Phase space of mechanical systems with a gauge group

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Publications on the structure of the physical phase space (PS) of dynamical systems with gauge symmetry are reviewed. The recently discovered phenomenon of reduction of the phase space of the physical degrees of freedom is studied systematically on mechanical models with a finite number of dynamical variables. In the simplest case of one degree of freedom this phenomenon consists of replacement of the phase space by a cone that is unfoldable into a half-plane. In the general case the reduction of the phase space is related with the existence of a residual discrete gauge group, acting in the physical space after the unphysical variables are eliminated. In "natural" gauges for the adjoint representation this group is isomorphic to Weyl's group. A wide class of modes with both the normal and Grassmann (anticommuting) variables and with arbitrary compact gauge groups is studied; the classical analysis and the quantum analysis are performed in parallel. It is shown that the reduction of the phase space radically changes the physical characteristics of the system, in particular its energy spectrum. A significant part of the review is devoted to a description of such systems on the basis of the method of Hamiltonian path integrals (HPIs). It is shown how the HPI is modified in the case of an arbitrary gauge group. The main attention is devoted to the correct formulation of the HPI with a poor choice of gauge. The analysis performed can serve as an elementary illustration of the well-known problem of copies in the theory of Yang–Mills fields. The dependence of the quasiclassical description on the structure of the phase space is demonstrated on a model with quantum-mechanical instantons.

1. INTRODUCTION

Gauge theories are now on center stage in the modern physics of the microworld. Judging from their successes, they will probably remain there during the next ten years. For this and other reasons, associated with their intrinsic merits (elegance, relative simplicity, clear geometric interpretation, etc.), gauge theories are under intensive study; quantum electrodynamics (QED)—a system with an Abelian gauge group—has now been studied for more than half a century.[1] At the same time new and sometimes unexpected aspects, previously neglected or simply unnoticed, are being discovered in these theories even now. This pertains both to fields with a non-Abelian gauge group, the Yang–Mills fields,[2] and to electrodynamics. Examples are the Higgs phenomenon,[3–8] asymptotic freedom,[9,10] and the problem of fixing the gauge in the Yang–Mills theory.[11]

The question of the structure of the phase space (PS) of gauge theories, i.e., the question of the peculiarities of the Hamiltonian dynamics of the physical variables, can also be included here. It is known that dynamical systems with gauge symmetry have unphysical variables, which change in time according to a law that is not contained in the equations of motion. Elimination of these variables means transferring from the initial phase space to a lower-dimensional physical phase space, which is usually assumed to be locally Euclidean.[1)] It has been found, however, that this is still not the end. Study of the simplest models has shown that a residual discrete gauge group, which brings about further reduction of the phase space, operates in the space of the physical variables (Refs. 12,13).[2)] This group, which in the simplest cases is identical to Weyl's group,[13,14] cannot lower the dimension of the physical phase space, but it can change its "volume." The essence of the matter lies in the well-known fact that points of configuration or phase space which are connected by a gauge transformation are physically indistinguishable. The residual discrete group identifies with one another some points of the space of physical variables and this is what leads to reduction of the space; for example, the phase plane is transformed into a cone that is unfoldable into a half-plane.[12] This circumstance brings about a change in the dynamics of the system.

Thus, in the case of a two-dimensional isotropic oscillator with the gauge group SO(2) the frequency of the oscillations is doubled (compared with the same oscillator without gauge symmetry). This result is easy to understand: if the gauge group is the group of rotations in the plane, then all points on a circle in the plane are physically indistinguishable and the only physical variable is r—the distance to the origin of coordinates. The motion of a material point now looks as follows: After the particle passes through the point $r = 0$ the subsequent motion is indistinguishable from the reverse motion, i.e., the motion along the path the particle has just traversed (see Sec. 2). As a result, the particle requires half the time to return to the starting point—this is the origin of the doubling of the frequency.

A different formulation of this phenomenon is also fruitful: Eliminating as unphysical one of the two Cartesian variables (by setting it equal to zero) we arrive at motion along an axis whose points, which are symmetric relative to the origin of coordinates (the point of intersection of a straight line with the gauge orbit, i.e., a circle), are physically indistinguishable. The residual gauge group Z_2 identifies these points with one another: The nontrivial element of this group is reflection relative to the origin of the coordinates. It is very important that this approach can be easily extended to arbitrary groups, and in the non-Abelian case Weyl's group plays the role of the group Z_2.[13,15]

This feature also appears in the quantum theory: Since the physical states are fixed by the requirement that the constraints vanish on them, some of the states of the initial space are eliminated. Only the states that are invariant under the

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residual gauge group, i.e., states whose energy spacings (frequencies) are doubled, remain. This situation occurs not only for the normal (commuting) but also for the Grassmann (anticommuting) variables.[15,16] The transition to an arbitrary group complicates the structure of the phase space without changing the essential nature of the problem.

The question of the formulation of such theories on the basis of the method of Hamiltonian path integrals (HPIs) is of independent interest. This problem is identical to problems in which it is required to take into account in a path integral zero boundary conditions (particle in a "box") or the topology of the configuration space.[17,18] The fundamental feature of this problem is that here the structure of the phase space must be taken into account. We emphasize that only physical variables are of concern here. This problem, which is quite trivial in the case of a three-dimensional isotropic oscillator (the SO(3) group[19]), becomes appreciably more complicated for an arbitrary simple Lie group.[13,15,20,21]

What are the consequences of changing the structure of the phase space? It was mentioned above that the spacing of the energy levels of a harmonic oscillator is doubled already in the case of an Abelian gauge group (or more complicated groups, but for representations with one physical variable). Transferring to groups of higher rank, generally speaking, increases the number of physical degrees of freedom, and the spectrum becomes more complicated. Thus for the adjoint representation of a simple compact group of rank l the spectrum is identical to that of a collection of noninteracting oscillators with the frequencies $r_\alpha\omega$, $\alpha = 1, 2, ..., l$, where ω is the frequency appearing in the Lagrangian and r_α is the degree of the independent Casimir operators in the group.[13] Reduction of the physical phase space leads to modification of the Hamiltonian path integrals. Therefore the quasiclassical description can be expected to change. However, this is already clear from analysis of the problem of an oscillator with gauge symmetry, for which the quasiclassical description is exact. In Ref. 22 it was shown for the example of quantum-mechanical instantons[23-27] that reduction of the phase space of physical variables brings about a change in the structure of the θ vacuum.[26,27]

This review is devoted to a range of questions connected with the phenomenon of the reduction of the phase space of the physical degrees of freedom in theories with gauge symmetry. Mechanical systems with a finite number of variables are studied; this stage is necessary in order to proceed to gauge field theories. Gauge field theories are very complicated, and if the properties and consequences of the reduction of the phase space are not understood for the simplest models, it will be impossible to understand and evaluate its role in field theory (the structure of the phase space of the Yang–Mills fields as well as the consequences of changing the phase space for some field systems are examined in Refs. 12, 15, 22, and 28). However, these models are of interest in themselves as the simplest examples of mechanical systems encountered in nature, but which have not been included in textbooks on theoretical mechanics.

Models with one physical degree of freedom are studied in Sec. 2. Models with an arbitrary gauge group and several degrees of freedom are studied in Sec. 3 and models with Grassmann (anticommuting) variables, which are encountered in the description of Fermi fields, are studied in Sec. 4. Section 5 is devoted to an analysis of Hamiltonian path integrals for models with a reduced phase space of the physical variables. The effect of the reduction of the phase space on the quasiclassical description is studied in Sec. 6. Finally, systems with several degrees of freedom and a phase space with a more complicated structure are studied in Sec. 7. These models make it possible to understand better the role of unphysical variables. In this paper we study only compact gauge groups. In the concluding section (Sec. 8), the investigations performed are summarized and the prospects for their application in field theory are discussed. Supplementary material is presented in the Appendix (Sec. 9). Our analysis of constrained systems is based on Dirac's scheme[29] (see Sec. 9.5 for a more detailed discussion).

2. GAUGE SYSTEMS WITH ONE PHYSICAL DEGREE OF FREEDOM

2.1. The simplest model (the SO(2) gauge group)

2.1.1. The classical theory[12]

Consider a dynamical system given by the Lagrangian function

$$L(\mathbf{x}, \dot{\mathbf{x}}, y, \dot{y}) = \frac{1}{2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} - yT\right)\mathbf{x}\right]^2 - V(\mathbf{x}^2), \qquad (2.1)$$

where the two-dimensional vector $\mathbf{x}$ and the scalar y are dynamical variables, $T = -i\tau_2$ (τ_2 is a Pauli matrix), and $(T\mathbf{x})_i = T_{ij}x_j$. The Lagrangian (2.1) describes a nonrelativistic particle of unit mass in a two-dimensional space. If we now transform to the complex quantities $\varphi = (x_1 + ix_2)/\sqrt{2}$, then L will assume the form

$$L(\varphi, \varphi^*, \dot{\varphi}, \dot{\varphi}^*, y, \dot{y}) = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} - iy\right)\varphi\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} - iy\right)\varphi\right]^* - V(2\varphi\varphi^*), \qquad (2.2)$$

whence it is obvious that the Lagrangian (2.2) is the Lagrangian of scalar electrodynamics in $(1+0)$ space-time, and $y = A_0(t)$ is the zeroth component of the vector potential A_μ. The Lagrangian (2.1) is invariant under the group of gauge transformations

$$\delta\mathbf{x} = \varepsilon T\mathbf{x}, \quad \delta y = \dot{\varepsilon}, \qquad (2.3)$$

where $\varepsilon = \varepsilon(t)$ is an arbitrary infinitesimal function of time. The transformation to the Hamiltonian formalism is determined by the equations $p^i = \partial L/\partial\dot{x}_i$, $\pi = \partial L/\partial\dot{y} = 0$, and

$$H = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{x}^2) + y\mathbf{p}T\mathbf{x}. \qquad (2.4)$$

It is easy to verify that the Lagrangian (2.1) defines a mechanical system with two constraints of the first kind:[29] $\pi = 0$, $\sigma \equiv \mathbf{p}T\mathbf{x} = 0$, and $\sigma = x_1p_2 - x_2p_1$ is the generator of rotations in two-dimensional space.[12] Only one of the three degrees of freedom y, x_1, and x_2 is physical. A description of the system using only physical variables can be obtained by two methods. The first method is explicitly invariant. Performing the canonical transformation $\mathbf{x}$, $\mathbf{p} \to \theta$, r, p_r, p_θ, where $r = (\mathbf{x}^2)^{1/2}$ and θ are the polar coordinates while $p_r = (\mathbf{p}, \mathbf{n}_r)$ and $p_\theta = (\mathbf{p}, \mathbf{n}_\theta)r$ are the momenta conjugate to them, $\mathbf{n}_r = \mathbf{x}/r$ and $\mathbf{n}_\theta$ are unit vectors, we verify that r and p_r are gauge invariants and can be taken as the physical variables. The second (noninvariant) method is as follows. Because of gauge invariance[2] the points of the circle $\mathbf{x}^2 = \text{const}$ are physically indistinguishable. As a representative one can

take from each circle a point that also lies on a line intersecting each circle only once, for example, the semiaxis $x_2 = 0$, $x_1 \geqslant 0$. It is more convenient, however, to assume that the physical variable x_1 runs over the entire real axis, on which the point x_1 is identified with the point $-x_1$. This is equivalent to the assertion of a transformation to a dynamical system with the gauge group Z_2 consisting of the two elements[4] 1 and $\hat{P}$, where $\hat{P}x_1 = -x_1$. It is obvious that both formulations are equivalent. In the case of more complicated groups and representations the relation between these approaches is no longer so simple.

The question of the structure of the phase space of a model is solved as follows. In the invariant approach $r \geqslant 0$, $-\infty < p_r < \infty$, so that the physical phase space is a half-plane. A more careful investigation[12] shows that on the straight line $r = 0$ the point p_r must be identified with the point $-p_r$. This is most easily seen by studying trajectories in the phase space (r, p_r) for some simple potential, for example, $V = \mathbf{x}^2/2$ in Eq. (2.4) (y plays the role of a Lagrange multiplier). It is easy to verify that the solutions of the corresponding Hamiltonian equations of motion and constraints $\sigma = 0$ have the form

$$\mathbf{x}(t) = \cos t \cdot \exp\left(T \int_0^t d\tau y(\tau)\right) \mathbf{x}(0)$$

and $\mathbf{p}(t) = -(\operatorname{tg} t)\mathbf{x}(t)$. Then $r(t) = |\cos t| r(0)$ and $p_r(t) = -(\operatorname{tg} t)|\cos t| r(0)$. It is obvious that at time $t = \pi/2$ ($r = 0$) the trajectory in the plane (r, p_r) has a discontinuity and p_r changes abruptly. It is clear from physical considerations, however, that there cannot be any jumps in the phase space of this mechanical system (the potential is regular). For this reason the points $(r = 0, p_r)$ and $(r = 0, -p_r)$ in the phase space must be identified with one another. This means that the phase space of the system is a cone that is unfoldable into a half-plane. In the noninvariant approach this result is obtained automatically. From the equation $\sigma = 0$ it follows that $\mathbf{p} = \lambda \mathbf{x}$, where λ is a function of time. For this reason, if $x_2 = 0$, then $p_2 = 0$. Since the laws of the gauge transformations for $\mathbf{p}$ and $\mathbf{x}$ are identical, p_1 also changes sign under the action of the group $\mathbb{Z}_2$, i.e., the gauge group $\mathbb{Z}_2$ operates in the phase space according to the rule

$$(x_1, p_1) \to (-x_1, -p_1).$$

Therefore the points (x_1, p_1) and $(-x_1, -p_1)$ of the phase plane are physically indistinguishable, i.e., the physical phase space is a cone that is unfoldable into a half-plane. For the model (2.1) both methods (the invariant method and the noninvariant method with the group Z_2) appear to be equivalent. The advantage of the noninvariant approach is seen in the case of more complicated groups or representations (see Secs. 3 and 7).

It may seem that the phase space of the polar variables r, p_r is also a half-plane in the absence of gauge symmetry, because $r \geqslant 0$. Analysis shows that this is not so. The question of the structure of the phase space of the variables in the polar coordinate system is studied in the Appendix (Sec. 9.1).

2.1.2. The quantum theory

The transition to the quantum description of the system (2.1) is made by transferring to operators $\mathbf{x}, \mathbf{p}, y, \pi \to \hat{\mathbf{x}}, \hat{\mathbf{p}}, \hat{y}, \hat{\pi}$ in the Hamiltonian and the constraints (the problem of ordering the operators does not arise here). In so doing the gauge must be fixed[30] by adding to the Lagrangian, for example, the term $\dot{y}^2/2$; then $H \to H + (\pi^2/2)$. In accordance with the standard recipe for quantizing such systems[29] the physical states are singled out by the conditions

$$\hat{\pi}\Phi = 0, \quad \hat{\sigma}\Phi = \hat{\mathbf{p}}T\hat{\mathbf{x}}\Phi = 0. \tag{2.5}$$

The first condition means that the physical state vectors do not depend on y. In what follows, we shall ignore this variable[5] (for example, in normalizing the states we shall not integrate over it). The second condition (2.5) means that Φ is invariant under the group of rotations in the plane, i.e., in the coordinate representation it does not depend on the angle θ and therefore $\Phi = \Phi(\mathbf{x}^2)$, since $\mathbf{x}^2$ is the only invariant that can be formed from the vector $\mathbf{x}$. The states are normalized as follows:

$$\frac{1}{2\pi}\int d^2x |\Phi|^2 = \int_0^\infty dr r |\Phi(r^2)|^2 = 1. \tag{2.6}$$

We note that although the theory is one-dimensional the integration is performed with a nontrivial measure. This is actually an invariant approach. Transferring in the Hamiltonian (2.4) to operators and rewriting it in polar coordinates $\hat{H} + (\hat{\pi}^2/2) = [(\hat{p}_r^2 + r^{-2}\hat{p}_\theta^2 - (4r^2)^{-1})/2] + (\hat{\pi}^2/2) + yp_\theta + V(r^2)$, we find the energy operator operating in the physical space ($\hat{p}_\theta \Phi = \hat{\pi}\Phi = 0$)

$$\hat{H}_{\text{ph}} = \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right) + V(r^2)$$
$$= \frac{1}{2}\left(p_r^2 - \frac{1}{4r^2}\right) + V(r^2); \tag{2.7}$$

here $\hat{p}_r = -ir^{-1/2}\partial r^{1/2}/\partial r$ and the wave functions are normalized according to Eq. (2.6).

The unphysical variables can be eliminated in an invariant manner even before quantization. In the variables r, p_r and θ, and p_θ the Hamiltonian (2.24) with the constraint $p_\theta = \sigma = 0$ has the simple form $(1/2)p_r^2 + V(r^2)$. But then the Hamiltonian operator obtained from it by making the substitution $p_r \to \hat{p}_r = -i\partial_r$, is not the same as the operator (2.7). This approach is acceptable *a priori* (see Sec. 9.5). However it has a number of inherent fundamental drawbacks. First, the theory is formulated on the semiaxis $r \geqslant 0$ and the operator $\hat{p}_r = -i\partial_r$ is not self-conjugate on the semiaxis and cannot be extended to a self-conjugate operator. Second, in order for the Schrödinger equation to make sense it must be supplemented with a boundary condition at the point $r = 0$, i.e., extraneous considerations must be invoked. Third, the choice of invariant variables is not unique. The form of the Hamiltonian depends on this choice (see Sec. 2.3.2). If, now, when quantizing, the canonical momentum in this Hamiltonian is replaced by the corresponding differentiation operator, then the quantum theories corresponding to different choices of the invariant variables will be different (unitarily nonequivalent; "quantization" can be performed only in Cartesian coordinates) (see Sec. 2.3.2).

Dirac's scheme[29] does not have these deficiencies (see once again Secs. 2.3.2 and 9.5). It can thus be used as a standard for comparing different quantization schemes.

The noninvariant analysis involves transforming in Eq. (2.4) to $x_2 = p_2 = 0$. However a direct approach cannot be used here. One cannot solve the constraint $x_1 p_2 = x_2 p_1$ for

p_2, substitute p_2 into Eq. (2.4) and set $x_2 = 0$ and then transfer to the quantum description. The fact that the constraints hold only on the physical states must be taken into account. For this reason, according to Ref. 17 the substitution $p_2 = (x_2/x_1)p_1$ in the term p_2^2 in the formula (2.4) can be made only for one momentum $p_2^2\Phi = p_2(x_2/x_1)p_1\Phi$, since the state obtained $(x_2/x_1)p_1\Phi$ will no longer be physical. The second momentum p_2 must first be transferred to the right of Φ and only then can the indicated substitution followed by elimination of x_2 ($x_2 = 0$) be made. Ultimately the term p_2^2 will transform into $(-i/x_1)p_1$; the Hamiltonian sought is

$$\hat{H}_{\text{ph}} = \frac{1}{2}\left(\hat{p}_1^2 - \frac{i}{x_1}\hat{p}_1\right) + V(x_1^2). \tag{2.8}$$

The operator (2.8) is defined on the entire axis, and we must take into account the residual gauge invariance associated with the group $\mathbb{Z}_2$, i.e., the physical states must be made to satisfy the gauge condition

$$\hat{P}\varphi(x_1) = \varphi(-x_1) = \varphi(x_1), \quad \hat{P}\hat{p}_1 = -\hat{p}_1\hat{P}. \tag{2.9}$$

(We recall that $\hat{P}$ is the only nontrivial element of the group $\mathbb{Z}_2$.) From Eqs. (2.9) we conclude that the physical vectors Φ are even functions of x_1, i.e., $\Phi = \Phi(x_1^2)$. However the normalization of Φ differs from that of the wave function of the one-dimensional problem with the Hamiltonian (2.8). In accordance with Eq. (2.6) we have

$$\frac{1}{2}\int_{-\infty}^{\infty} dx_1 |x_1| \, |\Phi(x_1^2)|^2 = 1. \tag{2.10}$$

The choice of the normalization condition in Eq. (2.10) requires explanation. The integration over the entire real axis corresponds to the formulation of the problem: x_1 assumes any real values, but the physical range of x_1 is the semiaxis $x_1 \geqslant 0$, whence appear the factor 1/2 and $|x_1|$ (instead of x_1) in Eq. (2.10). The facts that the Hamiltonians (2.7) and (2.8) are identical ($x_1 \leftrightarrow r$) and the corresponding problems are identical is obvious. We note that in choosing the normalization condition in the noninvariant approach it was necessary to resort to an invariant formulation.

The role of the discrete gauge group becomes clear from Eqs. (2.6), (2.9), and (2.10). Because of it: 1) Some of the eigenfunctions of the Hamiltonian are dropped and 2) the normalization condition changes (the factor 1/2 in Eq. (2.10)). There is one other feature. If the unphysical variables are eliminated before quantization is performed, then a quantum theory with the Hamiltonian $(1/2)p_1^2 + V(x_1^2)$ is obtained. Obviously, it differs from the theory given by Eqs. (2.8) and (2.10). Everything we have said above in our discussion of the invariant approach pertains also to the noninvariant method, with the exception of the fact that the operator $\hat{p}_1 = -i\partial_1$, in contrast to $\hat{p}_r$, is self-conjugate because $x_1 \in \mathbb{R}$. Here, as in the invariant approach, in order to prevent the quantum theory from depending on the gauge, Dirac's scheme (the noninvariant approach) must be used.

The example analyzed above illustrates the noncommutativity of the operations of quantization and elimination of unphysical variables in constrained systems.[12,17,32–35] The resulting theories can differ, first, by their Hamiltonians and, second, by the normalization condition. Of course, in particular cases it may happen that these operations commute (in the model (2.1) y can be eliminated at any stage after transferring to the Hamiltonian dynamics). The indicated noncommutativity is important for Yang–Mills fields.[32,36]

2.2. The harmonic oscillator (the gauge group (SO(*n*))

2.2.1. Coordinate representation

The elimination of unphysical variables in the classical theory for the model of Sec. 2.1 does not present any difficulty. Solving the constraint $p_2x_1 - x_2p_1 = 0$ for p_2, substituting the solution into Eq. (2.4), and then setting $x_2 = 0$ we obtain the Hamiltonian $(p_1^2/2) + V(x_1^2)$. The consequences of gauge invariance are taken into account by identifying with one another the points (x_1, p_1) and $(-x_1, -p_1)$ of the phase space. In the case of the harmonic oscillator ($V = x_1^2/2$) this results in doubling of the frequency of the oscillations, because in the configuration space after the particle passes through the point $x_1 = 0$ the reverse motion actually occurs (the points x_1 and $-x_1$ are identical), and the particle returns to the starting position in half the time. This fact should also be manifested in the quantum theory, i.e., the separations of the energy levels of the oscillator should be doubled.

We shall demonstrate this assertion for the example of an n-dimensional isotropic oscillator with the gauge group SO(n), when the gauge transformations are rotations of the n-dimensional vector $\mathbf{x}$.[12] The corresponding Lagrangian is an obvious generalization of the Lagrangian (2.1) ($yT \to y_a T_a$, $V = x^2/2$, where T_a^{ij} are antisymmetric matrices (the generators of SO(n)) and $a = 1, 2, \ldots, n(n-1)/2$). The secondary constraints are now generators of the group SO(n)

$$\sigma_a = p_i T_a^{ij} x_j = 0 \quad (i, j = 1, 2, \ldots, n), \tag{2.11}$$

where p_i are the momenta that are canonically conjugate to x_i. Since the stationary subgroup of an n-vector is SO($n-1$), from the $n(n-1)/2$ constraints only $n(n-1)/2 - (n-1)(n-2)/2 = n-1$ are independent, and the physical states are the functions that do not depend on the angular variables.

Schrödinger's equation for the radial functions in an n-dimensional spherical coordinate system is[12]

$$\left[-\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + \frac{l(l+n-2)}{r^2} + r^2\right]\Phi(r) = 2E\Phi(r), \tag{2.12}$$

where $l = 0, 1, \ldots$, whence by making the substitution $\Phi = r^2 \exp(-r^2/2) f(r)$ we obtain for $z(t) = f(r)$, $t = r^2$, the following equation

$$tz'' + (\alpha - t)z' - \beta z = 0, \tag{2.13}$$

in which $\alpha = l + (n/2)$ and $\beta = (\alpha - E)/2$. The solution of this equation that is regular at the origin is given by the confluent hypergeometric function

$$z(t) = {}_1F_1(\beta, \alpha; t). \tag{2.14}$$

From the condition that $\Phi(r)$ decay at infinity (from the condition that $z(t)$ is a polynomial, i.e., with $\beta = -k$) we find the energy spectrum:

$$E_k = 2k + l + \frac{n}{2}, \quad k = 0, 1, \ldots \tag{2.15}$$

Using the relation between the functions ${}_1F_1$ and the Laguerre polynomials $L_1^{\alpha}({}_1F_1(-k,\alpha+1;t) = L_k^{\alpha}(t)\Gamma(k+1)\Gamma(\alpha+1)/\Gamma(k+\alpha+1)$ (Ref. 37, p. 189), we obtain the sought solution of Eq. (2.12):

$$\Phi_{kl}(r) = \text{const}\cdot r^l L_k^{l-1+(n/2)}(r^2)\, e^{-r^2/2}. \tag{2.16}$$

The physical solutions are spherical symmetric functions Φ_{k0}. It is clear from Eqs. (2.15) and (2.16) that for $l=0$ the spacings between the levels double, and the physical states are even functions of r. The latter fact means, in particular, that in the noninvariant formulation the eigenfunctions of the Hamiltonian (2.8) ($V = x_1^2/2$) that are regular at the origin automatically satisfy the condition (2.9).

In conclusion, we shall discuss the meaning of the variable x_1. From the outset it was clear that x_1 is "not a completely" physical variable, because it changes under the gauge transformations ($x_1 \to -x_1$). It follows from Eqs. (2.9) and (2.10) that all matrix elements of x_1 between physical states are equal to zero

$$\frac{1}{2}\int_{-\infty}^{\infty} dx_1\, |x_1|\, \Phi_1^*(x_1^2)\, x_1 \Phi_2'(x_1^2) = 0. \tag{2.17}$$

The equation (2.17) is equivalent to the following assertion: The state with a definite nonzero value of x_1 cannot belong to the physical Hilbert space $\mathscr{H}_{\text{ph}}$, i.e.,

$$\hat{x}_1\psi = x_1\psi \to \psi \in \mathscr{H}_{\text{ph}}, \tag{2.18}$$

because in this case $\hat{P}\psi \neq \psi$. Indeed, the operators $\hat{P}$ and x_1 do not commute ($\hat{P}\hat{x}_1 = -\hat{x}_1\hat{P}$), so that the only eigenstate they can have in common is $\psi = 0$. Only even powers of x_1, i.e., variables of the form $|x_1| = x_1 \text{ sign } x_1, x_1^2, x_1^4, \ldots$ ($|x_1|$ can actually be represented by a series of even powers of x_1), are physically meaningful. This fact is important for the quantum field theory.[12,15,16,22,28]

We shall now summarize. After the unphysical variables are eliminated we obtain two equivalent formulations of the feature of interest of the dynamical system: 1) The system has a conical phase space and 2) the system is invariant under the discrete gauge group (in this case, the group $\mathbb{Z}_2$). The first formulation is useful for studying the classical Hamiltonian mechanics, while the second property is convenient in the Lagrangian approach, in quantum mechanics, and in the method of path integration. A physical consequence of this fact is that the spectrum of the Hamiltonian changes. In particular, the frequency of oscillations of the oscillator is doubled and the Hilbert space is reduced. We stress, however, that the condition (2.9) starts to play an important role only in the process of elimination of the unphysical variables (when transferring to one-dimensional motion in the problem (2.1)). Within the framework of Dirac's general approach[29] all information about the physical states is contained in the conditions (2.5).

2.2.2. The second-quantization representation

It is useful to represent in the second-quantized form the quantum-mechanical problem of Sec. 2.1 for the oscillator potential. Transforming in Eq. (2.4) from $V = \mathbf{x}^2/2$ to the new variables $\mathbf{a} = (\mathbf{p} - i\mathbf{x})/\sqrt{2}$ and $\mathbf{a}^+ = (\mathbf{p} + i\mathbf{x})/\sqrt{2}$, we write the Hamiltonian operator in the form

$$\hat{H} = \hat{\mathbf{a}}^+\hat{\mathbf{a}} + 1 + i\hat{\gamma}\hat{\mathbf{a}}^+ T\hat{\mathbf{a}}, \quad [a_i, a_j^+] = \delta_{ij}. \tag{2.19}$$

Taking into account the fact that the basis of the Hilbert space is given by the states $(\hat{a}_1^+)^m(\hat{a}_2^+)^n|0\rangle$, $m, n = 0, 1, \ldots$, we find the basis vectors of the physical space from the condition that the constraints $\hat{\sigma} = \hat{\mathbf{a}}^+ T\hat{\mathbf{a}}$ vanish on the physical states $\hat{\sigma}|\Phi\rangle = 0$:

$$|\Phi_k\rangle = (2^k k!)^{-1}\,[(\hat{\mathbf{a}}^+)^2]^k\,|0\rangle. \tag{2.20}$$

The result (2.20) is obvious, since $\hat{\sigma}$ is the generator of rotations in a plane: $[\hat{\sigma}, \hat{\mathbf{a}}^+] = -T\hat{\mathbf{a}}^+$, $[\hat{\sigma}, \hat{\mathbf{a}}] = -T\hat{\mathbf{a}}$; in addition, $\hat{\sigma}|0\rangle = 0$ and the physical basis is exhausted by vectors which are obtained by applying to $|0\rangle$ invariant polynomials of $\hat{\mathbf{a}}^+$. There exists only one independent invariant polynomial. This polynomial is $(\hat{\mathbf{a}}^+)^2$. This proves Eq. (2.20). Since

$$[\hat{\mathbf{a}}^+\hat{\mathbf{a}}, (\hat{\mathbf{a}}^+)^2] = 2(\hat{\mathbf{a}}^+)^2 \tag{2.21}$$

the spacings between the energy levels are doubled. In the case of an n-dimensional oscillator, whose physical states satisfy the conditions $\hat{\sigma}_a|\Phi\rangle = 0$ (see Eq. (2.11)), we have (Ref. 15, pp. 74 and 38)

$$|\Phi_k\rangle = \left(\frac{4^k k!\,\Gamma(k+n/2)}{\Gamma(n/2)}\right)^{-1/2}[(\mathbf{a}^+)^2]^k\,|0\rangle, \tag{2.22}$$

and by virtue of Eq. (2.21) the frequency of the oscillations is also doubled.

We note that in a physical state all n oscillators are excited and that the physical picture does not reduce to a one-dimensional oscillator with doubled frequency. The case $n = 3$ is an exception. In this case the normalization factor in Eq. (2.22) $[(2k)!]^{-1/2}$ admits such an interpretation. This can also be seen from Eq. (2.16) with $l = 0$, $n = 3$, and $rL_k^{1/2}(r^2) \sim H_{2k+1}(r)$ (Ref. 37, p. 193).

This representation has been found to be very useful in the case of more complicated groups for the adjoint representation (see Secs. 3.2 and 9.8).

2.3. The residual discrete group and the choice of physical variables ("choice of gauge")

2.3.1. The noninvariant approach

The foregoing presentation can create the impression that the residual discrete group $\mathbb{Z}_2$, being a subgroup of the gauge group, is an objective characteristic of a dynamical system (together with, say, the dimension of the physical space). In a certain sense this is true. The exact assertion, however, is as follows: The residual group is wholly determined by the choice of gauge[39] and the group $\mathbb{Z}_2$ is the simplest of the possible groups, as a result of which the gauge condition $x_2 = 0$ is the simplest method for eliminating the unphysical variable. A poor choice of physical variables can greatly complicate the residual gauge group. By its very nature this fact is related to the problem of nonuniqueness of the gauge, which was discovered in the study of Yang–Mills fields.[11] We shall now discuss the details.

Suppose that in the model (2.1) the physical variable is determined not by the straight line $x_2 = 0$ but rather by a line in the plane (x_1, x_2). There exist several fundamentally different possibilities, illustrated in Fig. 1 by the curves *1–3*. The concentric circles in the figure are orbits. The choice of *gauge* $x_2 = f_1(x_1)$ *(curve 1) is not acceptable at all, because* the physical region $0 \leqslant r < r_0$ ($r = |\mathbf{x}|$) is excluded (curve *1* is tangent to the circle $S(r_0)$ with radius r_0). The choice

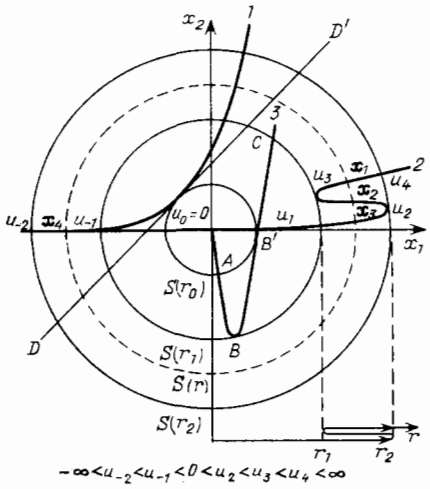

FIG. 1.

$x_2 = f_2(x_1)$ (curve *2*) means that orbits with $r<r_1$ and $r>r_2$ intersect twice while orbits with radii $r_1<r<r_2$ intersect four times. In contrast to the case $x_2 = 0$, transformations connecting the points of the curve 2 which lie on the same orbit do not form a subgroup of the gauge group. Indeed, by means of a rotation transformation the point $\mathbf{x}_2$ on the circle $S(r)$ (see Fig. 1) can be transferred into the point $\mathbf{x}_1$, but under such a rotation the point $\mathbf{x}_3$ is not transferred into $\mathbf{x}_2$, $\mathbf{x}_4$ into $\mathbf{x}_3$, and $\mathbf{x}_1$ into $\mathbf{x}_4$. For this reason the composition of two residual transformations, generally speaking, does not give a new residual gauge transformation. More elaborate curves can complicate even more the structure of the residual gauge group. Trajectories projected on the x_1 axis can have discontinuities; an example is curve 3 after the twice-traversed section $BB'C$ ($BB'\sim B'C$) is excluded. It is clear that when more realistic systems are studied all such possibilities must be taken into account.

This model is a good illustration of the problem of nonuniqueness in fixing the gauge. It is clear that a poor choice of the gauge condition can make the problem meaningless (curve *1*) and can even make it fantastically more complicated (the continuation of curve *2* with increasing amplitude of oscillations). We note that the recipe proposed in Ref. 11 for eliminating the nonuniqueness of the gauge in a path integral (integration over the region $\Delta>0$, where Δ is the Faddeev–Popov determinant) should at least not be regarded as a general method suitable for any choice of gauge. In the model at hand, for example, we fix the gauge by the condition $x_2 = 0$ for $x_1 > 0$ and $x_2 = f_3(x_1) = x_1(x_1 - a)$ for $x_1 \geqslant 0$. In this case the determinant Δ is $\det\{\chi,\sigma\} = \{x_2 - f_3(x_1), \mathbf{p}T\mathbf{x}\} = x_3' + f_3' f_3 = x_1(2x_1^2 - 3ax_1 + a^2 + 1)$. The equation $\Delta = 0$ gives the points at which the gauge orbits are tangent to the curve $x_2 = f_3(x_1)$; its solutions are $x_1^{(1,2)} = [3a \pm (a^2 - 8)^{1/2}]/4$ and $x_1^{(3)} = 0$. It is obvious that real solutions which are different from zero are possible only for $a^2 \geqslant 8$. In Fig. 1 these are points at which the curve 3 is tangent to the circles $S(r_1)$ and $S(r_0)$. It is also obvious that the determinant $\Delta = 2x_1(x_1 - x_1^{(1)})(x_1 - x_1^{(2)})$ is negative in the interval $(x_1^{(1)}, x_1^{(2)})$. According to the recipe of Ref. 11 precisely this interval—it corresponds to the motion between the points B and B'—is discarded. But from Fig. 1 it is clear that the interval (r_0, r_1) is thereby taken into account twice: once on the segment AB and again on the segment $B'C$. For the curve 2 this fact is illustrated by the symbolic contour of the variable $r = |\mathbf{x}|$: The segments with the arrows between the points r_1 and r_2, on which $\Delta > 0$, make identical contributions to the integral, i.e., they lead to double counting. The correct recipe takes into account all parts of the curve without exception; in this case, the determinant must be taken with the appropriate sign, so that the contributions from integration over segments in opposite directions would mutually cancel.

We note at the same time that although the nature of the phenomenon of nonuniqueness discovered in Ref. 11 is identical to that in the model (2.1) (the physical variables are singled out by transforming to curvilinear coordinates) these cases are concerned with different phenomena. Reference 11 is concerned with fixing the arbitrariness of the theory (in our language, fixing y), while here we are concerned with eliminating the unphysical variables, say, x_2 after y has been eliminated. We also note that the phenomenon of gauge nonuniqueness is usually attributed to the complexity of the gauge group, since this problem was encountered in the study of non-Abelian theories. It is clear from the example presented above that this problem can also arise in an Abelian theory (see also Ref. 40). In the non-Abelian theory the invariant gauges (for example, the Fermi gauge), which do not admit the existence of finite-dimensional residual gauge groups,[40] are "unsuccessful." This means that the relativistically invariant gauge condition is not natural for such systems (see also Ref. 41).

Remarks. 1) The determinant Δ does not permit judging the admissibility of the gauge condition. For the straight line DD' in Fig. 1 ($x_2 = x_1 + b$, $\Delta = 2x_1 + b$, $b>0$) $\Delta>0$ for $x_1 > -b/2$, though this is not at all a suitable method for fixing the gauge. 2) In this model the case $\Delta = \text{const}$ corresponds to an inadmissible gauge. Let $\Delta = x_1 + f'f = c$. Then $f = \pm[R^2 - (x_1 - c)^2]^{1/2} CF1$ *or* $x_2 \mp [R^2 - (x_1 - c)^2]^{1/2} = 0$; this is the equation of a circle, and this is inadmissible (for example, for $0<c<R$ the semiaxis $r > c + R$ is excluded). The assertion is apparently correct for any compact gauge group. To solve the question of admissibility of the gauge condition we turn to Dirac's scheme and the invariant approach.

2.3.2. The invariant approach[39]

We know (Sec. 2.1) that the gauge $x_2 = 0$ is directly related with the invariant description in terms of $r = (\mathbf{x}^2)^{1/2} = \text{inv}$. What do nontrivial gauge conditions of the type of the conditions prescribed by the curves *2* and *3* in Fig. 1 look like in the invariant approach? We transform from the variables x_1 and x_2 to the variables θ and u according to the rule[39,42]

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\theta T} \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix}, \tag{2.23}$$

where $f_i(u)$, $i = 1, 2$, are some functions of the parameter u. The case $f_1(u) = u$ and $f_2 = 0$ corresponds to transforming to polar coordinates $u^2 = x^2$; the matrix $e^{\theta T}$ in Eq. (2.23) rotates the ray $x_2 = 0$, $x_1 = r > 0$ so that it sweeps over the entire plane. For arbitrary f_i the equalities $x_1 = f_1(u)$ and $x_2 = f_2(u)$ give some line l in the plane $x_{1,2}$, the parameter u being an invariant of the gauge transformations ($f_1^2(u) + f_2^2(u) \equiv r^2(u) = \mathbf{x}^2$).

The mapping (2.23), generally speaking, does not necessarily determine the substitution of variables for all $(u,\theta)\in\mathbb{R}^2$. But this mapping will be a substitution of variables if a region $\subset\mathbb{R}^2$ is defined such that it prescribes a one-to-one correspondence between $(u,\theta)\in\tilde{K}$ and $\mathbf{x}\in\mathbb{R}^2$. This imposes certain restrictions on the curve l. In particular, the curve must pass through the origin of coordinates and for some u it must pass to infinity. For this reason, it is always possible to choose a parameter u such that for motion along the curve l it increases monotonically from $-\infty$ to $+\infty$, and $f_i(0)=0$ and $r^2(u)\to\infty$ as $u\to\infty$. We shall also assume that l is a continuous curve.

To determine the region K it is necessary to study the symmetry group $\tilde{S}$ of the mapping (2.23), i.e., the group of transformations of the plane $(u,\theta)\to(u_s,\theta_s)$ for which $\mathbf{x}$ in Eq. (2.23) does not change. For example, for a polar coordinate system ($f_1=u\equiv r$, $f_2=0$) this group consists of the two subgroups $\tilde{S}=S_\theta\times S$, where S_θ: $\theta\to\theta+2\pi n$, $r\to r,n$ is an integer and S: $r\to -r$, $\theta\to\theta+\pi$. The points in the plane $(r,\theta)\in\mathbb{R}^2$, that are connected by transformations from S correspond to one and the same point $\mathbf{x}\in\mathbb{R}^2$. Therefore $\tilde{K}$ is the fundamental region of the plane $\mathbb{R}^2(r,\theta)$ with respect to the action of the group S; any point $(r,\theta)\in\mathbb{R}^2$ can be obtained from a point $(r,\theta)\in\tilde{K}$ by the action of an appropriate element from $\tilde{S}$. For this reason, $\tilde{K}=\mathbb{R}^2/\tilde{S}$ is a strip, for example, $\theta\in[0,2\pi)$, $r\in[0,\infty)$. Analogously, for the variables u, θ in Eq. (2.23) we define $\tilde{S}=S_\theta\times S$, where S_θ: $\theta\to\theta+2\pi n$, $u\to u$, n is an integer and S: $u\to u_s=u_s(u)$, $\theta\to\theta+\theta_s(u)$. The general case differs from the case analyzed here by the fact that the transformation from S is a function of u, i.e., for different u the groups S can be different. The form of these functions is determined by the conditions of intersection of the curve l with circles. Indeed, by definition of S we have

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=e^{\theta T}\begin{pmatrix}f_1(u)\\f_2(u)\end{pmatrix}=e^{(\theta+\theta_s)T}e^{-\theta_s T}\begin{pmatrix}f_1(u)\\f_2(u)\end{pmatrix}=e^{(\theta+\theta_s)T}\begin{pmatrix}f_1(u_s)\\f_2(u_s)\end{pmatrix}. \tag{2.24}$$

The last equality means that the point $\mathbf{x}\in\mathbb{R}^2$ does not change when (u,θ) is replaced by $(u_s,\theta+\theta_s)$; this indicates that the transformation belongs to S. It is obvious that the transformations from S satisfy the group axioms, i.e., a composition of two transformations from S belongs to S and every transformation from S has an inverse transformation which also belongs to S.

The next to last equality in Eq. (2.24) shows that the points $f_i(u)$ and $f_i(u_s)$ lie on the same circle and transform into one another under a rotation by an angle θ_s. Therefore the transformation $u\to u_s$ connects all points of intersection of the curve l with a circle of radius $r(u)$. In particular, all functions u_s can be found by solving the equation

$$r^2(u_s(u))=r^2(u), \tag{2.25}$$

where $r^2(u)=f_i^2(u)=\mathbf{x}^2(u)$.

For given u Eq. (2.25) can have several solutions. Moreover, the structure of the solutions of Eq. (2.25) may be extremely complicated. For example, the curve l can intersect itself, it can intersect any circle an even number of times, etc. We will not analyze these cases here. We require only that, for the sake of simplicity, Eq. (2.25) have a finite number of solutions for every $u\in\mathbb{R}$, i.e., the curve l must intersect each circle centered at the origin of coordinates a finite number of times.

We divide the straight line $u\in\mathbb{R}$ into segments $\mathbb{R}_\alpha$ so that as u varies over $\mathbb{R}_\alpha$ the number N_α of solutions of Eq. (2.25) $u_s(u)$ is fixed; then, since $f_s(u)$ is continuous the $u_i(u)$ depend continuously on $u\in\mathbb{R}_\alpha$. We denote by S_α the group of transformations from S that act on $\mathbb{R}_\alpha$. Obviously, $S=\Pi_\alpha S_\alpha$. The fundamental region $\tilde{K}$ for the substitution (2.23) consists of $\theta\in[0,2\pi]$ and $u\in K$, and in addition $K=\cup_\alpha K_\alpha$, where $K_\alpha=\mathbb{R}_\alpha/S_\alpha$ is the fundamental region in $\mathbb{R}_\alpha$ relative to the group S_α.

For example, for curve 2 in Fig. 1 the entire axis $u\in\mathbb{R}$ separates into three regions: $\mathbb{R}_1=(u_{-1},0)\cup(0,u_1)$, $\mathbb{R}_2=(u_{-2},u_{-1})\cup(u_1,u_2)\cup(u_2,u_3)\cup(u_3,u_4)$ and $\mathbb{R}_3=(-\infty,u_{-2})\cup(u_4,\infty)$ (the points of the curve l $f_i(u_a)$, $a=-2,-1,\ldots,4$) are marked in Fig. 1 by the points u_a), which differ by the number of times this curve intersects the circles: $N_1=2$, $N_2=4$, and $N_3=2^6$. For $u\in\mathbb{R}_{1,3}$ Eq. (2.25) has a unique nontrivial solution, i.e., $u_s\neq u$. By definition the function $u_s(u)\in S_1$ prescribes a one-to-one mapping $u_s:\mathbb{R}_1\to\mathbb{R}_1$. According to Eq. (2.25) it consists of the one-to-one mapping $(0,u_1)\to(0,u_{-1})$ for $u\in(0,u_{-1})$ and the inverse mapping $(0,u_{-1}\to(0,u_1)$ for $u\in(0,u_{-1})$, so that $u_s(u)$ is continuous at the point $u=0$ ($u_s(0)=0$). The function $u_s(u)\in S_3$. is defined analogously. For this reason, the groups $S_{1,3}$ are isomorphic to $\mathbb{Z}_2$. For $u\in\mathbb{R}_2$ Eq. (2.25) has three nontrivial solutions for fixed u. The transformations from S_2 establish a one-to-one correspondence between points of the segments (u_{-2},u_{-1}), (u_1,u_2), (u_2,u_3) and (u_3,u_4). Here the functions u_s: $\mathbb{R}_2\to\mathbb{R}_2$ are all possible compositions of four one-to-one mappings: of the segment (u_{-2},u_{-1}) into any other segment in $\mathbb{R}_2$ for $u\in(u_{-2},u_{-1})$, of the segment (u_1,u_2) into any other segment in $\mathbb{R}_2$ with $u\in(u_1,u_2)$, and analogous transformations for (u_2,u_3) and (u_3,u_4). Obviously the functions u_s form the group of permutations of four segments in $\mathbb{R}_2$, i.e., S_2 is isomorphic to P_4—the permutation group. For this reason either of the two segments $(u_{-1},0)$ or $(0,u_1)$ can be taken as K_1; any semiaxis in $\mathbb{R}_3$ can be taken as K_3; and finally, any of the four segments constituting $\mathbb{R}_2$ can be identified with K_2. Thus here there are $2\cdot4\cdot2=16$ methods for choosing the fundamental region K. Fixing one of them and setting $u\in K$ we uniquely define the function $u_s(u)$: $K\to R$ from Eq. (2.25) or the definition (2.24). We note that the form of the functions $f_i(u)$ determines uniquely the form of the functions $u_i(u)$ and vice versa.

The substitution of variables (2.23), generally speaking, does not have a global coordinate grid in the plane $\mathbf{x}\in\mathbb{R}^2$. However this plane can be divided into rings in a manner so that such a coordinate grid will exist within each ring, i.e., a coordinate grid of the substitution of variables (2.23) is obtained by gluing together a finite and generally speaking countable number of maps (rings in this case). As an example, we consider once again the curve 2 in Fig. 1. Let $K_1=(0,u_1)$, $K_2=(u_1,u_2)$ and $K_3=(u_4,\infty)$. Then inside the circle $S(r_1)$ the coordinate grid is constructed from circles centered at the origin of coordinates and a fan of curves $l_1(\theta)$, which are obtained from a piece of the curve l, $u\in K_1$ by simultaneously rotating all its points by angles $\theta\in[0,2\pi)$. The coordinate grid in the ring between the circles $S(r_1)$ and $S(r_2)$ also consists of circles centered at the origin of coordinates and curves $l_2(\theta)$, which are obtained by simultaneous-

ly rotating all points of a piece of the curve l, $u \in K_2$ by angles $\theta \in [0, 2\pi)$. The coordinate grid outside the circle $S(r_2)$ is defined analogously.

The coordinate grid can be changed by changing the method for choosing K. For example, if in our example $K_2 = (u_1, u_2)$ is replaced by the segment (u_3, u_2), then the curves $l_2(\theta)$ of the new coordinate grid are obtained by rotating the piece of the curve l, $u \in (u_2, u_3)$. In so doing, however, only the method of gluing together local coordinate grids, which exist for $u \in K_\alpha$, i.e., giving the coordinate line $\theta = \text{const}$ as a continuous function of u, changes. The existence of such a gluing is guaranteed by the definition of the regions $\mathbb{R}_\alpha$. Thus for the example analyzed $K = (0, u_1) \cup (u_1, u_2) \cup (u_4, \infty)$, the line $\theta = \text{const}$ is obtained by gluing pieces of the curve l, $u \in (0, u_2)$ and $u \in (u_4, \infty)$, where the points u_4 and u_2 in Fig. 1 are made to coincide by rotating the second piece.

We are now ready to analyze the dynamics in the variables (2.23). For this we introduce the momenta that are canonically conjugate to the variables θ, u:

$$p_\theta = \mathbf{p}T\mathbf{x} = \sigma, \quad p_u = \frac{1}{2}(\mathbf{p}, \mathbf{x})\frac{\mathrm{d}}{\mathrm{d}u}\ln \mathbf{x}^2. \tag{2.26}$$

It is easy to check, using Eqs. (2.23), that $\{p_\theta, p_u\} = 0$, $\{\theta, p_\theta\} = \{u, p_u\} = 1$ (to obtain the last equality it is necessary to determine the derivative $\partial u/\partial x_i$, taking into account the equality $\mathbf{x}^2 = f_1^2(n) + f_2^2(u)$). It follows from Eq. (2.26) that θ is an unphysical variable, because $p_\theta = \sigma$ and σ is a secondary constraint, while u is a physical degree of freedom.

The points of the phase plane (u, p_u), connected by transformations from S: $u \to u_s$, $p_u \to p_{u_s} = (\mathrm{d}u_s/\mathrm{d}u)^{-1} p_u$ (in Eq. (2.26) $\mathrm{d}/\mathrm{d}u_s = (\mathrm{d}u_s/\mathrm{d}u)^{-1}\mathrm{d}/\mathrm{d}u$), correspond to states of the system which differ by values of the angle $\theta \to \theta + \theta_s(u)$, similarly to the manner in which in polar coordinates the states (r, p_r) and $(-r, -p_r)$ differ by the values of the angle, $\theta \to \theta + \pi$ (see Sec. 9.1). In the gauge theory θ is an unphysical variable, so that the points of the phase plane (u_s, p_{u_s}) are physically indistinguishable and should be identified with one another. This feature of the phase space (u, p_u) should be taken into account when describing the system. The last remark actually establishes a relation between the invariant and noninvariant methods for describing classical dynamics. Both methods are identical, if in the noninvariant method the reduction of the physical phase space is also taken into account: After all, the action of the group S on the invariant variable $(u \to u_s)$ is identical to that of the residual discrete gauge symmetry (RDGS), which reduces the physical phase space in the noninvariant description. Indeed, in the gauge $x_i = f_i(u)$ the points $x_i^s = f_i(u_s)$ lie on the same orbit (see Eq. (2.25)), i.e., they are related by RDGS transformations, and by construction u and u_s are related by a transformation from S. We stress, however, the fact that the nature of RDGS is different from S. RDGS is a residual symmetry in the noninvariant approach, which does not always form a subgroup of the gauge group, while S is a subgroup of the symmetry group of the mapping $\tilde{S}$. The group S can induce the action of RDGS only for a special substitution of variables in the invariant approach, when the surfaces of constant values of the physical variables are orbits of the gauge group while the surfaces of constant values of the unphysical variables are prescribed by the gauge conditions in the full configuration space. We shall say that such a substitution of variables is matched with the law of the gauge transformation and the gauge.

The last remark makes it possible to solve the question of admissibility of the gauge condition in the noninvariant approach. A given gauge is admissible if there exists a substitution of variables that is matched with it and the law of the gauge transformation.

We now proceed to the quantum theory. We shall show, in particular, that the physical amplitudes do not depend on the choice of functions f_i, if on quantization the curvilinear nature of the physical variable u and the reduction of the physical phase space are taken into account. The former is achieved by quantizing before eliminating the unphysical variable θ and the latter is achieved by defining the scalar product of the physical state vectors in accordance with the rules of the substitution of variables.

After quantizing the system (see Eq. (2.5)) we transform to the curvilinear coordinates (2.23). The second equality in Eq. (2.5) is equivalent to the equation $-i\partial/\partial\theta\Phi_{\text{ph}} = 0$, i.e., $\Phi_{\text{ph}}(u, \theta) = \Phi(u)$. Then, calculating the Laplace–Beltrami operator in the coordinates (2.23) and dropping in it the terms containing $\partial/\partial\theta$ we obtain the Schrödinger equation in $\mathscr{H}^f_{\text{ph}}$ (the index f indicates the choice of functions in Eq. (2.23))

$$\left[-\frac{1}{2\mu(u)}\frac{\mathrm{d}}{\mathrm{d}u}\circ\frac{r^2(u)}{\mu(u)}\frac{\mathrm{d}}{\mathrm{d}u} + V(r^2(u))\right]\Phi_E(u) = E\Phi_E(u), \tag{2.27}$$

where $\mu(u) = (1/2)(\mathrm{d}/\mathrm{d}u)r^2(u) = f_i(u)f_i'(u)$. The scalar product in $\mathscr{H}^f_{\text{ph}}$ has the form

$$\sum_\alpha \int_{K_\alpha} \mathrm{d}u\,|\mu(u)|\,\Phi_E^*(u)\,\Phi_{E'}(u) = \delta_{EE'}; \tag{2.28}$$

here we took into account the fact that $\mathrm{d}x_1\,\mathrm{d}x_2 = \mu(u)\,\mathrm{d}u\,\mathrm{d}\theta$. In $\mathscr{H}^f_{\text{ph}}$ the integration over θ gives a factor of 2π, which we included in the normalization of Φ_E. Generally speaking, K can be chosen so that on some K_α $\mu(u) < 0$, and for this reason the modulus[7] $|\mu|$ is inserted in Eq. (2.28).

We shall prove that $\mathscr{H}^f_{\text{ph}}$ for different f_i are isomorphic to one another. In Eq. (2.27) we make the substitution

$$\frac{\mathrm{d}}{\mathrm{d}u} = \frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r},$$

after which the Hamiltonian in Eq. (2.27) transforms into the Hamiltonian (2.7). Since Eq. (2.23) is a substitution of variables, in the equality (2.28)

$$\sum_\alpha \int_{K_\alpha} |\mu(u)|\,\mathrm{d}u$$

transforms into

$$\int_0^\infty r\,\mathrm{d}r,$$

i.e., Eq. (2.28) is equivalent to (2.6). From here it follows that all $\mathscr{H}^f_{\text{ph}}$ are isomorphic to one another, since they are isomorphic to the Hilbert space of the theory (2.7) and (2.6). From here it follows that according to the analysis made in Sec. 2.2.1 Φ_E in Eq. (2.27) are regular functions of $r^2 = f_i^2(u)$. Therefore the scalar products (amplitudes) in

$\mathscr{H}^{f}_{\text{ph}}$ do not depend on f_i, since

$$\Phi_E(u) = \widetilde{\Phi}_E(r^2(u)) = \widetilde{\Phi}_E(\mathbf{x}^2). \tag{2.29}$$

The S-invariance of the physical state factors follows from the equalities in Eqs. (2.29):

$$\Phi(u) = \Phi(u_s), \quad \Phi \in \mathscr{H}^{f}_{\text{ph}}. \tag{2.30}$$

All these features must be borne in mind when constructing a path integral. They are missed when simply transforming in the classical Hamiltonian (2.4) to the new variables (2.23) and (2.26) with $p_\theta = 0$ and then quantizing by making the substitution $p_u \to -i\partial_u$ (quantum theories corresponding to different f_i are unitarily nonequivalent). Elimination of unphysical variables in the path integral with the help of only δ functions from the constraints and additional conditions without transferring to H_{eff} corresponds precisely to such a quantization procedure. The correct method of constructing a Hamiltonian path integral with an arbitrary choice of physical variables is discussed in Secs. 5.5 and 5.6.

3. MODELS WITH AN ARBITRARY SIMPLE GAUGE GROUP

In spite of its simplicity the model studied in Sec. 2.1 contains all the basic features characteristic of models with both more complicated groups and a larger number of degrees of freedom (for example, field models). In this section the dynamical features of the simplest systems with an arbitrary gauge group are studied. The problem is important because before studying the dynamics of the physical variables of a system one must know how to separate these variables, and this is not easy to do in the case of an arbitrary group. However, the difficulties here are connected not so much with the complexity of the group as with the nontrivial nature of the representation. Thus in Sec. 2.2 a model with a group of arbitrary rank SO(n) was studied, and the example was a very simple one. This happened because the dynamical variables belonged to an elementary representation of the group which contained only one physical variable. We shall study primarily the case when the variables transform according to the adjoint representation; this is important for studying Yang-Mills fields.

3.1. The classical theory

3.1.1. Separation of the physical variables

We shall study the model given by the Lagrangian[13,21]

$$L(x, \dot{x}, y, \dot{y}) = \frac{1}{2}\operatorname{Tr}(D_t x)^2 - V(x), \tag{3.1}$$

where $D_t = \partial_t + [y, \ ]$, x and y are elements of the Lie algebra X of some simple compact group G, $x = x_a\lambda_a$ (and analogously for y) λ_a form a basis in X, and $\operatorname{Tr}\lambda_a\lambda_b = \delta_{ab}$, $[\lambda_a, \lambda_b] = F^c_{ab}\lambda_c$, $a, b, c = 1, 2, \ldots, N = \dim G$. The Lagrangian is invariant under the gauge transformations

$$x \to \Omega x \Omega^{-1}, \quad y \to \Omega y \Omega^{-1} + \Omega\partial_t\Omega^{-1}, \tag{3.2}$$

if $V(\Omega x \Omega^{-1}) = V(x)$. To simplify the notation we shall assume that $\operatorname{Tr} x^2 = x^2$ (and, in general, $px \equiv \operatorname{Tr} px$). The function L defines a system with N primary constraints $\partial L/\partial\dot{y}^a = \pi_a = 0$, the Hamiltonian

$$H = \frac{1}{2}p^2 + V(x) + y_a p T_a x \tag{3.3}$$

(here $p = \lambda_a p_a$, $p_a = \partial L/\partial\dot{x}_a$), and N secondary constraints $\sigma_a \equiv pT_a x = \operatorname{Tr}\lambda_a[p,x] = F^a_{bc}p_b x_c = 0$. All constraints are primary.[29] We are required to separate from the N variables the physical variables and to find them as functions of time. Here two approaches are likewise possible—invariant and noninvariant.

First we ask the question: how many physical variables are there in all? In the invariant approach the answer is simple: it is sufficient to find all independent gauge invariants—they will determine the set of physical variables. Their number, in turn, is determined by the number of independent invariant symmetric tensors in the adjoint representation, i.e., it is equal to the rank l of the group. Therefore the problem contains l physical degrees of freedom. This means, in particular, that among the N constraints $\sigma_a = 0$ only l are independent. However it is difficult to formulate a theory in terms of only invariant physical quantities, because the invariants are polynomials of x of different degrees and it is not at all simple to formulate a theory in the new variables (for groups of rank 2 this is done in Refs. 21 and 15). It is even more difficult to investigate next the dynamical features of the system (see Secs. 9.7 and 3.2.2).

The noninvariant approach rescues us. Let x be a given element of the algebra X; the key formula for what follows is (Ref. 14, p. 459)

$$x = S(z)\, hS^{-1}(z), \tag{3.4}$$

in which $h = h_i\lambda_i$, $i = 1, 2, \ldots, l$, $\{\lambda_i\}$ is the maximum set of mutually commuting generators (i.e., h is an element of the Cartan subalgebra H (Ref. 14)), and $S(z) \in G$ is obtained by exponential mapping of the element $z = z_\alpha\lambda_\alpha \in X \ominus H$ into the group $G(\alpha = l+1, l+2, \ldots, N)$, and h_i and z_α are real numbers (or functions of time). The variables h_i are analogous to x_1 from Sec. 2.1, and z_α can be eliminated by a gauge transformation. Indeed, since the variables x transform according to (3.2), then in accordance with Eq. (3.4) by means of an appropriate gauge transformation x can always be transformed into an element of the Cartan subalgebra:

$$h = \Omega x \Omega^{-1}, \quad \Omega = S^{-1}(z). \tag{3.5}$$

Thus any element of the algebra X is gauge equivalent to some element of the Cartan subalgebra (Ref. 43, p. 305). The dimension l of the maximum commutative subalgebra H determines the number of physical variables, for which the functions $h_i(t)$ can be taken. The formula (3.4) permits eliminating N—l unphysical variables. Although formally there are N secondary constraints, only N—l will be independent; this follows at least from the invariance of h relative to l—the parametric subgroup of transformations (3.4) with $z \in H$, i.e., l generators (constraints) identically vanish on the elements of H. What we have said can also be explained differently: It is well known that the generators of the group λ_a in the adjoint representation have l zero eigenvalues [44, p. 197], i.e., l of them vanish identically on l linearly independent vectors (in the example from Sec. 2.2 with the gauge group SO(n) $(n-1)(n-2)/2$ generators of the stationary subgroup of the vector vanished,[12] since a nonadjoint representation was studied there). Thus, the subspace of physical variables has been separated out. We shall now study its structure.

3.1.2. The structure of the physical subspace

Although we can no longer reduce the dimension of the physical subspace by means of gauge transformations, the physical subspace can be further reduced. As is well known (Ref. 14, p. 469), any Lie group contains a discrete finite subgroup W (Weyl's group), generated by mirror reflections relative to hyperplanes in H, orthogonal to the roots of the group (the algebra X is a linear space[44]). The group W is isomorphic to the group of permutations of the roots, i.e., the group preserving the system of roots. The existence of this discrete gauge group is what leads to further reduction of the space. While the group $\mathbb{Z}_2$ from Sec. 2.1 identified the mirror-symmetric points of the straight line x_1, i.e., it "reconstructed" the entire real axis x_1 based on the set $x_1 > 0$, Weyl's group reconstructs, based on some subspace $K^+ \subset H$ (called Weyl's chamber (Ref. 14, p. 470)), the entire space H (with the exception of the boundaries of the chamber—a set of measure zero).[8)] In what follows Weyl's chamber is the intersection of all positive half-spaces, bounded by hyperplanes orthogonal to simple roots ("positiveness" is defined by the direction of the roots; "Weyl's chamber is an open convex cone" (Ref. 45, p. 104), i.e., for any $h \in K^+$ the relation $(h,\omega) = h_i\omega_i > 0$, where ω belongs to the set of simple roots, is satisfied).

So, residual gauge transformations acting in H and forming a Weyl group make it possible to identify some subregions of the linear space H with one another. It is clear that this leads to reduction of the phase space. Indeed, the phase space in our case can be defined as an even-dimensional space $M_{2l}(h_i,p_i) = \Gamma^{2^l}$ (see Sec. 9.5)) provided with a symplectic structure.[46] Since $p_i = \partial L/\partial h_i$ and L is gauge invariant, p_i is also an object of the representation of Weyl's group and the physical phase space is reduced, because some of its points are identical to one another. It is precisely in M_2 (h_i, p_i) that the point (h, p) and (whw^{-1}, wpw^{-1}), where $p = \lambda_i p_i$, $w \in W$, are gauge equivalent. When they are identified with one another M_2 transforms into a $2l$-dimensional hypercone, which is unfoldable after an appropriate cut into $K^+ \otimes \mathbb{R}^l$.[13] As in the case of the simplest systems, reduction of the physical phase space radically changes the dynamics of the system. It often happens that it is simpler to study the quantum theory than the classical theory. The present problem confirms this observation. For this reason, we shall begin our study of the characteristics of the simplest mechanical system—the mechanical oscillator—with quantum mechanics.

3.2. Harmonic oscillator

3.2.1. Quantum theory

The quantum description of the model does not present any difficulties. Because of the antisymmetry of the matrices T_a (i.e., the structure constants), when transferring to the operators of the problem ordering of these matrices does not arise either in the constraints or in the Hamiltonian. In accordance with Dirac's recipe[29] the conditions

$$\hat{\pi}_a\Phi = 0, \quad \hat{\sigma}_a\Phi = 0, \quad \Phi \in \mathcal{H}_{\text{ph}}, \quad a = 1, 2, \ldots, N. \tag{3.6}$$

single out the physical Hilbert space. The first set of conditions is not interesting—it is equivalent to the assertion that Φ is independent of y^a (see Sec. 9.5). The second set is more informative, because it is related with the curvilinear coordinates. The transition to "radial" (transversal[43]) and angular (orbital) variables in the kinetic energy operator $\hat{p}_a^2$ (i.e., the transition to invariant and noninvariant variables) is a separate, quite complicated problem[43] (see Secs. 5.3 and 9.6), so that in order to elucidate the essence of the matter we shall confine our attention to the problem of a harmonic oscillator. Transferring to the operators $\hat{a}_b = (\hat{p}_b - i\hat{r}_b)/\sqrt{2}$ and $\hat{a}_b^+$ we write out the Hamiltonian (3.3) with $V = x_a^2/2$ and the constraints σ_b in the form

$$\hat{H} = \hat{a}^+\hat{a} + \frac{N}{2} + iy_b\hat{a}^+T_b\hat{a}, \quad \hat{\sigma}_b = \hat{a}^+T_b\hat{a}, \tag{3.7}$$

and

$$(\hat{a}_1^+)^{n_1}\ldots(\hat{a}_N^+)^{n_N}|0\rangle, \quad (\hat{a}_b|0\rangle \equiv 0), \quad n_i = 0, 1, \ldots, \tag{3.8}$$

form a basis of the Hilbert space. The operators $\hat{\sigma}_a$ satisfy the commutation relations

$$[\hat{\sigma}_a, \hat{\sigma}_b] = F_{ab}^c\hat{\sigma}_c, \tag{3.9}$$

i.e., $\hat{\sigma}_a$ are generators of the gauge group, so that from Eq. (3.6) and the equality $\sigma_a|0\rangle = 0$ it follows that the basis of the physical subspace is constructed by applying to $|0\rangle$ invariant polynomials of $\hat{a}_b^+$. The number of independent invariant polynomials is equal to the rank of the group l (Ref. 14, p. 573), and therefore the physical basis vectors are exhausted by the states[13]

$$P_{r_1}^{n_1}(\hat{a}^+)\ldots P_{r_l}^{n_l}(\hat{a}^+)|0\rangle, \tag{3.10}$$

where P_{r_i} are invariant homogeneous polynomials of degree $r_1,\ldots,r_l$ (r_i are the degree of the Casimir operators), and $n_i = 0, 1,\ldots$. The degrees r_i for all simple groups are given in Sec. 9.8 (see Table I). The homogeneity of the polynomials implies that

$$[\hat{a}^+\hat{a}, P_{r_i}(\hat{a}^+)] = r_iP_{r_i}(\hat{a}^+), \tag{3.11}$$

whence we find the energy spectrum of the physical states

$$E = \sum_{i=1}^{l} r_in_i + \frac{N}{2}. \tag{3.12}$$

It is clear that the spectrum (3.12) is degenerate. In a nongauge theory ($y^a = 0$ in Eqs. (3.1) and (3.3)) the spectrum would be given by the formula

$$E = \sum_{i=1}^{N} n_i + \frac{N}{2},$$

i.e., the pattern of energy levels changes radically. Since r_i are integers, the degree of degeneracy for fixed E is determined by the number of solutions of the Diophantine equation (3.2) with a fixed integer $E - N/2$. In order to visualize better the structure of the excited states we shall take in Eq. (3.10) the simplest invariant polynomial $P_2(\hat{a}^+) = \Sigma\hat{a}_a^{+2}$, which is a Casimir operator for all groups. It is obvious that the physical states correspond to excitation of all oscillators in the case of an arbitrary group also.

3.2.2. The classical theory. Analysis of the dynamics for groups of rank 2

It is obvious from the formula (3.12) that the spectrum of the system is determined by quite subtle characteristics of the gauge group (the degrees of invariant polynomials). Due

to the gauge invariance the number of physical degrees of freedom of the system (3.2) has been reduced from N ($=\dim G$) to the rank l ($=\dim H$) of the group. Judging from the basis of the physical states (3.10) and the energy spectrum (3.12), the resulting system can be represented as a collection of independent subsystems, each of which has the spectrum $E_i = nr_i + \text{const}$, $n = 0, 1, \ldots$, in which the spacings between the energy levels are r_i times greater than the spacing for a normal oscillator. What does this fact indicate?

To answer the question we shall analyze the dynamics in the case of groups of rank 2, when the oscillator has only two physical degrees of freedom h_1 and h_2. The physical configuration space, i.e., Weyl's chamber, is a sector in the plane (h_1, h_2). For the groups SU(3), SO(5) ~ Sp(4), and G_2 the aperture angles α of the sector are $\alpha = \pi/3$, $\pi/4$, and $\pi/6$, respectively. Figure 2 shows the trajectory of a material point in the physical configuration space for the group SU(3). For the sake of simplicity, the initial conditions were chosen so that the solutions of the equations of motion are $h_2(t) = A_2 \cos t$, $h_1(t) = A_1 \sin t$ and $A_2 > A_1$. Weyl's chamber in Fig. 1 is filled with dots. The ray OO'' is its symmetry axis. At the time $t = 0$ the particle is located at the point A. Then it moves along an ellipse, oriented along the axis h_2, and at time $t = \pi/6$ reaches the point B, i.e., the boundary ∂K^+. The further motion along the ellipse in the sector between the rays O_γ and O_{γ_1} is gauge-equivalent to the trajectory $B \to C$ in K^+. The particle is seemingly reflected from ∂K^+, moves along the trajectory $B \to C$, and reaches the point C at $t = \pi/3$. We note that the particle does not experience (the potential of the oscillator does not have singularities on ∂K^+!) any real "impact" against the wall (i.e., the action of a force), although at first glance it seems that on reflecting from the boundary ∂K^+ the momentum changes abruptly. The essence of the matter lies in the fact that the momentum before "impact" is gauge-equivalent to the momentum after "impact" (the momenta are related by a transformation from Weyl's group (see Sec. 3.1.2)), and for this reason no catastrophic change in the physical state of the particle occurs at the moment the particle reaches the boundary. In analyzing the quantum theory on the basis of a path integral (Sec. 5) we shall see that the behavior of the phase of the wave function in the case of such a reflection from the boundary of the physical configuration space differs from the behavior on reflection of a particle from an impenetrable barrier (as, for example, in the problems of a particle in a box, on a semiaxis, etc.). Let us proceed with the analysis. At the point C the particle undergoes another reflection and moves along the section of the ellipse $C \to D$ in K^+. Finally, at the time $t = \pi$ it reaches the point D, is reflected from the boundary, and passes along the trajectory in the opposite direction $D \to C \to B \to A$, ending up once again at the point A at the time $t = 2\pi$.

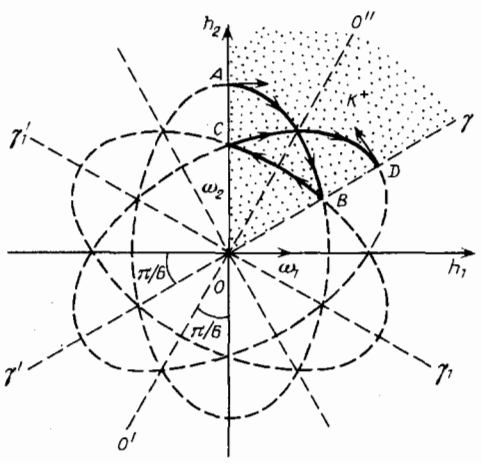

FIG. 2.

Let us see how the increase in the frequencies of the physical oscillations occurs. By definition the angular frequency is the quantity $2\pi/T$, where T is the time over which the system returns into the starting state. The state is determined by the values of the coordinate and momentum of the degree of freedom under study. Let us decompose the motion of the oscillator $h_{1,2}$ into oscillations along the axis $O'O''$ and rotation around the origin of coordinates O. After passing through the sections $A \to B \to C$ the angular degree of freedom returns to the starting state, since the angles $O''OC$ and $O''OA$ are equal to one another, and the fact that the canonically conjugate momentum is the same at these points follows from its conservation law. From the point C the angular variable once again passes along the "path" equivalent to the path studied ($C \to D \to C$), after which it returns to the point A along the path $C \to B \to A$. Thus the frequency of the oscillations of the angular variable is three times higher than the starting frequency. For oscillations along the axis $O'O''$, however, it is clear from Fig. 2 that the state of the radial degree of freedom at the point A is the same as at the point D. Therefore the frequency of the radial oscillations is twice the starting frequency.

The groups SO(5) ~ Sp(4) and G_2 could be studied in an analogous manner; in so doing we would find that the frequency of the radial oscillations (along the symmetry axis of K^+) is always doubled, while the frequencies of the oscillations of the angular variables (they are determined by the degrees of the second Casimir operators for these groups) increase by factors of 4 and 6, respectively (see Table I in Sec. 9.8). It is clear that this analysis can be extended to the case of a group of any rank, but this does not change the qualitative picture. In Sec. 9.7 explicit solutions of the classical and quantum equations of motion are given for the case just described above, and a relation is established between the variables studied here and the Casimir operators.

Thus in the classical theory the reduction of the physical phase space is manifested as the phenomenon of reflection of the particle trajectory from the boundary of the physical configuration space. In Sec. 5 we shall see that this fact must be taken into account when constructing a path integral: The reflected trajectories also contribute to the quasiclassical transition amplitude.

4. GAUGE SYSTEMS WITH GRASSMANN VARIABLES

4.1. The minimum model with an Abelian gauge group. Classical theory

Gauge theories with Fermi fields are employed just as often as theories with Bose fields. For this reason we shall now look at models with Grassmann variables. We shall first study a system which has the minimum number of anticommuting degrees of freedom and exhibits nontrivial dynamics. Let

$$L(\psi, \dot{\psi}, \psi^+, \dot{\psi}^+, y, \dot{y})$$
$$= \frac{i}{2}[\psi^+(\partial_t - iy\Gamma)\psi - (\partial_t + iy\Gamma)\psi^+\psi] - V(\psi^+\psi), \tag{4.1}$$

where the column vector $\psi = (\psi_1, \psi_2)$ is a two-component complex element of a Grassmann algebra, $\psi_\alpha = \theta_1^\alpha + i\theta_2^\alpha$, θ_j^α are "real" generatrices, and $(\theta_j^\alpha)^+ = \theta_j^\alpha$, $(\theta_j^\alpha)^2 = 0$, $\alpha_j = 1,2$; Γ is the "electric charge" operator, $\Gamma \equiv \tau_3$ (a Pauli matrix), and ψ^+ means that in the complex conjugate element ψ^* we have transferred from a column vector to a row vector. The Lagrangian (4.1) is invariant under the gauge transformations $\psi \to \exp(i\varepsilon\Gamma)\psi$, $y \to y + \dot{\varepsilon}$, where ε is an arbitrary function of time. The question of how Lagrangians which are linear in the velocities should be interpreted from the viewpoint of the Hamiltonian formalism ("formalism of the first order") is discussed in Sec. 9.2. From the analysis there it follows, in particular, that the model specified by the Lagrangian (4.1) can be regarded as a dynamical system with two primary constraints of the second kind[29] (constraints not in involution). It is easy to see that there is in addition a primary constraint of the first kind: $\pi_0 = \partial L/\partial\dot{y} = 0$. The transformation to the Hamiltonian formalism reduces to relaxing constraints of the second kind and eliminating some of the variables (actually, making the identification $\psi^+ \to \pi$; see Sec. 9.3) and replacing the Poisson brackets by Dirac brackets.[29] Having done all this, we arrive at the Hamiltonian

$$H = V(\psi^+\psi) - y\psi^+\Gamma\psi, \tag{4.2}$$

whence follows the existence of the secondary constraint of the first kind

$$\sigma \equiv \dot{\pi}_0 = \{\pi_0, H\} = \psi^+\Gamma\psi = 0 \tag{4.3}$$

(the Poisson brackets are determined by the formula (9.10)). The function V in Eq. (4.2) is specified by the expansion $V = V_0 + \omega\psi^+\psi + \Omega(\psi^+\psi)^2$, where V_0, ω, and Ω are constants. Since $\sigma = \psi_1^+\psi_1 - \psi_2^+\psi_2$, we have $(\psi_2^+\psi)^2 = -\sigma^2$. Setting $V_0 = 0$ we rewrite the Hamiltonian (4.2) in the form

$$H = \omega\psi^+\psi - y(t)\psi^+\Gamma\psi - \Omega(\psi^+\Gamma\psi)^2. \tag{4.4}$$

By virtue of the equalities (4.3) the term in the Hamiltonian that is quadratic in the constraints does not affect the equations of motion, which with $\Omega = 0$ have the form

$$\dot{\psi} = \{\psi, H\}_D = i(y\Gamma - \omega)\psi, \quad \dot{\psi}^+ = -i(y\Gamma - \omega)\psi^+ \tag{4.5}$$

and can be integrated in an elementary fashion

$$\psi(t) = \exp\left(-i\omega t + i\Gamma\int_0^t y(t')\,dt'\right)\psi(0). \tag{4.6}$$

Choosing $y(t) = -\omega$ we find

$$\psi_1(t) = e^{-2i\omega t}\psi_1(0), \quad \psi_2(t) = \psi_2(0), \tag{4.7}$$

whence we conclude that the dynamics of the degree of freedom ψ_2 is trivial and ψ_1 oscillates with twice the frequency. We know that this is an indication of reduction of the phase space (the concept of phase space for Grassmann variables is discussed in Sec. 9.3). We note that here the unphysical variables cannot be made to vanish by choosing the gauge. The difference lies in the fact that gauge arbitrariness is connected with multiplication of $\psi_{1,2}$ by ordinary complex functions, whose modulus is equal to unity, and such transformations cannot change the dimension of the Grassmann algebra. This is why an unphysical variable is manifested as a variable with trivial (or any prescribed beforehand, not determined by the Hamiltonian) dynamics.

The conclusion that the phase space of the physical variable is reduced in the model (4.1) can be arrived at by a different method. The equality $\psi_1(t) = \psi_2(t)$ ($y = 0$, $\psi_1(0) = \psi_2(0)$ in Eq. (4.6)) can be obtained by choosing the gauge and the initial conditions;[9] without violating this equality the sign of the physical component can be changed at any time $\psi_1 \to -\psi_1$ ($y(t') \to y(t') + \pi\delta(t - t')$). From here we conclude, in accordance with the definition of a phase space, that the phase space is $\Phi\Pi(\psi_1, \psi_1^+) = \mathrm{con}(\pi)$, because the pairs (ψ_1, ψ_1^+), $(-\psi_1, -\psi_1^+)$ are gauge-equivalent. By the way, it follows from here that in the expression $\psi(t) = z(t)\psi(0)$, where z is a diagonal matrix, $\arg z(t)$ for the physical variable is defined only mod (π) (see also Ref. 47).

4.2. The minimal model. Quantum theory

Quantization of the model (4.1) does not present any difficulties. According to (9.11) we transfer from the operators ψ_α, ψ_α^+ to the operators

$$[\hat{\psi}_\alpha, \hat{\psi}_\beta^+]_+ = i\{\psi_\alpha, \psi_\beta^+\}_D = \delta_{\alpha\beta}; \tag{4.8}$$

the brackets $\{\ ,\ \}_D$ are defined in Eq. (9.10). As always, the constraint $\pi_0 = 0$ can be ignored. The ground state of the system with the Hamiltonian (4.4) (the "vacuum" $|0\rangle$) is fixed by the condition $\hat{\psi}_\alpha|0\rangle = 0$. An arbitrary state $|\chi\rangle$ is defined as

$$\chi(\hat{\psi}^+)|0\rangle \equiv (\chi_0 + \chi_\alpha\hat{\psi}_\alpha^+ + \chi_3\hat{\psi}_1^+\hat{\psi}_2^+)|0\rangle = |\chi\rangle; \tag{4.9}$$

the conjugate state is

$$\langle\chi| = \langle 0|\chi(\hat{\psi}^+)^+ = \langle 0|(\chi_0^* + \chi_\alpha^*\hat{\psi}_\alpha + \chi_3^*\hat{\psi}_2\hat{\psi}_1), \tag{4.10}$$

and

$$\langle\varphi|\chi\rangle = \sum_{\alpha=0}^{3}\varphi_\alpha^*\chi_\alpha. \tag{4.11}$$

The physical states $|\Phi\rangle$ satisfy the condition that on them the constraints vanish:

$$\hat{\sigma}\,|\Phi\rangle = \hat{\psi}^+\Gamma\hat{\psi}\,|\Phi\rangle = 0. \tag{4.12}$$

Then among the eigenvectors $|0\rangle$, $\hat{\psi}_\alpha^+|0\rangle$, $\hat{\psi}_1^+\hat{\psi}_2^+|0\rangle$ of the Hamiltonian

$$\hat{H} = \omega(\hat{\psi}^+\hat{\psi} - 1)$$

(on quantizing we made the substitution $\psi^+\psi \to (1/2)(\hat{\psi}^+\hat{\psi} - \hat{\psi}\hat{\psi}) = \hat{\psi}^+\hat{\psi} - 1$ in order to take into account the noncommutability of the operators) only $|0\rangle$ and $\hat{\psi}_1^+\hat{\psi}_2^+|0\rangle$ satisfy the requirement (4.12). Indeed, $\hat{\sigma}\hat{\psi}_\beta|0\rangle = (\hat{\psi}^+\Gamma)_\beta|0\rangle \neq 0$, $\hat{\sigma}\hat{\psi}_1^+\hat{\psi}_2^+|0\rangle = (\hat{\psi}_1^+\hat{\psi}_2^+ + \hat{\psi}_2^+\hat{\psi}_2^+)|0\rangle = 0$. The spacing between the energy levels of the physical states is equal to $E_2 - E_0 = 2\omega$ in exact agreement with the classical picture.

4.3. Model with a derived gauge group. Adjoint representation

In elementary particle theory systems with anticommuting variables play no less an important role than systems with commuting variables. It is thus important to study gauge models with Grassmann variables and a non-Abelian gauge group. The difficulties here are connected not so much with the complexity of the group as with the complexity of the representation. As we have shown for the example of the group $SO(n)$ (see Sec. 2.2), it is quite simple to analyze the elementary[48] representations. The problem becomes nontrivial already for the adjoint representation. It may seem superfluous to study it for fermions, because Fermi fields usually transform according to elementary representations. This objection vanishes if we recall supersymmetry. At the same time, systems with Grassmann variables in an adjoint representation are quite informative and interesting in their own right.

Thus, we consider the Lagrangian (4.1), in which we set $V=\omega\psi^+\psi$, $\psi=\psi_a\lambda_a$, $\psi^+=\psi_a^+\lambda_a$, where λ_a is a basis in the Lie algebra of the group G from Sec. 3.1, and we replace the covariant derivative $\partial_t\psi - iy\Gamma\psi$ with $\partial_t\psi+[y,\psi]\,(y=\lambda_a y_a)$. As in Sec. 3.1 we assume that $\mathrm{Tr}\,\psi\psi^+=\psi_a\psi_a^+\equiv\psi\psi^+$. The transfer to the Hamiltonian formalism is made using the same rules as Sec. 4.2, the only difference being that now there will be $2N$ ($N=\dim G$) primary constraints (see Sec. 3.1)

$$\pi_a=\frac{\partial L}{\partial\dot\psi_a}=0,\quad \sigma_a=\psi^+T_a\psi=0,\ a=1,2,\ldots,N. \tag{4.14}$$

We turn immediately to the quantum description. Instead of the Hamiltonian (4.13) we have

$$\hat H=\omega\left(\hat\psi^+\hat\psi-\frac{N}{2}\right); \tag{4.13'}$$

Here $[\hat\psi_a,\hat\psi_b^+]_+=\delta_{ab}$, and the physical vectors satisfy conditions analogous to the conditions (4.12). As in the model with commuting variables (see Sec. 3.2), the physical states are obtained by applying to the "vacuum" invariant homogeneous polynomials $P(\hat\psi^+)$. Since the operators $\hat\psi^+$ anticommute, the polynomials P are determined by invariant *antisymmetric* tensors, which are expressed in terms of the structure constants and symmetric tensors. The polynomials sought have the form

$$P_k=\mathrm{Tr}\,(\lambda_{a_1}\ldots\lambda_{a_k})\,\hat\psi_{a_1}^+\ldots\hat\psi_{a_k}^+. \tag{4.15}$$

From Eq. (4.15) it follows that for even $kP_k\equiv 0$ (because $\mathrm{Tr}(\lambda_{a_1}\ldots\lambda_{a_k})=\mathrm{Tr}(\lambda_{a_2}\ldots\lambda_{a_k}\lambda_{a_1})$, while $\psi_{a_1}^+\ldots\hat\psi_{a_k}^+=-\hat\psi_{a_2}^+\ldots\hat\psi_{a_k}^+\hat\psi_{a_1}$). In the case of odd k we transform (4.15) as follows. We set $\hat\psi_a^+\hat\psi_b^+\lambda_a\lambda_b=1/2F_{ab}^c\lambda_c\hat\psi_a^+\hat\psi_b^+\equiv\lambda_cB_c=B$. Then

$$P_{2m+1}=\mathrm{Tr}\,(\lambda_{a_1}^{\mathrm{adj}}\ldots\lambda_{a_m}\lambda_a)\,B_{a_1}\ldots B_{a_m}\hat\psi_a^+. \tag{4.16}$$

The quantities B_a commute with one another, so that the trace in (4.16) can be symmetrized with respect to $a_1,\ldots,a_m$. We make the substitution in (4.16) $\lambda_{a_m}\lambda_a=\lambda_a\lambda_{a_m}+F_{a_ma}{}^b\lambda_b$; the term $F_{a_ma}{}^bB_{a_m}\hat\psi_a^+=(1/2)F_{a_ma}{}^bF_{cd}^{a_m}\hat\psi_a^+\hat\psi_c^+\hat\psi_d^+=0$ vanishes by virtue of Jacobi's identity for the structure constants[14]

$$-\tfrac{1}{6}F_{[ab}^cF_{a']c}^{b'}=F_{ab}^cF_{ca'}^{b'}+F_{ba'}^cF_{ca}^{b'}+F_{a'a}^cF_{cb}^{b'}=0,$$

where antisymmetrization is performed with respect to the indices in brackets. For this reason the trace in (4.16) can be symmetrized with respect to all indices $a_1,\ldots,a_m,a$, i.e., including a. This trace is a symmetric invariant tensor; it is expressed in terms of the irreducible symmetric invariants of the tensors $C_{a_1\ldots a_r}$, determining the Casimir polynomials C_r.

Any separation of the collection of indices $(a_1,\ldots,a_m,a)$ in (4.16) into subsets containing not less than two elements a_i, a (two because $r\geqslant 2$), necessarily generates Casimir polynomials of the form $P_{2r}(\hat\psi^+)=C_r(B)$, $r=2,\ldots,r_l$, $l=\mathrm{rank}\,G$. But we have established that the even invariant polynomials of $\hat\psi^+$ are equal to zero $C_r(B)=0$. The only remaining possibility is that $\mathrm{Tr}(\lambda_{a_1}\ldots\lambda_{a_m}\lambda_a)$ is an irreducible symmetric tensor (Casimir tensor) of rank $r=m+1$. Therefore the basis of the Hilbert space is generated by polynomials of the form $P_{2m+1}(\hat\psi^+)=P_{2r-1}(\hat\psi^+)$

$$[\mathrm{Tr}\,(\hat\psi^+)^{2r_1-1}]^{n_1}\ldots[\mathrm{Tr}\,(\hat\psi^+)^{2r_l-1}]^{n_l}\,|0\rangle, \tag{4.17}$$

where $n_i=0,1$ (the odd elements of the Grassmann algebra are nilpotent). The invariant antisymmetric tensors, according to (4.16) and the definition of B, have the form

$$C_{a_1\ldots a_{r-1}[a_r}F^{a_1}_{b_1c_1}\ldots F^{a_{r-1}}_{b_{r-1}c_{r-1}]}. \tag{4.18}$$

The simplest of the tensors (4.18) is identical to the structure tensor F_{bc}^a.

The basis (4.17) corresponds to the following spectrum of the Hamiltonian (4.13')

$$E=\omega\sum_{i=1}^{l}(2r_i-1)\,n_i-\frac{N\omega}{2},\quad n_i=0,1. \tag{4.19}$$

In cases of practical interest (i.e., in field theory) the problem of finding the invariant polynomials looks somewhat different, since the operators $\hat\psi$ can have additional indices, symbolizing the existence of other quantum numbers (for example, spin). They must also be taken into account in the process of antisymmetrization when constructing invariant polynomials. This question, however, falls outside the scope of this paper.

5. THE HAMILTONIAN PATH INTEGRAL FOR SYSTEMS WITH A REDUCED PHASE SPACE

In modern quantum physics it is virtually impossible to avoid the method of path integration. A Hamiltonian path integral (HPI), in which the integration extends over all paths in the phase space, is employed quite often, for example, in the study of constrained systems. It is clear that reduction of the phase space will change the HPI, so that it is important to know how. This is especially important for quantum field theory, because the path integral provides a natural possibility for going beyond the perturbation theory (classical theory). But before proceeding to field systems we must study the simplest models. This is what we shall now do.

5.1. The harmonic oscillator. The discrete gauge group $\mathbb{Z}_2$

In order to make clear the essence of the matter we shall consider the simplest model (the one-dimensional harmonic oscillator), in which the discrete gauge group $\mathbb{Z}_2$ is postulated, i.e., it appears as one of the characteristics of the phase space in the construction of the Hamiltonian formalism.

This problem can be solved in an elementary manner. The principal and simplest object of Hamiltonian path integrals is the kernel of the infinitesimal evolution operator $U_t(x,x') = \langle x|\exp(-i\hat{H}t)|x'\rangle = \langle x|\hat{U}_t|x'\rangle$, $t=\varepsilon\to 0$. In the case of a harmonic oscillator, there exists a simple formula for it (for any t) (Ref. 49, p. 218)

$$U_t(x,x') = \sum_{n=0}^{\infty} c_n^2 H_n(x) H_n(x') e^{-(x^2+x'^2)/2} e^{-iE_n t}, \quad (5.1)$$

where $E_n = n + 1/2$ is the spectrum of the energy operator; $H_n(x)$ are Hermite polynomials; the coordinates x and x' run over the entire real axis; and, c_n are normalization constants. As we determined in Sec. 2, the existence of the gauge group $\mathbb{Z}_2$ changes the spectrum of the Hamiltonian, since only even functions belong to the physical subspace. Therefore, in the case of a conical phase space (which is a consequence of the existence of the group $\mathbb{Z}_2$) the functions

$$\psi_{2n}(x) = \tilde{c}_{2n} H_{2n}(x) e^{-x^2/2}, \quad \tilde{c}_{2n} = \sqrt{2}\, c_{2n}, \quad n = 0, 1, \ldots \quad (5.2)$$

form a basis in the physical Hilbert space. The change in normalization $c_{2n}\to\tilde{c}_{2n}$ is related with the transfer in the normalization integral to integration over the semiaxis $[0, \infty)$. Therefore, instead of Eq. (5.1), we have for U_t^c (the index c indicates that the kernel pertains to a problem with a conical phase space)

$$\begin{aligned} U_t^c(x,x') &= \sum_{n=0}^{\infty} \tilde{c}_{2n}^2 H_{2n}(x) H_{2n}(x') \exp\left[-\frac{1}{2}(x^2+x'^2) - iE_{2n}t\right] \\ &= U_t(x,x') + U_t(x,-x'); \end{aligned} \quad (5.3)$$

x and x' in Eq. (5.3) run over the positive semiaxis. From here we conclude that

$$\psi_t(x) = \int_0^{\infty} dx' U_t^c(x,x')\psi_0(x') = \int_{-\infty}^{\infty} dx' U_t(x,x')\Psi_0(x'), \quad (5.4)$$

where

$$\begin{aligned} \Psi_0(x) &= \int_0^{\infty} dx' Q(x,x')\psi_0(x'), \\ Q(x,x') &= \delta(x-x') + \delta(x+x'), \end{aligned} \quad (5.5)$$

and $\psi_t(x) = \psi(x,t)$ is the wave function at time t. We note that ψ_t is normalized to unity on the entire axis only if this is true for Ψ_0. It is obvious from Eq. (5.5) that Ψ_0 is a function of ψ_0, continued in an even fashion into the unphysical region of negative values of x: $\Psi_0(x)\equiv\psi_0(x)$, $x>0$, $\Psi_0(x)\equiv\psi_0(-x)$, $x<0$, i.e., Ψ_0 is defined on the entire axis and the operator $\hat{Q}$ with the kernel (5.5) defines ψ_0 for $x<0$. From the formulas (5.4) and (5.5) there follows the representation

$$U_t^c(x,x') = \int_{-\infty}^{\infty} dx'' U_t(x,x'')Q(x'',x'), \quad (5.6)$$

from which it becomes clear how the Hamiltonian path integral is modified for the kernel of the evolution operator of the harmonic oscillator:

$$U_t(x,x') = \int \prod_{\tau=0}^{t} \left(\frac{dp(\tau)\,dx(\tau)}{2\pi}\right) \exp\left\{i\int_0^t d\tau\left[p\dot{x} - \frac{1}{2}(p^2+x^2)\right]\right\}, \quad (5.7)$$

where $x = x(t)$ and $x(0) = x'$. We shall make several remarks.

1. The formulas (5.1) and (5.7) completely solve the problem. It is helpful to verify that they are self-consistent. It is known that the representation of the operator U_t by a path integral is based on the formula $\hat{U}_{t+t'} = \hat{U}_t\hat{U}_{t'}$.[49] It is obvious that this formula holds for the operators U_t given by Eq. (5.1), while for $\hat{U}_t^c$ it must be checked: the equation

$$U_{t+t'}^c(x,x') = \int_0^{\infty} dx'' U_t^c(x,x'')U_{t'}^c(x'',x'), \quad (5.8)$$

must be satisfied, and the validity of this equation must be proved. In operator language the problem is formulated in terms of the self-consistency of the conditions

$$U_t^c = U_t\hat{Q} \text{ и } U_{t+t'}^c = U_t^c U_{t'}^c;$$

combining them, we obtain $\hat{U}_{t+t'}\hat{Q} = \hat{U}_t\hat{Q}\hat{U}_{t'}\hat{Q} = \hat{U}_t\hat{U}_{t'}\hat{Q}$, i.e., the condition

$$\hat{Q}U_t\hat{Q} = U_t\hat{Q} \quad (5.9)$$

must be satisfied. These operator equalities require explanation. The problem is that the range of the second argument of the kernel $Q(x,x')$ is the positive semiaxis, while the range of the first argument is the entire axis. Since the point of the operator $\hat{Q}$ is to continue the function in an even manner over the entire axis, $\hat{Q}$ operates on even functions as a unit operator, so that the content of Eq. (5.9) consists of requiring that the kernel $U_t\hat{Q}$ be even with respect to the first argument. From here and the fact that the kernel $Q(x,x')$ is even with respect to the first argument we find a sufficient condition for U_t: $\hat{U}_t(-x,x') = U_t(x,-x')$. In reality, the self-consistency of the apparatus of path integrals is guaranteed by the weaker condition

$$U_\varepsilon(-x,\ x') = U_\varepsilon(x,\ -x') + O(\varepsilon^2), \quad (5.10)$$

which we shall employ below.

2. From the formulas (5.1), (5.3), and (5.6) it is clear that the operator $\hat{Q}$ eliminates the contribution of unphysical states, associated with the odd functions, to the kernel $U_t^c(x,x')$. This is the point of $\hat{Q}$.

3. It is instructive to compare the role of $\hat{Q}$ in Eq. (5.6) with the role of the analogous operator in the problem of a particle on the semiaxis (with the zero boundary condition[17,18]), where it is given by the kernel $\tilde{Q}(x,x') = \delta(x-x') - \delta(x+x')$. The kernel $\tilde{Q}$ guarantees that the boundary condition at zero is satisfied; the purpose of the second δ function is to take into account also, in addition to the straight trajectory, the trajectory reflected from the boundary (approximating the trajectories by straight lines for U_ε). The difference in the signs in front of the second δ function in Q and $\tilde{Q}$ corresponds to a change in the physics of the problem. In the model at hand the motion for $x<0$ is indistinguishable from motion for $x>0$ in the opposite direction; at the point $x=0$ the particle is seemingly reflected, but without a change of phase, while in the problem of a particle on the semiaxis[17,18] the phase changes by π on reflection from the boundary. This can be interpreted as follows. In the case of an impenetrable wall the sum of the contributions of the direct trajectory and the trajectory reflected from the wall for an infinitesimal kernel U_ε^c is equal

to the sum of the contributions of the trajectories (direct) from x' and $-x'$ into the point x, and in addition the second contribution is taken with a negative sign. This guarantees that the zero boundary conditions are satisfied. In the model at hand it is also necessary to take the sum of the contributions of trajectories from the points x' and $-x'$ to x, but with the same signs. This guarantees that the wave function is even, i.e., the wave function is invariant under the gauge group $\mathbb{Z}_2$.

4. In spite of the reduction of the phase space, the measure in the Hamiltonian path integral (5.6) and (5.7) did not acquire a factor of 1/2, as one would expect. The reason is that the region of integration in the normalized integral changed: the physical variable x runs not over the entire axis but only over the semiaxis. This is why the substitution $c_n \to \tilde{c}_n = \sqrt{2}c_n$ was made in Eqs. (5.2) and (5.3).

In conclusion we call attention to the following. On transferring to polar coordinates an operator $\hat{Q}$ with the kernel[50]

$$Q(r, \varphi, r', \varphi') = \delta(r - r') \sum_{n=-\infty}^{\infty} \delta(\varphi - \varphi' + 2\pi n) + \delta(r + r') \sum_{n=-\infty}^{\infty} \delta(\varphi - \varphi' + \pi + 2\pi n).$$

also appears in the path integral. Does this result agree with the formula (5.5)? It is easy to see that it does. Since by virtue of Eq. (2.5) the physical functions do not depend on the angular variable, by integrating over φ' in

$$\int_0^{2\pi} d\varphi' \int_0^{\infty} dr' r' Q(r, \varphi, r', \varphi') \psi_0(r')$$

we eliminate the δ functions from the summand. The kernel so obtained is identical to the kernel (5.5) (to within the notation). This is actually a different method for deriving the relations (5.4) and (5.5),[10] underscoring the fact that the formulas found are completely equivalent to the consequences of the standard conditions on the physical state vectors (2.5) in Dirac's quantization scheme[29] (see the discussion at the end of Sec. 2.2.1).

5.2. Harmonic oscillator with the gauge group SO(3)

In Sec. 5.1 we studied, for pedagogical considerations, a model in which the discrete gauge group $\mathbb{Z}_2$ was postulated. We shall clarify the situation in models prescribed by a gauge-invariant Lagrangian. Consider the Lagrangian (2.1), in which $yT \to y^a T_a$, where T_a are generators of the group SO(3), $a = 1$, 2, and 3, and x and y are three-dimensional vectors. Since the constraints (2.11) $\sigma_a = \mathbf{p} T_a \mathbf{x}$ are components of the angular momentum ($F^a_{bc} = i\varepsilon_{abc}$, see Sec. 3.1), the Hamiltonian (3.3) with $V = -r^2/2$ corresponds to the operator (Ref. 51, p. 168; for the convenience of the reader we shall carry out the calculations explicitly, without referring to Sec. 2.2)

$$\hat{H} = \frac{1}{2}\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{\hat{\sigma}^2}{r^2} + r^2\right) + y\hat{\sigma} = \frac{1}{2}\left(\hat{p}_r^2 + \frac{\hat{\sigma}^2}{r^2} + r^2\right) + y\hat{\sigma}, \tag{5.11}$$

where $\hat{p}_r = -ir^{-1}\partial_r \circ r$. Since the physical states satisfy the condition $\hat{\sigma}\Phi = 0$, the problem reduces to the equation

$$\frac{1}{2}(\hat{p}_r^2 + r^2)\Phi = E\Phi; \tag{5.12}$$

We are interested in the solutions that are regular at the origin (see Sec. 2.1.2). The functions Φ_k, which behave at the origin as $\sim 1/r$, though they are normalizable (we recall that the integration over the semiinfinite interval is performed with a weight of r^2), satisfy the inhomogeneous Schrödinger equation with the δ function $\delta(x)$ on the right side (Ref. 52, p. 219). Transferring from Φ to $\psi = r\Phi$, we find that ψ are the eigenfunctions of the Hamiltonian of the one-dimensional oscillator, i.e., $\psi_n = c_n H_n(r) \exp(-r^2/2)$. From the condition that Φ is regular at zero we conclude that states with odd n form a physical basis, i.e.,

$$\Phi_{2k+1} = \tilde{c}_{2k+1} \frac{H_{2k+1}(r)}{r} e^{-r^2/2}, \quad k = 0, 1, \ldots \tag{5.12'}$$

and the energy spectrum is given by the formula $E_k = 2k + 3/2$, in agreement with Eq. (3.12). Turning once again to the formula (5.1), we find that the evolution operator of the problem is given by the equality[19]

$$U_t^c(r, r') = \frac{1}{rr'}(U_t(r, r') - U_t(r, -r')); \tag{5.13}$$

and $U_t(r,r')$ is the kernel (5.1) of the evolution operator of the one-dimensional harmonic oscillator. Finally, we arrive at formulas analogous to (5.4) and (5.5):

$$\psi_t(r) = \int_0^{\infty} dr' r'^2 U_t^c(r, r') \psi_0(r') = \int_{-\infty}^{\infty} dr' \frac{r'}{r} U_t(r, r') \Psi_0(r'), \tag{5.14}$$

$$\Psi_0(r) = \int_0^{\infty} dr' Q(r, r') \psi_0(r'),$$

where Q is given by (5.5). The difference from Eqs. (5.4) and (5.5) is that the integration measure is different. Thus the characteristic features of the initial space, including the physical and unphysical variables, are transferred into the space of physical variables. The unphysical degrees of freedom do not vanish without trace (as a consequence of the fact that they are related with the curvilinear coordinates; see also Sec. 5.6).

We note in conclusion that in the case of an arbitrary group SO(n) (see Sec. 2.2) the solution does not reduce to the one-dimensional harmonic oscillator.[19,21] In Eq. (5.12) the "quantum potential" $(n-1)(n-3)/8r^2 (\sim \hbar^2)$, which must be born in mind in the formula (5.7), is added to $r^2/2$. The integration measure in Eq. (5.14) also changes: $(r')^2 \to (r')^{(n-1)/2}$.

5.3. Model with an arbitrary gauge group. Adjoint representation

It is helpful to study the case of the arbitrary simple gauge group, which was studied in Sec. 3. To this end, in the problem with the Lagrangian (3.1) we single out the physical variables, i.e., we transfer from x to h, z (3.4), where h, according to Sec. 3.1.1, plays the role of the physical variable and belongs to the Cartan subalgebra H while z includes all unphysical variables, $z \in X \ominus H$. Mathematically the problem reduces to writing the Hamiltonian operator (3.3)

$$\hat{H} = \frac{1}{2}\mathrm{Tr}\,\hat{p}^2 + V(\hat{x}) + \hat{y}_a \hat{\omega}_a \tag{5.15}$$

in the curvilinear coordinates h, z and actually (since $-i\partial/\partial x_a \equiv \hat{p}_a$) to writing the N-dimensional Laplace operator $\Delta = \partial_a^2$ in the variables h and z. For this, it is necessary

to find the metric tensor in the new variables, i.e., it is necessary to calculate the square of the element of length $ds^2 = \mathrm{Tr}\, dx^2$. Using Eq. (3.4), we find $dx = dShS^{-1} + S\,dS^{-1} + Sh\,dS^{-1}$. Squaring dx and using the formula $dS^{-1} = -S^{-1}dSS^{-1}$, we find

$$ds^2 = \mathrm{Tr}\,\{(dh)^2 + 2[(hS^{-1}dS)^2 - h^2(S^{-1}dS)^2] + [S^{-1}dS,\ h]\,dh\}$$
$$= \mathrm{Tr}\,[(dh)^2 + [h,\ S^{-1}dS]^2] = dh_i^2 + \tilde{g}_{\alpha\beta}(h,\ z)\,dz_\alpha\,dz_\beta \tag{5.16}$$

(the trace in the last term in braces is equal to zero). The metric tensor sought has a block structure $g_{ab} = (\delta_{ij}, \tilde{g}_{\alpha\beta})$, $j, i = 1,2,..., l$; $\alpha, \beta = l+1, ..., N$. Using the representation $S^{-1}dS = \lambda_a F_\alpha^a(z)dz_\alpha$, where $F_\alpha^a(z)$ are unknown functions, and the fact that $[\lambda_i, \lambda_j] = 0$, $[\lambda_i, \lambda_\alpha^a] \in H$, we obtain $[h, S^{-1}dS] = \lambda_\gamma h_i F_{\delta\gamma}^i F_\alpha^\delta dz_\alpha$, i.e.,

$$\tilde{g}_{\alpha\beta} = F_\alpha^\gamma(z)\, g_{\gamma\delta}(h)\, F_\beta^\delta(z), \quad g_{\alpha\beta} = \omega_{\alpha\gamma}\omega_{\beta\gamma}, \quad \omega_{\alpha\beta}(h) = h_i F_{\alpha\beta}^i. \tag{5.17}$$

Therefore the Beltrami–Laplace operator has the form

$$\Delta = \frac{1}{g^{1/2}}\partial_a \circ g^{1/2} g^{ab}\partial_b = \varkappa^{-2}\partial_i \circ \varkappa^2 \partial_i + \frac{1}{g^{1/2}}\partial_\alpha \circ g^{1/2}\tilde{g}^{\alpha\beta}\partial_\beta, \tag{5.18}$$

where $g^{1/2} = \det \omega(h) \det F(z) \equiv \mu(h)\tilde{\mu}(z)$, $\mu(h) \equiv \varkappa^2(h)$, $g^{ac}g_{cb} = \delta_b^a$, $\partial_i = \partial/\partial h_i$. $\partial_\alpha = \partial/\partial z_\alpha$. We have the identity $\varkappa^{-2}\partial_i \circ \varkappa^2 \partial_i \equiv \varkappa^{-1}\partial_i^2 \circ \varkappa - \varkappa^{-1}(\partial_i^2 \varkappa)$; in Sec. 9.6 it is shown that $\varkappa(h) = \Pi_{\alpha>0}(h,\alpha)$ ($\alpha > 0$ means that the product extends over the positive roots), whence it follows that the "quantum potential" is $V_q = \varkappa^{-1}(\partial\varkappa)/2 = 0$ (see Eq. (9.25)). Based on what has been said, we arrive at the following expression for the Hamiltonian in the physical subspace:

$$\hat{H}_{\mathrm{ph}} = -\frac{1}{2}\varkappa^{-1}\partial_i^2 \circ \varkappa + V(h) \tag{5.19}$$

(according to Eq. (3.6) the physical states do not depend on z: the independent constraints are generators of translations along z_α). The scalar product of the physical state vectors is given by the formula

$$(\Phi_1, \Phi_2) = \int_{K^+} dh\mu(h)\,\Phi_1^*(h)\,\Phi_2(h), \tag{5.20}$$

in which K^+ indicates that the integration is performed inside Weyl's chamber (Ref. 44, p. 470). Here and below h is a vector in R with the components $(h_1,...,h_l)$ in the orthogonal basis: Instead of $d^l h$ or $\mathbf{dh}$ we shall write dh, instead of $\psi(\mathbf{h})$ we shall write $\psi(h)$, etc. The weight $\mu(h)$ is proportional to the volume of the gauge orbit. The narrowing of the range of the variables h_i from $H = \mathbb{R}^l$ to K^+ in (5.20) is connected, as explained in Sec. 3.1, with the action of Weyl's residual gauge group in H.[11)]

Thus, we have arrived at the quantum-mechanical problem of a particle in an l-dimensional space with unusual normalization of the state vectors. We must determine how the kernel of the evolution operator of the system with the Hamiltonian (5.19) and the scalar product (5.20) but without the symbol K^+ (integration over the entire space $\mathbb{R}^l$) changes when Weyl's gauge group is turned on, i.e., on transferring to the scalar product (5.20) with the symbol K^+. Although the Hamiltonian (5.19) is expressed in terms of the physical variables, not all solutions of the equation $\hat{H}_{\mathrm{ph}\psi} = E\psi$ will be physical solutions. This is illustrated in Sec. 5.2, where it is shown that only solutions of Eq. (5.12) (invariants of Weyl's group $\mathbb{Z}_2$) that are even in r (regular at the origin) belong to the physical Hilbert space $\mathscr{H}_{\mathrm{ph}}$.

Weyl's group W is a subgroup of the gauge group G. Hence from the fact that the physical states are stationary under G ($\hat{g}\Phi = \Phi$, $\hat{g} \in G$ and is generated by the operators σ_a; see Sec. 3.2.1) it follows that they are stationary under W ($\hat{w}\Phi = \Phi$, $\hat{w} \in W$). Consider any arbitrary function $\psi(h)$ that is normalized in the entire space. Its physical (invariant) component is

$$\Phi = N_W^{-1/2}\sum_w \hat{w}\psi(h) = N_W^{-1/2}\sum_w \psi(h_w). \tag{5.21}$$

We shall explain the notation and the meaning of Eq. (5.21). We shall write the transformation $h \to \overline{w}h\tilde{w}^{-1} = h_w$ as $h_w = wh$ (here w is an $l \times l$ matrix). By virtue of the gauge invariance of the initial Lagrangian the theory does not contain any constant noninvariant tensors. From here there follows the equality $\hat{w}\psi(h) = \psi(h_w)$, employed in Eq. (5.21). The summation in this formula extends over all elements of Weyl's group. The normalization factor is chosen from the following considerations. It is assumed that Φ is normalized according to Eq. (5.20), i.e.,

$$\int_{K^+} dh\mu\,|\Phi|^2 = 1, \quad \text{and} \quad \int_{\mathbb{R}^l} dh\mu\,|\psi|^2 = 1. \tag{5.22}$$

But the group W operates in $\mathbb{R}^l$ simply transitively on the set of Weyl's chambers,[45] so that for N_w elements of this group the symbolic equality $N_w = V_{\mathbb{R}^l}/V_{K^+}$, where $V_{\mathbb{R}}$ and V_K are the "volumes" of $\mathbb{R}^l$ and K^+ (they are infinite; we note at the same time that $N_w = r_1 ... r_l$ (Ref. 14, p. 568)), is satisfied. This explains the choice of the factor $N_w^{-1/2}$ in front of the summation sign in Eq. (5.21). It is clear that Φ is invariant: $w\Phi = \Phi$, because W is a group.

From here there follows a recipe for constructing the kernel of the physical evolution operator U_t^c. From the kernel U_t, constructed with the participation of all eigenfunctions $\psi_{(n)}$ of the operator (5.19) (analog of the kernel (5.1)), it is necessary to single out a term containing only the physical function $\Phi_{(n)}$ (analog of (5.3); here (n) symbolizes the complete set of quantum numbers specifying Φ). Using Eq. (5.21) we obtain

$$U_t^c(h, h') = \sum_{(n)} \Phi_{(n)}(h, t)\,\Phi_{(n)}^*(h', t)$$
$$= N_W^{-1}\sum_{w, w'} U_t(h_w, h'_{w'}), \quad h, h' \in K^+, \tag{5.23}$$

where $U_t(h,h')$ is the kernel of the evolution operator in the problem with the Hamiltonian (5.19) and with normalization in the entire space $\mathbb{R}^l$. We can verify directly that as $\varepsilon \to 0$

$$U_\varepsilon(h, h') = (\varkappa\varkappa')^{-1}\int \frac{dp}{(2\pi)^l}\exp\left\{i\left[p(h-h') - \varepsilon\left(\frac{p^2}{2} + V(h)\right)\right]\right\},$$
$$\varkappa' \equiv \varkappa(h'); \tag{5.24}$$

the appearance of the factor $(\varkappa\varkappa')^{-1}$ in Eq. (5.24) is connected with the fact that the eigenfunctions of the Hamiltonian $\hat{H}' = -(1/2)\partial_i^2 + V$ are, according to Eq. (5.19), $\varkappa\psi_{(n)}$. It is easy to prove that the kernel (5.24) has the prop-

erty

$$U_\varepsilon(h_w,\ h') = U_\varepsilon(h,\ h'_w) + O(\varepsilon^2) \tag{5.25}$$

for any potential $V(h)$; if, however, the potential is invariant $V(h_w) = V(h)$, then in Eq. (5.25) the term $O(\varepsilon^2)$ can be dropped. This assertion can be checked by making the substitution of variables $p \to wp$ in the integral taking into account the invariance of dp and p^2, which follows from the orthogonality of the matrices w. Based on Eq. (5.25), we rewrite the kernel (5.23) at $t = \varepsilon$ in the form

$$U^c_\varepsilon(h,\ h') = \sum_w U_\varepsilon(h, h'_w) = \int dh'' U_\varepsilon(h, h'')\, Q(h'', h'), \tag{5.26}$$

$$Q(h, h') = \sum_w \delta(h - h'_w); \tag{5.27}$$

the integration extends over the entire space, unless stipulated otherwise. The formula (5.26) is an extension of the formula (4.6) to the case of an arbitrary group. As in the formula (5.6), the kernel Q here plays a dual role: on the one hand, Q eliminates the contribution of unphysical states in U^c_t while on the other hand, based on formulas analogous to (5.4)–(5.5), it extends the initial function $\Phi_0(h)$ to the entire space $\mathbb{R}^l$. Since

$$Q(h_w,\ h') = Q(h,\ h') \tag{5.28}$$

(which is a consequence of Eq. (5.27) and the obvious equality $\delta(h_w) = \delta(h)$, which follows from the orthogonality of the transformation w), Q extends Φ_0 outside K^+ in a symmetric fashion. We shall write out the corresponding formulas explicitly:

$$\Phi_\varepsilon(h) = \int\limits_{K^+} dh' \mu(h')\, U^c_\varepsilon(h, h')\, \Phi_0(h')$$

$$= \int dh' \mu(h')\, U_\varepsilon(h, h')\, \Phi_0(h'), \tag{5.29}$$

$$\Phi_0(h) = \int\limits_{K^+} dh' Q(h, h')\, \Phi_0(h'). \tag{5.30}$$

In deriving Eq. (5.30) we employed Eq. (5.26) and the invariance of $\mu(h)$ under Weyl's group. Obviously, $\Phi_0(h \in K^+) = \Phi_0(h)$; the range of the first argument of Q is the entire space and the range of the second argument is K^+. By virtue of the formulas (5.25) and (5.27) we have the analog of the formula (5.9) for U_ε. Then repeated use of U^c_ε reduces to the operator $\hat{U}^c_t = \hat{U}_t \hat{Q}$, in which the kernel U_t is given by the path integral

$$U_t(h, h')$$

$$= (\varkappa\varkappa')^{-1} \int \prod_{\tau=0}^{t} \frac{dp(\tau)\, dq(\tau)}{(2\pi)^l} \exp\left\{ i \int_0^t d\tau \left[p\dot{q} - \frac{p^2}{2} - V(q) \right] \right\}, \tag{5.31}$$

where $h \equiv g(t)$, $h' \equiv g(0)$, $\varkappa' = \varkappa(h')$. The relations (5.26), (5.27), (5.29), and (5.31) together with the equality $\hat{U}^c_t = \hat{U}_t \hat{Q}$ and the definition of the scalar product (5.20) solve the problem.

In the formulas presented above the integration over h is performed with the weight $g^{1/2} \to \mu = \varkappa^2$. The integral (5.30) (or (6.26)) is an exception. It is easy to see that by redefining Q as $Q \to Q_\varkappa = (\varkappa')^{-1} \tilde{Q}$, we achieve complete uniformity:

$$\int\limits_{K^+} dh' Q(h, h')\, \Phi_0(h') = \int\limits_{K^+} dh' \mu(h')\, Q_\varkappa(h, h')\, \Phi_0(h'). \tag{5.32}$$

The explicit form of the kernel $\tilde{Q}$ is determined by the transformation properties of the function $\varkappa$ relative to Weyl's group (compare Eq. (5.5) and the representation with the help of $\tilde{Q}$ (5.13); see also Sec. 9.6).

5.4. Model with Grassmann variables

We shall now determine how the path integral changes when a gauge group is included in models with Grassmann variables.

By definition, the matrix element of the evolution operator is

$$\langle \varphi | e^{-i\hat{H}t} | \chi \rangle = \int d\theta\, d\theta' \langle \varphi | \theta \rangle \langle \theta | e^{-i\hat{H}t} | \theta' \rangle \langle \theta' | \chi \rangle \tag{5.33}$$

(see Sec. 9.4; we denoted the variable ξ^* by θ and we treat θ as a real variable), and in addition

$$U_t(\theta, \theta') = \langle \theta | e^{-i\hat{H}t} | \theta' \rangle = \sum_E \langle \theta | E \rangle e^{-iEt} \langle E | \theta' \rangle, \tag{5.34}$$

where the summation extends over the spectrum of the Hamiltonian. We shall find the kernel $U^c_t(\theta, \theta')$ for the model of Sec. 4.1 ($\theta = (\theta_1, \theta_2)$; see Sec. 4.1). The spectrum of the Hamiltonian (4.4) under the condition (4.12) consists of two points 0 and 2ω (the energy is measured from the energy of the "vacuum": $E \to E - E_0$)

$$U^c_t(\theta, \theta') = \langle \theta | 0 \rangle \langle 0 | \theta' \rangle + e^{-2i\omega t} \langle \theta | 2\omega \rangle \langle 2\omega | \theta' \rangle. \tag{5.35}$$

We shall choose the representation introduced in Sec. 9.4: $|\chi\rangle \to \langle \theta | \chi \rangle$; $\hat{\psi}^+_\alpha \to \theta_\alpha$, $\hat{\psi}_\alpha \to \partial / \partial\theta_\alpha$. Then, according to Sec. 9.4 (see Eq. (9.17)) $\langle \theta | 0 \rangle = 1$, $\langle \theta | \omega \rangle_1 = \theta_1$, $\langle \theta | \omega \rangle_2 = \theta_2$, $\langle \theta | 2\omega \rangle = \theta_1 \theta_2$, and in accordance with Eqs. (9.17b) and (9.19)

$$\langle 0 | \theta \rangle = \theta_1 \theta_2,\ \langle \omega | \theta \rangle_1 = \theta_2,\ \langle \omega | \theta \rangle_2 = \theta_1,\ \langle 2\omega | \theta \rangle = 1.$$

As a result we have

$$U^c_t(\theta, \theta') = \theta'_1 \theta'_2 + e^{-i2\omega t} \theta_1 \theta_2. \tag{5.36}$$

For the model without gauge symmetry (i.e., without the condition (4.12)) the terms

$$e^{-i\omega t} (\langle \theta | \omega \rangle_1 \langle \omega | \theta' \rangle_1 + \langle \theta | \omega \rangle_2 \langle \omega | \theta' \rangle_2),$$

would have been added on the right side of Eq. (5.36), i.e., we would have obtained

$$U_t(\theta, \theta') = \theta'_1 \theta'_2 + e^{-i\omega t} (\theta_2 \theta'_1 - \theta_1 \theta'_2) + e^{-2i\omega t} \theta_1 \theta_2. \tag{5.37}$$

It turns out that, as in Sec. 5.1, the kernels U_t and U^c_t are related by the formula

$$U^c_t(\theta, \theta') = \int d\theta''_2\, d\theta''_1 U_t(\theta, \theta'')\, Q(\theta'', \theta'), \tag{5.38}$$

where

$$Q(\theta, \theta') = \tfrac{1}{2} (\delta(\theta - \theta') + \delta(\theta + \theta')), \tag{5.39}$$

and $\delta\theta = \theta_1 \theta_2$, i.e., $\theta(\theta, \theta') = \theta_1 \theta_2 + \theta'_1 \theta'_2$ (δ functions of Grassmann variables are discussed in Refs. 53–56). The validity of the recipe (5.38) and (5.39) can be proved directly, for example,

$$\int d\theta''_2\, d\theta''_1 e^{-i\omega t} (\theta_2 \theta''_1 - \theta_1 \theta''_2) (\theta''_1 \theta''_2 + \theta'_1 \theta'_2) = 0.$$

We call attention to the appearance of the factor 1/2 in Eq. (5.39), as compared with the analogous formula for normal

variables (5.5), where the absence of this factor is connected with the transfer to integration over the semiaxis. For Grassmann variables the concept "semiaxis," obviously, does not exist; this is what explains the appearance of the 1/2 in Eq. (5.39).

It is easy to check that the kernel (5.36) has the necessary property

$$\int d^2\theta'' U_t^c(\theta, \theta'') U_{t'}^c(\theta'', \theta') = U_{t+t'}^c(\theta, \theta'). \tag{5.40}$$

The formulas (5.37)–(5.39) essentially solve the problem, since the construction of the path integral for U_t is standard.

We shall give the expression for the Hamiltonian path integral in the holomorphic representation. The holomorphic representation for fermions is introduced in Sec. 9.4 (see Eq. (9.13)). The physical states in this representation, satisfying the condition (4.12), are $\langle\psi^*|0\rangle = 1$, $\langle\psi^*|2\omega\rangle = \psi_1^*\psi_2^*$. Repeating the arguments leading to the formulas (5.37)–(5.39), we find

$$U_t^c(\psi^*, \psi) = \int \prod_{\alpha=1}^{2} (d\tilde{\psi}_\alpha^* d\tilde{\psi}_\alpha) \exp(-\tilde{\psi}^*\tilde{\psi}) U_t(\psi^*, \tilde{\psi}) Q(\tilde{\psi}^*, \psi), \tag{5.41}$$

where

$$Q(\psi^*, \psi) = \frac{1}{2}(e^{\psi^*\psi} + e^{-\psi^*\psi}) = \operatorname{ch}\psi^*\psi, \tag{5.42}$$

and $U_t(\psi^*, \psi)$ is the kernel of the evolution operator neglecting the gauge symmetry, for which the standard representation of the Hamiltonian path integral is valid:

$$U_t(\psi^*(t), \psi(0)) = \int \prod_{\tau=0}^{t} (d\psi^*(\tau) d\psi(\tau)) \exp\left(\frac{1}{2}\phi\right) \times \exp\left\{i \int_0^t d\tau \left[\frac{1}{2i}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - V(\psi^*\psi)\right]\right\}, \tag{5.43}$$

in which $\Phi = \psi^*(t)\psi(t) + \psi^*(0)\psi(0)$. The kernel Q in the holomorphic representation for Bose variables is derived in Sec. 9.8.

In accordance with the rule (9.16) the kernels of the operators (5.24) and (5.43) are related by the following expression:

$$U_t(\theta, \theta') = P \int \exp(-\psi'^*\psi') U_t(\psi^*, \psi') d\psi_1' d\psi_2' \Big|_{\substack{\psi'^* = \theta' \\ \psi^* = \theta}}, \tag{5.44}$$

which is also valid for other operators (for example, $\hat{Q}$).

5.5. The Hamiltonian path integral in an arbitrary gauge (the gauge group SO(2))[39]

In Sec. 2.3 it was shown that the discrete gauge group operating in the phase space of physical degrees of freedom can have a quite complicated structure if the gauge (the physical variables) are poorly chosen. In realistic theories the gauge is usually fixed not based on the structure of the orbits of the gauge group but rather based on other requirements, for example, the requirement of Lorentz covariance, which engenders the well-known problem of nonuniqueness.[11,41]

We shall show what the Hamiltonian path integral looks like for an arbitrary choice of physical variables. We shall give the solution of this problem for not too complicated, but quite representative gauge conditions for the example of Sec. 2.3. The extension of the recipe to an arbitrary group is studied in the next section.

The kernel of the unit operator $\langle u|u'\rangle_{ph}$ in the quantum theory, given by Schrödinger's equation (2.27) and the scalar product (2.28), is defined for $u, u' \in K$. However, because of the property of S-invariance (2.30) of the physical wave functions, this kernel can be analytically continued into the unphysical region $u \in \mathbb{R}$ (by complete analogy to Eqs. (5.5), (4.6), and (5.13) or (5.27))

$$\langle u|u'\rangle_{ph} = \sum_E \Phi_E(u) \Phi_E^*(u') = \int_{-\infty}^{\infty} \frac{du''}{(\mu(u)\mu(u''))^{1/2}} \delta(u - u'') Q(u'', u'), \tag{5.45}$$

where $u, u' \in \mathbb{R}u', \in K$ and

$$Q(u, u') = \sum_S \delta(u - u_s'). \tag{5.46}$$

The functions $u'_s \equiv u_s(u')$ were determined in Sec. 2.3.2.

We shall use the standard procedure for deriving the Hamiltonian path integral for an infinitesimal evolution operator. In our case the corresponding formulas are

$$U_\varepsilon^{ph}(u, u') = \langle u|e^{-i\hat{H}\varepsilon}|u'\rangle_{ph} = (1 - i\varepsilon\hat{H}(u))\langle u|u'\rangle_{ph} + O(\varepsilon^2), \tag{5.47}$$

where $\varepsilon \to 0$, and the Hamiltonian $\hat{H}(u)$ is given in Eq. (2.27). We substitute into Eq. (5.47) the expression (5.45), replacing in it the δ function by the integral $(2\pi)^{-1} dp \exp[ip(u - u'')]$. In writing out Eq. (5.47) the order in which the operators in $E(u)$ follow one another must be taken into account.[17,39] To this end, we shall rewrite the Hamiltonian in Eq. (2.27) in terms of the Hermitian momentum operators $\hat{p}_u = -i\mu^{-1/2} \times \partial_u \circ \mu^{1/2}$,[17,39] after which we perform the differentiation in Eq. (5.47). In the expression obtained $u - u''$ can be replaced by $\dot{u}''\varepsilon$. As a result, up to terms of order $O(\varepsilon^2)$ we have

$$U_\varepsilon^{ph}(u, u') = \int_{-\infty}^{\infty} \frac{du''}{(\mu(u)\mu(u''))^{1/2}} U_\varepsilon^{eff}(u, u'') Q(u'', u'), \tag{5.48}$$

where

$$U_\varepsilon^{eff}(u, u'') = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp[i\varepsilon(p\dot{u}'' - H^{eff}(u, p))]. \tag{5.49}$$

The effective Hamiltonian H^{eff} is obtained from the Hamiltonian in Eq. (2.27) by replacing all derivatives $-i\partial$ by p and adding terms that take into account the order of the operators. Ultimately V is replaced by $V + V_q$, where[39]

$$V_q = \frac{i}{2} p(\partial_u \gamma) + \frac{1}{2\mu^{-1/2}} \partial_u(\gamma \partial_u \mu^{1/2}). \tag{5.50}$$

Here $\gamma = r'^2(u)/\mu^2(u)$.

If $f_1 = u$ and $f_2 = 0$ in Eq. (2.27), then $V_q = -(8r^2)^{-1}$, where u coincides with $r = |\mathbf{x}|$ in the polar coordinate system.

In constructing the Hamiltonian path integral the ker-

nels (5.49) can be used instead of the kernels (5.48). In view of the invariance of Eq. (2.27) under the group S it can be proved,[39] as in Sec. 5.1, that $\hat{U}_\varepsilon^{\rm ph}\hat{U}_\varepsilon^{\rm ph} = \hat{U}_\varepsilon^{\rm eff}\hat{Q}\hat{U}_\varepsilon^{\rm eff}\hat{Q} = \hat{U}_\varepsilon^{\rm eff}\hat{U}_\varepsilon^{\rm eff}\hat{Q} = \hat{U}_{2\varepsilon}^{\rm eff}\hat{Q}$, where

$$U_{2\varepsilon}^{\rm eff}(u,u') = \int_{-\infty}^{\infty} du''\, U_\varepsilon^{\rm eff}(u,u'')\, U_\varepsilon^{\rm eff}(u'',u') \qquad (5.51)$$

(compare Eqs. (5.8)–(5.10)). The composition of the kernels $U_\varepsilon^{\rm ph}$ is determined in accordance with the scalar product (2.28). As a result, we obtain that the formula (5.48) is valid for a finite time interval τ with the kernel $U_t^{\rm eff}(u,u')$, given by the standard Hamiltonian path integral

$$U_t^{\rm eff}(u,u') = \int \prod_{\tau=0}^{t} \left(\frac{dp(\tau)\,du(\tau)}{2\pi}\right) \exp\left[i\int_0^t d\tau\,(p\dot{u} - H^{\rm eff}(p,u))\right], \qquad (5.52)$$

where $u(0) = u'$ and $u(t) = u$.

It follows from Eqs. (5.48), (5.51), and (5.52) that taking into account the so-called "copies," i.e., gauge-equivalent points on the axis $u\in\mathbb{R}$, comes down not to reduction of the region of integration in the functional integral,[11] but rather to symmetrization of the kernel $U_t^{\rm eff}(u,u')$, defined by the Hamiltonian path integral with the standard measure, with respect to the residual gauge group S at the point u' or u.

In conclusion, we note that in accordance with the results of Sec. 2.3.2 taking into account the curvilinearity of the physical variables and the reduction of their phase space guarantees that all physical amplitudes will be explicitly gauge invariant (independent of f_i). The amplitude $U_t^{\rm ph}(u,u')$ is not an exception. Indeed, let us represent it as a spectral sum over the orthonormal solutions of Eq. (2.27). Then, according to Eq. (2.29), we conclude that $U_t^{\rm ph}(u,u')$ is an analytic function of two variables $f_i^2(u) = \mathbf{x}^2$ and $f_i^2(u') = \mathbf{x}'^2$, i.e., it is explicitly gauge-invariant.

5.6. The operator Q and the Hamiltonian path integral in an arbitrary gauge (arbitrary gauge group)

The reduction of the phase space of physical variables is taken into account in the Hamiltonian path integral by the operator Q, which symmetrizes the kernel of the evolution operator with respect to the residual discrete gauge group. The explicit form of $\hat{Q}$, generally speaking, depends on the method by which the physical variables are chosen (see Sec. 5.5). On the other hand, it was shown for the example of the simple model in Sec. 2.3.2 that the amplitudes (scalar products) do not depend on the choice of physical variables, since the physical state vectors are functions of invariants of the starting variables.

We shall show below that the operator $\hat{Q}$ does not depend on the dynamics of systems and is completely determined by the gauge group, its representation, and the choice of physical variables. Using this property of $\hat{Q}$, we shall give a recipe for constructing the Hamiltonian path integral for an arbitrary choice of physical variables; the recipe agrees with Dirac's quantization scheme (taking into account the curvilinearity of the physical variables and the structure of their phase space).[39,42]

We shall study the quantum theory given by the Schrödinger equation

$$\left(-\frac{1}{2}\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right\rangle + V(x)\right)\psi_E = E\psi_E \qquad (5.53)$$

and the normalization condition

$$\int d^N x \psi_E^*(x)\psi_{E'}(x) = \delta_{EE'}. \qquad (5.54)$$

Here x realizes some linear representation of the compact group G,

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$$

is the invariant scalar product in the space of the representation, where x_i and y_i are real, i.e., the integration in Eq. (5.54) extends over $\mathbb{R}^N$, and V is a G-invariant potential.

The quantum theory with gauge symmetry is obtained from Eq. (5.53) and (5.54) by imposing the requirement[29] that $\hat{\sigma}_a\Phi_{\rm ph}(x) = 0$ (analogously to Eq. (3.6)); these conditions single out the physical subspace of the states of $\mathscr{H}_{\rm ph}$. The operators $\hat{\sigma}_a$ generate the gauge transformations from the group G, $\exp(\omega_a\hat{\sigma}_a)\psi(x) = \psi(T(\omega)x)$, where $T(\omega)$ is an element of the representation of the group G. Since $\sigma_a\Phi_{\rm ph} = 0$, we have

$$\Phi(T(\omega)x) = \Phi(x), \qquad (5.55)$$

i.e., $\Phi_{\rm ph}(x)$ are gauge invariants. The relations (5.53)–(5.55) prescribe the quantum gauge theory in the full configuration space.

The group averaging operation

$$\hat{P}_G\psi(x) = \frac{1}{V_G}\int_G dg(\omega)\,\psi(T(\omega)x), \qquad (5.56)$$

where V_G is the volume of the group space and $dg(\omega)$ is the invariant measure on G, is the projector onto $\mathscr{H}_{\rm ph}$. We shall now study the kernel of the unit operator in the full Hilbert space $\mathscr{H}$:

$$\langle x\,|\,x'\rangle = \sum_E \Phi_E(x)\Phi_E^*(x') + \sum_{\tilde{E}}\psi_{\tilde{E}}(x)\psi_{\tilde{E}}^*(x') = \delta^N(x-x'), \qquad (5.57)$$

where Φ_E and $\psi_{\tilde{E}}$ form bases of the physical subspace $\mathscr{H}_{\rm ph}$ and the unphysical subspace $\mathscr{H}_{\rm nph} = \mathscr{H}\ominus\mathscr{H}_{\rm ph}$, respectively.[12] The first term in Eq. (5.57) is the kernel of the unit operator in $\mathscr{H}_{\rm ph}$. Since $\hat{P}_G$ is the projector onto $\mathscr{H}_{\rm ph}$, operating with it on the equality (5.57) we obtain the kernel of the unit operator in $\mathscr{H}_{\rm ph}$:

$$\langle x\,|\,x'\rangle_{\rm ph} = \langle x\,|\,\hat{P}_G\,|\,x'\rangle = V_G^{-1}\int_G dg(\omega)\,\delta^N(x - T(\omega)x'). \qquad (5.58)$$

The kernel (5.58) is obviously gauge-invariant.

The formula (5.58) makes it possible to write down the gauge-invariant kernel of the evolution operator without resorting to separating out the physical variables explicitly. The kernel of the evolution operator for the problem (5.53) and (5.54) is represented by the integral

$$U_t(x,x') = \int\prod_{\tau=0}^{t}\left(\frac{d^N p(\tau)\,d^N x(\tau)}{(2\pi)^N}\right) \times \exp\left[i\int_0^t d\tau\left(\langle p,\dot{x}\rangle - \frac{1}{2}\langle p,p\rangle - V(x)\right)\right], \qquad (5.59)$$

where $x(0) = x'$ and $x(t) = x$. On the other hand, by defini-

tion we have

$$U_t(x, x') = \sum_E \Phi_E(x)\,\Phi_E^*(x')\,e^{-iEt} + \sum_{\tilde{E}} \psi_{\tilde{E}}(x)\,\psi_{\tilde{E}}^*(x')\,e^{-i\tilde{E}t}, \tag{5.60}$$

where the first sum is the kernel of the evolution operator $U_t^{\text{ph}}(x,x')$ in $\mathscr{H}_{\text{ph}}$. By virtue of the projection properties of the operator $\hat{P}_G$ we obtain

$$U_t^{\text{ph}}(x, x') = \int \mathrm{d}^N x'' U_t(x, x'') \langle x'' \mid \hat{P}_G \mid x' \rangle. \tag{5.61}$$

In Sec. 9.8 analogous formulas are derived for the holomorphic representation and the kernels (5.58) for the models from Secs. 2.2, 3.1, and 6.1 are also explicitly calculated (see also Refs. 39, 42, and 57).

The formula (5.58) shows that the kernel $\langle x|x'\rangle_{\text{ph}}$ has a universal structure, i.e., it depends only on the group G and its representation. We shall establish the relation between (5.58) and the operator $\hat{Q}$. For this, we must specify the physical variables and transfer from the description in the full configuration space to the description in its physical subspace. Let the number of physical variables be M. Then the number of independent constraints is $N - M$. We introduce curvilinear coordinates

$$x = T(\theta)\,\bar{x}, \quad \bar{x} = \bar{x}(u), \tag{5.62}$$

where the components $\bar{x}$ identically satisfy the additional $N - M$ conditions $\chi_a(\bar{x}) = 0$, i.e., all N components of the element $\bar{x}$ are functions $u \in \mathbb{R}^M$ such that $\chi_a(x(\bar{u})) \equiv 0$ (analogously to Eq. (2.23)). The additional conditions χ_a must be chosen so that the equality (5.62) would specify the mapping $(\theta,u) \in \mathbb{R}^{N-M} \otimes \mathbb{R}^M \to x \in \mathbb{R}^N$ (see Sec. 2.3.2 regarding the admissibility of the gauge condition). Then there exists in $\mathbb{R}^N$ a region K such that the mapping (5.62) $(\theta,u) \in \tilde{K} \to x \in \mathbb{R}^N$ defines a substitution of variables. In the new variables the physical wave functions $\tilde{\Phi}_{\text{ph}}(x(u,\theta)) = \Phi_{\text{ph}}(u,\theta) = \Phi_{\text{ph}}(u)$ do not depend on θ, since σ_a are generators of translations θ.

A basis $\Phi_E(u)$ in $\mathscr{H}_{\text{ph}}$ is constructed from the solutions of Eq. (5.53) in the curvilinear coordinates (5.62). Since Eq. (5.62) is a substitution of variables, the states from $\mathscr{H}_{\text{ph}}$ are normalized according to (5.54) with the measure

$$\int \mathrm{d}^N x = |V_G V_H^{-1}| \int_K \mathrm{d}^M u \mu(u),$$

where V_G is the volume of the group space and V_H is the volume of the group space of the stationary subgroup H of the element $\bar{x}$ (actually $V_G V_H^{-1} \mu(u)$ is the volume of the orbit of G for the element $\bar{x}$ (Ref. 58)). This representation follows from the obvious relation $\tilde{K} = G/H \cup K$, where $\theta \in G/H$, $u \in K$. If the substitution (5.62) is such that $H = 1$, then $V_H = 1$.

To determine K, we shall study the symmetry group of the mapping (5.62) $\tilde{S} = S_\theta \times S$, where S_θ consists of translations of θ by amounts that are multiples of the periods of the compact manifold G/H and S_θ does not change u. The group S consists of the transformations $\theta \to \theta'_s$, $u \to u_s$ such that x in Eq. (5.62) does not change. Then, analogously to Eq. (2.24), we have $T(\theta'_s) = T(\theta)\,T^{-1}(\theta_s)$, where the transformation $T(\theta_s)$ must satisfy the condition $\chi_a(\bar{x}) = \chi_a(T(\theta_s)\bar{x}) = 0$, i.e., it does not violate the additional conditions $\chi_a = 0$. Therefore such transformations of $\bar{x}$ induce the transformations $u \to u_s$: $T(\theta_s)\bar{x}(u) = \bar{x}(u_s)$, $u_s = u_s(u)$, $u \in \mathbb{R}^M$. They form a group isomorphic to the group of permutations of the points u_s in $\mathbb{R}^M$. From the definition of the group $\tilde{S}$ we have the equality $\tilde{K} = \mathbb{R}^N/S = \mathbb{R}^{N-M}/S_\theta \cup \mathbb{R}^M/S = G/H \cup K$, i.e., K is a fundamental region in $\mathbb{R}^M(u)$ relative to the action of the group S: $k \to u_s(u)$ $(u_s: \mathbb{R}^M \to \mathbb{R}^M)$.

We also recall that the group S can depend on the point u, i.e., for different u it can have a different number of elements. In this case $\mathbb{R}^M(u)$ must be divided into $\mathbb{R}_\alpha^M$ so that for $u \in \mathbb{R}_\alpha^M$ the number of elements $S(u) = S_\alpha(u)$ is fixed, i.e., $S = \Pi_\alpha S_\alpha$ and $K = \cup_\alpha K_\alpha$, where $K_\alpha = \mathbb{R}_\alpha^M/S_\alpha$.

Thus, the scalar product in $\mathscr{H}_{\text{ph}}$ is given by the formula (2.28), in which $\mathrm{d}u \to \mathrm{d}^M u$ and the factor $V_G V_H^{-1}$ is included in the normalization of $\Phi_E(u)$. If $\psi_E(u)$ is a basis in $\mathscr{H}$, then the basis $\Phi_E(u)$ in $\mathscr{H}_{\text{ph}}$ can be defined as an induced basis, using the projection formula (5.56) and the decomposition $\mathscr{H} = \mathscr{H}_{\text{ph}} \otimes \mathscr{H}_{\text{nph}}$, i.e.,

$$\Phi_E(u) = \hat{P}_G \psi_E(x) = \hat{P}_G \psi_E(T(\theta)\,\bar{x}(u)) = \hat{P}_G \psi_E(\bar{x}(u)). \tag{5.63}$$

The last equality in Eq. (5.63) follows from the properties of invariance of the measure $g(\omega)$ in Eq. (5.56). Due to the invariance of the measure $\mathrm{d}g(\omega)$ and the definition $x(u_s) = T(\theta_s)\bar{x}(u)$, from Eq. (5.63) follows the S-invariance of the physical states $\Phi_E(u_s) = \Phi_E(u)$, i.e., the equality (2.30) is valid in the general case; it permits determining the analytical continuation of the kernel of the unit operator $\langle u|u'\rangle_{\text{ph}}$ in $\mathscr{H}_{\text{ph}}$ into the unphysical region $u \in \mathbb{R}^M$, analogously to Eq. (5.45), where $\mathrm{d}u'$ and $\delta(u - u'')$ must be replaced by $\mathrm{d}^M u''$ and $\delta^M(u - u'')$, respectively. The operator $\hat{Q}$ also has the form of Eq. (5.46). If the measure $\mu(u) = \varkappa^2(u)$, where $\varkappa(u)$ is a real analytic function in $\mathbb{R}^M$, then in Eq. (5.45) $(\mu\mu'')^{1/2} \to (\varkappa\varkappa'')$ (an example of such a theory is presented in Sec. 5.3). Using the method of Sec. 5.5, it is possible to construct a Hamiltonian path integral in the physical phase space. The Hamiltonian $\hat{H}(u)$ in Eq. (5.47) is calculated by transferring in Eq. (5.53) to curvilinear coordinates (5.62) and dropping terms containing the derivatives $\partial\theta_a$ $(\Phi_{\text{ph}}(u,\theta) = \Phi_{\text{ph}}(u))$. The obtained Hamiltonian path integral has the form (5.52). The effective quantum correction V_q to the potential will have a more complicated form than (5.50).[39] The complication is connected with the fact that in the general case the system contains several physical degrees of freedom. But the method for calculating V_q does not change.

The kernel of the operator $\hat{Q}$ can also be determined from the projection formula (5.58), substituting for x and x', respectively, $T(\theta)\bar{x}(u)$ and $T(\theta')\bar{x}(u')$, and carrying out the integration over the group. As a result

$$\langle u \mid u' \rangle_{\text{ph}} = V_H^{-1} \int_G \mathrm{d}g(\omega) \delta^N(\bar{x}(u) - T(\omega)\,\bar{x}(u')); \tag{5.64}$$

because of the invariance of the measure $\mathrm{d}g(\omega)$ the dependence on θ and θ' is eliminated by performing a group translation. The factor V_H^{-1} in Eq. (5.64) is connected with the difference of the normalizations of the kernels (5.58) and (5.64). The kernel (5.64) is normalized in the physical configuration space with the measure

$$\int_K \mathrm{d}^M u \mu(u)$$

similarly to Eq. (2.28). For this reason, it can be obtained from the kernel (5.58) by multiplying the latter by the volume of the unphysical configuration space $V_G V_H^{-1}$. Comparing Eq. (5.45) (definition of the operator $\hat{Q}$) and Eq. (5.64), we find a universal expression for the kernel of the operator $\hat{Q}$ that is suitable for any group with any (reasonable) gauge

$$Q(u, u') = \frac{|\mu(u)|}{V_H} \int_G dg(\omega)\, \delta^N(\bar{x}(u) - T(\omega)\bar{x}(u')), \quad (5.65)$$

where $u \in \mathbb{R}^M$, $u' \in K$. We note that the calculation of the integral in Eq. (5.65) automatically reproduces the sum (5.46), i.e., it determines the group S.

The formula (5.65) shows that the operator $\hat{Q}$ does not depend on the dynamics of the gauge system, but rather it is completely determined by the gauge group, its representation, and the choice of physical variables.

What is the result? We can say that if the gauge is poorly chosen the correct description of the system on the basis of a Hamiltonian path integral may turn out to be a problem of incredible complexity. As is well known, invariant gauges of the Fermi type do not adequately describe the physics of a system in the theory of Yang–Mills fields (by virtue of their nonuniqueness[11,41]), so that the problem of correct and constructive formulation of this theory with the help of explicitly Lorentz-invariant path integrals is apparently hopeless.

6. REDUCTION OF THE PHASE SPACE AND QUASICLASSICAL DESCRIPTION

The formalism of path integration makes it possible to determine in a natural manner the quasiclassical transition amplitude. The problem reduces to calculating the path integral by the method of stationary phase in the neighborhood of the classical trajectory.

In Secs. 2.2.1, 3.2.2, and 5.1 we saw that a classical trajectory is sensitive to the structure of the phase space, so that it can be expected that the quasiclassical description will depend on the structure of the phase space.

Two problems are usually distinguished in the quasiclassical approach:[27] determination of the spectrum of the system by using periodic classical solutions (WKB quantization) and description of tunneling effects with the help of the solutions of Euclidean equations of motion (classical trajectories in imaginary time—instantons). We shall examine these questions for systems with a conical phase space $\mathrm{con}(\pi)$. More complicated systems are discussed in Refs. 15, 20, and 22.

6.1. The WKB method

Let the potential of the system V be such that there exists a periodic solution of the classical equations of motion. It is obvious that the period $T = T(E)$ is a function of the energy of the system E.

According to the Bohr–Sommerfeld quantization rule, the quantum energy levels can be determined by solving the equation

$$W(E) \equiv \oint p\, dq = \int_0^T p\dot{q}\, dt = 2\pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, \ldots, \quad (6.1)$$

for E. The integral in Eq. (6.1) is taken along a classical trajectory.

What changes if the phase space is $\mathrm{con}(\pi)$? It is easy to see that the period of the oscillations of the system T^c with the phase space $\mathrm{con}(\pi)$ is one-half the period of a system with the phase space $\mathbb{R}^2$ and the same energy E:

$$T^c(E) = \frac{1}{2} T(E). \quad (6.2)$$

The reason is simple. Since the potential is an even function $V(g) = V(-g)$ (a consequence of $\mathbb{Z}_2$ invariance (see Sec. 2.1)), the time the particle spends in the region $q < 0$ is equal to the time that it spends in the region $q > 0$. Since the phase space is finite the motion with $q < 0$ is indistinguishable from the motion with $q > 0$. From here the halving of the physical period follows immediately, and therefore for the conical phase space $\mathrm{con}(\pi)$

$$W^c(E) = \int_0^{T/2} p\dot{q}\, dt = \frac{1}{2} W(E). \quad (6.3)$$

From the quantization condition $W^c(E) = 2\pi(n + 1/2)$, $n = 0, 1, \ldots$, it follows that the quasiclassical spectrum of a system whose phase space is $\mathrm{con}(\pi)$ contains half the number of levels of a spectrum whose phase space is $\mathbb{R}^2$. The physical energy levels in Eq. (6.1) correspond only to even n.

In the case of systems with several physical degrees of freedom the difficulties of taking into account the reduction of the phase space in the WKB method are connected with singling out the independent modes of oscillation. As was shown in Sec. 3.2.2, this problem is far from simple. The essence of the matter, however, does not change, only the technical details become more complicated. For example, in Refs. 15 and 22 the spectrum of internal excitations of a quantum soliton, whose phase space is $\mathrm{con}(\pi)$, was found. The result agrees with Eq. (6.3).

6.2. Quantum-mechanical instantons

We shall study the simplest example of the effect of reduction of the phase space on instanton calculations. We recall that instantons are employed in quantum theory for calculating tunneling effects.[26,27,58] For example, consider a one-dimensional quantum system with a periodic potential.[27] Then the ground state in the neighborhood of each local minimum of the potential is degenerate. The degeneracy is removed by tunneling effects, and in addition the ground state becomes a band. It turns out that knowing the solutions of the Euclidean equations of motion (the equations of motion with imaginary time $t \to -i\tau$) makes it possible to calculate the energy levels in the band and find approximately the corresponding wave functions (θ vacuums).

Consider the model from Sec. 3 with the group $SO(3) \sim SU(2)$ and the periodic potential $V(\mathbf{x}^2) = 1 - \cos(\mathbf{x}^2)^{1/2}$. An analogous one-dimensional model has been studied many times (see Ref. 27 and the literature cited there), i.e., with $\mathbb{R}^2$ as the phase space. In our case the phase space with $r = (\mathbf{x}^2)^{1/2}$ as the only physical variable is $\mathrm{con}(\pi)$.

We shall study a Euclidean variant of the theory: $t \to i\tau$, $y \to iy$ in Eq. (3.1), and then $L \to L_E = (1/2)(D_t x)^2 + V(x^2)$. We shall take the solutions of the classical equations of motion

$$\partial_\tau \frac{\partial L_E}{\partial \dot{x}} = \frac{\partial L_E}{\partial x}, \quad \dot{x} = \partial_\tau x, \tag{6.4}$$

in the form[15,22] $x_{inst}(\tau) = r_{inst}(\tau)\lambda_1$, where

$$r_{inst}(\tau) = 4 \operatorname{arctg} \exp(\tau - \tau_c) + 2\pi m. \tag{6.5}$$

Here m is an integer, $y(\tau) \in H$ and λ_1 is the "only" basis element in H, and $\tau_c = \text{const}$. The solution (6.5) is determined to within the gauge transformation (3.2). It relates the local minima of the potential: $x^2_{inst} \to (2\pi m)^2$, $\tau \to \infty$ and $\to (2\pi(m-1))^2$, $\tau \to -\infty$.

We note that setting in Eq. (6.4) $x = \lambda_1 r \in H$ and $y = 0$ we obtain an equation for r that is identical to the equation of motion for an analogous system whose phase space is $\mathbb{R}^2$; its solution is given by Eq. (6.5). For a one-dimensional model with a flat phase space the calculations are performed as follows.[27] The transition amplitude $U_\tau(2\pi m, 2\pi m')$ between two local minima of the potential is calculated by the saddle-point method; in so doing, the instanton solution serves as the stationary point.[13)] In the limit $\tau \to \infty$ the states from the bottom band (states with higher energy make an exponentially small contribution) make the main contribution to this amplitude:

$$U_\tau(2\pi m, 2\pi m') \equiv \langle 2\pi m \mid \exp(-\tau \hat{H}) \mid 2\pi m' \rangle \approx \int_0^{2\pi} d\theta \langle 2\pi m \mid \theta \rangle \langle \theta \mid 2\pi m' \rangle \exp(-\tau E_\theta), \quad \tau \to \infty,$$

where the parameter θ enumerates the energy levels E_θ in the lower band. The amplitude $\langle 2\pi m | \theta \rangle$ is obtained by calculating the path integral by the saddle-point method. The corresponding calculations were performed in detail in Ref. 27; it was found that

$$U_\tau(2\pi m, 2\pi m') = \int_0^{2\pi} \frac{d\theta}{2\pi^{3/2}} e^{-i(m-m')\theta} e^{-\tau E_\theta}, \quad \tau \to \infty, \tag{6.6}$$

$$E_\theta = \frac{1}{2} - e^{-S_0} S_0^{1/2} K \cos\theta, \tag{6.7}$$

where S_0 is the action for the solution (6.5) and K is some number that does not depend on θ (instanton determinant[27]). The quantity $\langle 2\pi m | \theta \rangle \sim \exp(-im\theta)$ gives the value of the wave function of the θ vacuum $\langle r | \theta \rangle$ in the neighborhood of a local minimum of the potential $r = 2\pi m$, so that the approximate expression for $\langle r | \theta \rangle$ has the form[27]

$$\langle r \mid \theta \rangle = \text{const} \cdot \sum_{m=-\infty}^{\infty} e^{-im\theta} \langle r \mid 2\pi m \rangle, \tag{6.8}$$

where $\langle r | 2\pi m \rangle \sim \exp[-(1/2)(r - 2\pi m)^2]$ is the wave function of the ground state in the neighborhood of each local minimum of the potential.

How are the calculations modified if the phase space is $\operatorname{con}(\pi)$? Obviously, now the amplitude $U^c_\tau(2\pi m, 2\pi m')$ must be used instead of the amplitude $U_\tau(2\pi m, 2\pi m')$. The relation between these amplitudes is given by the formula (5.13) with the substitution $t \to -i\tau$, according to which we obtain

$$U^c_\tau(2\pi m, 2\pi m') = \int_0^{2\pi} \frac{d\theta}{\pi^{3/2}} \frac{\sin m\theta \cdot \sin m'\theta}{(2\pi)^2 m m'} e^{-\tau E_\theta}. \tag{6.9}$$

Therefore reduction of the phase space does not change the energy levels in the bottom band. However the form of the coefficients $\langle 2\pi m | \theta \rangle$ changed, so that the form of the wave function (6.8) of the θ vacuum also changes:

$$\langle r \mid \theta \rangle^c = \text{const} \cdot \sum_{m=-\infty}^{\infty} \frac{\sin m\theta}{2\pi m} \langle r \mid 2\pi m \rangle. \tag{6.10}$$

In view of the obvious equality $\langle -r | 2\pi m \rangle = \langle r | -2\pi m \rangle$ the function (6.10), in contrast to the function (6.8), is even $\langle -r | \theta \rangle^c = \langle r^c | \theta \rangle^c$, i.e., $\mathbb{Z}_2$ is invariant. We also note that the function (6.10) is normalized according to (5.20), while the function (6.8) is normalized as in the one-dimensional theory with the phase space $\mathbb{R}^2$.

The energy levels E_θ are independent of the structure of the phase space, generally speaking, only for the continuous spectrum. Here the analogy with the case of a free particle is pertinent. The reduction of the phase space $\mathbb{R}^2 \to \operatorname{con}(\pi)$ does not change the spectrum of the system. On the other hand, the discrete spectrum (for example, the oscillator spectrum) is sensitive to reduction. This is also true for instantons, as can be easily verified by studying a model with the potential $V = (x^2 - a^2)^2/4!$ and the gauge group $SO(3) \sim SU(2)$ (the one-dimensional analog was studied in Ref. 26). In this model the vacuum is doubly degenerate: $x = r\lambda_1 \in H$ and $r = \pm a$. The band contains two levels, and the wave function of the lower level is odd and the wave function of the upper level is even.[14)] Taking into account the structure of the phase (which is $\operatorname{con}(\pi)$) eliminates the contribution of the $\mathbb{Z}_2$-odd state.

7. MORE COMPLICATED MECHANICAL SYSTEMS. CHARACTERISTIC FEATURES OF THE DYNAMICS

Thus far we have studied systems in which all physical variables had a reduced phase space. Increasing the number of degrees of freedom while maintaining constant the number of gauge parameters leads to models of a new type, in which, seemingly, some physical variables must have a flat phase space while others must have a reduced phase space. The situation, however, is more complicated. The gauge group engenders a specific relationship between the physical degrees of freedom which often has a purely kinematic character and prevents such a simple interpretation of the structure of the phase space.[39]

7.1. Two particles in a two-dimensional space

We shall study a system of two particles in a two-dimensional space. If we take for the Lagrangian the sum of Lagrangians of the form (2.1)

$$L' = \frac{1}{2}[(\partial_t - y_1 T)\mathbf{x}_1]^2 + \frac{1}{2}[(\partial_t - y_2 T)\mathbf{x}_2]^2 - V_1(\mathbf{x}_1) - V_2(\mathbf{x}_2), \tag{7.1}$$

where y_1 and y_2 change independently under gauge transformations, then a simple analysis shows that the phase space of each particle is a cone $\operatorname{con}(\pi)$ and the energy of the system is the sum of the energies of each subsystem. Let us now narrow the gauge group of the Lagrangian (7.1) $SO(2) \otimes SO(2)$ to $SO(2)$, making the identification $y_1 \equiv y_2 = y$, i.e., we transform to the Lagrangian[39]

$$L = \frac{1}{2}[(\partial_t - yT)\mathbf{x}_1]^2 + \frac{1}{2}[(\partial_t - yT)\mathbf{x}_2]^2 - V_1(\mathbf{x}_1) - V_2(\mathbf{x}_2), \tag{7.2}$$

which is invariant under gauge transformations

$$\delta \mathbf{x}_i = \varepsilon T \mathbf{x}_i, \quad \delta y = \dot{\varepsilon}, \quad \delta V_i = 0, \quad i = 1, 2. \tag{7.3}$$

The standard analysis shows that the Lagrangian (7.2) prescribes a system with one primary and one secondary constraint

$$\pi = \frac{\partial L}{\partial \dot{y}} = 0, \quad \sigma = \mathbf{p}_1 T \mathbf{x}_1 + \mathbf{p}_2 T \mathbf{x}_2 = 0 \tag{7.4}$$

and the Hamiltonian

$$H = \frac{1}{2}(\mathbf{p}_1^2 + \mathbf{p}_2^2) + V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2) - y(t)\sigma \equiv H_1 + H_2, \tag{7.5}$$

where each H_i is identical to (2.4). The constraint σ generates gauge transformations of $\mathbf{x}_i$. What can we say about the physical phase space of each variable $\mathbf{x}_1, \mathbf{x}_2$? If we choose the gauge $x_1^{(2)} = 0 (\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)}))$, then in view of the remaining gauge arbitrariness, associated with the discrete group $\mathbb{Z}_2$ (see Sec. 2.1.1), it would seem natural to conclude that $\mathrm{PS}(x_1^{(1)}) = \mathrm{con}(\pi)$, while the phase space of the two remaining physical degrees of freedom is flat: $\mathrm{PS}(\mathbf{x}_2) = \mathbb{R}^2 \times \mathbb{R}^2$. However instead of $x_1^{(2)}$ we could have eliminated $x_2^{(2)}$ and drawn the conclusion that $\mathrm{PS}(x_2^{(1)}) = \mathrm{con}(\pi)$. Therefore, although according to (7.5) $H = H_1 + H_2$ and $\{H_1, H_2\} = 0$, i.e., although the systems described by the Hamiltonians H_i appear to be dynamically independent, the question of the structure of the physical phase space of each system separately is meaningless. We can talk only about the structure of the phase space of the system as a whole. Thus the Lagrangian (7.2) is another example of the unusual nature of the properties of theories with gauge symmetry: The dynamical characteristics of two systems, which by the standard measures are independent, are found to be interdependent, one independent subsystem affecting the other through the "physically meaningless" degree of freedom y (formally, through the constraint $\sigma = 0$ in Eq. (7.4)).

We shall now proceed to the quantum description.[39] Once again we shall confine our attention to the case of oscillator potentials $V_i = \omega_i^2 \mathbf{x}_i^2/2$, $\omega_1 \neq \omega_2$. In the second-quantization representation the Hamiltonian operator $\hat{H}$ and the constraint $\hat{\sigma}$ (see Eqs. (7.4) and (7.5)) are written in the form

$$\hat{H} = \sum_{j=1,2} [\omega_j \hat{\mathbf{a}}_j^+ \hat{\mathbf{a}}_j + y(\hat{\mathbf{a}}_j^+ T \hat{\mathbf{a}}_j) + \omega_j], \tag{7.6}$$

$$\hat{\sigma} = \hat{\mathbf{a}}_1^+ T \hat{\mathbf{a}}_1 + \hat{\mathbf{a}}_2^+ T \hat{\mathbf{a}}_2, \tag{7.7}$$

where $\hat{\mathbf{a}}_j = (p_j - s\mathbf{x}_j)/\sqrt{2}$ and $\hat{\mathbf{a}}_j^+ = (\mathbf{p}_j + s\mathbf{x}_j)/\sqrt{2}$. The physical states are generated by invariants of the rotation group which are composed of the vectors $\hat{\mathbf{a}}_1^+$ and $\hat{\mathbf{a}}_2^+$:

$$\hat{b}_1 = (\hat{\mathbf{a}}_1^+)^2, \quad \hat{b}_2 = (\hat{\mathbf{a}}_2^+)^2, \quad \hat{b}_3 = (\hat{\mathbf{a}}_1^+ \hat{\mathbf{a}}_2^+), \quad \hat{b}_4 = \varepsilon_{ij} \hat{a}_1^{(i)+} \hat{a}_2^{(j)+}; \tag{7.8}$$

ε_{ij} is the unit antisymmetric tensor and $\varepsilon_{12} = 1$. Here the following should be noted. The operators (7.8) are invariant under the group SO(2). All of them, except for $\hat{b}_4$, are invariant under the larger group $\mathrm{O}(2) = \mathrm{SO}(2) \otimes \mathbb{Z}_2$ (the nontrivial element of $\mathbb{Z}_2$ corresponds to reflection of one of the coordinate axes; the operator $\hat{b}_4$ changes sign in the process). Question: Should the operator $\hat{b}_4$ be included among the operators that generate the basis of the Hilbert space? In other words, what is the gauge group of the model (7.2): SO(2) or O(2)? Formally, all information about the dynamics and constraints of the system is contained in the Lagrangian (7.2). The standard analysis[29] gives the constraints—the generators of the gauge group. But the generators make it possible to reconstruct only the constrained component of the identity element of the group,[14,44] so that one can talk only about the gauge group SO(2). The existence of a discrete gauge group for the Lagrangian (7.2) cannot be established.[39] According to Sec. 5.1, the requirement of such a symmetry can be formulated only in the form of an independent condition that is not contained in the Lagrangian. The other possibility is to regard this problem as the result of reduction of a larger configuration space and the group O(2) as the subgroup of a larger group, for example, SO(3). The latter group contains elements which change the direction of any axis in the plane. In any case, additional assumptions are required in order to transfer to discrete gauge transformations. We shall adhere to this point of view, i.e., we shall include $\hat{b}_4$ among the generating operators.

Because of the identity $\varepsilon_{ij}\varepsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ the operator $\hat{b}_4^2$ can be expressed in terms of the other operators $\hat{b}_i$ $(i = 1, 2, 3)$, so that the basis of the physical space $\mathscr{H}_{\mathrm{ph}} = \mathscr{H}_{\mathrm{ph}}^0 \oplus \mathscr{H}_{\mathrm{ph}}'$ is given by the vectors

$$\hat{b}_1^{n_1}\hat{b}_2^{n_2}\hat{b}_3^{n_3}|0\rangle \in \mathscr{H}_{\mathrm{ph}}^0, \quad \hat{b}_1^{n_1}\hat{b}_2^{n_2}\hat{b}_3^{n_3}b_4|0\rangle \in \mathscr{H}_{\mathrm{ph}}', \quad n_i = 0, 1, \ldots; \quad i = 1, 2, 3, \tag{7.9}$$

$n_i = 0, 1, \ldots$ and $i = 1, 2, 3$, and the energy spectrum is given by the formula

$$E = 2n_1\omega_1 + 2n_2\omega_2 + n_3(\omega_1 + \omega_2) + n_1(\omega_1 + \omega_2) + \omega_1 + \omega_2, \tag{7.10}$$

$n_1 = 0, 1.$

The structure of the spectrum gives no basis for assigning to some degree of freedom a definite phase space. For example, we cannot say that the phase space of one variable is $\mathrm{con}(\pi)$ and that the phase spaces of the other variables are planes. We can talk only about the structure of the phase space as a whole.

The foregoing example shows the specific nature of the effect of unphysical variables on the physical variables: A fundamental characteristic of a dynamical system—the phase space—changes. Conversely, the effect of the physical degrees of freedom on one another is not so dramatic: only the equations of motion change. Thus if a term $V(\mathbf{x}_1, \mathbf{x}_2)$ that is invariant under the group $\mathrm{SO}(2) \otimes \mathrm{SO}(2)$ is added to the Lagrangian (7.1), then the phase space of each subsystem $\mathbf{x}_1, \mathbf{x}_2$ remains unchanged, though the subsystems will strongly affect one another.

What is the role of the residual discrete gauge group (or the corresponding group S) in such systems? The formula (5.65) makes it possible to calculate S in any theory. In particular, if in the system under study the physical variables are separated by the condition $x_1^{(2)} = 0$, then, as the calculations (9.46)–(9.49) show (see Sec. 9.8), $S = \mathbb{Z}_2$, and S operates on all physical variables simultaneously: $x_1^{(s)}, x_2^{(1)} \to x_1^{(1)}$, $-x_2^{(i)}$ (analogously for the canonically conjugate momenta). For this reason the physical phase space is $(\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2)/\mathbb{Z}_2$. Obviously, it is different from

$\mathrm{con}(\pi) \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, where $\mathrm{con}(\pi) = \mathbb{R}^2/\mathbb{Z}_2$. This fact is of a general character,[39,42] i.e., the residual discrete gauge group operates simultaneously on all physical variables. For this reason, the physical phase space is reduced as a whole, thereby giving rise to a specific purely kinematic relationship between the physical degrees of freedom.

7.2. System with Bose and Fermi degrees of freedom

It is instructive to study a mixed system which includes Grassmann variables together with the normal variables. The simplest system of this type is given by the Lagrangian (more complicated systems are studied in Ref. 42)

$$L = \frac{1}{2}\{[(\partial_t - yT)\mathbf{x}]^2 - \omega_1^2\mathbf{x}^2\} + \boldsymbol{\psi}^+(i\partial_t + y\Gamma)\boldsymbol{\psi} - \omega_2\boldsymbol{\psi}^+\boldsymbol{\psi}, \tag{7.11}$$

which is essentially the sum of the Lagrangians (2.1) and (4.1) with specific potentials V. An analysis analogous to that performed in Secs. 2 and 4 shows that the Lagrangian (7.11) defines a system with one primary constraint $p_y = \partial L/\partial\dot{y} = 0$, the Hamiltonian

$$H = \frac{1}{2}(\mathbf{p}^2 + \omega_1^2\mathbf{x}^2) + \omega_2\boldsymbol{\psi}^+\boldsymbol{\psi} + y(t)(\mathbf{p}T\mathbf{x} - \boldsymbol{\psi}^+\Gamma\boldsymbol{\psi}) \tag{7.12}$$

and one secondary constraint

$$\sigma = \mathbf{p}T\mathbf{x} - \boldsymbol{\psi}^+\Gamma\boldsymbol{\psi} = 0. \tag{7.13}$$

The uniqueness of the model lies in the fact that the constraint σ is an even element of the Grassmann algebra. The vanishing of these elements is equivalent to several equalities (similarly to the manner in which the vanishing of a vector means that all its components vanish), i.e., in classical mechanics (7.13) is equivalent to two conditions:

$$\sigma_1 = \mathbf{p}T\mathbf{x} = 0, \quad \sigma_2 = \boldsymbol{\psi}^+\Gamma\boldsymbol{\psi} = 0. \tag{7.14}$$

There arises the question: Does this mean that new independent constraints appear? At first glance the answer depends on whether a classical or quantum theory is under study. In the classical theory the Hamiltonian (the "energy") of such systems is an even element of the Grassmann algebra; according to the Hamiltonian (7.12), the normal and Grassmann variables do not mix in the process of the motion, i.e., both subsystems evolve in time independently. For this reason, the conditions (7.14) can be interpreted as an indication of reduction of the phase space in both subsystems, because, for example, the equality $\sigma_2 = \psi_1^+\psi_1 - \psi_2^+\psi_2 = 0$ is impossible for independent variables ψ_1 and ψ_2. In the quantum theory, however, the canonical variables of both types are, on the contrary, operators and the condition $(\hat{\sigma}_1 - \hat{\sigma}_2)\Phi = 0$ does not lead to the equalities $\hat{\sigma}_i\Phi = 0$, $i = 1, 2$ since the eigenvalues of $\hat{\sigma}_i$ are c numbers.

What we have said above agrees with the definition of δ functions of even elements of a Grassmann algebra,[56] in this case the function $\delta(\sigma)$. The appearance of this function is unavoidable in the formulation of the quantum theory on the basis of a Hamiltonian path integral (when the unphysical variables are eliminated). But classical variables appear in the Hamiltonian path integral, so that it could shed light on the situation in the classical theory. In Ref. 56 it is shown that $\delta(\sigma)$ must be understood as a Taylor series

$$\delta(\sigma_1 - \sigma_2) = \delta(\sigma_1) + \sum_{n=1}^{k}\delta^{(n)}(\sigma_1)(-\sigma_2)^n, \tag{7.15}$$

which cuts off at the k th term such that $\sigma_2^{k+1} = 0$. The δ function defined in this manner has the necessary property

$$\int f(\hat{\xi})\,\delta(\hat{\xi} - \hat{\xi}_0)\,d\hat{\xi} = f(\hat{\xi}_0),$$

where $\hat{\xi}$ is an even element of the Grassmann algebra (definitions of integrals over the even elements of a Grassmann algebra as well as more general integrals on such an algebra are given in Refs. 56 and 59). From Eq. (7.15) it is clear that the appearance of the δ function $\delta(\sigma_1 - \sigma_2)$, where $\sigma_{1,2}$ are classical quantities (7.14), does not mean that the equality $\sigma = \sigma_1 + \sigma_2 = 0$ is satisfied term by term in the process of evolution of the system. Thus, in the quantum theory there is only one secondary constraint, and since the classical picture is obtained from the quantum picture by passing to a limit the quantum picture also will correspond to dynamics characterized by one secondary constraint. This conclusion becomes obvious, when one takes into account the fact that the real gauge arbitrariness is determined by the number of independent arbitrary parameters (functions of time) in front of the generators of the gauge transformations. The Hamiltonian (7.12), which contains all information about the dynamics of the system, contains only one such parameter $y(t)$. Fixing this parameter, for example, by the condition $x_2 = 0$, exhausts the gauge arbitrariness (to within a discrete subgroup; the residual gauge subgroup $\mathbb{Z}_2$ cannot change the dimension of the phase space).

We can now find the spectrum of the Hamiltonian (7.12). We write it in the second-quantized form as

$$\hat{H} = \omega_1\hat{\mathbf{a}}^+\hat{\mathbf{a}} + \omega_2\hat{\boldsymbol{\psi}}^+\hat{\boldsymbol{\psi}} + y(t)(i\hat{\mathbf{a}}^+T\hat{\mathbf{a}} - \hat{\boldsymbol{\psi}}^+\Gamma\hat{\boldsymbol{\psi}}) + (\omega_1 - \omega_2) \equiv \hat{H}_1 + \hat{H}_2, \tag{7.16}$$

where $\hat{H}_1$ with $\omega_1 = 0$ is identical to (2.19) and $\hat{H}_2$ is identical to (4.4), if in the latter we set $\omega = \omega_2$ and $\Omega = 0$. Obviously, $[\hat{H}_1, \hat{H}_2] = 0$, i.e., according to the standard criteria the subsystems are dynamically independent. The basic gauge-invariant operators, which generate the physical Hilbert space, are given by the formulas

$$\hat{b}_1 = (\hat{\mathbf{a}}^+)^2, \quad \hat{b}_2 = \hat{\mathbf{a}}^+\hat{\boldsymbol{\psi}}^+, \quad \hat{b}_3 = \hat{\boldsymbol{\psi}}^+T\hat{\boldsymbol{\psi}}^+, \quad \hat{b}_4 = \hat{\mathbf{a}}^+T\hat{\boldsymbol{\psi}}^+ \quad (\boldsymbol{\psi} = (\psi_1, \psi_2)). \tag{7.17}$$

The operator $\hat{b}_4$ appears here for the same reason as in (7.8): Enlarging the gauge group up to O(2) eliminates it. We note that such an enlargement would also eliminate the operator $\hat{b}_3$—this is another argument in favor of the group SO(2).[15] The state vectors obtained by applying all possible nonnegative integer powers of the operators (7.17) to the ground state form the basis of the physical Hilbert space (analogously to (7.9)). They are all eigenvectors of the operator $\hat{H}$. This makes it possible to write out the energy spectrum:

$$E = 2n_1\omega_1 + n_2(\omega_1 + \omega_2) + 2n_3\omega_2 + n_4(\omega_2 + \omega_1) + \omega_1 - \omega_2. \tag{7.18}$$

In contrast to (7.10), here only n_1 runs through all the positive integers $n_1 = 0, 1, 2, \ldots$, while n_i, $i = 2, 3$, and 4, assume the values 0 and 1, since for these values of $i\hat{b}_i^2 = 0$, and in addition $n_2 + n_3 + n_4 = 1$. The latter restriction follows from the identities for the operators (7.17): $\hat{b}_2\hat{b}_4 = \hat{b}_1\hat{b}_3$ and $\hat{b}_{2,4}\hat{b}_3 = 0$. We note that for $\omega_1 = \omega_2$ the energy of the

ground state is equal to zero (supersymmetry). From the structure of the spectrum (7.18) we conclude that, as in the model (7.2), here it is impossible to single out the degree of freedom whose excitation does not depend on the residual subsystem while the spectrum would indicate a conical or flat phase space.

8. CONCLUSIONS

We shall now summarize what we have said. We have reviewed the results of the investigation of the phenomenon of the reduction of the phase space of physical variables in theories with gauge symmetry. This phenomenon was studied for the simplest models with a finite number of degrees of freedom. The reason for the reduction of the phase space was determined (the existence—after elimination of the unphysical variables—of a residual discrete group, isomorphic to Weyl's subgroup of the complete gauge group), and we established some of its consequences (change in the energy spectrum of oscillators and modification of the Hamiltonian path integral). This phenomenon is of a general character and occurs both in theories with normal (commuting) variables and in theories with Grassmann (anticommuting) variables.

We studied only the simplest mechanical systems. We purposefully did not discuss the theory of gauge fields, which we had at the back of our minds when we wrote this review. This is a subject for an independent and more extensive investigation, which is still far from completion. However the results already obtained in this direction make it possible to talk about the possibility of studying the phenomenon for the theory of elementary particles. In Ref. 12, in which scalar electrodynamics was studied, in particular, the Higgs phenomenon, it was concluded that the phase space of the Higgs field is reduced. In Refs. 15, 20, and 22 the sine-Gordon model with electromagnetic interaction was investigated. It was shown that the existence of a discrete gauge group changes the spectrum of the breather excitations. In Refs. 15 and 28 the phase space of the Yang-Mills fields was studied. It was shown that the residual Weyl group changes the phase space of the physical components of the gauge field as well as the fields interacting with it. The most important consequences, following from these investigations, concern the nature of the Higgs field and the problem of confinement. In Refs. 15, 16, and 28 it is concluded from the fact that the phase space of the Higgs field is reduced (in the standard approach) that the Higgs field is of a composite nature, i.e., it is concluded that this field is not an elementary field. The importance of the phenomenon of reduction of the phase space for understanding confinement is pointed out in the preprints of Refs. 60 and 61. One would think that by eliminating in the quantum electrodynamics all unphysical degrees of freedom ("fixing the gauge") we would arrive at the standard field theory with the normal canonical variables. There then arises the question: what prevents the corresponding particles (excitations of these fields) from flying apart? It turns out that it is precisely the residual gauge symmetry, acting in the space of the physical variables, that prevents them from separating. Only composite objects, which are invariant under these residual gauge transformations, can be regarded as legitimate physical objects.[60,61]

Many questions were not studied in this review. Consideration of the problem of the structure of the phase space of field systems, especially systems with several fields, including spinor and scalar, opens up an extensive field for research. Other questions, for example, questions connected with quasiclassical calculations, also were not studied. This is work for the future. We believe that the solution of the problems considered will make it possible not only to understand better the nature of dynamical systems with gauge symmetry but it will also lead to progress in describing the properties and interactions of so-called "elementary particles."

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9. APPENDIX

9.1. The phase space in polar coordinates

The phase space of each of the two degrees of freedom of a particle moving in a plane in Cartesian coordinates is a plane: $\mathrm{PS}(x_i,p_i)=\mathbb{R}^2$, $i=1, 2$. In polar coordinates each variable r and θ takes on values which lie only on a part of the real axis. What then are the phase spaces of the corresponding degrees of freedom? The fact that r takes on values on the semiaxis $r\geqslant 0$ and θ in the segment $[0, 2\pi]$ suggests that $\mathrm{PS}(r,p_r)$ is the half-plane and $\mathrm{PS}(\theta,p_\theta)$ is a strip of width 2π, since p_r and p_θ are unbounded. A more careful analysis shows that in both cases the phase spaces are actually the full planes.

To prove this result for r and p_r, we shall study in parallel the motion of a particle through the origin in Cartesian and polar coordinates. Let the particle move along the x_1 axis. As long as the particle moves along the positive semiaxis the equality $x_1=r$ is satisfied and no paradoxes arise. As the particle moves through the origin x_1 changes sign, r does not change sign, and θ and p_r change abruptly: $\theta\to\theta+\pi, p_r=|p|\cos\theta\to -p_r$. Although these jumps are not related with the action of any forces they are consistent with the equations of motion. Meanwhile, the kinematics of the system admits an interpretation that permits avoiding them. Indeed, the formulas for the transformation to polar coordinates $x_1=r\cos\theta$ and $x_2=r\sin\theta$ are invariant under the substitutions $\theta\to\theta+\pi$ and $r\to -r$. This means that the motion with values of the polar coordinates $\theta+\pi$ and $r>0$ is not distinguishable from motion with values of the polar coordinates θ and $r<0$. As expected, the phase trajectory will be identical in both the (r, p_r) *plane* and the (x_1, p_1) plane.

In other words $\mathrm{PS}(r, p_r)$ is the plane folded in half along the axis $r=0$. Each half of the plane corresponds to values of θ differing by π, so that the physical states in these planes are different, and prior to being combined with the first half $r>0$ the second half $r<0$ is turned by an angle π around the axis $p_r=0$ (in view of the jump in the momentum $p_r\to -p_r$). Thus to each point of the phase space $r\geqslant 0$, $p_r\in\mathbb{R}$ there correspond two different physical states (they differ by the values of the angle θ), and therefore such a half-plane is isomorphic to the phase plane $(r,p_r)\in\mathbb{R}^2$, to each point of which there corresponds only one physical state of the system, i.e., by definition of the phase space it is precisely this plane that must be identified with $\mathrm{PS}(r,p_r)$. In the presence of gauge symmetry θ becomes an unphysical variable

(unobservable variable), so that the difference between the two superposed half-planes vanishes and $PS(r,p_r) = con(\pi)$ (the rotation by an angle π of the second half of the phase space $r<0$ prior to combining with the first half $r>0$ is precisely what makes it possible to identify them with one another in agreement with ideology of the gauge group $\mathbb{Z}_2$, operating in the phase space of the physical variables).

By studying the rotation of a particle around the origin of the coordinates we can verify that in order to describe the motion uniquely the angular variable must be unbounded $\theta\in\mathbb{R}$, if the number of rotations completed by the particle around the origin is taken into account. Therefore $PS(\theta,p_\theta)$ is also a plane.

9.2. First-order formalism and Hamiltonian mechanics

In theories with Fermi fields the Lagrangian is usually linear in the velocities. In order to elucidate the content of these theories from the viewpoint of the Hamiltonian formalism, we shall study a model with the Lagrangian

$$L=\frac{1}{2}\alpha^{ij}\xi_i\dot{\xi}_j - V(\xi), \quad i,j=1,2,\ldots,2n, \quad \alpha^{ij}=-\alpha^{ji}. \tag{9.1}$$

The essence of the matter will not change, if we set for simplicity $n=1$ and $\alpha^{12}=-1$. Then, transferring to the Hamiltonian formalism, we have $\pi_1=\partial L/\partial\dot{\xi}_1=\xi_2/2$, $\pi_2=\partial L/\partial\dot{\xi}_2=-\xi_1/2$, i.e., we have two primary constraints of the second kind: $\Phi_1=\pi_2+\xi_1/2=0$, $\Phi_2=\pi_1-\xi_2/2=0$, $\{\Phi_1,\Phi_2\}=1$. The variables Φ_1 and Φ_2 are, essentially, a pair of canonically conjugate variables. Another such pair is $\xi=-(\pi_2-\xi_1/2)$, $\pi=(\pi_1+\xi_2/2)$, and in addition $\{\xi,\Phi_i\}=\{\pi,\Phi_i\}=0$. According to Dirac[29] the constraints can be relaxed (the unphysical variables can be eliminated), replacing at the same time the Poisson brackets $\{\ ,\ \}$ by Dirac brackets

$$\{f,g\}_D=\{f,g\}-\{f,\Phi_i\}\{\Phi_i,\Phi_j\}^{-1}\{\Phi_j,g\}. \tag{9.2}$$

We have $\xi\approx\xi_1$, $\pi\approx\xi_2$ and $\{\xi_1,\xi_2\}_D=1$ (the symbol $\approx$ symbolizes an equality taking into account the constraints, $\{\Phi_i,\Phi_j\}^{-1}$ is a 2×2 matrix that is the inverse of the matrix $\{\Phi_i,\Phi_j\}$). Finally,

$$H(\xi,\pi) = \pi_1\dot{\xi}_1+\pi_2\dot{\xi}_2-\frac{1}{2}(-\xi_1\dot{\xi}_2+\xi_2\dot{\xi}_1)+V(\xi_1,\xi_2)\approx V(\xi_1,\xi_2), \tag{9.3}$$

i.e., Eq. (9.1) is essentially the "density" of the action in the Hamiltonian form: The action $A=S_H=\int L\,dt$, and V is the Hamiltonian. Thus ξ_1 and ξ_2 are actually canonically conjugate variables. All calculations can be easily extended to any integer n, since by linear transformation of ξ the matrix α can be reduced to the standard block-diagonal form with the matrices $-\tau\equiv-i\tau_2$ along the diagonal (τ_2 is a Pauli matrix). The conclusion is that the first-order formalism is essentially a Hamiltonian formalism: If the matrix α is reduced to the standard block-diagonal form, then $\xi_{2i-1}=q_i$, $\xi_{2i}=p_i$, $i=1,2,\ldots,n$, $V(\xi)=H(p,q)$, and the action $A=\int L(\xi,\dot{\xi})dt$ is the action in the Hamiltonian form

$$A=S_H=\int dt\left[\frac{1}{2}(p_i\dot{q}_i-q_i\dot{p}_i)-H(p,q)\right];$$

the equations of motion in a theory with the Lagrangian (9.1) are identical to the Hamiltonian equations of motion $\dot{q}_i=\partial H/\partial p_i$, $\dot{p}_i=-\partial H/\partial q_i$.

9.3. Dynamical systems with Grassmann variables

A Lagrangian system is usually given in the form[16)]

$$L=\frac{i}{2}\theta_j\dot{\theta}_j-H(\theta). \tag{9.4}$$

The equations of motion have the customary form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_j}=\frac{\partial L}{\partial\theta_j}; \tag{9.5}$$

but the type of derivative in them must be defined more precisely (left or right[53,54]). In this paper left derivatives are employed. Taking the simplest case, when $j=1,2$, we find the equations of motion for a "Grassmann" harmonic oscillator[17)]

$$\dot{\theta}_1=\omega\theta_2,\quad \dot{\theta}_2=-\omega\theta_1, \tag{9.6}$$

whose solution can be written in the form

$$\theta(t)=\exp(i\omega\tau_2 t)\theta(0), \tag{9.7}$$

where θ is the column vector (θ_1,θ_2). Writing the solution (9.7) in the more detailed form

$$\theta_1(t)=\theta_1(0)\cos\omega t+\theta_2(0)\sin\omega t,$$
$$\theta_2(t)=\theta_2(0)\cos\omega t-\theta_1(0)\sin\omega t$$

shows that the motion reduces to rotation of the two-dimensional "vector" (θ_1,θ_2) with the angular velocity ω.

The transfer to the Hamiltonian formalism is made in the same manner as for the normal variables, studied above. The Poisson brackets are defined as[53]

$$\{f,g\}=-i\frac{\partial}{\partial\theta'_j}\frac{\partial}{\partial\theta_j}f(\theta)g(\theta')|_{\theta'=\theta}. \tag{9.8}$$

The equations of motion have the standard form: $\dot{\theta}_j=\{\theta_j,H\}$.

Having in mind subsequent quantization, it is convenient to transform in the Lagrangian (9.4) to the complex variables $\eta=(\theta_1+i\theta_2)/\sqrt{2}$ and $\eta^+=(\theta_1-i\theta_2)/\sqrt{2}$ (it is assumed that the number of generators is even):

$$L=\frac{i}{2}(\eta^+\dot{\eta}-\dot{\eta}^+\eta)-\omega\eta^+\eta. \tag{9.9}$$

The Poisson brackets in the new variables η_j, η_j^+ assume the form

$$\{f(\eta,\eta^+),g(\eta,\eta^+)\} = -i\left(\frac{\partial}{\partial\eta'_j}\frac{\partial}{\partial\eta_j^+}+\frac{\partial}{\partial\eta_j'^+}\frac{\partial}{\partial\eta_j}\right)f(\eta,\eta^+)g(\eta',\eta'^+)|_{\eta'=\eta}, \tag{9.10}$$

and in addition $\{\eta_i,\eta_j^+\}=-\delta_{ij}$. The complex generators η,η^+, as is easily shown, play the role of canonical variables and are convenient for transforming to the quantum description[53,54]

$$[\hat{\eta}_i,\hat{\eta}_j^+]_+=i\{\eta_i,\eta_j^+\}=\delta_{ij}, \tag{9.11}$$

where $[\hat{A},\hat{B}]_+$ is the anticommutator of the operators $\hat{A}$ and $\hat{B}$. The equations of motion for η and η^+ are

$$\dot{\eta}=-i\omega\eta\quad \dot{\eta}^+=i\omega\eta^+, \tag{9.12}$$

i.e., $\eta(t) = \exp(-i\omega t)\eta(0) \equiv z(t)\eta(0)$.[18] We can now introduce the concept of phase space for a Grassmann oscillator. We define it as a two-dimensional plane in which $\mathrm{Re}\, z(t) = \cos t$ is plotted along one axis and $\mathrm{Im}\, z(t) = -\sin t$ is plotted along the other axis. This agrees with the definition of the phase space for the normal oscillator, where $a = (q + ip)/\sqrt{2}$, $a^* = (q - ip)/\sqrt{2}$ and $q = \mathrm{Re}\, a\sqrt{2}$, $p = \mathrm{Im}\, a\sqrt{2}$ play the role of η and η^+ (if $H = \omega a^* a$, then the equations of motion are $\dot{a} = -i\omega a$ and $\dot{a}^* = -i\omega a^*$ and their solutions are $a(t) = \exp -(i\omega t)a(0) = (q(t) + ip(t)/\sqrt{2}$ and $a^*(t)$; we plot $\cos\omega t$ and $-\sin\omega t$ along the q and p axes, if $p(0) = 0$ and $q(0) = 1$).

9.4. Quantum description of a system with Grassmann variables

Aside from the description in the "coordinate-free" (Dirac) formalism presented in Sec. 4.2, it is helpful to have a specific realization of the algebra (9.11), which is necessary for constructing a path integral. The following choice is natural and convenient. Let ξ_α and ξ^*_α, $\alpha = 1, 2, \ldots, n$, be generators of the complex Grassmann algebra, $\xi_\alpha = \theta^\alpha_1 + i\theta^\alpha_2$, $\xi^*_\alpha = \theta^\alpha_1 - i\theta^\alpha_2$, $(\theta^\alpha_i)^* = \theta^\alpha_i$ $(i = 1, 2)$. Then to an arbitrary state $|\chi\rangle$ there corresponds an element of the Grassmann algebra

$$\langle \xi | \chi \rangle = \sum_{k=0}^{n} \sum_{\{\alpha\}} \chi_{\alpha_1 \ldots \alpha_k} \xi^*_{\alpha_1} \ldots \xi^*_{\alpha_k} = \chi(\xi^*),$$

and to the conjugate state $\langle\chi|$ there corresponds the element

$$\langle \chi | \xi \rangle = \sum_{k=0}^{n} \sum_{\{\alpha\}} \chi^*_{\alpha_1 \ldots \alpha_k} \xi_{\alpha_k} \ldots \xi_{\alpha_1} = \overline{\langle \xi | \chi \rangle} \equiv \chi^*(\xi).$$

The operator $\hat{\xi}^+_\alpha$ is realized as a multiplication operator $\hat{\xi}^+_\alpha \chi(\xi^*) = \xi^*_\alpha \chi(\xi^*)$, and $\hat{\xi}_\alpha$, according to Eq. (9.11), is realized as a differentiation operator: $\hat{\xi}_\alpha \chi(\xi^*) = (\partial/\partial\xi^*)\chi(\xi^*)$. The scalar product is defined as follows:[53]

$$\langle \varphi | \chi \rangle = \int e^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} \varphi^*(\xi)\chi(\xi^*) \prod_\alpha (d\xi^*_\alpha d\xi_\alpha) = \varphi^*_0\chi_0 + \varphi^*_\alpha\chi_\alpha + \ldots, \tag{9.13}$$

with

$$\int d\xi = \int d\xi^* = 0, \quad \int \xi d\xi = \int \xi^* d\xi^* = 1, \quad \int \xi\xi^* d\xi^* d\xi = 1. \tag{9.14}$$

This definition is a concrete representation of the formulas (4.9)–(4.11). In the formula (9.13) we can integrate over $\Pi_\alpha d\xi_\alpha$. Then the scalar product will assume the new form

$$\langle \varphi | \chi \rangle = \int \bar{\varphi}(\xi^*)\chi(\xi^*) \prod_\alpha d\xi^*_\alpha, \tag{9.15}$$

where

$$\bar{\varphi}(\xi^*) = P \int e^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} \varphi^*(\xi) \prod_\alpha d\xi_\alpha, \tag{9.16}$$

and P is the parity of the permutation:

$$\prod_\alpha (d\xi^*_\alpha d\xi_\alpha) = P \prod_\alpha d\xi_\alpha \prod_\beta d\xi^*_\beta.$$

It can be verified that precisely in the sense of the metric (9.13) (or (9.15) and (9.16)) the operations of multiplication by ξ^*_α and differentiation $\partial/\partial\xi^*_\alpha$ are Hermitian conjugates. The operation (9.16) evidently corresponds to Martin's definition of conjugation.[54] For example, for $n = 2$ we have

$$\langle \xi | \chi \rangle = \chi(\xi^*) = \chi_0 + \chi_\alpha \xi^*_\alpha + \chi_3 \xi^*_1 \xi^*_2, \tag{9.17a}$$

$$\bar{\varphi}(\xi^*) = \varphi^*_0 \xi^*_1 \xi^*_2 + \varphi^*_1 \xi^*_2 + \varphi^*_2 \xi^*_1 + \varphi^*_3 \equiv \langle \varphi | \xi \rangle \tag{9.17b}$$

and

$$\langle \varphi | \chi \rangle = \int \bar{\varphi}(\xi^*)\chi(\xi^*)\, d\xi^*_2\, d\xi^*_1 = \sum_{\alpha=0}^{3} \varphi^*_\alpha \chi_\alpha. \tag{9.18}$$

The variable ξ^* in formulas of the type (9.15) can be regarded as a real Grassmann variable θ, since in the formalism (9.15) only ξ^* is employed and since the complex conjugate transforms (9.15) into

$$\langle \varphi | \chi \rangle^* = \langle \chi | \varphi \rangle = \int \bar{\varphi}^*(\xi)\chi^*(\xi) \prod_\alpha d\xi_\alpha = \int \bar{\varphi}^*(\xi^*)\chi^*(\xi^*) \prod_\alpha d\xi^*_\alpha = \int \bar{\chi}(\xi^*)\varphi(\xi^*) \prod_\alpha d\xi^*_\alpha. \tag{9.19}$$

The second equality is a consequence of (9.14) (the fact that the algebras $\{\xi\}$ and $\{\xi^*\}$ are identical to one another), and the last equality follows from Eq. (9.15) and (9.16). The variables ξ and ξ^* must be distinguished in an algebra with involution, i.e., with the generators $\{\xi, \xi^*\}$.

9.5. Quantization of constrained dynamical systems

Gauge theories are a typical, though not the most general, example of constrained dynamical systems. The problem of quantizing them was first encountered in the description of the electromagnetic field. Already in the 1920s Heisenberg and Pauli[65] and Fermi[66] (see Ref. 30 for a more detailed discussion) formulated a procedure for quantizing such systems. This problem later arose in all its magnitude in the study of gravitation and the Yang–Mills fields. Dirac[67,29] and Bergmann[68] studied the problem in its general form, they gave a classification of constraints, and they formulated a general procedure for quantizing the constraints. An extensive literature was subsequently devoted to this question (see, in particular, Refs. 17, 18, 31–34, and 69–78).

By quantization we mean making a transition to a quantum description of a dynamical system, i.e., to a description of its evolution with the help of probability amplitudes. In this paper we adhere to the standard procedure formulated by Dirac in Ref. 29 for quantizing constrained systems. The study of specific models on the basis of general rules of Ref. 29 has raised a number of questions that were not studied previously. We shall briefly formulate Dirac's procedure,[29] discuss the questions associated with it, and indicate specific methods (invariant and noninvariant) for implementing the general scheme.

In the process of making a transition from the Lagrangian to the Hamiltonian formalism there can arise constraints, i.e., expression of the form $\varphi(q, p) = 0$, which establish relations between the canonical variables (generalized coordinates and momenta). When transferring to the quantum description it is necessary to distinguish between constraints of the first and second kind. The Pois-

son brackets (PB) of the latter constraints are different from zero (even *weakly*, i.e., taking into account the constraints), while the PB of constraints of the first kind are equal to zero in the weak sense. Simplifying the situation somewhat, we can regard constraints of the second kind as conditions on some generalized coordinates and momenta $\hat{\mathcal{Q}}_i = 0$, $\hat{\mathcal{P}}_i = 0$, and constraints of the first kind as conditions on the generalized momenta.[79] On quantization the canonical variables become operators, which satisfy the standard commutation relations. From here it is clear that the constraint conditions cannot be transformed into operator equalities (the condition $[\hat{\mathcal{Q}}_i, \hat{\mathcal{P}}_j] = i\delta_{ji}$ is violated). It remains to select state vectors on which the constraints vanish; the corresponding linear space is called a physical Hilbert subspace. But this is admissible only with respect to constraints of the first kind, because these states that would be eigenstates for both canonically conjugate variables $\hat{\mathcal{Q}}_j\psi = 0$, $\hat{\mathcal{P}}_j\psi = 0$ do not exist (they are not consistent with the uncertainty relation). For this reason the only remaining possibility for constraints of the second kind is to relax them, i.e., to eliminate the corresponding unphysical variables, prior to quantization. Thus, in the process of quantization Dirac treats the constraints of the first and second kind differently,[29] though both types of constraints refer to unphysical variables.

The formulation of the theory in the language of Hamiltonian path integrals[69] creates the impression (erroneous, see Refs. 17 and 54 (p. 172)) that in this manner it is possible to do away with the problem of noncommutability of the canonically conjugate quantities, problems of ordering such quantities, etc., i.e., in some manner it is possible to work entirely on the basis of the classical Hamiltonian formalism. But then any distinction between the unphysical variables which correspond to constraints of the first or second kind is also lost. Their elimination by substituting into the path integral the corresponding δ-functions (in the case of constraints of the first kind so-called additional conditions are also added;[69] in so doing, the entire set of conditions is equivalent to constraints of the second kind) leads to a quantum theory that is identical to the theory obtained by quantizing after the unphysical variables are excluded. The corresponding quantum picture can differ from the picture given by Dirac's procedure.[29] This fact has been pointed out by many authors;[17,32-35] the operations of quantization and elimination of unphysical variables (constraints of the first kind) are, generally speaking, not commutative. The reason for the inconsistency is buried in the nature of quantum mechanics: canonical quantization cannot be performed in curvilinear coordinates,[51,52] and it is precisely such coordinates that one must usually use when transferring to the physical variables (see Sec. 2.1.2). If, however, the unphysical variables are related with the Cartesian coordinates (for example, $y(t)$ in the model (2.1)), then these operations do commute.

Which approach is the correct one? The objection to Dirac's scheme[29] is that it is unsuitable, since physical states satisfying conditions of the type $\hat{p}_i\Phi = 0$ are not normalizable. Meanwhile, in the alternative approach this problem does not arise. In Ref. 60 it is shown (see also Ref. 31, p. 665) that this objection to the procedure of Ref. 29 is unfounded, at least, for gauge theories. We note first that the collection of constraints of the first kind as generators of translations must be divided into two classes, related with compact and noncompact groups of transformations. The physical states on which the generators of compact groups vanish are normalizable in quantum mechanics. There remain generators that are noncompact on their range of operation. Both are encountered in gauge theories. But in gauge theories the unphysical variables with a noncompact domain (such as $y(t)$ in Secs. 2 and 3 or A_0 in the Yang–Mills theory) belong to a subspace of the Euclidean space, orthogonal to the "physical" space (i.e., containing the physical variables; in the model (2.1) this is the subspace $\mathbb{R}^1$ of the variable y in the full configuration space $\mathbb{R}^3$). But in this case it makes no difference whether they are included before or after quantization. For this reason, they can be ignored (there is no need to integrate over them when normalizing the vectors[31]) or eliminated before quantization. Since the results are the same, either approach can be used, depending on the desire or circumstances. In both cases the problem of nonnormalizability does not arise. The general case of dynamical systems with constraints which are generators of arbitrary compact and noncompact groups has not been studied and we do not examine it.

We have shown that Dirac's method is well-grounded for gauge theories. Are there any arguments which show that it is preferable? Such arguments do exist. We recall that although we must usually reconstruct the quantum picture starting from the classical picture, the quantum theory is the true theory. It must be explicitly relativistically invariant and therefore, in application to gauge fields, it must be formulated with the participation of all components of the vector fields A_μ, including also unphysical fields. This means that the requirement of explicit Lorentz invariance leads to the appearance of operators corresponding to unphysical variables. Their presence makes reduction of the full Hilbert space (transition to the physical subspace) unavoidable. This reduction is achieved by requiring that the constraints (generators of the gauge transformations) vanish on the physical vectors. The necessity of using Dirac's scheme[29] in physics appears to us to be indisputable.

We wish to say a few words about applications of general procedures. In both schemes (Dirac and the alternative one) specific theories can be constructed in two ways—invariant and nonvariant. In the invariant approach only explicitly gauge-invariant variables are employed, while the noninvariant approach involves the use of gauge-noninvariant quantities; this approach is also noninvariant in form (outwardly). Since both approaches are embodiments of the same theory which differ in form, physically (according to their content) they must be indistinguishable. Unfortunately, in both schemes these possibilities have not been studied in general form, so that we shall direct the reader to specific models. In the alternative scheme both approaches were studied in Sec 2.1 for the example of the model (2.1). The additional conditions that must be imposed in order for both approaches to be physically identical were also indicated there. These approaches are realized on the basis of Dirac's method in Secs. 2.1.2, 3.2, and 9.7.

In conclusion we shall give the definition of a physical phase space in the invariant and noninvariant approaches. In the first case, the physical phase space of a system is obtained by identifying with one another all points that are connected by a gauge transformation, i.e., the physical phase space is the factor space of the full (enveloping) phase space under the gauge group. We have in mind the *extended gauge*

group[29] G, generated by all constraints of the first kind (primary and secondary, independently). In the second approach, one first transfers from the starting phase space $\Gamma^{2N} = \mathbb{R}_q^N \otimes \mathbb{R}_p^N$ to a smaller space $\Gamma^{2M} = \mathbb{R}_q^M \otimes \mathbb{R}_p^M$, if $N - M$ is the total number of independent constraints of the first kind. In Secs. 2 and 3 it is shown that, as a rule, a residual discrete gauge group S operates in this reduced space and leads to additional reduction of the reduced space. Factorization of the space Γ^{2M} according to this group gives the physical phase space.[19] This can all be written down symbolically in the form $(\mathrm{PS})_{\mathrm{ph}} = \Gamma^{2N}/\widehat{G} = \Gamma^2/S$.

9.6. Properties of the measure $\varkappa^2(h)$ (Sec. 5.3)

The measure $\mu = \varkappa^2$ can be calculated explicitly.[13,15,43] To this end, we introduce in the Lie algebra X the Cartan–Weyl basis[14,44]

$$[h, e_\alpha] = (h, \alpha)\, e_\alpha, \quad [e_\alpha, e_{-\alpha}] = \alpha, \quad [e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}, \quad (9.20)$$

where h, $\alpha \in H$, $\alpha > 0$ are positive roots, e_α are the corresponding root vectors, $N_{\alpha\beta}$ are nonzero numbers if $\alpha + \beta$ is a root of the algebra; $(\ ,\ )$ is the Cartan–Killing form $(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$, where ad^x is an operator of the adjoint representation for the element x, operating in G according to the rule $\mathrm{ad}\, x\ (y = [x, y]$, $\forall x, y \in X$. For compact Lie groups the normalization of the structure constants can be chosen so that $(x,y) = x_a y_a$. Obviously, in the basis (9.20) any element $z \in X \ominus H$ can be written in the form $z = \Sigma_{\alpha>0} (z_\alpha^+ e_\alpha + z_\alpha^- e_{-\alpha})$. However a basis in $X \ominus H$, constructed from $e_\alpha, e_{-\alpha}$, is not orthogonal, since the square of the vector $x = h + z$, $h \in H$, $z \in X \ominus H$ in this basis is (Ref. 47, p. 217)

$$(x, x) = (h, h) + \sum_{\alpha>0} z_\alpha^+ z_\alpha^-. \quad (9.21)$$

To calculate $\det \omega$, in Eq. (5.17) it is necessary to introduce an orthogonal basis in $X \ominus H$

$$z = \sum_{\alpha>0} (z_\alpha^c c_\alpha + z_\alpha^s s_\alpha), \quad (9.22)$$

where

$$c_\alpha = \frac{1}{\sqrt{2}} (e_\alpha + e_{-\alpha}), \quad s_\alpha = \frac{1}{\sqrt{2}} (e_\alpha - e_{-\alpha}). \quad (9.23)$$

Then the second term on the right side of (9.21) is replaced by

$$\sum_{\alpha>0} [(z_\alpha^c)^2 + (z_\alpha^s)^2].$$

In the basis (9.23) it is easy to find the matrix $\omega_{\alpha\beta}(h)$. By definition (5.17) $[h, \lambda_\alpha] = \omega_{\alpha\beta}(h) \lambda_\beta$, where λ_α form an orthogonal basis in $X \ominus H$. For $\{\lambda_{l+1}, \lambda_{l+2}, \ldots, \lambda_N\}$ we choose the set $\{c_{\alpha_1}, s_{\alpha_1}, c_{\alpha_2}, s_{\alpha_2}, \ldots\}$. Then from Eqs. (9.23) and (9.20) there follows $[h, c_\alpha] = (h, \alpha) s_\alpha$, $[h, s_\alpha] = (h, \alpha) c_\alpha$. Therefore, in this basis the matrix $\omega_{\alpha\beta}$ consists of the blocks $\tau_1 (h, \alpha)$, where τ_1 is a Pauli matrix and α runs over the set of all positive roots. Therefore, to within a sign we have (in the Cartan–Weyl basis; $\omega_{\alpha\beta}$ are real)

$$\varkappa(h) = (\det \omega)^{1/2} = \prod_{\alpha>0} (\alpha, h), \quad (9.24)$$

i.e., $\varkappa(h)$ is a polynomial of h_i of degree $(N - l)/2$. By construction of (5.16) and (5.17) the quantity $V_G V_H^{-1} \mu(h)$ is the volume of the orbit of the gauge group of the element h.[59] Here V_G is the volume of the group space and V_H is the volume of the stationary subgroup of the element h, i.e., $V_H = (2\pi)^l$ (Cartan's subgroup is isomorphic to $(\otimes U(1))^l$.

Using Eq. (9.24), it is not difficult to check that the "quantum potential" $V_q = \varkappa^{-1}(\partial_i^2 \varkappa)/2)$ is equal to zero:

$$V_q = \frac{1}{2} \sum_{\alpha \neq \beta > 0} \frac{(\alpha, \beta)}{(\alpha, h)(\beta, h)} = \frac{1}{2} \sum_{\substack{\text{over all} \\ \text{planes}}} \sum_{\substack{\alpha \neq \beta > 0, \\ \text{in one} \\ \text{plane}}} \frac{(\alpha, \beta)}{(h, \alpha)(h, \beta)} = 0. \quad (9.25)$$

Here we divided the sum over the positive roots $\alpha \neq \beta > 0$ into a sum over the positive roots lying in one plane and a sum over all such planes. The sum in one plane can be calculated explicitly, since the mutual arrangement of the positive roots in it is determined by the four cases $\cos^2 \theta_{\alpha\beta} = (\alpha,\beta)^2 (\alpha,\alpha)^{-1} (\beta,\beta)^{-1} = 0,\ 1/4,\ 1/2,\ 3/4$,[14,44] i.e., actually V_q for a group of rank l is determined by V_q for a group with $l = 2$: SU(3), Sp(4) ~ SO(5), and G_2. It can be checked by explicit calculation that for them $V_q = 0$. The same result can be obtained by algebraic-geometric methods.[43] (theorem A.5.33).

We shall discuss the question of the poles of the physical wave functions (5.21). The eigenfunctions of the operator (5.19) are represented in the form $\psi = \varkappa^{-1} \varphi$, where φ is the solution of Schrödinger's equation with the Hamiltonian $-\partial_i^2/2 + V(h)$. The functions ψ can have singularities at the points where $\varkappa(h) = 0$. We shall show that the physical functions (5.21) are regular at these points. We base our argument on the following assertion (Ref. 43, p. 403): Any polynomial $p(h)$, $h \in H$, which has the property $p(wh) = \det w\, p(h)$, can be represented in the form

$$p(h) = \varkappa(h) q(h), \quad (9.26)$$

where q is an invariant of the group W.

Since Weyl's group is the group generated by reflections of the roots, $\mu = \varkappa^2$ is an invariant of W (see Eq. (9.24)), i.e.,

$$\varkappa(wh) = \pm \varkappa(h) = \det w \varkappa(h), \quad \det w = \pm 1. \quad (9.27)$$

Then we have Φ (5.21)

$$\Phi(h) = N_W^{-1/2} \varkappa^{-1}(h) \sum_w \det w \varphi(wh) \equiv \varkappa^{-1} \tilde{\varphi}(h). \quad (9.28)$$

Obviously, $\tilde{\varphi}(wh) = \det w \tilde{\varphi}(h)$. We expand $\tilde{\varphi}(h)$ in a series in powers of h_i and collect together the terms of the series into homogeneous polynomials $p_n(h)$ with a fixed total degree n, i.e.,

$$\tilde{\varphi}(h) = \sum_{n=0}^{\infty} \tilde{\varphi}_n p_n(h).$$

Since the group W operates uniformly on h, $p_n(wh) = \det w p_n(h)$. According to Eq. (9.26) $p_n(h) = \varkappa(h) q_n(h)$, where $q_n(wh) = q_n(h)$ and $\tilde{\varphi}_n = 0$ for $n < (N - l)/2$—the degree of the polynomial $\varkappa(h)$. Therefore $\tilde{\varphi}(h) = \varkappa(h) \chi(h)$, and

$$\chi(h) = \sum_{n=0}^{\infty} \tilde{\varphi}_n q_n(h)$$

is a W-invariant function. The conclusion is that the factor

$\varkappa^{-1}(h)$ in Eq. (9.28) cannot produce poles in the functions Φ.

9.7. Invariant coordinates for groups of rank 2[15,21]

In Sec. 3 it was pointed out that independent excitation of "Cartesian" physical variables h is impossible as a result of reduction of their phase space. On the other hand, in the second-quantized representation (see Sec. 3.2.1) independent degrees of freedom can be separated out. They are found to be collective excitations of the initial degrees of freedom. Such variables can also be introduced in the coordinate representation. The wave functions which depend on them are explicitly gauge-invariant.

We introduce in the Hamiltonian (3.3) for groups of rank $l = 2$ the gauge-invariant variables

$$\Phi_1 = (\mathrm{Tr}\, x^2)^{1/2},\ \Phi_2 = \Phi_1^{-r}\,\mathrm{Tr}\, x^r, \tag{9.29}$$

where r is the degree of the second independent Casimir operator (see Table I), and the momenta canonically conjugate to them

$$\pi_i = \mathrm{Tr}\, pe_i\,(\mathrm{Tr}\, e_i^2)^{-1},\ i = 1, 2. \tag{9.30}$$

By direct calculation we verify that the elements

$$e_i = \frac{\partial}{\partial x}\Phi_i\left(\frac{\partial}{\partial x} = \lambda_\alpha \frac{\partial}{\partial x_\alpha}\right)$$

have two properties: $\mathrm{Tr}\, e_1 e_2 = 0$ and $[e_1, e_2] = 0$. For this reason they can serve as a local basis in H. It can be shown that $\mathrm{Tr}\, e_1^2 = 1$, $\mathrm{Tr}\, e_2^2 = r^2\Phi_1^{-2}(c_2 + c_1\Phi_2 - \Phi_2^2) \equiv r^2\Phi_1^{-2}(\alpha - (\Phi_2 - \beta)^2)$, where $\beta = c_1/2$, $\alpha = c_2 + c_1^2/4$, and the constants $c_{1,2}$ depend on the structure constants and determine the decomposition of the polynomial $(\mathrm{Tr}\,(\lambda_\alpha x^{r-1}))^2 = (c_1\Phi_2 + c_2)\Phi_1^{2(r-1)}$. For example, for SU(3) $c_1 = 0$ and $c_2 = 1/6 (r = 3)$.

In the new variables $p = \pi_i e_i + \tilde{p}$, where $\mathrm{Tr}\, e_i\tilde{p} = 0$, so that the solution of the equations of constraint $[p, x] = 0$ is $p = 0$, since e_i form a basis in H. Then in the case of the oscillator $V = (1/2)\Phi_1^2$ we have the physical Hamiltonian

$$H_{\mathrm{ph}} = \frac{1}{2}\pi_1^2 + \frac{r^2\pi_2^2}{2\Phi_1^2}[\alpha - (\beta - \Phi_2)^2] + \frac{1}{2}\Phi_1^2, \tag{9.31}$$

and the condition for the norm to be positive $\mathrm{Tr}\, e_2^2 \geqslant 0$ implies that $-1 \leqslant (\Phi_2 - \beta)/\sqrt{\alpha} \leqslant 1$. The Hamiltonian equations of motion

$$\dot{\pi}_1 = \{\pi_1, H_{\mathrm{ph}}\}$$

$$= -\Phi_1 + \frac{r^2\pi_2^2}{\Phi_1^3}[\alpha - (\beta - \Phi_2)^2],\quad \dot{\Phi}_1 = \{\Phi_1, H_{\mathrm{ph}}\} = \pi_1,$$

$$\dot{\pi}_2 = \{\pi_2, H_{\mathrm{ph}}\}$$

$$= \frac{r^2\pi_2^2}{\Phi_1^2}(\Phi_2 - \beta),\quad \dot{\Phi}_2 = \{\Phi_2, H_{\mathrm{ph}}\} = \frac{r^2\pi_2}{\Phi_1^2}[\alpha - (\beta - \Phi_2)^2] \tag{9.32, 9.33}$$

have oscillating solutions independently for each degree of freedom:

$$\Phi_2 = \pi_2 = 0,$$
$$\Phi_1(t) = A\,|\cos t\,|,\ \pi_1(t) = -A\sin t\,\varepsilon(\cos t), \tag{9.34}$$

where $A = \mathrm{const}$, ε is the sign function, and

$$\Phi_1(t) = v = \mathrm{const},\quad \pi_1 = 0,$$
$$\Phi_2(t) = \beta + \sqrt{\alpha}\cos rt,\quad \pi_2(t) = -\frac{v^2}{r\alpha^{1/2}\sin rt}. \tag{9.35}$$

The modulus in Eq. (9.34) is inserted in view of the fact that Φ_1 is positive. From the solutions (9.34) and (9.35) it is clear that the independent frequencies are 2 and r, as was established in Sec. 3.2.1. Obviously, $\arccos(\Phi_2 - \beta)/\sqrt{\alpha}$ can be related with the angular variable introduced in Sec. 3.2.2.

The variables (9.29) permit constructing explicitly gauge-invariant wave functions of the oscillator. For this purpose, it is necessary to transform in Schrödinger's equation with the Hamiltonian (5.19) to the new variables $\Phi_1 = (\mathrm{Tr}\, h^2)^{1/2}$, $\Phi_2 = (\mathrm{Tr}\, h^r\Phi_1^{-r} - \beta)/\sqrt{\alpha}$. By explicit calculation we verify that $\mu = \varkappa^2 = \mathrm{const}\,\Phi_1^{2r}(1 - \Phi_2^2)$; after the substitution $\psi_E = \varkappa^{-1}\varphi_E$, this equation assumes the form ($V = \Phi_1^2/2$)

$$\left(-\frac{1}{\Phi_1}\frac{\partial}{\partial\Phi_1}\Phi_1\frac{\partial}{\partial\Phi_1} - \frac{r^2}{\Phi_1^2}(1 - \Phi_2^2)^{1/2}\frac{\partial}{\partial\Phi_2}(1 - \Phi_2^2)^{1/2} \times \frac{\partial}{\partial\Phi_2} + \Phi_1^2\right)\Phi_E = 2E\Phi_E. \tag{9.36}$$

We seek the solution in the form $\varphi_E = F(\Phi_1)f(\Phi_2)$. Then Eq. (9.36) is equivalent to

$$-(1 - \Phi_2^2)f'' + \Phi_2 f' + cf = 0, \tag{9.37}$$

$$-F'' - \frac{1}{\Phi_1}F' - \left(\frac{r^2c}{\Phi_1^2} - \Phi_1^2 + 2E\right)F = 0, \tag{9.38}$$

where c is the separation constant. Since ψ_E must be finite at the boundary of Weyl's chamber (see Sec. 9.5), it is necessary to impose the boundary conditions $f(\pm 1) = 0$ ($\mu = 0$ for $\Phi_2 = \pm 1$). From here we find the solution of Eq. (9.37) $f = (1 - \Phi_2^2)^{1/2}U_m(\Phi_2) = \sin[(m + 1)\arccos\Phi_2]$, where U_m are Chebyshev polynomials of the second kind ($m = 0, 1, \ldots$), and $c = -(m + 1)^2$. The substitutions $F(\Phi_1) = \Phi_1^{r(m+1)}\exp(-\Phi_1^2/2)g(\Phi_1)$ and $\Phi_1^2 = t$ reduce Eq. (9.38) to the standard equation (2.13),

TABLE I.

Group	$A_l \sim \mathrm{SU}(l+1)$	$B_l \sim \mathrm{SO}(2l+1)$	$C_l \sim \mathrm{Sp}(2l)$	$D_l \sim \mathrm{SO}(2l)$
r_i, $i = 1, 2, \ldots, l$	$i+1$	$2i$	$2i$	$2i$, $l\,(i \neq l)$

Group	G_2	F_4	E_6	E_7	E_8
r_i, $i = 1, 2, \ldots, l$	2, 6	2, 6, 8, 12	2, 5, 6, 8, 9, 12	2, 6, 8, 10, 12, 14, 18	2, 8, 12, 14, 18, 20, 24, 30

and in Eq. (2.13) we must set $\alpha = r(m+1)+1$ and $\beta = -(E-\alpha)/2$. As a result we find the final expression for the wave functions that are invariant under the group W:

$$\psi_{nm} = \text{const}\, \Phi_1^{rm} U_m(\Phi_2) L_n^{r(n_i+1)}(\Phi_1^2) \exp\left(-\frac{1}{2}\Phi_1^2\right). \tag{9.39}$$

It is clear from the expression (9.39) that ψ_{nm} depend only on $p_{r_1}(h)$ and $p_{r_2}(h)$ ($r_1 = 2$, $r_2 = r$), i.e., the expression in front of the exponential in Eq. (9.39) is a polynomial in $p_{r_i}(h)$. The latter fact is obvious for G_2 and $\mathrm{Sp}(4) \sim \mathrm{SO}(5)$, since $r = 6$ and 4, respectively. For SU(3) $r = 3$ and for odd $m\Phi_1^{3m}$ is proportional to $(\mathrm{Tr}\, h^2)^{1/2}$. However it was pointed out above that for SU(3) $\beta = 0$. Therefore in this case $\Phi_2 = \sqrt{6}\ \mathrm{Tr}\, h^3 \Phi_1^{-3}$ and the singularity in the expression $\Phi_1^{3m} U_m(\Phi_2)$ vanishes.

Based on a theorem of Chevalley[44] on the analytical continuation of W-invariant polynomials in H we conclude that the explicitly gauge-invariant wave functions in the full configuration space have the form (9.39), where $p_{r_i}(h)$ are replaced by $p_{r_i}(x)$, $x \in X$.

9.8. The operator $\hat{Q}$ for an arbitrary group

In the formulation of the quantum theory a holomorphic representation, in which the state vectors are functions of the complex variable $a_j = (x_j + ip_j)/\sqrt{2}$ (j enumerates the degrees of freedom) and the operators $\hat{a}_j = (\hat{x}_j + i\hat{p}_j)/\sqrt{2}$ and $\hat{a}_j^+$ are, respectively, the differentiation operator $\partial/\partial a_j$ and the operator of multiplication by a_j, is often employed. In this representation it is also possible to construct the kernel of the projector onto the physical subspace $P_G(a,a^*)$ and thereby define the Hamiltonian path integral without explicitly separating out the physical variables. This is most simply done with the help of the spectral decomposition for $\hat{P}_G$. For example, the states (2.22) $\langle a|\Phi_k\rangle = \Phi_k(a)$ ($\langle a|\Phi_0\rangle = 1$) form a basis in $\mathscr{H}_{\mathrm{ph}}$. Therefore[38]

$$P_G(\mathbf{a},\mathbf{a}^*) = \sum_{k=0}^{\infty} \Phi_k(\mathbf{a})\Phi_k^*(\mathbf{a}) = \Gamma\left(\frac{n}{2}\right)\xi^{1-(n/2)} I_{(n/2)-1}(2\xi), \tag{9.40}$$

where $\xi = 1/2(a^2 a^{*2})^{1/2}$, and $I_\nu(x) = J_\nu(ix)$ is a Bessel function of imaginary argument. The formula (9.40) shows that $\hat{P}_G$ is an analytic function of the invariant of the theory $a^2 a^{*2}$.

The laws of the gauge transformations of x_j and a_j are identical, so that $\Phi_{\mathrm{ph}}(a) \in \mathscr{H}_{\mathrm{ph}}$ are functions of invariant polynomials constructed from a_j. From here it follows that the projection formula (5.56) is also valid for the holomorphic representation, if in it x is replaced by a. A characteristic feature of the holomorphic transformation is that the kernel of the identity operator is $\exp\langle a,a^*\rangle = \exp(a_i a_i^*)$ instead of a δ-function. Then, instead of Eq. (5.58) we have

$$P_G(\mathbf{a},\mathbf{a}^*) = \frac{1}{V_G}\int dg(\omega) \exp\langle \mathbf{a}^*, T(\omega)\mathbf{a}\rangle. \tag{9.41}$$

One can see that P_G does not depend on the dynamics and has a universal character.

In many cases the integration over the group in Eq. (9.41) or (5.58) can be performed explicitly. For example, for the model from Sec. 7.1 we have $\langle a^*, T(\omega)a\rangle = \Sigma_i (a_i^* \exp(\omega T) a_i)$ ($i = 1,2$), $V_G = 2\pi$, and

$$\int_G dg(\omega) = \int_0^{2\pi} d\omega;$$

then Eq. (9.41) gives

$$P_G(a,a^*) = \frac{1}{2}(I_0(\xi_+) + I_0(\xi_-)), \tag{9.42}$$

where $\xi_\pm = \Sigma_i (a_i^* a_i \pm a_i T a_i^*)$. The corresponding formulas in the coordinate representation are also of interest. To calculate the integral (5.58) we construct the matrix $\tilde{x}$ whose columns are x_i ($\tilde{x}_{ij} = x_i^{(j)}$). Then

$$P_G(\tilde{x},\tilde{x}') = \frac{1}{2\pi}\int_0^{2\pi} d\omega \delta^4(\tilde{x} - e^{\omega T}\tilde{x}'); \tag{9.43}$$

here the δ-function of a matrix is interpreted as the product of the δ-functions of its elements. After the substitution of variables $\tilde{x} = e^{\theta T}\rho$, $\tilde{x}' = e^{\theta T}\rho'$, where ρ and ρ' are upper-triangular matrices, the integral can be easily calculated, and the result is[39,20]

$$P_G(\tilde{x},\tilde{x}') = P_G(\rho,\rho') = \frac{1}{\pi}|\det\rho|\,\delta^3(\rho^{\mathrm{T}}\rho - \rho'^{\mathrm{T}}\rho')\left(1 + \frac{\det\rho}{\det\rho'}\right) \tag{9.44}$$

$$= \frac{1}{\pi}|\det\tilde{x}\det\tilde{x}'|^{1/2}\,\delta^3(\tilde{x}^{\mathrm{T}}\tilde{x} - \tilde{x}'^{\mathrm{T}}\tilde{x}')\left(1 + \frac{\det\tilde{x}}{\det\tilde{x}'}\right). \tag{9.45}$$

The equality (9.44) makes it possible to find the kernel $Q(\rho,\rho')$. Taking into account the difference of the normalizations of $\langle\rho|\rho'\rangle_{\mathrm{ph}}$ (see Eq. (5.64)) and the kernel P_G, we find

$$Q(\rho,\rho') = \delta^3(\rho - \rho') + \delta^3(\rho + \rho'), \tag{9.46}$$

which is in complete agreement with the analysis of Sec. 7.1. The residual gauge group $S = \mathbb{Z}_2$ operates simultaneously on all components of ρ. We note that the representation (9.45) for P_G makes it possible to calculate $\hat{Q}$ for any method of prescribing the physical variables, i.e., instead of $\rho_{12} = 0$ any condition on the components of ρ can be employed. It is sufficient to set in Eq. (9.45) $\tilde{x} = \exp(T\theta)\rho$ (analogously for x'), where the components of ρ satisfy the chosen condition, to find $\langle\rho|\rho'\rangle_{\mathrm{ph}}$, based on which it is easy to reconstruct $\hat{Q}$ from the formula (5.65).

In the case of an arbitrary group (the model from Sec. 3) the kernel $P_G(x,x')$, where $x, x' \in X$ and X is a Lie algebra, is determined by Casimir operators. To prove this we shall construct the analytic continuation of the kernel of the identity operator for the model from Sec. 5.3

$$\langle h|h'\rangle_{\mathrm{ph}} = \frac{1}{\varkappa^2(h)}\sum_w \delta^l(h - wh'), \quad h \in H, \quad h' \in K^+ \tag{9.47}$$

in the full configuration space X (we note that the kernel (9.47) can be derived by explicit calculation from the formula (5.64), if the theorem (A.5.35) from Ref. 43 is employed and if the fact that the stationary subgroup of an arbitrary element $x \in X$ is a Cartan subgroup, i.e., $V_H = (2\pi)^l$, $l = \mathrm{rank}\, G$, is taken into account). We shall use the formula (III.3.7) from Ref. 43

$$\det\left\|\frac{\partial P_{r_i}(h)}{\partial h_j}\right\| = c\varkappa(h), \quad c = \text{const}, \tag{9.48}$$

where $P_{r_i} = \mathrm{Tr}\, h^{r_i}$, r_i is the degree of the independent Casimir operators (see Table I), and $\varkappa(h)$ is defined in Eq. (9.24). Using the rule for substituting for the argument in a multidimensional δ-function we shall represent (9.47) in the following form[57]

$$\langle h \mid h' \rangle_{\mathrm{ph}} = c\,(\varkappa^2(h)\varkappa^2(h'))^{1/4} \prod_{i=1}^{l} \delta\,(P_{r_i}(h) - P_{r_i}(h')). \quad (9.49)$$

Here we have employed the invariance of the measure $\varkappa^2(h)$ under the Weyl group W (see Sec. 9.5) and the definition $|\varkappa(h)| = (\varkappa^2(h)^{1/2})$. The polynomial $\varkappa^2(h)$ can be expressed in terms of the polynomials $P_{r_i}(h)$, as can any W-invariant polynomial in H.[44] In view of the equality $P_{r_i}(x) = P_{r_i}(S(z)hS^{-1}(z)) = P_{r_i}(h)$ (see Eq. (3.4)) the elements h, $h' \in H$ in Eq. (9.49) can be replaced by x and x', respectively. Now, in order to find the kernel $P_G(x,x')$ for an arbitrary group G, Eq. (9.49) must be multiplied by $V_H V_G^{-1} = (2\pi)^l V_G^{-1}$. As one can see from Eq. (9.49), the kernel P_G is determined only by the characteristics of the gauge group—the Casimir operators—and does not depend on the dynamics.

[1)] The exact definition of the concept of a physical phase space is given in Sec. 9.5.

[2)] This phenomenon has also been observed and is under intensive study in the modern theory of strings (see, for example, Ref. 80).

[3)] By gauge group we mean the full group generated by both the primary and secondary constraints.[29]

[4)] The normal (nongauge) dynamics can be given a discrete gauge symmetry only "by hand." It cannot be prescribed by a Lagrangian which does not contain unphysical variables. Here $\mathbf{Z}_2$ is a subgroup of a gauge group.

[5)] See Sec. 9.5 of the Appendix as well as Ref. 31.

[6)] Generally speaking, the points u_a cannot belong to any of the sets $\mathbf{R}_{1,2,3}$, since for $u = u(a_a \neq 0)$ Eq. (2.25) has three solutions (for $u = u_0 = 0$ it generally becomes an identity $0 \equiv 0$). The set $\cup u_a$ has measure zero, so that it is not significant for determining the scalar product of physical state vectors. Formally it cannot be included in the set $\mathbf{R}_\alpha$ (the open sets in $\mathbf{R}_{1,2,3}$ cannot be closed), since the corresponding functions $u_s(u)$ cannot always be continuously continued into $\cup u_\alpha$ (see below).

[7)] However, this need not be done, if it is agreed that the orientation of K_α be chosen in accordance with the sign of μ. For example, if $K_2 = (u_2, u_3)$ (see curve *2* in Fig. 1), then $\mu(u) < 0$, but

$$\int_{u_2}^{u_3} du\,|\mu(u)| = \int_{u_3}^{u_2} du\,\mu(u).$$

[8)] The gauge $x = h$ is called an incomplete global gauge with residual group symmetry $W \subset G$ (see Ref. 40).

[9)] The equality $\psi_1(0) = \psi_2(0)$ actually follows from Eq. (4.3) after substitution of Eq. (4.6).

[10)] The characteristic features associated with the curvilinearity of the physical variable in the model of Sec. 2.1 are examined in Sec. 5.2.

[11)] The equality (3.4) determines the substitution of variables with $x \in X \sim \mathbf{R}^N$ and $h \in K^+$, $S(z) \in G/G_H$, and G_H is a Cartan subgroup.[43]

[12)] Since $[\hat{o}_a, \hat{H}] = 0$, where $\hat{H}$ is defined in Eq. (5.3), $\mathscr{H} = \mathscr{H}_{\mathrm{ph}} \oplus \mathscr{H}_{\mathrm{nph}}$.

[13)] More accurately, a linear combination of the solutions of Eq. (6.5), such that τ_c individual instantons are separated by quite large intervals and as $\tau \to \pm\infty$ this combination approaches $2\pi m$ and $2\pi m'$, respectively, serves as a stationary point. This is the instanton-gas approximation.[27]

[14)] In the corresponding one-dimensional model[26] the opposite is true. The difference is connected with the existence of the measure $\mu(r) = r^2$, on which the wave functions depend (see the analysis of Eq. (5.12)).

[15)] The state $\hat{b}_3|0\rangle$ is the only physical state in the model from Sec. 4.2.

[16)] Generally speaking, there exist consistent theories with Grassmann variables, whose Lagrangian is quadratic in the velocities (see, for example, Ref. 62). In this paper this case is not studied.

[17)] This is also true in the case when the residual gauge transformations do not form a subgroup of the gauge group.

[18)] The general case of systems with Grassmann variables was studied in Refs. 63 and 64.

[19)] We note that the transformation $\theta_i \to \lambda\theta_i$, where λ is an arbitrary real number ($\lambda \neq 0$), does not change Grassmann algebras. In this case $z(t)$ is a function of time: It carries information about the dynamics of the system.

[20)] Here and in Eq. (9.48) the δ-function of a symmetric 2×2 matrix is interpreted as the product of δ-functions of the three independent components of this matrix.

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