# Spectral density of eigenvalues of the wave equation and vacuum polarization 

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The effective Lagrange function of vacuum polarization is expressed in terms of the spectral density of the eigenvalues of the wave equation and the related five-dimensional Green's function introduced by Fock in his fifth-coordinate method. The method can be used for arbitrarily strong external fields, but the interaction of the vacuum fields is neglected. The vacuum polarization by the gravitational and electromagnetic fields is calculated.

## I. INTRODUCTION

This paper augments methodologically and mathematically my papers (Refs. 1 and 2) in which I put forward a conjecture about the Lagrangian of the gravitational and the electromagnetic field (in the second case the basic idea is due to Pomeranchuk and Landau, Fradkin, and Zel'dovich ${ }^{3}$ ). In its simplest form, the conjecture states that the Lagrange function of boson fields (gravitational, electromagnetic, and meson) is generated by vacuum polarization effects of fermions. In this work the term vacuum polarization is used in a wider sense than usual-the Lagrange functions of free boson fields and even the cosmological constant are attributed to polarization.

In this paper, a method of calculating the effective polarization Lagrange function is developed on the basis of the concept of the spectral density of the wave equation (Sec. II). In Sec. III, the spectral density is related to the Green's function defined in five-dimensional space (physical space augmented by a fifth auxiliary coordinate). The auxiliary fifth coordinate ("proper time") was first introduced by Fock in 1937.9 This method was further developed by Schwinger and others. ${ }^{10}$ Here, the general expression (25) for the effective polarization Lagrange function is derived differently.

In Sec. IV, the general method is applied to the gravitational field. In a model theory a formal cutoff of divergent integrals is used to find an expression for the gravitational constant, which has the correct sign ( $G>0$ ). In Sec. V, the method is illustrated for the example of the electromagnetic field, and the well-known expressions for the vacuum polarization by the electromagnetic field are again obtained. The signature of the metric tensor is ( +--- ), and gravitational units, in which $G=\hbar=c=1$, are used.

## II. SPECTRAL DENSITY OF THE EIGENVALUES OF THE WAVEEQUATION

Ref. 2 contains a sketch of the idea behind the new method of calculating vacuum polarization. So as to make the exposition comprehensive, some of this section [Eqs. (1) to (5)] repeats these ideas with some necessary refinements.

We consider the effect of vacuum polarization by an external field $\psi(x)$, which we assume fixed. The elementary fields with which $\psi$ interacts will be denoted by $\varphi_{1}, \varphi_{2}, \ldots$, $\varphi_{j}, \ldots$. We.ignore the interaction of the fields $\varphi$ with one another. This is the main assumption of the paper, and is equivalent to the restriction to single-loop diagrams. Because of this assumption, the paper does not have great phys-
ical significance but is rather methodological, or mathematical, in character. To illustrate the idea of Ref. 2 in the simplest way possible, the $\varphi$ 's will be assumed to be neutral scalar fields, and the field $\psi$ a given neutral tensor $g_{i k}$ or vector $A_{i}$ field; $g_{i k}$ is the metric tensor of the gravitational field and $A_{i}$ the potential of the electromagnetic field.

Suppose that the field $\psi(x)$ is defined in some fourdimensional volume $V$. The total action of the fields $\varphi_{j}$ in this volume is a functional $S(\psi)$. We denote the value of the functional for $\psi=0$ (vacuum value) by $S(0)$. Obviously, $S(\psi)-S(0)$ is the vacuum polarization effect due to the field $\psi$. By the conjecture, this difference is the effective action of the field $\psi$. The functional $S(\psi)=\Sigma S_{j}$ is the sum of the functionals for the individual fields $\varphi_{j}$. We calculate one of these terms (omitting for brevity the subscript $j$ ). We expand $\varphi$ in a series in eigenfunctions of the wave equation (in the simplest case, this is simply an expansion in a four-dimensional Fourier series):

$$
\begin{align*}
& \varphi=\sum z_{i} \varphi_{i}, \quad \square_{\psi} \varphi_{i}+\left(m^{2}+\Lambda_{i}\right) \varphi_{i}=0,  \tag{1}\\
& \int \mathrm{~d} x(-g)^{1 / 2} \varphi_{i} \varphi_{i}^{+}=\delta_{i i}^{\prime} . \tag{2}
\end{align*}
$$

Here the symbol $\square_{\psi}$ stands for the generalized d'Alembertian in the presence of the given field $\psi$. If $\psi=0$, than $\square_{0}=\partial^{2} / \partial t^{2}-\partial^{2} / \partial x_{1}^{2}-\partial^{2} / \partial x_{2}^{2}-\partial^{2} / \partial x_{3}^{2}$. If $\psi=A_{i}$ is the electromagnetic field, then $\nabla \rightarrow \nabla_{\psi}=\nabla-i e A$. If $\psi=g_{i k}$, then $\square_{\psi}$ is the Beltrami operator ${ }^{1)} ; \psi_{i^{\prime}}^{+}$is an eigenfunction of the adjoint equation; $\Lambda_{i}$ is an eigenvalue of the wave equation, and every $\Lambda_{i}$ is a functional of $\psi$. If $\psi=0$, $\Lambda_{i}=\omega_{i}^{2}-k_{i}^{2}-m^{2}$. We assume that the classical action for the field $\varphi$ is

$$
\begin{equation*}
S_{\mathrm{cl}}=\sum_{i} S_{i}, \quad S_{l}=\frac{z_{i}^{2}}{2} \Lambda_{i} \tag{3}
\end{equation*}
$$

This accords with the classical equations of motion. We find the action of the quantum fluctuations as the phase of the functional integral in the case of variation of $\varphi$ :

$$
\begin{equation*}
S=\arg \int\{\delta \varphi\} e^{i S_{\phi \varnothing}}=\sum_{i} \arg \int_{-\infty}^{+\infty} d z_{i} e^{i i_{i}^{2} \Lambda_{i} / 2}=\frac{\pi}{4} \sum \operatorname{sign} \Lambda_{i} \tag{4}
\end{equation*}
$$

The integral with respect to $\mathrm{d} z_{i}$ is calculated by the change of variables

$$
i z_{i}^{2}=-\zeta_{i}^{2} \operatorname{sign} \Lambda_{i}, \quad \mathrm{~d} z_{i}=d \zeta_{i} \mathrm{e}^{\mathrm{i}(\pi / 2) \operatorname{sign} \Lambda_{\mathrm{i}}} .
$$

Generalizing (4) to the presence of spinor fields $\phi_{j}$ and taking into account the statistical weight $g_{j}$, we obtain the gen-
eral expression

$$
\begin{equation*}
S(\psi)=\frac{\pi}{4} \sum_{J} g_{j} C_{j} \sum_{i} \operatorname{sign} r \Lambda_{l r} \tag{5}
\end{equation*}
$$

Here

$$
C_{j}= \begin{cases}+1, & \text { if } \mathscr{\varphi}_{j} \text { is a boson field }  \tag{5a}\\ -1, & \text { if } \varphi_{j} \text { is a fermion field } .\end{cases}
$$

The factor $C_{j}$ takes into account the fact that for spinor fields the contribution to the action has the opposite sign. In the sum (5) we have formally introduced a convergence factor of unknown physical origin. The "cutoff" weight function is

$$
\operatorname{sign} r \Lambda=\left\{\begin{array}{lll}
\operatorname{sign} \Lambda & \text { for } & |\Lambda|<\Lambda_{0}  \tag{5b}\\
0 & \text { for } & |\Lambda|>\Lambda_{0}
\end{array}\right.
$$

or

$$
\begin{equation*}
\operatorname{sign} r \Lambda=e^{-|\Lambda| / \Lambda_{0}} \operatorname{sign} \Lambda, \tag{5c}
\end{equation*}
$$

where $\Lambda_{0}$ is the square of the cutoff mass. We assume $\Lambda_{0} \sim 1$ in gravitational units.

For the following discussion we consider not the particular form of the function sign $r \Lambda$, but a sum with arbitrary function $\Phi(\Lambda)$ :

$$
\begin{equation*}
\Sigma_{\Phi}=\sum_{i} \Phi\left(\Lambda_{i}\right) \tag{6}
\end{equation*}
$$

The sum $\Sigma_{\Phi}$ diverges, since in any interval ( $\Lambda$, $\Lambda+d \Lambda$ ) there are infinitely many eigenvalues $\Lambda_{i}$. For example, in the case $\psi=0$, taking $V$ to be a parallelepiped, we find that the points $k_{10}, k_{i 1}, k_{i 2}, k_{i 3}$ form an infinite periodic four-dimensional lattice. Between the two hyperboloids $\Lambda=$ const and $\Lambda+d \Lambda=$ const there is an infinite volume containing infinitely many sites of the lattice. The situation is similar in the general case.

We now introduce the concept of "conditional convergence" of the sum $\Phi$. For simplicity, we restrict ourselves to the case when the volume $V$ is topologically equivalent to a four-dimensional cube. Deforming $V$ continuously into a cube $L^{4}$ and taking the limit as $\psi \rightarrow 0$, we map the functions $\varphi_{i}$ into functions of the form $\exp (2 \pi i / L)$ ( $n_{0} x_{0}-n_{1} x_{1}-n_{2} x_{2}-n_{3} x_{3}$ ). We define an invariant of the deformation process: $J(i)=n_{0}^{2}+n_{1}^{2}+n_{2}^{2}+n_{3}^{2}$. The $\operatorname{sum} \Sigma_{\boldsymbol{\Phi}}$ converges conditionally if the following limit exists:

$$
\begin{equation*}
\lim _{J_{0} \rightarrow \infty} \sum_{i} e^{-J(i) / J_{0} \Phi\left(\Lambda_{i}\right)} \stackrel{\text { def }}{=} \Sigma_{\Phi} . \tag{7}
\end{equation*}
$$

Other equivalent definitions are possible.
The sum (6) is conditionally convergent if the summability conditions

$$
\begin{align*}
& \int_{-\infty}^{+\infty} \mathrm{d} \Lambda \Phi(\Lambda)=0  \tag{8}\\
& +\infty \\
& \int_{-\infty}^{+\infty} \mathrm{d} \Lambda \Lambda \Phi(\Lambda)=0 \tag{9}
\end{align*}
$$

are satisfied and the function $\Phi(\Lambda)$ decreases sufficiently rapidly as $\Lambda \rightarrow \infty$; the idea behind the proof of this assertion is sketched in the next section.

We now define the spectral density $P(\Lambda)$ of the eigenvalues of the wave equation by requiring

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{k}}\left\{\sum_{i} \Phi_{k}\left(\varepsilon \Lambda_{i}\right)-\int_{-\infty}^{+\infty} \Phi_{k}(\varepsilon \Lambda) P(\Lambda) \mathrm{d} \Lambda\right\}=0 . \tag{10}
\end{equation*}
$$

The functions $\Phi_{k}$ satisfy certain conditions, which depend on $k$. For $k=1,2$, we assume that $\Phi(\varepsilon \Lambda) / \varepsilon$ does not tend to infinity as $\delta \rightarrow 0$.

By virtue of the conditions (8) and (9), the function $P(\Lambda)$ is not defined uniquely by (10), but only up to the addition of an arbitrary linear function of $\Lambda$. The integral $\int_{-\infty}^{+\infty} \Phi(\Lambda) P(\Lambda) d \Lambda$ does not change under the transformation

$$
\begin{equation*}
P(\Lambda) \rightarrow P(\Lambda)+A \Lambda+B . \tag{11}
\end{equation*}
$$

Representing the function $P(\Lambda)$ as the series

$$
\begin{equation*}
P(\Lambda)=C_{0} \lambda \ln \frac{|\lambda|}{\lambda_{0}}+C_{1} \ln \frac{|\lambda|}{\lambda_{1}}+\frac{C_{2}}{\lambda}+\frac{C_{3}}{\lambda^{2}}+\ldots \tag{12}
\end{equation*}
$$

where $\lambda=m^{2}+\Lambda$, we determine the coefficients $C_{0}, C_{1}$, etc., successively from (10). These coefficients do not depend on the form of the function $\Phi$. The coefficients $\lambda_{0}$ and $\lambda_{1}$ are arbitrary in accordance with (11).

## III. GREEN'S FUNCTION METHOD IN FIVE-DIMENSIONAL SPACE

In Ref. 4, McKean and Singer considered the Helmholtz equation in $n$-dimensional Riemannian space with definite metric (we change their notation slightly):

$$
\begin{equation*}
\Delta \varphi_{i}+\lambda_{i} \varphi_{i}=0 . \tag{13}
\end{equation*}
$$

They show that the sum

$$
\begin{equation*}
\Sigma(\tau)=\sum_{i} e^{-i_{i} \tau} \tag{14}
\end{equation*}
$$

can be calculated by means of a Green's function in an auxiliary space of $n+1$ dimensions. Here, the Green's function $G\left(x_{0}, x_{1}, \tau\right)$ is a normalized singular solution of the equation

$$
\begin{equation*}
\Delta G=\partial G / \partial \tau . \tag{15}
\end{equation*}
$$

In Ref. 4 it is shown that

$$
\begin{equation*}
\Sigma(\tau)=\int \mathrm{d} x(g)^{1 / 2} G(x, x, \tau), \quad x=x_{0}=x_{1} . \tag{16}
\end{equation*}
$$

McKean and Singer point out that M. Kac was one of the originators of the basic idea.

We apply here a similar method to find the density function $P(\Lambda)$ in the case of real physical space, i.e., for the wave equation (1). Denoting the auxiliary fifth variable by the letter $l$, we write down by analogy with (15) an equation for the five-dimensional Green's function:

$$
\begin{equation*}
\square_{\psi} G=i \partial G / \partial l . \tag{17}
\end{equation*}
$$

An equation of the type (17) was first introduced by Fock. ${ }^{9}$ The Green's function $G$ depends on the nine variables $t_{0}, x_{0}$, $y_{0}, z_{0}$ (abbreviated $x_{0}$ ), $t_{1}, x_{1}, y_{1}, z_{1}$ (abbreviated $x_{1}$ ), and $l=l_{1}-l_{0}$. If $\varphi$ has many components, $G$ also depends on the discrete numbers of the initial and final state with respect to $l$, i.e., it is a matrix $G_{m_{0} m_{1}}$ ( the use of the same notation for the discrete variable and the field mass $m$ should not cause confusion). The Green's function defined by (17) satisfies an integral relation similar to (16):

$$
\begin{equation*}
\Sigma(l)=\sum_{i} e^{\lambda_{i} l}=\int(\mathrm{d} x)(-g)^{1 / 2} \mathrm{Sp} G_{m_{0} m_{1}}(x, x, l), \quad x=x_{1}=x_{0} \tag{18}
\end{equation*}
$$

where $\lambda_{i}$ are the eigenvalues of the wave equation

$$
\square_{\psi} \varphi_{i}+\lambda_{i} \varphi_{i}=0, \quad \lambda_{i}=\Lambda_{i}+m^{2} .
$$

Equation (18) is proved by representing the Green's function as the conditionally convergent sum [see (7) above]

$$
\begin{align*}
& G_{m_{0} m_{1}}\left(x_{0}, x_{1}, l\right) \\
& \quad=\lim _{J_{0} \rightarrow \infty} \sum_{i} \exp \left[-\frac{J}{J_{0}}+i u_{i}\right\} \varphi_{i}\left(x_{1}, m_{1}\right) \varphi_{i}^{+}\left(x_{0}, m_{0}\right) . \tag{19}
\end{align*}
$$

The function defined by (19) satisfies Eq. (17) and goes over into the four-dimensional $\delta$-function as $l \rightarrow 0$ on account of the orthogonality relations

$$
\lim _{J_{0} \rightarrow \infty} \sum_{i} e^{-J / J_{0}} \varphi_{i}\left(x_{1}, m_{1}\right) \varphi_{i}^{+}\left(x_{0}, m_{0}\right)=\delta\left(x_{0}-x_{1}\right) \delta_{m_{0} m_{1}}
$$

Setting $x_{1}=x_{0}=x$ and $m_{1}=m_{0}$ in (19), we integrate over $x$ and sum over $m$. We arrive at (18), in which the sum over $i$ is also understood in the sense of (7). The fact of convergence of the sum accords with the fact that the function $e^{i \lambda l}$ satisfies the conditions (8) and (9) when $l \neq 0$. We now show how these conditions are related to the convergence of the sum (7) for arbitrary weight function $\Phi(\Lambda)$. We set

$$
\Phi(\lambda)=\int_{-\infty}^{+\infty} f(l) e^{i \lambda l_{\mathrm{d}} l}, \quad f(l)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Phi(\lambda) e^{-i l l_{\mathrm{d}} \lambda .}
$$

It follows from (8) and (9) that $f(0)=(d f / d l)(0)$ $=0$, which allows convergence of the integral $I=\int d l d x T r$ $G(x, x, l) \dot{f}(l)$ in the neighborhood of $l=0$ despite $G(l)$ 's having a singularity at $l=0$ of the form $\alpha /(l|l|)+\beta /|l|$ [see (23) below]. The integral converges in this case if there are additional but hardly restrictive conditions on the function $\Phi(\Lambda)$ [for example, if $\Phi(\Lambda)$ is bounded in modulus and decreases as $\lambda \rightarrow \pm \infty$ in such a way that its integral is absolutely convergent].

It follows from (18) that

$$
I=\int \mathrm{d} l \lim _{J_{0} \rightarrow \infty} \sum_{i} \exp \left\{-\frac{J}{J_{0}}+u_{i} l\right\} f(l)
$$

or, reversing the order of summation and integration,

$$
\lim _{J_{0} \rightarrow \infty} \sum_{i} e^{-J / J_{0}} \Phi\left(\lambda_{i}\right)=I
$$

Thus, the sum (7) is reduced to the finite value $I$.
We now introduce the local spectral density $\rho_{m}(\lambda, x)$ of the wave equation, which is related to the integral density $P(\lambda)$ by

$$
\begin{equation*}
\sum_{m} \int \mathrm{~d} x(-g)^{1 / 2} P_{m}(\lambda, x)=P(\lambda) \tag{20}
\end{equation*}
$$

(recall that $\lambda=m^{2}+\Lambda$ ).
From (18) and (20),

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \rho(\lambda, x, m) e^{j \ell l} \mathrm{~d} \lambda=G_{m m}(x, x, l) . \tag{21}
\end{equation*}
$$

We find the form of $G$ in flat space for $\psi=0$. We have

$$
\begin{aligned}
& G_{0}=G_{t} G_{x} G_{y} G_{z}, \quad G_{t}=a_{t} \frac{1}{(4 \pi l)^{1 / 2}} e^{-i t^{2} / 4 l}, \\
& G_{x}=\frac{a_{x}}{(4 \pi l)^{1 / 2}} e^{i x^{2} / 4 l}
\end{aligned}
$$

and similarly for $y$ and $z$. The normalization coefficients can be determined from the conditions $\int_{-\infty}^{+\infty} d t G_{t}(t, t, l)=1$, etc., separately for $l>0$ and $l<0$.

We find

$$
\begin{equation*}
G_{0}=-\frac{i \operatorname{sign} l}{(4 \pi l)^{2}} e^{-i\left(t^{2}-x^{2}\right) / 4 l} \tag{22}
\end{equation*}
$$

Writing $\rho(\lambda)$ and $G(l)$ as series, we have
$\rho_{m}=a_{0} \lambda \ln \frac{|\lambda|}{\lambda_{0}}+a_{1} \ln \frac{|\lambda|}{\lambda_{1}}+a_{2} \lambda^{-1}+a_{3} \lambda^{-2}+\ldots$,
$G_{m m}=\operatorname{sign} l\left[A_{0} l^{-2}+A_{1} l^{-1}+A_{2}+A_{3} l+\ldots\right\}$,
$a_{0}=i A_{0} / \pi=1 / 16 \pi^{3}, \quad a_{1}=-A_{1} / \pi, \quad a_{2}=-i A_{2} / \pi$,
$a_{3}=-A_{3} / \pi, \quad a_{4}=2 i A_{4} / \pi \quad$ etc.
This connection between $a$ and $A$ follows from (21). It follows from (5) that the increment of the effective Lagrangian due to the vacuum polarization of the particles of the field $\varphi$ in the presence of some external field $\psi$ is

$$
\begin{equation*}
\Delta \mathscr{Z}_{j}=\frac{\pi C_{j}}{4} \sum_{m} \int_{-\infty}^{+\infty} \mathrm{d} \lambda \operatorname{sign} r \Lambda \cdot\left(\rho_{\psi}-\rho_{0}\right) . \tag{24}
\end{equation*}
$$

Applying to (24) and (20) the convolution theorem from the theory of Fourier integrals, we represent $\Delta \mathscr{L}$, as an integral with respect to the auxiliary variable $l$,
$\Delta \mathscr{Z}_{j}=\frac{\pi C_{j}}{4} \int_{-\infty}^{+\infty} \mathrm{d} l Z(l) R_{\psi}(l)$,
$R_{\psi}(l)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} \sum_{m}\left(\rho_{\psi}-\rho_{0}\right) e^{i \lambda l} \mathrm{~d} \lambda=\frac{1}{\sqrt{2 \pi}}\left(\operatorname{Sp} G_{\psi}-\operatorname{Sp} G_{0}\right)$,
$Z(l)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} \operatorname{sign} r\left(\lambda-m_{j}^{2}\right) e^{-i \lambda l_{\mathrm{d}} \lambda}=-\frac{2 i}{\sqrt{2 \pi}} e^{-i m_{\mathrm{j}}^{2} l} \operatorname{Reg}(1 / l)$, where $\operatorname{Reg}\{1 / l\}$ is the function "cut off" for $|l| \leqslant 1 / \Lambda_{0}$ [using (5c), we have $\left.\operatorname{Reg}\{1 / l\}=l / l^{2}+\Lambda_{0}^{-2}\right]$. Substituting this, we find

$$
\begin{align*}
& \Delta \mathcal{Z}_{j}=-\frac{i C_{j}}{4} \int_{-\infty}^{+\infty} \mathrm{d} l \operatorname{Reg}(1 / l) e^{-i m_{j}^{2} l}\left(\operatorname{Sp} G_{\psi}-\operatorname{Sp} G_{0}\right) \\
& =\frac{C_{j}}{2} \int_{1 / \Lambda_{0}}^{\infty} \frac{\mathrm{d} l}{l} \operatorname{Im}\left\{e^{-i m_{j}^{2} l}\left(\operatorname{Sp} G_{\psi}-\operatorname{Sp} G_{0}\right)\right\} \\
& =\frac{C_{j}}{2} \int_{1 / \Lambda_{0}}^{\infty} \frac{\mathrm{d} l}{l}\left\{\cos m_{j}^{2} l\left(\operatorname{Im~Sp} G_{\psi}-\operatorname{Im~Sp} G_{0}\right)\right. \\
& \left.\quad-\sin m_{j}^{2} R \operatorname{ReSp} G_{\psi}\right\} . \tag{25}
\end{align*}
$$

## IV. COSMOLOGICAL CONSTANT AND POLARIZATION OF THE VACUUM BY THE GRAVITATIONAL FIELD

The Lagrange function $\mathscr{L}_{0}$ of flat space for $\psi=0$ is (up to sign) Einstein's cosmological constant (compare this with the formulation of the problem and the calculations of Zel'dovich ${ }^{5}$ ). If the summation conditions (8) and (9) are
satisfied, then on the basis of (5),

$$
\begin{align*}
\mathcal{Z}_{0} & =\frac{\pi}{4} \sum_{j} g_{j} C_{j} \sum_{i} \operatorname{sign} r \Lambda_{i j} \\
& =\frac{\pi}{4} \int_{-\infty}^{+\infty} d \lambda \sum_{j} C_{j} \rho_{0 j}(\lambda) \operatorname{sign} r\left(\lambda-m_{j}^{2}\right), \tag{26}
\end{align*}
$$

where $\rho_{0}=g_{j}\left[\lambda \ln \left(|\lambda| / \lambda_{0}\right)\right] / 16 \pi^{3}$.
Recall that $C_{j}= \pm 1$ for bosons (fermions). The summability condition (8) is satisfied automatically [since $S$ $\operatorname{sign} r(\Lambda) d \lambda=0$ ], and the condition (9) leads to

$$
\begin{equation*}
\sum_{j} g_{j} C_{j}=0 . \tag{27}
\end{equation*}
$$

The integral (26) subject to (27) can be calculated by an equation of the type (25):

$$
\begin{equation*}
x_{0}=-\frac{1}{2} \int_{1 / \Lambda_{0}}^{\infty} \frac{\mathrm{d} l}{l} \sum_{j} g_{j} C_{j} \frac{\cos m_{j}^{2} l}{16 \pi^{2} l^{2}} . \tag{28}
\end{equation*}
$$

To logarithmic accuracy,

$$
\begin{equation*}
x_{0}=\frac{1}{64 \pi^{2}} \sum g_{j} C_{j} m_{j}^{4} \ln \frac{\Lambda_{0}}{m_{j}^{2}} . \tag{29}
\end{equation*}
$$

In reality, it is well known that $\mathscr{L}_{0}$ (up to sign it is Einstein's cosmological constant) is either zero or extremely small. In a model theory of noninteracting particles $\mathscr{L}_{0}$ can vanish as a result of the composition of the contributions of bosons and fermions. In a more realistic theory with allowance for interaction and spontaneous symmetry breaking the vanishing of $\mathscr{L}_{0}$ must be regarded as a physical condition imposed on the constants in the unrenormalized Lagrangian. ${ }^{2)}$

In curved Riemannian space (i.e., in a space in which the Riemann tensor $R_{i k l m}$ is nonzero), $\rho(\lambda), G(l)$, and $\mathscr{L}$ are changed (polarization of the vacuum by the gravitational field).

We represent the Green's function in the form

$$
\begin{align*}
& G_{R j}=G_{0 j}\left(1+i Q R l-\left[U_{j} R^{2}+V_{j} R^{i k} R_{i k}\right.\right. \\
& \left.\left.+W_{j} R^{i k l m} R_{i k l m}+Y \square R\right] l^{2}+\ldots\right) \tag{30}
\end{align*}
$$

The coefficients $Q, U$, etc., are found in the Appendix for scalar fields $\varphi_{j}$.

Substituting into (25) and using $G_{0 j}=-i g_{j}$ sign $l / 16 \pi^{2} l^{2}$, we obtain the first-order correction

$$
\begin{equation*}
\Delta \mathcal{Z}_{1 j}=-\frac{R Q C_{j} g_{j}}{2} \int_{1 / \Lambda_{0}}^{\infty} \frac{\mathrm{d} l}{l} \frac{\sin m_{j}^{2} l}{16 \pi^{2} l} . \tag{31}
\end{equation*}
$$

The similar expression for the second-order correction is

$$
\begin{equation*}
\Delta \mathcal{Z}_{2 j}=\frac{\left[U_{J} R^{2}+V R^{i k}+\ldots\right]}{2} \int_{1 / \Lambda_{0}}^{\infty} \frac{\mathrm{d} l}{l} \frac{\cos m_{j}^{2} l}{16 \pi^{2}} \tag{32}
\end{equation*}
$$

Calculation of the integrals in (31) and (32) to logarithmic accuracy gives

$$
\begin{align*}
& \Delta \mathscr{I}_{1 j}=\frac{R Q_{j} C_{j} g_{j}}{32 \pi^{2}} \ln \frac{\Lambda_{0}}{m_{j}^{2}}  \tag{31a}\\
& \Delta \mathscr{I}_{2 j}=\frac{\left[U_{j} R^{2}+\ldots\right]}{32 \pi^{2}} \ln \frac{\Lambda_{0}}{m_{j}^{2}} \tag{32a}
\end{align*}
$$

Equating $\Sigma_{j} \Delta \mathscr{L}_{1 j}=-R / 16 \pi G$ in accordance with the conjecture, we find that the gravitational constant is

$$
\begin{equation*}
G=2 \pi\left(\sum_{j} Q_{j} C_{j} g_{j} m_{j}^{2} \ln \frac{\Lambda_{0}}{m_{j}^{2}}\right)^{-1} \tag{33}
\end{equation*}
$$

Thus, in the model theory with formal cutoff at $|\Lambda| \sim \Lambda_{0}$ we have found the correct sign $G>0$ with allowance for $Q_{j} C_{j}>0$. Equation (33) gives the correct numerical value of $G$ (equal to 1 in the chosen units) if the mass spectrum of the elementary fields extends to $m_{j} \sim G^{-1 / 2}$. The expression (32a) diverges for particles with rest mass $m_{j}=0$ (neutrino, photon, graviton; the last case requires special treatment).

The quadratic correction $\mathscr{L}_{2 j}$ for gravitons has been considered by De Witt. ${ }^{6} \mathrm{He}$ assumed that for particles of zero mass the logarithmic divergence is cut off in the infrared limit at lengths that depend on the characteristic scale $L$ of the problem. Since our method does not require an expansion in a series in powers of the curvature tensor, it automatically leads to a cutoff of the infrared divergence at $l \sim L^{2}$.

Let us demonstrate this for the example of a scalar field without the conformal term in the equation of motion. We consider a space with the metric

$$
\mathrm{d} s^{2}=\mathrm{d} t^{2}-\mathrm{d} r^{2}-L^{2} \mathrm{sh}^{2} \frac{r}{L}\left(d \vartheta^{2}+\sin ^{2} \vartheta \mathrm{~d} \varphi^{2}\right) .
$$

Applying the method described in the Appendix, we find $G(0,0, l)=G_{0}(0,0, l) e^{-i l / L^{2}}$. Substituting into (25), we have

$$
\begin{equation*}
\Delta \mathscr{Z}=\frac{1}{32 \pi^{2}} \int_{1 / \Lambda_{0}}^{\infty} \mathrm{d} l \frac{\left(1-\cos \frac{l}{L^{2}}\right)}{l^{3}}=\frac{\ln \left(L^{2} \Lambda_{0}\right)}{64 \pi^{2} L^{4}} . \tag{34}
\end{equation*}
$$

The coefficient in the expression with allowance for $U=1 / 72$ and $6 / L^{2}=R$ corresponds to Eq. (32a) with the cutoff $l_{\max }=L^{2}, l_{\text {min }}=1 / \Lambda_{0}$.

## V.VACUUM POLARIZATION BY THE ELECTROMAGNETIC FIELD

Another example of application of the general method is the vacuum polarization by a given electromagnetic field. We assume that the vectors $\mathbf{E}$ and $\mathbf{H}$ do not depend on the coordinates. If either of the invariants $J_{1}=\mathbf{E}^{2}-\mathbf{H}^{2}$ or $J_{2}=(\mathbf{E H})^{2}$ is nonzero, there exists a Lorentz transformation as a result of which $E \rightarrow \mathbf{E}_{0}, \mathbf{H} \rightarrow \mathbf{H}_{0}$, so that $\mathbf{E}_{0} \| \mathbf{H}_{0}$ (and to be specific we take them along the $x$ axis). The vector potential in this frame of reference has components $A_{t}=-x E_{0} / 2, A_{x}=t E_{0} / 2, A_{y}=-z H_{0} / 2, A_{z}=y H_{0} / 2$. We calculate $\Delta \mathscr{L}$ at the point $(0,0,0,0)$. We consider first a complex scalar field $\varphi$. The equation for the Green's function,

$$
\begin{equation*}
\square_{A} G_{A}=i \frac{\partial G_{A}}{\partial l}, \quad \square_{A}=\left(\frac{\partial}{\partial t}-i e A_{t}\right)^{2}-\left(\frac{\partial}{\partial \mathbf{x}}-i e \mathbf{A}\right)^{2}, \tag{35}
\end{equation*}
$$

has the solution $G_{A}=G_{1}(0,0, x, t, l) G_{2}(0,0, x, z, l)$. Substituting into (35),

$$
\begin{aligned}
& G_{1}=\frac{1}{4 \pi l} \exp \left\{-i\left(t^{2}-x^{2}\right) \alpha(l)+\beta(l)\right\}, \\
& G_{2}=-\frac{i}{4 \pi l} \exp \left\{i\left(y^{2}+z^{2}\right) \gamma(l)+\beta(l)\right\} \operatorname{sign} l,
\end{aligned}
$$

we find

$$
\begin{array}{ll}
\alpha=\frac{e E_{0}}{4} \operatorname{cth}\left(e E_{0} l\right), & \beta=\ln \frac{e E_{0} l}{\operatorname{sh}\left(e E_{0} l\right)}, \\
\gamma=\frac{e H_{0}}{4} \operatorname{ctg}\left(e H_{0} l\right), & \delta=\ln \frac{e H_{0} l}{\sin \left(e H_{0} l\right)} .
\end{array}
$$

Hence, the Green's function of the scalar complex field for $x_{1}=x_{0}=(0,0,0,0)$ is

$$
\begin{equation*}
G_{A}=-\frac{i \operatorname{sign} l}{(4 \pi)^{2}} \frac{e E_{0}}{\operatorname{sh}\left(e E_{0} l\right)} \frac{e H_{0}}{\sin \left(e H_{0} l\right)} . \tag{36}
\end{equation*}
$$

For a charged spinor field, Eq. (35) must be regarded as a four-row equation with the substitution ( $\Sigma_{x}$ and $\alpha_{x}$ are Dirac matrices)

$$
\begin{align*}
& \square_{A} \rightarrow \square_{A^{\prime}}=\delta_{m_{0} m_{1}} \square_{A}+\Sigma_{x} e H_{0}+i \alpha_{x} e E_{0}  \tag{36a}\\
& G_{A^{\prime}}(0,0, l)=\delta_{m_{0} m_{1}} G_{A} \exp l\left\{-i e H_{0} \Sigma+e E_{0} \alpha\right\}
\end{align*}
$$

The contribution of the spinor with mass $m=m_{j}$ on the basis of (36a) and (25) is

$$
\begin{equation*}
\Delta \mathcal{L}_{j A}=\frac{1}{8 \pi^{2}} \int_{1 / \Lambda_{0}}^{\infty} \frac{\mathrm{d} l}{l} \cos m^{2} l\left\{\frac{e^{2} E_{0} H_{0}}{\operatorname{tg}\left(e H_{0} l\right) \operatorname{th}\left(e E_{0} l\right)}-\frac{1}{l^{2}}\right\} \tag{37}
\end{equation*}
$$

(cf. Ref. 8).
Expanding coth and cot in series ${ }^{3)}$ and expressing the polynomials in $E_{0}$ and $H_{0}$ in terms of the invariants $J_{1}$ and $J_{2}$, we obtain for the expression in the square brackets in (37)

$$
\begin{aligned}
\{\ldots\} & =\frac{e^{2} J_{1}}{3}-e^{4}\left(\frac{J_{1}^{2}}{45}+\frac{7 J_{2}}{45}\right) l^{2} \\
& +e^{6}\left(\frac{2}{945} J_{1}^{3}+\frac{13}{945} J_{2} J_{1}\right) l^{4}+\ldots
\end{aligned}
$$

After integration of (37),

$$
\begin{equation*}
\Delta \mathscr{L}_{j A}=\frac{e^{2} J_{1}}{24 \pi^{2}} \ln \frac{\Delta}{m^{2}}+\frac{e^{4}\left(J_{1}^{2}+7 J_{2}\right)}{360 m^{4} \pi^{2}}+\frac{e^{6}\left(2 J_{1}^{3}+13 J_{2} J_{1}\right)}{1260 \pi^{2} m^{8}}+\ldots \tag{38}
\end{equation*}
$$

The sum of the logarithmic terms for all charged particles must in accordance with the conjecture be $\Delta \mathscr{L}_{1}=\left(\mathbf{E}^{2}-\mathbf{H}^{2}\right) / 8 \pi$, and from this "sum rule" one can find the fine structure constant $e^{2}$ (Landau and Pomeranchuk ${ }^{3}$ ). The second term in (38) is the vacuum polarization found by Weisskopf in 1936. ${ }^{7}$ The third term describes sixphoton processes.

## VI. CONCLUSIONS

We have rederived the expressions of the proper-time method of Fock and Schwinger based on the concept of the spectral density of a wave equation. We have found an expression for the polarization by the gravitational field of the vacuum of scalar, spinor, and vector particles to terms quadratic in the components of the curvature tensor; for zero-mass particles we have obtained an expression that does not use an expansion in powers of the curvature tensor and does not contain an infrared divergence. The method has also been illustrated by the example of the electromagnetic field. The method can be readily generalized to any pro-
cesses that can be described by single-loop diagrams. For example, the method is fully applicable to the calculation of the effective density of the Lagrange function of boson fields and also fermion fields with nonzero masses and charges, and to the calculation of the radiative corrections to the magnetic moment of particles with spin in an arbitrarily strong external field (i.e., to the calculation of not only the "intrinsic" magnetic moment, but also the polarizability), to the calculation of the effective Lagrangian of vector fields of the Yang-Mills type, etc. However, all these calculations presuppose a restriction to single-loop diagrams if the method is used unmodified. How to extend the method to diagrams of more general form is not clear.

I thank the participants of the Theoretical Seminar of the P. N. Lebedev Physics Institute for discussing a first version of the work in June 1970, and also Ya. B. Zel'dovich for numerous discussions of the basic ideas. His paper on the cosmological constant ${ }^{5}$ and the Lagrangian of the electromagnetic field ${ }^{3}$ were important stimuli for this work. I thank I. M. Gel'fand for discussion and for drawing my attention to the work of McKean and Singer, ${ }^{4}$ and also giving me a photocopy of Ref. 4.

## APPENDIX

Following McKean and Singer's method, we find the Green's function of $n$-dimensional Riemannian space with definite metric for a scalar, a spinor, and a vector field.

We represent the Green's function of Eq. (15) in the form

$$
\begin{align*}
G_{R}= & G_{0}\left(1+Q R \tau+\left[U R^{2}+V R^{i k} R_{i k}\right.\right. \\
& \left.\left.+W R^{i k l m} R_{i k l m}+Y \square R\right\rfloor \tau^{2}+\ldots\right) \tag{A1}
\end{align*}
$$

The coefficients $Q, U$, etc., in this equation do not, according to Ref. 4, depend on the dimensionality $n$ of the space. Therefore, all the coefficients except $Y$ can be found by studying the solutions for three spaces of different dimensionality and constant curvature (for example, a sphere $S^{2}$, a hypersphere $S^{3}$, and supersphere $S^{4}$ ).

For each of these spaces one finds the eigenfunctions $\varphi_{i}$, determines $g_{i}$ and $\lambda_{i}$, and forms the sum

$$
\begin{equation*}
G(\tau)=G_{0}(\tau)\left(1+a_{1} \tau+a_{2} \tau^{2}+\ldots\right)=\frac{1}{V} \sum g_{i} e^{-\lambda_{i} \tau} \tag{A2}
\end{equation*}
$$

The sums are calculated by means of the asymptotic series

$$
\begin{equation*}
\sum_{0}^{\infty} f(n)=\int_{0}^{\infty} f(n) \mathrm{d} n+\frac{f(0)}{2}-\frac{f^{\prime}(0)}{12}+\frac{f^{\prime \prime \prime}(0)}{720}+\ldots \tag{A3}
\end{equation*}
$$

We denote the coefficients for a sphere by $a_{1}^{(2)}$ and $a_{2}^{(2)}$ and similarly for the hypersphere and supersphere.

We find the coefficient $Q$ as $a_{1}^{(2)} / R^{(2)}$, or as $a_{1}^{(3)} / R^{(3)}$, or as $a_{1}^{(4)} / R^{(4)}$ [using the values of the curvature $R^{(2)}=2$, $R^{(3)}=6, R^{(4)}=12$, where here and below the radius of the sphere is 1]. Of course, the result is the same. We find the coefficients $U, V, W$ from the three linear equations with three unknowns

$$
a_{2}^{(n)}=U R^{(n)^{2}}+V R^{(n)_{i k} R_{i k}^{(n)}+W R^{(n)_{\mathrm{ik} \mid m}} R_{i k l m}^{(n)}}
$$

(equations for $n=2,3,4$ ).
In Ref. 4 approximately the same method was used to consider the scalar field without the additional term $-R \varphi / 6$ in the equation of motion.

The method of Ref. 4 differs from ours in that $S^{4}$ is not considered, but instead a relation between $U$ and $Q$ is added ( $U=Q^{2} / 2$ ). For a scalar we readily find

| $g$ | $\lambda$ |
| :--- | :--- |
| $S^{2} 2 j+1$ | $j(j+1)$ |
| $S^{3}(J+1)^{2}$ | $\cdot J(J+2)$ |
| $S^{4} \frac{I}{6}(I+1)(I+2)(2 I+3)$ | $I(I+3)$. |

The summation for $S^{3}$ is particularly simple, the difference terms vanish identically, and $G=G_{0} e^{\tau}$.

## Author's Remark ${ }^{4}$

Unfortunately there are some errors in the paper. The consideration of vector and spinor cases is incorrect. This part of the paper is omitted in the present publication. The values of coefficients $Q$ to $Y$ for the scalar equation (without the addition term $-R \varphi / 6$ ) are

| $Q$ | $U$ | $V$ | $W$ | $Y$ |
| :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{6}$ | $\frac{1}{72}$ | $-\frac{1}{180}$ | $\frac{1}{90}$ | $\frac{1}{30}$ |

The coefficient $Y$ is determined by considering $G(0)$ with metric having $R \neq$ const:

$$
d s^{2}=d r^{2}+S(r) d \phi^{2}
$$

Of greatest importance is the erroneous sign in the formula (33) for the gravitational constant. This error originated in the author's incorrect passage from the euclidean to the pseudoeuclidean case in the formula (30) (the sign in front of the term $i Q$ ). After correcting this error, one sees that the increase of $1 / G$ has the sign opposite to that of $C_{i} Q_{i}$. Thus we arrive at a considerable difficulty with the sign of the gravitational constant in the theory of the zero Lagrangian.
${ }^{1)}$ Generally speaking, the equation for the scalar $\varphi$ may contain an additional term ( $-R \varphi / 6$, where $R$ is the trace of the Ricci tensor), which in the case $m=0$ makes the theory conformally invariant. For the discussion of the basic ideas, these details are not important.
${ }^{2)} \mathscr{L}_{0}=0$ in theories with supersymmetry and possibly in theories with spontaneous breaking of supersymmetry (communication of V.I. Ogievetskii).
${ }^{3)}$ This is done conveniently on the basis of the formulas coth $(2 x)=\frac{1}{2}(\operatorname{coth} x+1 / \operatorname{coth} x), \cot (2 x)=\frac{1}{2}(\cot x-1 / \cot x)$ by the method of undetermined coefficients.
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