar{\Omega}=m \Delta \omega$.

In other words, an additional effective waveguide channel of width $\Delta I$ is formed along the axis of the unperturbed ray with action $I_{0}$. The number of these additional waveguide channels is determined by the number of possible nonoverlapping resonances of the type (3.2).

We note that our approximation of an isolated nonlinear resonance is valid only when the width of a resonance $\Delta \omega$ is much smaller than the separation $\delta \omega$ between neighboring resonances:

$$
\begin{equation*}
K=\left(\frac{\Delta \omega}{\partial \omega}\right)^{2} \ll 1 \tag{3.9}
\end{equation*}
$$

### 3.2. Propagation velocity of a signal along resonant rays

We consider the effect of spatial nonlinear resonance of rays in a waveguide on the important characteristics of the radiation propagating in the waveguide, such as the transmission time and propagation velocity of a signal along the rays. Let $t(z)$ be the transmission time of a signal along a ray from the plane $z=0$ to the plane $z=$ const and let $v(z)$ be the local velocity of propagation of the signal along the ray in the direction of the waveguide axis:

$$
\begin{equation*}
t(z)=\int_{\gamma} \frac{\mathrm{d} \sigma}{c(\mathrm{r}, \mathrm{z})}, \quad v(z)=\frac{c_{0}}{\mathrm{~d} t(\mathrm{z}) / \mathrm{d} z}, \tag{3.10}
\end{equation*}
$$

where the integration goes along the arc of ray $\gamma$ and $\mathrm{d} \sigma=\left(\mathrm{d} R^{2}\right)^{1 / 2}$.

Using the Hamiltonian representation (2.7) with $z$ as the independent variable, the transmission time (3.10) of the signal can be written in the form ${ }^{11}$

$$
\begin{equation*}
t(z)=c_{0}^{-1} \int_{0}^{z}\left(\mathbf{p}_{\perp} \frac{\partial H}{\partial \mathbf{p}_{\perp}}-H\right) \mathrm{d} \mathrm{z} \tag{3.11}
\end{equation*}
$$

The integrand in (3.11) is invariant to the choice of the canonical variables ( $\mathbf{r}, \mathbf{p}_{\mathrm{A}}$ ). Choosing the action-angle variables $(I, \vartheta)$ as the canonical variables, we obtain

$$
\begin{equation*}
t(z)=c_{0}^{-1} \int_{0}^{z}\left(\mathbf{l}\left(z^{\prime}\right) \frac{\partial H}{\partial \mathrm{I}}-H\right) \mathrm{d} z^{\prime} \tag{3.12}
\end{equation*}
$$

In the special case of an unperturbed waveguide which is homogeneous in the $z$ direction such that the system has the integrals of the motion $I=\left(I_{1}, I_{2}\right)$, we have

$$
\begin{align*}
& t_{0}(\mathrm{I}, z)=\left(\mathrm{I} \omega(\mathrm{I})-H_{0}(\mathrm{I})\right) \frac{z}{c_{0}}  \tag{3.13}\\
& \frac{c_{0}}{v_{0}(\mathrm{I})}=\mathrm{I} \omega(\mathbf{I})-H_{0}(\mathbf{I})
\end{align*}
$$

This last expression for the propagation velocity of a signal is identical to (2.29), which was obtained using a different method.

We consider the behavior of the optical path length of the ray $S(I, z)=c_{0} t(z)$ and the local velocity of the signal $v(z)$ in the neighborhood of an isolated resonance. Let $I_{0}$ be the value of the action $I$ for which the nonlinear resonance condition (3.2) is satisfied for a certain pair of numbers ( $m, s$ ). Then, using the solution of Hamilton's equations (3.4) and (3.5), it is not difficult to obtain the following expressions for $S(z)$ and $v(z)$ in the neighborhood of an isolated nonlinear resonance:

$$
\begin{align*}
S(I, z)= & S_{0}\left(I_{0}, z\right)-\frac{\bar{H}}{m^{2} \omega^{\prime}\left(I_{0}\right)} z+\frac{2 I_{0}}{m} \arccos \mathrm{dn}\left(\bar{\Omega}_{z, \rho}\right) \\
& +\left(1+\frac{I_{0} \omega^{\prime \prime}\left(I_{0}\right)}{2 \omega^{\prime}\left(I_{0}\right)}\right) \Delta I_{m s} \frac{\rho}{\bar{\Omega}_{z}} \int_{0}^{\mathrm{En}^{2}(u, \rho) \mathrm{d} u}  \tag{3.14}\\
\frac{c_{0}}{v(I, z)}= & \frac{c_{0}}{v_{0}\left(I_{0}\right)}+\frac{I_{0} \bar{\Omega}}{m} \operatorname{cn}(\bar{\Omega} z, \rho)-\frac{\bar{H}}{m^{2} \omega^{\prime}\left(I_{0}\right)} \\
& +\left(1+\frac{I_{0} \omega^{\prime \prime}\left(I_{0}\right)}{2 \omega^{\prime}\left(I_{0}\right)}\right) \Delta I_{m s} \frac{\bar{\Omega} \rho^{2}}{m} \mathrm{cn}^{2}(\bar{\Omega} z, \rho) \tag{3.15}
\end{align*}
$$

where $\mathrm{cn}(u, \rho)$ and $\mathrm{dn}(u, \rho)$ are Jacobian elliptic functions with modulus $\rho=\left[\left(\bar{H}+\bar{\Omega}^{2}\right) / 2 \bar{\Omega}^{2}\right]^{1 / 2} ; \bar{H}=H_{u}$ is of the order of the small perturbation parameter $\varepsilon \ll 1 ; \Delta I_{m s}$ is the width of the nonlinear resonance.

It is evident from (3.14) and (3.15) that the transmission time of the signal $t(z)$ and the local velocity of the signal $v(z)$ are modulated along the direction of propagation $z$ of the wave with a spatial frequency of order $\bar{\Omega}$. The differences $S(I, z)-S_{0}\left(I_{0}, z\right)$ and $c / v(I, z)-c / v_{0}\left(I_{0}\right)$ vary only slightly when the initial coordinates of the ray are changed. According to (3.15), the average propagation velocity of a signal along the ray

$$
\begin{equation*}
\dot{\bar{v}}=\lim _{z \rightarrow \infty} \frac{z}{t(z)}, \tag{3.16}
\end{equation*}
$$

remains equal to its unperturbed value $v_{0}\left(I_{0}\right)$ at the resonance value of the action $I_{0}$, to within terms of the order of the small perturbation parameter $\varepsilon \ll 1$. But the propagation velocity of a signal along the unperturbed ray $v_{0}(I)$ varies much more strongly within the width of the resonance $\Delta I_{m s}$ about the resonance action $I_{0}$. Indeed, from (3.3), (3.8), and (3.13) we have
$\Delta \frac{c_{0}}{v_{0}(I)}=\left(\frac{d}{\mathrm{~d} I} \frac{c_{0}}{v_{0}(I)}\right) \Delta I_{m s}=I \frac{\mathrm{~d} \omega}{\mathrm{~d} I} \Delta I_{m s} \propto(\varepsilon \alpha)^{1 / 2}>\varepsilon \propto \Delta \bar{v}$,
where $\Delta \bar{v}$ is the corresponding variation of the average velocity of the signal (3.16) of resonant rays over the width $\Delta I_{m s}$.

This result means that when a signal propagates along rays trapped in resonance, the broadening of a pulse is much less than in the case of propagation along unperturbed rays.

This effect can be important in practice in suppressing the broadening of signals caused by intermode dispersion in problems of wave propagation to great distances.

### 3.3. Examples

We discuss several examples of waveguide channels perturbed by periodic inhomogeneities along the $z$ axis.
3.3.1. We first consider a waveguide whose axis deviates periodically from a straight line along $z$. The index of refraction is given by the expression ${ }^{17}$

$$
\begin{align*}
& n(x, z)=n_{0}(x-f(z)), \\
& n_{0}^{2}(x)=n_{\infty}^{2}+\frac{n_{0}^{2} \Delta}{\operatorname{ch}^{2}(x / a)}, \tag{3.18}
\end{align*}
$$

where $f(z)=f_{0} \cos \Omega z$ describes the periodic deviation of the waveguide axis from the $z$ axis. For small deviations ( $\varepsilon=f_{0} / a \ll 1$ ) an expression for the perturbation can be found from (2.18) and (2.20) and has the form

$$
\begin{equation*}
\varepsilon V(x, p, z)=\frac{\varepsilon n_{1}(x, z)}{2 H_{0}}=-f(z) \frac{\mathrm{d} p}{\mathrm{~d} z} \tag{3.19}
\end{equation*}
$$

In the absence of the perturbation, the trajectory of the ray is periodic with spatial period $L=2 \pi / \omega(I)$, where

$$
\begin{align*}
& \omega(I)=\frac{I_{s}-I}{a^{2}\left|H_{0}(I)\right|}  \tag{3.20}\\
& H_{0}(I)=-\left[n_{\infty}^{2}+n_{0}^{2} \Delta\left(1-\frac{I}{I_{s}}\right)^{2}\right]^{1 / 2}, \quad I_{s}=n_{0} \Delta a
\end{align*}
$$

The quantities $I=I_{s}$ and $H_{s}=H_{0}\left(I_{s}\right)=-n_{\infty}$ correspond to the separatrix.

The condition for nonlinear resonance in this case takes the form

$$
\begin{equation*}
(2 m+1) \omega(I)=\Omega, \tag{3.21}
\end{equation*}
$$

since the expansion of the generalized momentum $p$ in a Fourier series contains only odd harmonics.

The widths of the nonlinear resonance near the separa$\operatorname{trix}\left(I \rightarrow I_{s}\right)$ in the action $I$ and frequency $\omega$ are given by

$$
\begin{align*}
& \frac{\Delta I}{I_{s}}=4\left(\frac{2}{\pi}\right)^{1 / 2}\left(\frac{\varepsilon \omega n_{\infty}}{\omega_{0} n_{0}}\right)^{1 / 2},  \tag{3.22}\\
& \frac{\Delta \omega}{\omega_{0}}=4\left(\frac{2}{\pi}\right)^{1 / 2}\left(\frac{\varepsilon \omega n_{0}}{\omega_{0} n_{\infty}}\right)^{1 / 2}, \quad \omega_{0}=\omega(0) .
\end{align*}
$$

3.3.2. As a second example we consider a homogeneous waveguide with perfectly reflecting walls, one of which is periodically corrugated (Fig. 2). The deviation of the corrugated wall from the unperturbed state is specified by the periodic function $f(z)=f(z+l)$ with spatial period $l$.

Rays in the unperturbed waveguide are straight lines which reflect periodically from the waveguide walls. The spatial period of the ray is

$$
\begin{equation*}
L=\frac{2 \pi}{\omega(I)}=\frac{2 \pi I_{0}^{2}\left|H_{0}(I)\right|}{I}=2 a \operatorname{ctg} \theta \tag{3.23}
\end{equation*}
$$

where

$$
\begin{align*}
& \omega(I)=\frac{\mathrm{d} H_{0}(I)}{\mathrm{d} I}=\left(n_{0}^{2}-\frac{2 I}{a_{0}}\right)^{-1 / 2} a_{0}^{-1},  \tag{3.29}\\
& H_{0}(I)=-\left(n_{0}^{2}-\frac{2 I}{a_{0}}\right)^{1 / 2}
\end{align*}
$$

are the unperturbed oscillation frequency of the ray along the $z$ axis and the Hamiltonian, respectively.

The periodic perturbation will have the strongest effect on rays whose frequencies are near trajectory parametric resonance

$$
\begin{equation*}
2 \omega(I)=\Omega . \tag{3.30}
\end{equation*}
$$

Therefore we can neglect the rapidly oscillating terms in (3.28). Then, keeping only the resonant terms, (3.28) can be written in the form of Hamilton's equations

$$
\begin{equation*}
\frac{\mathrm{d} I}{\mathrm{~d} z}=-\frac{\partial \bar{H}}{\partial \psi}, \quad \frac{\mathrm{~d} \psi}{\mathrm{~d} z}=\frac{\partial \bar{H}}{\partial I}, \tag{3.31}
\end{equation*}
$$

where
$\bar{H}(I, \psi)=2 H_{0}(I)-I \Omega-\varepsilon I \omega(I) \sin \psi, \quad \psi=2 v-\Omega z$.
We note that $H(\bar{I}, \psi)$ is an integral of the motion. We consider the motion of the ray in the phase plane ( $I, \psi$ ). We introduce the dimensionless variables $\xi=2 I / a_{0}, x=a_{0} \Omega / 2$ and we also put $n_{0}=1$. Let $I_{0}$ be the value of the action $I$ satisfying the condition (3.30).

In Fig. 3 the contours correspond to constant values of the Hamiltonian $\bar{H}(I, \psi)=$ const in the phase plane $(\xi, \psi)$ and were obtained numerically for the perturbation parameter $\varepsilon=0.01$ and for three values of the dimensionless quantity $x=1.05,1.4$, and 3.0 , which correspond to the resonance values of the dimensionless action $\xi_{0}=1-\varkappa^{-2}=0.093,0.49$, and 0.889 . It is evident from Fig. 3 that inside a small neighborhood around the resonance value of the action $\xi_{0}=2 I_{0} / a_{0}$ the ray dynamics is fundamentally different from the unperturbed case. Some of the rays begin to perform finite phase oscillations in $\psi$ (curves $I$ and 2), i.e. they are trapped in a resonance. The rays not trapped by the resonance correspond to curves of the type 4, which can be obtained by a simple deformation of the unperturbed curves $\xi=$ const. These two types of curves are separated by the separatrix 3 .

Therefore a periodic perturbation causes modulations in the oscillation amplitude of a ray trapped in the parametric resonance ( 3.30 ). The amplitude oscillates about the resonance amplitude $d_{0}=\left(2 I_{0} a_{0}\right)^{1 / 2}$. We estimate the width of the parametric resonance. For small values of the resonance action $\xi \ll 1$ the width of the parametric resonance is

$$
\begin{equation*}
\Delta \xi=2\left[8(\kappa-1) \varepsilon+\varepsilon^{2}\right]^{1 / 2} . \tag{3.32}
\end{equation*}
$$

For resonance values $\xi_{0}$ close to unity, the width of the resonance is

$$
\begin{equation*}
\Delta \xi=2\left[2 \varepsilon\left(1-x^{-2}\right)\right]^{1 / 2} / x \tag{3.33}
\end{equation*}
$$

It is not difficult to show that this estimate corresponds to the width (3.8) in the approximation of nonlinear resonance.

These results show that the nonlinearity of the ray oscillations leads to a restriction on the rolling amplitude of the ray and prevents the leakage of radiant energy from the


FIG. 3. Phase portrait of rays, showing parametric resonance of rays in a parabolic waveguide with periodically varying width. Curves 1,2 ) rays trapped in resonance, 3) separatrix, 4) nontrapped rays.
waveguide channel due to the periodic inhomogeneities along its axis.

## 4. NUMERICAL ANALYSIS. "DEVIL'S STAIRCASE"

4.1.

We consider the example of Sec. 3 of wave propagation in a waveguide with a corrugated wall with the help of the discrete mapping method.

Let the source of the radiation be located in the upper wall of the waveguide at $z=0$. Let $z_{n}$ be the longitudinal coordinate of the ray at the point of $n$th reflection of the ray from the upper unperturbed wall of the waveguide and let $\theta_{n}$ be the angle between the ray and the $z$ axis after the $n$th reflection. The relation between the quantities $\left(z_{n}, \theta_{n}\right)$ and ( $z_{n+1}, \theta_{n+1}$ ) after a single reflection from the corrugated wall is determined directly from the geometry of the problem and is given by the exact mapping:

$$
\begin{align*}
& \theta_{n+1}=\theta_{n}-2 \operatorname{arctg} f^{\prime}\left(\psi_{n}\right) \\
& \psi_{n}=z_{n}+\left(a+f\left(\psi_{n}\right)\right) \operatorname{ctg} \theta_{n},  \tag{4.1}\\
& z_{n+1}=\psi_{n}+\left(a+f\left(\psi_{n}\right)\right) \operatorname{ctg} \theta_{n+1}, \quad f^{\prime}(z)=\frac{\mathrm{d} f}{\mathrm{~d} z},
\end{align*}
$$

where the longitudinal coordinate $\psi_{n}$ is the point at which the ray is reflected from the lower periodically corrugated wall after its $n$th reflection from the upper wall.

The mapping (4.1) assumes that the ray is reflected from the perturbed wall only once per period $l$. However, when the ray propagates at small angles $\theta$ to the $z$ axis the ray can be reflected more than once per period. This case is described in Ref. 18.

It can be shown directly that the mapping ( $E_{n}, z_{n}$ ) to $\left(E_{n+1}, z_{n+1}\right)$, where $E_{n+1}=\cos \theta_{n+1}$, preserves area, i.e. the Jacobian is


FIG. 4. Fractal dependence of the normalized spatial frequency $x$ on the angle of emergence $\theta_{0}$ of the rays from the source calculated using the step size $\Delta \theta_{0}=0.01$. Insert: tenfold magnification of the rectangular region calculated with the step size $\Delta \theta_{0}=10^{-3}$. The waveguide parameters were $a / l=1 / 3$ and $b / a=3 \cdot 10^{-3}$.

$$
\left|\frac{\partial\left(E_{n+1}, z_{n+1}\right)}{\partial\left(E_{n}, z_{n}\right)}\right|=1
$$

It is important to note that the mapping (4.1) reduces the exact Ulam mapping in the problem of Fermi acceleration ${ }^{2,3}$ when the angle and the perturbation are small.

The results of a numerical analysis of the mapping (4.1) are shown in Fig. 4. The dependence of the normalized spatial oscillation frequency of the ray along the $z$ axis

$$
\begin{equation*}
\kappa=\lim _{N \rightarrow \infty} \frac{N l}{z_{N}} \tag{4.2}
\end{equation*}
$$

is shown as a function of the initial angle of departure from the source $\theta_{0}$, as calculated with a step size of $\Delta \theta_{0}=0.01$. The insert to Fig. 4 shows the area enclosed by the rectangle magnified by a factor of ten. The step size for the calculations shown in the insert was $\Delta \theta_{0}=10^{-3}$. The waveguide parameters were chosen as $a / l=1 / 3$ and $b / l=10^{-2}$. We note that the spatial oscillation frequency $\varkappa$ of the ray is equivalent to the so-called rotation angle for the dynamical system.

In the unperturbed waveguide the dependence of $x$ on $\theta_{0}$ is given by the smooth function

$$
x=\frac{l}{2 a} \operatorname{tg} \theta_{0}
$$

which follows from (3.23) and (4.2). This dependence changes completely in the presence of a periodic perturbation. The calculations show that there exist intervals $\Delta \theta(P / Q)$ of angles of departure of the ray from the source for which the frequency $\varkappa$ is constant and rational $\varkappa=P / Q$, where $P$ and $Q$ are integers.

Another feature of $\chi\left(\theta_{0}\right)$ is that it is a self-similar function of the step size.


FIG. 5. Dependence of $N\left(\Delta \theta_{0}\right)$ on the step size of the calculation $\Delta \theta_{0}$.

The dependence of the spatial frequency $\varkappa$ on the angle $\theta_{0}$ shown in Fig. 4 is called a "devil's staircase". There are a number of other problems of the Hamiltonian type in which a devil's staircase appears in the problem (examples are the one-dimensional Ising model ${ }^{48}$ and the Frenkel-Kontorova model. ${ }^{49}$ The devil's staircase is a special case of a fractal object. ${ }^{52-54}$ The fractal behavior of the scattering angle as a function of impact parameter is observed in the classical scattering of a particle by a potential field. ${ }^{50,51}$

The fractal dimensionality of the dependence of $\varkappa$ on $\theta_{0}$ is given by the exponent $D$ in the power-law dependence of the number of gaps $N\left(\Delta \theta_{0}\right)$ on the step size $\Delta \theta_{0}$ in the graph of $\psi$ against $\theta_{0}:^{52}$

$$
\begin{equation*}
N\left(\Delta \theta_{0}\right)=\left(\frac{1}{\Delta \theta_{0}}\right)^{D} \tag{4.3}
\end{equation*}
$$

We determine the fractal dimensionality $D$ in the dependence of $\varkappa$ on $\theta_{0}$ for a waveguide with the parameters $a / l=1 / 3$ and $b / l=0.001$. The dependence $N\left(\Delta \theta_{0}\right)$ on step size $\Delta \theta_{0}$ is shown in Fig. 5 for $0.2<\theta_{0}<0.3$. It follows from this graph that $D=0.45$.

The physical cause of this phenomenon is nonlinear resonance. It is not difficult to see from Fig. 4 that the steps in the dependence of $\varkappa$ on $\theta_{0}$ lie in the neighborhood of the resonance angles $\theta^{(m, s)}$ defined by the nonlinear resonance condition (3.25). It follows from (3.2), (3.23), and (3.24) that the value of the spatial frequency $\psi$ corresponding to the resonance ( $m, s$ ) is determined by $m$ and $2 s+1$ :

$$
\begin{equation*}
x=\frac{\omega(I) l}{2 \pi}=\frac{l}{2 a} \operatorname{tg} \theta=\frac{m}{2 s+1} . \tag{4.4}
\end{equation*}
$$

The width of a step $\Delta \theta(P / Q)$ should be determined by the width of the corresponding nonlinear resonance (3.26). Rays with the oscillation frequency $\varkappa=1 / 1$ have the largest width $\left(\Delta \theta(1 / 1)=0.260\right.$ for the step size $\left.\Delta \theta_{0}=10^{-2}\right)$. From (3.26) we obtain for this resonance ( $m=1, s=0$ ) the next estimate $\Delta \theta^{(1,0)}=0.245$ for the waveguide parameters $b / l=10^{-2}$ and $a / l=1 / 3$. This estimate agrees closely with the numerical result.

Ray dynamics in the neighborhood of a nonlinear resonance can be studied with the help of the portrait of the mapping (4.1) in the phase plane ( $z, \theta$ ). Figure 6 shows the phase portrait for the waveguide parameters $b / l=5 \cdot 10^{-3}$ and $a / l=1 / 3$. The coordinate $\xi_{n} \equiv\left\{\left(z_{n} / l\right)+(1 / 2)\right\}$, is plotted along the vertical axis and $\theta_{n}$ is plotted along the horizontal. We see that the coordinates of rays trapped in nonlinear resonance form regular closed curves.

A more detailed analysis of Figs. 4 and 6 shows the existence of a random distribution of spatial frequencies $\varkappa$


FIG. 6. Phase portrait of the mapping (4.1) in the $\xi=\{(z / l)+(1 / 2)\}, \theta$ plane for 15 initial angles of emergence from the source (angular width $\Delta \theta=0.1$ ).
and coordinates $\left(\xi_{n}, \theta_{n}\right)$ for small angles of departure $\theta_{0}$ of the ray from the source. This random distribution is associated with a stochastic instability of rays in this region of angles and will be discussed in more detail in Sec. 6.

## 4.2.

We consider the effect of inhomogeneities in the index of refraction of the medium along the transverse $x$ axis. Let

$$
\begin{equation*}
n_{0}(x)=\left(1-\frac{a^{2} x^{2}}{a^{2}}\right)^{1 / 2}(0<x<a+f(z)) \tag{4.5}
\end{equation*}
$$

where $\alpha<1$ is a constant characterizing the inhomogeneity of the index of refraction.

In this case rays leaving the upper unperturbed wall of the waveguide at small, angles $\theta$ less than the value $\theta_{c}=\arcsin \alpha$ do not reach the lower corrugated wall (curves of type 1 in Fig. 7). Rays begin to be reflected from the lower wall when $\theta>\theta_{\mathrm{c}}$ (type 2 curves). In this case the exact mapping (4.1) has the form
$\psi_{n}=z_{n}+\frac{a H_{n}}{\alpha} \arcsin \frac{a\left(1+f\left(\psi_{n}\right) / a\right)}{\left(1-H_{n}^{2}\right)^{1 / 2}}$,
$H_{n+1}=n_{0}\left(a+f\left(\psi_{n}\right)\right) \cos \left[\arccos \frac{H_{n}}{n_{0}\left(a+f\left(\psi_{n}\right)\right)}-2 \operatorname{arctg} f^{\prime}\left(\psi_{n}\right)\right]$,

$$
\begin{equation*}
z_{n+1}=\psi+\frac{a H_{n+1}}{a} \arcsin \frac{a\left(1+f\left(\psi_{n}\right) / a\right)}{\left(1-H_{n+1}^{2}\right)^{1 / 2}} \tag{4.6}
\end{equation*}
$$



FIG. 7. Types of rays in a waveguide with a nonuniform profile (4.5): 1 ) rays which do not reach the corrugated wall, 2) rays reflected from the corrugated wall.
where $H_{n}=n_{0} \cos \theta_{n}$ is conserved between reflections from the corrugated wall, and the coordinates $\psi_{n}, z_{n}$, and $z_{n+1}$ are shown in Fig. 7.

Figure 8 shows the dependence of the reciprocal of the normalized spatial frequency of the ray $f=x^{-1}$ on the quantity $H=\cos \theta_{0}$, where $\theta_{0}$ is the initial angle of departure of the ray from a source located in the upper wall. The mapping was analyzed numerically for the waveguide parameters $a / l=0.4, \alpha=0.01$, and $b / l=0.01$. The perturbation was of the form (3.24). Each of the steps in the devil's ladder lies in the neighborhood of a resonance value of $H$ given by the nonlinear resonance condition

$$
\begin{equation*}
m \omega(H)=s \cdot \frac{2 \pi}{l}, \tag{47}
\end{equation*}
$$

where

$$
\omega(H)=\frac{\pi a}{a H \arcsin \left[\alpha\left(1-H^{2}\right)^{1 / 2}\right]}
$$

is the spatial oscillation frequency of the ray in a waveguide with the profile (4.5) in the absence of the perturbation $(f(z) \equiv 0)$. An expression for $\omega(I)$ can be obtained by intro-


FIG. 8. Fractal dependence of the quantity $x^{-1}$ on $H=\cos \theta_{0}$, where $\theta_{0}$ is the initial angle of emergence of the ray from the source.
ducing action-angle variables and using the relations given in Sec. 2.5.

The width of the nonlinear resonance in $H$ is found from (3.8)
$\Delta H=4 \sqrt{2} \frac{m}{s}\left(\frac{b a}{a q}\right)^{1 / 2}$

$$
\begin{equation*}
\times\left[\frac{\left(1-H^{2}\right)^{1 / 2}}{\pi^{2} m^{2}-q^{2}} \frac{1-(-1)^{m} \cos q}{1+\frac{k}{q\left(1-k^{2}\right)^{1 / 2}} \frac{H^{2}}{1-H^{2}}}\right]^{1 / 2}, \tag{4.8}
\end{equation*}
$$

where $k=\alpha /\left(1-H^{2}\right)^{1 / 2}, q=\arcsin k$. This analytical estimate for the resonance width is in satisfactory agreement with the numerical results.

An aspect of this problem that is different from the case of a waveguide with a uniform profile is the absence of a random distribution of the quantity $x$ in the small-angle $\left(\theta_{0}\right)$ region.

## 4.3.

As noted in Sec. 3.2, the most important characteristic of radiation in a waveguide is the propagation velocity of a signal along the waveguide axis. We analyze this quantity numerically for a homogeneous waveguide with a corrugated wall.

Let $S\left(\theta_{0}, z\right)$ be the optical path length along the ray as a function of the angle $\theta_{0}$ of deprature of the ray from a source located at the point $(x=0, z=0)$. Here $z$ is the distance to the plane of observation $z=$ const.

Figure 9 shows the numerical results for the pathlength difference

$$
\begin{equation*}
\Delta S\left(\theta_{0}, z\right)=S\left(\theta_{0}, z\right)-z \tag{4.9}
\end{equation*}
$$

as a function of the angle $\theta_{0}$ for the waveguide parameters $a / l=1 / 3, b / l=0.005$, and $z=50 l$. The smooth curve shows the dependence in the unperturbed case ( $b=0$ ).

We see from Fig. 9 that there exist intervals of angles $\Delta \theta(P / Q)$ for which the optical path $S\left(\theta_{0}, z\right)$ remains nearly constant, whereas in the unperturbed case the optical path varies significantly within these intervals. This behavior of $S\left(\theta_{0}, z\right)$ is analogous to the devil's staircase behavior of the spatial frequency of the ray considered in Sec. 4.1.

The largest value of the width $\Delta \theta(P / Q)$ corresponds to the fundamental resonance ( $m=1, s=0$ ) and is equal to $\Delta \theta(1 / 1)=0.19=10.9^{\circ}$ for the perturbation parameter $b / l=0.005$. A qualitative estimate of the width of the fundamental resonance ( 1,0 ) according to ( 3.25 ) gives the value $\Delta \theta^{(1,0)}=0.173=10.0^{\circ}$, which is in satisfactory agreement with the numerical result. The maximum difference of the optical path lengths corresponding to rays of the fundamental resonance is nearly two orders of magnitude smaller than the difference for an unperturbed waveguide. This result is consistent with the results of Sec .4 for the transmission time of a signal and its propagation velocity along resonant rays.

## 5. RAY DYNAMICS IN AN INHOMOGENEOUS MOVING MEDIUM

In this Section we consider some features of the ray trajectories in inhomogeneous moving media. In sound propagation in the atmosphere and ocean, which are exam-


FIG. 9. Fractal dependence of the optical path difference $\Delta S\left(\theta_{0}, z\right)=S\left(\theta_{0}, z\right)-z$ on the initial angle $\theta_{0}$. The curve describes an unperturbed waveguide.
ples of moving media, the typical linear dimension characterizing the variation of the parameters of the medium in the horizontal direction is usually much larger than the corresponding linear dimension in the vertical direction. Therefore many of the features of sound propagation in the atmosphere and ocean can be explained in terms of the model of a stratified medium, in which the parameters of the medium depend only on the vertical coordinate. Then the ray equations with the Hamiltonian (2.11) describing sound propagation in the moving medium are completely integrable, since there are three integrals of the motion: the Hamiltonian $H=0$ and the two horizontal components of the generalized momentum $\mathbf{P}$. The corresponding ray equations can be reduced to quadratures in this case (see the review article of Ref. 28).

When the parameters of the medium vary along the horizontal coordinates exact solutions of the ray equations do not exist. Approximate methods must be used to study the ray trajectories in this case.

We consider sound propagation in a model of the ocean in which the speed of sound and the flow velocity vary slowly in the horizontal direction. ${ }^{30}$ The solution is found with the help of the adiabatic invariant method of classical mechanics. ${ }^{35}$

For simplicity we consider sound propagation at small angles to the $z$ axis. Then we can use the paraxial approximation for the ray equations (2.14) and (2.15). The $x$ axis is taken along the vertical. We assume that the parameters of the medium are slowly varying functions of the horizontal coordinate $z$ defining the main direction of propagation of the wave:

$$
\bar{\varepsilon}=\bar{\varepsilon}(x, z), \quad \mathrm{v}=\mathrm{v}(x, z)
$$

The quantity $\bar{\varepsilon}(x, z)$ can be expressed in terms of the inhomogeneous part of the speed of sound $\Delta c(x, z)=c(x, z)-c_{0}$ :

$$
\bar{\varepsilon}=-\frac{2 \Delta c(x, z)}{c_{0}}
$$

We consider horizontal flow and assume that it also varies slowly along the $z$ axis:

$$
\begin{equation*}
v(x, z)=\left(0, v_{y}(x, z), v_{z}(x, z)\right) . \tag{5.1}
\end{equation*}
$$

Since the parameters of the medium are independent of the transverse coordinate $y$, it follows from the equations of motion (2.15) that the $y$ component of the generalized momentum is conserved:

$$
\begin{equation*}
P_{y}=p_{y}-v_{y}(x, z)=P_{y}^{0}=\text { const. } \tag{5.2}
\end{equation*}
$$

Hence the Hamiltonian system (2.14) and (2.15) with two degrees of freedom can be reduced to a system with one degree of freedom with the Hamiltonian

$$
\begin{equation*}
\bar{H}=\frac{1}{2} P_{x}^{2}+U(x, z), \tag{5.3}
\end{equation*}
$$

where

$$
U(x, z)=\frac{1}{c_{0}}\left(\Delta c(x, z)+v_{z}(x, z)\right)+\frac{1}{c_{0}} P_{y} v_{y}(x, z)
$$

We consider waveguide propagation. Assume that the typical linear dimension $l$ characterizing the variation of the speed of sound $c(x, z)$ and flow velocity $v(x, z)$ along the propagation direction $z$ is much larger than the cycle length $L$ of the ray: $l \gg L$. Then the action, defined by the relation
$I=\frac{1}{2 \pi} \oint_{p_{x}}(x, z) \mathrm{d} x, \quad p_{x}(x, z)=[2(\bar{H}-U(x, z))]^{1 / 2}$,
where the integration goes along the $x$ axis for one cycle, is an adiabatic invariant and is constant in the horizontal direction $z$. In terms of the new action-angle variables $(I, \vartheta)$, where the angular variable $\vartheta$ is defined by the relation

$$
\vartheta=\frac{\partial}{\partial I} \int_{x_{0}}^{x} p\left(x^{\prime}, z\right) \mathrm{d} x^{\prime}
$$

the Hamiltonian $\bar{H}=\bar{H}(I, z)$ is a function of the action $I$, while $z$ appears as a parameter. The ray trajectories $x=x(I, \vartheta)$ and $p=p(I, \vartheta)$ are periodic functions of the angular variable $\vartheta=\omega(I ; z) z+\vartheta_{0}$ with period $2 \pi$. The spatial oscillation frequency of the ray $\omega(I ; z)=d \bar{H}(I, z) / d I$ and the spatial period $L=2 \pi / \omega(I ; z)$ are local functions of the longitudinal coordinate $z$.

The coordinates of the ray along the $y$ axis are obtained from (5.2) and the relation $d y / d z=p_{y}$ :

$$
\begin{align*}
y & =y_{0}+P_{y}^{0}\left(z-z_{0}\right)+\frac{1}{c_{0}} \int_{z_{0}}^{z} v_{y}(x, z) \mathrm{d} z \\
& =y_{0}+p_{y}^{0}\left(z-z_{0}\right)+\frac{1}{c_{0}} \int_{z_{0}}^{z}\left(v_{y}(x, z)-v_{y}\left(x_{0}, z_{0}\right)\right) \mathrm{d} z \tag{5.5}
\end{align*}
$$

where $p_{y}^{0}$ and $P_{y}^{0}$ are the $y$ components of the tangent vector to the ray $p$ and the normal vector to the wave front in the initial plane $z=z_{0}$, and ( $x_{0}, y_{0}$ ) are the initial coordinates of the ray in this plane.

In the case of uniform flow $v=$ const with a nonzero transverse component $v_{y}$, the ray will lie in a vertical plane

$$
\begin{equation*}
y=y_{0}+\left(P_{y}+\frac{v_{y}}{c_{0}}\right)\left(z-z_{0}\right) \tag{5.6}
\end{equation*}
$$

making a certain angle with the propagation direction of the wave along the $z$ axis.

If the flow is nonuniform with depth, but $v_{y}=0$, then the moving medium is equivalent to a nonmoving medium with an effective speed of sound

$$
\Delta c_{\mathrm{eff}}=\Delta c(x, z)+v_{z}(x, z) c_{0}^{-1}
$$

In this case the rays will lie in the vertical plane.
The two-dimensional property of the rays is lost when the transverse component $v_{y}$ is nonzero. In this case the ray trajectories will be three-dimensional curves in space.

We consider a model of the ocean with linear profiles for the speed of sound $c(x, z)$ and flow velocity $v(x, z)$ along the vertical coordinate $x$ :

$$
\begin{equation*}
c(x, z)=c_{0}\left(1+g_{0}(z) x\right), \quad \mathbf{v}(x, z)=v_{0}(z)\left(1-\frac{x}{h}\right) \tag{5.7}
\end{equation*}
$$

where the speed of sound increases with depth, while the flow velocity decreases. Here $h$ is the depth of the ocean and the gradients of the speed of sound $g_{0}(z)$ and flow velocity $v_{0}(z)$ are slowly varying in the $z$ direction. In this medium a waveguide channel is formed near the surface and sound waves propagate inside the channel and are reflected repeatedly from the surface of the ocean.

The ray trajectories in the $x$ direction are given by

$$
\begin{align*}
& x=\frac{1}{2} g(z) L^{2}(I ; z)\left[\frac{\vartheta}{2 \pi}-\left(\frac{\vartheta}{2 \pi}\right)^{2}\right],  \tag{5.8}\\
& p_{x}=g(z) L(I ; z)\left(\frac{1}{2}-\frac{\vartheta}{2 \pi}\right),
\end{align*}
$$

where

$$
\begin{align*}
& \vartheta=\omega(I ; z) z+\vartheta_{0} \\
& L(I ; z)=\frac{2 \pi}{\omega(I ; z)}=\frac{\pi g(z)}{\sqrt{2}}\left(\frac{I}{I_{0}(z)}\right)^{-1 / 3},  \tag{5.9}\\
& I_{0}(z)=\frac{2 \sqrt{2}}{3 \pi g(z)} \\
& g(z)=g_{0}(z)-\left(v_{0 x}(z)+p_{y}^{0} v_{0 y}(z)\right)\left(c_{0} h\right)^{-1}
\end{align*}
$$

The ray coordinates in the $y$ direction are found by substituting (5.8) into (5.5). The action I is related to the initial coordinates of the ray:

$$
\begin{equation*}
\left.I=\mathfrak{K}\left(p_{x}^{0}\right)^{2}+2 g\left(z_{0}\right) x_{0}\right]^{3 / 2}\left(3 \pi g\left(z_{0}\right)\right)^{-1} \tag{5.10}
\end{equation*}
$$

It follows from (5.5) and (5.8) that in the presence of transverse flow the ray is a three-dimensional curve whose projection onto the $x, z$ plane is a parabola which is periodically reflected from the surface of the ocean $x=0$ (Fig. 10). The maximum distance of the ray from the surface of the ocean is $x_{m}=g(z) L^{2}(I ; z) / 8$. A ray reflected from the surface of the ocean at an angle $\theta_{0}=\arcsin p_{x}^{0}$, where $p_{x}^{0}=(2 h g(z))^{1 / 2}$, reaches the bottom of the ocean (the angle $\theta$ is measured with respect to the surface of the ocean $x=0$ ).

We see from Fig. 10 that after one cycle the ray is displaced along the transverse direction $y$ by a distance $\Delta y$ from the initial plane of departure of the ray from the source. This effect is called transverse drift of the ray. It is analogous to the drift of a charged particle in a transverse magnetic field


FIG. 10. Trajectory of a ray in a model of the ocean with linear speed of sound and flow velocity profiles. The initial coordinates of the ray are $x_{0}=y_{0}=p_{y}^{0}=0$.
in the presence of a reflecting surface. ${ }^{2}$
The transverse displacement $\Delta y$ of the ray per cycle from the initial plane of departure of the ray can be calculated from (5.5):

$$
\begin{align*}
\Delta y & =\frac{1}{c_{0}} \int_{z}^{z+L}\left(v_{y}\left(x\left(z^{\prime}\right), z^{\prime}\right)-v_{y}(x(z), z)\right) \mathrm{d} z^{\prime} \\
& =\frac{1}{c_{0}} \oint\left(v_{y}(x, z)-v_{y}\left(x_{0}, z_{0}\right)\right) p_{x}^{-1}(x, z) \mathrm{d} x \tag{5.11}
\end{align*}
$$

where

$$
\begin{align*}
p_{x}(x, z)= & (2(\bar{H}-U(x, z)))^{1 / 2} \\
= & \sqrt{2}\left[\bar{H}(z)-2\left(\Delta c(x, z)+v_{z}(x, z)\right) c_{0}^{-1}\right. \\
& \left.-2 P_{y}\left(v_{y}(x, z)-v_{y}\left(x_{0}, z_{0}\right)\right) c_{0}^{-1}\right]^{1 / 2} \tag{5.12}
\end{align*}
$$

Using (5.3) and (5.9), the transverse displacement of the ray per cycle can be rewritten in the form

$$
\begin{equation*}
\Delta y=-2 \pi \frac{\partial I}{\partial P_{y}} \tag{5.13}
\end{equation*}
$$

where $I$ is the action given by (5.3).
For an ocean with linear profiles of the speed of sound and flow velocity (5.7), it is not difficult to obtain from (5.9) and (5.10) the following expression for the transverse displacement of the ray:

$$
\begin{equation*}
\Delta y=-\frac{v_{0 y}}{c_{0} h} g(z) L^{3}(I ; z) \tag{5.14}
\end{equation*}
$$

for a ray with initial coordinates $x_{0}=p_{y}^{0}=0$.
We consider the transverse drift of a ray in an underwater waveguide channel. In this case the ray does not reach the surface or the bottom of the ocean and a transverse drift of the ray along the $y$ axis only occurs when the second derivative of the flow velocity with respect to the vertical coordinate $x$ is nonzero: $\mathrm{d}^{2} v_{y}(x) / \mathrm{d} x^{2} \neq 0$, which implies that the vortex vector of the moving medium

$$
\mathscr{H}=-\operatorname{rot} \mathbf{v}
$$

is nonuniform along the vertical coordinate $x$. Indeed, we have from (5.11)

$$
\begin{align*}
& \Delta y(z)=\frac{1}{c_{0}} \oint\left(v_{y}(x, z)-v_{y}\left(x_{0}, z_{0}\right)\right) p_{x}^{-1}(x, z) \mathrm{d} x . \\
& \quad \approx \oint \mathrm{d} x p_{x}^{-1}(x, z)\left[\left.\frac{\mathrm{d} v_{y}}{\mathrm{~d} x}\right|_{x=x_{0}}\right. \\
& \left.\times\left(x-x_{0}\right)+\left.\frac{\mathrm{d}^{2} v_{y}}{\mathrm{~d} x^{2}}\right|_{x=x_{0}}\left(x-x_{0}\right)^{2}+\ldots\right] . \tag{5.15}
\end{align*}
$$

The integral of the first term in (5.15) vanishes, therefore the transverse shift $\Delta y$ of the ray per cycle along the $y$ axis is determined mainly by the second term:

$$
\begin{equation*}
\Delta y(z)=\frac{1}{c_{0}} \frac{\mathrm{~d}^{2} v_{y}}{\mathrm{~d} x^{2}} \oint \mathrm{~d} x \frac{\left(x-x_{0}\right)^{2}}{p_{x}(x, z)} \tag{5.16}
\end{equation*}
$$

This type of transverse displacement of the ray is not associated with the reflection of the ray from the surface or the bottom of the ocean, but is determined by the rate of change of the vortex vector along the vertical axis. It is analogous to the drift of charged particles in a nonuniform transverse magnetic field. ${ }^{4}$

We note that in contrast to a stratified moving medi$u m,{ }^{46}$ in the case of a continuous inhomogeneity in the propagation direction of the wave, the transverse drift of the ray per cycle $\Delta y$ is not the same for all cycles, but varies slowly from cycle to cycle.

The adiabatic approximation used here becomes inapplicable when the linear dimension characterizing the inhomogeneities of the medium along the propagation direction are of the order of the cycle length $L$ of the ray. Then one can have a resonant interaction between the ray and the inhomogeneities, which significantly changes the nature of ray propagation in an inhomogeneous moving medium.

## 6. DYNAMICAL CHAOS OF RAYS

One of the most important properties of a dynamical system is the possibility of random motion induced by regular (nonrandom) forces. This phenomenon is called dynamical chaos, ${ }^{1-4}$ and is also observed in the ray theory of wave propagation in regularly inhomogeneous waveguide media. ${ }^{17-26}$

It has been shown (see Refs. 1-4, for example) that in the neighborhood of the separatrix a periodic perturbation forms a so-called stochastic layer in which the particle trajectories are random. The most important feature of this phenomenon is that the stochastic layer is formed for arbitrary (shape and magnitude) periodic perturbations and only the width of the layer is determined by the nature of the perturbation.

The fact that rays show the same properties in regularly inhomogeneous waveguide media was noted for the first time in Refs. 17 and 21. In particular, it was shown in Refs. 18 and 20 that a stochastic layer is formed not only near the separatrix, but also in other regions of the ray phase space far from the separatrix.

Below we consider several examples of regular waveguide channels in which there is stochastic instability of rays. The condition for a chaotic instability of rays can be
studied qualitatively using the condition of overlap of nonlinear resonances: the motion of the system becomes chaotically unstable when $K \geqslant 1$, where $K$ is defined in (3.9) (the Chirikov criterion ${ }^{1-4}$ ).

### 6.1. Regular wavegulde with index of refraction (3.18)

It follows from (3.21) that in the neighborhood of the separatrix $\omega \rightarrow 0$ the distance between neighboring resonances is $\delta \omega \approx 2 \omega^{2} / \Omega$. It then follows from this relation and (3.32) that when the separatrix is approached $(\omega \rightarrow 0)$ the quantity $\delta \omega$ goes to zero more rapidly than the resonance width $\Delta \omega$. Hence $K$ reaches unity at a certain value $\omega_{c}$ and the condition $K \geqslant 1$ is satisfied. Therefore a stochastic layer forms near the separatrix and its width $\omega_{c}$ is determined by the condition $K=1$ :

$$
\begin{equation*}
\frac{\omega_{\mathrm{c}}}{\omega_{0}}=\left[4 \pi^{-1 / 2} \varepsilon\left(\frac{\Omega}{\omega_{0}}\right)^{2} \frac{n_{0}}{n_{\infty}}\right]^{1 / 3} . \tag{6.1}
\end{equation*}
$$

From (3.20) and (6.1) we obtain the following estimates for the width of the stochastic layer in terms of the action $I$ and H:

$$
\begin{align*}
& \frac{\delta I}{I_{\mathrm{s}}}=\frac{n_{\infty} \omega_{\mathrm{c}}}{n_{0} \omega_{0}}=\left[4 \pi^{-1 / 2} \varepsilon\left(\frac{\Omega}{\omega_{0}}\right)^{2} \frac{n_{\infty}^{2}}{n_{0}^{2}}\right]^{1 / 3},  \tag{6.2}\\
& \frac{\delta H}{\left|H_{\mathrm{s}}\right|}=\left(\frac{\delta I}{I_{\mathrm{s}}}\right)^{2} .
\end{align*}
$$

If the initial state of the ray lies within the stochastic layer (6.2), its path in space along the $z$ axis will be like that in diffusion. Because of diffusion, the ray reaches the unperturbed separatrix and escapes from the waveguide channel. Therefore the inhomogeneities of the medium, like the perturbation, lead to an effective decrease in the width of the waveguide channel.

We note that from (6.2) the width $\delta I$ of the stochastic layer is proportional to the cube root of the small perturbation parameter. Therefore even a small perturbation can lead to a stochastic layer of significant width.

### 6.2. Waveguide with a corrugated wall

According to (3.25) and (3.26), the distance between neighboring resonances $\delta \theta$ decreases more rapidly than the resonance width $\Delta \theta$ when the angle $\theta$ approaches zero. Hence the nonlinear resonances overlap in the region $\theta<\theta_{c}$. The critical angle $\theta_{c}$ is defined by the condition $K=1$

$$
\begin{equation*}
\theta_{c}=\arcsin \left[\frac{8}{\pi}\left(\frac{a b}{l^{2}}\right)^{1 / 2}\right] . \tag{6.3}
\end{equation*}
$$

Hence it follows that rays propagating at small angles $\theta$ to the $z$ axis become chaotically unstable. The stochastic layer is defined by the inequality $0<\theta<\theta_{c}$. This result explains the numerical results of Sec. 4 using the mapping (4.1). Figure 4 shows that the spatial frequency distribution $x$ is random in the small-angle region $0<\theta_{c}<0.16$ of propagation. The value of $\theta_{c}$ obtained from the analytical result (6.3) is $\theta_{c}=0.148$ for the waveguide parameters considered above; this is in satisfactory agreement with the numerical analysis.

Figure 11 shows the phase portrait of the mapping (4.1) in the phase plane ( $\xi, \theta)$ in the interval $0<\theta_{0}<0.3$ for the waveguide parameters $a / l=1 / 3$ and $b / a=1.5 \cdot 10^{-2}$. The trajectory points determined by the mapping (4.1) are randomly distributed in the ( $\xi, \theta$ ) plane in the small-angle region, where the rays are chaotically unstable.

This is also consistent with the ray divergences calculated for three values of the initial coordinates of the ray in the phase plane ( $\xi, \theta$ ). In Fig. 12 we show the dependence of the quantity

$$
\begin{equation*}
\frac{\ln \left(D_{n} / D_{0}\right)}{\ln \left(1 / D_{0}\right)} \tag{6.4}
\end{equation*}
$$

where

$$
D_{n}=\left[\left(\xi_{n}-\bar{\xi}_{n}\right)^{2}+\left(\theta_{n}-\bar{\theta}_{n}\right)^{2}\right]^{1 / 2}
$$

on the step $n$ of the mapping. Here ( $\xi_{n}, \theta_{n}$ ) and ( $\bar{\xi}_{n}, \bar{\theta}_{n}$ ) are the values of the variables in the $n$th step of the mapping for two trajectories with close initial conditions ( $\xi_{0}, \theta_{0}$ ) and ( $\bar{\xi}_{0}, \bar{\theta}_{0}$ ). Curve 1 corresponds to a highly unstable ray with initial coordinates ( $\xi_{0}=0, \theta_{0}=0.03$ ), curve 2 corresponds to a slightly unstable ray on the boundary of the stochastic


FIG. 11. Phase portrait of the mapping (4.1) in the ( $\xi, \theta$ ) plane for small angles. The waveguide parameters were $a / l=1 / 3$ and $b / a=1.5 \cdot 10^{-2}$.


FIG. 12. Dependence of the logarithm of the relative divergence of the rays $\ln \left(D_{n} / D_{0}\right) / \ln \left(1 / D_{0}\right)$ on the mapping step $n$ for different values of the angle $\theta_{0}: 1$ ) $\left.\left.\theta_{0}=0.03,2\right) 0.12,3\right) 0.25$.
region ( $\xi_{0}=0, \theta_{0}=0.12$ ), and curve 3 corresponds to a regular ray ( $\xi_{0}=0, \theta_{0}=0.25$ ).

Finally, we consider the propagation time of a signal along a ray in the stochastic region. Figure 13 shows the dependence of the optical path difference $\Delta \boldsymbol{S}\left(\theta_{0}, z\right)$ along the ray on the angle of departure $\theta_{0}$ of the ray from the source. It is obvious that this dependence is random. The $\Delta S$ points are distributed about the unperturbed "regular" dependence

$$
\Delta S_{0}\left(\theta_{0}, z\right)=\frac{z}{\cos \theta_{0}}-z
$$

on $\theta_{0}$ (curve). The waveguide parameters were $a / l=1 / 3$, $b / a=1.5 \cdot 10^{-2}$, and $z=50 l$.

## 6.3.

The chaotic behavior of an acoustic ray in a horizontally inhomogeneous model of the ocean was analyzed numerically in Ref. 23. The Hamiltonian system of rays was considered in the paraxial approximation with a Hamiltonian of the type (2.9). The deterministic function $\bar{\varepsilon}(x, z)=-V(x, z)$, where $x$ is vertically downward and $z$ is


FIG. 13. Dependence of the optical path difference (4.9) on $\theta_{0}$ for small angles. The curve corresponds to the unperturbed case.
along the propagation direction of the wave, was chosen to have the form

$$
\begin{equation*}
V(x, z)=\frac{1}{2}\left(1-\frac{c_{0}^{2}}{c^{2}(x)}\right)+\sqrt{2} A e^{-3 x / 2 B} \sin \frac{2 \pi z}{R} \tag{6.5}
\end{equation*}
$$

where

$$
c(x)=c_{\mathrm{a}}\left[1+\varepsilon\left(e^{-\eta}+\eta-1\right)\right], \quad \eta=2\left(x-x_{\mathrm{a}}\right)^{-1}
$$

The expression (6.5) describes sound propagation in an underwater acoustic channel with small periodic inhomogeneities along the horizontal coordinate $z$ with period $R$.

Numerical calculations were carried out for the following values of the parameters of the ocean model: $c_{0}=c_{a}=1.5 \mathrm{~km} / \mathrm{sec}, x_{a}=B=1 \mathrm{~km}, \varepsilon=0.0057, R=1.0$ km . The quantity $A$ is the mean-square deflection of the acoustic ray along the surface of the ocean and is of order $10^{-3}$ for many regions of the ocean.

In Fig. 14 the Poincaré sections of regular and chaotic rays are shown in the phase plane $(x, \phi=\arctan p)$, where $p=\mathrm{d} x / \mathrm{d} z$, for a series of initial values of the ray. We see that a stochastic layer in the phase plane of the ray develops as the perturbation amplitude $A$ increases.

## 7. WAVEGUIDES WITH COMPLEX CROSS SECTIONS

Waveguides with complex cross-section shapes are most often encountered in fiber optics, where there is increased interest in nontraditional fiber optical waveguides with noncircular cross sections. The main results and an extensive bibliography of papers on waveguides with noncircular cross sections are contained in Refs. 56-60.

The main difficulty in studying the structure of the field in such waveguides is the fact that separation of variables cannot be used in the wave equation. For a certain class of waveguides with two-dimensional index of refraction profiles which are nearly quadratic functions of the transverse coordinates $(x, y)$, the most effective method of solution is apparently the Birkhoff-Gustavson ${ }^{40-45}$ normal mode method, which is well known in classical mechanics. Below we consider the dynamical phenomena in the ray theory of wave


FIG. 14. Poincaré sections for a series of rays with initial coordinates $x_{0}=1 \mathrm{~km}$ and $\phi_{0}=1.5^{\circ}, 3.0^{\circ}, 4.5^{\circ}, 6.0^{\circ}, 7.5^{\circ}$, and $9.0^{\circ}$ for different values of the perturbation parameter $A$. In each square the arrow corresponds to the ray with angle $\phi_{0}=7.5^{\circ}$, which becomes stochastic when $A \geqslant 4.612 \cdot 10^{-3}$.
propagation in waveguides with a two-dimensional cross section and nonseparable variables. ${ }^{19}$

We consider a waveguide medium which is homogeneous along the waveguide axis (the $z$ axis) and whose index of refraction is $n=n(x, y)$. The ray equations have the form (2.7) with the Hamiltonian function

$$
\begin{equation*}
H=H\left(r, \mathrm{p}_{\perp}\right)=-\left(n^{2}(x, y)-\mathrm{p}_{\perp}^{2}\right)^{1 / 2} \tag{7.1}
\end{equation*}
$$

This system corresponds to a dynamical system with two interacting degrees of freedom. For general shapes of the waveguide cross section one can have the phenomenon of nonlinear resonance and chaotic instability caused by the interaction of the different degrees of freedom of the ray. ${ }^{19}$ These phenomena occur even when the index of refraction $n(x, y)$ does not vary along the $z$ axis.

### 7.1. Internal nonlinear resonance of rays

The index of refraction of a waveguide with a nonuniform cross section is written in the form

$$
\begin{equation*}
n^{2}(\mathrm{r})=n_{0}^{2}(\mathrm{r})+\varepsilon n_{1}(\mathrm{r}) \tag{7.2}
\end{equation*}
$$

where $n_{0}(\mathbf{r})$ corresponds to the integrable case, when the Hamiltonian of the system in terms of the action-angle variables ( $I_{1}, I_{2}, \vartheta_{1}, \vartheta_{2}$ ) has the form $H_{0}\left(I_{1}, I_{2}\right)$. The quantity $\varepsilon n_{1}(\mathbf{r})$ describes the small perturbation of the cross section. When $\varepsilon \ll 1$ we have

$$
\begin{equation*}
H=H_{0}\left(I_{1}, I_{2}\right)+\varepsilon V\left(I_{1}, I_{2}, \vartheta_{1}, v_{2}\right) . \tag{7.3}
\end{equation*}
$$

The perturbation $\varepsilon V$ describes the interaction between the different degrees of freedom and can be represented as a Fourier series:

$$
\begin{align*}
& \varepsilon V\left(I_{1}, I_{2}, \vartheta_{1}, \vartheta_{2}\right)=\frac{\varepsilon}{2} \sum_{m_{1}, m_{2}} V_{m_{1} m_{2}}\left(I_{1}, I_{2}\right) \\
& \times \exp \left(i m_{1} \vartheta_{1}+i m_{2} \vartheta_{2}\right)+\text { c.c. } \tag{7.4}
\end{align*}
$$

The perturbation will have the strongest effect on the ray when

$$
\begin{equation*}
m_{1} \omega_{1}\left(I_{1}, I_{2}\right)+m_{2} \omega_{2}\left(I_{1}, I_{2}\right)=0 \tag{7.5}
\end{equation*}
$$

Let $\left(I_{1}^{0}, I_{2}^{0}\right)$ be the resonance values of the action variables $I_{1}, I_{2}$ satisfying the resonance condition (7.5) for certain values of the numbers ( $m_{1}, m_{2}$ ). Then in a small neighborhood of the resonance action variables $\left(I_{1}^{0}, I_{2}^{0}\right)$ the different degrees of freedom of the spatial oscillations of the ray along the $z$ axis are synchronized, as in the case of a plane waveguide with longitudinal periodic inhomogeneities (Sec. 3.1). This effect is called internal spatial nonlinear resonance of rays ${ }^{19}$ and is analogous to internal nonlinear resonance in classical mechanics. ${ }^{2}$ In other words, an effective waveguide channel is formed in the neighborhood of the resonance values of the action variables ( $I_{1}^{0}, I_{2}^{0}$ ), and the width of the channel in ( $I_{1}, I_{2}$ ) is given by the relation ${ }^{19}$
$\Delta I_{k}=\left[4 \varepsilon\left|V_{m_{4} m_{2}}\left(\omega_{k}^{2}\left(I_{1}^{0}, I_{2}^{0}\right) B\left(I_{1}^{0}, I_{2}^{0}\right)\right)^{-1}\right|\right]^{1 / 2} \quad(k=1,2)$,
where
where
$B\left(I_{1}, I_{2}\right)=\sum_{l_{1}, l_{2}}(-1)^{l_{1}+l_{2}\left(\omega_{l_{1}} \omega_{l_{2}}\right)^{-1}} \frac{\partial^{2} H_{0}}{\partial I_{l_{1}} \partial I_{l_{2}}} \quad\left(l_{1,2}=1,2\right)$.

The following relation exists between the action variables $I_{1}$ and $I_{2}$ in the neighborhood of each resonance

$$
\begin{equation*}
\mathscr{T}=m_{2} I_{1}-m_{1} I_{2}=\text { const } \tag{7.7}
\end{equation*}
$$

The quantity $\mathscr{T}$ is an integral of the motion, in addition to the usual energy integral $H$.

As in the case of a waveguide with longitudinal periodic inhomogeneities (see Sec. 3) the propagation of a signal along rays trapped in an effective waveguide channel is fundamentally different from the case of an unperturbed waveguide. The average propagation velocity $v\left(I_{1}, I_{2}, z\right)$ of the signal (3.16) is, to within terms of order $\varepsilon \ll 1$, equal to the velocity of a signal along an unperturbed ray $v_{0}\left(I_{1}^{0}, I_{2}^{0}\right)$ given by (3.13), where $I_{1}^{0}$ and $I_{2}^{0}$ are the resonance values of the action variables, for all rays trapped in the given resonance. ${ }^{19}$

### 7.2. Chaotic instability of rays

For most waveguide cross-section shapes there exist regions of space in which the ray trajectories become chaotically unstable. In these regions the additional integral $\mathscr{T}$ given by (7.7) does not exist and the only constant of the motion is the energy integral $H$. As in the case of a waveguide with longitudinal periodic inhomogeneities, the propagation times of rays along chaotic rays will be randomly distributed.

It is useful to note the analogy between ray dynamics in a wave-guide with a uniform index of refraction over its cross section and the motion of particles in billiards. ${ }^{2,61-63}$ The possibility of stochastic motion in scattering billiards (Sinaĭ billiards) has been studied in the work of Sinaĭ (see Ref. 61, for example) and in nonscattering billiards in Ref. 62. The conditions for stochastic motion in certain special cases were derived in Ref. 2.

In contrast to the formation of a stochastic layer in the neighborhood of the separatrix in a waveguide with longitudinal periodic inhomogeneities (see Sec. 6), in this case the energy of the wave is not radiated from the stochastic region, since $H=$ const is an integral of the motion and the rays are distributed uniformly inside the stochastic layer with a given value of $H=$ const.

The propagation velocity of a signal along the rays depends in this case only on the integral of the motion $H=$ const and is given by ${ }^{65}$

$$
\begin{equation*}
\frac{c}{v}=\frac{\iint n^{2}(x, y) \mathrm{d} x \mathrm{~d} y}{|H| \iint \mathrm{d} x \mathrm{~d} y} \tag{7.8}
\end{equation*}
$$

in which the integration goes over the entire classically allowed region $n^{2}(x, y)-H^{2}>0$.

A similar distribution of rays over one of the variables when the other is conserved is observed in so-called quasiregular optical waveguides, in which weak random inhomogeneities of the medium lead to strong diffusion in one of the variables (for example, in circular waveguides the diffusion is strong along the angular variable and is much weaker along the radial direction). ${ }^{38}$

## 7.3.

We consider a waveguide whose cross section is close to the stadium shape of Fig. 15. The index of refraction is $n_{0}=$ const within the cross section and is equal to $n_{\infty}$ out-


FIG. 15. Stadium cross section.
side it ( $n_{\infty}<n_{0}$ ). The function $\varepsilon f(x)$ describes the deformation of the sides $A B$ and $C D$, where $\varepsilon f(x)=d \cos (\pi x / 2 a)$ ( $d \ll a, b$ ).

In the unperturbed case $(\varepsilon=0)$ the Hamiltonian $H_{0}\left(I_{1}, I_{2}\right)$ in terms of the action-angle variables $\left(I_{1}, I_{2}, \vartheta_{1}, \vartheta_{2}\right)$ is

$$
\begin{align*}
& H_{0}\left(I_{1}, I_{2}\right)=-\left(n_{0}^{2}-\frac{I_{1}^{2}}{I_{10}^{2}}-\frac{I_{2}^{2}}{I_{20}^{2}}\right)^{1 / 2}, \quad I_{k}=\frac{2 a_{k}}{\pi}, \\
& \left(a_{1}, a_{2}\right)=(a, b), \quad k=1,2 \tag{7.9}
\end{align*}
$$

Deformation of the sides $A B$ and $C D$ leads to interaction of the different degrees of freedom. Resonance occurs when

$$
\omega_{1}\left(I_{1}, I_{2}\right)=(2 m+1) \omega_{2}\left(I_{1}, I_{2}\right)
$$

or

$$
\begin{equation*}
\frac{I_{1}}{I_{10}}=(2 m+1) \frac{a}{b} \frac{I_{2}}{I_{20}} \tag{7.10}
\end{equation*}
$$

In Fig. 16 we show the cons̈tant curves $E=H_{0}\left(I_{1}, I_{2}\right)$ (the solid curves $I$ ) and the resonance lines (7.10) in the $I_{1}$, $I_{2}$ plane. We see that the distance between neighboring resonances decreases as the number $m$ of the resonance increases. The width of the resonances in the action $I_{1}$ is

$$
\begin{equation*}
\frac{\Delta I_{1}}{I_{10}}=\left\{4 d\left[b\left(n_{0}^{2}-E^{2}\right)\right]^{-1}\right\}^{1 / 2} \cdot\left(\frac{I_{2}}{I_{20}}\right)^{2} . \tag{7.11}
\end{equation*}
$$

The width is greatest for the $m=1$ resonance, which corresponds to large values of $I_{2} / I_{20}$ (see Fig. 16).

Compression of the resonances occurs for large $m$, i.e. as the ratio $I_{2} / I_{20}$ decreases. The distance between neighboring resonances decreases as

$$
\begin{equation*}
\frac{\delta I_{1}}{I_{10}}=2 a\left[b\left(n_{0}^{2}-E^{2}\right)\right]^{-1}\left(\frac{I_{2}}{I_{20}}\right)^{3} \tag{7.12}
\end{equation*}
$$

Then from the resonance overlap condition we obtain chaotic behavior of rays in the region $I_{2}<I_{20}$, where

$$
\begin{equation*}
\frac{I_{2 \mathrm{c}}}{I_{20}}=\left[\left(n_{0}^{2}-E^{2}\right) b d a^{-2}\right]^{1 / 2} \tag{7.13}
\end{equation*}
$$

The stochastic region is shown in Fig. 17 in the ( $\eta_{1}, \eta_{2}$ ) plane. The critical angle $\varphi_{c}$ determines the region of chaos $0<\varphi<\varphi_{c}$, where $\varphi=\arctan \left(\eta_{1} / \eta_{2}\right)$. We obtain the following relation for $\varphi_{\mathrm{c}}$ :

$$
\begin{equation*}
\varphi_{c}=\arcsin \left(\frac{d b}{a^{2}}\right)^{1 / 2} \tag{7.14}
\end{equation*}
$$



FIG. 16. Curves of constant $H_{0}\left(I_{1}, I_{2}\right)$ and resonance lines (7.10) ( $m=1,2,3, \ldots$ ) in the plane of the normalized actions $\eta_{1}, \eta_{2}$

When the waveguide parameters are such that $d b / a^{2} \geqslant 1$, nearly all waveguide rays become chaotic. This condition is the same as the condition for chaotic motion of particles in stadium billiards. ${ }^{2}$ Chaotic dynamics of rays in inhomogeneous resonant cavities have been studied in Ref. 64.

## 8. RAY DYNAMICS AND THE SPECKLE STRUCTURE OF THE WAVE FIELD IN MULTIMODE WAVEGUIDES

An interesting and important problem is the study of the connection between ray dynamics and the structure of the wave field in regularly inhomogeneous waveguides. A characteristic feature of wave fields in multidimensional waveguides is that the field consists of irregularly distributed spots. This structure of the field is called the speckle structure. A typical example is the speckle structure formed when a coherent laser beam is reflected from a rough surface in which the size of the inhomogeneities exceeds the wavelength. ${ }^{67,68}$

The speckle structure of the field in multidimensional waveguides is formed as follows. A large number of rays arrive at the point of observation from the excitation plane. Since the optical paths traveled by the different rays are in general different, the field at the observation point is the result of interference between a large number of fields with different phases, corresponding to these rays. The resulting speckle structure will be regular if the ray trajectories in the waveguide are regular and will be random if the rays are chaotic.


FIG. 17. Stochastic region of the rays (shaded region).

We turn to the quasiclassical representation of the wave field $u(\mathbf{r}, \boldsymbol{z})$. The waveguide is excited by the coherent field $u_{0}(\mathrm{r})$ in the plane $z=0$. The wave field in the region $z>0$ is then given by ${ }^{16,66}$

$$
\begin{equation*}
u(\mathrm{r}, z)=\sum_{j=1}^{N} u_{0}\left(\mathrm{r}_{0 j}, z\right) \mathscr{T}_{j}^{1 / 2} \exp \left(i k S_{j}(\mathrm{r}, z)+\frac{1}{2} i \pi \mu_{j}\right) \tag{8.1}
\end{equation*}
$$

where $u_{0}(\mathbf{r})=u(\mathbf{r}, 0)$ and $S_{j}(\mathbf{r}, z)$ is the action along the $j$ th ray:

$$
\begin{equation*}
S_{j}=\int_{\gamma_{j}} n(\mathrm{r}, z) \mathrm{d} \sigma, \tag{8.2}
\end{equation*}
$$

where $\mathrm{d} \sigma$ is the element of are length of ray $\gamma_{j}$ and $\mathbf{r}_{0 j}$ is the coordinate of the ray in the plane $z=0$. The quantity

$$
\mathscr{T}_{j}=\frac{\mathrm{d} a}{\mathrm{~d} a_{0}}
$$

is the divergence of the ray. Here $\mathrm{d} a_{0}$ and $\mathrm{d} a$ are the crosssectional areas of an elementary ray tube in the planes $z=0$ and $z=$ const, respectively, $\mu_{j}$ is the Morse index for the $j$ th ray, and $k=v / c_{0}$ is the wavenumber. The summation in (8.1) goes over all rays passing through the observation point ( $r, z$ ).
8.1.

We first consider a waveguide channel which is homogeneous along the $z$ axis and where the ray dynamics is regular. Then the phases of the waves $\phi=k S_{j}$ are determined by (3.13) in terms of the action-angle variables ( $I, \vartheta)$. The speckle structure of the field begins at a distance $z$ where the phase difference along neighboring rays $\Delta \phi=k\left(S_{j}-S_{j+1}\right)$ becomes of order $2 \pi$. We have from (3.13)

$$
\begin{equation*}
\Delta \phi=k \Delta I \frac{\mathrm{~d} S}{\mathrm{~d} I}=k z I \Delta \frac{I \mathrm{~d} \omega}{\mathrm{~d} I} \geq 2 \pi . \tag{8.3}
\end{equation*}
$$

Hence we obtain the following estimate for the critical distance $z_{0}$ for the formation of speckle structure

$$
\begin{equation*}
z_{0} \approx 2 \pi(k \omega \alpha \Delta I)^{-1} \tag{8.4}
\end{equation*}
$$

where $\Delta I$ is the difference in the actions corresponding to neighboring rays and $\alpha$ is the nonlinearity parameter (3.3). It follows from (8.4) that speckle structure is formed only in waveguides with intermode dispersion:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} l} \frac{c}{v(I)}=\alpha \omega \neq 0 \tag{8.5}
\end{equation*}
$$

An expression for $z_{0}$ in terms of wave number $k_{m}$ can also be obtained. It follows from (2.27) and (2.28) that

$$
\begin{equation*}
z_{0} \approx 2 \pi\left[m\left|\frac{\mathrm{~d}^{2} k_{m}}{\mathrm{~d} m^{2}}\right|\right]^{-1} \tag{8.6}
\end{equation*}
$$

Therefore speckle structure is possible only in waveguides with nonequidistant wave numbers $k_{m}$, i.e. when $d^{2} k_{m} / d m^{2} \neq 0$ (Ref. 75). The speckle structure of the field will be regular in this case and is reversible in principle.

The statistical properties of a wave field with speckle structure were analyzed in Ref. 75, in particular the distribution of dislocations of the wave field over the cross section of a regular fiber optical waveguide.

### 8.2. Chaotic instability ${ }^{20}$

In this case, because of the strong exponential instability of rays at sufficiently small distances from the plane of entry of the radiation into the waveguide, the divergence of the rays $\mathscr{T}_{j}$ and the number of rays $N$ arriving at the observation point increase as $\exp (h z)$, where $h$ is the instability growth factor. Hence the optical path length $S_{j}(r, z)$ will vary irregularly from one ray to another and will be randomly distributed in the stochastic region (see Fig. 13). One then expects that the phases of the waves $\phi_{j}=k S_{j}(\bmod 2 \pi)$ along the rays will be uniformly (or nearly uniformly) distributed in the interval $(0,2 \pi)$.

Hence the wave field ( 8.1 ) is the sum of a large number of quasiplane waves whose phases are uniformly distributed in the interval $(0,2 \pi)$. Because of the randomness of the phases, the resulting interference pattern of the wave field will be irregular with the field maxima and minima distributed randomly in space.

Because of the exponential growth of the chaotic instability of the rays, the speckle structure forms over a very short distance scale of the order of the phase correlation length of the ray oscillations $z_{r} \approx 1 / \omega \ln K$, where $K \gg 1$ is the parameter defined by (3.9). This leads to a loss of information about the details of the structure of the wave front of the original field, i.e. to irreversibility of the wave field. A rigorous treatment of the formation of speckle structure of a wave field is given in Ref. 20.

We note that a system of periodic ray instabilities can also lead to the formation of a regular speckle structure, in spite of the fact that neighboring rays are stochastic. This phenomenon is called 'scarring' in problems of quantum chaos, ${ }^{76}$ although it has still not been studied.

It is useful to point out an analogy between the problem considered here and similar problems in quantum mechanics. The possibility of quasi-random wave functions in quantum $K$ systems (systems with dynamical chaos in the quasiclassical limit) has been discussed in Refs. 2, 69, 70, 76. A pattern of nodes of the wave functions in quantum $K$ systems resembling speckle structure was obtained by numerical methods in Refs. 71 and 72.

## 8.3.

We next consider the difference between the speckle structures formed in a waveguide with regular ray dynamics and in a waveguide where the rays are stochastically unstable.

In the case of a stochastic instability of rays the typical linear scale of the speckle structure $z_{r}$ is much smaller than the corresponding length $z_{0}$ in a regular waveguide: $z_{r} \ll z_{0}$. A weak partial coherence of the initial wave in space or time leads to additional blurring of the speckle structure. Suppose the initial radiation is a spectral line of width $\Delta v$ about the central frequency $v_{0}$. Then in a regular waveguide with intermode dispersion, the blurring of the speckle structure occurs over a distance $z_{c} \approx z_{0} v_{0} / \Delta v \gg z_{0}$ (Refs. 74 and 75), where $z_{0}$ is the typical linear scale of the speckle structure (8.4). On the other hand, in the case of chaotic instability one expects blurring of the speckle structure over very short distances of order $z_{s} \sim z_{r} \ln \left(v_{0} / \Delta v\right)$, where $z_{s} \ll z_{c}$. However, this question is actually quite complicated and requires a rigorous treatment of wave effects.

We see that stochastic instability of rays in waveguide channels leads to two effects:

1) the distribution of speckles becomes random;
2) any slight broadening of the initial radiation leads to an irreversible blurring of the image.

The exponential blurring of the speckle structure of the field in the case of weak incoherence of the initial field is important in suppressing noise caused by the speckle structure of the field in information transfer systems using waveguide channels, especially in fiber-optical communication lines.

## 9. CONCLUSION

The results discussed and reviewed here were obtained in the framework of geometrical optics. Hence it is necessary to discuss the restrictions imposed by wave effects. A necessary condition for the approximation of geometrical optics to be valid is that the index of refraction of the medium $n(r)$ must vary slowly over a scale of the order of a wavelength.

Additional restrictions occur during wave propagation because of the accumulation of wave corrections due to spreading of a wave packet either because of a slight nonlinear divergence of the oscillations of the rays, or because of strong stochastic divergence.

The first case is quite common. It is associated with the nonlinearity of the ray dynamics and is analogous to the quantum spreading of a wave packet corresponding to a particle moving in an anharmonic potential. The distance $z_{D}$ below which wave effects appear is determined by the nonlinear oscillation frequency of the ray $\omega(I)$ and is equal to ${ }^{17}$

$$
\begin{equation*}
z_{\mathrm{D}} \approx 2 \pi k\left[\frac{\mathrm{~d} \omega(I)}{\mathrm{d} I}\right]^{-1} \tag{9.1}
\end{equation*}
$$

where $k=2 \pi / \lambda$ is the wave number. For a waveguide with the index of refraction profile (3.18), the value of $z_{D}$ is of order

$$
\begin{equation*}
z_{\mathrm{D}} \approx 2 \pi k a^{2}\left[n_{\infty}^{2}+n_{0}^{2} \Delta\left(\frac{I_{s}-I}{I_{\mathrm{s}}}\right)^{2}\right]^{1 / 2} n_{0}^{-1} \tag{9.2}
\end{equation*}
$$

The minimum value of $z_{D}$ is of the order of the diffraction length for a ray whose radius is of the order of the effective width of the waveguide. Since $k a \gg 1$, it follows from (9.2) and the results of Sec. 4 that the fractal properties of the rays (the propagation time of a signal, for example) begin to appear earlier than wave effects.

As in the corresponding problems in quantum mechanics, ${ }^{2}$ wave effects begin to appear in the case of stochastic instability of rays at much closer distances because of the exponential spreading of the wave packet. One of the strongest manifestations of stochasticity is as follows. The time over which the quasiclassical approximation in quantum mechanics begins to break down has been studied quite extensively ${ }^{2,69,70,77}$ There are two time scales over which the quasiclassical approximation of a quantum system breaks down for a system which is chaotic in the classical limit. Over very short times $t_{0}$ of order const $\cdot \ln (1 / \hbar)$ the exponential growth of the divergence of nearby trajectories stops. ${ }^{72}$ But classical diffusion in the system continues up to the relaxation time $t_{R}$

$$
t_{\mathrm{R}} \approx \hbar \rho=\frac{\hbar}{\Delta}
$$

where $\rho$ is the density of the quasi-energy spectrum of the quantum system and $\Delta$ is the average distance between energy levels. ${ }^{77}$

Something analogous should exist for the distance over which the approximation of geometrical optics is valid. In quantum mechanics the small parameter of the problem is $\hbar / I_{0} \quad$ (see (1.1)), wher $E_{0}$ is the action of the problem. For rays this parameter is $1 / k l$, where $l$ is the linear scale of the problem. Hence, as in quantum mechanics, one expects that the longitudinal distance $z_{0}$ where the simple geometrical picture of the ray dynamics breaks down will be of order const $\cdot \ln$ (const $/ k l$ ), while ray diffusion continues over the relaxation distance $z_{R} \sim \rho$, where $\rho$ is the density of quasiwave numbers of a waveguide with periodic inhomogeneities along its axis. The calculation of the constants is a very complicated problem and there are no results on this question available at the present time.

We emphasize the deep analogy between problems of stochastic ray dynamics and quantum chaos. The analogs of chaos suppression in quantum mechanics (in particular, the phenomenon of "scarring",76) may be important in applied problems in acoustics and wave propagation in inhomogeneous media. These problems require a special discussion, and are outside the scope of the present review article.

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${ }^{2)}$ At the time of publication.
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