Formation energy of a waveguide as a measure of its cutoff frequency

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It turns out that the cutoff frequency of a waveguide, multiplied by Planck's constant, and the rest mass of a photon in the waveguide, multiplied by the velocity of light squared, are equal to the work performed in opposition to the radiation-pressure forces of zero-point vacuum fluctuations during the formation of the waveguide from free space.

1. The analogy between the dispersion relation for a waveguide mode,

$$\omega^2 = \omega_n^2 + (ck_n)^2 \tag{1}$$

and the relativistic expression for the Hamiltonian of a free particle,

$$E^2 = (m_0c^2)^2 + (cp)^2 \tag{2}$$

has been pointed out by various investigators a number of times over the course of many years (Ref. 1, for example). Here ω and k_n are the frequency and propagation constant of a wave in the waveguide, for the mode of index n and cutoff frequency ω_n; E and p are the total energy and momentum of the particle, whose rest mass is m_0; and c is the velocity of light. Here quantities are being associated with each other in pairs: the frequency with the total energy, the cutoff frequency with the rest mass, and the propagation constant with the momentum.

Pursuing this analogy, De Broglie wrote[1] that everything proceeds as if the photon had an intrinsic mass determined by the shape of the waveguide and by the particular eigenvalue under consideration. He wrote that a photon might have a number of intrinsic masses in a given waveguide. Unfortunately, he went on to write that he was going to put aside all these considerations, which were distracting him from his subject.

One senses in this analogy a heuristic content[2] which might in fact bring us closer to an understanding of the essence of the matter.

Formally, of course, the cutoff frequency ω_n for a mode is found as an eigenvalue of the wave equation under the boundary conditions imposed by the cross-sectional shape of the waveguide.

A graphic kinematic representation of the cutoff frequency is usually constructed in terms of an interference of partial waves (which would be plane waves in the simple case of a rectangular cross section) whose superposition creates the field of the mode which satisfies the boundary conditions. As the wave frequency ω approaches the cutoff value ω_n, the wave vector of the partial wave also approaches the direction normal to the longitudinal axis of the waveguide. At $\omega = \omega_n$, this wave vector is in fact directed strictly along this normal, and the propagation of the longitudinal traveling wave is cut off ($k_n = 0$).

However, there is further meaning in the concept of a waveguide cutoff frequency ω_n, as we will see below: The energy corresponding to the cutoff frequency ($\hbar\omega_n$) is numerically equal to the least amount of work (w_n) which would be performed in forming a waveguide of finite cross section from unbounded free space:

$$\hbar\omega_n = w_n. \tag{3}$$

The rest mass of a photon in the waveguide is

$$m_n = \frac{w_n}{c^2}. \tag{4}$$

2. To demonstrate the meaning of these statements, we consider the very simple example of a plane electromagnetic waveguide formed by two unbounded parallel metal planes separated by a vacuum gap of thickness a. The field in such a waveguide may be thought of as a superposition of two partial plane waves, reflected from the metal bounding planes at angles of incidence θ. The boundary conditions are satisfied as a result of an interference of the partial waves if

$$\cos\theta_n = \frac{\omega_n}{\omega}, \tag{5}$$

where

$$\omega_n = \frac{\pi cn}{a} \qquad (n = 0, 1, 2, \ldots) \tag{6}$$

are the cutoff frequencies of the modes of the two possible polarizations: TE_n ($n = 1, 2,\ldots$) and TM_n ($n = 0, 1, 2,\ldots$); (the TM_0 mode is discussed separately below).

What transformations does the electromagnetic field of the mode undergo as the gap thickness a is varied, i.e., as the metal planes are moved toward each other or away from each other at constant velocities $\pm c\beta$ with respect to the symmetry plane of the waveguide? It follows from the Lorentz transformations that upon each reflection of a partial wave at the waveguide boundary there are changes in both the wave frequency ω (this is the Doppler effect) and the angle of incidence (or reflection) θ:

$$\omega^{\mathrm{ref}} = \omega^{\mathrm{inc}}(1+\beta\cos\theta^{\mathrm{inc}})^2(1-\beta^2)^{-1}, \tag{7}$$

$$\cos\theta^{\mathrm{ref}} = \frac{\cos\theta^{\mathrm{inc}} + 2\beta(1+\beta^2)^{-1}}{2\beta(1-\beta^2)^{-1}\cos\theta^{\mathrm{inc}} + 1} \tag{8}$$

[$\beta \gtrless 0$ as the metal planes are moved toward each other (or away from each other)].

Are these changes in frequency and angle of incidence compatible with waveguide conditions (5) and (6)? In other words, does the waveguide interference structure of the field

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remain the same as the reflecting surfaces are moved? Working from (7) and (8), one can show that, over one cycle of the propagation of the partial wave from one reflecting surface to the other, the gap thickness a acquires an increment

$$\Delta a = -2a\beta \frac{1 + 2\beta\cos\theta + \beta^2}{(1 + 3\beta)\cos\theta + \beta(3 + \beta^2)}, \tag{9}$$

and the changes in the frequency ω and the angle of incidence θ are given by

$$\Delta\omega = \omega\beta \frac{2\cos\theta + \beta(1 + \cos^2\theta)}{1 - \beta^2}, \tag{10}$$

$$\Delta \cos\theta = 2\beta \sin^2\theta \cdot (1 + 2\beta\cos\theta + \beta^2)^{-1}. \tag{11}$$

The total changes in the frequency ω and the angle θ as the result of multiple reflections as the gap thickness changes from a_0 to a can be found by going over from finite differences to a system of differential equations,

$$\frac{d\theta}{da} = \frac{\sin\theta}{a} \frac{\cos\theta + 3\beta + 3\beta^2\cos\theta + \beta^3}{(1 + 2\beta\cos\theta + \beta^2)^2}, \tag{12}$$

$$\frac{d\omega}{da} = -\frac{\omega}{2a} \frac{2\cos\theta + \beta(1 + \cos^2\theta)}{1 - \beta^2} \frac{\cos\theta + 3\beta + 3\beta^2\cos\theta + \beta^3}{1 + 2\beta\cos\theta + \beta^2} \tag{13}$$

and by integrating these equations.

The zeroth approximation in β of the integrals of system (12), (13) corresponds to the first-order Doppler effect upon a slow movement of the reflecting planes ($\beta \ll 1$). In this approximation we find

$$\operatorname{tg}\theta = \frac{a}{a_0} \operatorname{tg}\theta_0, \tag{14}$$

$$\omega = \omega_0 \frac{\sin\theta_0}{\sin\theta} = \omega_0 \frac{a_0}{a} \frac{\cos\theta_0}{\cos\theta}, \tag{15}$$

where ω_0 and θ_0 are the initial values of the frequency and the angle at $a = a_0$.

We can draw several conclusions at this point. As $\beta \to 0$, the frequency ω and the angle of incidence θ do not remain constant (as one might expect at first glance). Their instantaneous values are determined unambiguously by the waveguide gap a. If waveguide condition (5) holds in the initial position ($a = a_0$), then it also holds at arbitrary instantaneous values of a. We thus find an affirmative answer to the question of whether the interference structure of the waveguide field remains the same as the reflecting surfaces forming the waveguide are slowly moved: The radiation frequency and the angle of incidence θ continuously *adjust* to accommodate the waveguide conditions by virtue of the Lorentz transformations. If the displacement velocity β is not small, condition (5) does not remain satisfied for the instantaneous values of ω and θ, and the steady-state interference structure is disrupted. Here we find a separate and interesting problem: the scattering of the photons of one mode of a uniform waveguide into another mode, with indices n, cutoff frequencies ω_n, and photon masses m_n different from their initial values.

3. When the angle θ is eliminated from Eqs. (14), (15), we find a linear equation for the squared frequencies in a waveguide with moving reflecting surfaces:

$$\omega^2 - \omega_n^2 = \omega_0^2 - \omega_{n0}^2 = \text{const}, \tag{16}$$

where ω_{n0} is the cutoff frequency for mode n at $a = a_0$. This equation means that the propagation constant k_n remains constant. This point is also obvious from kinematic considerations concerning an unbounded uniform waveguide. We thus find expressions for the phase velocity ($v_n/\omega = v_{n0}/\omega_0 = \text{const}$) and the group velocity ($u_n\omega = u_{n0}\omega_0 = \text{const}$), where v_{n0} and u_{n0} are the initial values of these velocities, at $a = a_0$.

An important particular case is that in which the initial waveguide has an infinite transverse dimension ($a_0 \to \infty$), i.e., the case of *free space*. In this case we have $\omega_{n0} \to 0$ according to (6), and the two initial partial waves merge to form a single plane wave, whose wave vector is parallel to the longitudinal axis of the waveguide:

$$\omega^2 = \omega_0^2 + \omega_n^2. \tag{17}$$

When a field is compressed from unbounded free space into a waveguide of finite cross section, the waveguide turns out to be filled with a field whose frequency ω is the sum of the squares of the initial frequency ω_0 and the instantaneous cutoff frequency ω_n.

Let us assume that the initial field is a purely *static* field, with an initial frequency $\omega_0 = 0$. In this limit, a *wave* field with a nonvanishing frequency equal to the cutoff value,

$$\omega = \omega_n, \tag{18}$$

arises in the waveguide. In other words, the static field of free space serves as a generator with respect to the wave field of a cutoff waveguide of finite cross section.

When events proceed in the opposite direction, from a finite waveguide to free space, we would of course observe a degradation of the wave field to the point that it becomes a static field.

4. It is fairly obvious that the energy of the electromagnetic field which arises in a waveguide with moving reflecting surfaces originates from the work performed in opposition to the radiation-pressure forces acting on these surfaces.

The total radiation-pressure force is

$$F = \frac{W}{a}\cos^2\theta, \tag{19}$$

where W is the energy of the waveguide field, and the work performed on changing the gap thickness a is

$$\Delta W = -\frac{W}{a}\cos^2\theta\Delta a. \tag{20}$$

Alternatively, we could use (5), (6), and (16) and go over from finite differences to a differential equation:

$$\frac{dW}{da} = -\frac{W/a}{1 + (a/a_0)^2[(\omega_0/\omega_{n0})^2 - 1]} \tag{21}$$

a solution of this equation is

$$\frac{W}{\omega} = \frac{W_{n0}}{\omega_0}, \tag{22}$$

where W_{n0} is the initial value of the electromagnetic energy of the waveguide mode, at $a = a_0$.

It has been shown here that the cutoff frequency ω_n is determined as the frequency ω of the field which arises during the compression of an unbounded initial waveguide with a zero initial frequency. In other words, according to (18) and (22), we have

$$\omega_n = \omega = \frac{\omega_0}{W_{n0}} W. \tag{23}$$

Here we need to resolve the indeterminate form W_{n0}/ω_0.

Our first step in this direction must be to alter classical electrodynamics and—an important point—invoke the language of quantum mechanics. The field energy of the mode of the initial waveguide (i.e., of unbounded space) is

$$W_{n0} = \hbar\omega_0\left(f_{n0} + \frac{1}{2} + \frac{1}{2}\right), \tag{24}$$

where f_{n0} is the photon filling number of the mode, and the one $(1 = 1/2 + 1/2)$ embodies the contribution from the zero-point vacuum fluctuations of the two possible propagation directions in the waveguide. The ratio in question is thus given by

$$\frac{W_{n0}}{\omega_0} = \hbar\,(f_{n0} + 1), \tag{25}$$

and the energy corresponding to the cutoff frequency of the finite waveguide is

$$\hbar\omega_n = \frac{W}{f_{n0} + 1} \equiv w_n. \tag{26}$$

Here w_n is the work per photon of the field of the initial waveguide, found with allowance for vacuum fluctuations, which are also responsible for the Casimir effect.

The energy corresponding to the cutoff frequency ω_n of a waveguide of finite cross section is thus numerically equal to the work performed in order to form this waveguide from unbounded free space, in opposition to the radiation-pressure force of a single photon of the given mode. Here we have no need for real initial photons: All that is necessary for the process to occur is the energy $\hbar\omega_0 = (\hbar\omega_0/2) + (\hbar\omega_0/2)$ (always present) of the zero-point vacuum fluctuations, which furthermore have a zero frequency. The process amounts to a raking together (compression) of an initial static fluctuational field from the entire free space and the raising of the frequency of this field from zero to ω_n.

The thermodynamics of the compression process, in the course of which the performance of work is accompanied by a change in the entropy of the electromagnetic field, deserves separate study.

5. The TM_0 mode of a plane waveguide, which has a zero cutoff frequency, $\omega_n = 0$, at any value of the gap thickness a, requires a special discussion. In complete accordance with the arguments above, and in confirmation of them, a simple circumstance is responsible here: The polarization of the TM_0 mode is such that the electric vector is always perpendicular to the reflecting plane, so no radiation pressure acts on this plane. As a result, both the work w_n and the cutoff frequency ω_n $(n = 0)$ are zero.

We should also mention that the approach developed above could be taken in order to decipher the meaning of the concept of the resonant frequencies of a resonator.

6. In summary, it turns out that the concept of a cutoff frequency of a waveguide, which can be determined exhaustively within the framework of classical electrodynamics, requires a switch to quantum-mechanical terminology as soon as we take up the question of the evolution of the entities "free space" and "a waveguide of finite cross section." Here we are essentially seeing the uncertainty relation at work.

The cutoff frequency ω_n in (3) and the rest mass of a photon in the waveguide, m_n in (4), ultimately come from the very simplest form of electromagnetic matter: the static fluctuational field of vacuum.

The raking together of this static field, devoid of wave characteristics, from unbounded free space imparts wave properties and a finite mass-energy to this field.

This entire analysis has used a simple plane model, but this model will still convey the basic characteristics of waveguides with a cross section of arbitrary complexity. One might thus expect that a more general analysis would not result in any significant corrections to the conclusions drawn here. Furthermore, these results could apparently be extended to a long list of problems involving eigenvalues differing in physical nature.

Finally, going back to the analogy with which we started this paper, we might venture to pursue it under the assumption that the rest masses of particles originate from work performed in compressing the matter which makes up these particles.

[1] L. De Broglie, *Electromagnetic Waves in Waveguides and Cavity Resonators* [Russ. transl. IL, M., 1948].

[2] L. A. Rivlin, Kvant. Elektron. (Moscow) **6**, 1087 (1979) [Sov. J. Quantum Electron. **9**, 640 (1979)].

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