# Transition radiation of a relativistic particle moving along a curve 

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## 1. INTRODUCTION

The transition radiation that arises upon motion of charges in inhomogeneous and/or non-steady-state media has been intensively studied starting in the mid-forties. ${ }^{1}$ At present it constitutes an extensive, independent field of radiation theory. ${ }^{2}$ The emission from charges when they pass through phase boundaries of different media, while they move in periodic media, the so-called resonance radiation ${ }^{3}$ when they move in media having broad-spectrum random inhomogeneities, ${ }^{2,4}$ etc., are the topic. The latter case is most interesting from the standpoint of elucidating the physical meaning of transition radiation (which in this case is also called transition scattering ${ }^{2}$ ), since in the presence of a broad spectrum of inhomogeneities of the medium having arbitrary scales, the passing particle itself selects the harmonics with which it interacts most effectively in emitting waves of frequency $\omega$ in the direction $n$.

Transition radiation can be calculated in various ways. Depending on the approach, this radiation arises either as emission from the particle under study, ${ }^{3}$ or as emission from the medium in which this particle moves. ${ }^{2}$ Actually, if we specify the permittivity of the medium as depending on the coordinate $\varepsilon_{i j}(\mathbf{r})$, and calculate the corresponding Green's function $G_{i j}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)$ of the inhomogeneous medium, then the radiation field will be determined by the current of the particle $\mathrm{j}^{\ell}$. In this case all the polarization currents of the medium are fully taken into account in the Green's function, and the resulting formulas describe both the transition radiation involving the inhomogeneities of the medium and other types of radiation in the case in which the trajectory of the particle differs from rectilinear. ${ }^{3}$ This means that the transition radiation appears here as a part of the total emission from the particle.

On the other hand, one can calculate the current (polarization) induced in the medium by the field of the relativistic particle moving in it, and directly find the radiation from this current in the medium. ${ }^{2}$ In this case the transition radiation is emitted by the medium itself, while the field of the relativistic particle plays the role of a perturbing factor.

Thus the question arises: what is emitting in transition radiation-the particle or the medium? To answer this question, we note that the propagation in the medium can be described by using the exact Green's function (which, indeed, cannot always be found in explicit form). Then the resulting emission, which takes account of the inhomogeneities of the medium via the Green's function $G_{i j}$, will amount to the emission from the particle. Moreover, the energy being emitted is drawn from the kinetic energy of the particle, rather than from the energy of the medium, which can exist in the ground state, both before and after emission.

Moreover, when the corrections to the current that arise from the inhomogeneities of the medium can be described by perturbation theory, one can separate them from the current of the particle. If the particle being studied moves rectilinearly, while the condition for Vavilov-Cherenkov radiation is not satisfied, then only the stated current in the medium will make a nonzero contribution to the emission; in this sense we can treat the transition radiation as emission from the medium itself. However, we must bear in mind the fact that this treatment of transition radiation is possible only when perturbation theory is applicable, and the emission process can be graphically represented by a Feynman diagram (Fig. 1), in which the emitting agent is an electron of the medium. The resulting transition radiation comes from coherent addition of the emissions from individual particles of the medium. ${ }^{6}$

The following sections of this article will discuss the formation of transition radiation by a relativistic particle moving along a curvilinear trajectory in a randomly inhomogeneous medium. Since in this case we are interested in frequencies considerably exceeding the plasma frequency $\omega_{p}$, we shall use a very simple plasma formula to describe the dielectric properties of the medium

$$
\begin{equation*}
\varepsilon(\omega)=1-\frac{\omega_{p}^{2}}{\omega^{2}} \tag{1}
\end{equation*}
$$

and call such a medium a plasma.
Section 2 describes the procedure of perturbation theory applied for calculating the nonlinear plasma current in the case of arbitrary curvilinear motion of the particle, and discusses the defects of the method of the reaction of the mean field. ${ }^{7}$ After this, examples are discussed that demonstrate the role of the curvature of the particle trajectory in different cases. Thus, in Sec. 3 the transition radiation is calculated for a particle moving along a helix in a magnetic field, and in Sec. 4, the transition radiation of a particle per-


FIG. 1. Feynman diagram for transition radiation.
forming random walks in a medium owing to Coulomb collisions, or owing to scattering by random magnetic and electric inhomogeneities, which are always present in a randomly inhomogeneous plasma. Section 5 presents some final clarifying remarks.

## 2. GENERAL THEORY OF TRANSITION RADIATION OF ACCELERATED PARTICLES

Let us proceed to derive the general formulas that describe the transition radiation of a relativistic particle moving along an arbitrary trajectory. If we know the exact Green's function that describes the propagation of a photon in the given medium, then the energy of the radiation associated with the current of the external relativistic particle $\mathbf{j}_{\omega, \mathbf{k}}^{Q}$ is: ${ }^{8}$

$$
\begin{equation*}
\mathscr{E}_{\mathbf{n}, \omega}=(2 \pi)^{4} \int \mathbf{j}_{\omega, \mathbf{k}}^{*} \mathbf{E}_{\omega, \mathbf{k}} k^{2} \mathrm{~d} k, \tag{2}
\end{equation*}
$$

where $\mathbf{E}_{\omega, \mathbf{k}}$ is the field of the current $\mathbf{j}_{\omega, \mathbf{k}}^{Q}$ in the medium. Here Eq. (2) contains all types of radiation, in particular, also those not associated with the variable polarization of the medium, but arising from the curvature of the trajectory of the particle. Since, for media with random density inhomogeneities, the exact Green's function is not known, we must use the procedure of approximate calculation of the transition radiation based on perturbation theory.

Let us represent the electric field in the form of a power series in the amplitude of the density inhomogeneities of the plasma

$$
\begin{equation*}
\mathbf{E}_{\omega, \mathbf{k}}=\mathbf{E}_{\omega, \mathbf{k}}^{(0)}+\mathbf{E}_{\omega, \mathbf{k}}^{(1)}+\mathbf{E}_{\omega, \mathbf{k}}^{(2)} \ldots \tag{3}
\end{equation*}
$$

This corresponds to expansion of the current in the medium

$$
\begin{equation*}
\mathbf{j}_{\omega, \mathbf{k}}=\mathbf{j}_{\omega, \mathbf{k}}^{(0)}+\mathbf{j}_{\mathbf{j}_{\omega, \mathbf{k}}^{(1)}}^{(1)}+\mathbf{j}_{\omega, \mathbf{k}}^{(2)}+\cdots \tag{4}
\end{equation*}
$$

Then we can rewrite the expression for the total emitted energy (2) in the form:

$$
\begin{align*}
\mathscr{e r}_{n, \omega}= & (2 \pi)^{4} \operatorname{Re} \int\left(\mathbf{j}_{\omega, \mathbf{k}}^{*} \mathbf{E}_{\omega, \mathbf{k}}^{(0)}+\left\langle j_{\omega, \mathbf{k}}^{*(1)} \mathbf{E}_{\omega, \mathbf{k}}^{(1)}\right\rangle\right. \\
& \left.+\mathbf{j}_{\omega, \mathbf{k}}^{*(Q}\left\langle\mathbf{E}_{\omega, \mathbf{k}}^{(2)}\right\rangle+\left\langle\mathbf{j}_{\omega, \mathbf{k}}^{*(2)}\right\rangle \mathbf{E}_{\omega, \mathbf{k}}^{(0)}\right) k^{2} \mathrm{~d} k . \tag{5}
\end{align*}
$$

The first term in the parentheses in Eq. (5) describes the radiation involving the curvature of the trajectory of the particle or the Vavilov-Cherenkov radiation, while the transition radiation is contained in the subsequent terms, which are quadratic in the amplitude of the inhomogeneities $\delta N_{\omega, \mathbf{k}}$. The angle brackets denote averaging over the spectrum of the inhomogeneities, whereby the terms linear in $\delta N_{\omega, \mathbf{k}}$ vanish. Since $\mathbf{j}_{\omega, \mathbf{k}}^{*}$, and $\mathbf{E}_{\omega, \mathbf{k}}^{(0)}$ do not contain $\delta N_{\omega, \mathbf{k}}$, they are removed outside the averaging in (5).

Now let us proceed to find the fields and currents that enter into (5). Upon eliminating the magnetic field from the two vector equations of Maxwell, ${ }^{2}$ we obtain the relationship between the transverse electric field and the current that gives rise to it:

$$
\begin{equation*}
\left(c^{2} k^{2}-\omega^{2}\right) E_{\omega, \mathbf{k}}^{i}=4 \pi i \omega\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right) j_{\omega, \mathbf{k}}^{j} . \tag{6}
\end{equation*}
$$

Here we start from the microscopical equations of the electromagnetic field, while the current $j_{\omega, \mathbf{k}}^{j}$ contains within it both the current of the particle $j_{\omega, \mathrm{k}}^{\varrho}$ and the current involving the motion of the particles of the medium, in particular, the ordinary polarization current described by the permittivity
of the plasma of (1). Upon substituting (3) and (4) into (6), we obtain the following equation:

$$
\begin{align*}
& \left(c^{2} t^{2}-\omega^{2}\right)\left(E_{\omega, \mathbf{k}}^{(0) i}+E_{\omega, \mathbf{k}}^{(1) i}+E_{\omega, \mathbf{k}}^{(2) i}+\ldots\right) \\
& \quad=4 \pi i \omega\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right)\left(j_{\omega, \mathbf{k}}^{(0) j}+j_{\omega, \mathbf{k}}^{(1) j}+j_{\omega, \mathbf{k}}^{(2) j}+\ldots\right) . \tag{7}
\end{align*}
$$

Here we must perform an iteration procedure, i.e., interrelate the fields and currents of a given order. Then $E_{\omega, k}^{(0) i}$ will be determined by the current $j_{\omega, k}^{(0) j}$, the field $E_{\omega, \mathbf{k}}^{(0) i}$ by the current $j_{\omega, k}^{(0) j}$. The latter, in turn, is expressed from the kinetic equation in terms of the field $E_{\omega, \mathbf{k}}^{(0) i}$, etc. Thus we have a closed procedure that enables us to seek any of the corrections to the fields and currents.

It is important to note that here the denominator of the right-hand side of (7) repeatedly contains the difference ( $c^{2} k^{2}-\omega^{2}$ ) or ( $c^{2} k^{2}-\omega^{2} \varepsilon$ ) in taking account of the polarization current of the unperturbed plasma. For waves that propagate in the medium, i.e., ordinary transverse electromagnetic waves, this difference vanishes, which leads to divergent expressions for $E_{\omega, \mathbf{k}}^{(1) i}, E_{\omega, \mathbf{k}}^{(2) i}$, etc. Therefore, to make the procedure of perturbation theory valid, we must apply it only for the virtual component of the field, for which ( $c^{2} k^{2}-\omega^{2} \varepsilon$ ) does not vanish. Such a difficulty did not arise in studying the transition radiation of a rectilinearly moving particle, ${ }^{2}$ since the entire field of such a particle is virtual (apart from the cases in which the condition for VavilovCherenkov radiation is satisfied). Yet if the particle moves along a curve (e.g., along a helix), the remark that we have made proves to be essential, since now the field of the particle consists of two components-the radiation field (e.g., synchrotron radiation) and the virtual field proper, to which perturbation theory is applicable.

To find the currents that arise in a plasma with random inhomogeneities as a relativistic particle moves in it, we shall use the kinetic equation for a cold, collision-free plasma ${ }^{2}$

$$
\begin{equation*}
f_{\omega, k}(\mathbf{p})=\frac{e}{i \omega} \int \mathbf{E}_{\omega-\omega^{\prime}, \mathbf{k}-\mathbf{k}^{\prime}} \frac{\partial}{\partial \mathbf{p}} f_{\omega^{\prime}, \mathbf{k}^{\prime}}(\mathbf{p}) \mathrm{d}\left(\omega^{\prime} \mathbf{d} \mathbf{k}^{\prime} .\right. \tag{8}
\end{equation*}
$$

Here $e$ is the charge of an electron. Upon solving this equation by perturbation theory and taking account of the expansion of the electric field in (3), we find the corrections to the unperturbed distribution function

$$
\begin{align*}
& f_{\omega, \mathbf{k}}^{(0)}(\mathbf{p})=f(\mathbf{p}) \delta(\omega) \delta(\mathbf{k}),  \tag{9}\\
& f_{\omega, \mathbf{k}}^{(1)}=\frac{e \mathbf{E}_{\omega, \mathbf{k}}^{(0)}}{i \omega} \frac{\partial f(p)}{\partial \mathbf{p}}+\delta f_{\omega, \mathbf{k}}(\mathbf{p}) . \tag{10}
\end{align*}
$$

Here $f(\mathbf{p})$ is normalized to the particle concentration $N_{0}$, while the increment $\delta f_{\omega, \mathbf{k}}$ describes the inhomogeneities of the electron density of the plasma:

$$
\begin{equation*}
\int f(\mathbf{p}) \frac{d \mathbf{p}}{(2 \pi)^{3}}=N_{0}, \quad \int \delta f_{\omega, \mathbf{k}}(\mathbf{p}) \frac{d \mathbf{p}}{(2 \pi)^{3}}=\delta N_{\omega, \mathbf{k}} . \tag{11}
\end{equation*}
$$

In the following orders we have

$$
\begin{align*}
f_{\omega, \mathbf{k}}^{(2)}= & \frac{e \mathbf{E}_{\omega, \mathbf{k}}^{(\mathrm{k})}}{i \omega} \frac{\partial f(p)}{\partial \mathbf{p}}+\frac{e}{i \omega}-\int \mathbf{E}_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{\mathbf{1}}}^{(0)} \frac{\partial}{\partial \mathbf{p}} \delta f_{\omega_{1}, \mathbf{k}_{1}} \mathrm{~d} \omega_{1} \mathrm{~d} \mathbf{k}_{1} \\
& +\frac{e}{i \omega} \int \mathbf{E}_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0)} \frac{\partial}{\partial \mathbf{p}} \frac{e \mathbf{E}_{\omega_{1}, \mathbf{k}_{1}}^{(0)}}{i \omega_{\mathbf{1}}} \frac{\partial f}{\partial \mathbf{p}} \mathrm{~d} \omega_{1} \mathrm{~d} \mathbf{k}_{\mathbf{1}},  \tag{12}\\
f_{\omega \cdot \mathbf{k}}^{(3)}= & \frac{e E_{\omega \cdot \mathbf{k}}^{(2)}}{i \omega} \frac{\partial f}{\partial \mathbf{p}}+\frac{e}{i \omega} \int \mathbf{E}_{\omega-\omega_{2}, \mathbf{k}-\mathbf{k}_{\mathbf{z}}}^{(1)} \frac{\partial}{\partial \mathbf{p}} f_{\omega_{2}, \mathbf{k}_{2}}^{(1)} \mathrm{d} \omega_{2} \mathrm{~d} \mathbf{k}_{2} \\
& +\frac{e}{i \omega} \int \mathbf{E}_{\omega-\omega_{2}, \mathbf{k}-\mathbf{k}_{\mathbf{z}}}^{(0)} \frac{\partial}{\partial \mathbf{p}} f_{\omega_{2}, \mathbf{k}_{\mathbf{2}}}^{(2)} \mathrm{d} \omega_{2} \mathrm{~d} \mathbf{k}_{2} . \tag{13}
\end{align*}
$$

Equations (10)-(13) enable us to calculate the plasma currents of interest to us according to the formula

$$
\begin{equation*}
\mathbf{j}_{\omega, \mathbf{k}}=e \int \mathbf{v} f_{\omega, \mathbf{k}}(\mathbf{p}) \frac{\mathrm{d} \mathbf{p}}{(2 \pi)^{3}} . \tag{14}
\end{equation*}
$$

Upon substituting (10), (12), and (13) into (14) and calculating the integrals, part of which vanish, we find the following expressions for the currents in the plasma:

$$
\begin{align*}
& \mathbf{j}_{\omega, \mathbf{k}}^{(0)}= \mathbf{j}_{\omega, \mathbf{k}}^{Q}+\frac{i e^{2} N_{0}}{m \omega} \mathbf{E}_{\omega, \mathbf{k}}^{(0)}(\theta(\omega-\mathbf{k})+\theta(\mathbf{k} \mathbf{v}-\omega)), \\
& \mathbf{j}_{\omega, \mathbf{k}}^{(1)}= \frac{i e^{2} N_{0}}{m \omega} \mathbf{E}_{\omega, \mathbf{k}}^{(1)} \\
&+\frac{i e^{2}}{m \omega} \int \mathbf{E}_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0)} \delta N_{\omega_{1}, \mathbf{k}_{\mathbf{1}}} \theta\left(\omega-\omega_{1}-\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}\right) \\
& \times \mathrm{d} \omega_{1} \mathrm{~d} \mathbf{k}_{1} \\
&+\frac{i e^{2}}{m \omega} \int \mathbf{E}_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0)} \delta N_{\omega_{1}, \mathbf{k},} \theta\left(\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}-\left(\omega-\omega_{1}\right)\right) \mathrm{d} \omega_{1} \mathrm{~d} \mathbf{k}_{\mathbf{1}},  \tag{16}\\
&(16  \tag{17}\\
& \mathbf{j}_{\omega, \mathbf{k}}^{(2)}= \frac{i e^{2} N_{0}}{m \omega} \mathbf{E}_{\omega, \mathbf{k}}^{(2)}+\frac{i e^{2}}{m \omega} \int \mathbf{E}_{\omega-\omega_{2}, \mathbf{k}-\mathbf{k}_{2}}^{(1)} \delta N_{\omega_{2}, \mathbf{k}_{2}} \mathrm{~d} \omega_{2} \mathrm{~d} \mathbf{k}_{2} .
\end{align*}
$$

In Eq. (15) the electric field is divided into two parts: $\mathbf{E}_{\omega, \mathbf{k}}^{(0)} \theta(\omega-\mathbf{k v})$ and $\mathbf{E}_{\omega, \mathbf{k}}^{(0)} \theta(\mathbf{k v}-\omega)$, where $\theta(x)$ is the theta function of Heaviside, and $m$ is the mass of an electron. Since $\theta(x)+\theta(-x) \equiv 1$, this separation does not violate the validity of (15). At the same time, it allows us to take separate account in (7) of the real $(\theta(\omega-\mathbf{k v})$ ) and the virtual $(\theta(\mathbf{k v}-\omega))$ fields of the particle. Let us study the integral

$$
\mathbf{J}_{\mathrm{r}}=\int \mathbf{E}_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0)} \delta N_{\omega_{1}, \mathbf{k}_{1}} \theta\left(\omega-\omega_{\mathbf{1}}-\left(\mathbf{k}-\mathbf{k}_{\mathbf{1}}\right) \mathbf{v}\right) \mathrm{d} \omega_{1} \mathrm{~d} \mathbf{k}_{1}
$$

Since it contains the real radiation field, whose wavelength is much smaller than the characteristic dimensions of the inhomogeneities of the medium $|\mathbf{k}| \gg\left|\mathbf{k}_{1}\right|$, if we are not interested in effects of the type of Mandel'shtam-Brillouin scattering, we can obtain approximately

$$
\begin{align*}
\mathbf{J}_{r} & \approx \mathbf{E}_{\omega, \mathbf{k}}^{(0)} \int \delta N_{\omega_{1}, \mathbf{k}_{1}} \theta\left(\omega-\omega_{1}-\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}\right) \mathrm{d} \omega_{1} \mathrm{~d} \mathbf{k}_{1} \\
& =\mathbf{E}_{\omega, \mathbf{k}}^{(0)} \Delta N . \tag{18}
\end{align*}
$$

Then the expression (16) for the current $\mathbf{j}_{\omega, \mathbf{k}}^{(1)}$ is transformed as follows:
$\mathbf{j}_{\omega, \mathbf{k}}^{(1)}=\frac{i e^{2} N_{0}}{m \omega} \mathbf{E}_{\omega, \mathbf{k}}^{(1)}+\frac{i e^{2} \Delta N}{m \omega} \mathbf{E}_{\omega, \mathbf{k}}^{(0)}$
$+\frac{i e^{\mathbf{2}}}{m \omega} \int \mathbf{E}_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0)} \delta N_{\omega_{1}, \mathbf{k}_{1}} \theta\left(\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}-\left(\omega-\omega_{1}\right)\right) d \omega_{\mathbf{1}} \mathbf{d} \mathbf{k}_{1}$.

We note that in the third term of (19), which contains the virtual field, we must not ignore the shift of the arguments $\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}$, since the nonvanishing of this term is due exclusively to this variation in the arguments $\omega$ and $\mathbf{k}$. Let us substitute the obtained expressions for the currents (15), (17), and (19) into Eq. (7). Then we have
$\left(c^{2} k^{2}-\omega^{2}\right)\left(E_{\omega, \mathbf{k}}^{(0) i}+E_{\omega, \mathbf{k}}^{(1) i}+E_{\omega, \mathbf{k}}^{(2) i}+\ldots\right)=4 \pi i \omega\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right)$

$$
\times\left(j_{\omega, \mathbf{k}}^{Q j}+\frac{i e^{2}\left(N_{0}+\Delta N\right)}{m \omega} E_{\omega, \mathbf{k}}^{(0) j}+\frac{i e^{2} N_{0}}{m \omega} E_{\omega, \mathbf{k}}^{(1), j}+\frac{i e^{2} N_{0}}{m \omega} E_{\omega, \mathbf{k}}^{(2) j}\right.
$$

$$
+\frac{i e^{2}}{m \omega} \int E_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0) j} \delta N_{\omega_{1}, \mathbf{k}_{1}} \theta\left(\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}-\left(\omega-\omega_{1}\right)\right) \mathrm{d} \omega_{\mathbf{1}} \mathrm{d} \mathbf{k}_{1} .
$$

$$
\begin{equation*}
\left.+\frac{i e^{2}}{m \omega} \int E_{\omega-\omega_{\varepsilon}, \mathbf{k}-\mathbf{k}_{2}}^{(1) j} \delta N_{\omega_{2}, \mathbf{k}_{\mathbf{2}}} \mathrm{d} \omega_{2} \mathrm{~d} \mathbf{k}_{\mathbf{2}}+\ldots\right) . \tag{20}
\end{equation*}
$$

Let us take up the analysis of this expression in greater detail. Terms of the type ( $i e^{2} N / m \omega$ ) $E_{\omega, \mathrm{k}}^{j}$, which are linear
in the electric field, describe the polarization of the unperturbed plasma and lead to a change in the dispersion law of waves in the medium. Actually, if we transfer the stated terms to the left-hand side of (20), then instead of $\left(c^{2} k^{2}-\omega^{2}\right)$, we obtain ( $c^{2} k^{2}-\omega^{2} \varepsilon$ ), where $\varepsilon(\omega)$ is the permittivity of the cold isotropic plasma. In a magnetoactive plasma, as is known, Eq. (1) is valid under the condition

$$
\begin{equation*}
\frac{\omega_{\mathrm{p}}}{\omega_{\mathrm{Bc}}} \gg 1, \tag{21}
\end{equation*}
$$

which we shall assume below to be fulfilled. In the zeroorder term we can neglect the quantity $\Delta N$ in comparison with $N_{0}$, since taking it into account when $\Delta N \ll N_{0}$ does not lead to qualitatively new effects. Finally we note that, after the electric field has been separated into real and virtual components, this separation occurs automatically in the subsequent terms. Therefore, the last term of (20) contains simply the electric field $E_{\omega, k}^{(1) j}$.

After performing the cited transformations of (20), we shall write the expression for the electric field in the different orders of perturbation theory:
$\left.E_{\omega, \mathbf{k}}^{(0) i}=G_{i j}^{t}(\omega, \mathbf{k})\right)_{\omega, \mathbf{k}}^{Q j}$,
$E_{\omega, \mathbf{k}}^{(\alpha) i}=G_{i j}^{t}(\omega, \mathbf{k}) j_{\omega, \mathbf{k}}^{(\alpha) j} \quad(\alpha=1,2)$,
$j_{\omega, \mathbf{k}}^{(1)}$
$=\frac{i e^{2}}{m \omega} \int E_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(0) j} \delta N_{\omega_{1}, \mathbf{k}_{1}} \theta\left(\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}-\left(\omega-\omega_{1}\right)\right) \mathrm{d} \omega_{1} \mathrm{dk}_{1}$,
$j_{\omega, \mathbf{k}}^{(2) j}=\frac{i e^{\mathbf{2}}}{m \omega} \int E_{\omega-\omega_{1}, \mathbf{k}-\mathbf{k}_{1}}^{(1) j} \delta N_{\omega_{2}, \mathbf{k}_{2}} \mathrm{~d} \omega_{2} \mathrm{~d} \mathbf{k}_{2}$,
where

$$
\begin{equation*}
G_{i j}^{t}(\omega, \mathbf{k})=\frac{4 \pi i \omega}{c^{2} k^{2}-\omega^{2} \varepsilon}\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right) \tag{26}
\end{equation*}
$$

is the transverse Green's function in the medium.
Let us first study the last ("cross") terms in (5). To do this we shall calculate $\left\langle E_{\omega, \mathbf{k}}^{(2) i}\right\rangle$, which enters into (5). Upon using (23)-(26), we find:

$$
\begin{align*}
& \left\langle E_{\omega, \mathbf{k}}^{(2) i}\right\rangle=\left(\frac{4 \pi e^{2}}{m}\right)^{2} \frac{\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right)}{c^{2} k^{2}-\omega^{2} \varepsilon} \\
& \times \int \frac{\left[\delta_{j l}-\frac{\left(\mathbf{k}-\mathbf{k}_{2}\right)_{j}\left(\mathbf{k}-k_{2}\right)_{l}}{\left(\mathbf{k}-\mathbf{k}_{2}\right)^{2}}\right]\left\langle\delta N_{\omega_{1}, \mathbf{k}_{1}} \delta N_{\omega_{2}, \mathbf{k}_{2}}\right\rangle}{c^{2}\left(\mathbf{k}-k_{\mathbf{2}}\right)^{2}-\left(\omega-\omega_{2}\right)^{2} \varepsilon} E_{\omega-\omega_{2}-\omega_{\mathbf{2}}, \mathbf{k}-\mathbf{k}_{\mathbf{k}}-\mathbf{k}_{\mathbf{2}}}^{(0) l} \\
& \quad \times \theta\left(\left(\mathbf{k}-\mathbf{k}_{\mathbf{1}}-\mathbf{k}_{2}\right) \mathbf{v}-\left(\omega-\omega_{1}-\omega_{2}\right)\right) \mathrm{d} \omega_{1} \mathrm{~d} \omega_{2} \mathrm{~d} \mathbf{k}_{\mathbf{1}} \mathrm{d} \mathbf{k}_{\mathbf{2}} . \tag{27}
\end{align*}
$$

To describe the correlators of the density inhomogeneities we shall use the random-phase approximation:

$$
\begin{align*}
& \left\langle\delta N_{\omega_{1}, \mathbf{k}_{1}} \delta N_{\omega_{2}, \mathbf{k}_{2}}\right\rangle=|\delta N|_{\omega_{1}, \mathbf{k}_{1}}^{2} \delta\left(\omega_{1}+\omega_{2}\right) \delta\left(\mathbf{k}_{1}+\mathbf{k}_{2}\right),  \tag{28}\\
& \left\langle\delta N_{\omega_{1}, k_{1}} \delta N_{\omega_{2}, \mathbf{k}_{2}}^{*}\right\rangle=|\delta N|_{\omega_{1}, \mathbf{k}_{1}}^{2} \delta\left(\omega_{1}-\omega_{2}\right) \delta\left(\mathbf{k}_{1}-\mathbf{k}_{2}\right) . \tag{29}
\end{align*}
$$

Upon substituting (28) into (27) and integrating over $\mathrm{d} \omega_{2} \mathrm{~d} \mathbf{k}_{2}$ with account taken of the $\delta$-function, we obtain

$$
\begin{align*}
& \left\langle E_{\omega, \mathbf{k}}^{(2) i}\right\rangle=\left(\frac{4 \pi e^{2}}{m}\right)^{2} E_{\omega, \mathbf{k}}^{(0) l} \frac{\theta(\mathbf{k} \mathbf{k}-\omega)}{c^{2} k^{2}-\omega^{2} \varepsilon}\left(\delta_{i j}-\frac{k_{1} k_{j}}{k^{2}}\right) \\
& \quad \times \int\left[\delta_{j l}-\frac{\left(\mathbf{k}+\mathbf{k}_{1}\right)_{j}\left(\mathbf{k}+\mathbf{k}_{\mathbf{1}}\right)_{l}}{\left(\mathbf{k}+\mathbf{k}_{\mathbf{1}}\right)^{2}}\right] \frac{|\delta N|_{\omega_{i}, \mathbf{k}_{1}}^{2} \mathrm{~d} \omega_{\mathbf{j}} \mathrm{d} \mathfrak{l}_{\mathbf{l}}}{c^{2}\left(\mathbf{k}+\mathbf{k}_{1}\right)^{2}-\left(\omega+\omega_{1}\right)^{2} \mathrm{e}} . \tag{30}
\end{align*}
$$

We see that the mean field in (30) contains as a coefficient the $\theta$-function $\theta(\mathbf{k v}-\omega)$, which vanishes for $\omega$ and $\mathbf{k}$
related by the dispersion law in the plasma. Thus we have $\left\langle\mathbf{E}_{\omega, \mathbf{k}}^{(2)}\right\rangle=0$, and for the same reason $\left\langle\mathbf{j}_{\omega, \mathbf{k}}^{(2)}\right\rangle=0$. Hence the terms $\mathbf{j}_{\omega, \mathbf{k}}^{Q}\left\langle\mathbf{E}_{\omega, \mathbf{k}}^{(2)}\right\rangle$ and $\left\langle\mathbf{j}_{\omega, \mathbf{k}}^{(2)}\right\rangle \mathbf{E}_{\omega, \mathbf{k}}^{(0)}$ in (5) do not contribute to the intensity of the emission from the relativistic particle in the medium. Since we can treat the stated terms as interference terms, we arrive at the conclusion that interference is absent between the "intrinsic" radiation of the particle (the first term in (5)) and its transition radiation, which now involves only the second term in (5).

Upon integrating the second term in (5) over $\mathrm{d} k$, we represent the energy of the transition radiation in the form ${ }^{4,8}$

$$
\begin{equation*}
\left.\mathscr{E}_{\mathbf{n}, \omega}^{\mathrm{m}}=\left.(2 \pi)^{\boldsymbol{\theta}} \frac{\omega^{\mathbf{2}}}{c^{s}}\langle |\left[\mathbf{n} \mathbf{j}_{\omega, \mathbf{k}}^{(1)}\right]\right|^{2}\right\rangle \tag{31}
\end{equation*}
$$

where $\mathbf{n}$ is a radiation direction, and $\mathbf{j}_{\omega, \mathbf{k}}^{(1)}$ current is determined by Eq. (24). $E_{\omega, \mathbf{k}}^{(0) i}$ electric field is connected via Eq. (22) with $\mathbf{j}_{\omega, \mathbf{k}}^{Q}$ relativistic particle current, which can be expressed via its $\mathbf{v}(t)$ velocity and $\mathbf{r}(t)$ trajectory:

$$
\begin{equation*}
\mathbf{j}_{\omega, \mathbf{k}}^{Q}=Q \int_{-\infty}^{\infty} \mathbf{v}(t) \exp (-i \mathbf{k r}(t)+i \omega t) \frac{\mathrm{d} t}{(2 \pi)^{4}} \tag{32}
\end{equation*}
$$

where $Q$ is the charge of the relativistic particle.
Upon substituting (32) and (26) into (22), (22) into (24), and (24) into (31), and averaging the obtained expression over the random phases by using (29), we find that

$$
\begin{align*}
\mathscr{E}_{\mathbf{n}, \boldsymbol{\omega}}^{\mathrm{m}} & =\frac{8 Q^{2} e^{\mathbf{4}}}{m^{2} c^{3} \omega^{2}} \int_{V} \mathrm{~d} \mathbf{k}_{\mathbf{1}} \int_{-\infty}^{\infty} \mathrm{d} t \operatorname{Re} \int_{0}^{\infty} \mathrm{d} \tau \\
& \times \frac{|\delta N|_{\mathbf{k}_{\mathbf{1}}}^{2} \exp \left[i \omega \tau-i\left(\mathbf{k}-\mathrm{h}_{1}\right)(\mathbf{r}(t+\tau)-\mathbf{r}(t))\right]}{\left[1-\frac{\left.\left(\mathbf{k}-\mathbf{k}_{\mathbf{1}}\right)^{2} c^{2}\right]^{2}}{\omega^{2} \varepsilon(\omega)}\right]^{2}}\{[\mathbf{n v}(t)] \\
& \left.+\frac{\left[\mathbf{n k _ { 1 } ] ( \mathbf { k } - \mathbf { k } _ { \mathfrak { y } } ) \mathbf { v } ( t )}\right.}{\left(\mathbf{k}-\mathbf{k}_{\mathbf{1}}\right)^{2}}\right\} \\
& \times\left\{[\mathbf{n}, \mathbf{v}(t+\tau)]+\frac{\left[\mathbf{n k}_{\mathbf{1}}\right]\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}(t+\tau)}{\left(\mathbf{k}-\mathbf{h}_{1}\right)^{2}}\right\} \tag{33}
\end{align*}
$$

Here we have used the assumption that the inhomogeneities are quasistatic, $|\delta N|_{\omega, \mathrm{k}}^{2}=|\delta N|_{\mathbf{k}}^{2} \delta(\omega)$, while the sign of $\mathbf{v}$ implies that we must perform the integration with respect to $\mathrm{d} \mathbf{k}_{1}$ over the region of parameters $\omega-\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v} \leqslant 0$, that corresponds to the virtual electric field of the particle. This expression was applied in Ref. 2 for the case of a rectilinearly moving particle (see also the references in Ref. 2). We shall return to analyzing (33) in the other sections, where we shall study the effect of the curvature of the trajectory of the particle in a magnetic field (Sec. 3) and in a scattering medium (Sec. 4) on the formation of transition radiation. But here we shall discuss another method of calculating transition radiation-the so-called method of the reaction of the mean field. ${ }^{7}$ The difference of this method from that presented above consists in the following. In studying Eq. (7), the separation of the field into real and virtual fields is not performed. Then, after substituting the corresponding currents into (7) and averaging over the spectrum of inhomogeneities, it is not complicated instead of (20) to derive

$$
\begin{align*}
& \left(c^{2} k^{2}-\omega^{2} \varepsilon\right)\left(E_{\omega, \mathbf{k}}^{(0) i}+\left\langle E_{\omega, \mathbf{k}}^{(2) i}\right\rangle\right) \\
& \quad=4 \pi i \omega\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right) j_{\omega, \mathbf{k}}^{Q j}+E_{\omega, \mathbf{k}}^{(0) i}\left(\frac{4 \pi e^{2}}{m c}\right)^{2} \int \frac{|\delta N|_{\mathbf{k}_{1}}^{2} d \mathbf{k}_{1}}{k_{\mathbf{1}}^{2}-2 \star \mathbf{k}_{1}} . \tag{34}
\end{align*}
$$

Of course, owing to the averaging, all the terms have
dropped out of (34) that are linear in the amplitude of the inhomogeneities.

The last term in (34) differs from (30) only in that it does not contain the factor $\theta(\mathbf{k v}-\omega)$ that makes (30) vanish. Therefore, in this approach $\left\langle E_{\omega, \mathbf{k}}^{(2) i}\right\rangle$ differs from zero, and moreover, the calculation of the radiation intensity of the type

$$
\int \mathbf{j}_{\omega, \mathbf{k}}^{* Q}\left\langle\mathbf{E}_{\omega, \mathbf{k}}^{(2)}\right\rangle k^{2} \mathrm{~d} k
$$

leads to a divergent result, in agreement with what we said at the beginning of this section.

In the method being discussed this difficulty is overcome as follows. The replacement is performed in the last term of (34):

$$
\begin{equation*}
E_{\omega, \mathbf{k}}^{(0) i} \rightarrow E_{\circlearrowleft, \mathbf{k}}^{(0) i}+\left\langle E_{\omega, \mathbf{k}}^{(2) i}\right\rangle \cong\left\langle E_{\omega, \mathbf{k}}^{i}\right\rangle \tag{35}
\end{equation*}
$$

This small change in Eq. (34) fundamentally changes its structure. Actually, now by regrouping the terms we can reduce (34) to a form that coincides with the analogous equation for $\mathbf{E}_{\omega, \mathbf{k}}$ in a homogeneous plasma, whose polarization properties are described by the so-called effective permittivity:

$$
\begin{equation*}
\varepsilon^{\mathrm{eff}}=\varepsilon-\left(\frac{4 \pi e^{2}}{m \omega}\right)^{2} \frac{\left\langle\Delta N^{2}\right\rangle}{4}-\frac{i \pi^{2}}{\omega c}\left(\frac{4 \pi e^{2}}{m}\right)^{2} \int_{0}^{\infty}|\delta N|_{k_{1}}^{2} k_{1} d k_{1} \tag{36}
\end{equation*}
$$

Here the imaginary component arose from using the rule of passing around the pole in the integral in (34). The existence of the imaginary component in $\varepsilon^{\text {eff }}$ enables us to calculate the radiation by the known formulas for the energy losses of a particle in an absorbing medium ${ }^{8}$ having the permittivity of (36). However, the substitution (35) is ill-grounded. The point is that, to introduce the permittivity of (36), one would have to write out the corrections to the field at least to the fourth order (strictly speaking, one must consider all orders of perturbation theory). As a result fourth-order density correlators would arise, which in the general case do not reduce to the product of two second-order correlators. Thus the substitution (35) is an overshoot of exactness that takes account of only a fraction of the fourth-order terms. We note that the need to take account of the higher-order correlators to substantiate the method being discussed has already been pointed out in Ref. 2.

In closing, let us formulate the fundamental defects that have been noted in the method of the reaction of the mean field. ${ }^{7}$ First, perturbation theory is applied to the total field, which includes in the general case both the real and the virtual field. As we have already noted, this is incorrect. Second, it is not the quadratic quantity of (5) that is averaged, but the field itself of (34), whereby the terms of the type $\left\langle\mathbf{j}_{\omega, \mathbf{k}}^{(1) *} \mathbf{E}_{\omega, \mathbf{k}}^{(1)}\right\rangle$ drop out. Third, an overshoot of exactness was allowed in forming the permittivity of (36), owing to the substitution (35).

As we see it, the mentioned defects do not allow one to consider the results of calculation of transition radiation by the method of the reaction of the mean field as being reliable.

## 3. TRANSITION RADIATION OF A PARTICLE MOVING ALONG A HELIX

Now let us proceed to analyzing concrete situations in which the particle generating the transition radiation moves
along a curvilinear trajectory. To understand better the effects that arise here, we shall first examine qualitatively the role of the curvature of the trajectory.

To do this, we note that the transition radiation is formed over some finite length called the coherence length or the radiation-formation zone, ${ }^{2}$ which has the following form in a plasma for a rectilinearly moving particle:

$$
\begin{equation*}
l_{\mathrm{c}}=\frac{2 c}{\omega} \gamma^{2}\left(1+\frac{\omega_{\mathrm{p}}^{2} \gamma^{2}}{\omega^{2}}\right)^{-1} \tag{37}
\end{equation*}
$$

Here $\gamma=\mathscr{E} / M c^{2}$ is the Lorentz factor of the particle, $\mathscr{E}$ and $M$ are its energy and mass, and $\omega_{\mathrm{p}}$ is the Langmuir frequency. The fundamental fraction of the energy emitted at the frequency $\omega$ is emitted in a narrow cone along the velocity of the particle, with the opening angle

$$
\begin{equation*}
\theta_{\mathrm{c}} \approx\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right)^{1 / 2} \tag{38}
\end{equation*}
$$

In the presence in the plasma of density inhomogeneities with a broad spectrum, the radiation at the frequency $\omega$ arises from the interaction of the field of the relativistic particle with those harmonics for which $\tilde{k}^{-1} \sim l_{\mathrm{c}}(\omega)$ in (37). ${ }^{2}$ Let the curvature of the trajectory be such that the direction of the velocity of the particle varies by the angle $\theta_{\mathrm{c}}$ of (38) over a certain length $l^{\prime}<l(\omega)$. Then the narrow directionality of the radiation causes the spectrum of the emitted transition quanta to be formed over the short length $l^{\prime}$. In turn this implies that the inhomogeneities with scales $\tilde{k}^{-1} \sim l_{\mathrm{c}}(\omega)>l^{\prime}$ with which the transition radiation at the frequency $\omega$ is associated will interact less effectively with the field of the relativistic particle, and the corresponding radiation will prove to be strongly attenuated.

If the curvature of the trajectory of the particle arises from the homogeneous magnetic field $B$, then we have $l^{\prime}=l_{\mathrm{s}}=M c^{2} / Q B_{\perp}=c / \omega_{\mathrm{B}_{1}}$, where $\omega_{\mathrm{B}_{1}}=Q B_{\perp} / M c$ is the gyrofrequency of the particle, while the maximum value of $l_{c}(\omega)$ is attained when $\omega=\omega_{\mathrm{p}} \gamma ; l_{\mathrm{c}}^{\max }=c \gamma / \omega_{\mathrm{p}}$. Then the condition $l^{\prime} \ll l_{\mathrm{c}}$ in the given case has the form

$$
\begin{equation*}
\gamma \gg \gamma^{*}=\frac{\omega_{\mathrm{p}}}{\omega_{\mathrm{B}_{\perp}}} \tag{39}
\end{equation*}
$$

Thus the curvature of the trajectory of the particle in the magnetic field is essential when (39) is satisfied. In this region of the parameters it leads to suppression of the transition radiation (Fig. 2).

Let us proceed now to calculate the transition radiation of a relativistic particle moving along a helix in a plasma having random heterogeneities of electron concentration. To do this we must substitute into (33) the known expressions of the trajectory and the velocity $\mathbf{v}(t)$ of the particle in the magnetic field. ${ }^{9}$ Since, for an ultrarelativistic particle the formation zone of the radiation is considerably smaller than the Larmor radius, the arguments of the trigonometric functions in $\nabla(t)$ and $r(t)$ prove to be small, $\omega_{\mathrm{B}_{1}} \tau / \gamma \ll 1$. This allows us to expand the sines and cosines in a power series in the small quantity ( $\omega_{\mathbf{B}_{1}} \tau / \gamma$ ). However, to take correct account of the deviation of the trajectory of the particle from rectilinear, we must retain the next terms after the linear ones in the expansion, as is usually done in studying synchrotron radiation: ${ }^{10}$


FIG. 2. Spectrum of transition radiation in the presence of a magnetic field under the condition (39). The dotted line indicates the spectrum of a rectilinearly moving particle.

$$
\begin{align*}
& \mathbf{v}(t)=\mathbf{n} v\left(1-\frac{\theta^{\mathbf{2}}}{2}\right)+\mathbf{\theta} \mathbf{v}, \quad \mathbf{v}(t+\tau)=\mathbf{v}(t)+v[\mathbf{n}, \mathbf{\Omega}] \tau  \tag{40}\\
& \mathbf{r}(t+\tau)-\mathbf{r}(t) \\
& \quad \approx \mathbf{v}(t) \tau-v[\mathbf{n} \mathbf{\Omega}] \frac{\tau^{2}}{2}+v[\mathbf{\Omega} \theta] \frac{\tau^{2}}{2}-v[\mathbf{\Omega}[\mathbf{n} \mathbf{\Omega}]] \frac{\tau^{3}}{6} .
\end{align*}
$$

Here $\boldsymbol{\Omega}=Q \mathbf{B} / M c \gamma$, while the two-dimensional vector $\boldsymbol{\theta}$ is equal in order of magnitude to the angle between $n$ and $v(t)$.

Upon substituting (40) into (33), we go from the total emitted energy to the energy of emission per unit time (intensity of emission); upon taking the outer integral with respect to $\mathrm{d} t$ and neglecting the second terms in the curly brackets of (33), which are small in comparison with [nv], we find

$$
\begin{align*}
& I_{\omega}^{\mathrm{m}}=\frac{2 \pi Q^{2^{2}}}{m^{2} c^{3}} \int \mathrm{~d} \theta^{\prime} \int \frac{\mathrm{d} \mu \mathrm{~d} k_{1}|8 N|_{\mathbf{k}_{1}}^{2}}{\mu^{\mathbf{2}}} \\
& \quad \times \operatorname{Re} \int_{-\infty}^{\omega} \mathrm{d} \tau\left(\left[\mathrm{n}, \theta^{\prime}\right]^{2}+\theta^{\prime}[\mathrm{n} \Omega] \tau\right) \\
& \times \exp \left\{i\left(\omega-\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}\right) \tau-i \omega \theta^{\prime}[\mathbf{n} \Omega] \frac{\tau^{2}}{2}+i \omega \frac{[\mathbf{n} \Omega]^{2} \tau^{3}}{6}\right\} . \tag{41}
\end{align*}
$$

Here $\mu=\cos \vartheta=\mathbf{k} \mathbf{k}_{1} / k k_{1}$.
In Eq. (41) we transformed to the radiation in the total solid angle $I_{\omega}^{\mathrm{m}}$, since this quantity does not depend on the time. In studying the radiation in a fixed direction, we would have to deal with repeating short pulses of radiation, as in the case of synchrotron radiation. ${ }^{10}$

To integrate (41) with respect to the time $\tau$, it is convenient to make the substitution of the angular variable $\boldsymbol{\theta}^{\prime}=\boldsymbol{\theta}+[\mathbf{n} \boldsymbol{\Omega}] \tau$. Since the integration over the angles can be performed within infinite limits owing to the rapid convergence of the integrand, the stated substitution will not alter the limits of integration over $\mathrm{d} \theta$ :

$$
\begin{align*}
& I_{\omega}^{\mathrm{m}}=\frac{2 \pi Q^{2} e^{4}}{m^{2} c^{3}} \int \mathrm{~d} \theta \int \frac{\mathrm{~d} \mu \mathrm{~d} k_{1}|\delta N|_{\mathbf{k}_{1}}^{2}}{\mu^{2}} \\
& \quad \times \operatorname{Re} \int_{-\infty}^{\infty} \mathrm{d} \tau\left(\theta^{2}+2 \theta[\mathrm{n} \Omega] \tau+\frac{3}{4}[\mathrm{n} \Omega]^{2} \tau^{2}\right) \\
& \times \exp \left[-\frac{i \omega \tau}{2}\left(\gamma^{-2}+\frac{\omega_{p}^{2}}{\omega^{3}}+\theta^{2}+\frac{2 k_{1} \mathrm{v}}{\omega}\right)+\frac{i \omega \omega_{\mathrm{B}_{1}}^{2} \tau^{3}}{24 \gamma^{2}}\right] . \tag{42}
\end{align*}
$$

The inner integral can be expressed in terms of the Airy function $\operatorname{Ai}(\xi)$. Upon performing the substitution of variable $X=\left(\omega \omega_{\mathrm{B}_{1}}^{2} / \gamma^{2}\right)^{1 / 3} \tau / 2$ and taking account of the fact that $\mathrm{Ai}^{\prime \prime}(\xi)=\xi \mathrm{Ai}(\xi)$, while the term containing $\mathrm{Ai}^{\prime}(\xi)$ makes no substantial contribution, we find (with the function $\mathrm{Ai}(\xi)$ normalized to unity):

$$
\begin{align*}
I_{\omega}^{\mathrm{m}} & =\frac{8 \pi^{2} Q^{2} e^{4}}{\omega m^{2} c^{3}}\left(\frac{\omega \gamma}{\omega_{\mathrm{B}_{\perp}}}\right)^{2 / 3} \int \mathrm{~d} \theta \int \frac{\mathrm{~d} \mu \mathrm{~d} k_{1}|\delta N|_{\mathcal{K}_{1}}^{2}}{\mu^{2}} \\
& \times\left[\theta^{2}-3\left(\frac{\omega_{\mathrm{B}_{\perp}}}{\omega \gamma}\right)^{2 / 3} \xi\right] \mathrm{Ai}(\xi), \tag{43}
\end{align*}
$$

where

$$
\begin{equation*}
\xi=\frac{\gamma^{-2}+\left(\omega_{\mathrm{p}}^{2} / \omega^{2}\right)+\theta^{2}+\left(2 \mathbf{k}_{\mathbf{1}} \mathbf{n} / \omega\right)}{\left(\omega_{\mathrm{B}_{\perp}} / \omega \gamma\right)^{2 / 3}} \tag{44}
\end{equation*}
$$

while the integration in (43) must be performed over the region of parameters $\xi \leqslant 0$, which corresponds to scattering by the virtual field (see Sec. 2).

To proceed further in the calculations of the transition radiation, we must concretize the form of the spectrum of the inhomogeneities $|\delta N|_{\mathbf{k},}^{2}$. We shall assume that this spectrum is described by the power function relationship:

$$
\begin{equation*}
|\delta N|_{\mathbf{k}_{1}}^{2}=\frac{v-1}{4 \pi} \frac{k_{0}^{v-1}\left\langle\Delta N^{2}\right\rangle}{k_{1}^{v+2}} . \tag{45}
\end{equation*}
$$

Here $L_{0}=2 \pi / k_{0}$ is the fundamental scale, while $\left\langle\Delta N^{2}\right\rangle$ is the mean square of the inhomogeneities. Upon substituting (45) into (43) and writing in explicit form the limits of integration with account taken of $\xi \leqslant 0$, we obtain

$$
\begin{align*}
I_{\omega}^{\mathrm{m}}= & \frac{2 \pi(\nu-1) Q^{2} e^{4} k_{0}^{\nu-1}\left\langle\Delta N^{2}\right\rangle}{\omega m^{2} c^{3}}\left(\frac{\omega \gamma}{\omega B_{\perp}}\right)^{2 / 3} \int d \theta \\
& \times \int_{k_{\min }}^{\infty} \frac{d k_{1}}{k_{1}^{\nu+2}} \int_{-1}^{-\frac{k_{\min }}{k_{1}}} \frac{\mathrm{~d} \mu}{\mu^{2}}\left[\theta^{2}-3\left(\frac{{ }^{\omega} \mathrm{B}_{\perp}}{\omega \gamma}\right)^{2 / 3} \xi\right] \mathrm{Ai}(\xi) \tag{46}
\end{align*}
$$

where we have

$$
\begin{equation*}
k_{\min }(\theta)=\frac{\omega}{2 c}\left(\gamma^{-2}+\theta^{2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right) . \tag{47}
\end{equation*}
$$

First of all we note that Eq. (46) allows taking the limit in the case of zero magnetic field. Actually, as $\omega_{B_{1}} \rightarrow 0$, we have

$$
\begin{equation*}
\lim _{\omega_{B_{\perp}} \rightarrow 0} \frac{\mathrm{Ai}\left(\frac{\omega-\left(\mathbf{k}-\mathbf{k}_{\mathrm{j}}\right) \mathbf{v}}{\left(\omega_{\mathrm{B}_{\perp}} / \omega \gamma\right)^{2 / 3}}\right)}{\left(\omega_{\mathrm{B}_{\perp}} / \omega \gamma\right)^{2 / \mathbf{3}}}=\delta\left(\omega-\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}\right) \tag{48}
\end{equation*}
$$

and the integrals in (46) over $\mathrm{d} \mu$ and $\mathrm{d} k_{1}$ are easily calculated. Then, upon taking the outer integral with respect to $\mathrm{d} \theta$, in line with (2), we obtain the spectral-angular intensity of the transition radiation from a rectilinearly moving particle

$$
\begin{equation*}
I_{\mathrm{n}, \omega}^{1}=\frac{8 \pi(\nu-1) Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{v c m^{2} \omega^{3}}\left(\frac{2 k_{0} c}{\omega}\right)^{\nu-1} \frac{\theta^{2}}{\left[\gamma^{-2}+\theta^{2}+\left(\omega_{\mathrm{p}}^{2} / \omega^{2}\right)\right]^{\gamma+2}} . \tag{49}
\end{equation*}
$$

We shall return to analyzing Eq. (46). It is important to note that, in the case of curvilinear motion of the particle, taking the outer integral with respect to $\mathrm{d} \theta$ does not yield, strictly speaking, the intensity of radiation in the given direction n . This involves the fact that now the angle $\theta$ is referred
to some instantaneous value of the velocity, whereas the radiation is collected from a finite (albeit small) region of the trajectory of the particle over which the direction of its velocity varies. This essential difference involves the change in the symmetry properties of the system being studied-in motion of the particle along a helix the problem ceases to be axially symmetric with respect to the direction of its velocity. Running ahead (Sec. 4), we shall say that, if the particle is scattered in a medium isotropic on the average, all the directions of change of its velocity are equally probable, and the problem remains axially symmetric with respect to the direction of the velocity of the particle.

Nevertheless, we shall find it convenient for a while to omit in (46) the integral over d $\theta$, so as to have the possibility of comparing the derived expressions with (49).

Despite the fact that the analysis of Eq. (46) in general form is difficult, one can find asymptotic dependences that adequately describe the emission spectrum in individual frequency ranges, and which coincide well with the results of numerical integration of (46).

First let us study the frequencies for which

$$
\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}} \gg\left(\frac{\omega_{\mathrm{B}_{1}}}{\omega \gamma}\right)^{2 / 3},
$$

which corresponds to the intervals

$$
\begin{align*}
& \omega \ll \omega_{*}=\omega_{\mathrm{p}}\left(\frac{\omega_{\mathrm{p}} \gamma}{\omega_{\mathrm{B}_{\perp}}}\right)^{1 / 2},  \tag{50}\\
& \omega \gg \omega_{\mathrm{B}_{\perp}} \gamma^{2} . \tag{51}
\end{align*}
$$

In this case the role of the curvature of the particle trajectory is small, and the Airy function can be replaced by the $\delta$ function of (48). However, in the second term in the brackets in (46) we must take account of the finite width of the stated function, since when $\theta \rightarrow 0$ its contribution proves to be the major one. The finiteness of the width of the function $\left(\omega \gamma / \omega_{\mathrm{B}_{1}}\right)^{2 / 3} \mathrm{Ai}(\xi)$ has the result that $\xi$ effectively varies within the bounds $-1 \lesssim \xi \leqslant 0$. Therefore, for a rough estimate we can assume, e.g., that $\xi=-1 / 3$, whereupon we find
$I_{\mathbf{n}, \omega}^{2}$
$=\frac{8 \pi(\nu-1) Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{\nu c m^{2} \omega^{3}}\left(\frac{2 k_{0} c}{\omega}\right)^{\nu-1} \frac{\theta^{2}+\left(\omega_{B_{\perp}} / \omega \gamma\right)^{2 / 3}}{\left[\gamma^{-2}+\theta^{2}+\left(\omega_{\mathrm{p}}^{2} / \omega^{2}\right)\right]^{\nu+2}}$.

This expression differs from (49) only in the additional term $\left(\omega_{\mathrm{B}_{1}} / \omega \gamma\right)^{2 / 3}$ in the numerator, whose presence, as will be seen below, weakly alters the radiation intensity in the total solid angle under the conditions (50) and (51).

In the other case in which $\gamma^{-2}+\omega_{p}^{2} / \omega^{2} \ll\left(\omega_{\mathbf{B}_{1}} / \omega \gamma\right)^{2 / 3}$, or

$$
\begin{equation*}
\omega_{*} \ll \omega \ll \omega_{B_{\perp}} \gamma^{2} \tag{53}
\end{equation*}
$$

the curvature of the trajectory is essential, and the argument of the Airy function effectively proves to be small: $|\xi| \ll 1$ (this involves the rapid convergence of the integral over $\left.\mathrm{d} k_{1}\right)$. This allows us to expand $\operatorname{Ai}(\xi)$ in a series and keep only the first term of the expansion $\mathrm{Ai}(\xi) \approx \mathrm{Ai}(0) \approx 1 / 3^{2 / 3} \Gamma(2 / 3)$. Then the integration over $\mathrm{d} \mu$ and $\mathrm{d} k_{1}$ is performed without difficulty and yields:

$$
\begin{align*}
I_{\mathrm{n}, \omega}^{3}= & \frac{24 \pi(v-1)}{v^{2}(v+1)} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle \operatorname{Ai}(0)}{c m^{2} \omega^{3}} \frac{\left(2 k_{0} c / \omega\right)^{v-1}\left(\omega \gamma / \omega_{\mathrm{B}_{\perp}}\right)^{2 / 3}}{\left[\gamma^{-2}+\theta^{2}+\left(\omega_{\mathrm{p}}^{2} / \omega^{2}\right)\right]^{v}} \\
& \times\left\{1+\frac{\nu \theta^{2}}{3\left[\gamma^{-2}+\theta^{2}+\left(\omega_{\mathrm{p}}^{2} / \omega^{2}\right)\right]}\right\} \tag{54}
\end{align*}
$$

It is important to note that the possibility of replacing the Airy function $\mathrm{Ai}(\xi)$ with $\mathrm{Ai}(0)$ implies a substantial change in the character of the transition radiation. In particular, the interaction lacks resonance character, which for a rectilinearly moving particle was ensured by the $\delta$-function $\delta\left(\omega-\left(\mathbf{k}-\mathbf{k}_{1}\right) \mathbf{v}\right)$.

We see from (54) that the intensity of the transition radiation decreases with increasing magnetic field: $I_{n, \omega}^{3} \sim \omega_{B_{1}}^{-2 / 3}$ (in contrast to what occurs for synchrotron radiation). That is, transition radiation is suppressed by the magnetic field.

Upon integrating (52) and (54) over the angles, we find the spectral distribution of the transition radiation in the intervals (50), (51), and (53) that are usually of greatest interest:

$$
\begin{align*}
I_{\omega}^{2} & =\frac{8 \pi^{2}(v-1)}{v^{2}(v+1)} \frac{Q^{2} \varepsilon^{4}\left\langle\Lambda N^{2}\right\rangle}{c m^{2} \omega^{3}}\left(\frac{2 k_{0} c}{\omega}\right)^{v-1}\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right)^{-v} \\
& \times\left[1+\frac{v\left(\omega_{\mathrm{B}_{\mathrm{I}}} / \omega \gamma\right)^{2 / 3}}{\gamma^{-2}+\left(\omega_{\mathrm{p}}^{2} / \omega^{2}\right)}\right]  \tag{55}\\
I_{\omega}^{3} & =\frac{16 \pi^{2}(2 \nu+1) \mathrm{Ai}(0)}{v^{2}(v+1)^{2}} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{c m^{2} \omega^{3}}\left(\frac{\omega \gamma}{\omega_{\mathrm{B}_{\perp}}}\right)^{2 / 3} \\
& \times\left[\frac{\omega}{2 k_{0} c}\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right)\right]^{1-v} \tag{56}
\end{align*}
$$

As we see from (55) and (56), the complete spectrum of the radiation is divided into four power-function regions: $I_{\omega}^{\mathrm{m}} \sim \omega^{\nu-2} \quad$ when $\quad \omega \ll \omega_{*}, \quad I_{\omega}^{\mathrm{m}} \sim \omega^{\nu-(10 / 3)} \quad$ when $\omega_{*}<\omega \ll \omega_{\mathrm{p}} \gamma, I_{\omega}^{\mathrm{m}} \sim \omega^{-\nu-(4 / 3)}$ when $\omega_{\mathrm{p}} \gamma<\omega<\omega_{\mathrm{B}_{1}} \gamma^{2}$, and $I_{\omega}^{\mathrm{m}} \sim \omega^{-v-2}$ when $\omega \gg \omega_{\mathrm{B}_{1}} \gamma^{2} \quad$ (see Fig. 2). Thus, when $\omega>\omega_{*}$, the intensity of the transition radiation rapidly declines, while the fundamental fraction of the radiation is concentrated in the frequency region $\omega \lesssim \omega_{*}(50)$. This effect is observed if $\omega_{\neq}<\omega_{p} \gamma$, which coincides with the condition (39) obtained from qualitative considerations. Consequently the curvature of the trajectories of particles of sufficiently high energy (39) completely changes the character of their transition radiation by suppressing radiation at frequencies $\omega \lesssim \omega_{\mathrm{p}} \gamma$, where a rectilinearly moving particle would emit the main fraction of the energy. At the same time the characteristic angle within which the transition radiation is emitted increases, while remaining small in comparison with unity. Upon substituting $\omega_{*}$ into (38), we find

$$
\begin{equation*}
\theta_{\mathrm{CB}} \approx\left(\frac{\omega_{\mathrm{B}_{\perp}}}{\omega_{\mathrm{p}} \gamma}\right)^{1 / 2}>\gamma^{-1} \tag{57}
\end{equation*}
$$

Since a particle moving along a helix in a plasma with random inhomogeneities also generates synchrotron radiation along with transition radiation, we note that the former at frequencies $\omega<\omega_{\mathrm{p}} \gamma$ is exponentially small owing to the effect of density (the Razin-Tsytovich effect), whereas the transition radiation is concentrated precisely in this frequency region. The effect of suppression of transition radiation by a magnetic field occurs at frequencies $\omega_{\neq} \leqslant \omega<\omega_{p} \gamma$. The synchrotron radiation of protons and other nuclei is very small owing to their large mass, so that transition radiation can dominate for them over synchrotron
radiation even when $\omega>\omega_{\mathrm{p}} \gamma$. Consequently the suppression of transition radiation by a magnetic field occurs at the frequencies at which it proves to be the fundamental mechanism of emission, both from electrons and heavy particles.

Now let us study the integral intensity over all frequencies of the transition radiation:

$$
\begin{equation*}
I_{\mathfrak{t o t}}^{\mathrm{m}}=\int_{\omega_{\mathrm{p}}}^{\infty} I_{\omega}^{\mathrm{m}} \mathrm{~d} \omega \tag{58}
\end{equation*}
$$

In the absence of a magnetic field (or under the condition opposite to (39)), the fundamental contribution to the integral of (58) comes from the frequency region $\omega \leqslant \omega_{\mathrm{p}} \gamma$, since, when $\omega>\omega_{\mathrm{p}} \gamma$, the intensity $I_{\omega}^{\mathrm{m}}$ rapidly declines:

$$
\begin{align*}
I_{\mathrm{tot}}^{\mathrm{m}} & \approx \int_{\omega_{\mathrm{p}}}^{\omega_{\mathrm{p}}^{\gamma}} I_{\omega}^{2} \mathrm{~d} \omega \\
& =\frac{8 \pi^{2}}{v^{2}(v+1)} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{c m^{2} \omega_{\mathrm{p}}^{2}}\left(\frac{2 k_{0} c}{\omega_{\mathrm{p}}} \gamma\right)^{\nu-1} \sim \gamma^{v-1} \tag{59}
\end{align*}
$$

Yet if the magnetic field differs from zero and the condition (39) is satisfied, the transition radiation begins to decline sharply, even at frequencies $\omega_{*} \leqslant \omega<\omega_{\mathrm{p}} \gamma$, and hence now we have

$$
\begin{align*}
I_{\mathrm{tot}}^{\mathrm{m}} & \approx \int_{\omega_{\mathrm{p}}}^{\omega_{\mu}} I_{\omega}^{2} \mathrm{~d} \omega \\
& =\frac{8 \pi^{2}}{v^{2}(v+1)} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{c m^{2} \omega_{\mathrm{p}}^{2}}\left(\frac{2 k_{0} c}{\omega_{\mathrm{p}}}\right)^{v-1}\left(\frac{\omega_{\mathrm{p}} \gamma}{\omega_{\mathrm{B}}}\right)^{\frac{v-1}{2}} \sim \gamma^{\frac{v-1}{2}} . \tag{60}
\end{align*}
$$

In the special case of a system of sharp boundaries $v=2$ (pile of plates, ensemble of shock waves, etc.), instead of the linear dependence $I_{\text {tot }}^{\mathrm{m}} \sim \gamma$ we obtain the square-root dependence $I_{\text {tot }}^{\mathrm{m}} \sim \gamma^{1 / 2}$. This substantial change involves the fact that (as we have already stated), in a magnetic field under the condition (39), the transition radiation is suppressed at the frequencies at which the main fraction of the energy would be emitted when $B=0$.

For orientation we shall give some illustrative estimates of the magnitude of $\gamma_{e, p}^{*}$ (for electrons and protons) in (39) in various physical situations. In a laboratory plasma with $n_{\mathrm{e}} \sim 10^{14} \mathrm{~cm}^{-3}\left(\omega_{\mathrm{p}} \sim 5 \times 10^{11} \quad \mathrm{~s}^{-1}\right)$ and $B \sim 10^{4} \quad G$ $\left(\omega_{\mathrm{Be}} \sim 2 \cdot 10^{11} \mathrm{c}^{-1}, \quad \omega_{\mathrm{Bp}} \sim 10^{8} \mathrm{c}^{-1}\right)$, we have $\gamma_{\mathrm{e}}^{*} \sim 2.5$, $\gamma_{\mathrm{p}}^{*} \sim 5 \cdot 10^{3}$. In the active regions on the Sun (arcs) with $n_{\mathrm{e}} \sim 2.5 \times 10^{9} \mathrm{~cm}^{-3}\left(\omega_{\mathrm{p}} \sim 2.5 \times 10^{9} \mathrm{~s}^{-1}\right)$ and $B \sim 10^{2} \mathrm{G}$ $\left(\omega_{\mathrm{Be}} \sim 2 \cdot 10^{9} \quad \mathrm{c}^{-1}, \quad \omega_{\mathrm{Bp}} \sim 10^{6} \quad \mathrm{c}^{-1}\right), \quad$ we find $\gamma_{\mathrm{e}}^{*} \sim 1$, $\gamma_{\mathbf{p}}^{*} \sim 2 \cdot 10^{3}$. In interplanetary space $\left(n_{\mathrm{e}} \sim 4 \mathrm{~cm}^{-3}\right.$, $\left.B \sim 5 \times 10^{-5} \mathrm{G}\right)$, correspondingly we have $\gamma_{\mathrm{e}}^{*} \sim 10^{2}$, $\gamma_{\mathrm{p}}^{*} \sim 2 \cdot 10^{5}$, while in radio galaxies $\left(n_{\mathrm{e}} \sim 10^{-2} \mathrm{~cm}^{-3}\right.$, $B \sim 10^{-5} \mathrm{G}$ ), we have $\gamma_{\mathrm{e}}^{*} \sim 30, \gamma_{\mathrm{p}}^{*} \sim 5 \cdot 10^{4}$. Thus the situation proves typical in which the transition radiation of all or of a considerable fraction of the electrons is strongly suppressed, whereas for heavy particles the influence of the magnetic field is not essential up to rather high energies.

Let us make another interesting estimate. Let the particle move along a line of force of a very strong magnetic field $B \sim 10^{12} \mathrm{G}$, which is realized, in particular, in neutron stars (this problem has been treated in Ref. 2 under the assumption that the field is homogeneous). Since the magnetic field in the vicinity of the neutron star is not homogeneous, the trajectory of the particle will trace the curved line of force to
which it is strongly bound. In this case Eq. (39) must be somewhat modified. Actually, now the role of the Larmor radius is played by the radius of curvature of the line of force $R_{l} \rightarrow R_{\text {cur }}$. Therefore, upon substituting into (39) the frequency $\omega_{\text {cur }}=c \gamma / R_{\text {cur }}$ instead of the gyrofrequency $\omega_{\mathbf{B}_{1}}$, we obtain

$$
\begin{equation*}
\gamma^{*}=\left(\frac{\omega_{\mathrm{p}} R_{\mathrm{cur}}}{c}\right)^{1 / 2} \tag{61}
\end{equation*}
$$

Upon assuming that $\omega_{\mathrm{p}} \sim 10^{12} \mathrm{~s}^{-1}$ and $R_{\text {cur }} \sim 10^{7} \mathrm{~cm}$, we find $\gamma^{*} \sim 10^{4}$. However, as will be shown in the next section, this estimate can be lowered even further upon taking account of the scattering of the relativistic particle by random electromagnetic fields, which are apparently present in the plasma near a pulsar.

## 4. TAKING ACCOUNT OF MULTIPLE SCATTERING (STOCHASTIC TRAJECTORY)

Let us study the transition radiation of a relativistic particle that performs a random walk in a medium with random inhomogeneities. This character of motion can arise from collisions of the particle with Coulomb centers (atoms, nuclei) in a condensed medium, or from its scattering in random fine-scale magnetic or electric fields present in a turbulent plasma. It is important to take account of these fields, in particular, when the particle is moving along the axis of a regular (quasihomogeneous) field and is not subject to its influence.

The trajectory of the particle in the given case is a random function. Therefore now (33) must be averaged over the corresponding ensemble of realizations:
$\mathscr{E}_{\mathrm{n}, \omega}^{\mathrm{m}}=\frac{8 Q^{2} e^{4}}{m^{2} c^{3} \omega^{2}} \int \frac{|\delta N|_{\mathbf{k}_{1}}^{2} \mathrm{~d} \mathbf{k}_{\mathbf{1}}}{\left[1-\frac{\left(\mathbf{k}-\mathbf{k}_{\mathbf{1}}\right)^{2} c^{2}}{\omega^{2} e(\omega)}\right]^{2}} \int_{-\infty}^{\infty} \mathrm{d} t \operatorname{Re} \int_{0}^{\infty} d \tau e^{i \omega \tau}$
$\times\left\langle[\mathbf{n v}(t)][\mathbf{n v}(t+\tau)] \exp \left[-i\left(\mathbf{k}-\mathbf{k}_{1}\right)(\mathbf{r}(t+\tau)-\mathbf{r}(t))\right]\right\rangle$.

The averaging denoted in (62) by angle brackets can be performed by using the distribution function:

$$
\begin{align*}
\langle\ldots\rangle= & \int \mathrm{dr} \mathbf{d} \mathbf{r}^{\prime} \mathrm{d} \mathbf{v} \mathrm{~d} \mathbf{v}^{\prime}\left[\mathbf{n v ^ { \prime }}\right][\mathbf{n v}] \\
& \times \exp \left[-i\left(\mathbf{k}-\mathbf{k}_{\mathbf{1}}\right)\left(\mathbf{r}^{\prime}-\mathbf{r}\right)\right] F(\mathbf{r}, \mathbf{v}, t) \\
& \times W\left(\mathbf{r}, \mathbf{v}, \mathbf{r}^{\prime}, \mathbf{v}^{\prime}, \tau\right)=\int \mathrm{d} \mathbf{v}^{\prime}[\mathbf{n v}]\left[\mathbf{n} \mathbf{v}^{\prime}\right] W_{\mathbf{k}-\mathbf{k}_{1}}\left(\mathbf{v}, \mathbf{v}^{\prime}, \tau\right) \tag{63}
\end{align*}
$$

Here $F(\mathbf{r}, \mathbf{v}, t)$ is the distribution function of the particle at the instant of time $t$. Owing to the normalization to unity upon integration over $\mathrm{d} \boldsymbol{r} \mathbf{d v}$, this yields the unit coefficient $W$ ( $\mathbf{r}, \mathbf{v}, \mathbf{r}^{\prime}, \mathbf{v}^{\prime}, \tau$ )-the conditional probability that the particle goes in the time $\tau$ from the point of phase space ( $\mathbf{r}, \mathbf{v}$ ) to the point ( $\mathbf{r}^{\prime}, \mathbf{v}^{\prime}$ ), while $W_{\mathbf{k}-\mathbf{k}_{1}}$ is its spatial Fourier transform.

The function $W$ (which satisfies the kinetic equation) has been calculated as applied to radiation problems, both for Coulomb collisions ${ }^{11}$ and for scattering in random fields, ${ }^{4}$ and can be represented as follows:

$$
\begin{align*}
& W_{\mathbf{k}-\mathbf{k}_{1}}\left(\mathbf{v}_{0}, \mathbf{v}, \tau\right) \\
& \quad=v^{-2} \delta\left(v-v_{0}\right) \\
& \quad \times \exp \left\{-i\left[\frac{\omega v}{c}\left(1-\frac{\omega_{\mathrm{p}}^{2}}{2 \omega^{2}}\right)+\mathbf{k}_{1}^{\prime} \mathbf{n} v\right] \tau\right\} U\left(\boldsymbol{\theta}_{0}, \boldsymbol{\theta}, \tau\right) . \tag{64}
\end{align*}
$$

Here the vectors $\boldsymbol{\theta}_{0}$ and $\boldsymbol{\theta}$ are defined analogously to (40) by the relationships
$\mathbf{v}_{0}=\mathbf{n} v\left(1-\frac{\theta_{0}^{2}}{2}\right)+\theta_{0} v, \quad \mathbf{v}=\mathbf{n} v\left(1-\frac{\theta^{2}}{2}\right)+\theta v$,
while the function $U\left(\theta_{0}, \theta, \tau\right)$ satisfies the equation

$$
\begin{equation*}
\frac{\partial U}{\partial \tau}-\frac{i \omega}{2} \theta^{2} U=q \Delta_{\theta} U \tag{66}
\end{equation*}
$$

Here $q=q_{0} \gamma^{-2}$ is the frequency of collisions of the relativistic particle with the Coulomb centers or with fine-scale magnetic fields. In the former case we have ${ }^{11}$

$$
\begin{equation*}
q_{0}=2 \pi N\left(\frac{Z e^{2}}{M c^{2}}\right)^{2} \ln \frac{183}{Z^{1 / 3}}, \tag{67}
\end{equation*}
$$

where $N$ and $Z e$ are the concentration and the charge of the nuclei in the medium, while in the latter case we have ${ }^{12}$

$$
\begin{equation*}
q_{0}=\frac{L_{0}}{3 c} \frac{Q^{2}\left\langle B_{\mathrm{gt}}^{2}\right\rangle}{M^{2} c^{2}}=\frac{\omega_{\mathrm{st}}^{2}}{3 \omega_{0}}, \quad \omega_{\mathrm{st}}<\omega_{0} \tag{68}
\end{equation*}
$$

Here $L_{0}$ and $\left\langle B_{\mathrm{st}}^{2}\right\rangle$ are the correlation length and the magnitude of the magnetic inhomogeneities, $\omega_{\mathrm{st}}=Q\left\langle B_{\mathrm{st}}^{2}\right\rangle^{1 / 2} /$ $M c, \omega_{0}=c / L_{0}$, while the field is assumed to be fine-scaled if $\omega_{\mathrm{st}}<\omega_{0}$. In the case of electric inhomogeneities, Eq. (66) remains as before, while in defining $q_{0}$ we must make the replacement $\left\langle B_{\text {st }}^{2}\right\rangle \rightarrow\left\langle E_{\text {st }}^{2}\right\rangle$.

The solution of (66) with account taken of the obvious initial condition $U\left(\boldsymbol{\theta}_{0}, \boldsymbol{\theta}, 0\right)=\delta\left(\boldsymbol{\theta}-\boldsymbol{\theta}_{0}\right)$ is

$$
\begin{align*}
& U\left(\theta_{0}, \theta, \tau\right) \\
& \quad=\frac{x}{\pi \operatorname{sh} z \tau} \exp \left[-x\left(\theta^{2}+\theta_{0}^{2}\right) \operatorname{cth}(z \tau)+2 x \theta \theta_{0} \operatorname{sh}^{-1}(z \tau)\right] \tag{69}
\end{align*}
$$

where

$$
\begin{equation*}
=1-i)\left(\frac{\omega}{16 q}\right)^{1 / 2}, \quad z=(1-i)(\omega q)^{1 / 2} \tag{70}
\end{equation*}
$$

Upon substituting (64) and (69) into (63) and (62) and transforming to the intensity of radiation, as in the previous section, we find

$$
\begin{align*}
& I_{\mathrm{n}, \omega}^{\mathrm{m}} \\
& =\frac{8 Q^{2} e^{4}}{m^{2} c \omega^{2}} \int \frac{|\delta N|_{\mathfrak{k}_{1}}^{2} \mathrm{~d} l_{1}}{\left[1-\frac{\left(\mathrm{k}-\mathrm{k}_{1}\right)^{2} c^{2}}{\omega^{2} \mathrm{E}(\omega)}\right]^{2}} \operatorname{Re} \int_{0}^{\infty} \mathrm{d} \tau \int \mathrm{~d} \theta\left(\theta \cdot \theta_{0}\right) \frac{x}{\pi \operatorname{sh}(z \tau)} \\
& \quad \times \exp \left[\frac{i \omega \tau}{2}\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}+\frac{2^{\prime} \mathrm{n} c}{\omega}\right)\right. \\
& \left.\quad-x\left(\theta^{2}+\theta_{0}^{2}\right) \operatorname{cth}(z \tau)+!2 x \theta \theta_{0} \operatorname{sh}^{-1}(z \tau)\right] \tag{71}
\end{align*}
$$

It is important that, in the presence of multiple scattering, the treatment of the intensity of radiation in the given solid angle in (71) is quite correct, in contrast to the situation treated in Sec. 3. This involves the fact that now the problem is axially symmetric (on the average) with respect to the initial direction of the particle, since all directions of change of its velocity are equally probable, while the angle $\theta_{0}$ in (71) is referred to the initial velocity $\mathbf{v}_{0}$. Of course, the stated symmetry holds for an ensemble (beam) of particles, whereas all the individual particles move along their individual random trajectories, which do not possess axial symmetry.

Since the integrand in (71) is a Gaussian function of the
angle, the integration over $\mathrm{d} \theta_{x}$ and $\mathrm{d} \theta_{y}$ can be performed over the region $]-\infty ;+\infty[$. The possibility of such a choice of limits of integration is ensured by the rapid convergence of the integrals over $\mathrm{d} \theta$, which involves the sharp directionality of the radiation along the velocity of the particle:

$$
\begin{align*}
\mathrm{m}, \omega & =\frac{8 Q^{2} e^{4} \theta_{0}^{2}}{m^{2} c \omega^{2}} \int \frac{|\delta N|_{k_{1}}^{2} \mathrm{~d} k_{1}}{\left[1-\frac{\left(\mathrm{k}-\mathbf{k}_{1}\right)^{2} c^{2}}{\omega^{2} \varepsilon(\omega)}\right]^{2}} \operatorname{Re} \int_{0}^{\infty} \frac{d \tau}{\operatorname{ch}^{2}(z \tau)} \\
& \times \exp \left[\frac{i \omega \tau}{2}\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}+\frac{2 \mathrm{~h}_{1} n c}{\omega}\right)-x \theta_{0}^{2} \operatorname{th}(z \tau)\right] . \tag{72}
\end{align*}
$$

As in the previous section, to perform the further integration we shall study the different limiting cases of (72). If we have

$$
\begin{equation*}
|z| \preccurlyeq \frac{\omega}{2}\left(\gamma^{-2}+\frac{\omega_{p}^{2}}{\omega^{2}}\right), \tag{73}
\end{equation*}
$$

then the arguments of the hyperbolic functions are small: $|z \tau| \ll 1$, th $z \tau \approx z \tau$, ch $z \tau \approx 1$, and (72) is reduced to the known expression (49) for a rectilinearly moving particle. The inequality (73) is satisfied in the low- and high-frequency regions

$$
\begin{align*}
& \omega \gtrless \omega_{* *}=\frac{\omega_{\mathrm{p}}}{2}\left(\frac{\omega_{\mathrm{p}}}{q_{0}}\right)^{1 / 3} \gamma^{2 / 3},  \tag{74}\\
& \omega \gg 8 q_{0} \gamma^{2} . \tag{75}
\end{align*}
$$

In the intermediate frequency region in which

$$
|z| \gg \frac{\omega}{2}\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right),
$$

or, as is the same,

$$
\begin{equation*}
\omega_{* *} \leqslant \omega \leqslant 8 q_{0} \gamma^{2} \tag{76}
\end{equation*}
$$

multiple scattering plays an essential role. Here the hyperbolic functions also are simplified: th $z \tau \approx 1$, $\operatorname{ch}^{2} z \tau \approx \exp (2 z \tau) / 4$. Thereupon the integration over the time yields the following expression (we omit the subscript " 0 " in $\theta_{0}$ ):

$$
\begin{equation*}
I_{\mathrm{n}, \omega}^{\mathrm{m}}=\frac{8 Q^{2} e^{4} \theta^{2}}{m^{2} c \omega(\omega q)^{1 / 2}} \int \frac{|\delta N|_{\mathbf{k}_{1}}^{2} d \mathrm{k}_{1}}{\left[1-\frac{\left(\mathbf{k}-\mathbf{k}_{1}\right)^{2} c^{2}}{\omega^{2} \varepsilon(\omega)}\right]^{2}} . \tag{77}
\end{equation*}
$$

The integration in (77) over $\mathrm{dk}_{1}$ with account taken of the spectrum of inhomogeneities of (45) is performed by analogy with (46) and yields
$I_{\mathrm{n}, \omega}^{\mathrm{m}}$
$=\frac{4(v-1)}{v(v+1)} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{c m^{2} \omega^{2}(\omega q)^{1 / 2}}\left(\frac{2 h_{0} c}{\omega}\right)^{v-1} \frac{\theta^{2}}{\left[\theta^{2}+\gamma^{-2}+\left(\omega_{p}^{2} / \omega^{2}\right)\right]^{v+1}}$.

We note that here, in contrast to the case of motion of a particle in a regular magnetic field, the vanishing of the radiation in the direction of the initial velocity of the particle $\theta=0$ is preserved. This involves the differing symmetry properties in the two problems. Upon integrating (78) over the angles, we find the spectrum of the transition radiation at the frequencies of (76):
$I_{\mathrm{\omega}}^{\mathrm{m}}=\frac{2}{\boldsymbol{v}^{2}(\nu+1)} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{c m^{2} \omega^{2}(\omega q)^{1 / 2}}\left[\frac{\omega}{2 k_{0} c}\left(\gamma^{-2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right)\right]^{1-\nu}$.
Thus, in the region (76) the radiation spectrum consists of two power-function regions $I_{\omega}^{\mathrm{m}} \sim \omega^{\nu-3.5}$ for $\omega_{* *} \ll \omega \ll \omega_{\mathrm{p}} \gamma$,


FIG. 3. Spectrum of transition radiation in the presence of multiple scattering under the condition (80). The dotted line indicates the spectrum of a rectilinearly moving particle.
and $I_{\omega}^{\mathrm{m}} \sim \omega^{-v-1.5}$ when $\omega_{\mathrm{p}} \gamma \ll 8 q_{0} \gamma^{2}$. Consequently, when $\omega \gtrsim \omega_{* *}$, the transition radiation is considerably weakened (see Fig. 3) and declines with increasing frequency of collisions, $I_{\omega}^{\mathrm{m}} \sim q^{-1 / 2}$.

The effect of suppression of transition radiation by multiple scattering in (79) occurs if the frequency region of (76) exists, i.e., $\omega_{* *} \ll 8 q_{0} \gamma^{2}$. When we take (74) into account, this is valid for particles of high enough energy:

$$
\begin{equation*}
\gamma \gg \gamma^{* *}=\frac{\omega_{\mathrm{p}}}{8 q_{0}} . \tag{80}
\end{equation*}
$$

The inequality (80) is an analog of the condition of suppression of transition radiation by a magnetic field in (39).

Let us study the dependence of the total energy emitted by the transition mechanism on the energy of the particle under the condition (80). Upon integrating (55) over the frequency up to the value $\omega_{* *}$, we obtain by analogy to (60)

$$
\begin{align*}
I_{\mathrm{tot}}^{\mathrm{m}} & \approx \int_{\omega_{\mathrm{p}}}^{\omega_{*}^{*}} I_{\omega}^{2} \mathrm{~d} \omega \mid \\
& =\frac{16 \pi^{2}}{2^{v} v^{2}(v+1)} \frac{Q^{2} e^{4}\left\langle\Delta N^{2}\right\rangle}{c m^{2} \omega_{\mathrm{p}}^{2}}\left(\frac{\omega_{\mathrm{p}} \gamma^{2}}{q_{0}}\right)^{\frac{\nu-1}{3}} \sim \gamma^{\frac{2}{3}(v-1)} \tag{81}
\end{align*}
$$

We see that this case is intermediate in a certain sense between the rectilinear motion of the particle in (59) and its turning in the magnetic field in (60). The dependence of $I_{\text {tot }}^{\mathrm{m}}$ • on $\gamma$ for these cases is shown schematically in Fig. 4.

We note moreover that the distribution function of (64) is known also in the case of joint action of a regular field and its random inhomogeneities on the motion of a particle. ${ }^{4}$ Hence it should be possible to take account of both factors simultaneously. However, as we see it, in the given case this would only complicate the formulas without adding clarity to the essence of the effects that arise here.

Since in condensed media under normal conditions the effect of suppression of transition radiation by multiple scattering by Coulomb centers is inessential up to values $\gamma_{\mathrm{e}}^{* *} \sim 10^{15}$, which far exceed the bounds of the possibilities of


FIG. 4. Dependence of the total energy of transition radiation on the energy of the emitting particle for various types of motion. 1-rectilinear trajectory, 2 -multiple scattering (random walk), 3-helix.
accelerator technology, we proceed directly to astrophysical estimates. Let us study for illustration an interplanetary plasma, whose parameters are well known: ${ }^{13}$ $\left\langle B_{\mathrm{st}}^{2}\right\rangle \approx 3.6 \cdot 10^{-10} \Gamma \mathrm{c}^{2}, L_{0} \sim 3 \cdot 10^{11} \mathrm{~cm}$ or $\omega_{\mathrm{st}}=0.25 \mathrm{c}^{-1}$, $\omega_{0}=0.1 \mathrm{c}^{-1}$. Upon substituting (68) into (80), we find

$$
\begin{equation*}
\gamma_{\mathrm{e}}^{* *} \approx \frac{\omega_{\mathrm{p}} \omega_{0}}{2 \omega_{\mathrm{st}}^{2}} \sim 10^{5} . \tag{82}
\end{equation*}
$$

Now let us return to analyzing the situation that arises in the vicinity of a neutron star (see the end of Sec. 3). If weak field inhomogeneities exist over the course of a line of force of the magnetic field along which the particle is moving (e. g., Alfven waves), of flux density $B_{\text {st }}=\left\langle B_{\mathrm{st}}^{2}\right\rangle^{1 / 2} \approx 5 \Gamma \mathrm{c}$ $\left(\sim 10^{-11} B_{o}\right)$ and $L_{0} \sim 1 \mathrm{~m}=10^{2} \mathrm{~cm}$, then $\omega_{\mathrm{st}} \approx 10^{8} \mathrm{c}^{-1}$, $\omega_{0} \sim 3 \cdot 10^{8} \mathrm{c}^{-1}$, and by using (82) we find

$$
\begin{equation*}
\gamma_{\mathrm{e}}^{* *} \approx 10^{2} \tag{83}
\end{equation*}
$$

Since in the vicinity of a neutron star the plasma is relativistic and $\gamma \gtrsim 10^{2}$ for practically all particles, in the given case the effect of suppression of transition radiation is important for most electrons and positrons that are present. Moreover, even in the complete absence of inhomogeneities of the magnetic field, an important role can be played by multiple scattering of particles by radio waves (with $\lambda \leqslant 1$ m ), which are generated in the magnetosphere of pulsars. What we have said here, and also at the end of Sec. 3, implies that the application of the theory of transition radiation to real objects having a strong magnetic field requires great caution, and in particular, taking scrupulous account of all factors that lead to bending of the trajectories of the emitting particles.

## 5. CONCLUSION

In the examples that we have discussed (Secs. 3 and 4), curvature of the trajectory leads to a decrease in the transition radiation. However, this does not always happen. We shall point out only one case, which was discussed in Ref. 3. If a relativistic particle crosses the phase boundary between a medium and vacuum, then the intensity of its emission can be represented in the form of the square of the difference between the coherent lengths in the vacuum

$$
\begin{equation*}
l_{\mathrm{v}}=\frac{c}{\omega}\left(\gamma^{-2}+\theta^{2}\right)^{-1} \tag{84}
\end{equation*}
$$

and in the medium

$$
\begin{align*}
& l_{\mathrm{c}}=\frac{c}{\omega}\left(\gamma^{-2}+\theta^{2}+\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}}\right)^{-1}  \tag{85}\\
& I_{\mathrm{n}, \omega}^{\mathrm{m}} \sim\left(l_{\mathrm{v}}-l_{\mathrm{c}}\right)^{2} \tag{86}
\end{align*}
$$

It is not complicated to see that at frequencies $\omega \ll \omega_{\mathrm{p}} \gamma$ we have $l_{v} \gg l_{\mathrm{c}}$; in this region transition radiation is effectively generated. However, when $\omega \gg \omega_{\mathrm{p}} \gamma$, for which the term $\omega_{\mathrm{p}}^{2} / \omega^{2}$ is relatively small, we have $l_{\mathrm{v}} \approx l_{\mathrm{c}}$, and the transition radiation rapidly declines with increasing frequency. Now let the particle undergo multiple scattering in the medium. This scattering leads to a decrease in the coherent length $l_{\mathrm{c}}$ as compared with (85), while the difference $l_{v}-l_{c}$ will differ appreciably from zero, including even the case $\omega>\omega_{\mathrm{p}} \gamma$, since in the vacuum the particle does not undergo multiple scattering. This means that the transition radiation is enhanced in the frequency region $\omega>\omega_{\mathrm{p}} \gamma$ as compared with the case of rectilinear motion of the particle. In the same situation the curvature of the trajectory in a magnetic field would act differently-it would suppress the transition radiation by decreasing both coherence lengths $l_{\mathrm{c}}$ and $l_{\mathrm{v}}$. Without dwelling further on a discussion of concrete problems and situations, we shall make some closing remarks for greater precision.

Let us turn our attention to the fact that the procedure of separating the electric field of a relativistic particle into virtual and real components is not always obligatory. We need not resort to it, apart from the trivial case of uniform rectilinear motion of the particle, when all its field is virtual, and in treating resonance radiation, ${ }^{3.5}$ since in the quasiclassical approximation one can construct a Green's function that describes the propagation of quanta in a periodic medium. This allows us, by using Eq. (2), to seek the total radiation generated by both the transition and the bremsstrahlung mechanisms. However, the stated separation occurs in such a case automatically. Actually, in Refs. 5 and 3 the regions of resonance quanta correspond to negative values of the "coherence lengths" used by Ter-Mikaélyan (somewhat differing from (37))-an analog of the condition $\omega-\left(\mathbf{k}-\mathbf{k}_{\mathrm{t}}\right) \mathbf{v} \leqslant 0$, while in the final expressions for the emission spectrum in a periodic medium $\theta$-functions arise that separate the region of resonance quanta from the region of bremsstrahlung quanta.

Thus the approach proposed in Sec. 2 for calculating the transition radiation agrees with the other methods applicable for this purpose, ${ }^{2,3}$ while possessing certain advantages. Among these advantages are the possibility of calculating the transition radiation of a particle moving along an arbitrary trajectory in an arbitrary inhomogeneous medium, as well as the fact that, by adopting the proposed method, one can systematically separate the transition radiation in the formulas from other mechanisms of emission. One can distinctly see from the results of Sec. 2 that one must take transition radiation to mean (as compared with bremsstrahlung) an additional channel of formation of transverse quanta that arise from the virtual intrinsic field of the particle owing to its interaction with the inhomogeneities of the medium.

The effects discussed in this paper of suppression of
transition radiation can prove to be substantial and highly useful in various physical situations. As we see from the estimates given at the end of Sec. 3, the case is typical in which the transition radiation of light particles is strongly suppressed practically throughout the existing range of energies, while the radiation from heavy particles is not suppressed over a broad range. In principle this enables one to obtain direct information on the nuclear component of relativistic particles in remote cosmic radio sources, ${ }^{14}$ even if the number of nuclei does not exceed the number of electrons.

The effect of suppression of transition radiation by a magnetic field can be used for selecting particles with different masses in using transition counters, ${ }^{3}$ for minimizing the total energy losses of particles during acceleration in a laboratory plasma, and for other purposes. Also special experimental studies of transition radiation of relativistic particles moving along curvilinear trajectories in randomly inhomogeneous media seem to be of great current interest. ${ }^{15}$
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