Neutron optics and ultracold neutrons*

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1. INTRODUCTION

Ultracold neutrons with velocity $v < v_0 = v_{\lim}$ can, as is well known, undergo virtually total reflection from the surface of many substances,[1-3] whose nuclei have a positive coherent scattering length b. The quantity v_0 is equal to

$$v_0 = \frac{\hbar}{m}\left(\frac{Nb}{\pi}\right)^{1/2}. \tag{1.1}$$

Here h is Planck's constant, m is the neutron mass, and N is the number of nuclei per unit volume.

For solids the velocity v_0 is of the order of several meters per second (for copper $v_0 = 5.7$ m/s), and it corresponds to energy $E \sim 10^{-7}$ eV.

Neutrons with velocities $v \leqslant v_0$ form a group whose properties are qualitatively different from those of other groups. It is entirely natural to introduce for such neutrons a special name. In the literature, neutrons whose energies are less than 10^{-4} eV are often called ultracold. It is reasonable to divide this energy range into two subranges:

—very cold neutrons, whose energies are $E < 10^{-4}$ eV, and

—ultracold neutrons, whose energies are $E < 10^{-7}$ eV.

I was happy to learn that Dr. Steyerl is already using in his lecture the terminology that I am advocating.[1)]

In my lecture I shall examine the optical properties of ultracold and, in part, very cold neutrons. It is well known that even thermal neutrons incident at a glancing angle on the surface of many substances (for which $b > 0$) undergo total internal reflection. For neutrons both the angle of total internal reflection and its dependence on the wavelength λ are virtually identical to these same quantities for x-rays. Here there is manifested a deep analogy to the optics of x-rays, which was first analyzed by Fermi.

For long wavelengths, corresponding to cold and very cold neutrons, the range of angles at which total reflection occurs increases smoothly with λ, and for $v = v_0$ even neutrons at normal incidence are almost totally reflected. This dependence of the angle of total reflection on λ for the same wavelengths ($\lambda \approx 10$–10^3 Å) does not have a simple analogy to optics. This difference is determined, as will be seen from what follows, by the dispersion relation of neutron waves, which is not applicable for light at optical frequencies. As regards the reflection of ultracold neutrons, it is analogous in many ways to the reflection of visible light from a metallic mirror.[4]

The optical properties of very cold neutrons can be studied by two methods: One can base the analysis on the average potential in which the neutrons in the medium move or one can proceed directly to find the index of refraction of the neutron waves, analogously to the manner in which this is done in optics. Neither method gives an exact solution of the problem, and both methods can be regarded as being independent to a certain extent. Both methods will be employed in this lecture.

In his review[3] F. L. Shapiro employs the first method, i.e., he employs the average potential which is directly related with Fermi's quasipotential[6] and is equal to

$$U = \frac{\hbar^2}{2\pi m} Nb. \tag{1.2}$$

If this potential is positive, then the energy of a neutron decreases by an amount U when it enters the medium:

$$\frac{mv_1^2}{2} = \frac{mv^2}{2} - U, \tag{1.3}$$

or

$$v_1^2 = v^2 - v_0^2,$$

where v_0^2 is given by Eq. (1.1). Thus neutrons with velocity $v < v_0$ cannot penetrate into the medium at any angle of incidence, since the square of their velocity becomes less than zero. In reality, Eq. (1.3) contains a more stringent requirement. Indeed, the force acting on a neutron at the boundary of the medium is directed along the normal to the surface of the medium (the z axis). For this reason, of the three components of the velocity ($v^2 = v_x^2 + v_y^2 + v_z^2$) only v_z will vary. Thus

$$v_1^2 = v_z^2 + v_y^2 + (v_z^2 - v_0^2), \tag{1.4}$$

$$v_{1z}^2 = v_z^2 - v_0^2.$$

Therefore, in order for the neutron wave to penetrate into the medium it is necessary that $v_z > v_0$, and the equality $v_z = v_0$ determines the angle of total internal reflection for neutrons for any value of v. Thus far the phenomenon of total reflection has been discussed in terms of a corpuscular formulation, which shows that for this phenomenon only the z component v_z of the neutron velocity is significant. The cosine of the angle of incidence is obviously equal to $\cos\theta = v_z/v$. Setting $v_z = v_0$ we find that total reflection occurs if $\cos\theta \leqslant v_0/v_z$, and therefore for $v < v_0$ reflection should occur for any angle θ. The arguments presented here are probably the simplest path to finding the angle of total reflection. It is very significant that the quantity v_0 is the same for both thermal and ultracold neutrons, even though their energies E differ by a factor of 10^5.

Thus for low-energy neutrons the coherent scattering length b is, to a high degree of accuracy, independent of the energy.

In order to find the reflection coefficient and also the neutron flux and neutron density in the medium the corpuscular formulation of the problem must be replaced by the wave formulation. This is especially necessary because the potential U is in reality a complex quantity:

$$U = U' - iU'' = \frac{h^2}{2\pi m} N(b' - ib''), \tag{1.5}$$

since the quantity b is complex,

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$$b = b' - ib''. \quad (1.6)$$

The imaginary part b'' is usually very small compared with the real part b'. Indeed, in many cases b' is close to 10^{-12} cm (for example, for copper $b' = 0.79 \cdot 10^{-12}$ cm). As regards b'', with the help of the optical theorem it can be related with the effective cross section

$$\sigma(v) = \frac{4\pi}{k} b'', \quad (1.7)$$

where k is the wave number. Setting $\sigma \sim 10$ b for thermal neutrons ($\sigma \approx 10^{-23}\mathrm{cm}^2$), we obtain $b'' \sim 0.3 \cdot 10^{-15}$ cm, which is less than one-thousandth of b'. For this reason, in most phenomena associated with scattering of slow neutrons and in order to determine v_0 it can be assumed that b is a real quantity. For ultracold neutrons, however, b'' cannot be set equal to zero when calculating the reflection coefficient, because in this case the reflection coefficient is equal to unity, since it is b'' that contains the neutron capture cross section. This is extremely important for the physics of ultracold neutrons, and for this reason in what follows we shall not neglect b''. We must also be mindful to the fact that in the medium b' and especially b'' are somewhat different from b_0' and b_0'' for an isolated nucleus. For this reason, σ in Eq. (7) is not equal to the total cross section σ_t for the interaction of neutrons with a nucleus, as one would expect on the basis of the optical theorem. The question of how b'' is related with b_0'' will be discussed below (see Sec. 3), where it will be shown that for very cold neutrons σ in Eq. (1.7) should be equal to the cross section for moderation of the neutron flux in matter (see Sec. 4). Here, however, for the time being we make only one, but very important, assumption, namely, we assume that not only b' but also b'' is independent of the neutron velocity and therefore σ in Eq. (1.7) satisfies the $1/v$ law, so that $\sigma(v)v = \mathrm{const}$.

Since the potential U in Eq. (1.5) is complex, instead of Eq. (1.4) we must write

$$v_1^2 = v^2 - v_0^2 + iv_i^2, \quad (1.8)$$

$$v_{1z}^2 = v_z^2 - v_0^2 + iv_i^2, \quad (1.9)$$

where

$$v_0^2 = \frac{h^2}{m^2}\frac{Nb'}{\pi}, \quad v_i^2 = \frac{h}{m^2}\frac{Nb''}{\pi} = \frac{h}{m} N\sigma(v)v. \quad (1.10)$$

The possibility of v_1^2 being negative for $v < v_0$ and the complexness of v_1^2 can be understood physically by switching from velocities to wave numbers

$$k_1^2 = \frac{m^2 v_1^2}{\hbar^2}, \quad k^2 = \frac{m^2 v^2}{h^2},$$

$$k_0^2 = \frac{m^2 v_0^2}{\hbar^2} = 4\pi Nb', \quad k_i^2 = \frac{m^2 v_i^2}{h^2} = 4\pi Nb''. \quad (1.11)$$

Thus Eqs. (1.8) and (1.9) assume the form

$$k_1^2 = k^2 - k_0^2 + ik_i^2 = k^2 - 4\pi Nb, \quad (1.12)$$

$$k_{1z}^2 = k_z^2 - k_0^2 + ik_i^2 = k_z^2 - 4\pi Nb. \quad (1.13)$$

It is obvious that the quantities k_1^2 and k_{1z}^2 described here satisfy Schroedinger's equation for a wave with fixed k and k_z, refracted from vacuum into a medium in which the potential U (1.5) is complex. The fact that, in accordance with Eq. (1.4), of the three components of the vector k only the component oriented along the normal changes when the wave is refracted is a general property of wave processes. However there is the significant feature that for $b = \mathrm{const}$ the change in k_z is a function only of k_z and does not depend on k. From the quantities k_1^2 and k^2 we immediately find the squared index of refraction of the waves $n^2 = \varepsilon$, i.e., a quantity analogous to the permittivity ε for light. By definition,

$$n^2 = \varepsilon = \frac{k_1^2}{k^2} = \frac{v_1^2}{v^2}, \quad (1.14)$$

or, using Eqs. (1.10) and (1.11),

$$n^2 = 1 - \lambda^2 \frac{Nb}{\pi}, \quad b = b' - ib'',$$

or, written in a different form,

$$n^2 = 1 - \frac{v_0}{v^2} + i\frac{v_i}{v^2}, \quad (1.15)$$

$$v_0^2 = \frac{\hbar^2}{m^2}\frac{Nb'}{\pi}, \quad v_i^2 = \frac{\hbar^2}{m^2}\frac{Nb''}{\pi} = \frac{h}{m}N\sigma(v)v.$$

Since only k_z is important for the reflection and refraction of waves, instead of a wave which has a vector k and is incident at an angle θ we can study a wave which propagates along the normal and has the wave vector $k_z = k\cos\theta$. Then, comparing Eqs. (1.12) and (1.13) we can see that analogously to Eq. (1.14) to such a wave we must associate the squared index of refraction n_z^2, which is obtained from Eq. (1.15) by substituting v_z for v:

$$n_z^2 = \frac{k_{1z}^2}{k_z^2} = 1 - \frac{v_0^2}{v_z^2} + i\frac{v_i^2}{v_z^2}, \quad (1.16)$$

so that, analogously to the relation $k_1 = kn$, we have

$$k_{1z} = k_z n_z. \quad (1.17)$$

Here, as previously, it is assumed that b' is a positive quantity. If the imaginary part of n^2 is neglected, then as v decreases the quantity n^2 decreases and becomes negative for $v < v_0$, i.e., n becomes an imaginary quantity. This is what determines the characteristic features of the optics of ultracold neutrons. For nuclei for which the scattering length is negative a plus sign must be inserted in front of v_0^2 or v_0^2 must be assumed to be negative. The quantity n^2 then increases as v^2 decreases.

2. THE OPTICAL ANALOGY AND THE CHARACTERISTICS OF THE DISPERSION OF NEUTRON WAVES

As we have already mentioned above, the second approach to the optics of ultracold neutrons consists of determining the quantity n^2 directly from the constants governing the interaction of neutrons with the nuclei in the medium without finding the potential U. Then, the amplitude and phase of the reflected and refracted waves are found from the boundary conditions which are imposed on the waves and which lead to relations that are analogous to the Fresnel coefficients.[4]

The index of refraction for neutron waves is of the same nature as for light waves. On scattering the incident wave gives rise to secondary waves and the coherent superposition of the secondary waves gives rise to the refracted and reflected waves. The difference from the case of light lies in the fact that it is not atoms, but rather nuclei that are primarily responsible for the scattering. Taking this into consideration, the index of refraction for neutron waves can be written by analogy to light.

Indeed, as is well known from optics, the index of refraction for light, which is close to unity, is determined by the formula

$$n^2 = 1 + 4\pi N\alpha, \quad |n^2 - 1| \ll 1, \tag{2.1}$$

where α is the polarizability of the atoms of the medium. Multiplying Eq. (2.1) by $k^2 = \omega^2/c^2$, where ω is the frequency of the light, and substituting $k_1^2 = k^2n^2$, we obtain

$$k_1^2 = k^2 + 4\pi N\frac{\omega^2\alpha}{c^2}. \tag{2.2}$$

As is well known, an electric field $E'e^{i\omega t}$ induces in an atom a dipole moment with amplitude $p = \alpha E'$, which generates in the direction of the primary wave an oscillating electric field

$$E_1e^{-i\omega t} = \frac{\omega^2}{c^2}\alpha\frac{E'}{r}e^{ikr-i\omega t}. \tag{2.3}$$

Thus the amplitude of the field of the wave scattered in the forward direction is equal to $A = (\omega^2/c^2)\alpha$. It is obvious that in the case of neutrons it must be replaced in Eq. (2.2) by $-b$. Then we obtain for k_1^2 a formula that is identical with Eq. (1.12)

$$k_1^2 = k^2 - 4\pi Nb, \tag{2.4}$$

or for n^2, in agreement with Eq. (1.14),

$$n^2 = 1 - \lambda^2\frac{Nb}{\pi}, \quad b = b' - ib''. \tag{2.5}$$

In 1944 E. Fermi, who was probably the first to use the concept of index of refraction to describe total internal reflection of neutrons, employed a formula identical to Eq. (2.5). In so doing, he restricts himself by the remark: "From theoretical considerations it follows ... ."[7,2)] These considerations are, in principle, probably analogous to those given here, since in lectures on neutron physics in 1945 (Ref. 8) he talks about analogies between the index of refraction for neutrons and x rays.

The formula (2.3) for the scattering of scalar waves (for example, sound waves) was derived by Foldy[9] in 1945. An extension of this formula and a discussion of it, including its application to neutrons, is contained in papers by Lax, published in 1951–1952.[10] A number of questions associated with these problems were discussed in other, later publications,[11-14] and will probably be discussed further.[3)]

Here I want to call attention to a characteristic feature of the dispersion of neutron waves. For them $k_1^2 - k^2 = -4\pi Nb$ (see Eqs. (2.4) and (1.12)) and the same relation also holds for $k_{1z}^2 - k_z^2 = -4\pi Nb$ (see Eq. (1.13)). Let us see what this corresponds to in the wave language. For any waves, provided that the angle of incidence θ is related with the angle of refraction θ_1 by the expression $\sin^2\theta_1 = n^{-2}\sin^2\theta$, the relation

$$k_z^2 - k_{1z}^2 = k^2(1 - n^2). \tag{2.6}$$

is satisfied.[18] If the difference on the left-hand side is required to be independent of k^2, which should be the case for $b = \text{const}$, then we immediately find that $(1 - n^2)$ is proportional to λ^2, as is the case in Eq. (2.5).

We thus arrive at the conclusion that if the dispersion relation (2.5) is correct, then for the analysis of the reflection and refraction of neutron waves only the component v_z of the neutron velocity, for which the index of refraction n_z (1.16) can be introduced, is important. The facts that the value of the angle of total internal reflection for thermal neutrons and the value of v_0 for ultracold neutrons, as we have already mentioned above, agree with the theory show that this assertion and the dispersion relation are correct to within at least several percent for values of λ^2 covering a range of five orders of magnitude (from thermal to ultracold neutrons).

For wavelengths corresponding to x-ray and vacuum-UV radiation there exists a direct analogy with light, since the same dispersion relation holds:

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2} \tag{2.7}$$

(here $\omega_p^2 = 4\pi Ne^2/m$ is the so-called plasma frequency). Herein lies one of the analogies between the optics of thermal neutrons and the optics of x-rays. In this case, however, two conditions that are necessary for electromagnetic waves are satisfied immediately—n^2 is close to unity and, moreover, ω is higher than the characteristic atomic frequencies. For wavelengths equal to the wavelengths of ultracold neutrons, i.e., for light in the optical range, the formula (2.7) is obviously incorrect.

In conclusion we call attention to the fact that only the v_z component of the velocity is important in other optical phenomena associated with neutrons, just as for reflection and refraction of waves. It is obvious that the Bragg–Wolf condition for reflection of neutron waves by crystals, if it is expressed in terms of v_z, has the form[4,5]

$$mv_z2d = nh. \tag{2.8}$$

This is a quantum-mechanical expression, which is completely familiar from the viewpoint of the Bohr condition of quantization $\int pdq = nh$. It is convenient for studying the method of neutron diffraction based on the time of flight.

3. THE EFFECTIVE FIELD OF NEUTRON WAVES

In deriving the formula (2.5) for the squared index of refraction of neutron waves we started from the analogy with light, for which the relation (2.1) is applicable. However Eq. (2.1) is valid only for values of n close to unity. For n appreciably different from unity Eq. (2.1) must be replaced by the Lorentz–Lorenz formula[4)]

$$n^2 = 1 + \frac{4\pi N\alpha}{1 - (4\pi/3)N\alpha}, \tag{3.1}$$

which is identical with Eq. (2.1) only for $N\alpha \ll 1$. If we employ the formula (3.1) as the starting point for finding n^2 for neutrons, then we obtain a different dispersion relation and a different value of v_0. In this case, v_{1z} will no longer be the component that is independent of v in the case of interactions with the medium.

It is therefore necessary to understand the reason why the analogy with light breaks down. In electrodynamics the formula (3.1) is obtained, as is well known, in an elementary way,[5)] if one takes into consideration the fact that in the medium the electric field E' acting on an atom is not equal to the external field E, i.e.,

$$E' = E + \frac{4}{3}\pi P, \quad P = N\alpha E', \tag{3.2}$$

whence

$$E' = \frac{E}{1 - (4\pi/3)N\alpha}, \tag{3.3}$$

and this must obviously be taken into account. For this reason, the formula (2.2), which, as Lax showed,[10] in its general form is applicable to both light and neutrons, has the form

$$k_1^2 = k^2 + 4\pi NCf(0). \tag{3.4}$$

Here $f(0)$ is the amplitude of the wave scattered elastically in the forward direction, and

$$C = \frac{\text{effective field}}{\text{coherent field}}.$$

Thus in the case of light the expression (3.4) differs from Eq. (2.2) by the presence of the coefficient C. Setting $C = E'/E$, we obtain immediately from Eq. (3.3) the Lorentz–Lorenz formula (3.1). In order to obtain Eq. (2.4) for neutrons it is evidently necessary, in contrast to light, to set $C = 1$. The problem of the difference between Eqs. (2.5) and (3.1) is associated with the distinction between the effective field in both of these cases.

This is easy to understand qualitatively. Indeed, aside from the field in the wave zone (2.3), proportional to $k^2 = \omega^2/c^2$ and decaying with distance as $1/r$, an electric dipole generates in the near zone an electric field that does not depend on ω. This field is proportional to the dipole moment and decreases with distance as $1/r^3$. Thus the polarization of dipoles which are located next to the given dipole generates an additional electric field; this corresponds to Eq. (3.2). It does not depend on the frequency of the oscillations. The situation is different in the case of a scalar wave, such as a neutron wave. In this case the amplitude of the scattered wave beyond the range of the nuclear forces decreases as $1/r$ and the field is proportional to be^{ikr}/r, i.e., the resulting field is obtained entirely from the coherent superposition of the waves.[6)]

Can we assert, however, that for neutron waves C must be equal to unity?[7)] Apparently, the answer is yes, on the basis of both theory and experiment. From this standpoint the papers of Refs. 11 and 12, where the same method is used to determine the index of refraction by coherent superposition of multiply scattered waves, are interesting. In the case of electromagnetic waves the Lorentz–Lorenz formula was obtained, while in the case of neutrons the formula (2.5) was obtained. We have already mentioned the agreement with experiment. It should be noted, however, that the values of the angle of total internal reflection and the threshold velocity v_0 are related with the real part of the scattering length b' and therefore the real part of C is indeed close to unity. As regards the imaginary part, we can only assert that it is small compared with unity. In addition, the quantity C not only can have, but is known to have, a small imaginary part, and it must be taken into account.

Admitting this and extending the formula (2.4) in accordance with Lax's result, we obtain

$$k_1^2 = k^2 - 4\pi N(C' - iC'')(b_0' - ib_0''), \tag{3.5}$$

where b_0' and b_0'' are the scattering lengths for an isolated nucleus. Returning to the starting formula (2.4) or (1.15), we find that the effective scattering lengths in the medium are equal to

$$b = C'b_0', \quad b = C''b_0' + C'b_0'', \tag{3.6}$$

and since b_0'' and C'' are small, we neglect their product. Since, as we have seen, the quantity b''/b' is of the order of 10^{-3}, it is sufficient for C'' to be small compared with C' for b'' to change appreciably compared with b_0''.

We shall show that this is, to some extent, true. In contrast to b'' (see the formula (1.7)) the optical theorem is obviously applicable to b_0''. Thus

$$b_0'' = \frac{k}{4\pi}\sigma_t = \frac{k}{4\pi}(\sigma_{ec} + \sigma_{en} + \sigma_c), \tag{3.7}$$

where σ_{ec} is the cross section for elastic coherent scattering, σ_{en} is the cross section for elastic noncoherent scattering, and σ_c is the cross section for neutron capture.

As for b'', it must obviously have a different form:

$$b'' = \frac{k}{4\pi}\sigma t = \frac{k}{4\pi}(\beta\sigma_{en} + \sigma_c + \sigma_n). \tag{3.8}$$

For very cold and ultracold neutrons it should contain σ_n, the cross section for heating of the neutrons by means of inelastic interaction with the medium. The quantity b_0'' does not take this process into account. On the other hand, in a perfectly uniform medium the quantity b'' for very cold (and, obviously, ultracold) neutrons should not contain the cross section for elastic coherent scattering σ_{ec}. Indeed, if this process were the only form of interaction, then for neutrons which are faster than ultracold neutrons the wave with wave vector k would simply transform, as a result of the coherent superposition of the scattered waves, into a wave propagating with wave vector k_1 in the medium. From the law of conservation of particles it follows that the wave should not decay in this case. Therefore, when only coherent scattering takes place, b'', in contrast to b_0'', should be zero. This feature was pointed out in Ref. 11 (see also Ref. 18).

Elastic noncoherent scattering in the medium could also be less important[14] (this is why the coefficient β appears in front of σ_{en} in Eq. (3.8)). The quantity σ for very cold neutrons, to a significant extent confirming what we have said, was measured and discussed in Refs. 17.

In the case of ultracold neutrons, b'' can be determined directly by determining their storage time in a closed vessel. The probability that an ultracold neutron with a prescribed velocity will vanish in the process of reflection from the walls of the vessel should theoretically be directly proportional to the ratio b''/b'. The anomaly observed in this case, manifested as a difference between the experimentally measured value of b'' and the theoretical value, which it exceeds, will be discussed in the concluding section of this lecture.

4. INDEX OF REFRACTION OF NEUTRON WAVES

Since the squared index of refraction $n^2 = \varepsilon$ is complex, where ε is analogous to the permittivity in optics

$$n^2 = \varepsilon = \varepsilon' + i\varepsilon'', \tag{4.1}$$

the question of the real n' and imaginary n'' parts of the index of refraction

$$n^2 = (n' + in'')^2 = (n'^2 - n''^2) + 2in'n'' \tag{4.2}$$

merits a separate analysis. Comparing with Eq. (4.1) and using Eq. (1.15) we obtain hence

$$\varepsilon' = (n'^2 - n''^2) = 1 - \frac{v_0^2}{v^2},$$
$$\varepsilon'' = 2n'n'' = \frac{v_\mathrm{i}^2}{v^2} = \frac{\hbar N}{mv^2}\sigma(v)v. \tag{4.3}$$

For $v < v_0$ the quantity ε' is negative, i.e., the imaginary part of n is greater than the real part $n'' > n'$. This feature is characteristic for the optics of metals. As regards the imaginary part of ε, in the optics of metals $\varepsilon'' = 4\pi\sigma/\omega$, where σ is the conductivity. Since $2\hbar/mv^2 = 1/\omega$, where ω is the frequency of the neutron waves, we obtain from Eq. (4.3) that in the case of neutron waves $N\sigma(v)v/8\pi$ plays the role of the conductivity. For n'^2 and n''^2 we obtain from Eq. (4.3)

$$n'^2 = \frac{\varepsilon'}{2} + \frac{1}{2}(\varepsilon'^2 + \varepsilon''^2)^{1/2},$$
$$n''^2 = -\frac{\varepsilon'}{2} + \frac{1}{2}(\varepsilon'^2 + \varepsilon''^2)^{1/2}, \tag{4.4}$$

where the square root must be taken with a plus sign. These formulas are also familiar in the optics of metals.[8)] The analogy, associated with the properties noted above, between the optics of metals and the reflection and absorption of ultracold neutrons was already mentioned in Ref. 4.

For our purposes, wave propagation characterized in the medium by the component k_z (directed normal to the surface of the medium) of the vector k is important. As we have seen above (see Sec. 1), we can use the index of refraction n_z, so that $k_{1z} = k_z n_z$, where n_z is given by Eq. (1.16). We find the real and imaginary parts of n_z from Eq. (4.4), using the relations (4.3) and replacing in them v by v_z. This is precisely what we shall do in this and the following sections of this lecture. We shall assume conventionally that very cold neutrons are neutrons for which $v_z > v_0$, and all other neutrons, for which $v_z < v_0$, are ultracold.[9)]

Then we obtain from Eqs. (4.4)

$$n_z'^2 = \frac{1}{2v_z^2}\left\{(v_z^2 - v_0^2) + \left[(v_z^2 - v_0^2)^2 + v_\mathrm{i}^4\right]^{1/2}\right\},$$
$$n_z''^2 = \frac{1}{2v_z^2}\left\{(v_0^2 - v_z^2) + \left[(v_z^2 - v_0^2)^2 + v_\mathrm{i}^4\right]^{1/2}\right\} \tag{4.5}$$

(the positive branch of the square root is taken). As we have already mentioned, v_i^2 for most substances is at least three orders of magnitude smaller than v_0^2, and thus, with the exception of a very narrow range of values of v_z, $(v_z^2 - v_0^2) \gg v_i^4$. Under this assumption it is easy to derive approximate values of $n_z'^2$ and $n_z''^2$.

4.1. Very cold neutrons

$v_z > v_0$; the region $(v_z^2 - v_0^2)^2 \gg v_i^4$. From Eq. (4.5) we have

$$n_z'^2 = \frac{v_z^2 - v_0^2}{v_z^2} = \frac{v_1^2}{v_z^2}, \tag{4.6}$$

$$n_z''^2 = \frac{v_\mathrm{i}^4}{4v_z^2(v_z^2 - v_0^2)} = \frac{\hbar^2}{4m^2v_z^2}\frac{(N\sigma(v)v)^2}{v_1^2}. \tag{4.7}$$

Here v_1 is the component of the neutron velocity which in the medium is directed along the normal to the interface ($v_1^2 = v_z^2 - v_0^2$). Therefore $k'_{1z} = k_z n'_z = mv_1/\hbar$ is obtained from k_z by substituting v_1 for v_z.

The quantity n_z^2 in Eq. (4.7) has a similar obvious meaning. If, as we have assumed, σ satisfies the $1/v$ law, then $\sigma(v)v = \sigma(v_1)v_1$ and therefore $\sigma(v_1) = \sigma(v)v/v_1$.

Thus n''_z in Eq. (4.7) is equal to

$$n''_z = \frac{1}{2k_z}N\sigma(v_1). \tag{4.8}$$

Since the ψ function decays as $\exp(-k_z n''_z z)$, the neutron density (and the flux) should decay as the square of this quantity:

$$\rho_1(z) = \rho_1 e^{-2k_z n''_z z} = \rho_1 e^{-N\sigma(v_1)z}. \tag{4.9}$$

We recall that the quantity $\sigma(v)$ was introduced, essentially, from dimensional considerations (see the formula (1.7)). From Eq. (4.9) it is obvious that it indeed determines the macroscopic cross section for the attenuation of a beam of very cold neutrons. The fact that this cross section should correspond to the velocity v_1 is almost obvious and is confirmed well by experiment.[17] It is obvious that in this theory it is precisely the experimental values of $\sigma(v_1)$ that should be used for n''_z.

4.2. Ultracold neutrons

$v_z < v_0$. We start from the assumption that the same value of $\sigma(v)v$ that was used for very cold neutrons can also be used for ultracold neutrons. Then we obtain from Eq. (4.5), under the condition that $(v_z^2 - v_0^2)^2 \gg v_i^4$,

$$n_z'^2 = \frac{v_\mathrm{i}^4}{4v_z^2(v_0^2 - v_z^2)} = \frac{\hbar^2}{4m^2v_z^2}\frac{(N\sigma(v)v)^2}{(v_0^2 - v_z^2)}, \tag{4.10}$$

$$n_z''^2 = \frac{v_0^2 - v_z^2}{v_z^2}. \tag{4.11}$$

Thus n'^2 and n''^2 are interchanged as compared with the case $v_z > v_0$ (with $v_z^2\ v_0^2$ replaced by $v_0^2 - v_z^2$). Here only the real part of the index of refraction depends on σ, and if $\sigma = 0$, then $n' = 0$ and the index of refraction becomes purely imaginary. Obviously, a wave cannot propagate in the medium and is not absorbed in it. The neutron density, in this case, decreases exponentially with increasing distance from the interface, and this decay does not depend on $N\sigma(v)$:

$$\rho = \rho_1\exp(-2k_z n''_z z) = \rho_1\exp\left[-\frac{2m}{\hbar}\left(v_0^2 - v_z^2\right)^{1/2} z\right]. \tag{4.12}$$

In the classical limit $\hbar = 0$ the density drops to zero already at the interface $z = 0$.

For $\sigma \neq 0$ the real part of n is not equal to zero and is proportional to the macroscopic cross section. Indeed, since the neutron density at the boundary of the medium is in a steady state, absorption should lead to the existence of a flux from the surface of the medium into the volume of the medium, and this flux should be all the larger the larger is the cross section $N\sigma$. In addition, the decay of the neutron density, determined by the formula (4.12), for not very large values of $N\sigma$ remains the same as in the case when there is no absorption.

We can interpret a nonzero value of n'_z, by analogy to the case $v_z > v_0$, as the presence of a real neutron velocity v'_1 in the medium:

$$n_z'^2 = \frac{k_{1z}^2}{k_z^2} = \frac{v_1'^2}{v_z^2}. \tag{4.13}$$

Therefore we can think of the situation as follows: If absorption is present, then the neutron velocity in the medium does not approach zero as $v_z \to v_0$, but rather there exists a finite value, even if $v_z < v_0$, namely

$$v_1' = \frac{v_i^2}{2(v_0^2 - v_z^2)^{1/2}} = \frac{\hbar}{2m} \frac{N\sigma(v)v}{(v_0^2 - v_z^2)^{1/2}} \tag{4.14}$$

(we recall that the formula is true when $(v_0^2 - v_z^2)^2 \gg v_i^4$ and therefore $v'_1 \ll v_i$). Obviously, we obtain the maximum value of v'_1 by setting $v_z = v_0$. Then we immediately obtain from Eq. (4.5) $v'_1 = v_i/\sqrt{2}$. This quantity is small compared with v_0.

For ultracold neutrons the formula (4.9) is also obviously correct, if in it v' is replaced with v'_1. Indeed, substituting v'_1 from Eq. (4.14) into the exponent in Eq. (4.9) we obtain Eq. (4.12):

$$N\sigma(v_1') = \frac{N\sigma(v)v}{v_1'} = \frac{2m}{\hbar}\left(v_0^2 - v_z^2\right)^{1/2}. \tag{4.15}$$

Thus the velocity v'_1 of ultracold neutrons in a medium can be given physical meaning by examining the propagation and absorption of ultracold neutrons in the medium. This velocity v'_1, as one can see from Eq. (4.14), does not vanish, even for $v_z = 0$, i.e., it can even exceed v_z in vacuum. Since the velocity v'_1 obviously has a real meaning, we must acknowledge that a neutron appearing with velocity v'_1 in the medium is capable of leaving the medium, though outside the medium its velocity will be less than v_0.

This assertion, by the way, is almost obvious: If neutrons with velocity less than v_0 flow into a medium which absorbs them, then a flux of opposite sign is therefore also possible. It is obvious from the formula (4.12) that the decay constant of the density does not contain the capture cross section, i.e., within certain limits it does not depend on the quantity $N\sigma(v)$. Thus a neutron in the medium must be ascribed a velocity v'_1 that is small enough so that the cross section $\sigma(v')$ corresponds to the index of refraction (4.15) that is the same as the attenuation coefficient of the density of ultracold neutrons (4.12). Does this mean that the absorption of ultracold neutrons in reality does not depend on $\sigma(v)$ and is determined only by the quantity $(v_0^2 - v_z^2)^{1/2}$ in Eq. (4.12)? It is easy to verify that the answer is no. In order to determine the absorption it is necessary to know the relative magnitude of the neutron flux in the medium. The neutron flux, which is carried by a wave arriving from the vacuum at the boundary, is obviously equal to (neglecting the reflected wave)

$$S_0 = v_z\rho_0 = v_z|\psi_0|^2, \tag{4.16}$$

where ψ_0 is the amplitude of the incident wave. It is easy to show that the flux of ultracold neutrons in the medium is equal to[19]

$$S_1(z) = v_z'\rho_1(z) = v_1'|f|^2|\psi_0|^2\exp\left[-\frac{2m}{\hbar}\left(v_0^2 - v_z^2\right)^{1/2} z\right], \tag{4.17}$$

where v'_1 is the velocity of the neutrons in the medium (4.14) and f is the Fresnel coefficient for the wave passing into the medium.[10)] For ultracold neutrons $|f|^2 = 4v_z^2/v_0^2$ (see the next section).

For an infinitely thick medium the entire flux passing into the medium is absorbed in it. Thus the fraction of neutrons that are absorbed in one reflection is equal to

$$\alpha = \frac{S_1(0)}{S_0} = \frac{v_1'|f|^2}{v_z} = n_z'|f|^2. \tag{4.18}$$

The absorption of ultracold neutrons, even though it depends on v_z, is all the smaller, the smaller is the value of $\sigma(v)v$, since n'_z (see Eq. (4.10)) is proportional to this quantity.

As far as the decay of the density in Eq. (4.12) is concerned, it determines not the magnitude of the absorption, but rather only the depth distribution of the absorption, which in reality is proportional to the neutron density at a given location.

Indeed, the decay of the flux in the presence of absorption is equal to

$$\frac{dS}{dz} = -\frac{\rho_1(z)}{T} = -N\sigma(v)v\rho_1(z). \tag{4.19}$$

It is proportional to the neutron density and inversely proportional to the average lifetime T of a neutron in the medium, with $T^{-1} = N\sigma(v)v$. If $\sigma(v)$ satisfies the $1/v$ law, then T does not depend on v. Since we assumed that for ultracold neutrons $\sigma(v)v$ is the same as for very cold neutrons, Eq. (4.19) should also be satisfied for them (we recall that $\sigma(v)$ includes not only the capture cross section, but also the cross section for heating of neutrons owing to inelastic scattering). Differentiating Eq. (4.17) with respect to z and using Eq. (4.14), it is not difficult to verify that Eq. (4.19) is indeed satisfied.[19]

5. REFLECTION AND TRANSMISSION OF NEUTRON WAVES

When neutron waves are reflected and refracted by a plane vacuum-medium interface the boundary conditions are the same as for light whose electric vector E lies in a plane perpendicular to the plane of incidence.[11)] For this reason, we can write for the reflected and refracted wave (with angle of incidence θ) the Fresnel coefficients r and f simply by analogy to light:[4]

$$r = \frac{\cos\theta - (n^2 - \sin^2\theta)^{1/2}}{\cos\theta + (n^2 - \sin^2\theta)^{1/2}}, \tag{5.1}$$

$$f = \frac{2\cos\theta}{\cos\theta + (n^2 - \sin^2\theta)^{1/2}}. \tag{5.2}$$

Substituting n^2 from Eq. (1.15) we immediately find that the angle θ is eliminated, but v_z plays the role of v. As expected, the Fresnel coefficients are the same as for waves at normal incidence on a medium whose index of refraction $n_z = n_z^+ + in_z''$:

$$r = \frac{(1 - n_z') - in_z''}{(1 + n_z') + in_z''}, \tag{5.3}$$

$$f = \frac{2}{(1 + n_z') + in_z''}. \tag{5.4}$$

The fraction of neutrons absorbed in the medium is obviously equal to

$$\alpha = 1 - |r|^2 = \frac{4n_z'}{(1 + n_z')^2 + n_z''^2} = n_z'|f|^2. \tag{5.5}$$

As expected, the formula (5.5) is identical with the formula (4.18). Using Eq. (4.5), we obtain

$$\alpha = 4v_z\{(1/2)(v_z^2 - v_0^2) + (1/2)[(v_z^2 - v_0^2)^2 + v_i^4]^{1/2}\}^{1/2}$$
$$\times [v_z^2 + [(v_z^2 - v_0^2)^2 + v_i^4]^{1/2} + 2v_z\{(1/2)(v_z^2 - v_0^2)$$
$$+ (1/2)[(v_z^2 - v_0^2)^2 + v_i^4]^{1/2}\}]^{-1}, \tag{5.6}$$

where the positive branch of each square root is taken. As we have already mentioned, in determining n' and n'' the inequality $(v_z^2 - v_0^2)^2 \gg v_i^4$ holds over a wide range of values of v_z. This is also true for ultracold neutrons, since $v_0^2 \gg v_i^2$. For this reason, F. L. Shapiro, who derived the formula (5.6) from somewhat different considerations, presents in his report[3] only the approximate formula[3,4] for ultracold neutrons which follows from it:[12)]

$$\alpha = \frac{2v_i^2 v_z}{v_0^2(v_0^2 - v_z^2)^{1/2}} = \frac{2b''}{b'}\frac{v_z}{(v_0^2 - v_z^2)^{1/2}} = \frac{2v_z}{mv_0^2}\frac{\hbar N\sigma(v)v}{(v_0^2 - v_z^2)^{1/2}}. \tag{5.7}$$

Here we employed Eq. (1.15) for writing down the different forms of Eq. (5.7). As one can see, the fraction of neutrons absorbed on reflection vanishes if $v_z = 0$. This does not contradict the fact that $n' = v_1'/v_z$ increases with decreasing v_z, since $|f|^2 = 4v_z^2/v_0^2$ decreases as v_0^2. As v_z approaches v_0 the quantity α increases, remaining much less than unity, since the formula holds only if $(v_0^2 - v_z^2)^2 \gg v_i^4$.

At the threshold itself $(v_0^2 - v_z^2)^2 \ll v_i^4$ we have[4]

$$\alpha = \frac{2\sqrt{2}v_i}{v_0}, \tag{5.8}$$

and since $v_i \ll v_0$, the reflection coefficient $R = 1 - \alpha$ is close to unity, though it is less than for ultracold neutrons. However it decreases rapidly in the region above threshold, i.e., for $v_z > v_0$. When $(v_z^2 - v_0^2)^2 \gg v_i^4$, we obviously obtain for the Fresnel coefficient r the real quantity

$$r = \frac{v_z - (v_z^2 - v_0^2)^{1/2}}{v_z + (v_z^2 - v_0^2)^{1/2}}, \tag{5.9}$$

and the coefficient of reflection $R = r^2$. For $v_z \gg v_0$ the quantity r^2 decreases rapidly.

For very cold neutrons, in addition to measuring the reflection coefficients, it is possible to measure directly neutron transmission through a thin foil. When calculating the transmission the fact that in the foil the neutron wave can be repeatedly reflected from the walls of the foil must be taken into account. This was investigated experimentally and theoretically by Steyerl.[17,20] An analogous analysis was performed in Ref. 19.

6. ANOMALOUS ABSORPTION OF ULTRACOLD NEUTRONS

It is well known that the confinement time of ultracold neutrons is found to be systematically smaller than the computed value. At the present time it cannot be excluded that this is caused by defects of the reflecting surface or contamination of the surface layer.[13)]

The universality of this effect and its quite significant strength more likely indicate, however, that ultracold neutrons can be absorbed or heated by yet another mechanism. The associated decrease of the reflection coefficient α_0 is equal to $3\cdot 10^{-4}$, and therefore, as one can see from Eq. (5.7), it corresponds to an additional quantity $\Delta b''$, equal to $\sim 3\cdot 10^{-4} b'$.[14)]

Having accepted the possibility of this additional absorption, we apparently must assume the following:

1. This absorption is not related directly with the capture and inelastic scattering cross section, since it is especially pronounced in substances with low absorption.

2. It must be specific to ultracold neutrons, since if it were to decrease as $1/v$ in the region of thermal neutrons it would correspond to an easily noticeable cross section of the order of 10^{-23} cm^2. Such an additional absorption cross section should be observable in the region of very slow neutrons and it is not likely that it could remain undetected in Steyerl's experiments.[17]

Two obvious assumptions characteristic only for ultracold neutrons and for this reason satisfying these conditions can be made:

a) Quasielastic scattering occurs, in which the velocity of a neutron changes on reflection by an amount v_0 with a probability of a few ten thousandths or by an even smaller amount but with a correspondingly large probability. In reality, the requirement that the velocity can change is even less stringent. Indeed, strictly speaking, a neutron is no longer ultracold if its velocity in the medium becomes significantly higher than the maximum value v_1' in the medium, equal to $v_i/\sqrt{2} \ll v_0$ (see Sec. 4). It is obvious that in the region of cold and very cold neutrons such a small change in the velocity will lead only to an insignificant broadening of the monoenergetic-neutron line, and such broadening is difficult to observe.

b) The second assumption can be justified by the fact that in the case of ultracold neutrons, in contrast to very cold neutrons, the neutron density in the medium is not uniform but rather decays rapidly with increasing depth. Near the surface of the medium neutrons are scattered in a nonuniform neutron field. As a result it is possible that sufficiently good averaging of the scattering over the coordinates of the vibrating nuclei does not occur, and an additional effect can arise in the scattering. The reasons a) and b) could be related with one another, since in both cases the motion of the nuclei accompanying the scattering of the neutrons must be taken into account. Therefore we can assume that in the region $v_z < v_0$ there appear some additional corrections $\Delta\varepsilon''$ to the imaginary part of ε; this corresponds to an increase of the imaginary part of the scattering length b'' by $\Delta b''$. This means that the imaginary part of the effective field C'' (3.6) changes in the case of ultracold neutrons. This effect can be formally taken into account by introducing some additional anomalous cross section σ_a. As we have seen above, the quantity σ appears in ε'' and b'' in the form of the product $\sigma(v)v$ (see Eqs. (1.15) and (4.3)). In contradistinction to this, in the case at hand it can be conjectured that, based on dimensional considerations, the correction will have the form $\sigma_a(v_0^2 - v_z^2)^{1/2}$. Since no hypotheses are made regarding the velocity dependence of σ_a, here there are no other physical assumptions, aside from the assumption that σ_a is significant only in the region $v_z < v_0$.[15)] Then we have (see Eqs. (4.3) and (3.8))

$$\Delta\varepsilon'' = \frac{\hbar N}{m v_z^2}\sigma_a\left(v_0^2 - v_z^2\right)^{1/2}, \tag{6.1}$$

$$\Delta b'' = \frac{k_z}{4\pi}\frac{\sigma_a\left(v_0^2 - v_z^2\right)^{1/2}}{v_z}. \tag{6.2}$$

Of course, the simplest assumption is that σ_a depends weakly on the velocity.

We recall that the decay of the UCN density (4.12) does not depend on the cross section σ and is proportional to

$$2k_z n_z'' = \frac{2m}{\hbar}\left(v_0^2 - v_z^2\right)^{1/2}. \tag{6.3}$$

In addition, $\varepsilon'' = 2n'n''$ (see Eq. (4.3)) and therefore σ_a changes n_z' by the amount $\Delta n'$

$$n_z' + \Delta n_z' = \frac{\hbar N}{2m v_z}\left[\frac{\sigma(v)v_z}{(v_0^2 - v_z^2)^{1/2}} + \sigma_a\right]. \tag{6.4}$$

Independently of the assumptions concerning the nature of σ_a the neutron velocity v_1' in the medium (see Eq. (4.14)) changes by the small amount

$$\Delta v_1' = \frac{\hbar N \sigma_a}{2m}. \tag{6.5}$$

The probability of absorption on reflection is equal to $|f|^2 = 4v_z^2/v_0^2$

$$\Delta\alpha = \Delta n_z''|f|^2 = \frac{2v_z}{m v_0^2}\hbar N \sigma_a. \tag{6.6}$$

More detailed experimental data and their theoretical analysis should make it possible to justify the hypothesis that there exists an anomalous cross section σ_a and to determine the nature of this cross section.

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1) See Ref. 21. Since 1974 this classification, which was proposed by I. M. Frank and A. Steyerl, has become standard (A. I. Frank).

2) Fermi uses the scattering cross section for b: $b = \sqrt{(\sigma/4\pi)}$. In addition, he studied only thermal neutrons, for which $n - 1$ is close to zero (of the order of 10^{-6}) so that $n + 1$ can be taken to be equal to 2.

3) In this connection, see Refs. 22 and 23 (A. F.)

4) Indeed, if Eq. (3.1) is solved for $N\alpha$, then we obtain the standard formula

$$\tfrac{4}{3}\pi N\alpha = \frac{n^2 - 1}{n^2 + 2}.$$

5) See, for example "Optics" by Max Born (1937 Russian edition),[16] where, in particular, a derivation of the Lorentz–Lorenz formula from the interference of coherently scattered waves is presented. This book is still of interest even now, in spite of the appearance of its several subsequent editions.

6) I am grateful to M. I. Podgoretskiĭ for a discussion of this question.

7) The important question of electromagnetic interactions of the neutron is not discussed here.

8) See, for example, M. Born and E. Wolf, Ref. 16, p. 673 of Russian edition.

9) With this classification, neutrons, whose velocity is $v > v_0$ and the angle of incidence is so large that $v_z < v_0$, can be considered to be ultracold.

10) The continuity of the flux at the boundary of the medium is guaranteed by the fact that aside from the flux of the incident wave there is also the flux carried off by the reflected wave $v_z|r|^2|\psi_0|^2$, where r is the Fresnel coefficient for the reflected wave.

11) Indeed, the continuity of the ψ function requires that $1 + r = f$ and $k \sin\theta = k_1 \sin\theta_1$, and from the continuity of the derivative it follows that $k\cos\theta(1 - r) = k_1 f \cos\theta_1$, whence Eqs. (5.1) and (5.2) are obtained immediately.

12) The same result is immediately obtained from Eq. (5.5), since $|f|^2$ in Eq. (5.4), as it is easy to show, is equal to $4v_z^2/v_0^2$ for $v_z < v_0$ and $(v_0^2 - v_z^2) \gg v_i^4$, since $n'' \gg n'$.

13) Although many years have passed since this lecture was written, the assertion made above has largely remained valid. The discrepancy remains even now, though measures taken to reduce the role of surface contaminants have significantly increased the confinement time of ultracold neutrons (A.F.).

14) The value $\Delta b'' \approx 6\cdot 10^{-6} b'$ was given recently in Ref. 24 (A.F.).

15) There is no need to assume that $\sigma_a(v_0^2 - v_z^2)^{1/2}$ is inapplicable in the region $v_z > v_0$. For $v_z > v_0$ this quantity is imaginary and for this reason it contributes not to b'', but rather to b', and the contribution is very small at that.

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