Interaction of the electron and the positron in pair production

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The influence of the interaction of the components on the differential probability of pair production is studied.

I.

In calculations of the probability of pair production, allowance is never made for the interaction of the components of the pairs. Some authors (Heitler[1]) are of the opinion that it is impossible to solve this problem in the framework of hole theory.

Our method is based on the following remarks:

1. The interaction of the components has a significant influence on the differential probability only when the components have a small relative velocity in the final state. Therefore, such interaction can be treated in the center-of-mass system of the electron and positron as a simple Coulomb interaction e^2/r.

2. This interaction cannot be repeated as a perturbation but must be taken into account in the calculation of the eigenfunctions of the electron-positron system, as can be seen from the correction factor (8) which we obtain for the probability, the electron charge occurring in this expression in an essentially transcendental manner.

II.

We use below a system of units in which $\hbar = c = m$ (mass of the electron) $= 1$.

The remaining notation is as follows: $e = 137^{-1/2}$ is the electron charge, E_+, E_- and $\mathbf{p}_+$, $\mathbf{p}_-$ are the energy and momenta of the positron and electron in the laboratory coordinate system, $\mathbf{k}_+$ and $\mathbf{k}_-$ are the momenta of the positron and electron in the coordinate system in which $\mathbf{k}_+ \approx \mathbf{k}_-$ (center-of-mass system), and v is the relative velocity of the electron and the positron. It is readily seen that

$$v = \sqrt{1 - (E_+E_- - p_+p_-)^{-2}} \approx \sqrt{p^2\vartheta^2 + (|p_+| - |p_-|)^2E^{-2}}. \tag{1}$$

Here, ϑ is the angle between the electron and the positron; the approximate expression for v corresponds to the case $v \ll p/E$ (i.e., $\mathbf{p}_+ \to \mathbf{p}_-$).

The wave function describing the electron and the positron is $\Psi_{\mathbf{p}_+\mathbf{p}_-}(q)$, where q denotes the arguments of the wave function; in the usual formulation of hole theory, this is the set of coordinates of the electrons of all occupied levels.

We use a system of eigenfunctions that go over into plane waves at the infinity of the coordinate space of the electron and positron. The indices $\mathbf{p}'_+$ and $\mathbf{p}'_-$ are the momenta corresponding to these plane waves (the momenta at infinity); $\Psi_0(q)$ is the wave function that describes the vacuum; V is the transition matrix element, and $\tilde{V}$ is the matrix element calculated without allowance for the interaction of the components. Quite generally, we use the tilde to denote the values of quantities calculated without allowance for the interaction of the components, in contrast to the "exact" values.

The required correction factor for the differential probability $d\omega$ is

$$T = d\omega/d\tilde{\omega} = |V/\tilde{V}|^2. \tag{2}$$

III.

In perturbation theory calculations, the Hamiltonian is represented as a sum of two terms: $H = H_0 + H_1$. The first term is used to calculate the eigenfunctions $H_0\Psi = E_0\Psi$, and the second is used to calculate the matrix element. For first-order processes,

$$V_{\mathbf{p}'_+\mathbf{p}'_-} = \int \Psi_0^* H_1 \Psi_{\mathbf{p}'_+\mathbf{p}'_-}\, dq. \tag{3}$$

(The following section is devoted to processes of higher order.) We shall assume that the interaction W of the components is included in H:

$$H_0 = \tilde{H}_0 + W.$$

The eigenfunctions Ψ of the operator H are linear combinations of the eigenfunctions $\tilde{\Psi}$ of the operator $\tilde{H}_0$. We have

$$\Psi_{\mathbf{p}_+\mathbf{p}'_-} = \int d^3\mathbf{p}_+ d^3\mathbf{p}_- (\rho_+\rho_- |c| \mathbf{p}'_+\mathbf{p}'_-)\tilde{\Psi}_{\mathbf{p}_+\mathbf{p}_-} \tag{4}$$

(for brevity, the spin variables are omitted here and below). In Eq. (4), c is some unitary singular matrix very close to a δ matrix. Indeed, in the limiting case $e \to 0$,

$$c \to \tilde{c} = \delta(\mathbf{p}_+ - \mathbf{p}_+')\delta(\mathbf{p}_- - \mathbf{p}_-').$$

The exact form of c will be found later.

Substituting (4) and (3) and reversing the order of integration over q and p, we obtain

$$V_{\mathbf{p}'_+\mathbf{p}'_-} = \int d^3\mathbf{p}_+ d^3\mathbf{p}_- (\mathbf{p}_+\mathbf{p}_- |c| \mathbf{p}_+'\mathbf{p}_-')\tilde{V}_{\mathbf{p}_+\mathbf{p}_-}. \tag{5}$$

Because of the δ-like nature of c, we can take $\tilde{V}$ in front of the integral sign and write

$$J = V/\tilde{V} = \int d^3\mathbf{p}_+ d^3\mathbf{p}_- \cdot c. \tag{6}$$

Further, we can go over to the center-of-mass system of the electron and the positron:

$$\mathbf{p}_+ \to \mathbf{k}_+, \quad \mathbf{p}_+' \to \mathbf{k}_+' \quad \text{etc.} \quad \mathbf{k}_+' \approx -\mathbf{k}_-'$$

$$(\mathbf{k}_+\mathbf{k}_- |c_k| \mathbf{k}_+'\mathbf{k}_-') = \sqrt{\lambda\lambda'}(\mathbf{p}_+\mathbf{p}_- |c| \mathbf{p}_+'\mathbf{p}_-').$$

Here, $\lambda(\mathbf{p}_+, \mathbf{p}_-)$ and $\lambda'(\mathbf{p}'_+, \mathbf{p}'_-)$ are the Jacobians of the transformation, and the factor $\sqrt{\lambda\lambda'}$ ensures that the unitar-

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ity of the matrix c is conserved:

$$\int d^6\mathbf{p}\cdot c^*c = \delta(\mathbf{p}_+' - \mathbf{p}_+'')\delta(\mathbf{p}_-' - \mathbf{p}_-''),$$

$$\int d^6\mathbf{k}(\mathbf{k}|c_k^*|\mathbf{k}')(\mathbf{k}|c\,|\mathbf{k}'') = \delta(\mathbf{k}_+' - \mathbf{k}_+'')\delta(\mathbf{k}_-' - \mathbf{k}_-'').$$

In calculating J [Eq. (6)], we can assume that $\lambda' \approx \lambda$ (because of the δ-like nature of c) and write

$$J = \int \lambda^{-1} c_k d^6\mathbf{p} = \int c_k d^6\mathbf{k}.$$

Finally, we can make the Fourier transformation

$$\int c_k d^6\mathbf{k} = (2\pi)^3(00|c_x|\mathbf{k}_+'\mathbf{k}_-'). \tag{7}$$

Here, c_x is the wave function in the coordinate space of the electron and positron, which satisfies (in the nonrelativistic approximation) the Schrödinger equation

$$-(1/2)(\Delta_+ + \Delta_-)c_x - (e^2/r)c_x = (1/2)(\mathbf{k}_+'^2 + \mathbf{k}_-'^2)c_x.$$

We introduce relative coordinates in the usual manner and solve the resulting equation with reduced mass $\frac{1}{2}$ by separating the variables in parabolic coordinates (cf. Bethe's solution to the problem of electron scattering[2]). We obtain (normalization to volume $d^3\mathbf{k}'_+ \times d^3\mathbf{k}'_-$)

$$c_x = \frac{F(i\varepsilon, 1, i[(\mathbf{k}_+' - \mathbf{k}_-')(\mathbf{x}_+ - \mathbf{x}_-) + |\mathbf{k}_+' - \mathbf{k}_-'||\mathbf{x}_+ - \mathbf{x}_-|])}{|F(i\varepsilon, 1, i\infty)|}\tilde{c}_x,$$

$$\tilde{c}_x = (2\pi)^{-3}\exp[i(\mathbf{k}_+'\mathbf{x}_+ + \mathbf{k}_-'\mathbf{x}_-)].$$

Here, F is the hypergeometric function, $\varepsilon = e^2/v$, where v is the relative velocity of the electron and the positron in the final state (1), and $\mathbf{x}_+$ and $\mathbf{x}_-$ are radius vectors.

On the basis of (7),

$$J = |F(i\varepsilon, 1, i\infty)|^{-1} = [2\pi\varepsilon/(1 - e^{-2\pi\varepsilon})]^{1/2}.$$

Finally, on the basis of (2),

$$T = J^2 = 2\pi\varepsilon/(1 - e^{-2\pi\varepsilon}). \tag{8}$$

IV.

The generalization of this derivation to processes of second (and higher) order is not difficult. The matrix element is calculated in the form of a sum (or integral, for generality) over so-called intermediate states $\mathbf{p}^i$. Instead of (3) for first-order processes, we obtain for second-order processes

$$V_{\mathbf{p}'} = \int d^3\mathbf{p}_+^i d^3\mathbf{p}_-^i \{\int \Psi_0^* H_1 \Psi_i dq\}\{\int \Psi_i^* H_1 \Psi_{\mathbf{p}'} dq\}\Delta_i^{-1}, \tag{9}$$

where Δ_i is the change in the energy in the intermediate state compared with the initial state. The final function $\Psi_{\mathbf{p}'} = \Psi\mathbf{p}'_+\,\mathbf{p}'_-(q)$ occurs in this formula linearly. The interaction of the components of the pair in the final state is taken into account in exactly the same way as in the case of first-order processes, and again leads to (8).

We now consider the influence of the interaction of the components in the intermediate state on the differential probability. Even without calculation it is clear that the point at which the relative velocity v^i in the intermediate state vanishes cannot be a singular point for the correction factor T (in contrast to the point of vanishing of v in the final state, which is such a singular point). The point is that the relative velocity v^i in the intermediate state is not a relativistically invariant quantity. If $v^i = 0$ in one frame of reference, then it will be nonzero in other frames of reference.

We shall show by a typical example that the interaction in the intermediate state is in fact unimportant. Let us consider pair production by a photon in the field of a nucleus, i.e., the term of the matrix element V due to the chain

$$\mathbf{p}_\gamma \to \mathbf{p}_+ + \mathbf{p}_-^i \to \mathbf{p}_+ + \mathbf{p}_- + \mathbf{q},$$

where $\mathbf{p}_\gamma$ is the momentum of the photon, and $\mathbf{q}$ is the momentum transferred to the nucleus. Here $v^i = 0$ for $\mathbf{p}_+ = \mathbf{p}_\gamma/2$, and it is obvious that this condition is relativistically noninvariant.

Substituting (4) in (9), we obtain

$$V = \int d\mathbf{p}_+^i d\mathbf{p}_-^i \Delta_i^{-1}\{\int \tilde{V}_1(\mathbf{p}^1|c|\mathbf{p}^i)\delta(\mathbf{p}_+^1 + \mathbf{p}_-^1 - \mathbf{p}_\gamma)d\mathbf{p}^1\}$$
$$\times\{\int \tilde{V}_2(\mathbf{p}^2|c^*|\mathbf{p}^i)\delta(\mathbf{p}_+^2 - \mathbf{p}_+)d\mathbf{p}^2\}.$$

Taking the slowly varying factors in front of the integral sign, we obtain $V = \tilde{V}J$, where $\tilde{V} = \tilde{V}_1\tilde{V}_2/\tilde{\Delta}_i$, and

$$J = \int d\mathbf{p}^i d\mathbf{p}^1 d\mathbf{p}^2 \delta(\mathbf{p}_+^2 - \mathbf{p}_+)\delta(\mathbf{p}_+^1 + \mathbf{p}_-^1 - \mathbf{p}_\gamma)$$
$$\times\ (\mathbf{p}^1|c|\mathbf{p}^i)(\mathbf{p}^2|c^*|\mathbf{p}^*). \tag{10}$$

Because of momentum conservation, the matrix c contains a δ-like factor, which it is expedient to split off. We set $\mathbf{p}_+^i + \mathbf{p}_-^i = \mathbf{p}_\sigma^i$, $\mathbf{p}_+^1 + \mathbf{p}_-^1 = \mathbf{p}_\sigma^1$, etc. We have

$$(\mathbf{p}_+^1\mathbf{p}_-^1|c|\mathbf{p}_+^i\mathbf{p}_-^i) = \delta(\mathbf{p}_\sigma^1 - \mathbf{p}_\sigma^i)(\mathbf{p}_-^1|d|\mathbf{p}_-^i), \tag{11}$$

where the new matrix d is also unitary:

$$\int d\mathbf{p}_-^i(\mathbf{p}_-^1|d|\mathbf{p}_-^i)(\mathbf{p}_-^2|d^*|\mathbf{p}_-^i) = \delta(\mathbf{p}_-^1 - \mathbf{p}_-^2). \tag{12}$$

Substituting (11) in (10), we find

$$J = \int d\mathbf{p}_-^i d\mathbf{p}_-^1(\mathbf{p}_-^1|d|\mathbf{p}_-^i)(\tilde{\mathbf{p}}_-^i|d^*|\mathbf{p}^i),$$

which is equal to unity by virtue of (12). Thus, in the approximation in which the entire theory is constructed ($\tilde{V}$ and Δ taken in front of the integral sign), the interaction in the intermediate state is indeed unimportant, which agrees with the invariance requirements.

V.

Hitherto we have ignored spin and relativistic effects. Do they influence our results? Equation (5) remains exactly the same, but the form of the matrix c is changed somewhat, and in the summation over the spin variables it is necessary to take into account the dependence of $\tilde{V}$ on the spins. However, this last circumstance is unimportant, since the spin is conserved in the Coulomb interaction of slow particles, so that $\tilde{V}$ can be taken in front of the sign of the summation over the spins.

Equations (6) and (7) do not hold, since $(00|c_x|)$ becomes infinite. Instead of taking $\tilde{V}$ in front of the integral in (5), we can on the basis of a well-known theorem make a Fourier transformation of $\tilde{V}$ and c:

$$\int d^6\mathbf{p}\tilde{V}_p c_p = \int d^6\mathbf{x} V_x c_x. \tag{13}$$

Here V_x is some δ-like function smeared over the region of space responsible for the pair production. (In the case of pair production as a result of a nuclear transition with forbidden emission of photons, the initial angular momentum of the nucleus is $J = 0$; V_x corresponds to the oscillations of

the Coulomb potential, and $V_x \neq 0$ within the nucleus. See Sakharov,[3] and also Oppenheimer,[4] and Yukawa and Sakata.[5])

The function c_x in (13) has a "weak" pole with degree of order 137^{-2} (by analogy with the function of a single electron in a Coulomb field). Since $\tilde{V}$ is smeared over a region of order of the radius R of the nucleus or more, and c_x differs from its nonrelativistic value in regions of order of the electron radius $r_0 \ll R$, we can in the calculation of (13) use instead of the exact values of c_x its nonrelativistic value at the coordinate origin. We again arrive at (7).

VI.

We note that the region of quantitative applicability of Eq. (8) is limited to medium Z (the charge of the nucleus) and relativistic velocities of the electrons and positrons on account of the "Born" treatment of the Coulomb field of the nucleus. In contrast, the relative velocity of the components may be arbitrarily small, since to treat the interaction we have not used the Born approximation (in contrast to Ref. 6, in which Rudnitskii studied annihilation; of course, our results also apply to annihilation).

This paper forms part of my dissertation. I am pleased to express my gratitude to my supervisor Professor I. E. Tamm.

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