METHODOLOGICAL NOTES

Average radiation-reaction force in quantum electrodynamics

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Standard invariant perturbative methods are used to derive an expression for the radiation-reaction force acting on a charged particle in quantum electrodynamics. When the average change in the particle's momentum is found by taking an average over the states closest to the classical state, the result is an expression whose classical limit is the same as the classical expression for the radiation-reaction force.

1. INTRODUCTION

A charge in accelerated motion radiates electromagnetic waves, and as a result it experiences another force: the "radiation-reaction force" or "radiation-damping force." When this force is taken into account, the equation describing the motion of a charge e (for definiteness, we will speak in terms of an electron) in an external electric field $\mathbf{E}$ and an external magnetic field $\mathbf{H}$ (both fields are given fields) takes the following form in the nonrelativistic case:[1,2]

$$m\ddot{\mathbf{r}} = \mathbf{F}_0 + \frac{2}{3}e^2\dddot{\mathbf{r}}, \qquad \mathbf{F}_0 = e\mathbf{E} + e\mathbf{v}\times\mathbf{H} \tag{1}$$

(here and everywhere below, unless otherwise stipulated, we are using a system of units in which the velocity of light is one, $c = 1$, as is Planck's constant, $\hbar = 1$).

Equation (1) can be used without the risk of obtaining "self-accelerating" solutions (or of running into paradoxes associated with a violation of causality) if the second term in (1) is small in comparison with the first, which is the Lorentz force. Without going into that matter, which is analyzed in detail in Refs. 1–3, we would like to point out that if the radiation-reaction force is small it can be dealt with in (1) by perturbation theory. In a first approximation we find $m\mathbf{r} = \mathbf{F}_0$, and then

$$m\ddot{\mathbf{r}} = \mathbf{F}_0 + \delta\mathbf{F}, \qquad \delta\mathbf{F} = \frac{2e^3}{3m}\dot{\mathbf{E}} + \frac{2e^4}{3m^2}\mathbf{E}\times\mathbf{H}. \tag{2}$$

In the general relativistic case, Eq. (2) becomes[1]

$$\begin{aligned} m\frac{du^\mu}{ds} &= eF^{\mu\nu}u_\nu + \delta F^\mu, \\ \delta F^\mu &= \frac{2e^3}{3m}\frac{\partial F^{\mu\nu}}{\partial x^\lambda}u_\nu u^\lambda + \frac{2e^4}{3m^2}F_{\sigma\lambda}F^{\sigma\nu}u^\lambda(u_\nu u^\mu - \delta^\mu_\nu), \end{aligned} \tag{3}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, u^μ is the 4-velocity of the particle, and s is the proper time.

Equation (1) can be derived in classical electrodynamics by making use of the fact that a moving charge creates a self-electromagnetic field $\mathbf{E}_s$, $\mathbf{H}_s$. The equation of motion of the charge in the external fields $\mathbf{E}$, $\mathbf{H}$ then becomes

$$m\ddot{\mathbf{r}} = e(\mathbf{E} + \mathbf{E}_s) + e\mathbf{v}\times(\mathbf{H} + \mathbf{H}_s). \tag{4}$$

Expressing $\mathbf{E}_s$ and $\mathbf{H}_s$ in terms of $\mathbf{r}(t)$ with the help of the classical expression for the retarded potentials, and going through several straightforward calculations including an expansion in the retardation, we can put (4) in the form[2,4]

$$m\ddot{\mathbf{r}} = \mathbf{F}_0' + \frac{2}{3}e^2\dddot{\mathbf{r}} - \delta m\ddot{\mathbf{r}}. \tag{5}$$

We should stress that in deriving (5) it is necessary to carry out an expansion in the retardation [specifically, in the small parameter $e^2/(mc^3t)$] in (4). Generally speaking, therefore, we cannot accept even expression (1) as an "absolutely accurate" expression for the radiation-reaction force.

The appearance of a term $\delta m\mathbf{r}$ in (5) should be interpreted as the acquisition of an "additional" mass, of electromagnetic origin, by the particle. This additional mass is $\delta m \sim e^2/r_0c^2$, where r_0 is the "radius" of the particle at which the divergent integrals must be cut off. It thus becomes necessary to renormalize the mass even in a classical theory. This renormalization reduces to the declaration that the observable mass of the particle is the sum $m + \delta m$. It is difficult to introduce δm in a relativistically invariant fashion. In the case of a point particle ($r_0 = 0$), δm is a functional of $\mathbf{r}(t)$. Taking the limit $r_0 \to 0$ actually forces one to take a quantum-mechanical approach in order to examine the motion of the particle in the given external electromagnetic field (only rarely is that point mentioned). The reason is that classical electrodynamics runs into logical contradictions at $r_0 \sim e^2/(mc^2)$ (at the classical radius of the electron), while quantum effects come into play at far greater distances—at r of the order of the Compton wavelength $\hbar/(mc)$, which is larger than the classical radius of the electron by a factor $\hbar c/e^2 \approx 137$. Consequently, essentially the entire problem must be dealt with by quantum theory. We should also mention that only in quantum theory can a renormalization be carried out systematically and unambiguously. Attempts to carry out a renormalization in a classical theory (Ref. 5, for example) run into some natural difficulties: The renormalization of the mass depends on the nature of the motion of the particle $\mathbf{r}(t)$, and in order to satisfy Maxwell's equations one must introduce some additional fictitious currents[5] near the charge. These currents will depend on $\mathbf{r}(t)$. The procedure for carrying out the classical renormalization becomes ambiguous. (There are other difficulties.)

The arguments above naturally suggest the following problem: Derive an expression for the radiation-reaction force in (2) [or in (3), which is equivalent to (2) by virtue of the Lorentz invariance of the theory] in quantum electrodynamics (QED) with the help of a scattering S-matrix and

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then take the classical limit. Solving that problem is the purpose of the present paper.

In contrast with some earlier studies,[4,6] we will systematically go through the procedure of taking an average over the states of the system which are closest to the classical state (over coherent states, if we use the Furry picture in the external field). It has been found that the deviations from such expectation values, i.e., fluctuations, are extremely influential. This circumstance resolves questions associated with self-accelerating solutions of (1). A self-acceleration arises in (1) during the time interval $\Delta t \sim e^2/(\mathrm{mc}^3)$, which is smaller by a factor of $e^2/(\hbar c)$ than $\hbar/(\mathrm{mc}^2)$, which is the time scale of the quantum fluctuations.

The radiation-reaction force has been derived by perturbation theory in the problem as formulated here; the use of a perturbation theory presupposes that this force is small in comparison with the Lorentz force $\mathbf{F}_0$ in (1). Analysis of the applicability of expressions (2) and (3) in the case $\mathbf{F}_0 = 0$ and of the related problems of "self-accelerating" solutions, the violation of causality, and the satisfaction of momentum, energy, and angular-momentum conservation lie outside the scope of the present discussion (these topics are discussed at length in the literature, e.g., Refs. 2–4).

The literature reveals a number of previous attempts to derive a radiation-reaction force in the form in (1) in QED by nonperturbative methods (see, for example, Refs. 4 and 6 and the bibliography in Ref. 3). These attempts have been conceptually close to the method used to derive Eq. (1) in classical electrodynamics. When that approach is taken, however, problems which arise in connection with the need for a renormalization cannot be resolved correctly. The correct existing method for carrying out a renormalization in QED is based on the use of an invariant perturbation theory and the elimination of the divergences from the S-matrix expanded in a power series in the interaction constant e. Just how correctly (1) can be derived by nonperturbative methods in a quantum theory (or whether such a derivation is possible at all) remains an open question. For this reason, the entire discussion below is based on the use of an S-matrix. We will simply point out the need to take account of quantum fluctuations, so average equations over short time intervals are meaningless.

2. GENERAL RELATIONS

We assume that the 4-potential of the external field in which the electron is moving is given by

$$\varphi^\mu(t,\mathbf{r}) = \int dq^0 d\mathbf{q}\, \varphi^\mu(q^\nu) \exp(i\mathbf{q}\mathbf{r} - iq^0 t), \tag{6}$$

and we assume that this potential is given and is not quantized. In other words, we have a c-number [the Greek letters $\mu, \nu, \ldots$ take on the values 0, 1, 2, 3; we are using the 4-vector $q^\nu = (q^0, \mathbf{q})$; $\varphi^\mu = (\varphi^0, \varphi), \ldots$; and we are using a metric $g_{\mu\nu}$ in which $g_{00} = 1$, $g_{0i} = 0$, $g_{ij} = -\delta_{ij}$, where $i, j, \ldots$ take on the values 1, 2, 3 (or x, y, z); and δ_{ij} is the Kronecker delta].

We assume that this field is small in comparison with the value $m^2/e[=m^2c^3(e\hbar)]$, at which we would need to consider possible pair creation by the external field. In addition to the field φ^μ there is a quantized field $\hat{A}^\mu$; the interaction with the latter field corresponds to the possible emission and absorption of a photon by the electron, and it leads to radiative corrections. The field operator $\hat{A}^\mu$ is given by the following expression (in the three-dimensionally transverse gauge used below):

$$\hat{A}^0 = 0,$$
$$\hat{\mathbf{A}}(t,\mathbf{r}) = \sum_{\mathbf{k},\alpha} \left(\frac{4\pi}{2|\mathbf{k}|}\right)^{1/2} [\hat{c}_{\mathbf{k}\alpha}\mathbf{e}_{\mathbf{k}\alpha}\exp(i\mathbf{k}\mathbf{r} - i|\mathbf{k}|t) + \hat{c}^+_{\mathbf{k}\alpha}\mathbf{e}^*_{\mathbf{k}\alpha}\exp(-i\mathbf{k}\mathbf{r} + i|\mathbf{k}|t)], \tag{7}$$

Here the normalization volume V has been set equal to one; the index α corresponds to various polarizations of the photon and takes on the values 1, 2; $\hat{c}_{\mathbf{k}\alpha}$ and $\hat{c}^+_{\mathbf{k}\alpha}$ are operators which annihilate and create a photon with a momentum $\mathbf{k}$ and a polarization α; and the vector $\mathbf{e}_{\mathbf{k}\alpha}$ is a unit polarization vector ($\mathbf{k}\cdot\mathbf{e}_{\mathbf{k}\alpha} = 0$, $|\mathbf{e}_{\mathbf{k}\alpha}| = 1$.) Expression (7) has been normalized to "one particle in the volume $V = 1$."

A corresponding expression can be written for the operators representing functions of the electron-positron field $\psi_a(t,\mathbf{r})$:

$$\hat{\psi}_a(t,\mathbf{r}) = \sum_{\mathbf{p}\sigma} \frac{1}{(2\varepsilon_\mathbf{p})^{1/2}} [\hat{a}_{\mathbf{p}\sigma}u^{(+)}_{\mathbf{p}\sigma a}\exp(i\mathbf{p}\mathbf{r} - i\varepsilon_\mathbf{p}t) + \hat{b}^+_{\mathbf{p}\sigma}u^{(-)}_{-\mathbf{p}-\sigma a}\exp(-i\mathbf{p}\mathbf{r} + i\varepsilon_\mathbf{p}t)], \tag{8}$$

where the spinor index a takes on the values 1, 2, 3, 4 (and will frequently be omitted below); σ specifies the polarization state and takes on the values $\pm 1/2$; $\varepsilon_p = (\mathbf{p}^2 + m^2)^{1/2}$; the operator $\hat{a}$ annihilates an electron; and the operator $\hat{b}^+$ creates a positron. The spinor amplitudes $u^{(\pm)}$ have been normalized by the invariant condition $\bar{u}^{(\pm)}_{\pm\mathbf{p}\pm\sigma}u^{(\pm)}_{\pm\mathbf{p}\pm\sigma} = \pm 2m$.

The time evolution of the system is described by an S-matrix:

$$|\Phi(t_2)\rangle = \hat{S}(t_2,t_1)\,|\Phi(t_1)\rangle, \tag{9}$$

$$\hat{S}(t_2,t_1) = T\exp\left(i\int_{t_1}^{t_2} dt\,\hat{\mathcal{L}}(t)\right), \tag{9'}$$

where $|\Phi(t)\rangle$ is the state vector at the time t, and the S-matrix is a chronological exponential of the interaction Lagrangian, which is itself given by

$$\hat{\mathcal{L}}(t) = \hat{\mathcal{L}}_\varphi(t) + \hat{\mathcal{L}}_A(t) = -\int d\mathbf{r}\,\hat{j}_\mu(t,\mathbf{r})(\varphi^\mu(t,\mathbf{r}) + \hat{A}^\mu(t,\mathbf{r})),$$
$$\hat{j}^\mu(t,\mathbf{r}) = e:\hat{\bar{\psi}}(t,\mathbf{r})\gamma^\mu\hat{\psi}(t,\mathbf{r}):, \tag{10}$$

where j^μ is the current 4-vector, and the notation : . . . : means the normal product of the operators, in which all the creation operators stand on the left of all the annihilation operators. We will be making repeated use of formulas for summing over polarization:

$$\sum_\alpha e^i_{\mathbf{k}\alpha}e^{*j}_{\mathbf{k}\alpha} = \delta^{ij} - k^ik^j\mathbf{k}^{-2}, \tag{11}$$

$$\sum_\sigma u^{(\pm)}_{\mathbf{p}\sigma a}\bar{u}^{(\pm)}_{\mathbf{p}\sigma b} = \Lambda^\pm_{\mathbf{p}ab}, \qquad \Lambda^\pm_\mathbf{p} = \gamma^0\varepsilon_\mathbf{p} \mp \gamma\mathbf{p} \pm m. \tag{12}$$

The force acting on the electrons is the time derivative of the momentum of the electron. This momentum is given by the expectation value of the operator

$$\hat{\mathbf{p}} = \hat{\mathbf{P}} - e\varphi - e\hat{\mathbf{A}}, \tag{13}$$

where the first term is a generalized momentum, given by

$$\hat{\mathbf{P}}^i = \int d\mathbf{r} \hat{T}^{i0}, \qquad \hat{T}^{\mu\nu} = \frac{i}{2} : \left[\hat{\bar{\psi}} \gamma^\mu \frac{\partial \hat{\psi}}{\partial x_\nu} - \frac{\partial \hat{\bar{\psi}}}{\partial x_\nu} \gamma^\mu \hat{\psi} \right] :. \tag{14}$$

The quantity $\hat{T}^{\mu\nu}$ in the last expression is the energy-momentum tensor of the electron-positron field. Substituting (8) into (14), we find

$$\hat{\mathbf{P}} = \sum_{\mathbf{p}\sigma} \mathbf{p} \left(\hat{a}^+_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} + \hat{b}^+_{\mathbf{p}\sigma} \hat{b}_{\mathbf{p}\sigma} \right). \tag{15}$$

To take the classical limit, we need to take an average over the states which are the ones "closest to the classical description of the electron." These states are wave packets

$$|\Phi(\mathbf{p}_0; \mathbf{r}_0; \sigma)\rangle = \sum_{\mathbf{p}} c(\mathbf{p}) |\mathbf{p}\sigma\rangle = \int \frac{d\mathbf{p}}{(2\pi)^3} c(\mathbf{p}) |\mathbf{p}\sigma\rangle,$$
$$c(\mathbf{p}) = (4\pi\lambda)^{3/4} \exp\left[-\frac{\lambda}{2} (\mathbf{p} - \mathbf{p}_0)^2 - i (\mathbf{p} - \mathbf{p}_0) \mathbf{r}_0 \right] \tag{16}$$

[we have switched from a summation over momentum to an integration in (16)]. In the state $|\Phi(\mathbf{p}_0;\mathbf{r}_0;\sigma)\rangle$, an electron is characterized by a polarization σ, by the expectation value of the momentum $\mathbf{p}_0$, and by the expectation value of the coordinate $\mathbf{r}_0$. The state $|\mathbf{p}\sigma\rangle$ is a state with a momentum $\mathbf{p}$ and a polarization σ. There is an adjustable parameter λ in (16), which is a measure of the average spread of the momentum and the coordinate:

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0, \quad \delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0,$$
$$\langle(\delta\mathbf{p})^2\rangle = \langle\Phi|(\delta\mathbf{p})^2|\Phi\rangle = \frac{3}{2\lambda}, \tag{17}$$
$$\langle(\delta\mathbf{r})^2\rangle = \frac{3\lambda}{2}.$$

The amplitudes $c(\mathbf{p})$ in (16) are normalized by the condition

$$\sum_{\mathbf{p}} c(\mathbf{p}) \overset{*}{c}(\mathbf{p}) = 1.$$

The wave packets in (16) minimize the expectation value $\langle(\delta\mathbf{p})^2\rangle^{1/2}\langle(\delta\mathbf{r})^2\rangle^{1/2}$, which takes on the following value (for each of the three coordinates) in the case of (16): $\langle(\delta p_x)^2\rangle^{1/2}\langle(\delta x)^2\rangle^{1/2} = 1/2 (= \hbar/2)$. [We are dealing with both the field φ^μ and the field $\hat{A}^\mu$ by perturbation theory. If we were instead to take the external field φ^μ into account exactly while dealing with the field $\hat{A}^\mu$ by perturbation theory [if we were to take that approach, we should have used the Furry picture instead of the interaction picture to construct the S-matrix), we would have to take an average over coherent states of the electron in the external field φ^μ instead of proceeding as in (16).]

In calculating the force (see the discussion below) acting on the electron, we find that quadratic combinations $\overset{*}{c}(\mathbf{p}')c(\mathbf{p})$ and $u^{(+)}_{\mathbf{p}\sigma}\bar{u}^{(+)}_{\mathbf{p}'\sigma}$ appear in the equations. These combinations serve as a density matrix of the initial state of the electron. We accordingly introduce the matrices

$$M_{ab}(\mathbf{p}, \mathbf{p}'; \sigma) = u^{(+)}_{\mathbf{p}\sigma a} \bar{u}^{(+)}_{\mathbf{p}'\sigma b}, \tag{18}$$

$$M_{ab}(\mathbf{p}, \mathbf{p}') = \frac{1}{2} \sum_\sigma M_{ab}(\mathbf{p}, \mathbf{p}'; \sigma), \tag{19}$$

which serve as density matrices for a polarized state with a polarization σ and for an unpolarized state, respectively. Rewriting the four-component spinors in terms of nonrelativistic two-component spinors w,

$$u^{(+)}_{\mathbf{p}\sigma} = \left\| \begin{matrix} (\varepsilon_\mathbf{p} + m)^{1/2} w_\sigma \\ (\varepsilon_\mathbf{p} - m)^{1/2} w_\sigma \dfrac{\mathbf{p}\boldsymbol{\sigma}}{p} \end{matrix} \right\|, \qquad \overset{*}{w}_\sigma w_\sigma = 1, \tag{20}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, and choosing a basis for w in the form

$$w_{\sigma_z, +1} = \left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\|, \qquad w_{\sigma_z, -1} = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|,$$

we can write explicit expressions for matrices (18) and (19). We note that we have $M(\mathbf{p},\mathbf{p}) = -\Lambda^+_\mathbf{p}/2$.

3. THE FORCE ACTING ON THE ELECTRON IN FIRST-ORDER PERTURBATION THEORY: THE LORENTZ FORCE AND THE MAGNETIC MOMENT OF THE ELECTRON

We define the force $\mathbf{F}(t)$ acting on the electron as

$$\mathbf{F}(t) = \frac{d}{dt} \langle\Phi(t)|(\hat{\mathbf{P}} - e\varphi - e\hat{\mathbf{A}})|\Phi(t)\rangle$$
$$= \frac{d}{dt} \langle\Phi_0|\hat{S}^+ (\hat{\mathbf{P}} - e\varphi - e\hat{\mathbf{A}}) \hat{S}|\Phi_0\rangle, \tag{21}$$

where the initial state $|\Phi_0\rangle$ is chosen to be the wave packet in (16).

In first order in the interaction constant e, we can ignore the quantized field $\hat{\mathbf{A}}$, since in the vacuum kets ("vacuum" here means from the standpoint of the photon field) we have $|\Phi_0\rangle$: $\langle\Phi_0|\hat{c}|\Phi_0\rangle = \langle\Phi_0|\hat{c}^+|\Phi_0\rangle = 0$ and therefore $\langle\Phi_0|\hat{\mathbf{A}}|\Phi_0\rangle = 0$. The only possible nonvanishing vacuum expectation value would be that from the quadratic combination $\langle\Phi_0|\hat{\mathbf{A}}\hat{\mathbf{A}}|\Phi_0\rangle$, which is of higher order in e, since the field $\hat{\mathbf{A}}$ appears in the product $e\hat{\mathbf{A}}$ in (10) and (21).

Taking only the field φ^μ into account, we find the following result from (21), within $o(e)$:

$$\mathbf{F}(t) = \frac{d}{dt} \Big\langle \Phi_0 \Big| \Big(1 - i \int^t dt_1 \hat{\mathscr{L}}_\varphi(t_1)\Big)$$
$$\times (\hat{\mathbf{P}} - e\varphi) \Big(1 + i \int^t dt_2 \hat{\mathscr{L}}_\varphi(t_2)\Big) \Big| \Phi_0 \Big\rangle. \tag{22}$$

Differentiating with respect to the time in (22), and retaining only the terms linear in e, we find

$$\mathbf{F}(t) = \Big\langle \Phi_0 \Big| i [\hat{\mathbf{P}}, \hat{\mathscr{L}}_\varphi(t)] - e \frac{d\varphi}{dt} \Big| \Phi_0 \Big\rangle, \tag{23}$$

where $[\hat{\mathbf{P}}, \hat{\mathscr{L}}] = \hat{\mathbf{P}}\hat{\mathscr{L}} - \hat{\mathscr{L}}\hat{\mathbf{P}}$ is a commutator of operators.

The contribution from the term $(-e\varphi)$ in force (21) should be taken into consideration only in first order in e. The reason is that since φ_μ is a c-number it commutes with $\hat{S}$, so we have

$$\frac{d}{dt} \langle\Phi_0|\hat{S}^+(-e)\varphi\hat{S}|\Phi_0\rangle = \frac{d}{dt} \langle\Phi_0|\hat{S}^+\hat{S}(-e)\varphi|\Phi_0\rangle$$
$$= -e\Big\langle\Phi_0\Big|\frac{d\varphi}{dt}\Big|\Phi_0\Big\rangle,$$

since the S-matrix is unitary: $\hat{S}^+\hat{S} = 1$.

Substituting (6), (8), (15), and (16) into (23), we find that the first term in (23) reduces to

$$\langle\Phi_0| i [\hat{\mathbf{P}}, \hat{\mathscr{L}}_\varphi(t)] |\Phi_0\rangle$$
$$= -ie \int dq^0 d\mathbf{q} \varphi_\mu(q^0, \mathbf{q}) \sum_\mathbf{p} c(\mathbf{p}) \overset{*}{c}(\mathbf{p} + \mathbf{q})$$
$$\times \frac{1}{2(\varepsilon_\mathbf{p}\varepsilon_{\mathbf{p}+\mathbf{q}})^{1/2}} \mathbf{q} \exp[i(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p} - q^0)t] \operatorname{Tr}(M(\mathbf{p}, \mathbf{p} + \mathbf{q})\gamma^\mu) \tag{24}$$

both in the case of an unpolarized M and in the case of a polarized initial state $|\Phi_0\rangle$.

We now assume that the momentum transfer is small in comparison with $\mathbf{p}$ and m. From (16) we then have $c(\mathbf{p})^*c(\mathbf{p}+\mathbf{q}) \approx c(\mathbf{p})c(\mathbf{p})e^{i\mathbf{q}\mathbf{r}_0}$, within terms of the order of $\mathbf{q}$; $1/\varepsilon_\mathbf{p}\varepsilon_{\mathbf{p}+\mathbf{q}} \approx 1/\varepsilon^2_\mathbf{p}$; $\exp[i(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p} - q^0)t] \approx \exp[i(\mathbf{q}\mathbf{v} - q^0)t]$; and $\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q})\gamma^\mu) \approx \mathrm{Tr}(M(\mathbf{p},\mathbf{p})\gamma^\mu) = 2p^\mu = 2(\varepsilon_\mathbf{p},\mathbf{p}) = 2\varepsilon_\mathbf{p}(1,\mathbf{v})$. Expression (24) thus becomes

$$\langle\Phi_0|\,i\,[\hat{\mathbf{P}}, \hat{\mathscr{L}}_\varphi(t)]\,|\Phi_0\rangle$$
$$= -ie\int dq^0 d\mathbf{q} \sum_\mathbf{p} c(\mathbf{p})\overset{*}{c}(\mathbf{p})\mathbf{q}\exp\{i[\mathbf{q}(\mathbf{r}_0+\mathbf{v}t)-q^0t]\}$$
$$\times(\varphi^0(q^0,\mathbf{q}) - \mathbf{v}\boldsymbol{\varphi}(q^0,\mathbf{q}))$$
$$= -e\sum_\mathbf{p} c(\mathbf{p})\overset{*}{c}(\mathbf{p})\Big(\frac{\partial\varphi^0(t,\mathbf{r}_0+\mathbf{v}t)}{\partial\mathbf{r}} - v_i\frac{\partial\varphi_i(t,\mathbf{r}_0+\mathbf{v}t)}{\partial\mathbf{r}}\Big), \quad (25)$$

since in the process of taking the inverse Fourier transforms in the spatial components of the quantity $\varphi^\mu(t,\mathbf{r})$ a multiplication by $i\mathbf{q}$ corresponds to the differentiation operator $\partial/\partial\mathbf{r}$ [and $(-iq^0)$ correspondingly becomes $\partial/\partial t$].

The sum over momenta in (25) actually corresponds to an average over the momenta in packet (16). Summing over momenta in accordance with (16), we find

$$\langle\Phi_0|\,i\,[\hat{\mathbf{P}}, \hat{\mathscr{L}}_\varphi(t)]\,|\Phi_0\rangle$$
$$= -e\Big(\frac{\partial\varphi^0(t,\mathbf{r}_0+\mathbf{v}_0t)}{\partial\mathbf{r}} - v_i\frac{\partial\varphi_i(t,\mathbf{r}_0+\mathbf{v}t)}{\partial\mathbf{r}}\Big), \quad (26)$$

where $\mathbf{v}_0 = \mathbf{p}_0/\varepsilon_{p_0}$ is the expectation value of the velocity in state (16).

The second term in (23) is (for brevity, we are omitting the subscript 0 from $\mathbf{r}_0$ and $\mathbf{v}_0$)

$$\Big\langle\Phi_0\Big|(-e)\frac{d\boldsymbol{\varphi}}{dt}\Big|\Phi_0\Big\rangle = -e\Big(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial\mathbf{r}}\Big)\boldsymbol{\varphi}(t,\mathbf{r}+\mathbf{v}t). \quad (27)$$

Collecting results (26) and (27), we have

$$\mathbf{F}(t) = e\{\mathbf{E}(t,\mathbf{r}_0+\mathbf{v}_0t) + \mathbf{v}_0\times\mathbf{H}(t,\mathbf{r}_0+\mathbf{v}_0t)\},$$
$$\mathbf{E} = -\frac{\partial\varphi}{\partial\mathbf{r}} - \frac{\partial\boldsymbol{\varphi}}{\partial t}, \qquad \mathbf{H} = \mathrm{rot}\,\boldsymbol{\varphi}. \quad (28)$$

This result corresponds to the Lorentz force which acts on a particle as it moves in an external electric field $\mathbf{E}$ and an external magnetic field $\mathbf{H}$.

We will now carry out a more accurate expansion of expression (24) in the momentum transfer $\mathbf{q}$, retaining terms of order up to $\mathbf{q}^2$. For simplicity, we consider the additional force acting on the electron, $\Delta\mathbf{F}$, in a frame of reference moving with the electron, i.e., in a frame in which we have $\mathbf{v}_0 = 0$.

Before we expand expression (24) in $\mathbf{q}$, there is a point we need to note. In expression (24), we must expand in $\mathbf{q}$ quantities which depend on $(\mathbf{p}+\mathbf{q})$ (e.g., $\varepsilon_{\mathbf{p}+\mathbf{q}}$). At the same time, it is not difficult to see that the characteristic value $\delta p \sim \lambda^{-1/2}$ is [according to (17)] much greater than q, so we run into difficulties in an expansion in q. The reason is that the force acting on the particle is determined not by the potentials φ^μ themselves but by their gradients. If the particle is to "feel" these gradients, the characteristic spatial "blurring" $\delta r \sim \lambda^{1/2}$ of the particle must be much smaller than the "characteristic distance over which the potentials φ^μ vary," i.e., q^{-1}. Since we have $\delta p\cdot\delta r \sim 1$, the condition $\lambda^{1/2} \sim \delta r \ll q^{-1}$ is equivalent to $\lambda^{-1/2} \sim \delta p \gg q$. (Along with that condition, we need to assume $\delta p \ll m$ in order to go over to the classical limit; the latter assumption makes it possible for the parameter λ to fall in the interval $m^{-2} \ll \lambda \ll q^{-2}$.)

We thus need to expand in powers of q quantities which depend on $(\mathbf{p}+\mathbf{q})$, in which the uncertainty in the first term in the argument satisfies $\delta p \gg q$. Straightforward arguments show that the most convenient approach here is to expand not "at the point $\mathbf{p}$" but "at the point $\mathbf{f} = \mathbf{p} + (\mathbf{q}/2)$," for which expression (24) has a certain symmetry, depending only on the combinations $\mathbf{f} + (\mathbf{q}/2)$ and $\mathbf{f} - (\mathbf{q}/2)$. Working from Eq. (16) ($\mathbf{p}_0 = 0$), we have, for the first calculation method,

$$\overset{*}{c}(\mathbf{p})c(\mathbf{p}+\mathbf{q}) = c(\mathbf{p})\overset{*}{c}(\mathbf{p})\,e^{i\mathbf{q}\mathbf{r}_0}\exp\Big[-\lambda\Big(\mathbf{p}\mathbf{q} + \frac{\mathbf{q}^2}{2}\Big)\Big], \quad (29)$$

and, for the second,

$$c\Big(\mathbf{f}-\frac{\mathbf{q}}{2}\Big)\overset{*}{c}\Big(\mathbf{f}+\frac{\mathbf{q}}{2}\Big) = c(\mathbf{f})\overset{*}{c}(\mathbf{f})\,e^{i\mathbf{q}\mathbf{r}_0}e^{-\lambda\mathbf{q}^2/2}. \quad (30)$$

[In the preceding expansion, a cruder estimate of this expression, in the form $c^*c\exp(i\mathbf{q}\mathbf{r}_0)$, with the replacement of the last factor in (29), (30) by one, would have been sufficient for our purposes, since the question of a distinction between (29) and (30) did not arise.] In expression (29), we find $\mathbf{p}$ in the last factor. This situation presents certain difficulties when we subsequently take an average over momentum [as we do in evaluating the sums $\Sigma_\mathbf{p} c(\mathbf{p})^*c(\mathbf{p})\times\ldots$], while the last factor in (30) does not depend on $\mathbf{f}$ and does not affect the procedure of taking an average over momentum $\mathbf{f}$. Here and in the following section of this paper, we will accordingly use the second expansion method, with expression (30). According to the discussion above, it then becomes possible first to carry out the average over $\mathbf{f}$ in (24), rewritten in the new notation (in terms of $\mathbf{f}$). Taking that average reduces to a replacement by $\langle\mathbf{f}\rangle = 0$ (we recall that we are calculating $\Delta\mathbf{F}$ in the frame of reference which is moving with the electron). We then expand the resulting expression in q and retain terms of order up to $\mathbf{q}^2$. Going through this procedure, we find

$$\langle\Phi_0|\,i\,[\hat{\mathbf{P}}, \hat{\mathscr{L}}_\varphi(t)]\,|\Phi_0\rangle = -ie\int dq^0 d\mathbf{q}\,\varphi_\mu(q^0,\mathbf{q})\,e^{i\mathbf{q}\mathbf{r}_0}e^{-\lambda\mathbf{q}^2/2},$$
$$\frac{1}{(4\varepsilon^2_{\mathbf{q}/2})^{1/2}}\,\mathbf{q}\exp[i(\varepsilon_{\mathbf{q}/2} - \varepsilon_{\mathbf{q}/2} - q^0)t]\,\mathrm{Tr}\Big(M\Big(-\frac{\mathbf{q}}{2},\frac{\mathbf{q}}{2}\Big)\gamma^\mu\Big)$$
$$\approx -ie\int dq^0 d\mathbf{q}\,\varphi_\mu(q^0,\mathbf{q})$$
$$\times\exp(i\mathbf{q}\mathbf{r}_0 - iq^0t)\,\frac{\mathbf{q}}{2m}\,\mathrm{Tr}\Big(M\Big(-\frac{\mathbf{q}}{2},\frac{\mathbf{q}}{2}\Big)\gamma^\mu\Big), \quad (31)$$

since

$$\varepsilon_{\mathbf{q}/2} = \varepsilon_{-\mathbf{q}/2} = m + \frac{(\mathbf{q}/2)^2}{2m} + \ldots = m + o(q).$$

At an accuracy level sufficient for our purposes we have $\mathrm{Tr}(M\gamma^0) = 2m + o(q)$ (this result corresponds to the Lorentz force found earlier). For an unpolarized state M we have $\mathrm{Tr}(M\gamma) = 0 + o(q)$, while for a polarized M with a

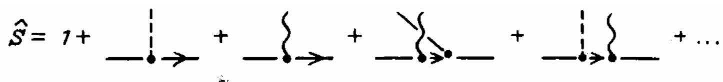

FIG. 1.

spin expectation value $\mathbf{s}$ we have $\mathrm{Tr}(M\gamma) = -2i\mathbf{q}\times\mathbf{s} + o(q)$. We thus find ($\varepsilon^{ijk}$ is the Levi–Civita density)

$$\Delta F^i = \frac{e}{m}\int dq^0 d\mathbf{q}\varphi^j(q^0,\mathbf{q})\exp(i\mathbf{q}\mathbf{r}_0 - iq^0 t)q^i\varepsilon^{jlm}q^l s^m, \quad (32)$$

i.e., when we switch from $i\mathbf{q}$ to $\partial/\partial\mathbf{r}$,

$$\Delta\mathbf{F} = \mathrm{grad}(\boldsymbol{\mu}\mathbf{H}), \quad \mathbf{H} = \mathrm{rot}\,\boldsymbol{\varphi}, \quad \boldsymbol{\mu} = \frac{e}{m}\mathbf{s}. \quad (33)$$

This result shows that we have $\Delta\mathbf{F} = 0$ in the unpolarized state, while in the polarized state the electron has a magnetic moment $\mu = e/ms = e/2m$ ($= e\hbar/2mc$ in ordinary units).

We complete our discussion of force (21) in first order in e with a convenient graphical interpretation of the results. We assume that a solid line (Fig. 1) corresponds to an electron, and a dashed line to an external field. Representing the first term in (13) by a 1, representing the second term by a vertical dashed line, and expanding the S-matrix in a series, we can rewrite (23) as a sum of three terms (Fig. 2).

4. RADIATION-REACTION FORCE

Expanding the S-matrix in a series in the electromagnetic interaction constant e (see Fig. 1, where a wavy line corresponds to a quantized field $\hat{A}^\mu$),

$$\hat{S}(t) \equiv \hat{S}(t,-\infty) = 1 + i\int_{-\infty}^{t} dt_1\hat{\mathscr{L}}_\varphi(t_1) + i\int_{-\infty}^{t} dt_1\hat{\mathscr{L}}_A(t_1) - \int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2\hat{\mathscr{L}}_\varphi(t_1)\hat{\mathscr{L}}_A(t_2) - \int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2\hat{\mathscr{L}}_A(t_1)\hat{\mathscr{L}}_\varphi(t_2) + \ldots, \quad (34)$$

we can derive expressions for the following terms of the expansion of force (21) in a power series in e. [The quantized field $\hat{A}^\mu$ "is turned on" in (34) at the time $t = -\infty$ (in other words, it always exists). For simplicity, we "turn on" the external field φ^μ adiabatically, also at the time $t = -\infty$.] Terms corresponding to the radiation-reaction force arise in third order in e ($\sim e^3$) and in higher orders. The classical limit of the first two terms in the radiation-reaction force ($\sim e^3$ and e^4) is given by expressions (2) and (3). We therefore take up the problem of finding the first term in (2), (3), which corresponds to the order e^3, by means of the methods developed above (the e^4 term could be derived in a corresponding way).

Strictly speaking, before we analyze the expression for the force acting on the electron which we find in third order in e, we should analyze the term $\mathbf{F}^{(2)}$ of the order of e^2, which is also present in the expansion of expression (21). It is clear at the outset that a term of this sort, of second order in e, could arise only in a quantum theory (i.e., it must vanish in the limit $\hbar\to 0$). It becomes clear from dimensionality considerations that this term may be of the form (in ordinary units)

$$\mathbf{F}^{(2)} = \mathrm{const}\cdot\frac{e^2\hbar}{m^2c^3}\,\mathbf{E}\times\mathbf{H}\left(\frac{\hbar q^0}{mc^2}\right)^n \quad (35)$$

(n is an integer), where the last factor corresponds to a possible dependence of $\mathbf{F}^{(2)}$ not on the fields $\mathbf{E}$, $\mathbf{H}$ themselves but on their time derivatives [as in Eq. (2)]. It is easy to see that expression (35) tends toward zero as $\hbar\to 0$. An exact calculation leads to the following result: $\mathbf{F}^{(2)}$ is strictly zero if the particle does not satisfy a condition for a Cherenkov resonance in the external field (more on this below).

Before we proceed, we would like to point out that by virtue of the relativistic invariance of the S-matrix it is most convenient to study the resulting expressions in a frame of reference moving with the electron (as at the end of the preceding section of this paper). To do so does not restrict the generality of our analysis, since the potential of the external field, φ^μ, is left totally arbitrary, and there is nothing to restrict its gauge. Relativistic invariance is restored most simply in the final result, after we have derived an expression for the radiation-reaction force $\delta\mathbf{F}$ in a frame of reference in which the expectation value of the velocity in wave packet (16) is $\mathbf{v}_0 = 0$.

We might add that we know from the very form of (2) that the expansion in the momentum transfer q^μ must be carried out to terms quadratic in q^μ, while in deriving the Lorentz force it was sufficient to use expressions linear in q^μ. In addition, we will ignore terms in expansion (34) which are of powers higher than the first in the external field φ^μ, in accordance with the expected result, (2), (3).

We thus expand the S-matrix in (21), retain terms up to order e^3 inclusively, and collect the terms of third order in e [as we mentioned in the preceding section of this paper, since φ^μ is a c-number and commutes with the S matrix, we do not have to consider the term $(-e\varphi)$ in the resultant momentum of the electron]. For clarity, we will discuss the terms $\delta\mathbf{F}$ which arise in the process in graphical terms.

We begin with the contribution $\delta\,\mathbf{F}_P$ from the first term (i.e., $\hat{\mathbf{P}}$) in (21). Terms of two types arise in third order in $\delta\,\mathbf{F}_P$. The terms of the first type correspond to the case in which $\hat{S}$ (or $\hat{S}^+$) is represented in (21) by a term of the order of e^3 in expansion (34), while $\hat{S}^+$ (or, correspondingly, $\hat{v}S$) is represented by a 1. Figure 3 shows half of these terms, for $\hat{S}\sim e^3$ and $\hat{S}^+ = 1$ (uncoupled diagrams are being eliminated).

The diagrams in Fig. 3 contain loops, which show that we need to carry out a renormalization in calculating the given contributions. Actually, it is not necessary to calculate these terms explicitly, by substituting (6)–(10) into (34) and then into (21). We see that in order to retain terms in Fig. 3 beyond first order (Fig. 2) all we need to do is replace the propagators of the electromagnetic and electron-positron fields and the vertex operator by the corresponding re-

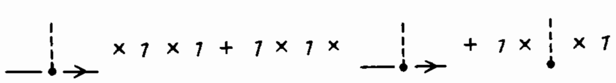

FIG. 2.

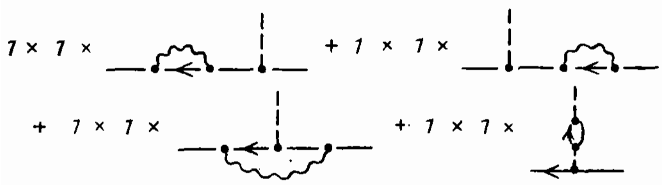

FIG. 3.

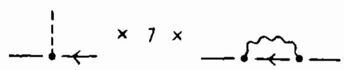

FIG. 4.

normalized expressions. Subtracting the divergences on the mass shell, we find[7,8] that the external electron lines in the first and second diagrams in Fig. 2 do not require renormalization, while in place of φ_μ, which corresponds to a line of an external photon field in Fig. 2, and in place of the vertex operator γ^μ in (24) we should substitute, respectively,

$$\varphi_\mu(q) \to \varphi_\mu(q) + \frac{1}{4\pi}\mathscr{D}_{\mu\nu}(q)\mathscr{P}^{\nu\lambda}(q)\varphi_\lambda(q), \tag{36}$$

$$\gamma^\mu \to \gamma^\mu f(q^2) - \frac{1}{2m}g(q^2)\frac{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu}{2}q_\nu, \tag{37}$$

where $q = q^\mu$, $q^2 = q_\mu q^\mu$. In this order in e, with an accuracy sufficient for our purposes at small values of $|q^2| \ll m^2$, the polarization operator is $\mathscr{P}_{\mu\nu}(q) = \mathscr{P}(q^2)[q_{\mu\nu} - (q_\mu q_\nu/q^2)]$, where $\mathscr{P}(q^2) \approx -e^2q^4/(15\pi m^2)$; the photon propagator is $\mathscr{D}_{\mu\nu}(q) = \mathscr{D}(q^2)[g_{\mu\nu} - (q_\mu q_\nu/q^2] + \mathscr{D}^{(1)}(q^2)(q_\mu q_\nu/q^2)$, where $\mathscr{D}(q^2) \approx 4\pi/q^2$; the form factors are $f(q^2) \approx 1 + (e^2q^2/3\pi m^2)\cdot[\ln(m/\lambda) - (3/8)]$; and $g(q^2) \approx e^2/2\pi$ (λ is an infinitely small mass which we assign to the photon in order to regularize the function f, which diverges in the infrared limit). Substituting this result in (24), and retaining terms with powers no higher than the second in (24) after an expansion in q, we find that there is no need to consider the radiation corrections to $\varphi_\mu(q)$ in (36). We can set $f(q^2) = 1$ in (37), retaining only the radiation correction $g(q^2)$. After several straightforward manipulations completely similar to those in §3, we find that the contribution to $\delta\mathbf{F}_P$ from (36), (37) (i.e., ultimately from the diagrams in Fig. 3) for an unpolarized state of the electron is zero, while for a polarized state this correction leads to the appearance of a Schwinger correction to the magnetic moment of the electron. This moment becomes

$$\mu = \frac{e}{2m}\left(1 + \frac{e^2}{2\pi}\right) = \frac{e\hbar}{2mc}\left(1 + \frac{e^2}{2\pi\hbar c}\right).$$

The contributions of the second type which arise in $\delta\mathbf{F}_P$ correspond to the circumstance that a term of the order of e comes from $\hat{S}^+$ (or $\hat{S}$) in (21), while a term of the order of e^2 comes from $\hat{S}$ (or, correspondingly, $\hat{S}^+$). These diagrams do not contain loops (since we are carrying out a subtraction on the mass shell, it is not necessary to take account of the diagram of the type in Fig. 4, which corresponds to a mass operator). Figure 5 shows some typical representatives of this second type of contribution.

It is not difficult to see that such contributions to (21) can be completely ignored. The reason is that both the state $\langle\Phi_0|\hat{S}^+$ and the state $\hat{S}|\Phi_0\rangle$ in (21) should correspond to some real state of the electromagnetic and electron-positron fields. One of the diagrams (that corresponding to $\hat{S}^+$ or $\hat{S}$) in each of the contributions in Fig. 5 corresponds to a process which is forbidden by 4-momentum conservation. The first diagram in Fig. 5, for example, represents the emission of a photon by a free electron.

In summary, incorporating the first term ($\hat{\mathbf{P}}$) in (13) in third order made it necessary to carry out a renormalization. After this renormalization, all the radiation corrections reduced to the anomalous magnetic moment of the electron.

It is not difficult to see why a radiation-reaction force could not arise from the term $\hat{\mathbf{P}}$ in (21). The reason is that the radiation-reaction force describes an additional "recoil" which an electron emitting a photon undergoes when it is in an external field. In diagrams of the type in Fig. 3, however, the photon is an intermediate, virtual photon, and it is not present in the final state. In the diagrams of the type in Fig. 5, in contrast, a real photon may be present in the final state $\hat{S}|\Phi_0\rangle$, but such processes are forbidden by 4-momentum conservation. We turn now to the contributions $\delta\mathbf{F}_A$ which arise in $\delta\mathbf{F}$ because of the third term, $-e\hat{\mathbf{A}}$, in (21) (this term is represented graphically by a wavy line). As in the incorporation of the term $\hat{\mathbf{P}}$, contributions of two types—the types shown in Figs. 6 and 7—arise when we substitute (34) into (21).

Processes corresponding to Fig. 6 are forbidden by 4-momentum conservation, so a radiation-reaction force could arise only from contributions of the type in Fig. 7, to which we now turn.

We write out explicitly the result of substituting (34) into (21), which corresponds to Fig. 7. The force which arises in third order in e can be written $\delta\mathbf{F}_A = \delta\mathbf{F}_1 + \delta\mathbf{F}_2$, where

$$\begin{aligned}\delta\mathbf{F}_1 = e\Big\langle\Phi_0\Big|\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\,(\hat{\mathbf{A}}(t)\hat{\mathscr{L}}_\varphi(t_1)\hat{\mathscr{L}}_A(t_2)\\ + \hat{\mathscr{L}}_A(t_2)\hat{\mathscr{L}}_\varphi(t_1)\hat{\mathbf{A}}(t)\\ + \hat{\mathbf{A}}(t)\hat{\mathscr{L}}_A(t_1)\hat{\mathscr{L}}_\varphi(t_2) + \hat{\mathscr{L}}_\varphi(t_2)\hat{\mathscr{L}}_A(t_1)\hat{\mathbf{A}}(t))\,|\,\Phi_0\Big\rangle,\end{aligned} \tag{38}$$

$$\begin{aligned}\delta\mathbf{F}_2 = e\Big\langle\Phi_0\Big|\int_{-\infty}^{t}dt_1\,(\hat{\mathbf{A}}(t)\hat{\mathscr{L}}_\varphi(t)\hat{\mathscr{L}}_A(t_1) + \hat{\mathscr{L}}_A(t_1)\hat{\mathscr{L}}_\varphi(t)\hat{\mathbf{A}}(t)\\ + \hat{\mathbf{A}}(t)\hat{\mathscr{L}}_A(t)\hat{\mathscr{L}}_\varphi(t_1) + \hat{\mathscr{L}}_\varphi(t_1)\hat{\mathscr{L}}_A(t)\hat{\mathbf{A}}(t))\,|\,\Phi_0\Big\rangle.\end{aligned} \tag{39}$$

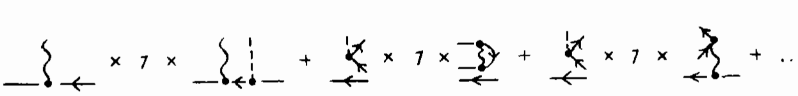

FIG. 5.

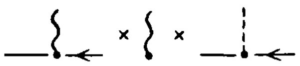

FIG. 6.

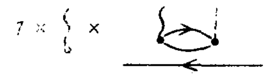

FIG. 8.

Substituting the expressions for the Lagrangians into (38) and (39), and omitting the lengthy intermediate calculations, we find (ignoring uncoupled diagrams of the type in Fig. 8)

$$\delta F_1^i$$

$$= e^3 \int dq^0 d\mathbf{q} \varphi_\mu (q^0, \mathbf{q}) \sum_{\mathbf{p},\mathbf{k}} c(\mathbf{p}) \overset{*}{c}(\mathbf{p}+\mathbf{q}-\mathbf{k}) \frac{4\pi(\delta^{ij} - k^i k^j/\mathbf{k}^2)}{8k(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}}\varepsilon_\mathbf{p})^{1/2}}$$

$$\times \exp[i\mathbf{k}\mathbf{r}_0 + i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} - q^0)t]$$

$$\times \Bigg\{ \frac{\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q}-\mathbf{k})\gamma^\mu \Lambda^+_{\mathbf{p}-\mathbf{k}}\gamma_j)}{\varepsilon_{\mathbf{p}-\mathbf{k}}}$$

$$\times \Bigg[\frac{-ik}{i(\varepsilon_{\mathbf{p}-\mathbf{k}} - \varepsilon_\mathbf{p} + k)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} + k - q_0 - i0)}$$

$$+ \frac{ik}{i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_{\mathbf{p}-\mathbf{k}} - q_0 - i0)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} - k - q_0 - i0)} \Bigg]$$

$$+ \frac{\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q}-\mathbf{k})\gamma_j \Lambda^+_{\mathbf{p}+\mathbf{q}}\gamma^\mu)}{\varepsilon_{\mathbf{p}+\mathbf{q}}}$$

$$\times \Bigg[\frac{ik}{i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_{\mathbf{p}+\mathbf{q}} - k)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} - k - q_0 - i0)}$$

$$+ \frac{-ik}{i(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p} - q_0 - i0)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} + k - q_0 - i0)} \Bigg]$$

$$- \frac{\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q}-\mathbf{k})\gamma^\mu \Lambda^-_{\mathbf{p}+\mathbf{q}}\gamma_j)}{\varepsilon_{\mathbf{p}+\mathbf{q}}}$$

$$\times \Bigg[\frac{-ik}{i(\varepsilon_{\mathbf{p}+\mathbf{q}} + \varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} + k)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} + k - q_0 - i0)}$$

$$+ \frac{ik}{-i(\varepsilon_\mathbf{p} + \varepsilon_{\mathbf{p}+\mathbf{q}} + q_0 + i0)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} - k - q_0 - i0)} \Bigg]$$

$$- \frac{\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q}-\mathbf{k})\gamma_j \Lambda^-_{\mathbf{p}-\mathbf{k}}\gamma^\mu)}{\varepsilon_{\mathbf{p}-\mathbf{k}}}$$

$$\times \Bigg[\frac{ik}{-i(\varepsilon_\mathbf{p} + \varepsilon_{\mathbf{p}-\mathbf{k}} + k)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} - k - q_0 - i0)}$$

$$+ \frac{-ik}{i(\varepsilon_{\mathbf{p}-\mathbf{k}} + \varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - q_0 - i0)\, i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} + k - q_0 - i0)} \Bigg] \Bigg\}, \tag{40}$$

$$\delta F_2^i$$

$$= e^3 \int dq^0 d\mathbf{q} \varphi_\mu (q^0, \mathbf{q}) \sum_{\mathbf{p},\mathbf{k}} c(\mathbf{p}) \overset{*}{c}(\mathbf{p}+\mathbf{q}-\mathbf{k}) \cdot \frac{4\pi(\delta^{ij} - k^i k^j/\mathbf{k}^2)}{8k(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}}\varepsilon_\mathbf{p})^{1/2}}$$

$$\times \exp[i\mathbf{k}\mathbf{r}_0 + i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} - q^0)t]$$

$$\times \Bigg\{ \frac{\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q}-\mathbf{k})\gamma^\mu \Lambda^+_{\mathbf{p}-\mathbf{k}}\gamma_j}{\varepsilon_{\mathbf{p}-\mathbf{k}}}$$

$$\times \Bigg[\frac{1}{i(\varepsilon_{\mathbf{p}-\mathbf{k}} - \varepsilon_\mathbf{p} + k)} + \frac{1}{i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_{\mathbf{p}-\mathbf{k}} - q_0 - i0)} \Bigg]$$

$$+ \frac{\mathrm{Tr}(M(\mathbf{p},\mathbf{p}+\mathbf{q}-\mathbf{k})\gamma_j \Lambda^+_{\mathbf{p}+\mathbf{q}}\gamma^\mu)}{\varepsilon_{\mathbf{p}+\mathbf{q}}}$$

$$\times \Bigg[\frac{1}{i(\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_{\mathbf{p}+\mathbf{q}} - k)} + \frac{1}{i(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p} - q_0 - i0)} \Bigg] - \dots \Bigg\}. \tag{41}$$

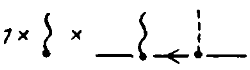

FIG. 7.

The first two terms $\delta \mathbf{F}_1$ in (40) correspond to the case in which the virtual particle in the diagram in Fig. 7 is an electron. The last two terms correspond to a virtual positron. In expression (41) for $\delta \mathbf{F}_2$ we have written out only the electron contributions; the corresponding positron contributions are represented by the ellipsis (since we will not be needing $\delta \mathbf{F}_2$ below).

One can show that the parity of the integrands in (40) and (41) has the consequence that the real parts of the energy denominators, $\mathrm{Re}\, 1/(E - i0) = \mathscr{P}(1/E)$, make a vanishing contribution. It thus becomes necessary to take the imaginary parts $\mathrm{Im}\, 1/E - i0 = \pi\delta(E)$ into account [the sign of the imaginary increment in the denominators in (40) and (41) is chosen in accordance with causality]. It should be noted in this connection that it is sufficient in practice to write the imaginary increment to only the frequency of the external field: $q^0 \to q^0 + i0$. Those denominators which do not contain q^0 (e.g., $\varepsilon_{\mathbf{p}-\mathbf{k}} - \varepsilon_\mathbf{p} + k$) never vanish (because a free electron could not radiate a photon), so their imaginary parts are zero.

Let us take a more detailed look at just which resonances can describe the denominators in expressions (40) and (41). Resonances of the type $\varepsilon_{\mathbf{p}-\mathbf{k}} - \varepsilon_\mathbf{p} + k = 0$ are ruled out by 4-momentum conservation, as we have already mentioned. To analyze the other possible resonances, it is most convenient to use the frame of reference moving with the electron. In this frame we have $p = \delta p \ll m$ and $q \ll m$; as we will show below, k is of the same order of magnitude as q^0, so we also have $k \ll m$. In a first approximation we thus write $\varepsilon_\mathbf{x} \approx m$ for an arbitrary argument $\mathbf{x}$ (which takes on the values $\mathbf{p}$, $\mathbf{p}-\mathbf{k}$, $\mathbf{p}+\mathbf{q}$, $\mathbf{p}+\mathbf{q}-\mathbf{k}$). Resonances of the type $\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - \varepsilon_\mathbf{p} + k - q^0 = 0$ thus reduce to the condition $k = q^0$ in this approximation. From the physical standpoint, this is a completely natural result: An electron oscillates in the external field, which has a frequency q^0, and it radiates a photon in the process, with the same frequency $k = q^0$. Resonances of this sort arise in the denominators in expression (40), and expression (40) does indeed describe the radiation-reaction force (as we will see below).

Denominators of the type $\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p} - q^0 = 0$, which are present in both (40) and (41), lead to the Cherenkov condition $\mathbf{qv} - \mathbf{q}^0 = 0$ when the particle is accelerated continuously by the external field $\varphi^\mu(q)$. In the case at hand we are not interested in that process, which has no bearing on the reaction of the radiation on a moving charge, so we eliminate such resonances from consideration [it can be assumed that the field $\varphi^\mu(q)$ does not satisfy the condition for a Cherenkov resonance].

The discussion above shows that a radiation-reaction force is present only in expression (40). In (40), we now take the classical limit, using Eq. (16) for $c(\mathbf{p})$, expanding (40) in powers of q, and retaining terms of up to second order inclusively. As in §3, in a calculation of the magnetic moment of the electron, we should carry out this expansion after we make the substitution $\mathbf{p} - \mathbf{f} - [(\mathbf{q} - \mathbf{k})/2]$, and we should then set $\mathbf{f} = 0$ (we will be omitting the subscript 0 from $\mathbf{r}_0$ below).

We begin with an analysis of expression (40) for an unpolarized M from (19).

We consider separately the contributions from the scalar potential φ^0 and the vector potential φ in (40). With $\varphi^\mu = \varphi^0$ we have, at an accuracy level sufficient for our purposes, $\mathrm{Tr}[M(-(\mathbf{q}-\mathbf{k})/2,\ (\mathbf{q}-\mathbf{k})/2)\gamma^0\Lambda^{\pm}_{(\mathbf{q}+\mathbf{k})/2)}\gamma_j] = 2mq_j + o(k;q)$; $\mathrm{Tr}[M(-(\mathbf{q}-\mathbf{k})/2,(\mathbf{q}-\mathbf{k})/2), \gamma_j\Lambda^{+}_{(\mathbf{q}+\mathbf{k})/2}\gamma^0] = -2mq_j + o(k;q)$; and $\varepsilon_{(\mathbf{q}-\mathbf{k})/2} = m + o(k;q) = \varepsilon_{(\mathbf{k}+\mathbf{q})/2}$. Consequently, the contribution $\delta\,\mathbf{F}_1^{(\mathrm{scal\cdot el})}$, from the scalar potential, in the electron terms in (40) is

$$\delta F_1^{i\,(\mathrm{scal\cdot el})} = e^3 \int dq^0 d\mathbf{q}\varphi^0(q^0,\mathbf{q}) \int \frac{d\mathbf{k}}{(2\pi)^3} \times \exp[i(\mathbf{q}-\mathbf{k})\mathbf{r}_0] \frac{4\pi}{8km}\left(\delta^i - \frac{k^i k^j}{k^2}\right)\exp(i\mathbf{k}\mathbf{r}_0 - iq_0 t) \times \left\{\frac{-2mq_j}{m}\left[\frac{(-ik)\,i\pi\delta(k-q^0)}{iki} + \frac{iki\pi\delta(k+q^0)}{i(-q_0)i}\right] + \frac{2mq_j}{m}\left[\frac{iki\pi\delta(k+q^0)}{i(-k)i} + \frac{(-ik)\,i\pi\delta(k-q^0)}{i(-q_0)i}\right]\right\}. \tag{42}$$

We carry out the summation over $\mathbf{f}$ as in §3, replacing $\Sigma_{\mathbf{k}}$ by $\int d\,\mathbf{k}/(2\pi)^3$. Evaluating

$$\int d\mathbf{n}\,(\delta^{ij} - n^i n^j) = \frac{8\pi}{3}\delta^{ij}$$

(where $|\mathbf{n}| = 1$, and the integration is over a unit sphere), we find

$$\int_0^{+\infty} k^2 dk\,[\delta(k-q^0) + \delta(k+q^0)] = (q^0)^2$$

(for both $q^0 \geqslant 0$ and $q^0 < 0$). Also using $q_j = -q^i$, we find, according to (42),

$$\delta\mathbf{F}_1^{(\mathrm{scal\cdot el})} = -\frac{2e^3}{3m}\int dq^0 d\mathbf{q} q^0 \mathbf{q}\varphi^0(q^0,\mathbf{q})\exp(i\mathbf{q}\mathbf{r}_0 - iq^0 t) = -\frac{2e^3}{3m}\frac{\partial^2\varphi^0(t,\mathbf{r})}{\partial t\partial\mathbf{r}}. \tag{43}$$

On the other hand, the contribution $\delta\,\mathbf{F}_1^{(\mathrm{scal\cdot pos})}$ from the positron terms vanishes in the case $\varphi^\mu = \varphi^0$, at the same accuracy level. The reason is that in place of the small k or q^0 in the denominators in (40) we find quantities which are equal to $2m + o(1)$, while the numerators, in the same approximation, are

$$\mathrm{Tr}\left[M\left(-\frac{1}{2}(\mathbf{q}-\mathbf{k}),\ \frac{1}{2}(\mathbf{q}-\mathbf{k})\right)\gamma^0\Lambda^{-}_{(\mathbf{k}+\mathbf{q})/2}\gamma_j\right] = \mathrm{Tr}\left[\frac{m(\gamma^0+1)}{2}\gamma^0 m(\gamma^0-1)\gamma_j\right] = 0 + o(1).$$

The contribution to (40) associated with the vector potential φ is also conveniently broken up into electron and positron components $\delta\,\mathbf{F}_1^{(\mathrm{vect\cdot el})}$ and $\delta\,\mathbf{F}_1^{(\mathrm{vect\cdot pos})}$. In the same approximation as was used in the calculation of (42), we find $\mathrm{Tr}[M\gamma_l\Lambda^-\gamma_j] = 4m^2\delta_{jl} + o(1) = \mathrm{Tr}[M\gamma_j\Lambda^-\gamma_l]$, $\mathrm{Tr}[M\gamma_l\Lambda^+\gamma_j] = 0 + o(k;q)$. Hence $\delta\,\mathbf{F}_1^{(\mathrm{vect\cdot el})} = 0$ and

$$\delta F_1^{i\,(\mathrm{vect\cdot pos})} = -e^3\int dq^0 d\mathbf{q}\varphi^l(q^0,\mathbf{q})\int\frac{d\mathbf{k}}{(2\pi)^3}\exp[i(\mathbf{q}-\mathbf{k})\mathbf{r}_0] \times \frac{4\pi}{8km}\left(\delta^i - \frac{k^i k^j}{\mathbf{k}^2}\right)\exp(i\mathbf{k}\mathbf{r}_0 - iq_0 t) \times\left\{\frac{4m^2\delta_{jl}}{m}\left[\frac{(-ik)\,i\pi\delta(k-q^0)}{i\cdot 2mi} + \frac{iki\pi\delta(k+q^0)}{(-i)\cdot 2mi}\right] + \frac{4m^2\delta_{jl}}{m}\left[\frac{iki\pi\delta(k+q^0)}{(-i)\cdot 2mi} + \frac{(-ik)\,i\pi\delta(k-q^0)}{i\cdot 2mi}\right]\right\}$$
$$= \frac{2e^3}{3m}\int dq^0 dq\,\exp(i\mathbf{q}\mathbf{r}_0 - iq^0 t)(q^0)^2\varphi^i(q^0,\mathbf{q}) = -\frac{2e^3}{3m}\frac{\partial^2\varphi^i(t,\mathbf{r})}{\partial t^2}. \tag{44}$$

Combining (43) and (44), we find

$$\delta\mathbf{F}_1 = \frac{2e^3}{3m}\frac{\partial}{\partial t}\left(-\frac{\partial\varphi^0}{\partial\mathbf{r}} - \frac{\partial\varphi}{\partial t}\right) = \frac{2e^3}{3m}\frac{\partial\mathbf{E}}{\partial t}, \tag{45}$$

which is the same as the classical expression, (2), for the radiation-reaction force.

Now that we have derived (45), we can restore the relativistic invariance of this expression for the reaction of radiation on a moving charge. The relativistically invariant formula corresponding to (45) is $\delta F^\mu = (2e^3/3m)(\partial F^{\mu\nu}/\partial x^\lambda)u_\nu u^\lambda$, where $F_{\mu\nu} = \partial\varphi_\nu/\partial x^\mu - \partial\varphi_\mu/\partial x^\nu$. This result is of course the same as the first term in (3).

It is not difficult to verify that when we use polarized state (18) in place of unpolarized state (19) in analyzing the radiation-reaction force in (40) we again find (43)–(45). This is as it should be, since in this approximation—third order in e, first order in the external field, and the maximum order of the derivatives of the external field, namely the first (in the quantum-mechanical approach, this situation corresponds to expressions which are quadratic in the momentum transfer)-the acquisition by a particle of a magnetic moment in classical electrodynamics again fails to give rise to new terms in the radiation-reaction force. The radiation-reaction force acting on the particle is given in this approximation by the same expression [the first term in (2)] as for a charged particle which has no magnetic moment.

Skipping over the detailed calculations, we note that the next term in expansion (40) in powers of q would be small with respect to (43)–(45) by a factor $\sim q^2\lambda \ll 1$ $[\lambda \sim \langle(\delta p)^2\rangle^{-1} \sim \langle(\delta r)^2\rangle]$. After inverse Fourier transforms were taken, this term would lead to an additional term $\mathrm{const}\cdot(e^3/m)\langle(\delta r)^2\rangle\ddot{\mathbf{E}}$ on the right side of (45). Such a term in (45) does indeed correspond to the first nonvanishing term in the expansion of the radiation-reaction force in powers of the "small parameter" $(\delta r\cdot\partial/\partial t)$ (separately in each order in e). A similar result is found in classical electrodynamics, in a discussion of the radiation-reaction force in the case of an extended charged particle with a "characteristic radius" L. Under these conditions, the "ordinary" expression $(2/3)e^2\dddot{\mathbf{r}}$ on the right side of (5) would be replaced by the infinite series[4] $(2/3)e^2(\dddot{\mathbf{r}} - (2/3)L\ddddot{\mathbf{r}} + (1/3)L^2\dddot{\mathbf{r}} - \ldots)$. This result is again an expansion in powers of $(L\partial/\partial t)$. This series was summed in Ref. 4. The equation describing the motion of the extended charged particle became a differential-difference equation.

5. CONCLUSION

This analysis shows that standard perturbative methods in QED can be used to derive an expression for the radi-

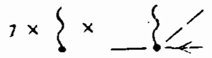

FIG. 9.

ation-reaction force in third order in e—an expression corresponding to the first term in (2), (3). In a corresponding way, one can use the methods which have been developed to derive an expression for the force acting on the electron in the next order, e^4 [this expression must of course correspond to the last part of (2), (3)], and also radiation corrections of higher orders.

Finally, we note that this entire discussion has been based on spinor electrodynamics. Consequently, the validity of result (45) has been proved only for particles with a spin $s = 1/2$. We would naturally expect, however, that the classical limit of the expression for the radiation-reaction force would not depend on the spin of the particle. The reason is that for a scalar particle (with a spin $s = 0$) one can construct a corresponding technique, which turns out to be similar to the methods presented above. The only distinction is that the Lagrangian of the scalar field is quadratic in the 4-momentum operator $P_\mu = i\partial_\mu$, while the Lagrangian of a spinor field is linear in P_μ. When a gauge interaction with an external electromagnetic field is taken into account, this circumstance leads to the appearance of a diagram with four external lines coming into one vertex (two of these lines correspond to the charged particle, and two to the electromagnetic field, A_μ or φ_μ), in addition to the diagrams in Fig. 1 which yield the expansion of the S-matrix. In studying the radiation-reaction force, we should thus now consider, in addition to the diagram in Fig. 7, one more diagram, which contains one dashed line and one wavy line (Fig. 9). In the same approximation which we used in the calculations above, we can show that for a scalar particle the contribution of the diagram in Fig. 7 reduces to (43), that the contribution of the new diagram in Fig. 9 is equal to expression (44), and that their sum is therefore equal to (45). We thus find that the classical limits of the expressions for the radiation-reaction force acting on scalar and spinor particles are the same.

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