Perturbation theory for resolvents as applied to problems in radiation theory

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The resolvent method, one of the most powerful methods of solving nonstationary problems in quantum mechanics, is presented. The use of this method has been hindered by its reputation for excessive complexity. In the proposed form, the resolvent method is no more complex than common perturbation theory. One of the attractive features of the method is the possibility of widespread use of the theory of functions of a complex variable. The main ideas of the method, which was initially developed for systems with a discrete spectrum, are generalized for systems with a continuous spectrum. The radiation from atoms in a waveguide and in free space is examined as examples of this method.

INTRODUCTION

At present, the number of problems confronting the field which is called nonrelativistic quantum electrodynamics (which studies the quantum processes of radiation by nonrelativistic systems of electrons, by atoms, by molecules, etc.) has greatly expanded. This is undoubtedly due to the tempestuous development of various laser methods of research. Among these problems one can isolate nonstationary problems of quantum electrodynamics, for which the process of development of the state of the system over time is important. One of the methods of solving these problems is the resolvent method, which is examined in this paper.

It is well known that the overwhelming number of problems in quantum electrodynamics are solved using perturbation theory methods. The most well-known method is the theory of transitions between discrete levels of quantum systems. The first orders of this theory of perturbations describe the behavior of a system only over a small time interval that is much smaller than the characteristic times of transitions. A consideration of higher orders is too laborious. Thus, even in the early stages of development of quantum electrodynamics various alternative methods were proposed to describe the development of the state of a system in substantial time segments. These methods include, for example, the Wigner–Weisskopf method.[1] This method was subsequently developed and acquired the form of what is called attenuation theory. However, the Wigner–Weisskopf method has a drawback: the solution is not sought in the method, but is actually guessed on the basis of some physical considerations; this approach is usually defined by the words "heuristic method." This same drawback is retained in attenuation theory, although in a more veiled form.

Like other "heuristic" approaches, this method has the drawback that it is not the first step of some consistent procedure to obtain a solution in the form of an expansion in terms of a small variable. Thus, it is difficult to evaluate the error of this approach or to obtain, when necessary, corrections to the approach.

In this paper, which is of a methodological character, we wish to draw the reader's attention to the resolvent method, a method which greatly surpasses the Wigner-Weisskopf method in capabilities, while retaining the consistent method of perturbation theory. Information about this method is spread among various sources, and the only textbook presentation that we know of is not very successful.[2] Apparently, widespread use of this method is hindered by its reputation for excessive complexity. However, this method can acquire a form which is no more complex than the usual perturbation theory. We hope that this article will be a step in this direction.

1. THE DISCRETE SPECTRUM

Let us first develop the resolvent method for quantum mechanical systems with a full infinite set of discrete levels, so that it can be later generalized to systems with a continuous spectrum.

1.1. Evolution operator

An important element of the resolvent method is the evolution operator, an operator which expresses the state $|\Psi(t)\rangle$ at an arbitrary time t using the state $|\Psi_0\rangle$ at the initial moment in time

$$|\Psi(t)\rangle = \hat{U}(t, t_0)\,|\Psi_0\rangle. \tag{1.1}$$

The trivial form of the evolution operator is

$$\hat{U}(t, t_0) = \exp\left[-\frac{i}{\hbar}\hat{H}(t-t_0)\right], \tag{1.2}$$

which can be easily verified by direct substitution of $|\Psi(t)\rangle$ into Schrödinger's equation. We note that Eq. (1.2) is of a formal character, which in itself usually does not permit one to calculate the operator $\hat{U}$ in an explicit form. This is also true of the next equation below, Eq. (1.3). In essence, the entire content of the article below is devoted to the construction of a form of operator $\hat{U}$ suitable for practical use.

Another form of the evolution operator which will be more important to us, is (assuming $t_0 = 0$)

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$$\hat{U}(t) = \frac{1}{2\pi i}\int_C dz\,\frac{e^{-izt/\hbar}}{z-\hat{H}}, \tag{1.3}$$

where integration is performed along contour C, which is illustrated in Fig. 1. (The expression $z - \hat{H}$ should have been written in the form $z\hat{I} - \hat{H}$, where $\hat{I}$ is an identity operator. However, in the majority of cases the operator $\hat{I}$ can be omitted without any loss of meaning.) We will show below, after several comments, that this is actually the evolution operator. The operator

$$\hat{R}(z) = \frac{1}{z-\hat{H}} \tag{1.4}$$

is called the resolvent operator. It is inverse in relation to the operator $z - \hat{H}$, that is,

$$(z-\hat{H})\hat{R}(z) = \hat{R}(z)(z-\hat{H}) = \hat{I}, \tag{1.5}$$

where $\hat{I}$ is an identity operator ("unity" operator). As will be obvious below, the operator $\hat{R}(z)$ is defined for all quantities z, including complex ones, as well as quantities which are equal to the eigenvalues of the Hamiltonian $\hat{H}$.

We note the relation of the resolvent operator to the Green's function. In a coordinate representation using the expansion of the identity operator $\hat{I} = \int |x'\rangle = dx'\langle x'|$ we obviously have

$$\langle x|\Psi(t)\rangle \equiv \Psi(x,t) = \langle x|e^{-i\hat{H}t/\hbar}|\Psi_0\rangle$$
$$= \int \langle x|\hat{U}(t)|x'\rangle\,dx'\,\langle x'|\Psi_0\rangle,$$

that is,

$$\Psi(x,t) = \int dx' G(x,t;x',0)\,\Psi(x',0). \tag{1.6}$$

Here the Green's function

$$G(x,t;x',0) = \langle x|\hat{U}(t)|x'\rangle \tag{1.7}$$

is the core of the evolution operator. In accordance with Eq. (1.3), we then obtain

$$G(x,t;x',0) = \frac{1}{2\pi i}\int_C dz e^{-izt/\hbar}\langle x|\hat{R}(z)|x'\rangle; \tag{1.8}$$

consequently, the Green's function and the resolvent operator are equivalent in description completeness.

Below we will make great use of the concept of a function of an operator. As usual, the function is defined by its series, and as a result the calculation of the function is reduced to the multiplication and addition of operators. The procedure is especially simplified if one knows the complete orthonormalized system of eigenvectors of the operator, which is the argument of the function.

Indeed, let

$$\hat{A}|n\rangle = A_n|n\rangle, \tag{1.9a}$$
$$\langle n'|n\rangle = \delta_{n',n}, \tag{1.9b}$$
$$\hat{I} = \sum_n |n\rangle\langle n|, \tag{1.9c}$$

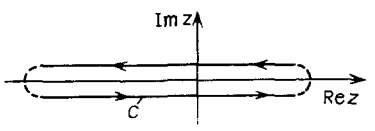

FIG. 1.

where $\hat{A}$ is the argument of some function $f(\hat{A})$ and $|n\rangle$ are its eigenvectors (states). Multiplying Eq. (1.9c) from the left by $\hat{A}$ and using Eq. (1.9a), we obtain

$$\hat{A} = \sum_n |n\rangle A_n \langle n|.$$

Raising this expression to the second, third, etc., power, we obtain, using Eq. (1.9b)

$$\hat{A}^2 = \sum_n |n\rangle A_n^2\langle n|, \quad \hat{A}^3 = \sum_n |n\rangle A_n^3\langle n|, \ldots,$$

then for an arbitrary analytical function $f(z)$ we obtain the formula

$$f(\hat{A}) = \sum_n |n\rangle f(A_n)\langle n|. \tag{1.10}$$

Now let $|n\rangle$ be eigenfunctions, and E_n be the eigenvalues of the Hamiltonian $\hat{H}$ (that is, $\hat{A} = \hat{H}$); then, according to Eq. (1.10) we have for $f(z-\hat{H}) = (z-\hat{H})^{-1}$

$$\hat{R}(z) = \sum_n \frac{|n\rangle\langle n|}{z-E_n}. \tag{1.11}$$

Let us substitute Eq. (1.11) into the right side of Eq. (1.3)

$$\frac{1}{2\pi i}\int_C dz e^{-izt/\hbar}\hat{R}(z) = \sum_n |n\rangle\left(\frac{1}{2\pi i}\int_C dz\,\frac{e^{-izt/\hbar}}{z-E_n}\right)\langle n|; \tag{1.12}$$

because the pole $z = E_n$ always lies within the contour of integration, the expression in square brackets is equal to $e^{-iE_nt/\hbar}$, and then according to Eq. (1.10) the right side of Eq. (1.12) is equal to

$$\sum_n |n\rangle e^{-iE_nt/\hbar}\langle n| = e^{-i\hat{H}t/\hbar}\sum_n |n\rangle\langle n| = e^{-i\hat{H}t/\hbar} = \hat{U}(t)$$

and, consequently, the correctness of expression (1.3) is shown for the resolvent operator.

1.2. Perturbation theory for the evolution operator

If the eigenstates and the corresponding eigenvalues of the Hamiltonian are known, then Eqs. (1.1) and (1.3), together with Eq. (1.11), solve the problem posed above of the development of a state over time. However, usually the Hamiltonian consists of unperturbed and perturbed parts

$$\hat{H} = \hat{H}_0 + V \tag{1.13}$$

and only the eigenstates and corresponding eigenvalues of the unperturbed Hamiltonian are known. In this case one can use perturbation theory.

However, before we develop this theory, let us note one essential circumstance. Up until now operators have been used which act in a Hilbert space of states of some quantum system. These operators were abstract in the sense that they had no relation to any specific reference system (basis). Hereinafter we will use a reference system of states formed by the eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ of the studied quantum system. The operators may then be given in the form of matrices, ket-vectors of states in the form of vector columns, and bravectors in the form of vector lines; the transition from the abstract operators and states to this specific representation is described by the equalities

$$O_{mn} = \langle m|\hat{O}|n\rangle, \quad \Psi_m = \langle m|\Psi\rangle, \quad \Psi_n^* = \langle\Psi|n\rangle,$$

where $\langle m|$ and $|n\rangle$ are the bra- and ket-eigenstates of the

unperturbed Hamiltonian $\hat{H}_0$. It is assumed that these states form a full orthonormalized system, that is, they satisfy Eqs. (1.9b) and (1.9c). When two or more eigenvectors belonging to one quantity E_α become degenerate, it is also assumed that they are orthogonal to each other and to all the remaining eigenvectors. When these assumptions are made, the operator equations (1.1), (1.2), (1.3), (1.5), (1.11), and (1.12) remain in effect, that is, in these equations the operators may be replaced by the corresponding matrices of the specific representation. In particular, the operator matrix which is the inverse of some given operator $\hat{O}$ is simply a matrix which is inverse to matrix $\|O_{mn}\|$.

Hereinafter we will not differentiate the operator and its corresponding matrix in the notation, bearing in mind that this matrix belongs to a reference system formed by the eigenstates of the unperturbed Hamiltonian $\hat{H}_0$.

Thus, operator $z - \hat{H}$, in matrix form, is equal to

$$z - \hat{H} = \begin{Vmatrix} z - H_{00} & -V_{01} & -V_{02} & -V_{03} & \dots \\ -V_{10} & z - H_{11} & -V_{12} & -V_{13} & \dots \\ -V_{20} & -V_{21} & z - H_{22} & -V_{23} & \dots \\ -V_{30} & -V_{31} & -V_{32} & z - H_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{Vmatrix}. \quad (1.14)$$

The matrix which is inverse to $Z - \hat{H}$, is constructed using known rules for the construction of inverse matrices

$$(z - \hat{H})^{-1} = \Delta^{-1} \begin{Vmatrix} D_{00} & D_{01} & D_{02} & D_{03} & \dots \\ D_{10} & D_{10} & D_{12} & D_{13} & \dots \\ D_{20} & D_{21} & D_{22} & D_{23} & \dots \\ D_{30} & D_{31} & D_{32} & D_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{Vmatrix}, \quad (1.15)$$

where Δ is the determinant of matrix $z - \hat{H}$, and D_{jk} is the algebraic complement to elements $(z - \hat{H})_{kj}$ of this matrix (we point out that the order of subscripts in D_{jk} is the reverse of the order of subscripts in the element $(z - \hat{H})_{kj}$, to which D_{jk} is an algebraic complement).

Because the determinant Δ enters into the denominator of Eq. (1.15), the matrix $(z - \hat{H})^{-1}$ is defined only for those values of z when $\Delta \neq 0$. The determinant Δ can be given in the form of a product

$$\Delta = \prod_m (z - E_m),$$

where E_m are the eigenvalues of the full Hamiltonian $\hat{H}$. Consequently, the complex variable z may acquire any value, in addition to the eigenvalues of the Hamiltonian H. We note that the eigenvalues E_m do not depend on the representation; thus, not only the matrix (1.15), but also the resolvent operator in general, Eq. (1.4), are defined for all values of z in addition to the eigenvalues E_m as was noted in the previous section.

The algebraic complements D_{jk}, as is well known, are also determinants, but of order lower by a unit than Δ_j; they are obtained from Δ by canceling the k th row and jth column and multiplying the resultant determinant by $(-1)^{k+j}$. As we will see, for any matrix element of the resolvent the poles are the same, since they are defined by the zeroes of the determinant Δ; they are usually spoken of as resolvent poles. However, the residues in these poles for different matrix elements may be different, since they depend on D_{jk}. For example, the zeroes of some D_{jk} may be close to some resolvent poles; then the contribution of these poles to the total amplitude will be small.

We note that all the eigenvalues E_m of the full Hamiltonian are real. Thus, the resolvent is a single-valued function of z and has only poles lying on the real axis, and this is a characteristic feature of a discrete spectrum.

The matrix element of the resolvent operator

$$R_{jk} = \frac{D_{jk}}{\Delta} \quad (1.16)$$

is the ratio of the algebraic complement D_{jk} to the determinant Δ. It should be noted that the numerator and denominator of R_{jk} are expanded below into perturbation theory series which are independent of each other. These expansions can and even should be made with a different accuracy, since their results affect different aspects of the process under study. The roots of the denominator define the indices of the exponentials in the residues, while the numerator yields only pre-exponential factors. Consequently, in order for the time interval (in which the obtained solution is close to the exact one) to be larger, one must determine the roots of the denominator with greater accuracy. Frequently, sufficient accuracy is obtained if the roots of the denominator are calculated in the second order of perturbation theory, while the algebraic complements are calculated in the first or even the zero order of perturbation theory.

Before we turn to a search for the aforementioned series of perturbation theory, we note that it is sufficient to find such a series only for the determinant Δ. Indeed, expansion of Δ in terms of the elements of any line has the form (n is the number of the line)

$$\Delta = \mathrm{Det}\,(z - \hat{H}) = \sum_m (z - \hat{H})_{nm} D_{mn}$$

and D_{mn} does not contain the matrix element $(z - \hat{H})_{nm}$ (in writing the determinant D_{mn} the element $(z - \hat{H})_{nm}$ is left out).

Thus, we have the expression

$$\frac{\partial \Delta\,(z)}{\partial\,(z - \hat{H})_{nm}} = \frac{\partial\,\mathrm{Det}\,(z - \hat{H})}{\partial\,(z - \hat{H})_{nm}} = D_{mn}. \quad (1.17)$$

Thus, if $\Delta(z)$ is known as a function of its matrix elements, D_{mn} can be obtained from $\Delta(z)$ by differentiation. We note that since $H_{nm} = H^0_{nm} + V_{nm}$, Eq. (1.17) can be written in a more convenient form

$$D_{mn} = -\frac{\partial \Delta\,(z)}{\partial V_{nm}}. \quad (1.18)$$

Then, taking Eq. (1.4) into consideration, we have

$$R_{jk} = -\frac{1}{\Delta\,(z)} \frac{\partial \Delta\,(z)}{\partial V_{kj}} = -\frac{\partial \ln \Delta\,(z)}{\partial V_{kj}}. \quad (1.19)$$

Thus, in the evolution operator $U(t)$ the matrix determinant $(z - \hat{H})$, which is given in the form of a function of its elements, is a key quantity, one which completely defines the process of the development of a state over time. Our next task is to represent $\Delta(z)$ in the form of a power series in terms of V_{kj}; after this, Eq. (1.19) will permit us to find R_{jk} as well.

We deduce the auxiliary expression linking the matrix determinant with the trace of its logarithm:

$$\mathrm{Det}\,\hat{A} = \exp\,(\mathrm{Sp}\,\ln \hat{A}). \quad (1.20)$$

The matrix (operator) $\ln \hat{A}$ is defined by the general equa-

tion (1.10). It can be easily calculated in a basis consisting of the eigenfunctions of the operator $\hat{A}$ (to do this such a basis should exist; if $\hat{A}$ is a Hermitian operator, then it always exists). To prove Eq. (1.20) we note that the determinant and trace of the matrix do not depend on the selection of the basis, they are invariants of the matrix. Thus, if Eq. (1.20) holds true in some basis, then it holds true in other bases. Let us examine a basis in which $\hat{A}$ is diagonal. In this case $\mathrm{Det}\,\hat{A} = \Pi_k A_k$, where $A_k \equiv A_{kk}$ are diagonal elements of $\hat{A}$ in this basis. According to Eq. (1.10) the matrix $\ln \hat{A}$ is also diagonal, and its matrix elements are equal to $\ln A_k$. Then,

$$\mathrm{Sp}\ln \hat{A} = \sum_k \ln A_k \text{ и } \exp(\mathrm{Sp}\ln \hat{A}) = \prod_k A_k = \mathrm{Det}\,\hat{A}.$$

Thus, it has been established that Eq. (1.20) holds true.

Further, for the purposes of perturbation theory, it is more convenient to write the operator $\hat{A}$ in the form

$$\hat{A} = \hat{I} - \hat{B},$$

and the logarithm of $\hat{A}$ in the form of a power series $\hat{B}$. It is obvious that the matrix $\hat{B}$ is diagonal if the matrix $\hat{A}$ is diagonal, and it follows from the last equation that

$$A_k = 1 - B_k.$$

If the moduli of all B_k are less than unity, then one can write a Taylor series for the logarithm

$$\ln A_k = \ln(1 - B_k) = -\sum_{N=1}^{\infty} \frac{1}{N}(B_k)^N. \tag{1.21}$$

Since $(B_k)^N = (\hat{B}^N)_{kk}$ is a diagonal element of the matrix $\hat{B}^N$, we have the equation

$$\ln \hat{A} = \ln(\hat{I} - \hat{B}) = -\sum_{N=1}^{\infty} \frac{1}{N}\hat{B}^N, \tag{1.22}$$

which, due to its matrix nature, is satisfied in any basis, and diagonal $\hat{A}$ and $\hat{B}$ are not obligatory (however, the condition that the moduli of all eigenvalues of B be less than unity is retained).

Now let us turn to the search for the perturbation theory series for $\Delta(z) = \mathrm{Det}\,(z - \hat{H})$. Writing $\hat{H}$ as the sum of the unperturbed Hamiltonian $\hat{H}_0$ and perturbation $\hat{V}$, we write the operator $z - \hat{H}$ in the form

$$z - \hat{H} = z - \hat{H}_0 - \hat{V} = (z - \hat{H}_0)[\hat{I} - (z - \hat{H}_0)^{-1}\hat{V}]. \tag{1.23}$$

The determinant of the matrix of this operator is obviously equal to

$$\Delta(z) = \mathrm{Det}\,(z - \hat{H}) = \mathrm{Det}\,(z - \hat{H}_0)\,\mathrm{Det}\,[\hat{I} - (z - \hat{H}_0)^{-1}\hat{V}]. \tag{1.24}$$

If the eigenvalues of the unperturbed Hamiltonian are known, that is, if $\hat{H}_0|n\rangle = E_n^0|n\rangle$, then

$$D_0(z) \equiv \mathrm{Det}\,(z - \hat{H}_0) = \prod_n (z - E_n^0). \tag{1.25}$$

We use Eqs. (1.20) and (1.22) to calculate the second factor in Eq. (1.24), setting

$$\hat{A} = \hat{I} - (z - \hat{H}_0)^{-1}\hat{V}, \qquad \hat{B} = (z - \hat{H}_0)^{-1}\hat{V}.$$

Then

$$\mathrm{Det}\,[\hat{I} - (z - \hat{H}_0)^{-1}\hat{V}] = \exp\left(-\sum_{N=1}^{\infty} \frac{1}{N}\mathrm{Sp}\,\hat{B}^N\right). \tag{1.26}$$

We introduce the notation

$$S_N = \mathrm{Sp}\,\hat{B}^N = \mathrm{Sp}\,[(z - \hat{H}_0)^{-1}\hat{V}]^N. \tag{1.27}$$

Then for $\mathrm{Det}\,(z - \hat{H}) = \Delta(z)$ we obtain

$$\Delta(z) = D_0(z)\exp\left(-\sum_{N=1}^{\infty} \frac{S_N}{N}\right). \tag{1.28}$$

The matrix elements of the operator $\hat{B} = (z - \hat{H}_0)^{-1}\hat{V}$ in a basis formed by the eigenstates of the unperturbed Hamiltonian can be easily calculated; placing the identity operator $\hat{I} = \Sigma_k |k\rangle\langle k|$ between the operators $(z - \hat{H}_0)^{-1}$ and $\hat{V}$ we have

$$B_{lm} = \langle l|(z - \hat{H}_0)^{-1}\hat{V}|m\rangle = \sum_k \langle l|(z - \hat{H}_0)^{-1}|k\rangle\langle k|\hat{V}|m\rangle = \sum_k (z - E_k^0)^{-1}\delta_{lk}V_{km} = \frac{V_{lm}}{z - E_l^0}. \tag{1.29}$$

Then we find

$$S_1 = \mathrm{Sp}\,\hat{B} = \sum_{l=0}^{\infty} B_{ll} = \sum_{l=0}^{\infty} \frac{V_{ll}}{z - E_l^0}. \tag{1.30}$$

For $(\hat{B}^2)_{lm}$, using Eq. (1.29) we obtain

$$(\hat{B}^2)_{lm} = \sum_{k=0}^{\infty} \frac{V_{lk}}{z - E_l^0}\frac{V_{km}}{z - E_k^0},$$

then

$$S_2 = \mathrm{Sp}\,\hat{B}^2 = \sum_{k,l} \frac{V_{lk}V_{kl}}{(z - E_l^0)(z - E_k^0)}. \tag{1.31}$$

In exactly the same way we obtain

$$(\hat{B}^3)_{lm} = \sum_{k,j} \frac{V_{lk}V_{kj}V_{jm}}{(z - E_l^0)(z - E_k^0)(z - E_j^0)},$$

$$S_3 = \sum_{k,j,l} \frac{V_{lk}V_{kj}V_{jl}}{(z - E_l^0)(z - E_k^0)(z - E_j^0)}. \tag{1.32}$$

From these formulas the manner in which S_N is constructed for any N is clear.

Now we can turn to Eq. (1.19) and use it to write an expression for the matrix element of resolvent R_{jk}. There are two paths one can take. In the first, one can use the formula

$$R_{jk} = -\frac{\partial \ln \Delta(z)}{\partial V_{kj}}.$$

Substituting Eq. (1.28) into it and taking into consideration that $D_0(z)$ does not depend on V_{kj}, we obtain

$$R_{jk} = \sum_{N=1}^{\infty} \frac{1}{N}\frac{\partial S_N}{\partial V_{kj}}.$$

From Eqs. (1.30)–(1.32) it is clear that S_N is a polynomial of degree N in V. Taking this into consideration we conclude that the last formula yields an expansion of the resolvent in powers of the perturbation, that is, a Born expansion.

The second method is associated with the use of Eq. (1.19) in the form

$$R_{jk} = -\frac{1}{\Delta(z)}\frac{\partial\Delta(z)}{\partial V_{kj}}, \tag{1.33}$$

into which we substitute the expression for $\Delta(z)$ expanded in terms of powers of V. Expanding the exponential in Eq. (1.18) into a series and grouping terms of the same order in V, we obtain

$$\Delta(z) = D_0(z)\Big[1 - S_1 + \frac{1}{2!}(S_1^2 - S_2) - \frac{1}{3!}(S_1^3 - 3S_1S_2 + 2S_3) + \frac{1}{4!}(S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4) + \ldots\Big]. \tag{1.34}$$

Then, substituting this into Eqs. (1.25), (1.30)–(1.32), we obtain an expansion of the denominator of the resolvent

$$\Delta(z) = \prod_l (z - E_l) - \sum_k \Delta k \prod_l (z - E_l) + \frac{1}{2!}\sum_{k,j}{}' \Delta_{kj} \prod_{l\neq k,\, j} (z - E_l) - \ldots, \tag{1.35}$$

where

$$\Delta_k = V_{kk}, \quad \Delta_{kj} = V_{kk}V_{jj} - V_{kj}V_{jk}, \ldots \tag{1.36}$$

are the principal minors of the perturbation matrix $\hat{V}$, that is, the determinants of the matrices obtained from matrix $\hat{V}$ by crossing out several lines and columns symmetrical to them (the subscripts to Δ indicate which lines and columns of matrix $\hat{V}$ form the given principal minor); the products and sums in this expansion are taken using all possible values of the subscripts from zero to infinity, excluding those values which are specially noted. The prime on the sum sign indicates that the subscripts over which the summation is carried out must necessarily be different. The denominator in Eq. (1.35) can be written in the form

$$\Delta(z) = \prod_l (z - E_l)\Big[1 - \sum_k \frac{\Delta_k}{z - E_k} + \frac{1}{2!}\sum_{k,j}{}' \frac{\Delta_{kj}}{(z - E_k)(z - E_j)} - \frac{1}{3!}\sum_{k,j,m}{}' \frac{\Delta_{kjm}}{(z - E_k)(z - E_j)(z - E_m)} + \ldots\Big]; \tag{1.37}$$

if one also uses the identity

$$\begin{aligned}[(z - E_1)(z - E_2)\ldots(z - E_n)]^{-1} &= [(z - E_1)(E_2 - E_1)\ldots(E_n - E_1)]^{-1} + \ldots + [(z - E_i)(E_1 - E_i)\ldots \times(E_n - E_i)]^{-1} \\ &\ldots + [(z - E_n)(E_1 - E_n)\ldots(E_{n-1} - E_n)]^{-1},\end{aligned} \tag{1.38}$$

then $\Delta(z)$ acquires the form

$$\Delta(z) = \prod_l (z - E_l)[1 - F(z)], \tag{1.39}$$

where

$$F(z) = \sum_k A_k (z - E_k)^{-1}, \tag{1.40}$$

$$A_k = \Delta_k - \sum_{j\neq k}\frac{\Delta_{jk}}{(E_k - E_j)} + \frac{1}{2!}\sum_{j,\, m\neq k}{}' \frac{\Delta_{jmk}}{(E_k - E_j)(E_k - E_m)} - \ldots \tag{1.41}$$

are coefficients which do not depend on z. Thus, the poles of $\Delta(z)$ lie at points $z = E_k$. At least in general form the coefficients A_k (that is, the residues at the poles of $\Delta(z)$) are all of the first order of smallness in terms of the perturbation V, although in some special cases individual Δ_k and other minors can, due to certain circumstances, become especially small or even equal to zero.

The function $1 - F(z)$ can be conveniently studied if one assumes that the interaction V goes to zero. Then the function $F(z)$ will be equal to zero virtually everywhere, except for the small (of the order of V) regions around the poles E_k. Consequently, the zeroes of the function $1 - F(z)$ also lie near these poles, as can be seen in Fig. 2. The reestablishment of V to a normal, finite (although small) value will shift the zeroes of $\Delta(z)$ somewhat, but they remain in the small neighborhood (of the order of V) of E_k, and do not change in number. A more complete study of the denominator of $1 - F(z)$ in the general form is hardly possible. Indeed, it actually includes information about all possible processes in the studied system for all possible initial conditions. In particular, as applied to radiation problems, which will be examined below, a full study of $1 - F(z)$ would indicate full knowledge of all possible processes of spontaneous and induced radiation, absorption, scattering, etc. In a general form this is hardly possible.

According to Eqs. (1.19) and (1.39), $R_{n,k}$, the matrix element of the resolvent operator, is equal to

$$R_{n,k} = \frac{\partial F(z)/\partial V_{k,n}}{1 - F(z)}. \tag{1.42}$$

The structure of the functions $\partial F(z)/\partial V_{kn}$ is approximately the same as for the function $F(z)$. They have poles at the same points E_k as does the function $F(z)$. Because of this, in particular, the poles of $\partial F(z)/\partial V_{kn}$ are not poles of the integrand as a whole. But the residues at these poles are of a different order in terms of perturbation. The poles of the numerator are next to the zeroes of the denominator, and so the value of the residue affects the amplitude of the corresponding contribution of the resolvent operator to the matrix element.

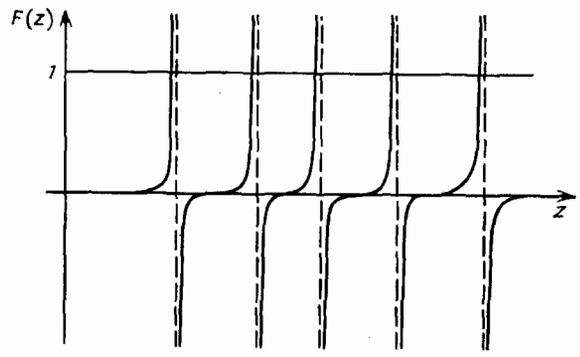

FIG. 2.

Differentiation of $F(z)$ with respect to V_{kn} includes differentiation of the minors $\Delta_{j\ldots t}$ with respect to V_{kn}. If k or n do not coincide with the subscripts of the minor, the derivative will be equal to zero. If both subscripts of V coincide with some subscripts of the minor, then the derivative will be equal to the algebraic complement of this minor to V_{kn}, multiplied by V_{nk}. Analogously, differentiation of the minor with respect to V_{nn} yields a minor of dimension lower by a unit (with deleted subscript n), if n coincides with some subscript of the minor. If there is no n among the subscripts of the minor, then its derivative with respect to V_{nn} is zero. Taking this into consideration, it is easy to give an estimate of the higher terms of the derivative of $F(z)$

$$\frac{\partial F(z)}{\partial V_{nm}} \approx \frac{V_{mn}}{(E_n - E_m)(z - E_n)}, \quad \frac{\partial F(z)}{\partial V_{nn}} \approx \frac{1}{z - E_n}. \qquad (1.43)$$

These estimates show that in calculation of, for example, the diagonal element $U_{n,n}(t)$, the zeroes of the denominator z_i closest to E_n are the most important ones because they are multiplied by a large quantity

$$\partial F(z)/\partial V_{nn}\,|_{z=z_i}. \qquad (1.43a)$$

Additional possibilities of classifying the singularities of the elements of the evolution operator appear after a transition to a continuous spectrum (see section 2). Here we note that the very possibility of a qualitative survey of these singularities is a strong feature of the resolvent method.

Expansion of Eq. (1.42) may also be examined as a Pade approximation[3] of the matrix elements of the resolvent. We also note an analogy between the expansion of Eq. (1.42) and the Fredholm expansions in the theory of integral equations. This analogy may be converted into a one-to-one correspondence if for individual elements of Eq. (1.41) one introduces the same Feynman diagrams as is done in Matthews and Walker's book[4] for the Fredholm expansions. It is known that Fredholm expansions, in contrast to a Neumann series, converge everywhere. Thus, the expansion of Eq. (1.42) is preferable to a Born expansion.

2. CONTINUOUS SPECTRUM. RADIATION PROBLEMS

As already mentioned, some dynamic systems, in particular, an electromagnetic field in free space, have a continuous spectrum. The presentation below will be dedicated exclusively to the electromagnetic field, its interaction with atoms, molecules, ions, and, in general, with quantum objects.

The continuous spectrum is frequently examined as a limiting case of a discrete spectrum. For the spectrum of an electromagnetic field to become discrete, the field must be imagined to be in some large resonator, where there are also some radiating objects. Then one can use the formulas derived above for a discrete spectrum, followed by letting the volume of the resonator go to infinity. The resultant limits, in particular for the matrix elements in Eq. (1.42), yield appropriate expressions for the continuous spectrum.

One should not think that the continuous spectrum leads only to complications; in terms of calculation it is even simplified, since many sums are replaced by integrals. To evaluate the latter, as will be seen below, one can use the theory of functions of a complex variable.

First we will examine one of the simplest examples of a radiation problem, the radiation of a two-level atom in a waveguide. In this example one can trace well the limiting transition from a discrete spectrum to a continuous one and understand all its special features.

Then we will construct a theory of the intrinsic width of a spectral line of spontaneous radiation which is free of the inconsistency which is inherent in the Wigner-Weisskopf theory and the attenuation theory.

2.1. Quantization of electromagnetic waves in a waveguide

We shall assume that in the examined waveguide, in the range which interests us, only one basic type of wave is propagated. If we take as an example a waveguide with a rectangular cross section (Fig. 3), then the critical frequencies of the waves of the fundamental type and of the next subsequent type differ only slightly, in all by only a factor of two. However, in microwave technology, there are approaches in which the critical frequency of the waves of the fundamental type can be substantially reduced. For example, if the waveguide is taken to be of the type shown in Fig. 3b, and the transverse dimensions are much less than the wavelength, then the critical frequency of the fundamental wave can be made much lower than the transition frequency of the radiating system, and the critical frequencies of the higher waves will be much greater than the transition frequency. The waveguide even retains its macroscopic character, since even in optics the wavelength of the radiation exceeds atomic dimensions by four orders of magnitude. For simplicity we will omit dispersion and absorption of the material from which the waveguide is made.

We will describe the fundamental TE-type wave in the waveguide using a vector potential

$$\mathbf{A} = G(a^+ e^{i\varkappa z} + a e^{-i\varkappa z})\,\varepsilon(x, y), \qquad (2.1)$$

where a^+ and a are operators for the creation and annihilation of photons in the examined wave. Since the vector potential in the gauge div $A = 0$ should satisfy the equation

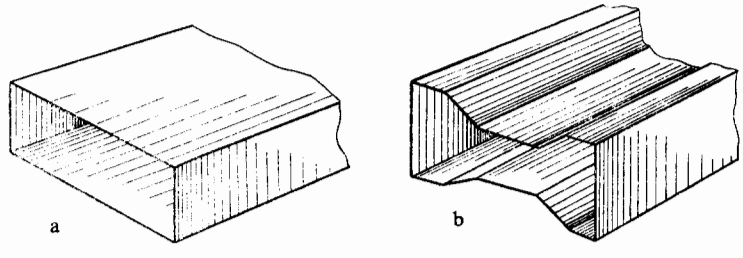

FIG. 3.

$$\Delta \mathbf{A} + k^2 \mathbf{A} = 0 \tag{2.2}$$

($k = \omega/c = 2\pi/\lambda$), then for a polarization vector $\boldsymbol{\varepsilon}(x, y) \perp \mathbf{e}_z$ we obtain the equation

$$\Delta_\perp \bar{\varepsilon} + (k^2 - \varkappa^2)\, \bar{\varepsilon} = 0, \tag{2.3}$$

for the condition $\mathrm{div}_\perp \boldsymbol{\varepsilon} = 0$. Moreover, the vector $\boldsymbol{\varepsilon}$ (like the vector potential A) should satisfy the boundary condition

$$\varepsilon_{\text{tang}} = 0 \tag{2.4}$$

at the surface of the waveguide. This problem has a solution only for the discrete set of positive values of the parentheses in Eq. (2.3).[5] The smallest of these quantities will be denoted by $\varkappa_c^2$. The field $\boldsymbol{\varepsilon}$ corresponding to this value of $\varkappa_c^2$ describes the transverse distribution of the fundamental TE-type wave in the waveguide. We have, consequently, the equation

$$k^2 = \varkappa^2 + \varkappa_{c_3}^2 \tag{2.5}$$

which defines the dependence of the frequency $\omega = ck$ of the fundamental wave on the propagation parameter $\varkappa = 2\pi/\lambda_{\text{wave}}$, which is inversely proportional to the waveguide wavelength. The condition $\varkappa = 0$ defines the critical frequency of the wave

$$\omega_c = c\varkappa_c. \tag{2.6}$$

Since Eq. (2.3) is linear with respect to the vector $\boldsymbol{\varepsilon}$, this vector may be subjected to the condition

$$\int \mathrm{d}s\varepsilon^2 = S, \tag{2.7}$$

where the integration is carried out along the cross section of the waveguide; it is also assumed that the average value of the square of the vector $\boldsymbol{\varepsilon}$ over the cross section of the waveguide is equal to unity. Then the normalization constant in the vector potential G is determined from the condition that the total energy of the wave be equal to $\hbar\omega\alpha^+\alpha$,

$$G = \left(\frac{2\pi\hbar c^2}{\omega SL}\right)^{1/2}. \tag{2.8}$$

As has already been stated, in order to study radiation starting with a discrete spectrum, one must limit the volume occupied by the field. We shall assume that the length of the examined waveguide is finite and equal to L, and the field at the ends of the waveguide will be subject to periodic boundary conditions. Then the propagation parameter acquires the following values

$$\varkappa_n = \frac{2\pi}{L} n, \tag{2.9}$$

where n is an integer which takes on the values $n = 0, 1, 2, \cdots$. Obviously, the adjacent quantities $\varkappa_n$ differ by $\Delta\varkappa = 2\pi/L$, that is, we have the equation

$$\frac{L}{2\pi} \Delta\varkappa = 1. \tag{2.10}$$

The imposition of periodic boundary conditions converts the waveguide into a resonator with a discrete spectrum, and this will be discussed below (section 2.3); however, considering that the length L will go to infinity, it makes sense to retain the term waveguide. Considering all that has been said, we conclude that the field in the waveguide is described by a vector potential

$$\mathbf{A}(x, y, z) = \boldsymbol{\varepsilon} \sum_j \left(\frac{2\pi\hbar c^2}{\omega_j SL}\right)^{1/2} (a_j^+ e^{i\varkappa_j z} + a_j e^{-i\varkappa_j z}). \tag{2.11}$$

This exhausts the basic information on the propagation of quantized waves in a waveguide which will be needed later.

2.2. The decay of the excited state of an atom in a waveguide

We turn now to the problem which interests us, the radiation from an atom. We will assume that in the initial moment the atom is in the excited state $|b\rangle$ and there are no photons, that is, all the fundamental waves or oscillators are in the ground vacuum state $|0\rangle$. All the remaining states of the atom are assumed to be at a higher energy than $|b\rangle$. The interaction Hamiltonian

$$V = -\frac{e}{mc} \mathbf{pA} \tag{2.12}$$

links the initial state $|b,0\rangle$ with a set of states $\langle \alpha_j|$ in which the atom is in the ground state, and there is one photon in the jth oscillator. One can calculate various characteristics of the process of radiation; we will discuss the decay law defined by the square of the modulus of the diagonal matrix element of the evolution operator (see Eqs. (1.3) and (1.37))

$$U_{b0,\,b0} = \frac{1}{2\pi i} \int_C \mathrm{d}z e^{-izt/\hbar} R_{b0,\,b0}. \tag{2.13}$$

As was shown in section 1.2, when one calculates the diagonal matrix element the largest amplitude is found in the residues at pole E_{b0} and its neighbors (which are separated from it by a distance of the order of V). It is easy to understand that in the examined case all such poles will be taken into account if, in addition to the pole at E_{b0} poles E_{aj} will also be retained in $F(z)$ which correspond in energy to the states $|a,j\rangle$, in which the atom is in the ground state and the jth oscillator has one photon. Thus, the function $1 - F(z)$ should be written in the form

$$1 - F(z) = 1 - \frac{A_{b0}}{z - E_{b0}} - \sum_j \frac{A_{aj}}{z - E_{aj}}, \tag{2.14}$$

where

$$A_{b0} = \sum_j \frac{|V_{b0,\,aj}|^2}{E_{b0} - E_{aj}} + \ldots, \quad A_{aj} = -\frac{|V_{b0,\,bj}|^2}{E_{b0} - E_{aj}}, \tag{2.15}$$

$$E_{b0} = E_b, \quad E_{aj} = E_a + \hbar\omega_j. \tag{2.16}$$

Thus, Eq. (2.14) acquires the form

$$1 - F(z) = 1 - \sum_j \frac{|V_{b0,\,aj}|^2}{(z - E_{b0})(z - E_{aj})}. \tag{2.17}$$

According to Eq. (1.43)

$$\frac{\partial F(z)}{\partial V_{b0,\,b0}} \approx \frac{1}{z - E_{b0}}. \tag{2.18}$$

The expansions of Eqs. (2.15) and (2.16) consider only those terms which make the main contribution to the evolution operator, and it is these terms which are considered heuristically in the Wigner-Weisskopf method. In the resolvent method, when necessary, these expansions can be refined, and even all the series can be written out completely (Sec. 1.2, Eqs. (1.39)–(1.41)).

Thus, in the examined case we obtain

$$U_{b_0,\,b_0}(t)=\frac{1}{2\pi i}\int\limits_C dz e^{-izt/\hbar}\left[(z-E_b)-\sum_j\frac{|V_{b_0,\,aj}|^2}{z-\hbar\omega_j}\right]^{-1},\tag{2.19}$$

where E_a is set to zero.

In the interaction Hamiltonian V there is an operator for the momentum of the atomic electron. The diagonal elements of this operator, which corresponds to transitions without a change in energy, are equal to zero; thus, for a two-level atom it has the form

$$\mathbf{p}=\mathbf{p}_0\hat{\sigma},\quad \hat{\sigma}=|b\rangle\langle a|+|a\rangle\langle b|.\tag{2.20}$$

For simplicity it is assumed that the parameter $\mathbf{p}_0$ is real and directed along $\boldsymbol{\varepsilon}$ at the location of the atom and its modulus is equal to $m\Omega r$, where r is the modulus of the matrix element of the coordinates for the transition $b\to a$, and Ω is the frequency of the transition $(E_b-E_a)/\hbar$. Then, taking into consideration Eqs. (2.12), (2.11), and (2.16), we obtain

$$V=-g\sum_j(\omega_jL_c)^{-1/2}(a_j^+e^{ix_jz}+a_je^{-ix_jz})\hat{\sigma},\tag{2.21}$$

where

$$g=\varepsilon\hbar\Omega\left(2\pi\alpha\frac{r^2}{S}\right)^{1/2}\quad\left(\alpha=\frac{e^2}{\hbar c}\right).\tag{2.22}$$

Consequently, for a matrix element linking the states $|b,0\rangle$ and $\langle a,j|$ in a dipole approximation in which the change in field within the atom is ignored, we obtain the expression

$$V_{a,j;\,b,0}=\langle a,j|V|b,0\rangle=-g\left(\omega_j\frac{L}{c}\right)^{-1/2}.\tag{2.23}$$

In the limiting transition $L\to\infty$, the sum in the integrand of Eq. (2.19), considering Eq. (2.10), changes to the integral

$$\sum_j\frac{|V_{a,j;\,b_0}|^2}{z-\hbar\omega_j}\Rightarrow I=\int\limits_0^\infty d\varkappa\frac{L}{2\pi}\frac{g^2c/\omega L}{z-\hbar\omega};\tag{2.24}$$

as we see, the dependence of the integral on the length of the waveguide L drops out, and the limiting value of the integral for $L\to\infty$ is obtained automatically

$$I=\frac{g^2c}{2\pi}\int\limits_0^\infty d\varkappa\frac{1}{\omega(z-\hbar\omega)}.\tag{2.24$'$}$$

This integral describes the interaction of the atom with the oscillators. The contribution of each oscillator, which is proportional to $|V_{a,j,b,0}|^2$, decreases as L increases, but the number of oscillators per unit interval $\varkappa$ increases, so the total interaction remains finite.

Let us consider the dependence of ω on $\varkappa$ (Eq. (2.5)

$$\varkappa=c^{-1}(\omega^2-\omega_c^2)^{1/2},\quad \omega_c=c\varkappa_c,\quad d\varkappa=d\omega\cdot\omega[c(\omega^2-\omega_c^2)^{1/2}]^{-1}\tag{2.25}$$

and let us introduce the dimensionless frequencies

$$\nu=\frac{\omega}{\Omega},\quad \nu_c=\frac{\omega_c}{\Omega},\quad d\nu=\frac{d\omega}{\Omega},\quad \zeta=\frac{z}{\hbar\Omega},\tag{2.26}$$

where Ω is the frequency of transition of the atom; then the integral in Eq. (2.24) acquires the form

$$I=\frac{g^2}{2\pi\hbar\Omega}\int\limits_{\nu_c}^\infty\frac{d\nu}{(\zeta-\nu)(\nu^2-\nu_c^2)^{1/2}}.\tag{2.27}$$

In evaluating this integral one must consider the fact that ζ is not a real number, because it lies on contour C (see Fig. 1). The integral in Eq. (2.27) is evaluated using standard methods

$$I=\frac{g^2}{2\pi\hbar\Omega}\frac{1}{(\zeta^2-\nu_c^2)^{1/2}}\ln\frac{(\zeta-\nu_c)^{1/2}+(\zeta+\nu_c)^{1/2}}{(\zeta-\nu_c)^{1/2}-(\zeta+\nu_c)^{1/2}}.\tag{2.28}$$

One can verify that the logarithmic expression, when $\nu_c\neq0$, does not go to zero or infinity for any finite ζ; consequently, the branching points of the logarithm play no role. The branching points of the roots are significant; the correct choice of the sheet of the Riemann surface for roots is determined by a condition which proceeds from the form of the integral in Eq. (2.27), and for real $\zeta<\nu_c$ the integral should be real, negative, and its modulus should decrease when $\zeta\to-\infty$. An investigation shows that Eq. (2.28) has only one branching point at $\zeta=\nu_c$ and a cut for real $\zeta>\nu_c$. Establishing these circumstances from Eq. (2.28) is a rather cumbersome matter, but they follow relatively easily from the form of the integral in Eq. (2.27); indeed, this integral is a Cauchy type integral, and it is known[6] that the Riemann surfaces of these integrals, have a cut along the integration contour, and the ends of this contour are branching points of the integral, which is examined as a function of the complex variable ζ.

Actually, the integral in Eq. (2.27) has no singularities for real $\zeta<\nu_c$ and for complex ζ if $\operatorname{Im}\zeta\neq0$. Consequently, it is an analytical function of ζ over the entire plane, with the exception of the cut from $\zeta=\nu_0$ to $\zeta\to\infty$ along the real axis.

At the upper edge of the cut ($\operatorname{Im}\zeta\downarrow0$) the integral acquires the following value ($\operatorname{Re}\zeta\to\nu_c$)

$$I=\frac{g^2}{2\pi\hbar\Omega}\frac{1}{(\zeta^2-\nu_c^2)^{1/2}}\left[\ln\frac{(\zeta+\nu_c)^{1/2}+(\zeta-\nu_c)^{1/2}}{(\zeta+\nu_c)^{1/2}-(\zeta-\nu_c)^{1/2}}-i\pi\right];\tag{2.29}$$

if in this expression the sign of i is reversed, one obtains the value of the integral at the lower edge of the cut ($\operatorname{Im}\zeta\uparrow0$). Equation (2.29) is a special case of the Sokhotskiĭ–Plemel formulas[6] for Cauchy type integrals. In many radiation problems, in particular in the one we are examining, the integral I may be analytically continued beyond the cut. This makes it possible to make broad use of the theory of functions of a complex variable in the calculations.

For real $\zeta<\nu_c$ the integral acquires the following value:

$$I=\frac{g^2}{2\pi\hbar\Omega}\frac{-2}{(\nu_c^2-\zeta^2)^{1/2}}\operatorname{arctg}\left(\frac{\nu_c+\zeta}{\nu_c-\zeta}\right)^{1/2},\quad -\nu_c<\zeta<\nu_c,\tag{2.30}$$

$$=\frac{g^2}{2\pi\hbar\Omega}\frac{-1}{(|\zeta|^2-\nu_c^2)^{1/2}}\ln\frac{(|\zeta|+\nu_c)^{1/2}+(|\zeta|-\nu_c)^{1/2}}{(|\zeta|+\nu_c)^{1/2}-(|\zeta|-\nu_c)^{1/2}},$$

$$\zeta<-\nu_c.\tag{2.31}$$

Let us return to an examination of the matrix element of the evolution operator of Eq. (2.13); we write it in the form

$$U_{b_0,\,b_0}(t)=\frac{1}{2\pi i}\int\limits_C d\zeta e^{-i\Omega\zeta t}F^{-1}(\zeta)\tag{2.32}$$

and recall that integration is performed along contour C (see Fig. 1).

As established above, the integral I, along with the inte-

grand in Eq. (2.32), has a cut from ν_c to ∞ along the real axis. This cut lies within the contour C. Separating the integration in Eq. (2.32) into upper (u) and lower (d) parts of the C contour, we obtain

$$U_{b0,\,b0}(t) = \frac{1}{2\pi i}\int\limits_{\hbar\omega_c}^{\infty} dz\,\frac{e^{-izt/\hbar}}{\Delta_{b0}^{(d)}} - \frac{1}{2\pi i}\int\limits_{\hbar\omega_c}^{\infty} dz\,\frac{e^{-izt/\hbar}}{\Delta_{b0}^{(u)}}.$$

The denominators $\Delta_{b0}^{(d)}$ and $\Delta_{b0}^{(u)}$, according to Eq. (2.29), can be written in the form

$$\Delta_{b0}^{(d,\,u)} = z - E_b - \Delta E_b(z) \mp i\Gamma(z),$$

where

$$\Delta E_b(z) = \frac{g^2}{4\pi}\,\frac{\ln\dfrac{(z+\hbar\omega_c)^{1/2}+(z-\hbar\omega_c)^{1/2}}{(z+\hbar\omega_c)^{1/2}-(z-\hbar\omega_c)^{1/2}}}{(z^2-\hbar^2\omega_c^2)^{1/2}},$$

$$\Gamma(z) = \frac{g^2}{4\,(z^2-\hbar^2\omega_c^2)^{1/2}}.$$

Consequently, the matrix element $U_{b0,b0}(t)$ can be written in the form

$$U_{b0,\,b0}(t) = \frac{-1}{\pi}\int\limits_{\hbar\omega_c}^{\infty} dz\,\frac{\Gamma(z)\,e^{-izt/\hbar}}{[z-E_b-\Delta E_b(z)]^2+\Gamma_b^2(z)}\ ; \quad (2.33)$$

in this form the spectral structure of the matrix element is clearly revealed. In particular it is obvious that the decay law is not strictly exponential; if ΔE_b and Γ_b did not depend on z, and the lower limit was equal to $-\infty$, the decay law would be strictly exponential. As we will see later, however, the deviation of the decay law from exponentiality is small.

The integration contour C may be deformed in different ways on the Riemann surface of the integral expression; because of this the matrix element $U_{b0,b0}(t)$ may be written in various forms. One of the methods of deformation is as follows. The function $F(\zeta)$ does not have zeroes in the lower half-plane and the exponent $e^{-i\Omega\zeta t}$ falls rapidly as $\mathrm{Im}\,\zeta \to -\infty$ ($t>0$), so by shifting the lower part of the contour C downward (Fig. 4) one can verify that the integral along this portion of the contour is equal to zero. An attempt to shift the upper part of the contour C in the same way is not successful, because for $\mathrm{Re}\,\zeta > \nu_c$ the contour will move to another sheet of the Riemann surface of the function $F(\zeta)$ (Fig. 5); this function, as well as the integral I contained in it, it has a branching point at $\zeta = \nu_c$. The upper part of contour C can be deformed only to a contour consisting of two parts (Fig. 6); contour C', which encloses the zero of the function $F(\zeta)$, that is, the pole of the integrand in Eq. (2.13), and contour C'', which encloses the branching point of function $F(\zeta)$ at $\zeta = \nu_c$. It should be stressed that the pole

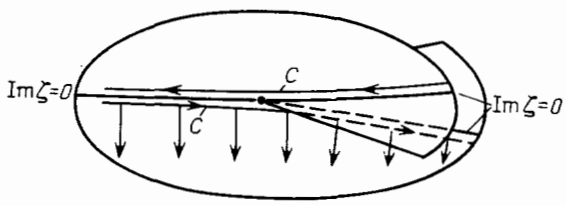

FIG. 4.

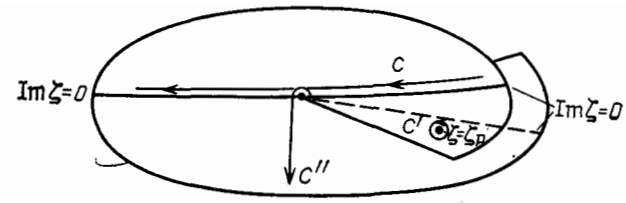

FIG. 5.

of the integrand in Eq. (2.33) [or the zero of $F(\zeta)$] lies on another sheet of the Riemann surface, and not on the one containing the lower part of contour C; thus the lower part of contour C can be shifted downward without "grazing" the pole at $\zeta = \zeta_p$.

It is easy to show that function $F(\zeta)$ has a zero which lies somewhat below the real axis if this function is written in the form

$$F(\zeta) = (\zeta - 1) - \frac{g^2}{4\pi\,(\hbar\Omega)^2}\,\frac{1}{(\zeta^2-\nu_c^2)^{1/2}} \times\left[\ln\frac{(\zeta+\nu_c)^{1/2}+(\zeta-\nu_c)^{1/2}}{(\zeta+\nu_c)^{1/2}-(\zeta-\nu_c)^{1/2}} - i\pi\right]. \quad (2.34)$$

We will examine the second term in this expression as a perturbation; then the unperturbed value of ζ, at which $F(\zeta) = 0$, is equal to unity, and in the first order of perturbation the value of ζ can be sought in the form

$$\zeta_p = 1 + \Delta\zeta + i\gamma,$$

where $\Delta\zeta$ and γ are small quantities. We substitute this value of ζ into the equation $F(\zeta) = 0$; one can then replace the term in parentheses, which is a perturbation, with the unperturbed quantity ζ, that is, unity:

$$\Delta\zeta + i\gamma - \frac{R}{(1-\nu_c^2)^{1/2}}\left[\ln\frac{(1+\nu_c)^{1/2}+(1-\nu_c)^{1/2}}{(1+\nu_c)^{1/2}-(1-\nu_c)^{1/2}} - i\pi\right] = 0, \quad (2.35)$$

$$R = \frac{g^2}{4\pi\,(\hbar\Omega)^2} = \frac{e^2}{\hbar c}\,\frac{r^2}{2S}. \quad (2.36)$$

Then we obtain

$$\Delta\zeta \approx -R\ln\nu_c, \quad \gamma \approx -\pi R. \quad (2.37)$$

We note that γ is negative, that is, the root of $F(\zeta)$ lies below the real axis. However, this root lies on a sheet of the Riemann surface of function $F(\zeta)$, which is the one containing the lower part of contour C since to find it the expression $F(\zeta)$ for the upper edge of the cut was used; it was analytically continued beyond the cut. In this case, because perturbation theory is used, the analytical continuation involves the assumption that the perturbation in Eq. (2.33) is the same as

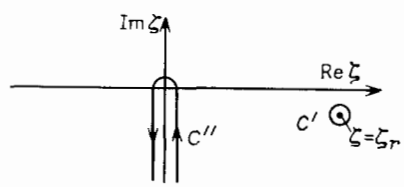

FIG. 6.

on the upper edge, that is, it is assumed that the perturbation is virtually unchanged when ζ is shifted downward by the small quantity γ.

For estimates we set

$$\frac{e^2}{\hbar c} \approx 10^{-2}, \quad r^2 \approx 10^{-16}\ \text{см}^2,$$

$$S \approx \lambda^2 \approx 10^{-8}\ \text{см}^2, \quad \nu_c \approx 10^{-3}, \quad \bar{\varepsilon}^2 \approx 1;$$

then

$$R = 0{,}5 \cdot 10^{-10}, \ \Delta\zeta = 3{,}5 \cdot 10^{-10}, \ \gamma = -1{,}6 \cdot 10^{-10}.$$

As will be seen below, the integral along the C'' contour, which encloses the branching point, describes small deviations of the decay law from exponentiality. However, these deviations are small, much smaller than can be observed experimentally. The main contribution to $U_{b0,b0}(t)$ yields a residue at point $\zeta = \zeta_p$

$$U_{b0,b0}(t) \approx \exp\left[\gamma\Omega t + i(1 + \Delta\zeta)\,\Omega t\right], \tag{2.38}$$

that is, the decay law for the excited state of the atom is found to be exponential to a high degree of accuracy.

The problem of the decay of the excited state of an atom in a waveguide is thereby solved. One should note an important feature of this solution. The solution is presented for the sake of this feature, which is that the resolvent is now represented by the function $F^{-1}(\zeta)$, which has more complicated singularities than in the case of a discrete spectrum; now the resolvent is a multivalued function of ζ (or z), has a branching point and poles which do not lie on the real axis. In the following section the origin of these singularities will be traced in more detail.

2.3. Limiting transition from a discrete spectrum to a continuous spectrum

Let us trace on an analyzed example how the more complex properties of the resolvent arise, in particular, its multivalued nature in the transition from a discrete spectrum to a continuous spectrum. To do this we again examine the denominator of the integrand in Eq. (2.19) for the case where the waveguide is assumed to be limited in length, and the spectrum of the entire system, consequently, is discrete.

The denominator of the integrand in Eq. (2.19) is written in the form

$$\hbar\Omega Y(\zeta) = \hbar\Omega\left[(\zeta - 1) - \frac{1}{2}\sum_j \frac{g^2/\nu_j L}{\hbar^2\Omega^3(\zeta - \nu_j)}\right] = \hbar\Omega\left[(\zeta - 1) - D\sum_j \frac{1}{\nu_j(\zeta - \nu_j)}\right], \tag{2.39}$$

where $D = R\lambda L^{-1}$. Figure 6 gives graphs of the first and second terms in square brackets in Eq. (2.19). The intersections of these graphs correspond to the roots of the function $Y(\zeta)$

$$Y(\zeta_n) = 0. \tag{2.40}$$

As we see, the roots of function $Y(\zeta)$ are situated exclusively on the real axis, and the function $Y(\zeta)$ itself is single-valued. Consequently, the integrand in Eq. (2.19) also is a single-valued function of ζ and has only poles, which are all located on the real axis.

The Riemann surface of the function $Y(\zeta)$, as in any single-valued function, has a single sheet; it retains its single sheet nature at any stage of the limiting transition $L \to \infty$, that is, for any density of the spectrum of eigenfrequencies of the waveguide and of the "picket fence" of the poles of the integrand in Eq. (2.13) on the real axis. Above we saw that for a waveguide of infinite length the function $F(\zeta)$, which is the limiting value of $Y(\zeta)$, is multivalued, and its Riemann surface has multiple sheets; consequently, the multivalued nature of $F(\zeta)$ and the multiplicity of the sheets of its Riemann surface arise discontinuously "at the end" of the limiting transition $L \to \infty$. Usually that sheet of the Riemann surface of the function $F(\zeta)$, which is obtained after the limiting transition from the single sheet of the Riemann surface of the function $Y(\zeta)$ is called a physical sheet. Additional sheets, which may be obtained after the limiting transition by analytically continuing the function $F(\zeta)$ beyond the cut, are called nonphysical sheets. Thus, one can say that nonphysical sheets of the Riemann surface arise abruptly in the limiting transition; correspondingly, the pole on one of the nonphysical sheets also arises discontinuously when $\zeta = \zeta_p$.

Abrupt changes in the process of the limiting transition are exceptional in physics; usually there are special reasons for such changes, and this is also true for the case we are examining. We shall show that the physical quantities in the case we are examining attain their limiting values without abrupt jumps. The abrupt jump occurs only in the calculation, in the mathematical formalism.

And so, we turn again to the matrix element in Eq. (2.19). The main contribution to it is made by residues at the roots near $\zeta = 1$. Indeed, the value of the residue is defined by the derivative $\partial Y/\partial\zeta$, which is taken in the appropriate root of the denominator; the derivative $\partial Y/\partial\zeta$ is the sum of the derivatives of the first and second terms in Eq. (2.39). The derivative of the first term is equal to unity; the derivative of the second term in the root of $Y(\zeta)$ goes to infinity as the root moves away from $\zeta = 1$, as can be seen in Fig. 7. Consequently, the residues taken at the roots of $Y(\zeta)$ that are rather distant from $\zeta = 1$, will be small, and will not make a noticeable contribution to the result.

The first term in the sum in Eq. (2.39), the frequency ν_j, changes only slightly as j changes in that region $\nu_j \sim \zeta = 1$, which makes the main contribution; thus, we will set it equal to unity. The function $Y(\zeta)$ acquires the following form

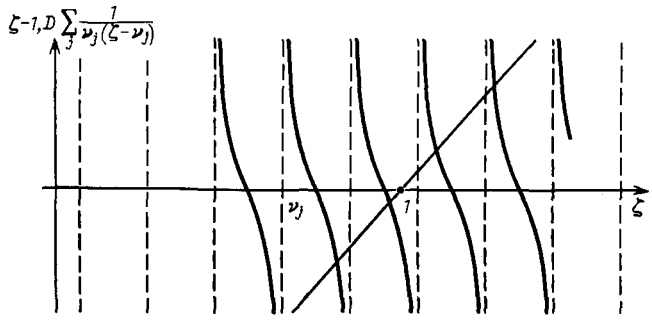

FIG. 7.

$$Y(\zeta) = (\zeta - 1) - D \sum_j \frac{1}{\zeta - \nu_j}. \qquad (2.41)$$

In the region $\nu_j \sim \zeta = 1$ the resonant frequencies of the waveguide are virtually uniformly distributed, so one can assume

$$\nu_j = 1 + \lambda L^{-1}\left(j + \frac{1}{2}\right), \quad j = 0, \pm 1, \pm 2, \ldots, \qquad (2.42)$$

consequently, for $Y(\zeta)$ we obtain the expression

$$Y(\zeta) = (\zeta - 1) - D \sum_j \left[(\zeta - 1) - \lambda L^{-1}\left(j + \frac{1}{2}\right)\right]^{-1}. \qquad (2.43)$$

Combining the terms corresponding to j and $-j-1$ in pairs, we obtain

$$Y(\zeta) = (\zeta - 1) - D \sum_{j=0}^{\infty} \frac{2(\zeta - 1)}{(\zeta - 1)^2 - (\lambda L^{-1})^2 (j + (1/2)^2} = (\zeta - 1) + \pi R \operatorname{tg}[\pi \lambda^{-1} L (\zeta - 1)], \qquad (2.44)$$

since[7]

$$\operatorname{tg} x = \sum_{j=0}^{\infty} \frac{2x}{\{\pi[j + (1/2)]\}^2 - x^2}.$$

Let us calculate the derivative of $Y(\zeta)$. As indicated it defines the value of the residue at the root of $Y(\zeta)$ in the integral in Eq. (2.19)

$$\frac{\partial Y}{\partial \zeta} = 1 + \pi^2 R \lambda^{-1} L \operatorname{tg}'[\pi \lambda^{-1} L (\zeta - 1)] = 1 + \pi^2 R \lambda^{-1} L \{1 + \operatorname{tg}^2[\pi \lambda^{-1} L (\zeta - 1)]\};$$

since the second term is proportional to L, as $L \to \infty$ the first unit in this expression can be ignored. Finally, then, we obtain

$$\left.\frac{\partial Y}{\partial \zeta}\right|_{\zeta = \zeta_n} \approx \pi^2 R \lambda^{-1} L \{1 + \operatorname{tg}^2[\pi \lambda^{-1} L (\zeta - 1)]\} = (R\lambda)^{-1} L [(\zeta_n - 1)^2 + (\pi R)^2], \qquad (2.45)$$

because in the roots, according to Eqs. (2.40) and (2.44) there is the equality

$$\operatorname{tg}[\pi \lambda^{-1} L (\zeta_n - 1)] = -(\pi R)^{-1} (\zeta_n - 1),$$

where n is the number of the root.

Consequently, after integration in Eq. (2.19) along the contour C we obtain

$$U_{b0,\,b0}(t) \approx (\hbar\Omega)^{-1} \sum_{n=-\infty}^{+\infty} \frac{e^{-i\Omega\zeta_n t}}{(\partial Y/\partial\zeta)_{\zeta=\zeta_n}} = R\lambda L^{-1} (\hbar\Omega)^{-1} \sum_n \frac{e^{-i\Omega\zeta_n t}}{(\zeta_n - 1)^2 + (\pi R)^2}; \qquad (2.46)$$

as we see, the spectrum of the matrix element $U_{b0,b0}(t)$ has a Lorentzian form. It is more accurate to say that the integrand in the following integral has a Lorentzian form

$$\int_{-\infty}^{+\infty} d\zeta \frac{dn}{d\zeta} \frac{e^{-i\Omega\zeta t}}{(\zeta - 1)^2 + (\pi R)^2},$$

for which the sum in Eq. (2.46) is an integral sum. Consequently, the sum is approximately equal to the integral for large L.

Let us turn again to Fig. 7 and examine the position of the roots of $Y(\zeta)$, that is, the intersections of the two graphs in this figure. If the root is far from $\zeta = 1$, then it is close to the corresponding resonance of the waveguide ν_j; this occurs when $|j| \gtrsim \pi R \lambda^{-1} L$. Correspondingly, in these regions the position of the roots is almost strictly equidistant. As the roots approach $\zeta = 1$ the distance between them gradually decreases and reaches a minimum when the roots are close to $\zeta = 1$. Then it is easy to see that in the region

$$-\pi R \lambda^{-1} L \lesssim j \lesssim \pi R \lambda^{-1} L$$

one extra root appears compared with the number of waveguide resonances ν_j in this region; thus, the average distance between roots in this region differs from the distance between the roots outside of this region by approximately

$$\frac{\lambda^2}{2\pi R L^2}$$

which is a quantity of the second order of smallness in terms of the parameter λ/L. This leads to the thought that the matrix element $U_{b0,b0}$ can be calculated if it is assumed that the roots ζ_n are equidistant, and if small deviations from equidistance are considered as a perturbation.

Thus, let the roots be equidistant

$$\zeta_n = \zeta_n^0 = 1 + n\lambda L^{-1}; \qquad (2.47)$$

obviously, the zero root corresponds to $\zeta_0 = 1$. Then we have

$$U_{b0,\,b0}(t) = R\lambda L^{-1} (\hbar\Omega)^{-1} \sum_{n=-\infty}^{+\infty} \frac{\exp[-i\Omega(1 + n\lambda L^{-1})t}{n^2 (\lambda L^{-1})^2 + (\pi R)^2} = \frac{\pi R L e^{-i\Omega t}}{2ch} \sum_n \frac{\exp[-2\pi i n c t/L]}{(n\pi)^2 + (\pi^2 R \lambda^{-1} L)^2}. \qquad (2.48)$$

One can verify that the sum is given by

$$\sum_{n=-\infty}^{+\infty} \frac{e^{-in\pi T^{-1}t}}{(n\pi)^2 + (\Gamma T)^2} = \frac{1}{\Gamma T} \frac{e^{-\Gamma(t-T)} + e^{\Gamma(t-T)}}{e^{\Gamma T} - e^{-\Gamma T}} \quad \text{for} \quad 0 \leqslant t \leqslant 2T, \qquad (2.49)$$

that is, it is an expansion into a Fourier series of a periodic function which coincides in the period $2T$ with the catenary (Fig. 8). Consequently, for $U_{b0,b0}(t)$ when $0 \leqslant t \leqslant 2T$, we obtain the expression

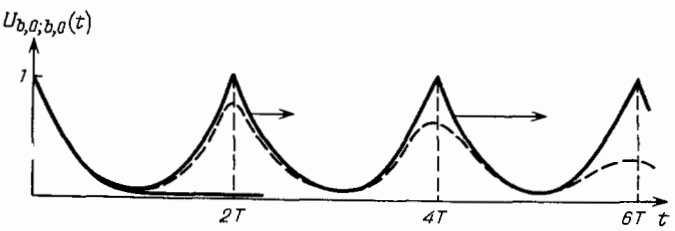

FIG. 8.

$$U_{b0,\,b0}(t)=\frac{e^{-i\Omega t}}{\hbar\Omega}\,\frac{e^{-\Gamma(t-T)}+e^{\Gamma(t-T)}}{e^{\Gamma T}-e^{-\Gamma T}}=\frac{e^{-i\Omega t}}{\hbar\Omega}\,\frac{e^{-\Gamma t}+e^{\Gamma t-2\Gamma T}}{1-e^{-2\Gamma T}}, \tag{2.50}$$

where

$$T=L/2c,\quad \Gamma=\pi\Omega R. \tag{2.51}$$

In the limit $T\to\infty$ $(L\to\infty)$, for finite t we have

$$U_{b0,\,b0}(t)=\frac{1}{\hbar\Omega}\,e^{-\Gamma t-i\Omega t}, \tag{2.52}$$

an exponential decay as obtained earlier.

Let us now give an estimate of those corrections to which we are led by a small deviation in the position of the roots of $Y(\zeta)$ from equidistance. Let us examine the difference between the exact and approximate expressions of Eq. (2.46)

$$\Delta U_{b0,\,b0}(t)=D(\hbar\Omega)^{-1}\times\sum_{n=-\infty}^{+\infty}\left[\frac{e^{-i\Omega\zeta_n t}}{(\zeta_n-1)^2+(\pi R)^2}-\frac{e^{-i\Omega\zeta_n^{(0)}t}}{(\zeta_n^{(0)}-1)^2+(\pi R)^2}\right]. \tag{2.53}$$

We expand the pre-exponential factor in the first term into a series in terms of small deviations of the roots from equidistant distribution, and we confine ourselves to the first approximation

$$\Delta U_{b0,\,b0}(t)=D(\hbar\Omega)^{-1}\sum_{n=-\infty}^{+\infty}e^{-i\Omega\zeta_n^{(0)}t}\left\{\frac{e^{-i\Omega\Delta\zeta_n t}-1}{(\zeta_n^{(0)}-1)^2+(\pi R)^2}-\frac{2e^{-i\Omega\Delta\zeta_n t}(\zeta_n^{(0)}-1)\,\Delta\zeta_n}{[(\zeta_n^{(0)}-1)^2+(\pi R)^2]^2}\right\}. \tag{2.54}$$

The numerator of the first term can be reduced to the form

$$-2ie^{(i/2)\Omega\Delta\zeta_n t}\sin\left(\frac{1}{2}\,\Omega\Delta\zeta_n t\right).$$

Since

$$\max|\Delta\zeta_n|\approx R^{-1}\left(\frac{\lambda}{\pi L}\right)^2,$$

for finite t, the numerator of the first term is a quantity of the order of $(\lambda/L)^2$. Since the numerator of the second term is proportional to a quantity of this same order of smallness $\Delta\zeta_n$, both terms are quantities of the second order of smallness relative to the parameter λ/L.

In the sum in Eq. (2.53) the number of significant terms is approximately equal to

$$2\pi R\lambda^{-1}L,$$

so this sum is of the order of λ/L. Consequently, corrections to the exponential law, Eq. (2.52), which are due to the nonequidistance of the spectrum, decrease inversely proportionally to L as the length of the waveguide increases.

Thus, the properties of the function $F^{-1}(\zeta)$ which lead to exponential decay, such as the existence of nonphysical sheets of the Riemann surface, a pole on this surface, etc., appear even before the end of the limiting transition $L\to\infty$, when the spectrum is already very dense, but is still, as initially, discrete, and the function $Y^{-1}(\zeta)$ is single-valued. Indeed, as we see, in the process of the limiting transition the decay law gradually approximates an exponential law, and its spectrum becomes Lorentzian. A Lorentz spectrum means that the pole lies next to the real axis. Figuratively speaking, the "picket fence" of poles of the function $Y^{-1}(\zeta)$, as it becomes more dense in the process of the limiting transition $L\to\infty$, becomes gradually transparent, and behind it there appears a nonphysical sheet of the Riemann surface of the function $F^{-1}(\zeta)$ with the singularities of this function lying on it.

One should also note that the pre-limit and post-limit behavior of the matrix element $U_{b0,b0}(t)$ are qualitatively different. Before the transition to the limit $U_{b0,b0}(t)$ behaves quasiperiodically, that is, in particular it approximately periodically repeats its initial value, which is equal to unity, with any pre-assigned accuracy. This means that excitation is quasiperiodically concentrated in the atom, in its excited state. This quasi-periodic behavior of the system is a manifestation of the well-known Poincaré cycle for quantum systems. As the spectrum condenses the length of the Poincaré cycle increases, and at the limit $L=\infty$ becomes infinite: the excitation never returns to the atom. In other words, when $L\to\infty$, the "reversibility" present for any finite L disappears. This is a characteristic of the continuous spectrum, which distinguishes it qualitatively from a discrete spectrum. It is only natural, therefore, that the mathematical descriptions of the pre-limit and post-limit behavior of $U_{b0,b0}(t)$ differ greatly.

2.4. Decay of an excited state of an atom in free space

As already noted, an electromagnetic field in free space has a continuous frequency spectrum, that is, waves of any frequency may be present in its spectrum. Systems with a continuous spectrum are examined in this paper as the limiting case of systems with a discrete spectrum in which the spectrum, depending on some parameter, becomes ever more dense. Thus, we will assume that in radiation problems the electromagnetic field and the radiating object (in this case, the radiating atom) are located in some cavity (resonator) of sufficiently large volume having resonant properties, that is, having a discrete spectrum. Then using the expressions for a discrete spectrum, we make the volume of the cavity tend to infinity, changing it in this way to a continuous spectrum. The interaction of the field of each individual mode (resonance) with the atom decreases, since the portion of this field in the volume of the atom decreases; however, the number of modes with which the atom interacts increases, so that the interaction with the field as a whole remains finite.

The simplest resonator is one in the form of a cube, with its own field subject to so-called periodic boundary conditions. Such a resonator has been repeatedly described in the literature in which radiation problems are examined[8] and, in particular, in the description of the radiation of an absolute blackbody.[9] Therefore we recall briefly only the basic facts associated with the description of such a resonator.

The vector potential operator of the quantized electromagnetic field of a cubic cavity with side L has the form $(V=L^3)$

$$\mathbf{A}(\mathbf{r}) = \sum_s \left(\frac{2\pi\hbar c^2}{\omega_s V}\right)^{1/2} (a_s^+ e^{-i\mathbf{k}_s\mathbf{r}} + a_s e^{i\mathbf{k}_s\mathbf{r}})\,\mathbf{e}_s, \tag{2.54a}$$

where the subscript s enumerates the fundamental oscillations (or oscillators) of the cavity, the wave vectors of which assume the values

$$\mathbf{k}_s = \left\{\frac{2\pi l_s}{L}, \frac{2\pi m_s}{L}, \frac{2\pi n_s}{L}\right\} \tag{2.55}$$

for integer l_s, m_s, and n_s. The frequency of the fundamental waves is determined by the condition

$$\omega_s = \frac{2\pi c}{L}(l_s^2 + m_s^2 + n_s^2)^{1/2}. \tag{2.56}$$

The fundamental waves may differ from each other not only in frequency and direction of the wave vector $\mathbf{k}$, but also in polarization. Two transverse and mutually orthogonal polarizations correspond to each frequency and wave vector (Coulomb gauge).

$$\mathbf{e}_{1,2}\mathbf{k} = 0, \quad \mathbf{e}_1\mathbf{e}_2 = 0. \tag{2.57}$$

Summation over all these oscillators is also performed in Eq. (2.54).

The operators for the creation and annihilation of photons a_s^+ and a_s obey the commutation relation

$$[a_s; a_{s'}^+] = \delta_{ss'}; \tag{2.58}$$

and the normalization constant in the vector potential is chosen so that the energy of the field is equal to

$$H_f = \sum_s \hbar\omega_s a_s^+ a_s. \tag{2.59}$$

The states of the atomic Hamiltonian H_a will be assumed to be known, and, without specifying them, we will denote them as $|g\rangle$. Thus the stationary states of the unperturbed Hamiltonian of the full system, atom + electromagnetic field, can be written in the form

$$|j\rangle = |g_j; n_j^{(1)}, n_j^{(2)}, n_j^{(3)} \ldots\rangle, \tag{2.60}$$

where the first subscript letter, g_j indicates the state of the atom, and the integer $n_j^{(k)}$ is equal to the number of photons in the k th oscillator. The energy of such a state is equal to

$$E_j = E_{gj} + \sum_k n_j^{(k)}\hbar\omega_k. \tag{2.61}$$

The process of spontaneous radiation corresponds to the initial state

$$|i\rangle \equiv |b; 0, 0, 0, \ldots\rangle \equiv |b, 0\rangle, \tag{2.62}$$

that is, in the initial time the atom is in an excited state $|b\rangle$, and photons are absent in all the oscillators $|0\rangle = |0,0,0,\ldots\rangle$.

Perturbation in the Hamiltonian consists of two terms

$$H' = V + W. \tag{2.63}$$

The first term defines the interaction of the quantum system (in this case, the atom) with the electromagnetic field, and has the form

$$V = -\frac{e}{mc}\mathbf{p}\mathbf{A}(\mathbf{r}) + \frac{e^2}{2mc^2}\mathbf{A}^2(\mathbf{r}); \tag{2.64}$$

below the main role belongs to the first term of this sum, the second will be omitted (discussion of the role of the second term of this sum may be found, for example, in Ref. 10).

The second term in the perturbation (2.63) is called a renormalization counterterm and has the form

$$W = \frac{1}{2\mu}\mathbf{p}^2, \tag{2.65}$$

that is, it has the same structure as the kinetic energy of the electron. The inclusion of this term in Eq. (2.63) is the initial stage of the process of mass renormalization. The need for this process arises due to the fact that the interaction in Eq. (2.64) of the atom with a transverse electromagnetic field leads to large corrections to the energy of the states of the atoms, which, however, have the structure of kinetic energy and thus can be interpreted as a change in the mass of the electron. This terminology is frequently used. The quantity M, which is defined by the equation

$$\frac{1}{M} = \frac{1}{m} + \frac{1}{\mu}, \tag{2.66}$$

is called the bare mass of the electron, and the mass m is the true or observed mass of the electron. How μ should be chosen will be made clear later.

A full program of renormalizations includes also renormalization of the charge and may be consistently implemented only in the framework of relativistic quantum electrodynamics. Here we will confine ourselves to only the initial stage of mass renormalization in the framework of nonrelativistic theory; fortunately, this requires borrowing of only one parameter from relativistic theory, the cut-off parameter or limit frequency up to which the electron interacts effectively with the electromagnetic field. Bethe[11] was the first to perform such a nonrelativistic renormalization.

Now we have all the necessary information to study the process of spontaneous radiation of an atom in free space. The decay law for the state [Eq. (2.62)] is described by the matrix element

$$U_{b0,\,b0}(t) = \langle b,\ 0\,|\,\hat{U}(t)\,|\,b,\ 0\rangle, \tag{2.67}$$

where $U(t)$ is the evolution operator of the system. For this matrix element, according to Eq. (2.13), we have the expression

$$U_{b0,\,b0}(t) = \frac{1}{2\pi i}\int_C dz e^{-izt/\hbar} R_{b0,\,b0}(z), \tag{2.68}$$

where, according to Eq. (1.19),

$$R_{b0,\,b0} = -\frac{\partial\Delta/\partial V_{b0,\,b0}}{\Delta}. \tag{2.69}$$

Let us first study the denominator of this expression, which is equal, according to Eq. (1.35), to

$$\Delta(z) = \prod_l (z - E_l^0) - \sum_k V_{kk}\prod_{l\neq k}(z - E_l) + \frac{1}{2!}\sum_{k,j}(V_{kk}V_{jj} - V_{kj}V_{jk})\prod_{l\neq k,\,j}(z - E_l^0) + \ldots \tag{2.70}$$

As already stated, perturbation in the Hamiltonian consists of two terms. It will be shown later that perturbation V has nondiagonal matrix elements which are proportional to the interaction constant (equal to the square root of the fine structure constant $e^2/\hbar c$), and perturbation W has diagonal

matrix elements proportional to the square of the interaction constant ($1/\mu \sim e^2/\hbar c$). If in Eq. (2.70) we confine ourselves to second-order terms in the interaction constant, then the equation acquires the following form

$$\Delta(z) = \prod_l (z - E_l^0) - \sum_k \left(V_{kk} + \sum_j \frac{V_{kj}V_{jk}}{z - E_j^0} \right) \prod_{l \neq k} (z - E_l^0) \ldots,$$

where the first term is of zero order in the interaction constant and the second, is the sum of two terms of second order in this constant. Thus, the denominator in Eq. (2.69) can be written in the form

$$\Delta(z) = \prod_l \Delta_l(z) \equiv \prod_l \left[z - E_l^0 - \left(V_{ll} + \sum_j \frac{V_{lj}V_{jl}}{z - E_j^0} \right) \right]. \tag{2.71}$$

For the numerator in $R_{b0,b0}$, according to Eq. (1.18), we have the expression

$$D_{b0,\,b0} = -\frac{\partial \Delta(z)}{\partial V_{b0,\,b0}} = \prod_{l \neq b_0} \left[z - E_l^0 - \left(V_{ll} + \sum_j \frac{V_{lj}V_{jl}}{z - E_j^0} \right) \right]. \tag{2.72}$$

Consequently, the matrix element of Eq. (2.68) is found to be equal to

$$U_{b0,\,b0}(t) = \frac{1}{2\pi i} \int_C dz \frac{e^{-izt/\hbar}}{\Delta_{b0}(z)}, \tag{2.73}$$

where

$$\Delta_{b0} = z - E_{b0} - \left(V_{b0,\,b0} + \sum_j \frac{V_{b0,\,j}V_{j,\,b0}}{z - E_j^0} \right). \tag{2.74}$$

As we can see, all the factors in the denominator of Eq. (2.71), except for one, cancel out the corresponding factors of the numerator. It is necessary here to make two comments. First, this cancellation is approximate, because the numerator has, strictly speaking, other higher-order terms of the interaction constant. If they were considered, then they would not be cancelled and it would be necessary to consider the poles of the integrand in Eq. (2.68) due to other factors in $\Delta(z)$; however, the residues at these poles would be small, because the main term in the numerator of Eq. (2.72) is equal to zero due to the presence in it of the same terms that are in the denominator. This justifies the cancellation performed above.

Second, we note that the factor remaining after the cancellation, $\Delta_{b0}(z)$ is characteristic specifically for the problem of spontaneous radiation; in another problem, for example, in the study of induced radiation, the main role is played by other factors in $\Delta(z)$.

Let us now turn to the evaluation and study of the denominator of Eq. (2.74) in the integrand of Eq. (2.73). The first term in parentheses in Eq. (2.74) is a diagonal matrix element of the second term in the perturbation of Eq. (2.63). Taking Eq. (2.65) into consideration, we obtain

$$V_{b0,\,b0} = \langle b, 0 | W | b, 0 \rangle = \frac{1}{2\mu} \langle b | \mathbf{p}^2 | b \rangle; \tag{2.75}$$

and we note that this matrix element does not depend on the state of the field. The matrix element of the square of the momentum is written in the form

$$\langle b | \mathbf{p}^2 | b \rangle = \sum_g \langle b | \mathbf{p} | g \rangle \langle g | \mathbf{p} | b \rangle = \sum_g |\mathbf{p}_{bg}|^2$$

and for $V_{b0,b0}$ we obtain the expression

$$V_{b0,\,b0} = \frac{1}{2\mu} \sum_g |\mathbf{p}_{bg}|^2. \tag{2.76}$$

To seek the second term in parentheses in Eq. (2.74) one must know the nondiagonal matrix elements of the operator

$$V = -\frac{e}{mc} \mathbf{p}\mathbf{A}(\mathbf{r}). \tag{2.77}$$

For a matrix element $V_{b0,gj}$ we obtain the expressions

$$V_{b0,\,gj} = \langle b, 0 | V | g, j \rangle = -\frac{e}{m} \left(\frac{2\pi\hbar}{\omega_j V} \right)^{1/2} ((\mathbf{p}e^{i\mathbf{k}_j\mathbf{r}})_{bg}\, \mathbf{e}_j); \tag{2.78}$$

it is easy to see that these matrix elements are nonzero only in the case when in the ket-state in one of the oscillators (in the jth oscillator) contains one photon, because the interaction is linear in the creation and annihilation operators of photons.

Let us discuss the question of dipole approximation. At first glance it appears that in the sum

$$S = \frac{1}{2!} \sum_{g,\,j} \frac{V_{b0,\,gj}V_{gj,\,b0}}{z - E_{gj}^0} = \frac{\pi e^2 \hbar}{mV} \sum_{g,j} \frac{((\mathbf{p}\mathbf{e}_j)\, e^{i\mathbf{k}_j\mathbf{r}})_{bg} (e^{-i\mathbf{k}_j\mathbf{r}} (\mathbf{p}\mathbf{e}_j))_{gb}}{\omega_j (z - E_g - \hbar\omega_j)} \tag{2.79}$$

the exponential becomes significant when the frequency increases, beginning with wavelengths comparable with the Bohr radius ($\sim 10^{-8}$ cm). In other words, it is assumed that the dipole approximation in which the exponential in Eq. (2.78) is replaced by unity, becomes unsuitable for wavelengths shorter than the Bohr radius. However, with respect to Eq. (2.79) this is not so.

Indeed, first, we note that in Eq. (2.79) only those values of g are significant for which the energies E_b and E_g are close, because as E_g increases with respect to E_b, the matrix elements of Eq. (2.78) decrease rather rapidly, since the $|g\rangle$ states become ever more widely distributed. As a result E_g cannot take on values which are too large.

Second, the frequencies which correspond to wavelengths less than the Bohr radius are so high that $\hbar\omega$ is approximately a factor of 10^4 larger than E_g; in the denominator of Eq. (2.80) one can ignore the term E_g compared to $\hbar\omega$; consequently, the denominator becomes independent of the subscript g and in the numerator we obtain the expression

$$\sigma = \sum_g |((\mathbf{p}\mathbf{e}_j)\, e^{i\mathbf{k}_j\mathbf{r}})_{bg}|^2 = \sum_g \langle b | (\mathbf{p}\mathbf{e}_j)\, e^{i\mathbf{k}_j\mathbf{r}} | g \rangle \langle g | e^{-i\mathbf{k}_j\mathbf{r}} (\mathbf{p}\mathbf{e}_j) | b \rangle.$$

Summation over g yields the identity operator, and for σ we obtain the expression

$$\sigma = \langle b \,|\, (\mathbf{pe}_j)\, e^{i\mathbf{k}_j\mathbf{r}} e^{-i\mathbf{k}_j\mathbf{r}}\, (\mathbf{pe}_j) \,|\, b\rangle = \sum_g \langle b \,|\, \mathbf{pe}_j \,|\, g\rangle \langle g \,|\, \mathbf{pe}_j \,|\, b\rangle$$

$$= \sum_g |\, (\mathbf{pe}_j)_{bg}\,|^2.$$

Consequently, for all values of ω, the exponential in Eq. (2.79) can be omitted and this expression is found to be equal to

$$S = \frac{\pi e^2 \hbar}{mV} \sum_{g,\,j} \frac{|\, (\mathbf{pe}_j)_{bg}\,|^2}{\omega_j (z - E_g - \hbar\omega_j)}\,, \qquad (2.80)$$

In this expression it is assumed that there is summation over all the field oscillators, including those which do not differ in their energy E^0_{gj}, but differ only in the direction of propagation of the corresponding waves and in their polarizations. Thus, summing the squares of the moduli in Eq. (2.78) over polarizations and directions

$$\sum_{\substack{\text{polariz.}\\ \text{direct.}}} |\, V_{b0,\,gj}\,|^2 \Rightarrow \frac{2e^2\hbar}{3\pi c m^2} |\, \mathbf{p}_{bg}\,|^2\, k\, dk = \frac{2e^2\hbar}{3\pi c^3 m^2} |\, \mathbf{p}_{bg}\,|^2\, \omega\, d\omega.$$

Taking into account the fact that the energy of the state E^0_{gj} is equal to $E^0_{gj} = E_g + \hbar\omega_j$, we obtain for the sum S the expression

$$S = \frac{e^2\hbar}{3\pi c^3 m^2} \sum_g |\, \mathbf{p}_{bg}\,|^2 \int_0^{\omega_{\max}} \frac{\omega d\omega}{z - E_g - \hbar\omega}\,. \qquad (2.81)$$

As already noted, the interaction of the electron with the field is described by the matrix element in Eq. (2.78) only to some limit frequency $\omega_{\max}$, and then quickly decreases; thus, integration in Eq. (2.81) is done to frequency $\omega_{\max}$. Bethe[11] has shown that this frequency corresponds to the Compton wavelength

$$\omega_{\max} = \frac{mc^2}{\hbar}\ (\sim 10^{21}\ \mathrm{c}^{-1}); \qquad (2.82)$$

at energies

$$\hbar\omega_{\max} = mc^2$$

the electron becomes a relativistic particle, so it is natural that nonrelativistic expressions become unsuitable. The parameter $\omega_{\max}$ is the only parameter which must be drawn from consistent relativistic theory in order that reasonable results should be obtained in the nonrelativistic theory.

Now, joining S and $V_{b0,b0}$ we assume that in the latter

$$\frac{1}{\mu} = \frac{2e^2}{3\pi c^3 m^2} \int_0^{\omega_{\max}} d\omega. \qquad (2.83)$$

As a result we obtain

$$V_{b0,\,b0} + S$$

$$= \frac{\alpha}{3\pi} \frac{\hbar}{m^2c^2} \sum_g |\, \mathbf{p}_{bg}\,|^2 \left(\int_0^{\omega_{\max}} d\omega + \int_0^{\omega_{\max}} d\omega\, \frac{\omega}{\zeta - \Omega_g - \omega} \right),$$

where $\alpha = e^2/\hbar c$ is the fine structure constant, $\zeta = z/\hbar$ and $\Omega_g = E_g/\hbar$. Joining the integrals in parentheses, we obtain for this sum the expression

$$V_{b0,\,b0} + S$$

$$= \frac{\alpha}{3\pi} \frac{\hbar}{m^2c^2} \sum_g |\, \mathbf{p}_{bg}\,|^2\, (\zeta - \Omega_g) \int_0^{\omega_{\max}} \frac{d\omega}{\zeta - \Omega_g - \omega}\,. \qquad (2.84)$$

It is easy to see that the integral in Eq. (2.84) diverges logarithmically as $\omega_{\max} \to \infty$, while the integral in Eq. (2.81), under the same conditions, diverges linearly, that is, more strongly. This circumstance and also the assumption, which was proven later, that in relativistic theory the integral in Eq. (2.84) will converge, were Bethe's motive for introducing the term W (Eq. (2.65)) into the perturbation when μ is defined by Eq. (2.83), which is the first step in mass renormalization.

The properties of the integral in Eq. (2.84) play an important role, so we will present the basic relevant mathematical facts.

In mathematics, an integral of the form

$$I(\zeta) = \int_a^b d\omega\, \frac{F(\omega)}{\omega - \zeta} \qquad (2.85)$$

is called a Cauchy integral.[6] This integral defines an analytical function of a complex variable ζ. The function does not have any singularities and is single-valued over the entire complex plane of this variable, and decreases and $|\zeta| \to \infty$. As ζ approaches the segment (a,b) from above or below (Fig. 9) the function $I(\zeta)$ tends to define limiting values, for which there exist very useful relations which are known as the Sokhotskiĭ–Plemel' relations

$$\lim_{\substack{\zeta \to \omega_0,\\ a < \omega_0 < b,\\ \mathrm{Im}\,\zeta > 0}} \int_0^b d\omega\, \frac{F(\omega)}{\omega - \zeta} = i\pi F(\omega_0) + \int_a^b d\omega\, \frac{F(\omega)}{\omega - \omega_0}\,, \qquad (2.86)$$

where ω_0 belongs to the segment (a,b) lying on the real axis, and the integral on the right side is understood in the sense of principal value. As ζ tends to the same value ω_0 from below ($\mathrm{Im}\,\zeta < 0$), the limiting value of the integral will be different, the sign in front of the imaginary unity will be negative.

Thus, the function $I(\zeta)$ has a cut along the segment (a,b), along which the integration is done. Strictly speaking, to satisfy the Sokhotskiĭ–Plemel' relations it is necessary for the function $F(\omega)$ to satisfy a special condition;[6] however, for all functions encountered in radiation problems, this condition is more than satisfied. For many simple functions $F(\omega)$ the integral in Eq. (2.85) may be evaluated explicitly (for example, for $F(\omega) = 1$, see also the case examined in section 2); in such cases the function $I(\zeta)$ may be analytically continued beyond the cut. Consequently, the function $I(\zeta)$ may be multivalued; the sheet of the Riemann surface of this function in which it goes to zero when $|\zeta| \to \infty$ is

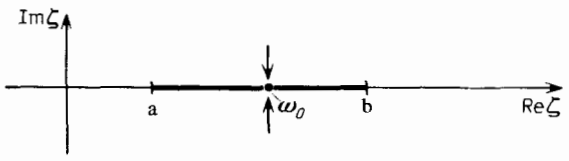

FIG. 9.

called the physical sheet, and the others are called nonphysical sheets.

Let us return to a study of the integral in Eq. (2.84)

$$I_g(\zeta) = \int_0^{\omega_m} d\omega \frac{1}{\omega - (\zeta - \Omega_g)}. \quad (2.87)$$

This integral, as a function of the complex variable ζ, has a cut along the segment $(\Omega_g, \Omega_g + \omega_m)$, which lies on the real axis. For real $\zeta < \Omega_g$ this function is equal to

$$I_g = \ln\left(1 - \frac{\omega_m}{\zeta - \Omega_g}\right) \quad (2.88)$$

and for ζ which have a large modulus and are negative it decreases approximately as

$$I_g(\zeta) \sim \frac{\omega_m}{\Omega_g - \zeta} \quad (\zeta \to -\infty).$$

In the interval $\Omega_g < \zeta < \Omega_g + \omega_m$, at the upper edge of the cut ($\operatorname{Im}\zeta = +0$), this function is equal to

$$I_g(\zeta) = i\pi + \ln\left(\frac{\omega_m}{\zeta - \Omega_g} - 1\right); \quad (2.89)$$

its value at the lower edge differs from Eq. (2.89) only in the sign of the imaginary unity. At $\zeta > \Omega_g + \omega_m$ the function in Eq. (2.87) is equal to

$$I_g(\zeta) = \ln\left(1 - \frac{\omega_m}{\zeta - \Omega_g}\right) \quad (2.90)$$

and for large ζ its modulus decreases approximately as

$$I_g(\zeta) \sim \frac{\omega_m}{\zeta - \Omega_g}.$$

Thus, taking Eq. (2.89) into consideration, we obtain for Δ_{b0} in the interval $0 < z < \hbar\omega_m$ the following expression

$$\Delta_{b0} = z - E_{b0} - \frac{\alpha}{3\pi} \sum_g \frac{|p_{bg}|^2}{m^2c^2} (z - E_g) \left[i\pi\theta(E_b - E_g) + \ln\left|\frac{\hbar\omega_m}{z - E_g} - 1\right| \right]. \quad (2.91)$$

We will now show that this expression goes to zero for some value of z lying on the nonphysical sheet of the Riemann surface $\Delta_{b0}(z)$ near the value $z = E_{b0}a$. The desired value of z will be assumed to be equal to

$$z_b = E_{b0} + \Delta E_b - i\Gamma_b, \quad (2.92)$$

where ΔE_b and Γ_b are small quantities subject to determination. We substitute this value of z into Δ_{b0} and set it equal to zero; thus, in the first term in Δ_{b0} we retain the corrections ΔE_b and Γ_b; in the second term (of the second order in the interaction constant) it is sufficient to assume that $z_b = E_{b0}$. Then, setting the real and imaginary parts individually equal to zero, we obtain

$$\Delta E_b = -\frac{\alpha}{3\pi} \sum_g \frac{|p_{bg}|^2}{m^2c^2} (E_b - E_g) \ln\left|\frac{\hbar\omega_m}{E_b - E_g} - 1\right|, \quad (2.93)$$

$$\Gamma_b = \frac{\alpha}{3} \sum_g{}' \frac{|p_{bg}|^2 (E_b - E_g)}{m^2c^2}, \quad (2.94)$$

in the latter equation summing is done only over states $|g\rangle$ for which $E_b - E_g > 0$, that is, those lying at a lower energy than $|b\rangle$, because for the remaining states $I_g(\zeta)$ does not contain an imaginary part (see Eq. (2.88)).

The value of ΔE_b defines the shift in the energy of state $|b,0\rangle$ and is usually called a Lamb shift; proceeding from Eq. (2.93), Bethe obtained the following approximate expression for the Lamb shift of level E_b

$$\Delta E_b = -\frac{2e^4\hbar\Psi_b^2(0)}{3m^2c^3} \left(\ln\frac{mc^2}{|E_g - E_b|_{av}}\right), \quad (2.95)$$

where $\Psi_b(0)$ is the amplitude of the wave function at the nucleus and

$$\left(\ln\frac{mc^2}{|E_g - E_b|}\right)_{av}$$

is the average value of the logarithm over all states $|g\rangle$. It follows from this expression that in this approximation the Lamb shift is nonzero only for S states in which the wave function does not vanish at the nucleus.

For the $2S$ state of the hydrogen atom

$$\Psi_{2S}^2(0) = \frac{m^3e^6}{8\pi\hbar^6}$$

and the average value of the logarithm, according to Bethe, is equal to

$$\left(\ln\frac{mc^2}{|E_g - E_{2S}|}\right)_{av} \approx 7{,}63;$$

consequently, for the Lamb shift of the $2S$ state we obtain the expression

$$\Delta E_{2S} = -\frac{7{,}63}{12\pi} \frac{e^{10}m}{c^3\hbar^5}. \quad (2.96)$$

On being expressed in frequency units, this Lamb shift is found to be approximately equal to 1040 MHz, which is very close to the experimental value.

The coefficient of attenuation $2\Gamma/\hbar$ (it also has the sense of the probability of transition per unit time) can be written in the form

$$\frac{2\Gamma}{\hbar} = \frac{4}{3} \frac{e^2\Omega_{ba}^3 |r_{ba}|^2}{\hbar c^3} = \frac{4}{3} \frac{e^2}{\hbar c} \frac{|r_{ba}|^2}{\lambda^2} \Omega_{ba}, \quad (2.97)$$

where the following known expression is used

$$p_{ba} = \langle b|p|a\rangle = im\Omega_{ba}\langle b|r|a\rangle = im\Omega_{ba}r_{ba}$$

and it is assumed that below the state $|b\rangle$ there lies only one state, $|a\rangle$. The quantity $\hbar/2\Gamma$ is the decay half-life of the excitation.

Thus, it has been established that the integrand in Eq. (2.74) has a pole at some $z = z_b$. We note that this pole lies on a nonphysical sheet of the Riemann surface of the function $\Delta_{b0}(z)$. Indeed, $\operatorname{Im} z_b = -i\Gamma_b < 0$, while to construct the equation $\Delta_{b0}(z) = 0$ a value of $I_g(\zeta)$ was used which belongs to the upper edge of the cut; this means that the function $\Delta_{b0}(z)$ is analytically continued beyond the cut, that is, into a nonphysical sheet.

The pole at $z = z_b$ is not the only singularity of the integrand in Eq. (2.74); it also has branching points at the same points as in the integrals $I_g(z/\hbar)$.

Returning to the matrix element in Eq. (2.73)

$$U_{b0,\,b0}(t) = \frac{1}{2\pi i} \int_C dz e^{-izt/\hbar} \Delta_{b0}^{-1}, \quad (2.98)$$

we note first that the lower part of contour C may be shifted downward in the complex plane, where, due to the presence

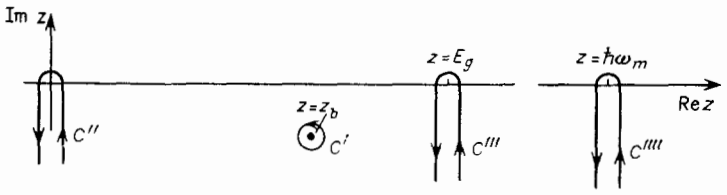

FIG. 10.

of an exponential the integral expression decreases quickly; consequently, the integral along the lower part of the C contour simply equals zero. The upper part of the contour C can be deformed only as shown in Fig. 10, since when it is shifted downward it encounters branching points. It is moved between these points along the nonphysical sheet of the Riemann surface where it also encounters a pole at $z = z_b$.

The main contribution to the matrix element $U_{b0,b0}(t)$ is made by the residue at the pole when $z = z_b$

$$U'_{b0,\,b0}(t) = \frac{\exp\{-i[(E_{b0}+\Delta E_b)t] - \Gamma_b t\hbar^{-1}\}}{(\partial\Delta_{b0}/\partial z)_{z=z_b}}. \qquad (2.99)$$

As we see, this part of the matrix element describes the exponential decay of excitation. The derivative

$$\left(\frac{\partial\Delta_{b0}}{\partial z}\right)_{z=z_b} = 1 - \frac{\alpha}{3\pi}\sum_g \frac{|\mathbf{p}_{bg}|^2}{m^2c^2}\left[i\pi + \ln\left(\frac{\hbar\omega_m}{E_b - E_g} - 1\right) + \frac{\hbar\omega_m}{\hbar\omega_m - (E_b - E_g)}\right]$$

is close to unity. Without calculating it in detail, we note only that in the second term, in addition to the small parameter α (the fine structure constant) there is a small parameter

$$\frac{|\mathbf{p}_{bg}|^2}{m^2c^2} \sim \frac{(2\pi a)^2}{\lambda^2} (\sim 3\cdot 10^{-7}),$$

which is approximately equal to the square of the ratio of the Bohr radius to the radiated wavelength. Then it is clear that the contributions from the integrals along the contours C'' and C''' are small.

The integral along contour C'' is a small deviation of the decay law from exponentiality, which was first noted by L. A. Khalfin[12] and which has been discussed numerous times in the literature.[13] Although this part is small in comparison to unity, at large t it decreases slowly according to a power law and becomes greater than the part of U' which decreases exponentially.

The integral along the contour C'' can be written in the form of a difference

$$U''_{b0,\,b0}(t) = \frac{-\hbar}{2\pi}\left(\int_0^\infty d\rho\, \frac{e^{-\rho t}}{\Delta^{(l)}_{b0}(\rho)} - \int_0^\infty d\rho\, \frac{e^{-\rho t}}{\Delta^{(r)}_{b0}(\rho)}\right), \qquad (2.100)$$

where ρ is the modulus of ζ ($\rho = -i\zeta$) on contour C'', $\Delta^{(l)}_{b0}(\rho)$ is the value of $\Delta^{(l)}_{b0}(z)$ on the left side of the contour C'', and $\Delta^{(r)}_{b0}(\rho)$ is the value of this function on the right side of this contour. At large t in these integrals it is sufficient to consider values of $\Delta^{(l)}_{b0}(\rho)$ and $\Delta^{(r)}_{b0}(\rho)$ at small ρ. In this case the difference

$$\Delta^{(r)}_{b0}(\rho) - \Delta^{(l)}_{b0}(\rho) = \frac{2\alpha}{3}\frac{|\mathbf{p}_{ba}|^2}{m^2c^2}\hbar\rho$$

is proportional to ρ, and as $z \to 0$, the product $\Delta^{(r)}_{b0}(\rho)\,\Delta^{(l)}_{b0}(\rho)$ tends to the constant E'^2_{b0}, which is equal to

$$E'_{b0} = E_{b0} + \frac{\alpha}{3\pi}\sum_{g\neq a}\frac{|\mathbf{p}_{bg}|^2}{m^2c^2}E_g\left[i\pi\theta(E_b - E_g) + \ln\left|\frac{\hbar\omega_m}{E_g} + 1\right|\right];$$

the subscript a denotes the ground state of the atom, whose energy is equal to zero.

Consequently, the part of the matrix element $U''_{b0,b0}$ (Eq. (2.73)) due to the contour C''

$$U''_{b0,\,b0}(t) = \frac{\alpha}{3\pi}\frac{|\mathbf{p}_{ba}|^2}{m^2c^2}\frac{\hbar^2}{E'^2_{b0}t^2}, \qquad (2.101)$$

actually decreases according to a power law at large t.

The contour C'' goes around the branching point of the term in Δ_{b0} which corresponds to $g = a$. Other terms corresponding to other g also have branching points at $z = E_g$. One such branching point at $z = E_g$ together with contour C''', which goes around it, is shown in Fig. 9. In the majority of cases the integral along contour C''' is small. It may become significant when E_g is close to E_b, that is, when levels intersect; however, this case, although it may be easily examined using the resolvent method, is all the same a special case and will not be examined in this article.

Integrals in Δ_{b0} are taken along a finite segment of the real axis from E_g to $E_g + \hbar\omega_m$; consequently, as a function of z they have branching points not only at the left end of the integration segment, but also on its right end. One such branching point and the integration contour which goes around it, C'''', are shown in Fig. 9. The integrals along these contours are in virtually all cases negligibly small; what is more significant is that a consideration of these integrals lies beyond the limits of applicability of nonrelativistic theory. Indeed, the right end of the integration segment corresponds to those energies at which the movement of an electron becomes relativistic, and its description requires a full measure of quantum electrodynamics. Fortunately, these corrections to the optical problems which interest us in this article are small.

Although in this section much attention has been devoted to calculation of the Lamb shift (basically, this was done to show that there is nothing complex in this problem), only in rare cases is knowledge of this shift significant; in the majority of optical problems, the Lamb shift can be assumed to be equal to zero.

CONCLUSION

The problems examined in this paper on radiation in a waveguide and in free space show that the resolvent method is very effective for the solution of radiation problems, espe-

cially when used in combination with the theory of functions of a complex variable. It can be applied to the problem of the intersection of levels taking attenuation into account, to the problem of the discovery of new channels of decay, to the investigation of induced radiation in complex cases, to the problem of the interaction of radiating atoms, etc. In our opinion the resolvent method is the most effective method of solving nonstationary problems in quantum electrodynamics selected from those based on perturbation theory.

One of the urgent tasks in the development of this method is the inclusion in it of a consistent renormalization procedure.

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