Basic properties of squeezed light

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The physical picture of the properties of light in squeezed and in other nonclassical states is presented and compared with the properties of light in the classical coherent state. The theoretical basis of the description and the generation of squeezed light is presented, and a practical scheme for obtaining light in other of its nonclassical states is discussed. The general nature of the phenomenon is emphasized, as well as the possibility of its transfer to other fields such as acoustics, mechanics, and various Bose fields.

1. INTRODUCTION

One of the most important events in the field of optics in recent years has been the experimental observation of squeezed states of light.[1,2] Even though these states were predicted theoretically a long time ago, the full importance of this event may perhaps be appreciated if one recalls that it was widely held among researchers in optics that the quantum nature of light gives rise to only small, noise-related corrections to phenomena that are described by the nonquantized Maxwell's equations. In essence, this point of view is the cornerstone of the so-called semi-classical theory, in which matter is treated quantum mechanically, while the field is not quantized, and to which laser optics and nonlinear optics in general are indebted for many successes. Now, with the observation of squeezed states it is quite clear that allowance for the quantum mechanical nature of light leads to qualitatively new phenomena similar to the squeezed states.

This methodological note is not intended as a review of work in the area of squeezed light (see the reviews in Refs. 3 and 4). Rather, it is focused on the physical picture of squeezed light, its theoretical descriptions, and, briefly, possible applications. By dealing principally with these topics, I had hoped that the reader might become aware on the one hand of the simplicity of this phenomenon, and on the other, of its extremely general significance. Indeed, the substance of this paper deals mainly with the states of the quantum-mechanical harmonic oscillator, and of course, is valid for any oscillator whose quantization is carried out according to the Bose scheme. Consequently, squeezed states can be found not only in optics, but in such widely separated fields as elementary particles (π_0 mesons), acoustics (phonons), and even mechanics (mechanical vibrations). One might therefore anticipate the observation of squeezed sound as well as squeezed light (although there may be difficulties in doing so; see the end of Section 4). In principle, squeezed states are even possible in the oscillations of such well-known and even ordinary objects such as pendulums and strings. Therefore, I have attempted to concentrate on the physical nature of the phenomenon of squeezed states, dispensing with a great deal of technical and mathematical detail.

2. QUANTUM MECHANICAL HARMONIC OSCILLATOR

The harmonic oscillator plays an enormously important role in quantum electrodynamics. The reason is that the Maxwell's equations that describe the electromagnetic field in a vacuum are linear, and because of this linearity, the electromagnetic field in vacuum can be considered as a collection of linear, or harmonic oscillators, for example, plane waves. The electromagnetic field retains its linear properties up to extremely high field strengths, where effects due to scattering of light by light, involving the creation of virtual electron-positron pairs, become significant.

Modern laser technology has shown that it is possible to excite one individual oscillator of the field, a single mode, in optical cavities. Because of the interaction of the field with the mirrors of the cavity the region of linearity of the oscillator is narrower than in free space, but nevertheless it is still very great. The range of applicability of the theory of the harmonic oscillator is thus very broad.

Let us consider free oscillations of a classical, unquantized field by means of the diagram shown in Fig. 1, which is convenient for going over to the quantum mechanical case. This diagram shows the probability of observing a particular value of the field at a particular time. Mathematically, this probability (strictly speaking, the probability density) is described by the quantity

$$|\psi(E)|^2 = \delta(E - E_0 \cos(\omega t + \varphi)), \tag{1}$$

which can be called the modulus squared of the "classical" wave function. The delta function in the expression allows the exactly determined quantities of the classical theory to be described in the language of probability, which is inherent to quantum mechanics. In fact, at a particular time t it is possible to observe only a single value of the potential,

$$E = E_0 \cos(\omega t + \varphi), \tag{2}$$

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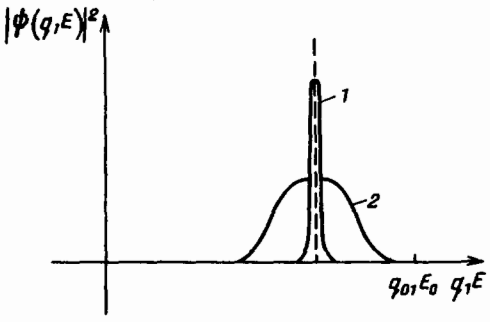

FIG. 1. Distribution of the possible values of the electric field in the classical case (*1*) and in a coherent state (*2*).

in which, as in Eq. (1), there are three parameters that describe the field of the classical harmonic oscillator, the amplitude E_0, the phase φ, and the frequency ω. Thus, the distribution of Eq. (1), shown in Fig. 1, moves to the right and to the left sinusoidally with an amplitude E_0 and frequency ω.

It is well known that in going over to quantum mechanics, such classical quantities as the coordinate and momentum, which had taken on specific values, lose their determinacy and are described by certain distributions. These distributions are described mathematically by the squared modulus of the wave function. For subsequent discussions the Gaussian form is important

$$|\psi(E)|^2 = A \exp\{-[E - E_0 \cos(\omega t + \varphi)]^2/D^2\}, \qquad (3)$$

as shown in Fig. 1 by curve *2*. The factor A is a normalization constant and D is that new factor that embodies the quantum mechanical description. It is the dispersion, that is, the rms spread of possible values of the electric field. The distribution (3) moves as a function of time just as distribution (1): to the right and to the left with an amplitude E_0 and frequency ω. If the dispersion D is small, then the situation is not very different from the classical situation.

Both theoretically and from the point of view of applications an entity that plays an important role is the so-called coherent state[1)] for which

$$D^2 = \frac{\hbar}{2\omega}. \qquad (4)$$

In this state both the electric and the magnetic fields have constant and small indeterminacies and therefore the coherent state is very similar to the classical state (1).

Let us discuss briefly the usual quantum mechanical interpretation of the distribution (3). It is assumed that in measurements on an oscillator in this state it is possible to obtain any value of the field. However, if many measurements are carried out with many identically prepared oscillators, then the results of the measurements will be described by the distribution (3). Repeated measurements on the same object is a much more complicated matter: it is one of the most difficult problems in the interpretation of quantum mechanics. It is not possible to proceed very far in this direction, but fortunately, these difficult problems can be set aside.

To do so we recall that the variance

$$D^2 = \langle E^2\rangle - \langle E\rangle^2 \qquad (5)$$

is equal to the difference between the mean value of the square of the electric field and the square of the mean value of the field. Now, writing Eq. (5) as

$$\langle E^2\rangle = \langle E\rangle^2 + D^2; \qquad (6)$$

we interpret the three terms in the following way (ignoring unimportant numerical factors): $\langle E^2\rangle$ is the total energy of the electric field, $\langle E\rangle^2$ is the classical part of the energy of the electric field, and D^2 is the energy of the quantum noise of the electric field. This energy interpretation of relation (6) greatly simplifies the matter, since energy characteristics are familiar and relatively easily measured.

In the case of the coherent state the energy of the quantum noise is generally negligible. For example, if the energy stored in an oscillator is one joule, this gives the ratio

$$D^2/\langle E^2\rangle \approx 10^{-19}. \qquad (7)$$

However, as we shall see later, for nonclassical states of the field this ratio can be completely different; in particular, it can be equal to unity.

Coherent states of the electromagnetic field apparently are obtained in lasers. This permits the wide use of the so-called semiclassical theory, in which the field is not quantized. The semiclassical theory has been the basis of all the progress attained in laser optics and, in particular, in nonlinear optics. It has been widely believed that the quantum theory of the electromagnetic field can give only small noise-related corrections, and that it is important only at low intensities.

However, the facts are quite otherwise. The equations of the quantum theory are vastly more complicated than the classical equations and it is naive to think that the only solutions that these equations have are those similar to the classical solutions. It is obvious that there must be solutions that are qualitatively different from the classical ones, including solutions in a macroscopic region; that is, at high energies. This can also be seen from the diagram of Fig. 1. Actually, it is a radical step from the δ function in the classical theory to the continuous distribution of the quantum theory, although the radicalness of this step is at first concealed by the sharpness of the continuous distribution in the coherent state, and by the consequently small difference of this state from the classical state. It is not necessary, however, that this state be narrow, and moreover, it can be arbitrary at the initial instant of time, $t = 0$. Subsequent development of the state, of course, is governed by the Schrödinger equation, while at the initial instant of time it is arbitrary. If this initial state is expanded in a basis set

$$\psi(E) = c_0\psi_0(E) + c_1\psi_1(E) + c_2\psi_2(E) + \dots, \qquad (8)$$

then the coefficients c_n are defined that characterize the state $\psi(E)$. These coefficients are in general infinite in number, and they all may be involved in some physical phenomena. Thus, in addition to the three parameters, amplitude, phase and frequency, which characterize the field in the classical theory, we have an entire Hilbert space of parameters that also characterize the same field. Consequently there is an immeasureable increase in the set of possible physical effects in the quantized field. The Hilbert space of parameters

mentioned above is, of course the well-known Hilbert space of the states of quantum mechanics.

We can therefore conclude that there exists a great multitude of quantum-mechanical, nonclassical states of the electromagnetic field. In this sense the observation of squeezed states is only the first swallow of summer; it is clear that in future the variety of nonclassical states will be considerably expanded.

In this connection it is relevant to recall the nonclassical states of the field, which have been known since the appearance of quantum mechanics. Their incorrect interpretation, however, was an obstacle to their identification as nonclassical states. The situation involved the so-called stationary states of the quantum mechanical harmonic oscillator, whose wave functions are shown in Fig. 2. It is easy to see that since the function $|\psi_n(E)|^2$ is symmetric, positive and negative values of the field are equally probable, and consequently the average value of the field E in this state is zero. This is a true sign that for the states that are close to the classical states the field strength must be proportional to the square root of the energy of the field. A state of this kind can moreover be a macroscopic state, since its energy is equal to

$$\left(n+\frac{1}{2}\right)\hbar\omega \tag{9}$$

and, by virtue of the factor $(n+1/2)$ it can assume macroscopic values; for example, it can be equal to one joule $(n \gg 1)$.

Pauling and Wilson in 1935 claimed that this state corresponds to the classical free oscillations of an oscillator on the basis of the observation that for large n the envelope of the probability distribution (Fig. 3) is similar to the function

$$W(q)=\frac{dt}{dq}=\frac{-1}{\omega(q_0^2-q^2)^{1/2}} \tag{10}$$

for a classical oscillator, which defines the time that a classical oscillator spends in a small part of its trajectory; here q is the coordinate of the oscillator

$$q(t)=q_0\cos\omega t\,. \tag{11}$$

The quantity $W(q)$ is sometimes taken to be the probability of observation of the oscillator in a particular point of its trajectory during classical motion, but this idea is, of course, wrong—the classical process is entirely deterministic and

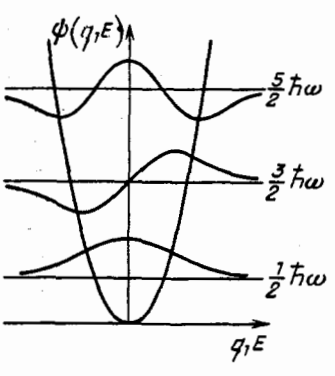

FIG. 2. Stationary states of an oscillator.

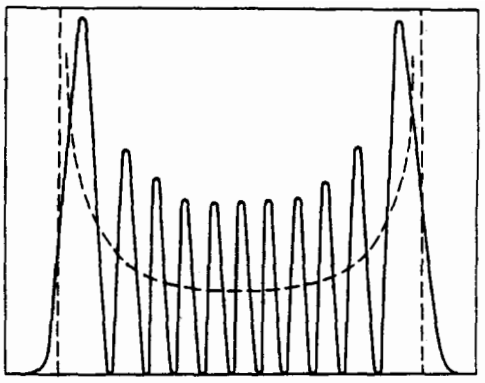

FIG. 3. Analogy between stationary states and classical oscillations.

the notion of probability does not arise. The similarity between the envelopes of the probability distribution (Fig. 3) and that of Eq. (10) is insufficient to reconcile the quantum mechanical and the classical pictures. First of all, the coordinate does not depend sinusoidally on the time in the quantum mechanical state. Moreover, it is easy to see that the indeterminacy in the coordinate and the momentum in the nth stationary state are equal, respectively to

$$\Delta q=[(2n+1)\hbar/2\omega]^{1/2},$$

and

$$\Delta p=[(2n+1)\hbar\omega/2]^{1/2}\,,$$

and their product is

$$\Delta q\cdot\Delta p=(2n+1)\hbar/2\,.$$

In this relation, taking the limit as $\hbar\to 0$ (since classically, nonquantized systems can have a finite energy, we must let $n\to\infty$ since $E=n\hbar\omega=\text{const}$) it is clear that the product of the indeterminacies

$$\lim_{\hbar\to 0}(\Delta q\cdot\Delta p)=E/\omega$$

remains finite as $\hbar\to 0$, while for any classical motion $\Delta q=\Delta p=0$.

Finally, as pointed out above, we have the coherent state, whose properties are very much like those of classical oscillations and which formally goes over into the classical states as $\hbar\to 0$. However, the incorrect ideas of Pauling and Wilson have penetrated into textbooks,[5,6] and held up for a long time the correct understanding of this state. Only in recent years has the issue been raised concerning the excitation of stationary states of the field regarded specifically as nonclassical states.

A short discussion of the mathematical apparatus for the description of the quantized harmonic oscillator is appropriate. The Hamiltonian of the oscillator has the form

$$H=\frac{1}{2}(p^2+\omega^2q^2)\,, \tag{12}$$

where p and q, the momentum and position operators, respectively, obey the commutation relation

$$qp-pq=i\hbar\,. \tag{13}$$

With the use of the annihilation and creation operators

$$a = \frac{1}{(2\hbar\omega)^{1/2}}(\omega q + ip),\ a^+ = \frac{1}{(2\hbar\omega)^{1/2}}(\omega q - ip),\ [a, a^+] = 1 \tag{14}$$

this Hamiltonian can be written as

$$H = \hbar\omega\left(a^+a + \frac{1}{2}\right). \tag{15}$$

The steady states of the oscillator, shown in Fig. 2, are designated by the conventional Dirac notation $|n\rangle$ (or $\langle n|$, where n is the number of photons in the oscillator). For these states the following relations hold:

$$a|n\rangle = n^{1/2}|n-1\rangle,$$

$$a^+|n\rangle = (n+1)^{1/2}|n+1\rangle,\ |n\rangle = \frac{a^{+n}}{(n!)^{1/2}}|0\rangle, \tag{16}$$

where $|0\rangle$ is the ground, or vacuum state of the oscillator, which contains no photons.

The coherent state

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\sum_n \frac{z^n}{(n!)^{1/2}}|n\rangle$$

$$= \exp\left(-\frac{1}{2}|z|^2\right)\sum_n \frac{(za^+)^n}{n!}|0\rangle \tag{17}$$

is a special form of superposition of stationary states, where for free oscillations

$$z = z_0 e^{i\omega t}. \tag{18}$$

The average value of the coordinate q in the coherent state is

$$\langle q\rangle = \left(\frac{2\hbar}{\omega}\right)^{1/2}|z_0|\cos(\omega t + \varphi),\ \ \varphi = \arg z_0; \tag{19}$$

and the dispersion of q

$$D = \left(\frac{\hbar}{2\omega}\right)^{1/2} \tag{20}$$

is equal to the dispersion of the coordinate in the vacuum state $|0\rangle$. An important property of the coherent state is that it is an eigenstate of the annihilation operator a

$$a|z\rangle = z|z\rangle \tag{21}$$

with an eigenvalue equal to the parameter z. Another important point is that in the coherent state the uncertainty relation is a minimum, that is,

$$\Delta p\cdot\Delta q = \frac{1}{2}\hbar.$$

In quantum mechanics the momentum and coordinate are mutually conjugate variables. The coordinate of the electromagnetic field is the vector potential, which in the case of a plane wave has the form

$$\mathbf{A}(\mathbf{r}, t) = \left(\frac{2\pi\hbar c^2}{\omega V}\right)^{1/2}(a^+e^{i\omega t - i\mathbf{kr}} + ae^{-i\omega t + i\mathbf{kr}})\mathbf{e}, \tag{22}$$

where $\mathbf{e}$ is the polarization vector, which is orthogonal to $\mathbf{k}$, and V is the volume occupied by the wave. The momentum that is conjugate to it is the electric field

$$\mathbf{E}(\mathbf{r}, t) = -i\left(\frac{2\pi\hbar\omega}{V}\right)^{1/2}(a^+e^{i\omega t - i\mathbf{kr}} - ae^{-i\omega t + i\mathbf{kr}})\mathbf{e}. \tag{23}$$

At time $t = 0$ and at the origin of the coordinate system, these quantities can be written

$$\mathbf{A}(0, 0) = \left(\frac{2\pi\hbar c^2}{\omega V}\right)^{1/2}(a^+ + a)\mathbf{e},\ \mathbf{E}(0, 0)$$

$$= -\left(\frac{2\pi\hbar\omega}{V}\right)^{1/2} i(a^+ - a)\mathbf{e}.$$

However, in experimental investigations in quantum optics, the vector potential (and the magnetic field) are not of any importance. Therefore, in the published literature the conjugate variables are ordinarily taken to be the two values of the electric field at different times such that the phase shift between the two is $\pi/2$. It is easy to see that one of these values of the field is exactly proportional to the vector potential. These two values of the field have been identified by the terminology "quadrature components." The convenience of the term quadrature components is due to the wide use of phase detection in experimental physics, which separates one of the quadrature components from the total signal.

3. SQUEEZED STATES

Squeezed states were discovered theoretically in 1970 by Stoler,[7] although they were in fact involved in the scientific literature beginning with the work of K. Husimi[9] in 1953. As I have noted above, the probability distribution at the initial instant of time can be arbitrary in shape. For the case of the coherent state it is a Gaussian distribution with a variance equal to the variance of the vacuum state. Therefore it is legitimate to ask how the state will develop that is described, as is the coherent state, by a Gaussian distribution but with different parameters.

The elegant mathematical apparatus, also developed by Stoler, has played a not insignificant role in the popularization of the topic of squeezed states, and eventually their discovery. According to this mathematical method, a squeezed state is defined as the eigenstate of the operator b

$$b|\zeta\rangle = \zeta|\zeta\rangle, \tag{1}$$

that is related (as is the conjugate operator b^+) to the annihilation and creation operators a and a^+ by

$$b = \mu a + \nu a^+,\ \ b^+ = \nu^* a + \mu^* a^+, \tag{2}$$

where

$$|\mu|^2 - |\nu|^2 = 1. \tag{3}$$

It is easy to see that like the operators a and a^+, the operators b and b^+ satisfy the commutation relation

$$[b, b^+] = bb^+ - b^+b = 1. \tag{4}$$

Along with the introduction of the operators b^+ and b comes the temptation, in complete analogy with the operators a^+ and a, to treat the state $|\zeta\rangle$ as the coherent state of a different oscillator, an oscillator with a different frequency $\omega' \neq \omega$. However, this interpretation is correct only if we conceive of an oscillator with a complex frequency. This point is particularly clear if we transform to the coordinate representation. First, let us construct the coordinate representation of a coherent state. By definition we have

$$a|z\rangle = z|z\rangle, \tag{5}$$

where z is an arbitrary complex number and

$$a = \left(\frac{\omega}{2\hbar}\right)^{1/2} q + \frac{i}{(2\hbar\omega)^{1/2}} p . \tag{6}$$

Multiplying (5) from the left by the vector $\langle q|$, an eigenvector of the operator q, we obtain the equation

$$\hbar \frac{\partial \psi_z}{\partial q} + \omega q \psi_z = (2\hbar\omega)^{1/2} z \psi_z, \quad \psi_z = \psi_z(q) = \langle q | z \rangle . \tag{7}$$

Solving this equation, we have

$$\psi_z(q) = A \exp\left\{ -\frac{\omega}{2\hbar}\left[q - \left(\frac{2\hbar}{\omega}\right)^{1/2} z \right]^2 \right\}, \tag{8}$$

where A is a normalization constant. In this expression the only term that can be complex is the one that is subtracted from q, while the coefficient in front of the square brackets is always negative ($\omega > 0$).

Now let us construct the coordinate representation of a squeezed state. Multiplying Eq. (1) from the left by the same vector $\langle q|$ we obtain the equation

$$\hbar(\mu - \nu) \frac{d\psi_\zeta}{dq} + \omega(\mu + \nu) q \psi_\zeta = (2\hbar\omega)^{1/2} \zeta \psi_\zeta . \tag{9}$$

Solving this equation, we have

$$\psi_\zeta(q) = A \exp\left\{ -\frac{\omega}{2\hbar} \frac{\mu + \nu}{\mu - \nu}\left[q - \left(\frac{2\hbar}{\omega}\right)^{1/2} \frac{\zeta}{\mu + \nu} \right]^2 \right\}, \tag{10}$$

where A is also a normalization constant. For the squeezed vacuum, ($\zeta = 0$) the normalization constant is

$$A = \left[\frac{\omega(|\mu| + |\nu|)^2}{\pi\hbar} \right]^{1/4} .$$

As we see, μ and ν are complex numbers, and consequently the factor in front of the square brackets is also complex, so that Eq. (10) is quite different from Eq. (8). It will be shown below that the phases of the parameters μ and ν can vary with time. Therefore, let us find the conditions for which the factor $(\mu + \nu)/(\mu - \nu)$ can be real. Let us write the factor in the form

$$\frac{\mu + \nu}{\mu - \nu} = \frac{|\mu| + |\nu| e^{-i\psi_0}}{|\mu| - |\nu| e^{-i\psi_0}} = \frac{|\mu| + |\nu| \cos\psi_0 - i|\nu| \sin\psi_0}{|\mu| - |\nu| \cos\psi_0 + i|\nu| \sin\psi_0}, \tag{11}$$

where

$$\psi_0 = \psi_\mu - \psi_\nu . \tag{12}$$

The argument ψ in expression (11) is determined by the relation

$$\begin{aligned} \mathrm{tg}\psi &= -\mathrm{tg}\left(\mathrm{arctg} \frac{|\nu| \sin\psi_0}{|\mu| + |\nu| \cos\psi_0} + \mathrm{arctg} \frac{|\nu| \sin\psi_0}{|\mu| - |\nu| \cos\psi_0} \right) \\ &= -\frac{|\mu||\nu| \sin\psi_0 - |\nu|^2 \sin\psi_0 \cos\psi_0 + |\mu||\nu| \sin\psi_0 + |\nu|^2 \sin\psi_0 \cos\psi_0}{|\mu|^2 - |\nu|^2 \cos^2\psi_0 - |\nu|^2 \sin^2\psi_0} \\ &= -2|\mu||\nu| \sin\psi_0 . \end{aligned} \tag{13}$$

Therefore the condition that expression (11) be real is

$$\psi_0 = n\pi, \quad n = 0, 1, 2, \ldots \tag{14}$$

When this condition is satisfied, the squeezed state, which is defined by Eq. (1), can be represented as the coherent state of an oscillator that is different from the initial oscillator (with frequency ω).

Let us consider now the temporal development of the squeezed state defined by Eq. (1). For this purpose we take it as the initial state and investigate how it is transformed with time. At time t the state has the form

$$\begin{aligned} |\zeta(t)\rangle &= e^{-i\omega t a^+ a} |\zeta\rangle = e^{-i\omega t a^+ a} e^{-(1/2)|\zeta|^2} \sum_m \frac{(\zeta b^+)^m}{m!} |0_b\rangle \\ &= e^{-(1/2)|\zeta|^2} e^{-i\omega t a^+ a} \sum_m \frac{1}{m!} \left[\zeta(\mu^* a^+ + \nu^* a) \right]^m e^{i\omega t a^+ a} e^{-i\omega t a^+ a} |0_b\rangle \\ &= e^{-(1/2)|\zeta|^2} \sum_m \frac{1}{m!} \left[\zeta(\mu^* e^{-i\omega t} a^+ + \nu^* e^{i\omega t} a) \right]^m e^{-i\omega t a^+ a} |0_b\rangle . \end{aligned} \tag{15}$$

It can thus be seen that the state $|\zeta(t)\rangle$ retains its identity as a squeezed state, since it remains coherent in the new basis set b' and b'^+, and only the coefficients μ and ν are modified

$$\mu' = \mu e^{i\omega t}, \quad \nu' = \nu e^{-i\omega t} \tag{16}$$

while preserving relation (3).

The state

$$|0_{b'}\rangle = e^{-i\omega t a^+ a} |0_b\rangle , \tag{17}$$

that enters into Eq. (15) is the vacuum state for the operators b'^+ and b'. It is obvious that we have the result

$$\begin{aligned} b'|0_{b'}\rangle &= (\mu' a + \nu' a^+) e^{-i\omega t a^+ a} |0_b\rangle \\ &= e^{-i\omega t a^+ a} (\mu a + \nu a^+) e^{i\omega t a^+ a} e^{-i\omega t a^+ a} |0_b\rangle = e^{-i\omega t a^+ a} b |0_b\rangle = 0; \end{aligned} \tag{18}$$

which shows that $|0_{b'}\rangle$ is the vacuum state of the operators b'^+ and b'.

Let us calculate further the variance of the squeezed state and its time dependence. The variance is defined by the equation

$$D^2 = \langle q^2 \rangle - \langle q \rangle^2 = \frac{\hbar}{2\omega} \left[\langle (a^+ + a)^2 \rangle - \langle (a^+ + a) \rangle^2 \right]. \tag{19}$$

Since

$$(a^+ + a)^2 = 1 + (a^{+2} + a^2 + 2a^+a) \tag{20}$$

and

$$\begin{aligned} a^{+2} &= \mu^2 b^{+2} + \nu^{*2} b^2 - 2\mu\nu^* b^+ b - \mu\nu^* , \\ a^2 &= \nu^2 b^{+2} + \mu^{*2} b^2 - 2\mu^*\nu b^+ b - \mu^*\nu , \\ a^+a &= -\mu\nu b^{+2} - \mu^*\nu^* b^2 + (|\mu|^2 + |\nu|^2) b^+ b + |\nu|^2 , \end{aligned} \tag{21}$$

averaging the operator $a^{+2} + a^2 + 2a^+a$ over $|\zeta\rangle$ we have

$$\begin{aligned} \langle a^{+2} + a^2 + 2a^+a\rangle &= \langle (\mu - \nu)^2 b^{+2} + (\mu^* - \nu^*)^2 b^2 \\ &\quad + 2(\mu - \nu)(\mu^* - \nu^*) b^+ b\rangle + 2|\nu|^2 - \mu\nu^* - \mu^*\nu \\ &= [(\mu - \nu)\zeta^* + (\mu^* - \nu^*)\zeta]^2 + 2|\nu|^2 - \mu\nu^* - \mu^*\nu . \end{aligned} \tag{22}$$

It is readily seen that the expression in the square brackets is simply the average value of the operator $a^+ + a$; that is, when the subtraction is carried out in (19) the square brackets disappear. It is thus clear that the variance of a squeezed state is the same as the variance of the vacuum state in the basis b. Thus, the square of the variance is

$$\begin{aligned} D^2 &= \frac{\hbar}{2\omega}\left[1 + 2|\nu|^2 - \mu\nu^* - \mu^*\nu\right] \\ &= \frac{\hbar}{2\omega}\left(|\mu|^2 - \mu\nu^* - \mu^*\nu + |\nu|^2\right) \\ &= \frac{\hbar}{2\omega}(\mu - \nu)(\mu^* - \nu^*) \\ &= \frac{\hbar}{2\omega}\left[|\mu|^2 + |\nu|^2 - 2|\mu||\nu|\cos\psi_0\right]. \end{aligned} \tag{23}$$

Let us now recollect that μ and ν vary with time according to Eq. (16). For the square of the variance we have

$$D^2(t) = \frac{\hbar}{2\omega}\left[|\mu|^2 + |\nu|^2 - 2|\mu||\nu|\cos(\psi_0 + 2\omega t)\right]. \tag{24}$$

Therefore, the variance takes on minimum values

$$D^2_{\min} = \frac{\hbar}{2\omega}\left(|\mu| - |\nu|\right)^2 \tag{25}$$

twice in a period at the times given by the relation

$$\psi_0 + 2\omega t = 2n\pi , \tag{26}$$

and maximum values

$$D^2_{\max} = \frac{\hbar}{2\omega}\left(|\mu| + |\nu|\right)^2 \tag{27}$$

also twice per period, for

$$\psi_0 + 2\omega t = (2n + 1)\pi . \tag{28}$$

It should be noted that the times of the minimum and the maximum variance coincide with the times [Eq. (14)] when the squeezed state $|\zeta\rangle$ can be considered as the coherent state of an oscillator. The frequencies of these oscillators are

$$\begin{aligned} \Omega' &= \omega(|\mu| + |\nu|)^2 = \frac{\omega}{(|\mu| - |\nu|)^2} , \\ \Omega'' &= \omega(|\mu| - |\nu|)^2 = \frac{\omega}{(|\mu| + |\nu|)^2} . \end{aligned} \tag{29}$$

As we see, the minimum variance corresponds to the higher frequency Ω' and the maximum variance to the lower frequency Ω'':

$$\Omega' > \omega > \Omega''. \tag{30}$$

As will be shown subsequently, the function (24) is determined experimentally and indicates the presence of squeezed states, since the minimum variance can be less than the variance of the vacuum or the coherent states.

The expression for the average value of the field is

$$\begin{aligned} \langle E\rangle &= \left(\frac{\hbar}{2\omega}\right)^{1/2}\langle(a^+ + a)\rangle = \left(\frac{\hbar}{2\omega}\right)^{1/2}[(\mu - \nu)\zeta^* + (\mu^* - \nu^*)\zeta] \\ &= \left(\frac{2\hbar}{\omega}\right)^{1/2}|\zeta|\,[|\mu|\cos(\psi_\mu - \psi_\zeta) - |\nu|\cos(\psi_\nu - \psi_\zeta)] . \end{aligned} \tag{31}$$

The time dependence in this expression can be taken into account if for ψ_μ we substitute $\psi_\mu + \omega t$ and for ψ_ν, we substitute $\psi_n - \omega t$. In this way the time dependence of the average value is sinusoidal. From relation (31) it can be seen that the maximum and minimum values of the variance can be obtained at any phase of the harmonic oscillations of the electric field. If we set $\psi_\mu = -\psi_\nu = \psi'$, then

$$\langle E\rangle \sim |\mu|\cos\left(\frac{n}{2}\pi - \psi_\zeta\right) - |\nu|\cos\left(\frac{n}{2}\pi + \psi_\zeta\right); \tag{32}$$

since here we have the free parameter ψ_ζ, then for a given difference $\psi_\mu - \psi_\nu$ that determines the position of the maximum and the minimum values of the variance, the phase of the field can vary arbitrarily.

Ordinarily, a squeezed state is characterized by the squeeze factor

$$K = \frac{D_{\text{vac}}}{D_{\min}} = \left(\frac{D_{\max}}{D_{\min}}\right)^{1/2} = |\mu| + |\nu| . \tag{33}$$

This factor varies from unity for the coherent state ($|\nu| = 0$) up to large values (in principle, up to infinity) for strongly squeezed states ($|\nu| \to \infty$). There are, however, energy limitations on this factor, since for a given average energy of the oscillations, the squeeze factor cannot assume values that are greater than some maximum value. To find this maximum value of the squeeze factor, we calculate the average number of photons in the squeezed state

$$\begin{aligned} N &= \langle\zeta|a^+a|\zeta\rangle = \langle\zeta|(\mu b^+ - \nu^* b)(\mu^* b - \nu b^+)|\zeta\rangle \\ &= \langle\zeta|[(|\mu|^2 + |\nu|^2)b^+b - \mu\nu b^{+2} - \mu^*\nu^* b^2 + |\nu|^2]|\zeta\rangle \\ &= |\zeta|^2[(|\mu| + |\nu|)^2\sin^2\psi + (|\mu| - |\nu|)^2\cos^2\psi] + |\nu|^2. \end{aligned} \tag{34}$$

It can be seen from Eq. (33) that the greater is the value of $|\nu|$ the greater is the squeezing. For a given number N of

photons the quantity $|\nu|$ is a maximum for $|\zeta|^2 = 0$, that is, in the squeezed vacuum state. Then

$$|\nu|^2 = N \tag{35}$$

and

$$K_{\max} = (N+1)^{1/2} + N^{1/2}; \tag{36}$$

which is also the maximum possible squeeze factor. For large values of $N \gg 1$, we have approximately

$$K_{\max} \approx 2N^{1/2}. \tag{37}$$

In the optical region, with an energy of the order of 1 J in the cavity, the squeeze factor can have values of the order of 10^{10}. This is dramatically different from the experimentally obtained values. At the present time the squeeze factors that have been obtained have been from a few per cent greater than unity to several times greater than unity. The reasons for this discrepancy are still not clear.

As noted above, the maximum value of the squeeze factor is obtained for $|\zeta| = 0$; that is, in the case where there are no oscillations of the field, $\langle E \rangle = 0$. These states are usually called the squeezed vacuum state. It should be remembered that the squeezed vacuum state has little in common with the vacuum state (the lowest state). The squeezed vacuum state can be a highly excited, high-energy state.

These arguments lead to a graphic picture (Fig. 4). Figure 4a shows the oscillations for the coherent state of the field; the thickness of the line indicates the time-invariant variance. As usual, for macroscopic energies the variance is small compared to the amplitude. Figure 4b shows the oscillations in the case of squeezed states of the field. Here the thickness of the line is comparable to the amplitude of the oscillations and changes with the time. The point with the lowest variance can have any phase relative to the oscillations of the field. Figure 4c shows the "oscillations" in the squeezed vacuum state. The word oscillations is in quotation marks, since now there are essentially no oscillations at the fundamental frequency ω. There is only a variation of the variance at double the frequency.

In the calculations of the squeezed states the unitary operator introduced by K. Stoler that transforms the coherent state into the squeezed state plays an important role. This operator has the form

$$U(z) = \exp\left[\frac{1}{2}\left(za^2 - z^* a^{+2}\right)\right], \tag{38}$$

where z is an arbitrary complex number. It is easy to show that the annihilation and creation operators are transformed by this operator in the following way:

$$\begin{aligned} b' &= U(z)aU^+(z) = a\cosh r + a^+ e^{-i\psi}\sinh r, \\ b'^+ &= U(z)a^+U^+(z) = ae^{i\psi}\sinh r + a^+\cosh r \end{aligned} \tag{39}$$

where $r = |z|$ and $\psi = \arg z$. As we see, this transformation is a special case of the transformation (2), which leads to squeezed states and corresponds to $\psi_\mu = 0$. It can be seen from the following argument that the operator $U(z)$ transforms a coherent state into a squeezed state. Let $|\sigma\rangle$ be a coherent state, that is, an eigenstate of the operator a. Writing the equation (39) in the form

$$U(z)a = b'U(z),$$

it is easy to see ($U(z)a|\sigma\rangle = \sigma U(z)|\sigma\rangle = b'U(z)|\sigma\rangle$) that the state $|\zeta\rangle = U(z)|\sigma\rangle$ is an eigenstate of the operator b, that is, a squeezed state.

Let us note some important features that squeezed states introduce into the polarization structure of light.[8] Consider a plane wave

$$\mathbf{E}(\mathbf{r}) = A[\mathbf{e}_x(a^+e^{-ikr} + \text{h.c.}) + \mathbf{e}_y(b^+e^{-ikr} + \text{h.c.})] \tag{40}$$

in which the oscillators a and b are in coherent states $|\xi,\eta\rangle$ with parameters ξ and η, respectively. Then the field $\mathbf{E}(\mathbf{r})$ can be expressed in terms of the polarization vectors

$$\begin{aligned} \mathbf{i}_1 &= (\xi^*\mathbf{e}_x + \eta^*\mathbf{e}_y)N, \ \mathbf{i}_2 \\ &= (-\eta\mathbf{e}_x + \xi\mathbf{e}_y)N \quad (N = (|\xi|^2 + |\eta|^2)^{-1/2}), \end{aligned} \tag{41}$$

of two orthogonal and in general elliptical polarizations

$$E(r) = A[(e^{-ikr}c^+\mathbf{i}_1 + \text{h.c.}) + (e^{-ikr}d^+\mathbf{i}_2 + \text{h.c.})], \tag{42}$$

where

$$c^+ = (\xi a^+ + \eta b^+)N, \ d^+ = (-\eta^* a^+ + \xi^* b^+)N, \tag{43}$$

are the creation operators for photons of elliptical polarization. It is now easy to see that the initial coherent state is an eigenstate of the annihilation operators c and d with eigenvalues $(|\xi|^2 + |\eta|^2)^{1/2}$ and 0. This means that when the states of polarized oscillators are coherent the total field can be reduced to a single excited oscillator, where the fields are calculated by the parallelogram rule.

A completely different picture emerges when the field oscillator is in the squeezed vacuum state. To demonstrate this, we use again expression (42) for the field, but assume now that ξ and η are not related to the state of the field, but are simply parameters that determine the representation of the field in the form (42). Ordinarily the polarization state is analyzed with phase plates and polarizers: the phase plate converts the elliptical polarization to linear polarization, which is easily discriminated with a polarizer. Let us consid-

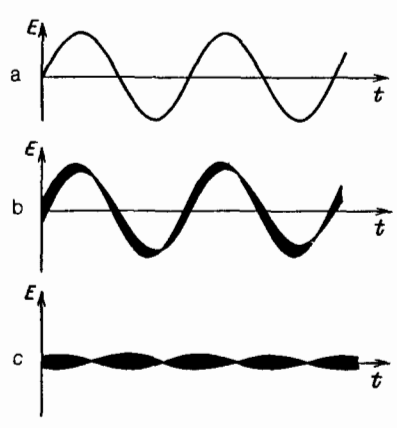

FIG. 4. Oscillations and indeterminacy of the field in a coherent state (a) in squeezed light (b), and for the squeezed vacuum state (c).

er to what extent this procedure is effective when the field oscillators are in the squeezed vacuum state.

After passing through the phase plate the wave (42) has the form

$$\mathbf{E}(\mathbf{r}) = A[(e^{-i\mathbf{kr}+i\delta}c^+\mathbf{i}_1 + \text{h.c.}) + (e^{-i\mathbf{kr}^-}d^+\mathbf{i}_2 + \text{h.c.})] , \tag{44}$$

where δ is the phase introduced into the polarization by the phase plate. We obtain the wave transmitted through the polarizer by projecting (44) on the direction of transmission of the polarizer, defined by the vector

$$\mathbf{e}_1 = \mathbf{e}_x \cos\alpha - \mathbf{e}_y \sin\alpha , \tag{45}$$

where α is the angle between $\mathbf{e}_1$ and the x axis, and

$$E_1(\mathbf{r}) = A\{[(e^{-i\mathbf{kr}+i\delta}c^+(\xi^* \cos\alpha - \eta^* \sin\alpha) + \text{h.c.}] - [e^{-i\mathbf{kz}}d^+(\eta \cos\alpha + \xi \sin\alpha) + \text{h.c.}]\}N . \tag{46}$$

The intensity of the transmitted light is

$$\begin{aligned} I = A^2N^2\{&(c^+c + cc^+)[|\xi|^2\cos^2\alpha + |\eta|^2\sin^2\alpha \\ &- (\xi^*\eta + \xi\eta^*)\sin\alpha\cos\alpha] \\ &+ (d^+d + dd^+)[|\xi|^2\sin^2\alpha + |\eta|^2\cos^2\alpha \\ &+ (\xi^*\eta + \xi\eta^*)\sin\alpha\cos\alpha] \\ &- [e^{i\delta}c^+d(\xi^*\cos\alpha - \eta^*\sin\alpha)(\eta^*\cos\alpha + \xi^*\sin\alpha) + \text{h.c.})\} . \end{aligned} \tag{47}$$

In averaging over the squeezed vacuum one must keep in mind that the operator $c^+d(c = \mu_1^* c' - \nu_1 c'^+, d = \mu_2^* d' - \nu_2 d'^+)$ operating on the squeezed vacuum state generates a state that is orthogonal to the latter, since c^+ and d operate on different components of the state $|0,0\rangle$. Therefore, the average value of the terms in the last square brackets in Eq. (47) is equal to zero. Consequently, the intensity of the light transmitted through the polarizer is

$$\begin{aligned} \langle I\rangle = A^2N^2\{&(2|\nu_1|^2 + 1)[|\xi|^2\cos^2\alpha + |\eta|^2\sin^2\alpha \\ &- (\xi^*\eta + \xi\eta^*)\sin\alpha\cos\alpha] \\ &+ (2|\nu_2|^2 + 1)[|\xi|^2\sin^2\alpha + |\eta|^2\cos^2\alpha \\ &+ (\xi^*\eta + \xi\eta^*)\sin\alpha\cos\alpha]\} \end{aligned} \tag{48}$$

and is independent of the phase δ introduced by the phase plate. An interesting point is that if oscillators c and d have an equal number of photons,

$$|\nu_1|^2 = |\nu_2|^2 = |\nu|^2 \tag{49}$$

the intensity no longer depends on the angle α of rotation of the polarizer

$$\langle I\rangle = A^2(2|\nu|^2 + 1) , \tag{50}$$

that is, the signal transmitted through the polarizer does not depend on the angle of rotation of the latter. This same situation is encountered in the circular polarization of ordinary light, but in that case there is a dependence on δ; when δ is changed the intensity again depends on α. In squeezed light, on the other hand, no change in δ can restore the dependence on α. The situation is similar to that for unpolarized light. However, this similarity is found only for changes over long periods of time, longer than the coherence time of the unpolarized light. For times less than the coherence time the polarization structure of ordinary light of necessity shows up. The polarization situation for squeezed light is therefore totally unusual. In the literature this light has received the not very apt name "scalar light."[8] An important consequence of this analysis is that in squeezed light it is not possible for two polarized oscillators to be reduced to a single one, and the parallelogram rule is not valid. It is also to be noted that other nonclassical polarization states are also possible.[8]

4. THE EXCITATION OF SQUEEZED STATES

The simplest means of exciting squeezed states, at least in principle, is parametric excitation of a harmonic oscillator. This is made possible by the existence of an exact, in a certain sense, solution of the corresponding mathematical problem (the meaning of these words will be revealed later). Let us first analyze this problem.

Parametric excitation is, by definition, the excitation of oscillations in an oscillator by varying (periodically, as a rule) one of its parameters such as the frequency. Therefore, the mathematical problem is reduced to solving the time-dependent Schrödinger equation with a time-dependent Hamiltonian

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = \frac{1}{2}\left[p^2 + \Omega^2(t)q^2\right]\psi(q,t) . \tag{1}$$

This equation has the solution[9,10]

$$\psi(q,T) = \frac{\Omega_0^{1/4}}{(\pi\hbar)^{1/4}(\varepsilon(t))^{1/2}} \times \exp\left[\frac{i\dot\varepsilon}{2\hbar\varepsilon}q^2 + \left(\frac{2\Omega_0}{\hbar}\right)^{1/2}\frac{\alpha}{\varepsilon}q - \frac{\alpha^2}{2}\frac{\varepsilon^*}{\varepsilon} - \frac{|\alpha|^2}{2}\right] , \tag{2}$$

where $\varepsilon(t)$ is the solution of the equation

$$\frac{d^2\varepsilon(t)}{dt^2} + \Omega^2(t)\varepsilon(t) = 0 , \tag{3}$$

that satisfies the initial conditions

$$\varepsilon(0) = 1,\ \dot\varepsilon(0) = i\Omega_0 , \tag{4}$$

where Ω_0 is a characteristic (e.g., average) frequency of the oscillator. It is readily seen that the quantity

$$\dot\varepsilon\varepsilon^* - \varepsilon\dot\varepsilon^* = 2i\Omega_0 \tag{5}$$

is independent of the time for this particular solution.

Therefore, solution (2) is an exact solution of Eq. (1). However, the function $\varepsilon(t)$ appears in it, and is a solution of Eq. (3), and the explicit form of this function is unknown. Nonetheless, solution (2) is very important. Indeed, it convinces us that the most subtle quantum mechanical part of the solution contains no uncertainties. Equation (3) has been well studied (for a periodic function $\Omega(t)$ it reduces to

the Hill or Mathieu equations) so that we do not overlook any essential features if we use its approximate solutions.

If we introduce the parameters

$$\mu = \frac{1}{2}\left(\frac{\dot{\varepsilon}}{i\Omega_0} + \varepsilon\right), \quad \nu = \frac{1}{2}\left(\frac{\dot{\varepsilon}}{i\Omega_0} - \varepsilon\right), \tag{6}$$

that manifestly satisfy relation (3.3), then the state (2) is brought to the form (3.10), which is characteristic of a squeezed state. At time $t = 0$, $\nu = 0$, and, consequently, at time $t = 0$ state (2) is the vacuum state. From relations (6) and (3.23) it follows that the variance of the state (2) is

$$D^2 = \frac{\hbar}{2\Omega_0}|\varepsilon(t)|^2. \tag{7}$$

Let us return now to the solution of Eq. (3), using as $\Omega^2(t)$ the function

$$\Omega^2(t) = \Omega_0^2(1 + \alpha \sin \omega t); \tag{8}$$

For this choice, Eq. (3) is the well-studied Mathieu equation. It is well known that depending on the parameters α and ω the solution of Eq. (3) can be bounded or increasing with time. The corresponding regions are shown in Fig. 5, where the hatched region is the most interesting one for our purposes. Here the solution is an increasing one, or more precisely, there are two independent solutions, one of which grows and the other decays.

Let us examine the region of small α, near $\omega \simeq 2\Omega_0$. In this region we can use perturbation theory and find approximate solutions of Eq. (3) in explicit form. According to the Floquet theorem the solution of Eq. (3) has the form

$$\varepsilon(t) = e^{i\lambda t}u(t), \quad u(t) = \sum_n \varepsilon_n e^{in\omega t}, \tag{9}$$

where $u(t)$ is a periodic function with the same period as the driving force (8) and thus can be expanded in a Fourier series (9). The parameter λ is a function of α and ω. An important question is whether it is purely real or it has an imaginary part. In the former case the solution is bounded, and in the latter case it grows or decays with time. Substituting solution (9) into Eq. (3) and setting the coefficients of the harmonics separately to zero, we obtain the system of equations

$$[1 - (\lambda + n\omega)^2]\varepsilon_n + \frac{\alpha}{2i}\left(\varepsilon_{n-1} - \varepsilon_{n+1}\right) = 0, \tag{10}$$

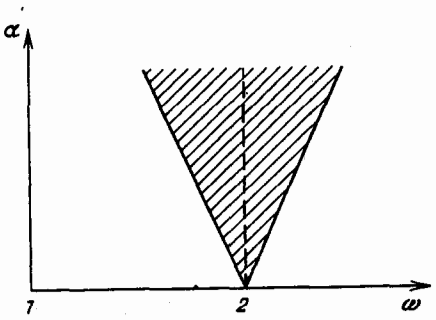

FIG. 5. Region of excitation of a parametric oscillator.

which determine the coefficients ε_n. The determinant of this linear homogeneous system must be zero, and provides the equation for the determination of λ.

For small α and ω close to $2\Omega_0$,

$$\omega = 2\Omega_0 + \Delta\omega \tag{11}$$

we can separate out the case where only two coefficients are important: ε_0 and ε_1. Here the parameter λ must be close to -1,

$$\lambda = -1 + \Delta\lambda, \tag{12}$$

where $\Delta\lambda$ is a small quantity. Then the system of equations (10) assumes the following simple form

$$2\Delta\lambda \cdot \varepsilon_0 + \frac{i\alpha}{2}\varepsilon_1 = 0, \quad \frac{i\alpha}{2}\varepsilon_0 + 2(\Delta\lambda + \Delta\omega)\varepsilon_1 = 0. \tag{13}$$

Setting the determinant of this system equal to zero, we obtain the possible values of $\Delta\lambda$:

$$\Delta\lambda = \frac{1}{2}\left[\pm\left(\Delta\omega^2 - \frac{1}{4}\alpha^2\right)^{1/2} - \Delta\omega\right]. \tag{14}$$

From this result it can be seen that in the sector $|\Delta\omega| < \alpha/2$, $\Delta\lambda$ is complex, and, consequently, there are two solutions, one of which grows with time and the other decays. However, outside of this sector, where $|\Delta\omega| > \alpha/2$, the solution is oscillating and bounded.

Substituting (14) into (13) we find the coefficients ε_0 and ε_1 corresponding to the two values of $\Delta\lambda$ and thus belonging to the two independent solutions of Eq. (3):

$$\varepsilon' = Ae^{-(1/4)\alpha t}\sin \Omega_0 t, \quad \varepsilon'' = Be^{(1/4)\alpha t}\cos \Omega_0 t. \tag{15}$$

The solution that satisfies conditions (4) is obtained as a linear combination of the solutions (15):

$$\varepsilon(t) = e^{-\alpha t/4}\cos \Omega_0 t - \frac{\alpha}{4\Omega_0}e^{-\alpha t/4}\sin \Omega_0 t + ie^{-\alpha t/4}\sin \Omega_0 t. \tag{16}$$

For the variance of state (2) at time t we find, according to Eq. (7), the expression

$$D^2 = \frac{\hbar}{2\Omega_0}\left[\left(e^{(1/4)\alpha t}\cos \Omega_0 t - \frac{\alpha}{4\Omega_0}e^{-(1/4)\alpha t}\sin \Omega_0 t\right)^2 + e^{-(1/2)\alpha t}\sin^2\Omega_0 t\right]. \tag{17}$$

It is easy to see that the variance varies from the minimum value

$$D^2_{\min} \approx \frac{\hbar}{2\Omega_0}e^{-(1/2)\alpha t} \tag{18}$$

to the maximum

$$D^2_{\max} \approx \frac{\hbar}{2\Omega_0}e^{(1/2)\alpha t}. \tag{19}$$

twice a period. The squeeze factor therefore increases exponentially with the time

$$K = D_{\text{vac}}/D_{\min} = e^{\alpha t/4}. \tag{20}$$

Consequently, parametric excitation of a harmonic oscillator does in fact generate a squeezed state with a squeeze factor that increases with time. In actual situations there is no need to restrict α to small values. The qualitative behavior of the parametric oscillator is the same over the entire hatched region in Fig. 5. It must be noted, however, that this theory is highly idealized. It does not take into account such important factors as the presence of losses in the harmonic oscillator or the reverse action of the oscillator on the pumping source. Both of these factors can cause the squeeze factor to saturate; that is, it will cease to increase after attaining some limiting value. As has been pointed out previously, there is a considerable difference between the possible values of K and those actually attained. It is possible that this discrepancy is due to just these factors that were ignored.

We should point out the important role of the initial state in parametric excitation of squeezed states. The squeezed vacuum state is generated only if the initial state is the vacuum state. If, however, the initial state is a coherent state, then it will develop further as a coherent state, increasing in amplitude. In optics, the initial state is automatically given as a vacuum state, since at room temperature $\hbar\omega \gg kT$, and other states are simply not populated. It is another matter for the low-frequency regions, for example, in acoustic or mechanical oscillations. There, any state in thermodynamic equilibrium is a mixed state, and this can cause a definite complication in the generation of squeezed states.

The parametric generation of squeezed states of light has been achieved by a group of investigators at the University of Texas.[2] Although this method was not the first one—squeezed states had previously been observed in a number of laboratories—in the opinion of the present author the method that they used is technically the most promising, and I will give a short description of it. The experimental apparatus is shown in Fig. 6. It consists of three main parts, a source of parametric pumping, the parametric oscillator itself, and a detector-analyzer of the squeezed state. The first two parts in their important aspects duplicate the device used in the first experiments on parametric oscillation.[11] The detector, however, is especially designed for the analysis of squeezed states. The theory of its operation is quite complicated, and will not be described here. It should only be noted that it operates on the principle of a stroboscope, and measures the variance of a light signal excited in the parametric oscillator only at those times that are close to the peak of the reference signal. Figure 7 shows the results of the observations. As we see, the variance of the signal varies twice over a period of the light field; the period of the variation of the variance is π and not 2π. It can be seen also that for some values of the phase θ of the signal the variance becomes less than the one that corresponds to the vacuum state (the dashed line). This is sure evidence that the light is in a squeezed state.

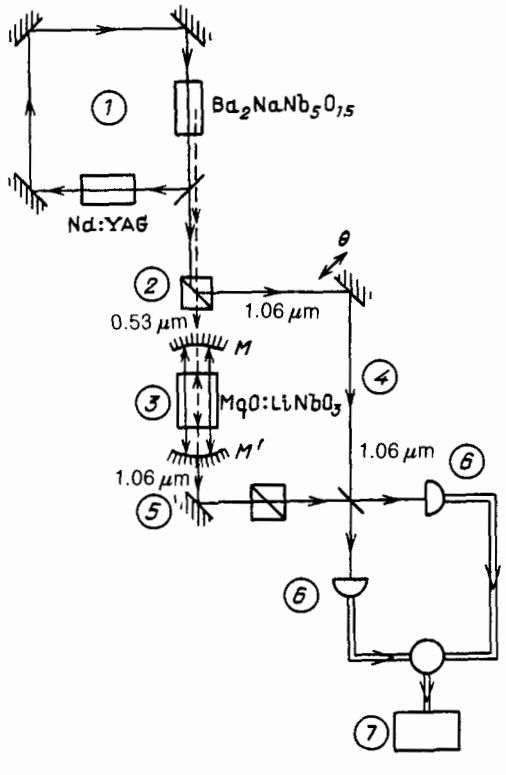

FIG. 6. Experimental setup for excitation and analysis of squeezed light. *1*) Ring laser with second harmonic generator for pumping; *2*) polarizer; *3*) optical parametric oscillator; *4*) reference signal in a coherent state; *5*) light signal in a squeezed state; *6*) photodiode; *7*) noise spectrum analyzer.

5. POSSIBLE APPLICATIONS OF SQUEEZED LIGHT

The applications of squeezed light have been discussed in the reviews of Refs. 3 and 4. By way of illustration three examples of these applications, involving interference measurements,[12] nonlinear phenomena,[13,14] and information transmission[15] are presented in this section. Considering the multitude of forms of interference phenomena, nonlinear phenomena, and information transmission lines, the examples quoted here will be very simple ones, but nonetheless, examples that bring out the fundamental advantages of the use of squeezed light.

5.1. Suppression of noise in a Mach–Zehnder interferometer

A diagram showing the principle of operation of the interferometer is shown in Fig. 8. Under ordinary conditions of operation, that is, without noise suppression by means of squeezed light, monochromatic light enters into the interferometer in channel b (for example, laser light in a coherent

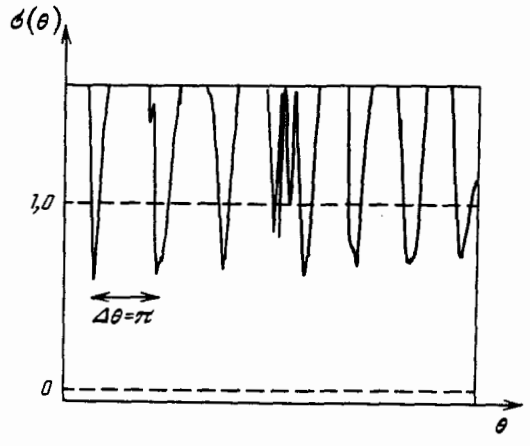

FIG. 7. Dependence of the variance on the phase θ.

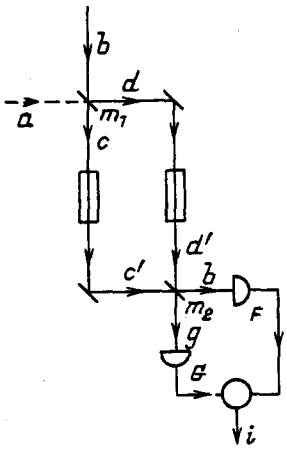

FIG. 8. Mach-Zehnder interferometer.

state). The semitransparent mirror m_1 splits the light into two beams which propagate along the arms of the interferometer as far as the semitransparent mirror m_2. The optical lengths of the two interferometer arms are different, and the phase difference generated in the arms is created ordinarily by the object studied and is the measured quantity. Passing through the semitransparent mirror m_2, the beams are incident on the detectors F and G; the current of the photodetectors is amplified and fed into a difference circuit.

Squeezed light, by means of which the noise in this device can be reduced, is fed into the free channel a of the interferometer. The transmission of the beams of light through the semitransparent mirrors (50% transmission) is described by the relation

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\rho} & 1 \\ -1 & e^{-i\rho} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{1}$$

where a, b, c, and d are the annihilation operators for photons in the corresponding beams of light (Fig. 8). It is easy to see that if we forget that these quantities are operators, then relation (1) is just the classical relation that connects the fields on the two sides of the mirror. In this relation the unimportant total phase shift of the light in the mirror is omitted, and only the phase difference 2ρ is retained. The unimportant total phase shift is also omitted in the subsequent equations.

The amplitudes c and d in the transmission through the arms of the interferometer acquire additional phase shifts ψ and $-\psi$, as described by

$$\begin{pmatrix} c' \\ d' \end{pmatrix} = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}. \tag{2}$$

The transmission of the light beams through the semitransparent mirror m_2 is described by the relation

$$\begin{pmatrix} g \\ f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\bar{\rho}} & 1 \\ -1 & e^{-i\bar{\rho}} \end{pmatrix} \begin{pmatrix} c' \\ d' \end{pmatrix}, \tag{3}$$

which is analogous to relation (2); the operators g and f are the annihilation operators of photons in the corresponding arms. Multiplying out the matrices, we obtain for g and f the following expressions:

$$\begin{aligned} g &= e^{i\zeta}(ie^{i\rho} \sin\sigma\cdot a + \cos\sigma\cdot b), \\ f &= e^{-i\zeta}(\cos\sigma\cdot a + ie^{-i\rho} \sin\sigma\cdot b), \end{aligned} \tag{4}$$

where

$$\zeta = \frac{1}{2}\left(\bar{\rho} - \rho\right), \quad \sigma = \frac{1}{2}\left(\bar{\rho} + \rho + 2\psi\right). \tag{5}$$

The currents in the detectors are proportional to the number of photons in arms g and f; for the photon number operators we have

$$\begin{aligned} N_f &= f^+ f = a^+ a \cos^2\sigma \\ &\quad + i(a^+ b e^{-i\rho} - a b^+ e^{i\rho}) \sin\sigma\cdot\cos\sigma + b^+ b \sin^2\sigma, \\ N_g &= g^+ g = a^+ a \sin^2\sigma \\ &\quad - i(a^+ b e^{-i\rho} - a b^+ e^{i\rho}) \sin\sigma\cdot\cos\sigma + b^+ b \cos^2\sigma. \end{aligned} \tag{6}$$

The difference in the number of photons is

$$\begin{aligned} \Delta N &= N_f - N_g \\ &= (a^+ a - b^+ b) \cos 2\sigma + 2i(e^{-i\rho} a^+ b - e^{i\rho} a b^+) \sin 2\sigma. \end{aligned} \tag{7}$$

To average these and the other operators over the squeezed light in arm a we introduce the operators A and A^+; see Eq. (3.2):

$$a^+ = \mu A^+ - \nu^* A, \quad a = -\nu A^+ + \mu^* A, \quad |\mu|^2 - |\nu|^2 = 1, \tag{8}$$

where the coefficients μ and ν characterize the squeezed state $|\zeta\rangle$ in arm a. Now, bearing in mind that the light in arm a is in the squeezed vacuum state ($\zeta = 0$), while the light in arm b is in the coherent state $|z\rangle$, we find the average value of the various operators

$$\begin{aligned} &\langle a^+ a\rangle = |\nu|^2, \ \langle a^2\rangle = -\mu^* a, \ \langle a^{+2}\rangle = -\mu\nu^*, \\ &\langle b^+ b\rangle = |z|^2, \ \langle a^+ a a^+ a\rangle = |\nu|^2(2|\mu|^2 + |\nu|^2). \end{aligned} \tag{9}$$

In this way, the interference pattern, which is defined as the dependence of the difference of the average number of photons in arms f and g on the phase σ is described by the expression

$$\langle \Delta N\rangle = \langle f^+ f - g^+ g\rangle = (|\nu|^2 - |z|^2) \cos 2\sigma, \tag{10}$$

(see Fig. 9). Since the intensity of the squeezed light is always low, then $|\nu|^2 \ll |z|^2$ (where $|\nu|^2$ and $|z|^2$ are, respectively, the number of photons in arms a and b), and the squeezed light has little effect on the interference pattern. For the noise, however, the picture is completely different.

The intensity of the current noise is proportional to

$$D = \langle \Delta N^2\rangle - \langle \Delta N\rangle^2. \tag{11}$$

According to Eq. (7), we have

$$\begin{aligned} \Delta N^2 &= (a^+ a a^+ a - 2a^+ a b^+ b + b^+ b b^+ b) \cos^2 2\sigma \\ &\quad - (e^{2i\rho} a^2 b^{+2} - e^{-2i\rho} a^{+2} b^2 - 2a^+ a b^+ b - a^+ a - b^+ b) \sin^2 2\sigma. \end{aligned} \tag{12}$$

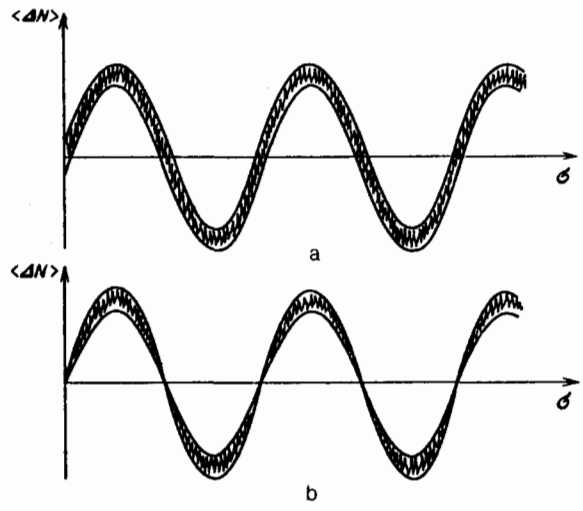

FIG. 9. a) Noise in the interference pattern without squeezed light in arm a; b) with light in the squeezed vacuum state in arm a.

Averaging this expression over the initial states and subtracting from it $\langle \Delta N^2 \rangle$, we obtain

$$P = (2|\mu|^2|\nu|^2 + |z|^2)\cos^2 2\sigma + |z|^2(|\nu|^2|z|^{-2} + 2|\nu|^2 + 1 + 2|\mu|\cdot|\nu|\cos\psi)\sin^2 2\sigma, \tag{13}$$

where

$$\psi = 2\rho - 2\psi_z - \psi_\mu + \psi_\nu .$$

In taking into account the time dependence, it must be kept in mind that ψ_z contains the term $-\omega t$, ψ_μ contains the term $-\omega t$, and ψ_ν contains the term ωt. Thus, the phase ψ does not depend on the time, and is a free parameter that can be controlled at the will of the observer by, for example, delaying the squeezed light relative to the coherent light.

Let us consider the noise at the points where the interference pattern, expression (10), has zeros, i.e., where $\cos 2\sigma = 0$. The first term in (13) is zero and plays no role. In the second term, $\sin 2\sigma = \pm 1$, and, consequently, the amount of noise can be reduced by an appropriate choice of the phase ψ in the coefficient of $\sin^2 2\sigma$. Since the first three terms in this coefficient are positive, the term is, obviously, a minimum for $\cos\psi = -1$. Then the round brackets in this coefficient can be written as

$$S = (|\nu|^2/|z|^2) + (|\mu| - |\nu|)^2 = (|\nu|^2/|z|^2) + (1/K)^2, \tag{14}$$

where

$$K = |\mu| + |\nu| = 1/(|\mu| - |\nu|)$$

is the squeeze factor of light in the arm a (the ratio of the variance of the vacuum state to the minimum variance). The coefficients $|\nu|^2$ and K^2 are related by

$$|\nu|^2 = (K^2 - 1)^2/4K^2 \approx K^2/4.$$

Therefore, as K increases, the first term in S increases and the second term decreases. The quantity S is a minimum for $K^2 = 2|z|$; here $S_{\min} = 1/|z|$. Since $S = 1$ in the absence of the squeezed light; that is, when $|\nu| = 0$, it can be seen that the noise power decreases by a factor of $|z| = N^{1/2}$, where N is the number of photons in arm b. The optimum number of photons in arm a is

$$|\nu|^2 = \frac{1}{2}|z| = \frac{1}{2}N^{1/2}.$$

Thus, the overall pattern of the noise in the case of interference is shown in Fig. 9. In the absence of squeezed light in arm a the noise power is the same at each point of the interference pattern. However, with squeezed light in arm a with the proper phase and amplitude, the noise power near the zeros of the interference pattern is significantly reduced, so that it is possible to measure with greater precision the spacing between the zeros, and, consequently, the phase difference of the two beams. These arguments were verified in the paper by Kimble and his coworkers.[16] In their experiment they measured the noise power at the frequency 1.6 MHz with a channel frequency width of 100 kHz. They turned on and off the high-frequency modulation of the phase in one of the arms of the interferometer at a frequency of 50 Hz. When the modulation was turned on the signal was fed to the detection circuit. The results of the measurements are shown in Fig. 10a,b. Figure 10a corresponds to the case where there is no squeezed light in arm a. It can be seen that the noise is at the level of the vacuum fluctuations (the dashed line). Figure 10b corresponds to the case where light in the squeezed

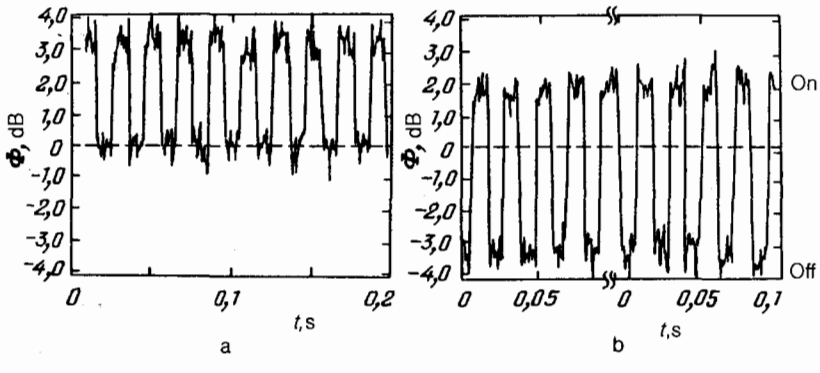

FIG. 10. Experimental observation of the suppression of noise in an interference pattern.

vacuum state is present in arm a. It can be seen that the noise is considerably reduced and the signal-to-noise ratio has increased.

5.2. Squeezed light in nonlinear phenomena

The generation of optical fields in squeezed states may have a large influence in nonlinear optics. To show how this is so, let us consider the nonlinear polarization of matter in a field in a squeezed state.[13,14]

It is well known that the polarization of a material is a nonlinear function of the field and can be represented as a series

$$P(E) = \alpha_1 E + \alpha_2 E^2 + \alpha_3 E^3 + \ldots = \sum_n \alpha_n E^n , \qquad (15)$$

where α_n are coefficients that characterize the nonlinear medium. The item of interest is the $2N$th harmonic of this polarization. This harmonic can arise from the term

$$P_{2N}(E) = \alpha_{2N} E^{2N} \qquad (16)$$

of the expansion given above. For simplicity, the field will be considered as a single-mode field. Then the electric field is

$$E = \beta(a^+ + a) .$$

To calculate the $2N$th harmonic it is necessary to know the quantity

$$\langle E^{2N}\rangle = \langle\zeta|E^{2N}|\zeta\rangle = \langle\zeta|(a^+ + a)^{2N}|\zeta\rangle = p_{2N} ,$$

where $|\zeta\rangle$ is the squeezed state. Since the operators a^+ and a are related to the operators b^+ and b by relation (3.2), we have

$$p_{2N} = \langle\zeta|[(\mu - \nu)b^+ + (\mu^* - \nu^*)b]^{2N}|\zeta\rangle , \qquad (17)$$

and the time dependence of the squeezed state will be taken into consideration if the parameters μ and ν are taken in the form

$$\mu e^{i\Omega t}, \quad \nu e^{-i\Omega t} .$$

Since the squeezed state $|\zeta\rangle$ is an eigenstate of the operator b, the average in expression (17) can be easily performed if we carry out the normal ordering of the operators b^+ and b in the operator

$$B_{2N} = [(\mu - \nu)b^+ + (\mu^* - \nu^*)b]^{2N} = (\gamma b^+ + \gamma^* b)^{2N} , \qquad (18)$$

where $\gamma = \mu e^{i\Omega t} - \nu e^{-i\Omega t}$. The average value of the normally ordered expression (18) is simply the $2N$th power of the expression

$$p_1 = \langle\zeta|\gamma b^+ + \gamma^* b|\zeta\rangle$$

$$= \gamma\zeta^* + \gamma^*\zeta = |\zeta|[(\mu e^{-i\varphi} - \nu^* e^{i\varphi})e^{i\Omega t} + (\mu^* e^{i\varphi} - \nu e^{-i\varphi})e^{-i\Omega t}] ,$$

where $\varphi = \arg\zeta$.

The operator B_M can be written in the form of the normally ordered series

$$B_M = (\gamma b^+ + \gamma^* b)^M = :(\gamma b^+ + \gamma^* b)^M:$$
$$+ X_M^1 \gamma^*\gamma :(\gamma b^+ + \gamma^* b)^{M-2}:$$
$$+ X_M^2 \gamma^{*2}\gamma^2 :(\gamma b^+ + \gamma^* b)^{M-4}: + \ldots ,$$

where

$$X_M^j = \frac{M!}{2^j j!(M-2j)!} , \quad 2j \le M; \quad X_M^j = 0, \quad 2j > M .$$

This expansion can be easily verified for small M, for example

$$B_1 = \gamma b^+ + \gamma^* b, \quad B_2 = :(\gamma b^+ + \gamma^* b)^2: + \gamma^*\gamma ,$$

$$B_3 = :(\gamma b^+ + \gamma^* b)^3: + 3\gamma^*\gamma(\gamma b^+ + \gamma^* b) \text{ etc.}$$

and for large M it can be demonstrated by induction.

Averaging B_{2N} over the squeezed state, we obtain

$$\langle B_{2N}\rangle = \langle\zeta|B_{2N}|\zeta\rangle = \sum_{j=0}^{N} X_{2N}^j (\gamma^*\gamma)^j p_1^{2(N-1)} . \qquad (19)$$

It should be mentioned that in the coherent state (that is, without squeezing), the $2N$th harmonic can be obtained only in the first term of expansion (19), since in this case $\gamma^*\gamma$ does not depend on the time ($|\mu| = 1, \nu = 0$), while the parameter p_1, which is time dependent, appears in all terms but the first in powers lower than $2N$. In the squeezed states, however, the factor $\gamma^*\gamma$ depends sinusoidally on the time, and, consequently, the $2N$th harmonic appears in all the terms of expansion (19). As we shall see later, in the squeezed vacuum state the most important term is not the first, but the last term of the expansion (19).

Let us express the quantities $\gamma^*\gamma$ and p_1 of expansion (19) in terms of the squeeze factor K. It is easy to see that

$$\mu = \frac{1}{2}\left(K + K^{-1}\right)e^{i\psi_\mu} , \quad \nu = \frac{1}{2}\left(K - K^{-1}\right)e^{-i\psi_\mu} .$$

So that

$$\gamma^*\gamma = \frac{1}{2}\left(K^2 + K^{-2}\right) - \frac{1}{4}\left(K^2 - K^{-2}\right)\left(e^{2i\Omega t + i\varphi} + e^{-2i\Omega t - i\varphi}\right) ,$$

$$p_1 = \frac{1}{2}|\zeta|[(K + K^{-1})e^{i\Omega t + i\psi'} - (K - K^{-1})e^{i\Omega t - i\psi''} + \text{c.c.}],$$

where

$$\psi' = \psi_\mu - \psi_\zeta, \quad \psi'' = \psi_\nu - \psi_\zeta, \quad \psi = \psi_\mu - \psi_\nu .$$

Consequently $\langle B_{2N}\rangle$ takes the following form:

$$\langle B_{2N}\rangle = \langle\zeta|(\gamma b^+ + \gamma^* b)^{2N}|\zeta\rangle$$

$$= \sum_{j=0}^{N} X_{2N}^j \left[\frac{1}{2}(K^2 + K^{-2}) - \frac{1}{4}(K^2 - K^{-2})(e^{2i\Omega t + i\psi} + \text{c.c.})\right]^j$$

$$\times \left(\frac{1}{2}|\zeta|\right)^{2(N-j)} [(K + K^{-1})e^{i\Omega t + i\psi'}$$

$$+ (K - K^{-1})e^{i\Omega t + i\psi''} + \text{c.c.}]^{2(N-j)} .$$

Let us separate out in this expression the terms with the frequency $2N\Omega$:

$$\langle B_{2N}\rangle_{2N\Omega} = \sum_{j=0}^{N} (-1)^j X_{2N}^j \left(\frac{1}{2}|\zeta|\right)^{2(N-j)}$$

$$\times\left\{\left[\frac{1}{4}(K^2 - K^{-2})\right]^j [(K + K^{-1})e^{-i\psi'}\right.$$

$$\left. - (K - K^{-1})e^{-i\psi''}]^{2(N-j)} e^{2iN\Omega t + ij\psi} + \text{c.c.}\right\}. \tag{20}$$

We shall consider the cases of the coherent state ($K = 1$) and the squeezed vacuum state ($K = (N+1)^{1/2} + N^{1/2}, |\zeta| = 0$). It is clear that in the first case only the first term in Eq. (20) remains:

$$\langle B_{2N}\rangle_{2N\Omega}^{\text{coh.}} = |\zeta|^{2N}(e^{2iN\Omega t + 2iN\psi'} + \text{c.c.}).$$

In the case of the squeezed vacuum state, only the last term in (19) remains:

$$\langle B_{2N}\rangle_{2N\Omega}^{\text{sq. vac.}} = (2N-1)!!\left[\frac{1}{4}(K^2 - K^{-2})\right]^N (e^{2iN\Omega t + 2iN\psi} + \text{c.c.}).$$

The quantities $|\zeta|^2$ and $\frac{1}{4}(K^2 - K^{-2})$ can be expressed in terms of the average number of photons in the cavity

$$|\zeta|^2 = n_0, \quad \frac{1}{4}\left(K^2 - K^{-2}\right) = n_0.$$

Therefore, for the same average number of photons in the cavity (in other words, for the same energy stored in the cavity) the $2N$th harmonic in the squeezed vacuum state is generated by a factor $(2N-1)!!$ more efficiently than that in the coherent state.[13]

This result may be of great significance for nonlinear optics and, in particular, for the generation of the higher harmonics. At the present time harmonics up to and including the 33rd have been obtained.[17] Consequently, the gain in efficiency of generating them may be very high. Of course, it must be remembered that the coefficients α_N fall off rapidly with increasing N. Which factor is the more important one—the advantage obtained from the squeezed states or the decrease in the coefficients α_N—is a question that can be resolved only with appropriate experiments. We recall, however, that the generation of the higher harmonics is associated with a certain expectation of obtaining coherent short-wavelength radiation up to the soft x-ray region. The use of squeezed light may revive some of these expectations.

Janszky *et al.*[14] have studied the multi-photon absorption of light in squeezed states. It is well known[18] that this absorption is proportional to $\langle a^{+N} a^N\rangle$. Those authors[14] have shown that in the squeezed vacuum state the absorption is a factor of $(2N-1)!!$ greater than in the coherent state, a result similar to that described above.

In connection with the advantage afforded by the use of squeezed light in nonlinear phenomena, for example in harmonic generation, it is necessary to say a few words about the relation between the classical and quantum phenomena. The point is that in the case of classical fields, where the field is nonmonochromatic, and consequently in the case of random, phase relations between the components is also stochastic, there is also a gain in the harmonic generation compared to monochromatic light by a factor of $m!$, where m is the index of the harmonic. Therefore it is tempting to consider squeezed light as a kind of stochastic light. This is also in accord with the terminology that has evolved, where the quantum-mechanical indeterminancy is called noise. However, this is only a superficial analogy. There is a great difference between squeezed light and stochastic light. Squeezed light can occur in the context of a single oscillator (or mode) whereas stochastic light is necessarily the superposition of several modes with various frequencies and random phases.

5.3. Squeezed light in communications systems

The use of squeezed light in communications systems has been the subject of many (until now, theoretical) publications.[15] It should be noted that squeezed light does not greatly extend the information capacity of a channel; the maximum increase is a factor of two. This is because in the usual information channels, without dividing the signal at the detector into quadrature components, the detector records only the amplitude variations. However, in systems with phase detection, the phase channel is also a carrier of information.

However, the capacity of an information channel is not the only important characteristic of a communications system. A very important factor is the probability of error, especially in computer communication lines. The advantage in the use of squeezed light in this application can be demonstrated in the example of a line in which the binary signals 0 and 1 are coded by signals that are shifted in phase by π (Fig. 11). In the case of the coherent state (Fig. 11a), the signal is described by relatively broad distributions, the overlap of which can give rise to quantum errors. In the case of squeezed light (Fig. 11b) of the same intensity the overlap of the distributions is considerably less. Calculations show that the probability of error in the coherent and squeezed state are

$$P_{\text{coh}} \approx \frac{1}{4}e^{-2\langle N\rangle}, \quad P_{\text{sq}} \approx \frac{1}{4}e^{-2\langle N\rangle(\langle N\rangle+1)},$$

where $\langle N\rangle$ is the average number of photons in the signal; that is, with squeezed light the probability of error is much lower.

6. CONCLUSIONS

The squeezed states of light, as well as many other nonclassical macroscopic states of light, are new and important objects of quantum optics. The properties of light in such states, as well as its interaction with matter, have important special features, and can have a great influence in the field of

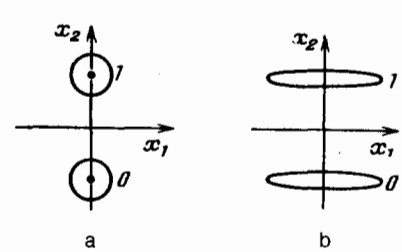

FIG. 11. Distribution of possible values of the field in the case of phase coding (a) in a coherent state, and (b) in a squeezed state.

optics. Important events would be the observation of squeezed and, in general, nonclassical states in acoustics and mechanics.

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[1)] The term "coherent state" is a generally accepted one, and I have also adopted it, although it is not a very apt term. In particular, it should be understood that a coherent state in the quantum mechanical sense may be incoherent in the classical sense.

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