Constrictions in a relativistic pinch with a longitudinal magnetic field

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1. INTRODUCTION

References 1–4 examined a hypothesis about the generation of galactic cosmic rays in cosmic plasma pinches which do not contain an internal magnetic field. It was shown that accelerated particles arise in plasma streams squeezed from pinches. In the range of ultrarelativistic energies $E \gg Mc^2$ the integral particle spectrum has the form $I = CE^{-\nu}$, with $\nu = \sqrt{3}$. This is close to the exponent of the observed spectrum of galactic cosmic rays, and increases the plausibility of the hypothesis.

However, the plasma fiber filaments observed in space contain, as a rule, an internal longitudinal magnetic field, which is evidenced by the polarization of its synchrotron radiation. Thus, this article examines the problem of constrictions in a pinch with a longitudinal field. One can predict that, in an ideal plasma with a fully "frozen-in" internal magnetic field, constrictions in the pinch will not be able to break off completely so that the forces of magnetic compression remain finite, and the yield of accelerated particles with large energies will be substantially reduced compared to the case of a pinch without a field. It is shown below that the exponent of the spectrum of particles in a pinch with a field is $\nu = (3+\alpha)^{1/2}$, where $\alpha = (cB_{\parallel}^{0}/B_{\perp}^{0} v_{\rm T}^{0})^2$, where $v_{\rm T}^{0}$ is the thermal speed of ions.

2. FOUR-DIMENSIONAL VECTORS OF THE ELECTRIC AND MAGNETIC FIELDS

If one assumes the absence of collisions which aid in the exchange of energy between various degrees of freedom of particles, one should use magnetic hydrodynamic equations with anisotropic plasma pressure. These equations of ideal relativistic anisotropic magnetic hydrodynamics are a relativistic generalization of the well-known Chew, Goldberger, and Low equations (see Ref. 5), which were examined earlier in Refs. 6–10. In all cases they are derived from a relativistic kinetic equation.

Here for simplicity we note a more concise derivation of the equations of ideal relativistic anisotropic magnetic hydrodynamics using a purely hydrodynamic approach. We note that in the system of coordinates moving with matter, the four-dimensional energy-momentum tensor of the plasma and the field should have the form

$$\tilde{T}^{ik} = \begin{pmatrix} e+\mu & 0 & 0 & 0 \\ 0 & p_{\perp}+\mu & 0 & 0 \\ 0 & 0 & p_{\perp}+\mu & 0 \\ 0 & 0 & 0 & p_{\parallel}-\mu \end{pmatrix}, \qquad (1)$$

where $e = \rho c^2 \langle \gamma \rangle$ is the energy density of the particles, ρ is the mass density, $\mu = \tilde{B}^2/8\pi$ is the magnetic energy density, and $p_{\parallel,\perp}$ are the components of pressure. To write this tensor in an arbitrary, in particular, a laboratory system of coordinates, it is useful first to examine the ancillary problem of movement in the fields **E** and **B** of a hypothetical particle such as a "Dirac monopole" with an electric charge q_e and a magnetic charge q_m. The equation of relativistic motion (see Ref. 11) should have the form $d\mathbf{p}/dt = \mathbf{F}_e + \mathbf{F}_m$, where $\mathbf{F}_e = q_e \mathbf{E}^*$, $\mathbf{E}^* = \mathbf{E} + [\vec{\beta}\mathbf{B}]$ is the well-known Lorentz force and $\mathbf{F}_m$ is the not so well-known "magnetic analog of the Lorentz force," which is presented, for example, in Ref. 11, and is equal to $\mathbf{F}_m = q_m \mathbf{B}^*$, $\mathbf{B}^* = \mathbf{B} - [\vec{\beta}\mathbf{E}]$, $\vec{\beta} = \mathbf{v}/c$.

Further, we recall that in the special theory of relativity in four-dimensional space with coordinates $\tau = ct = x^0$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and with a metric $(ds)^2 = (cdt)^2 - (d\mathbf{r})^2 = g_{ik}\,dx^i\,dx^k = (d\tau/\gamma)^2$, a four-velocity vector is introduced $u^i = dx^i/ds = (\gamma, \mathbf{u})$, $\mathbf{u} = \vec{\beta}\gamma$, $\gamma = 1/(1-\beta^2)$ as well as a four-acceleration $w^i = du^i/ds$. This can be used to write the equation of motion of a hypothetical particle presented above as

$$w^i = \xi_e e^i + \xi_m b^i, \quad e^i = (\mathbf{u}\mathbf{E}^*, \gamma\mathbf{E}^*), \quad b^i = (\mathbf{u}\mathbf{B}^*, \gamma\mathbf{B}^*), \qquad (2)$$

where $\xi_{e,m} = q_{e,m}/Mc^2$ are scalar. Then it is clear that e^i and b^i are correct relativistic four-vectors which are also used to construct the equations of relativistic magnetohydrodynamics.

3. EQUATIONS OF RELATIVISTIC ANISOTROPIC MAGNETOHYDRODYNAMICS

To derive the equations of ideal relativistic anisotropic magnetohydrodynamics, we note that for the infinite electric conductivity which we have assumed, we have the condition $\mathbf{E}^* = \mathbf{E} = [\vec{\beta}\mathbf{B}] = 0$, so that $e^i = 0$. Only the four-vector of the magnetic field b^i (Eq. (2)) remains, and with it the energy-momentum tensor can be written as

$$T^{ik} = S_1 u^i u^k + S_2 g^{ik} + S_3 b^i b^k, \qquad (3)$$

where $S_{1,2,3}$ are three scalars. Comparison of Eq. (1) with Eq. (3) yields their values

$$S_1 = e + p_{\perp} + 2\mu, \ S_2 = -p_{\perp} - \mu, \ S_3 = (p_{\parallel} - p_{\perp} - 2\mu)\tilde{B}^{-2}. \qquad (4)$$

If we then use the more convenient tensor

$$T_k^i = S_1 u^i u_k + S_2 \delta_k^i + S_3 b^i b_k, \quad b_k = g_{ki} b^i, \qquad (5)$$

the desired equations of ideal relativistic anisotropic magneto hydrodynamics are reduced to the equations

$$\nabla_i T_k^i = 0, \quad \nabla_i \rho u^i = 0, \quad \nabla_i (b^i u^k - b^k u^i) = 0, \qquad (6)$$

the first of which yields the laws of conservation of entropy, energy, and momentum. The second is the relativistic equa-

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tion of continuity, and the third, considering the expression $\mathbf{B} = \gamma \mathbf{b} - \mathbf{u} b^0$ is equivalent to the freezing-in equation

$$\operatorname{div} \mathbf{B} = 0\,, \quad \partial\widetilde{\mathbf{B}}/\partial t = \operatorname{curl}[\mathbf{vB}]\,. \qquad (7)$$

It is useful to note that the combination $u^k \nabla_i T^i_k = 0$ yields for the adiabatic curves equation

$$\rho \frac{d}{ds}(S_1/\rho) + \frac{d}{ds} S_2 + \frac{1}{2}\rho^2 S_3 \frac{d}{ds}(\widetilde{B}/\rho)^2 = 0, \qquad \frac{d}{ds} = u^i \nabla_i\,, \qquad (8)$$

and the combination $\nabla_i T^i_k - u_k u^i \nabla_j T^j_i = 0$ at $k = 0$ yields the law of conservation of energy

$$S_1 \frac{d\gamma}{ds} + \gamma\rho \frac{d}{ds}(S_1/\rho) + \frac{\partial}{\partial\tau} S_2 + \nabla_i (S_3 b^i b^0) = 0\,. \qquad (9)$$

4. EQUATIONS OF LONG-WAVELENGTH PERTURBATIONS OF A PINCH

Let us use the equations of ideal relativistic anisotropic magnetohydrodynamics which we have obtained for the problem that we have examined of constrictions in a pinch with a longitudinal magnetic field. Due to the complexity of the general equations we limit ourselves, as before, to an analysis of only long-wavelength perturbations with $\lambda \gg a$, where $a(t,z)$ is the radius of the pinch. For these perturbations one can use the "narrow jet approximation" in which the quantities $e, \mu, \rho, p_\parallel, p_\perp, v = v_z$ and $B = B_z$ are considered to be constant over the cross section of the pinch πa^2. The radial components are considered to be equal, respectively, to

$$v_r = r\dot{a}/a, \quad \dot{a} = \partial a/\partial t + v \partial a/\partial z\,, \quad B_r = -(r/2)\partial B_z/\partial z\,. \qquad (10)$$

Then, setting $u = \operatorname{sh} y$, $\rho_* = \rho a^2/\rho_0 a_0^2$ and introducing two convenient operators

$$\hat{P} = u^i \nabla_i = \frac{d}{ds} = \gamma \frac{\partial}{\partial\tau} + u \frac{\partial}{\partial z}\,, \quad \hat{Q} = u \frac{\partial}{\partial\tau} + \gamma \frac{\partial}{\partial z}\,, \qquad (11)$$

we can reduce the continuity equation (6) and the energy equation (9) to the form

$$\hat{Q}y = -\hat{P} \ln \rho_*, \quad (e + p_\parallel)\hat{P}y = -\hat{Q}p_\parallel + (p_\parallel - p_\perp)\hat{Q} \ln \widetilde{B}\,. \qquad (12)$$

In the derivation of the last equation from Eq. (9) it was assumed that in the narrow jet approximation one can set $b^i \nabla_i = \widetilde{B}\hat{Q}$, $\nabla_i b^i = \widetilde{B}\hat{P}y$. Further, we assume that in its own system of coordinates the plasma is nonrelativistic and $e = \rho c^2 + p_\perp + p_\parallel/2$. Then Eq. (8) yields two Chew, Goldberger, and Low adiabatic curves: $p_\parallel \sim \rho^3 \widetilde{B}^{-2}$, $p_\perp \sim \rho\widetilde{B}$, $T_\perp \sim \widetilde{B}$.

However, for simplicity, hereafter we will assume that $p_\parallel = 0$. This assumption is reasonable when applied to a hypothetical space pinch, whose formation should apparently be preceded by a stage of gradual piling up of the plasma into a cylinder. One can expect that in this stage the component $p_\perp$ will increase much more than the longitudinal pressure $p_\parallel$, which corresponds to $p_\perp \gg p_\parallel$. Assuming equal temperatures and densities of electrons and ions and substituting the adiabatic curve expression $p_\perp = p_e + p_i = p^0_\perp \rho_* (a_0/a)^4$ into the condition of equality of pressures $p_\perp + \widetilde{B}^2/8\pi = B^2_\varphi/8\pi$ at the boundary of the pinch $r = a$, we express the effective density in terms of the transverse temperature

$$\rho_* = \varepsilon(1 - x)/x(1 - \varepsilon), \quad \varepsilon = (B^0_\parallel/B^0_\varphi)^2, \quad x = \varepsilon T_\perp/T^0_\perp\,, \qquad (13)$$

and substituting it into Eq. (12), we finally find two equations

$$\hat{P}x = x(1 - x)\hat{Q}y, \quad \hat{P}y = -\nu\hat{Q}x, \quad \nu = 2T^0_\perp/\varepsilon Mc^2\,. \qquad (14)$$

5. SOLUTION USING THE HODOGRAPH METHOD WITH A LORENTZ TRANSFORMATION

To solve these nonlinear equations we first introduce the inverse functions $\tau = ct = T(x,y)$, $z = Z(x,y)$, which are the "laboratory" time and coordinate. Then we introduce another "accompanying" time and coordinate $\widetilde{T}(x,y)$, $\widetilde{Z}(x,y)$ which are linked with the laboratory quantities T and Z by the Lorentz transformation formulas: $\widetilde{T} = \gamma T - uZ$, $\widetilde{Z} = \gamma Z - uT$. Then it is easy to verify that this "hodograph transformation" with an additional Lorentz transformation can yield two equations from Eq. (14)

$$\widetilde{T}'_y + \widetilde{Z} + \widetilde{Z}_x/\nu = 0, \quad \widetilde{Z}'_y + \widetilde{T} - x(1 - x)\widetilde{T}_x{}' = 0\,, \qquad (15)$$

from which, taking into account the nonrelativistic nature of the quantity $\nu x = 2T_\perp/Mc^2 \ll 2$, we obtain

$$x(1 - x)\widetilde{T}''_{xx} + x(\nu - 2)\widetilde{T}'_x = \nu(\widetilde{T} - \widetilde{T}''_{yy})\,. \qquad (16)$$

Finally, introducing the convenient variable $\xi = 1 - 2x$, we finally obtain the "proper time equation"

$$\hat{\theta}\widetilde{T} = \nu(\widetilde{T} - \widetilde{T}''_{yy}), \quad \hat{\theta}\widetilde{T} = (1 - \xi^2)\widetilde{T}''_{\xi\xi} + (1 - \xi)(2 - \nu)\widetilde{T}'_\xi\,. \qquad (17)$$

For our purposes, only special solutions of this equation which describe perturbations that vanish in the opposite time limit $t \to -\infty$ are interesting. This "condition of spontaneity" of perturbations seems to simulate the preliminary stage of gathering of plasma into a cylindrical pinch, which is initially assumed to be in equilibrium without perturbations. The absence of perturbations corresponds to the "starting point" $\xi = \xi_0 = 1 - 2\epsilon$, $y = y_0$, and we require the function $\widetilde{T}(\xi,y)$ to have a property of the type of $\widetilde{T} \to -\infty$ at this point and to go to zero at all boundaries of the examined range of change in arguments $-1 < \xi < 1$, $-\infty < y < \infty$!. It is not difficult to verify that only a solution of the following type goes to zero at the boundaries

$$\widetilde{T} = \sum_{k=0}^{\infty} C_k \psi_k Y^{\beta_k}, \quad \hat{\theta}\psi_k(\xi) = \theta_k \psi_k, \qquad Y = e^{-|y|} = [\gamma + (\gamma^2 - 1)^{1/2}]^{-1}\,, \qquad (18)$$

where the eigenvalues are $\theta_k = -(1 + k)(k + \nu)$ and the eigenfunctions are the well-known Jacobi polynomials (see Ref. 12)

$$\psi_k = wP_k, \quad w = (1 - \xi)^\alpha (1 + \xi)^\beta, \quad P_k = P_k^{(\alpha,\beta)}(\xi), \qquad \alpha = 1, \quad \beta = \nu - 1 > 0\,, \qquad (19)$$

which have the property of orthogonality

$$\int_{-1}^{+1} wP_n P_k d\xi = h_k \delta_k^n, \quad h_k = 2^{\nu+1}(1+k)/(k+\nu)(2k+\nu+1)\,. \tag{20}$$

One can expand any function, including a delta function, in terms of the full set of polynomials P_k

$$\delta(\xi - \xi_0) = w(\xi)\sum_{k=0}^{\infty} P_k(\xi)P_k(\xi_0)h_k^{-1}\,, \tag{21}$$

which has the required property at the starting point $\xi = \xi_0 = 1 - 2\varepsilon > 0$. We note, by the way, that at $\varepsilon > 1/2$ the pinch is stable and constrictions cannot grow due to the stabilizing effect of the field $B_{\parallel}$. The full set of solutions of the spontaneous type that interest us is

$$\widetilde{T}^{(m,n)} \sim w(\xi)\sum_{k=0}^{\infty} h_k^{-1}P_k(\xi)\left(\frac{d}{d\xi_0}\right)^m P_k(\xi_0)\left(\frac{d}{dy}\right)^n y^{q_k}\,. \tag{22}$$

6. THE SPECTRUM OF PARTICLES ACCELERATED IN CONSTRICTIONS

The perturbations described by the solutions in Eq. (22) vanish as $t \to -\infty$, then gradually increase in the time interval $-\infty < t < 0$, and at the critical time $t = 0$ in the narrowest parts of the constrictions we have $\rho_* \to 0$ at the smallest possible but finite value of the radius $a = a_{\min} = a_0\varepsilon^{1/2}$. Plasma is squeezed from the constrictions into bulges which in our model with a "narrow jet approximation" have the form of thin disks, in which we have $\rho_* \to \infty$, $a \to \infty$, $x \to 0$. The momentum distribution function of particles $p = Mcu$ is found from the expression $dN = n\gamma\pi a^2 dz = F(u,t)du$. Taking into account Eq. (15) we obtain

$$F = (N_0/\nu)\rho_*[\widetilde{Z}_x'^2 + \nu x(1-x)\widetilde{T}_x'^2]/(\gamma\widetilde{T}_x' + u\widetilde{Z}_x'), \quad N_0 = \pi a_0^2 n_0\,. \tag{23}$$

In the limit $x \to 0$ we then have $F = \text{const}\,\gamma^{-1}(\widetilde{T}_x')_{x\to 0}$, and at $\gamma \gg 1$ in the series in Eq. (22) it is sufficient to retain the first nonvanishing term with $k = k_{\min}$, which yields asymptotically an ultrarelativistic particle spectrum of the form $F \sim dN/dE \sim E^{-s}$ with exponent $s = 1 + q_k$, where $k = k_{\min}$.

The first solution $\widetilde{T}^{(0,0)}$ from the set in Eq. (22) has a term with $k = 0$ with an exponent $q_0 = \sqrt{2}$ both in the presence and absence of a longitudinal field $B_{\parallel}$. The remaining exponents are equal to $q_k = [1 - (\theta_k/\nu)]^{1/2} = \{2 + k + [k(k+1)/\nu]\}^{1/2}$. However, it can be shown that the solution $\widetilde{T}^{(0,0)}$ describes perturbations which are periodic over the length of the pinch, and as they grow one must have periodic "priming" perturbations. One can suggest, as was done earlier in Refs. 1–4, that there are no visible reasons for the development of perturbations which are periodic over the length of the pinch in space, so that the solution $\widetilde{T}^{(0,0)}$ apparently is not realized "in practice."

All remaining solutions given by Eq. (22) describe not periodic but local perturbations of the type of "isolated constrictions," and apparently one should consider the most typical among them to be the solution $\widetilde{T}^{(1,0)}$, which begins with the term with $k = 1$ and has a spectral exponent $q_1 = [3 + (\varepsilon Mc^2/T_\perp^0)]^{1/2}$. In the absence of a longitudinal field $B_{\parallel} = 0$, we obtain the previous result $q_1 = \sqrt{3}$; however, at a nonrelativistic temperature $T_\perp^0 \ll Mc^2$ the addition of even a relatively small longitudinal field $B_{\parallel}^0$ yields a sharply falling off spectrum of accelerated particles.

7. CONCLUSION

We note that the case without a longitudinal field[1–4] is obtained from the formulas presented above with a field by passing to the limit $\varepsilon \to 0$, $\nu \to \infty$, when we have $\psi_k \to \eta e^{-\eta} L_k^{(1)}(\eta)$, where $\eta = 2T_\perp/Mc^2$, and $L_k^{(1)}$ is the Laguerre polynomial introduced earlier in Refs. 1–4.

It is also useful to note that in laboratory experiments with deuterium pinches conducted in thermonuclear studies, the addition of a longitudinal magnetic field substantially reduces the yield of accelerated deuterons and the neutrons of the nuclear reactions D + D which they produce; however, the stability of the pinches increases.

Finally, it was shown in Ref. 13 that the energy supply for galactic cosmic rays may be provided by the so-called cosmic "gamma bursts" which in our opinion are discharges like "cosmic lightning" with pinches similar to those examined in this article.

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