Nonlinear acoustics of superfluid helium

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Usp. Fiz. Nauk **160**, 51–95 (June 1990)

The results of investigations of nonlinear waves of first and second sound in HeII are reviewed. The equations of hydrodynamics of HeII in the standard and Hamiltonian formulations are presented. One-dimensional nonlinear waves and effects such as nonlinear conversion of waves into one another, formation of shock fronts, renormalization of the velocity of sound, and others are described. The effect of damping and dispersion is studied. A description of multidimensional waves is followed by a description of the phenomenon of self-focusing. Next, stability is discussed, the solution of the problem of the stability of a pressure shock wave is presented, and stochastic wave processes in acoustic turbulence are discussed. Experimental investigations of nonlinear waves in HeII are presented. In conclusion the paths for further development of the problems touched upon are discussed.

INTRODUCTION

In the last few years interest in intense sound waves in superfluid helium has increased. In my opinion, this is attributable to several factors. First, the theory of nonlinear waves is a rapidly developing field, in which a number of new general methods and results, which can be transferred to the case of helium II, have appeared. Second, superfluid helium, as is now being widely discussed, can be employed as a refrigerant in different cryogenic systems: solenoids, resonators, etc. This also stimulates the study of the hydrodynamics and acoustics of HeII. Finally, the increased interest in nonlinear waves is also due to, so to speak, "internal" factors. The point is that acoustic methods play an important role in the study of the properties of HeII. In this sense nonlinear acoustics is much more promising, since the physics of nonlinear phenomena is much more diverse than the linear theory and a large number of effects could be connected with the properties of helium. We note, by the way, that the question of the applicability of the linear theory can be resolved systematically only on the basis of nonlinear acoustics.

In superfluid helium, like in many other media, "standard" nonlinear wave phenomena, such as steepening of the wave profile, formation of discontinuities, self-focusing of wavy packets, etc., are observed. At the same time there are a number of specific features that are characteristic solely of this liquid. The distinctive feature of HeII from the viewpoint of nonlinear acoustics is that HeII has two coupled wave modes, one of which—temperature waves—is a unique phenomena with no classical analog. The existence of two or several waveguide modes, of course, is not an exception. The situation is the same for plasma waves or, for example, ocean waves. In contradistinction to these systems, however, waves in helium exhibit virtually no dispersion of the velocity of sound in a wide range of temperatures. In some situations, for example, in the study of one-dimensional waves, the absence of dispersion is a simplifying factor, since the propagation of waves is described by a system of first-order differential equations (see Secs. 2.1–2.5). In other cases, for example, when Hamiltonian methods are employed, the absence of dispersion complicates the analysis, since the small parameter in the perturbation theory is related with the dispersion.

A characteristic feature of HeII is that its properties are strongly temperature dependent. Since acoustic phenomena ultimately are determined by the "play" of thermodynamic quantities (from the formal viewpoint) they are also strongly temperature dependent. On the other hand this leads to interesting new effects, such as, for example, steepening of the trailing edge of waves of second sound (see Sec. 2.5). On the other hand, the strong temperature dependence of the acoustic properties makes it necessary to give a justification for whatever approximation is used in the equations of motion.

Another characteristic feature of helium is that for some fully achievable parameters of the sound waves the properties of helium change so sharply that the chosen hydrodynamic model becomes inadequate. For example, sufficiently powerful and prolonged heat pulses engender in helium a vortex structure that cannot be described on the basis of the classical equations of two-velocity hydrodynamics. A further increase of the amplitude or (and) duration can result in the fact that a film of vapor of HeI appears in helium II. A HeII–HeI phase transition can also occur in an intense pressure wave, since the λ curve has a finite slope of the order of 110 at/deg in p, T coordinates. Of course, strong waves can significantly change the properties of the usual media also. An example are cavitation phenomena in sound waves in a liquid or, say, ionization of a gas in a strong shock wave. These cases, however, refer to extremely (record) high values of the parameters of the sound waves, while in HeII the generation of quantum vortices is observed in heat pulses, which, as it was still recently believed, are described by irrotational equations of two-velocity hydrodynamics. The propagation of sound pulses, which engender vortices, has thus far been little studied, and there are virtually no theoretical works on the subject. We shall not discuss these questions in this review, rather we shall confine our attention to some remarks in the conclusions.

By nonlinear acoustics of HeII we shall mean below the theory of intense sound waves, whose propagation laws can be explained on the basis of classical two-velocity hydrodynamics. This review is devoted primarily to theoretical results. The reason for this is not only and even not so much because I am a theoretician but rather because the theory of nonlinear waves in HeII is a more or less developed field, which cannot be said about experiment. With few exceptions, the existing experimental works concern primarily the relatively particular question of the dynamics of intense pulses of second sound. In addition, the purely nonlinear

 0038-5670/90/060429-24$01.00

effects observed in the works cited are confused with phenomena connected with quantum vortices. As regards other questions concerning the dynamics of nonlinear waves in HeII, they have virtually not been studied at all by experimenters. For this reason I felt it would be best to discuss the questions connected with the experimental investigations in a separate section. It consists of two parts. In the first part the experimental investigations are briefly reviewed. The main purpose of this review is to describe how experimental investigations are developing and the main trends. Special attention is devoted to experimental studies that illustrate the results described in the first sections of this paper. The second part of this section is, so to speak, of a "promotional" character. In it a number of experiments that are either of interest in themselves or could serve as a tool for studying HeII by the methods of nonlinear acoustics are proposed.

This review is organized as follows. The equations of hydrodynamics of HeII in the standard and Hamiltonian formulations are given in Sec. 1 for reference purposes. The laws of propagation of one-dimensional nonlinear waves are studied in Sec. 2. Section 3 is devoted to the study of the evolution of weakly multidimensional wave packets. In Sec. 4 the stability of nonlinear waves is discussed and the solution of the problem of the stability of a pressure shock wave is presented. In Sec. 5 stochastic wave processes are discussed. Section 6, as I have already mentioned, is devoted to experiments.

1. THE EQUATIONS OF HYDRODYNAMICS OF HELIUM II

1.1. Classical two-velocity hydrodynamics

For purposes of continuity as well as to introduce the notation, we shall write out the equations of hydrodynamics of HeII in the standard and Hamiltonian variables. The hydrodynamic description of HeII is made with the help of a collection of eight variables. They can be, for example, the momentum density $\mathbf{j}(\mathbf{r},t)$, the velocity of the superfluid component $v_s(r,t)$, as well as the mass density $\rho(\mathbf{r},t)$ and the entropy density $S(\mathbf{r},t)$.

Taking dissipative effects into account the dynamical equations for the enumerated quantities have the following form:[1]

$$\frac{\partial \rho}{\partial t}+\operatorname{div}\mathbf{j}=0, \tag{1.1}$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial \Pi_{ik}}{\partial x_k}=\frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{ni}}{\partial x_k}\right)+\delta_{ik}\left(\zeta_1\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)+\zeta_2\operatorname{div}\mathbf{v}_n\right)\right], \tag{1.2}$$

$$\frac{\partial S}{\partial t}+\operatorname{div}S\mathbf{v}_n=\operatorname{div}\frac{\varkappa\nabla T}{T}+\frac{R}{T}, \tag{1.3}$$

$$\frac{\partial \mathbf{v}_s}{\partial t}+\nabla\left(\mu+\frac{\mathbf{v}_s^2}{2}\right)=\nabla\left[\zeta_3\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)+\zeta_4\operatorname{div}\mathbf{v}_n\right]. \tag{1.4}$$

The quantities appearing on the left sides of the relations (1.1)–(1.4) can be determined with the help of the expression for the energy density E_0 in the coordinate system in which there is no superfluid motion. The quantity E_0 depends on ρ and S as well as on the momentum density $\mathbf{j}_0$ in the superfluid system, related with the vector $\mathbf{j}$ by the expression

$$\mathbf{j}_0=\mathbf{j}-\rho\mathbf{v}_s. \tag{1.5}$$

The differential of the quantity $E_0(\rho,S,\mathbf{j}_0)$ is

$$dE_0=T\,dS+\mu\,d\rho+(\mathbf{v}_n-\mathbf{v}_s,d\mathbf{j}_0). \tag{1.6}$$

The last term serves as a definition for the velocity v_n of normal motion. It expresses the fact that the derivative of the energy with respect to the momentum is equal to, by definition, the velocity. Aside from v_n the relation (1.6) determines the temperature T and the chemical potential μ.

From symmetry considerations it follows that the vector $\mathbf{j}_0$ is related with the relative velocity v_n-v_s by the following relation:

$$\mathbf{j}_0=\rho_n(\mathbf{v}_n-\mathbf{v}_s), \tag{1.7}$$

which can be regarded as a definition of the density of the normal component ρ_n. The density of the superfluid component ρ_s is correspondingly equal to $\rho_s=\rho-\rho_n$. It follows from Eqs. (1.5) and (1.7) that

$$\mathbf{j}=\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s. \tag{1.8}$$

The momentum flux density tensor is defined as follows:

$$\Pi_{ik}=\rho v_{si}v_{sk}+v_{si}j_{0k}+v_{sk}j_{0i}+p\delta_{ik}, \tag{1.9}$$

or, equivalently,

$$\Pi_{ik}=\rho_s v_{si}v_{sk}+\rho_n v_{ni}v_{nk}+p\delta_{ik}; \tag{1.10}$$

here the pressure is, as usual, equal to the derivative of the total energy with respect to the volume,

$$p=-\frac{\partial E_0V}{\partial V}=-E_0+TS+\mu\rho+(\mathbf{v}_n-\mathbf{v}_s,\mathbf{j}_0). \tag{1.11}$$

The quantities T, μ, p, ρ_s, and ρ_n are functions of ρ, S, and the vector $\mathbf{j}_0$. For small values of the relative velocity $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$ dependence on $\mathbf{j}_0$ is replaced by dependence on $\mathbf{w}$ and can be determined from Maxwell's relations.[1]

The nondissipative equations of motion (1.1)–(1.4) lead to the law of conservation of energy E, in the laboratory system equal to

$$E=\frac{1}{2}\rho\mathbf{v}_s^2+\mathbf{v}_s\mathbf{j}_0+E_0(\rho,S,\mathbf{j}_0), \tag{1.12}$$

which has the following form:

$$\frac{\partial E}{\partial t}+\operatorname{div}\mathbf{Q}=0; \tag{1.13}$$

here Q is the energy flux density, equal to

$$\mathbf{Q}=\left(\mu+\frac{\mathbf{v}_s^2}{2}\right)\mathbf{j}+ST\mathbf{v}_n+\mathbf{v}_n(\mathbf{v}_n\mathbf{j}_0). \tag{1.14}$$

Terms related with dissipation enter on the right side of the expressions (1.1)–(1.4). The notation is as follows: η is the coefficient of shear viscosity; $\varkappa$ is the thermal conductivity; ζ_1, ζ_2, ζ_3, and ζ_4 are the coefficients of second viscosity, and owing to Onsager's symmetry principle $\zeta_1=\zeta_4$. The dissipative function R is a quadratic form of the gradients of the variables introduced with the coefficients η, $\varkappa$, and ζ_i; an explicit expression for R is given in Ref. 1.

1.2. The Hamiltonian form of the equations of motion

An alternative to the nondissipative system (1.1)–(1.4) is the representation of the equations of motion in the so-called Hamiltonian form. The method of the Hamiltonian formalism is very effective for studying nonlinear waves (see, for example, Refs. 2 and 3). For HeII the Hamiltonian representation of the equations of motion was obtained in Ref. 4. As shown in Ref. 4 the Hamiltonian variables are three canonically conjugate pairs (ρ, α), (S, β), and the so-called Clebsch variables (f, γ). These quantities are related with the variables introduced earlier as follows:

$$\mathbf{v}_s = \nabla\alpha, \quad \mathbf{j}_0 = S\nabla\beta + f\nabla\gamma. \tag{1.15}$$

In the new variables the equations of motion of HeII acquire the canonical form

$$\begin{aligned} \dot\rho &= \frac{\delta H}{\delta\alpha}, \quad \dot\alpha = -\frac{\delta H}{\delta\rho}, \quad \dot S = \frac{\delta H}{\delta\beta}, \\ \dot\beta &= -\frac{\delta H}{\delta S}, \quad \dot f = \frac{\delta H}{\delta\gamma}, \quad \dot\gamma = -\frac{\delta H}{\delta f}; \end{aligned} \tag{1.16}$$

here the Hamiltonian H is the energy E (1.12), expressed in canonical variables. It can be verified by direct calculations that the relations (1.16) are identical to the dissipation-free equations (1.1)–(1.14).

The canonical equations (1.16) can be derived in the standard manner from the Lagrangian formalism and the principle of least action for the hydrodynamics of a superfluid liquid, as described in Ref. 5.

To study problems in nonlinear acoustics it is convenient to change from the variables ρ, α, S, and β over to the so-called normal coordinates, separating in the linear case the first and second sound modes.[1)] This change is based on the fact that the canonically conjugate variables are not unique, and there exists an entire class of transformations, called canonical,[2] from one set to another, and in addition the Hamiltonian structure of the equations of motion remains. In particular, it is possible to change over to variables in which the quadratic part of the Hamiltonian H (corresponding to the linear equations) will be diagonal with respect to the variables characterizing first and second sounds. The change from the Fourier components of the quantities ρ, S, α, and β over to normal coordinates $a_{\mathbf{k}}^{\nu}(t)$ (the indices $\nu = \pm 1$ and ± 2 identify the wave mode and the minus sign indicates complex conjugation) is performed in Ref. 6.[2)] The normal coordinates $a_{\mathbf{k}}^{\nu}(t)$, which are also called complex amplitudes, satisfy the following equations:

$$i\frac{\partial a_{\mathbf{k}}^{\nu}}{\partial t} = \operatorname{sign}\nu \frac{\delta H}{\delta a_{\mathbf{k}}^{-\nu}} \quad (\nu = \pm 1, 2). \tag{1.17}$$

The Hamiltonian H is a series in integral powers of the variables $a_{\mathbf{k}}^{\nu}$. The quantity H has the following form up to third order inclusively:

$$\begin{aligned} H &= \int \omega_{\mathbf{k}}^1 a_{\mathbf{k}}^1 a_{\mathbf{k}}^{-1}\, d\mathbf{k} + \int \omega_{\mathbf{k}}^2 a_{\mathbf{k}}^2 a_{\mathbf{k}}^{-2}\, d\mathbf{k} \\ &+ \sum_{\nu_i = \pm 1, 2} \int V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3} a_{\mathbf{k}_1}^{\nu_1} a_{\mathbf{k}_2}^{\nu_2} a_{\mathbf{k}_3}^{\nu_3} \delta\left(\sum_{j=1}^{3} \mathbf{k}_j \operatorname{sign}\nu_j\right) d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3; \end{aligned} \tag{1.18}$$

here $\omega_{\mathbf{k}}^1 = c_1|\mathbf{k}|$ and $\omega_{\mathbf{k}}^2 = c_2|\mathbf{k}|$ are the dispersion laws of the two types of sound. The sound velocities c_1 and c_2 can be calculated in the standard fashion from the linearized equations (1.16), in which the Hamiltonian H is expressed in terms of the variables $\delta\rho$, δS, α, and β with quadratic accuracy with respect to these quantities. Neglecting the quantity

$$\frac{c_1^2 c_2^2}{c_1^2 - c_2^2}\frac{\beta_T^2}{\sigma_T}$$

(β_T is the coefficient of expansion), which is small in practically the entire range of temperatures, the velocities c_1 and c_2 are equal to, respectively,

$$c_1 = \left(\frac{\partial p}{\partial\rho}\right)^{1/2}, \quad c_2 = \left(\frac{\rho_s S^2}{\rho_n \rho}\frac{\partial T}{\partial S}\right)^{1/2}. \tag{1.19}$$

The coefficients $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3}$, which contain all information about the nonlinear interaction of waves (in this approximation), are called matrix elements or the vertex parts (vertices). They can be calculated in the standard fashion by expanding the energy $E(\rho, S, \mathbf{j}_0, \mathbf{v}_s)$ in the deviations from equilibrium and then changing over to the quantities $a_{\mathbf{k}}^{\nu}$. We shall not write out here the unwieldy expressions for the vertices V. We only note the important fact that they all have the following structural dependence on the arguments k:

$$\begin{aligned} V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3} &= (k_1k_2k_3)^{1/2}\left(P_1^{\nu_1\nu_2\nu_3} + P_2^{\nu_1\nu_2\nu_3}\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\right. \\ &\left. + P_3^{\nu_1\nu_2\nu_3}\frac{\mathbf{k}_2\mathbf{k}_3}{k_2k_3} + P_4^{\nu_1\nu_2\nu_3}\frac{\mathbf{k}_1\mathbf{k}_3}{k_1k_3}\right), \end{aligned} \tag{1.20}$$

i.e., they are homogeneous functions of degree 3/2. In what follows we shall employ the terminology introduced in Ref. 6 for the different types of nonlinear processes. If two of three indices ν_j are equal to ± 2, then such processes are called decomposition processes. If two of three indices ν_j are equal to ± 1, then the corresponding processes are called Cherenkov processes. If, finally, all indices are equal to ± 1 (or ± 2), then such processes are called characteristic nonlinear processes in the first (or second) sound modes, respectively.

2. ONE-DIMENSIONAL NONLINEAR WAVES

2.1. The characteristic form of the equations of nonlinear acoustics

In this section we shall study the laws of propagation of one-dimensional nonlinear waves. We shall start with the simplest case of dissipation-free equations (1.1)–(1.4). We assume that the waves propagate along the x axis. We choose the following quantities as the variables: the perturbation of the density ρ', the x-component of the mean-mass velocity v_x ($\mathbf{v} = \mathbf{j}/\rho$), the perturbation of the entropy per unit mass σ', and the x-component of the relative velocity w_x ($\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s$). This choice is convenient in that in the linear case the first pair ρ', v_x describes first sound and the second pair σ', w describes second sound. Up to terms of second order in these quantities (in this section we confine our attention to this accuracy everywhere except in Sec. 2.9) the equations of motion can be written as follows:

$$\frac{\partial\varphi_i}{\partial t} + \sum_j A_{ij}(\varphi)\frac{\partial\varphi_j}{\partial x} = 0 \quad (i, j = 1, 2, 3, 4); \tag{2.1}$$

here φ is a column vector consisting of the quantities $\varphi_1 = \rho'$, $\varphi_2 = v$, $\varphi_3 = \sigma'$, $\varphi_4 = \omega$ (we drop the index x). The dependence of the matrix elements $A_{ij}(\varphi)$ on the variables φ_k is no stronger than linear. The matrix $A_{ij}(\varphi)$ is written out in an explicit form in Ref. 7.[3)]

In the linear case (see Sec. 2.2) the matrix A_{ij} has a block-diagonal form and the system of equations (2.1) decomposes into two subsystems, each having wave solutions in which only the quantities ρ' and v (first sound) or σ' and w (second sound) oscillate. Any general solution is a superposition of these two sounds. In the nonlinear case the matrix $A_{ij}(\varphi)$ is not a block-diagonal matrix, the sounds are "entangled," the pairs ρ', v and σ', w no longer represent "pure" wave modes and the excitation of any mode results in oscillations of all variables. It is intuitively obvious that in this case both types of waves arise in helium, but what kind of waves they are, what they transport, how they propagate, interact, etc. remain open questions. Significant progress has been made in the study of this problem with the help of the method of Riemann invariants, which can be developed here for waves traveling in one direction along the x axis.

Following Ref. 8 we multiply Eqs. (2.1) by the left row eigenvector $l(\varphi)$, defined as follows, for the matrix $A_{ij}(\varphi)$: $\Sigma\, l_j(\varphi)A_{ij} = \xi l_i$, where ξ is an eigenvalue. For different eigenvectors $l^{(\mu)}(\varphi)$ we have

$$\sum_i l_i^{(\mu)}\left(\frac{\partial\varphi_i}{\partial t} + \xi^{(\mu)}\frac{\partial\varphi_i}{\partial x}\right) = 0 \quad (i,\ \mu = 1, 2, 3, 4). \qquad (2.2)$$

In order that Eqs. (2.2) not contain infinitesimals of order higher than second the dependence of the elements of the row vector $l_i(\varphi)$ on the quantities φ_k must not be stronger than linear. The form of the equations of motion presented above is distinguished by the fact that in each of the four relations (2.2) all variables are differentiated in the same direction in the x, t plane. These directions, called characteristics, are determined by the equalities $dx/dt = \xi^{(\mu)}(\varphi(x,t))$, $\mu = 1, 2, 3, 4$.

For different eigenvectors $l^{(\mu)}(\varphi)$ the characteristics $\xi^{(\mu)}(\varphi)$ have the following form:

$$\xi^{(1,2)} = v \pm \left(c_1 + \frac{1}{2c_1}\frac{\partial^2 p}{\partial\rho^2}\rho'\right), \qquad (2.3)$$

$$\xi^{(3,4)} = v + \left(\frac{2\rho_s}{\rho} + \frac{\sigma}{\rho_n}\frac{\partial\rho}{\partial\sigma}\right) w \pm \left(c_2 + \frac{\partial c_2}{\partial\sigma}\sigma' + \frac{\partial c_2}{\partial\rho}\rho'\right). \qquad (2.4)$$

If the Pfaffian $\Sigma_i l_i^{(\mu)}(\varphi)d\varphi_i$ can be integrated, i.e., it is a total differential of some quantity $I_\mu(\varphi)$, then Eqs. (2.2) can be further simplified:

$$\frac{\partial I_\mu}{\partial t_{j_1}} + \xi^{(\mu)}\frac{\partial I_\mu}{\partial x} = 0 \quad (\mu = 1, 2, 3, 4). \qquad (2.5)$$

The remarkable property of the system (2.5), which makes this equation exceptionally convenient for studying concrete problems, is that each of the equations (2.5) describes a conservation law of the quantity $I_\mu(\varphi)$ along the characteristic direction. The quantities $I_\mu(\varphi)$ are called Riemann invariants (RIs). If only one RI is given initially, then, as follows from Eq. (2.5), it will remain the only invariant. Thus the disturbances transported by RIs are in this sense independent. This fact as well as the relatively simple form of the system (2.5) permit regarding the waves described by the Riemann invariants I_1 and I_2 as well as I_3 and I_4 as analogs of first and second sounds.

2.2. Linear acoustics

In the limiting case of infinitesimal amplitudes, when in the quantities $l^{(\mu)}(\varphi)$ and $\xi^{(\mu)}(\varphi)$ the dependence on φ can be neglected, the Pfaffians are a sum of differentials $d\varphi_i$ with constant coefficients. Such forms, of course, can be integrated. The linear RIs I_μ^0 have the following form:

$$I_{1,2}^0 = \rho' \pm \frac{\rho}{c_1} v, \quad I_{3,4}^0 = \sigma' \pm \frac{\rho_s\sigma}{\rho c_2} w. \qquad (2.6)$$

The evolution of the quantities I_μ^0 is described by Eqs. (2.5), in which $\xi^{(1,2)} = \pm c_1$ and $\xi^{(3,4)} = \pm c_2$. Thus we have arrived at the classical result (see Ref. 1) that HeII supports two types of sound waves. One can see from the expressions (2.6) that with an initial perturbation of the density ρ' and (or) the velocity v only the first-sound wave propagates in the liquid. Conversely, with an initial perturbation of the entropy σ' and (or) the relative velocity w only a second-sound wave will propagate in helium. Thus, neglecting the terms connected with the coefficient of expansion β_T, the wave modes are independent.

The particular case of waves traveling in one direction can be obtained by setting the RIs I_2^0 and I_4^0 equal to zero. In this case there are no waves traveling to the left along the x axis, while a "hard" dependence between the quantities ρ' and v in first sound and σ' and v in second sound appears in waves traveling to the right:

$$\rho' = \frac{\rho}{c_1} v, \quad \sigma' = \frac{\rho_s\sigma}{\rho c_2} w. \qquad (2.7)$$

In the linear case it is not difficult to include in this scheme terms containing β_T. As the calculations show, for waves traveling to the right the RIs have the following form:

$$I_1^\beta = \rho' - \sigma'\frac{\partial\rho/\partial T}{\partial\sigma/\partial T}, \quad I_3^\beta = \sigma' + \frac{c_2^2}{\rho}\frac{\partial\rho}{\partial T}\rho'. \qquad (2.8)$$

Aside from RIs the velocities of sound c_1 and c_2 change by small amounts ($\propto\beta_T$). Unlike Eqs. (2.6) the expressions (2.8) contain cross terms (for example, the variable σ', characterizing second sound, appears in I_1^β), which are small terms of order β_T. The existence of these cross terms results in "coupling" of the two types of sound. For example, in the process of pulling out a piston it is possible to detect, in addition to the standard density wave, an entropy wave moving with velocity close to c_2. In addition, the density wave itself, moving with a velocity close to c_1, contains a small "admixture" of an entropy perturbation σ'. However it is more accurate to talk here not about coupling of the two types of sound but rather about a different representation of the sound modes. The independent oscillations are not the oscillations of the density ρ' and the entropy σ' but rather some combinations of them which, evidently, are identical to the RIs. In connection with the results described above we call attention to the recent paper,[81] in which a general formalism is proposed for describing linear "coupled" acoustic systems, in particular, for HeII.

2.3. Reimann invariants in the nonlinear case

In the case of finite (but small) amplitudes the Pfaffians are a sum of four differentials $d\varphi_i$ with coefficients that depend on φ_k (not more strongly than linearly). Such forms, generally speaking, cannot be integrated, and the RIs cannot be obtained in the general case. However it turned out that this can be done for waves traveling in one direction.[7]

We shall recall what waves traveling in one direction or simple waves, as they are called in standard gas dynamics, are. The apparatus of simple waves, developed by Reimann, played a very important role in the solution of different gas-dynamic problems (see Refs. 8 and 9). From the mathematical viewpoint simple waves are a particular case of the solution of Euler's equations, in which the unknown ρ' and v are related by some functional dependence $v = v(\rho')$. To obtain the evolutionary equations for simple waves the following device is employed. The function $v(\rho')$ is substituted into Euler's equation, and derivatives of the type $\partial v/\partial t$ are expanded as $(dv/d\rho)\partial\rho'/\partial t$, etc. This gives a system of algebraic equations for the derivatives $\partial\rho'/\partial t$ and $\partial\rho'/\partial x$. The condition that this system be compatible permits determining the function $v(\rho')$, with whose help it is easy to derive the evolutionary equation for the quantity $\rho'(x,t)$.

In the case of superfluid helium the analogous technique cannot be employed. Indeed, as shown in the preceding section, in the linear case the quantities ρ' and v as well as σ' and w in waves traveling to the right are related by a functional dependence of the following form (see Eq. (2.7)):

$$v=v(\rho'),\quad w=w(\sigma'). \tag{2.9}$$

In the linear case the matrix $A_{ij}(\varphi)$ is not a block-diagonal matrix. It contains off-diagonal elements, as a result of which cross terms, which are quadratic in the variables φ_k, appear in the equations of motion. It is thus natural to assume that in the linear case the relations (2.9) are functions of the following form:

$$v=v(\rho',\sigma'),\quad w=w(\sigma',\rho'), \tag{2.10}$$

and the dependence on the second argument is of second order.[4)] If, next, the functions $v(\rho',\sigma')$ and $\omega(\sigma',\rho')$ are sought from the condition that the algebraic equations (for the variables $\partial\rho'/\partial t$, $\partial\rho'/\partial x$, $\partial\sigma'/\partial t$, $\partial\sigma'/\partial x$ be compatible, then we obtain one condition for two functions, i.e., the dependences sought cannot be found in this manner.

Let us see what simple waves are from a somewhat more fundamental viewpoint. For the standard gas dynamics the scheme described above for deriving the RIs can be realized for isentropic flows. Indeed, in this case there are only two variables (ρ', v) and the Pfaffian form can always be integrated. As a result, in the standard gas dynamics there are two RIs, corresponding to two different characteristics. If one of them is identically equal to zero, then the remaining RI describes a simple wave. Thus the two basic properties of waves traveling in one direction (following trivially one from the other in the standard gas dynamics) are as follows: first, the existence of a functional dependence between the variables and, second, waves traveling in the other direction vanish identically. These two properties can be employed to extend the apparatus of simple waves to the case of superfluid helium. Referring the reader to Ref. 7 for the details of the calculations, we write out the final result.

Waves propagating in the positive direction along the x axis can be described with the help of the RIs I_1 and I_3, which are equal to

$$I_1=\rho'+\alpha_1(\rho')^2+\alpha_2(\sigma')^2, \tag{2.11}$$
$$I_3=\sigma'+\beta_1(\sigma')^2+\beta_2\rho'\sigma', \tag{2.12}$$

where we used the notation

$$\alpha_1=\frac{1}{2}\left(\frac{1}{2c_1^2}\frac{\partial^2 p}{\partial\rho^2}-\frac{1}{\rho}\right),$$

$$\alpha_2=\left(\frac{\rho c_2}{\rho_s\sigma c_1}\right)^2\left(\frac{2\rho_s\rho_n}{\rho}-\rho^2\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right),$$

$$\beta_1=\frac{1}{4}\left(\frac{\partial^2T/\partial\sigma^2}{\partial T/\partial\sigma}+\frac{2\rho_n}{\rho_s}\frac{\partial}{\partial\sigma}\frac{\rho_n}{\rho}\right),$$

$$\beta_2=\frac{1}{4\rho}\left[\frac{c_1+c_2}{c_1-c_2}-\frac{c_2\rho^2}{c_1-c_2}\left(\frac{1}{\rho_s}-\frac{1}{\rho_n}\right)\frac{\partial}{\partial\rho}\frac{\rho_s}{\rho}\right]. \tag{2.13}$$

The Riemann invariant I_1 and I_3 satisfy Eqs. (2.5), in which the characteristics $\xi^{(1)}$ and $\xi^{(3)}$ must be expressed in terms of I_1 and I_3.

Thus in the case of waves traveling in one direction the starting system of four equations (2.1) can be reduced to two equations of the type (2.5), each of which describes a conservation law for I_1 (or I_3) along the characteristic direction $\xi^{(1)}$ (or $\xi^{(3)}$). The asymmetric character of the dependence of the RIs I_1 and I_3 on the variables ρ' and σ' is a consequence of the fact that the thermodynamic quantities depend on the relative velocity w and, of course, they do not depend on the mean mass velocity v (see Sec. 1.1 and also Ref. 1).

2.4. The nonlinear decomposition of an entropy wave

The special form of the equations of motion written in the form of RIs significantly simplifies the study of different concrete problems.

As an example we shall study the propagation of waves in HeII in two cases: a) a density disturbance is created at the boundary $x = 0$ (a piston moves); b) an entropy disturbance is created at the boundary $x = 0$ (a wall is heated).

Before we attack this problem it is useful to express the dependence of ρ' and σ' on the RIs I_1 and I_3 in an explicit form. To the order considered the dependence sought will be as follows:

$$\rho'=I_1-\alpha_1I_1^2-\alpha_2I_3^2, \tag{2.14}$$

$$\sigma'=I_3-\beta_1I_3^2-\beta_2I_1I_3. \tag{2.15}$$

In the case a) we have at the boundary $x=0$ $\rho'(0,t)=\rho_0(t)$ and $\sigma'(0,t)=0$. In accordance with the formulas for the RIs (2.11)–(2.12) in this case only one of the RIs I_1 is different from zero. This invariant, in accordance with the first of the equations (2.5), remains constant along the characteristic $\xi^{(1)}$. Thus it describes a wave propagating with a velocity close to c_1. In this wave, as one can see from the inversion formulas (2.14)–(2.15), there is only a disturbance of the density ρ', while the disturbance of the entropy σ' is equal to zero. Thus in the case of a density disturbance at the boundary of the liquid only a wave of density ρ' (and, of course, pressure p' and velocity v) propagates in the volume of the helium.

In the case b) the situation turned out to be somewhat more interesting. If an entropy disturbance $\sigma'(0,t)=\sigma_0(t)$ is created at the wall $x=0$ (and $\rho'(0,t)=0$), then, as one can easily see, both RIs are different from zero. Since the waves related with I_1 and I_3 propagate with different velocities, in the x, t plane they separate, as shown in Fig. 1. The wave carrying the invariant I_1 leads with velocity $\xi^{(1)}\approx c_1$. In this wave, as one can see from the inversion formulas (2.14)–(2.15), the density disturbance ρ' is, in order of magnitude, $\rho'\sim\alpha_2\sigma'^2$, and the entropy disturbance $\sigma'=0$. A wave transporting I_3 moves with velocity $\xi^{(3)}\approx c_2$ immediately behind the wave transporting I_1. Both the density dis-

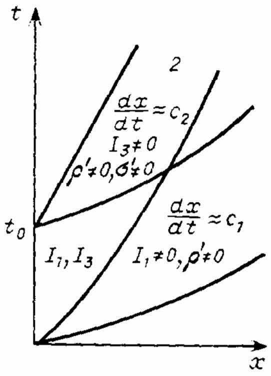


FIG. 1. Illustration of the nonlinear decomposition of an entropy pulse $\sigma'(0,t)$, arising when a wall is heated during a time t_0. The wave transporting the Riemann invariant I_1 propagates along the characteristics filling the band 1. In this wave, as follows from Eqs. (2.14) and (2.15), only the disturbance of the density is different from zero. The slope of the band 1 is equal to $dx/dt = \xi^{(1)} \approx c_1$. The wave transporting the Riemann invariant I_2 propagates along the characteristics filling the band 2. In this wave both the disturbance of the density and the disturbance of the entropy are different from zero.

turbance $\rho' \sim -\alpha_2\sigma_0^2$ and the entropy disturbance $\sigma' \sim \sigma_0$ are related with I_3.

Thus the following picture is observed in the case of pulsed heating of a wall. A "precursor"—a density wave—moves away from the wall with the velocity of first sound. A "mixture" of density and entropy waves follows immediately behind it with the velocity of second sound.

We shall examine the magnitude of the described effect. One can see from the expression (2.11) for the RI I_1 and the inversion formulas (2.14)–(2.15) that the measure of the transformation of the second sound into a "precursor" is determined quantitatively by the coefficient α_2 [see Eq. (2.13)]. Therefore the pressure δp in the "precursor" will be as follows:

$$\delta p = \frac{W^2}{2(\rho_s\sigma T)^2}\left(\frac{2\rho_s\rho_n}{\rho} - \rho^2\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right). \tag{2.16}$$

Numerical estimates show that the second term in the parentheses is always greater than the first term (compare with Ref. 12). In the presence of a heat pulse with amplitude W of the order of 10 W/cm^2 ($T \simeq 1.8$) the pressure in the precursor will be of the order of $\delta p \simeq -10^4$ g/cm$\cdot$s^2.

The question of the creation of a pressure wave accompanying the pumping of second sound into the system was discussed previously in Ref. 12. However the method of successive approximations in the form employed in Ref. 12 did not make it possible to describe the nonlinear distortion of the wave and separate the signal into a "precursor" and a main pulse.

As analysis shows, taking into account the thermal compressibility of helium β_T results in the fact that in both cases studied there arise two waves, in which there are disturbances of all hydrodynamic variables.[82] Two mechanisms for transfer of wave energy into a "foreign" mode can be distinguished: first, thermal decomposition, studied in Sec. 2.2, and second, the nonlinear decomposition described in this section. The coefficients α and β appearing in the expressions (2.11)–(2.12) for the RIs and describing the nonlinear decomposition change by amounts proportional to β_T. It is shown in Ref. 82 that under real experimental conditions it is important to take both mechanisms into account.

2.5. The evolution of intense waves

We shall study in greater detail the propagation of nonlinear waves in accordance with the evolutionary equations (2.5). We shall first consider second sound. As shown in the preceding section, under the conditions of an entropy disturbance at the boundary there arise two waves in helium. In the leading wave the amplitudes of the disturbances are small, as a result of which the characteristic $\xi^{(1)}$ is equal to c_1 to the order considered. As a result, the evolution of this wave is determined by the first of Eqs. (2.5), in which $\xi^{(1)} = c_1$, i.e., it is described by the usual linear theory. In the trailing wave, transporting I_3 (in this wave $I_1 = 0$), the characteristic $\xi^{(3)}$ differs from c_2 already within the chosen accuracy and depends on the values of the oscillating variables ρ', v, σ', and w. However not all these variables are independent. First of all, we have the functional relations $v = v(\rho',\sigma')$ and $\omega = \omega(\sigma',\rho')$ [see Eq. (2.10)]. Second, the fact that in the trailing wave the RI $I_3(\rho', \sigma')$ vanishes imposes a relation between ρ' and σ'. In reality in such a wave there is only one independent variable, for example, $w(x,t)$. It is more convenient, however, to work not with the variable w but rather with the related quantity v_n. Expressing the characteristic $\xi^{(3)}$ and the RI I_3 in terms of v_n and substituting them into the third equation of Eqs. (2.5) we arrive at the following result:

$$\frac{\partial v_n}{\partial t} + (c_2 + \alpha_2(T)v_n)\frac{\partial v_n}{\partial x} = 0; \tag{2.17}$$

here $\alpha_2(T)$ is the coefficient of nonlinearity of second sound, equal to

$$\alpha_2(T) = \frac{\sigma T}{C_1}\frac{\partial}{\partial T}\left(c_2^3\frac{\partial\sigma}{\partial T}\right)_p. \tag{2.18}$$

Analogous arguments lead to the conclusion that with an initial density perturbation the leading wave can be described by the single variable $v(x,t)$, the equation for which has the following form:

$$\frac{\partial v}{\partial t} + (c_1 + \alpha_1(T)v)\frac{\partial v}{\partial x} = 0; \tag{2.19}$$

here

$$\alpha_1(T) = \left(1 + \frac{\rho}{2c_1^2}\frac{\partial^2 p}{\partial\rho^2}\right) = 1 + \frac{\partial \ln c_1}{\partial \ln \rho}. \tag{2.20}$$

The results presented essentially are identical to the results obtained previously by Khalatnikov,[11] who found the nonlinear corrections to the velocities of the different types of sound (he actually calculated $\alpha_1(T)$ and $\alpha_2(T)$), using the method of simple waves for the starting system (2.1), i.e., under the assumption that all variables sought ρ', v, σ', and w are functionally related with one of them. In the method of RIs there are two such variables, but in the particular case described in this section, when one of the RIs is equal to zero, there arises an additional (with respect to the relations (2.10)) relation between the starting variables, as a result of which there remains one independent function. In other words, the vanishing of one of the RIs indicates a transition from simple waves of rank 2 (see footnote 4) to standard simple waves.

There is no special difficulty in solving the boundary-

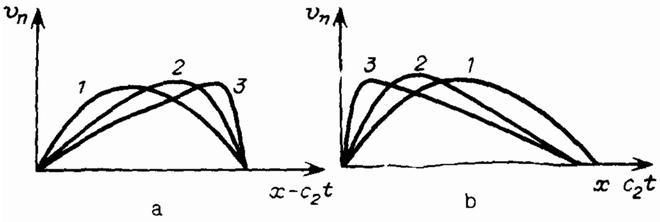


FIG. 2. Schematic diagram of the evolution of a pulse of first sound (a) and a pulse of second sound with a negative coefficient of nonlinearity $\alpha_2(T)<0$ (b). The wave profiles are shown in coordinate systems moving with the corresponding sound velocities. The numbers 1–3 enumerates sequential moments in time.

value problem for equations of the type (2.17)–(2.19) (see, for example, Refs. 8, 9, and 13). For greater clarity, however, we shall describe the solution starting from qualitative considerations. For infinitesimal amplitudes (the linear case) Eqs. (2.17) and (2.19) describe the propagation of initial disturbances as a whole with velocities c_2 and c_1, respectively. For finite but small amplitudes the expressions $\alpha_2(T)v_n$ [or $\alpha_1(T)v$] in Eqs. (2.17) and (2.19) can be interpreted as corrections to the velocities of propagation of the different sounds c_2 (or c_1), which, however, depend on the local value of the oscillating quantity v_n (or v). This means that different sections of the wave pulse move with different velocities and the wave profile (the form of the wave in the coordinates $v_n, x-c_2t$ or $v, x-c_1t$) becomes deformed. The wave steepens and a shock front forms, after which the equations are no longer valid and additional considerations regarding the further evolution of the waves are required.

The case of first sound is identical, right up to the notation, to the standard gas-dynamics. The quantity $\alpha_1(T)$ for helium is equal to approximately four, i.e., it is positive. As a result the pulse of first sound with a positive density disturbance $\rho'>0$ ($v>0$) behaves as follows. The more intense sections move with a high velocity, the hump catches up with the base of the wave, the leading edge becomes steeper, and a discontinuity forms on it. The evolution of pulses is shown schematically in Fig. 2a. The characteristic time of formation of a discontinuity τ_d is equal to $\tau_p \sim (\alpha_1(T)\nu k)^{-1}$, where k^{-1} is the spatial size of the disturbance. In the case of a negative pulse ($\rho'<0$, $v<0$) the hump lags behind, and a discontinuity forms on the trailing edge of the wave. In the Fourier representation this process describes the creation of higher-order harmonics.

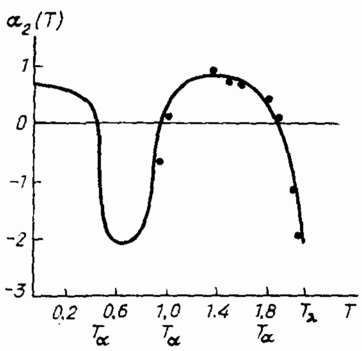


FIG. 3. The coefficient of nonlinearity $\alpha_2(T)$ as a function of the temperature.[11] The dots correspond to the experimental results of Ref. 15.

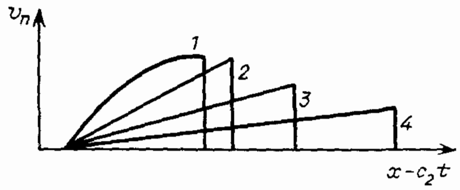


FIG. 4. Schematic diagram of the propagation of pulses of second sound after the formation of a discontinuity. The coefficient of nonlinearity $\alpha_2(T)>0$.

The case of second sound is somewhat more interesting. The coefficient $\alpha_2(T)$ is a complicated function of the temperature (Fig. 3). One can see from the figure that there are regions of both positive and negative values of $\alpha_2(T)$. If $\alpha_2(T)<0$, then the entropy pulse with $\sigma'>0$ ($v_n>0$) becomes steeper and a discontinuity forms on the trailing edge of the wave (Fig. 2b). For positive $\alpha_2(T)$ the situation is identical to the case of first sound (Fig. 2a). At temperatures $T_\alpha \approx 1.88$ K and $T_\alpha \approx 0.9$ K the quantity $\alpha_2(T)$ is equal to zero, the quadratically nonlinear term vanishes, and the evolution of the pulses is driven by the terms of the next higher (third) order (see Sec. 2.9). The formation of discontinuities on the trailing edge of the wave is a phenomenon specific to HeII.

The evolution of the pulse after the discontinuity forms can be described with the help of the conservation law for the quantity

$$\int_{-\infty}^{\infty} v_n\, dx,$$

which follows from Eq. (2.17) (for definiteness we shall consider the case of second sound). From this law it follows, in particular, that the discontinuity with amplitude Δv_n moves with the velocity U_d equal to

$$U_p = c_2 + \frac{\alpha_2(T)\,\Delta v_n}{2}\,. \tag{2.21}$$

The evolution of the wave is shown qualitatively in Fig. 4. Since the discontinuity moves (in the comoving system) with velocity $\alpha_2(T)\Delta v_n/2$ (2.21) the spatial extent of the pulse increases. From the conservation law for the quantity

$$\int_{-\infty}^{\infty} v_n\, dx$$

it follows that the amplitude of the discontinuity Δv_n should decrease. Asymptotically the solution is described by a triangle, whose amplitude decays with time as $t^{-1/2}$. We call attention to the fact that in spite of the decrease in the amplitude, it is still not possible to transfer to the linear case at any stage of the propagation of the wave. The described behavior of intense heat pulses has been observed in many experimental works.[13-19] Good quantitative agreement has been obtained for the quantity $\alpha_2(T)$ (Ref. 15) and for the transit time of the pulse, calculated according to Eq. (2.17).[13]

2.6. Damped waves

In this section we shall study the effect of the dissipative terms on the propagation of one-dimensional waves. The dissipative terms can be taken into account in the derivation of the equations for traveling waves on the basis of perturbation theory, regarding the waves as small disturbances. Actually, the nonlinear and viscous terms must be infinitesimals of the same order of magnitude. We shall derive the equation for the wave of second sound. Adding dissipative terms to the last two equations in the system (2.1) we obtain

$$\frac{\partial\sigma'}{\partial t}+A_{33}\frac{\partial\sigma'}{\partial x}+A_{34}\frac{\partial w}{\partial x}=\frac{\varkappa}{\rho C}\frac{\partial^2\sigma'}{\partial x^2}, \tag{2.22}$$

$$\frac{\partial w}{\partial t}+A_{43}\frac{\partial\sigma'}{\partial x}+A_{44}\frac{\partial w}{\partial x}=b\frac{\partial^2 w}{\partial x^2}; \tag{2.23}$$

here A_{ij} are the elements of the matrix $A_{ij}(\varphi)$ (see Sec. 2.1) and the quantity b is equal to

$$b=\frac{\rho_s}{\rho\rho_n}\left[\frac{4}{3}\eta-(\zeta_1+\zeta_4)\rho+\zeta_2+\zeta_3\rho^2\right].$$

In Eqs. (2.22)–(2.23) we drop terms including ρ' and v since they are third-order infinitesimals (see Sec. 2.5).

In the absence of dissipative terms Eqs. (2.22)–(2.23) have a solution in the form of a traveling wave, in which there is a functional relation between the variables $w=w(\sigma')$ [see Eq. (2.10)]; we recall that $I_1(\rho',\sigma')=0\Rightarrow\rho'=\rho'(\sigma')$. In the presence of dissipation this relation is not satisfied. It can be assumed, however, that viscosity will change this relation by some quantity $\psi(x,t)$, which is a second-order infinitesimal, i.e.,

$$w=w(\sigma')+\psi(x,t). \tag{2.24}$$

Solutions of the type (2.24) are called quasisimple waves.[20] The function $\psi(x,t)$ can be found by the methods of perturbation theory (see Ref. 20). Using next the dependence (2.24) in the system (2.22)–(2.23) we arrive at the following equation for v_n:

$$\frac{\partial v_n}{\partial t}+(c_2+\alpha_2(T)v_n)\frac{\partial v_n}{\partial x}=\mu_2\frac{\partial^2 v_n}{\partial x^2}, \tag{2.25}$$

where

$$\mu_2=\frac{\rho_s}{2\rho\rho_n}\left(\frac{4}{3}\eta+\zeta_2-2\rho\zeta_1+\rho^2\zeta_3+\frac{\rho_n\varkappa}{\rho_s C}\right).$$

An equation of the type (2.25) is often encountered in the theory of nonlinear waves and is called Burgers equation. In a reference system moving with the velocity c_2 $(x\to x-c_2t)$ Burgers equation can be reduced, by making the substitution

$$v_n=-\frac{2\mu_2}{\alpha_2(T)}\frac{\partial\ln\varphi}{\partial x}$$

to the heat-conduction equation

$$\varphi_t=\mu_2\varphi_{xx}.$$

As a result the problem of the evolution of a nonlinear wave in a viscous medium can be solved in a closed analytical form (see the monographs of Refs. 20 and 21). The calculations show that the initial behavior of the wave is close to the nondissipative case. For example, the nonlinear term $\alpha_2(T)v_n\partial v_n/\partial x$ describes, as before, the steepening of the wave profile. In contradistinction to the nonviscous case, however, this steepening does not result in the formation of a discontinuity. Indeed, as the steepening increases, the initially small dissipative term $\mu_2\,\partial^2 v_n/\partial x^2$ increases owing to the increase in the velocity gradient and prevents further distortion of the wave profile. Comparing the order of magnitude of the nonlinear and viscous terms in (2.25) we find that the characteristic extent of the wavefront is $\delta x\sim\mu_2/\alpha_2(T)\Delta v_n$, where Δv_n is the amplitude of the pulse. The starting stage of the evolution of the pulse is shown schematically in Fig. 5a. The further evolution of the wave occurs in accordance with the conservation law for the quantity

$$\int_{\infty}^{\infty} v_n\,dx$$

(see Sec. 2.5), which also holds for Burger's equation. The extent of the pulse increases and the amplitude of the pulse decreases in a manner such that the area of the wave profile remains constant. As the amplitude decreases the extent of the steep front increases $\delta x\propto 1/\Delta v_n$. The final stage of the evolution, when the "spreading" of the front δx becomes comparable to the size of the pulse, can be described on the basis of the linear theory. This is what makes the propagation of a damped wave different from the nondissipative case, where the linear limit does not exist. The evolution of a pulse after steepening of the leading edge is shown schematically in Fig. 5b.

2.7. Dispersion of second sound

The linearized Burgers equation (2.25) leads to the following dispersion law for ω_k^2 for a monochromatic wave in which $v_n\propto\exp[i(\omega_k^2t-\mathbf{kr})]$:

$$\omega_k^2=c_2|\mathbf{k}|+i\mu_2k^2. \tag{2.26}$$

The expression (2.26) can be regarded as the first terms of the expansion of the frequency ω_k^2 in powers of the wave vector $\mathbf{k}$. The second term, describing the damping, is an imaginary quantity. The next term in the expansion describes the dispersion of the velocity of sound. We shall write the frequency of second sound ω_k^2 in the following form, taking into account the dispersion of the velocity:

$$\omega_k^2=c_2|\mathbf{k}|+iD_2k^2+D_3k^3. \tag{2.27}$$

Unlike Eq. (2.25) here we denote the damping constant as D_2, having in mind the fact that it can differ from the quantity μ_2. Actually the dispersion (and, by the way, the damping also) is connected with the existence of an internal

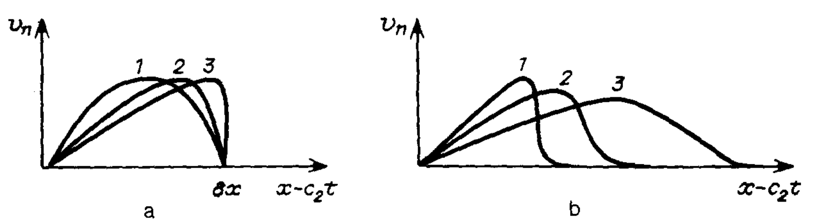


FIG. 5. The evolution of a nonlinear pulse of second sound in a dissipative medium. At the first stage (a) the wave profile becomes steeper and a "diffuse" shock front is formed. At the second stage (b) the duration increases and correspondingly the amplitudes decrease.

structure of the liquid (temporal or spatial) and the expression (2.27) is the expansion of the quantity ω_k^2/c_2k in powers of the dimensionless parameter kl, where l is the internal scale of the system. For example, in the standard acoustics l is determined by the mean free path of the molecules. The first term in Eq. (2.27) corresponds to the Euler approximation, the second term corresponds to the Navier-Stokes approximation, and the third term takes into account the Barnett corrections. In helium at low temperatures l can be the free path of the quasiparticles.

One region where the effect of dispersion could be significant is the neighborhood of the λ transition. Near T_λ the dispersion of the velocity of second sound is caused by relaxation effects and the interaction of sound waves with developed fluctuations of the thermodynamic quantities.[1,22,23] According to the fluctuation theory of phase transitions, the dispersion law for second sound is a function of the dimensionless parameter $k\xi$, where ξ is the correlation radius (see Refs. 22 and 23). We shall rewrite the expression (2.27), taking into account what was said above, in the following form:

$$\omega_{\mathbf{k}}^2 = c_2 |\mathbf{k}| (1 + iA_2\xi k + A_3\xi^2k^2). \qquad (2.28)$$

The coefficient A_2 can be determined from measurements of the damping of second sound[24-26] or experiments on the scattering of light,[27] and is of the order of $A_2 \sim 0.1$. The situation is more complicated in the case of A_3. The point is that the dispersion is very small and it is virtually impossible to measure the dispersion by the methods of linear acoustics. Indeed, even such record-high frequencies for second sound as $\nu \sim 10^4$ Hz result in a relative change of the velocity $\Delta c_2/c_2$ equal to 10^{-6} at $T_\lambda - T \sim 51$ mK (here the fact that A_3 is of the order of unity was employed for the estimate). In the nonlinear case, as one can see from the preceding presentation, high-frequency harmonics arise spontaneously in the wave and the dispersion can play a significant role in the evolution of sound pulses.

Taking the dispersion into account the evolutionary equation for a wave of second sound has the following form:

$$\frac{\partial v_n}{\partial t} + (c_2 + \alpha_2(T) v_n) \frac{\partial v_n}{\partial x} = D_2 \frac{\partial^2 v_n}{\partial x^2} - D_3 \frac{\partial^3 v_n}{\partial x^3}. \qquad (2.29)$$

It can be derived in a manner similar to Burgers equation (2.25), under the assumption that the terms on the right side do not exceed in magnitude the nonlinear term.[28] The relation (2.29) is also encountered in the theory of nonlinear waves and is called the Korteweig–de Vries Burgers equation (KdVB).

If the parameters of the waves are such that the first term on the right side of the KdVB equation is much greater than the second term, then dispersion can be neglected and evolution of the wave occurs in the manner described in the preceding section. In the reverse case, when the dissipation is small, the relation (2.29) reduces to the well-known KdV equation (see, for example, Refs. 20, 21, and 29). The KdV equation can be solved analytically with the help of the so-called inverse-problem method of the theory of scattering (IPMTS). One of the most remarkable results of IPMTS is the conclusion that the initial pulse decomposes into a number of separate soliton disturbances, whose form remains constant and which move with constant, amplitude-dependent, velocities. The details connected with the IPMTS and the solution of the KdV equation can be found in the monograph of Ref. 29.

In reality, for helium the starting pulses, as a rule, do not contain higher harmonics, and their evolution at the first stage is described by Burgers equation. As the profile of the wave becomes steeper, however, the term with the third-order derivative in Eq. (2.29) grows in magnitude and dispersion starts to affect the evolution of the wave. Numerical studies of the KdVB equation[20] show that the profile of the wave can acquire a soliton form, and the front acquires an oscillatory structure. The condition under which these soliton-like bursts can form can be derived by comparing the dissipative and dispersion terms in the KdVB equation (2.29). The term with the second-order derivative is of the order of $D_2\Delta v_n/(\delta x)^2$, where Δv_n is the amplitude of the pulse of second sound. The dispersion term is equal to $D_3\Delta v_n/(\delta x)^3$. Choosing for δx the magnitude of the "spreading" of the shock front $\delta x \sim D_2/\alpha_2(T)\Delta v_n$ we find that the dispersion and dissipative terms are comparable if

$$(\alpha_2(T) D_3 \Delta v_n)^{1/2} \sim D_2. \qquad (2.30)$$

The criterion under which the oscillatory structure of the front can be observed can be derived in a more rigorous fashion by studying the specific form of the initial perturbation. For example, for a steady-state shock wave a criterion of the type (2.30) is given in the next section.

In connection with the analysis of the propagation of second sound in the vicinity of the λ transition, it is important to study the behavior of the nonlinear term $\alpha_2(T) v_n \partial v_n/\partial x$ as a function of the temperature. It follows from the formula (2.18) and the scaling relations that $\alpha_2(T) \propto (T_\lambda T)^{-1}$ i.e., it increases rapidly as T_λ is approached (Fig. 6). In this connection it is necessary to discuss the correctness of the linear formulation of acoustic problems near the λ transition. As T_λ is approached the velocity of second sound c_2 decreases, $c_2 \propto |T_\lambda - T|^{1/3}$; therefore the parameter $\Delta v_n/c_2$, with respect to which the formal linearization of the equations is performed, increases (with a constant pump amplitude Δv_n). But even if the condition on $\Delta v_n/c_2$ is satisfied, the nonlinear term is still not small owing to the large value of $\alpha_2(T)$. This must be kept in mind in studying HeII in the region of a phase transition by acoustic methods. It is entirely possible that the overestimation of the value of the absorption coefficient for second sound, ob-

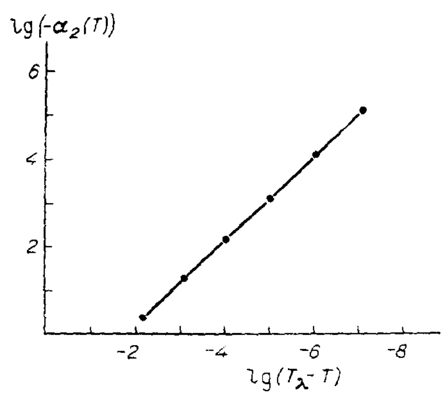


FIG. 6.

tained in Ref. 24, is due to an invalid linear interpretation of the experiment.

2.8. Stationary solutions of the Burgers and KdVB equations

The Burgers and KdVB equations derived above have steady-state solutions in the form of a traveling wave with a constant profile, $v_n = v_n(x - Ut)$. We shall first obtain this solution for Burgers equation. With the substitution $v_n = v_n(x - Ut)$ the relation (2.25) becomes an ordinary differential equation

$$(-U + c_2 + \alpha_2(T)\, v_n)\, v_n' = \mu_2 v_n'', \tag{2.31}$$

where the prime denotes differentiation with respect to the argument $\xi = x - Ut$. Integrating Eq. (2.31) with the boundary conditions $v_n(\infty) = 0$, $v_n'(\infty) = 0$ leads to the following result:

$$v_n(x, t) = \Delta v_n \left[1 + \left(\exp\frac{\alpha_2(T)\,\Delta v_n}{2\mu_2}\right)(x - Ut)\right]. \tag{2.32}$$

The solution (2.32) is a step with a "diffuse" front, moving from left to right with velocity V, and a jump Δv_n in the quantity v_n, related with U by the following expression $U = c_2 + \alpha_2(T)\Delta v_n/2$ [compare with Eq. (2.21)]. The width of the transitional region δx is equal to $2\mu_2/\alpha_2(T)\Delta v_n$ (compare with Sec. 2.6). The solution is shown schematically in Fig. 7. As $\mu_2 \to 0$ the solution transforms into a shock wave of second sound, described in Ref. 1. The same result was derived in Ref. 83 without using Burgers equation.

We shall now explain how dispersion affects the structure of the shock wave. Substituting the solution $v_n = v_n(x - Ut)$ into the KdVB equation (2.29) and integrating it once with the conditions $v_n = v_n' = v_n'' = 0$ as $x \to \infty$ we obtain the following second-order equation:

$$-D_3 v_n'' = \alpha_2(T)\,\frac{v_n^2}{2} + (c_2 - U)\, v_n - D_2 v_n'. \tag{2.33}$$

Equation (2.33) cannot be solved in the general form, but the solution can be studied qualitatively with the help of a mechanical analog based on the fact that Eq. (2.33) has the form of the equation of motion of a particle of mass D_3 in a field with the potential $P(v_n)$ given by

$$P(v_n) = \alpha_2(T)\frac{v_n^3}{6} + (c_2 - U)\frac{v_n^2}{2}, \tag{2.34}$$

and the force of friction $D_2 v_n$.[20] The form of the potential $P(v_n)$ is shown in Fig. 8. The quantity $-\xi = -x + Ut$ plays the role of time. At the time $t \to -\infty$ ($\xi = \infty$) the particle is located at the origin of coordinates ($v_n = 0$). By the time $t = \infty$ ($\xi = -\infty$) the particle, having undergone several oscillations, "drops" to the bottom of the well.

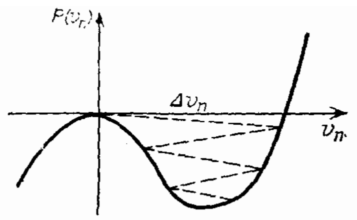


FIG. 8.

This behavior corresponds to the solution $v_n(x - Ut)$ shown in Fig. 9. It consists of a step with a jump in v_n equal to $\Delta v_n = v_n(-\infty) - v_n(\infty)$, which is related with the velocity of propagation by the well-known expression $U = c_2 + \alpha_2(T)(\Delta v_n/2)$. The shock front of this step has an oscillatory character. In the case of weak friction the oscillations are solitons with amplitude $3\alpha_2(T)\Delta v_n/2$ and moving, of course, with velocity U. If the viscosity is large, the motion of the particle will be aperiodic. This means that the wavefront will be monotonic, as in the purely dissipative case. The values of D_{2cr}, separating the oscillatory and monotonic cases, can be derived, as in the preceding section, by comparing the dissipative and dispersion terms in order of magnitude. This estimate gives, naturally, an expression of the type (2.30). More accurate calculations (see Ref. 20) change this estimate by a factor of $\sqrt{2}$ and thus

$$D_{2\text{кр}} = (2\alpha_2(T)\,\Delta v_n D_3)^{1/2}. \tag{2.35}$$

Using the definitions of D_2 and D_3 [see Eqs. (2.27) and (2.28)] we find that the relation (2.37) is equivalent to the following equality:

$$\frac{A_2^2}{A_3} = 2\alpha_2(T)\,\frac{\Delta v_n}{c_2}. \tag{2.36}$$

If the right side in Eq. (2.36) is less than the left side, then $D_2 > D_{2cr}$ and the shockfront has a monotonic form. In the opposite case $D_2 < D_{2cr}$ and oscillations can be observed on the shock wavefront.

2.9. Cubically nonlinear effects

The behavior of nonlinear waves of second sound, as shown previously, depends strongly on the temperature. Depending on the sign of $\alpha_2(T)$, steepening of both the leading and trailing edges of the wave is possible. For this reason the temperature range where α_2 vanishes is of special interest (see Sec. 2.5). In this case there are no quadratically nonlinear terms, and the evolution of the wave is determined by the next higher order (cubic) terms in the equations of motion. In this approximation the perturbations of the density ρ' and velocity v engendered by it affect the propagation of the

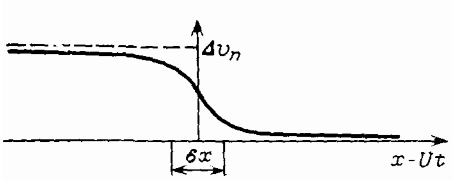


FIG. 7. The profile of a shock wave of second sound in a dissipative medium.

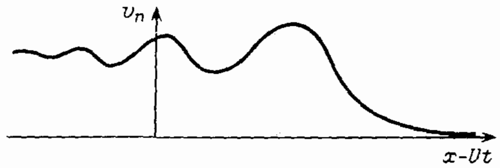


FIG. 9. The profile of a shock wave in a dispersive medium with coefficient of viscosity less than the critical value.

wave. The interaction with these disturbances as well as the nonlinear, third-order effects "inside" the second acoustic mode determine the structure of the wave packet. A cubically nonlinear medium has one remarkable property. It can support nonlinear steady-state (i.e., the profile does not change) monochromatic waves of the type $\sigma' \propto \exp[i(\tilde{\omega}^2_{k_0} t - \mathbf{k}_0 \mathbf{r})]$, where $\tilde{\omega}^2_{k_0}$ is the frequency renormalized owing to the nonlinearity. We recall that in a quadratically nonlinear medium a sinusoidal wave first becomes deformed and then transforms into a sawtooth wave.

It is convenient to solve the problem of a nonlinear monochromatic wave in the Hamiltonian form (see Sec. 1.2) for the quantity

$$\Psi(\mathbf{r}, t) = (2\pi)^{-3/2} \int a_{\mathbf{k}}^{\pm 2} \exp i(\mathbf{k} - \mathbf{k}_0)\mathbf{r}\, d\mathbf{k},$$

which is the complex envelope of the wave packet. The transformation from the variables $a^{\pm 2}_{\mathbf{k}_0}$ to the function $\Psi(r,t)$ is canonical, which makes it possible, given the Hamiltonian $H(\Psi)$, to write out directly the equations of motion.[30] The quantity H is found by simply going through all contributions to the energy in the given approximation. For example, the contribution of the interaction of the starting wave with the first sound engendered by it can be taken into account as follows (compare with Ref. 31). In the presence of first sound the frequency $\omega^2_{\mathbf{k}_0}$ of the second sound changes by the amount $\delta\omega^2_{\mathbf{k}_0}$, equal to

$$\delta\omega^2_{\mathbf{k}_0} = \frac{\partial\omega^2}{\partial\rho}\rho' + \frac{\partial\omega^2}{\partial \mathbf{v}}\mathbf{v}. \qquad (2.37)$$

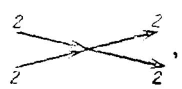
The last term describes the Doppler shift and is obviously equal to $\mathbf{k}_0 \cdot \mathbf{v}$. The change in the frequency $\delta\omega^2$ corresponds to the following change in the quadratic part of the Hamiltonian:

$$\delta H = \int \delta\omega^2 a^2_{\mathbf{k}} a^{-2}_{\mathbf{k}}\, d\mathbf{k} \underset{a_{\mathbf{k}}^{\pm 2} \to \Psi(\mathbf{r},t)}{=} \int \delta\omega\, |\Psi|^2\, d\mathbf{r}. \qquad (2.38)$$

Having gone through in this manner all possible processes it is possible to write out the Hamiltonian $H(\Psi)$ and to obtain from it, according to general rules, the following equation for the function $\Psi(x,t)$:

$$i\frac{\partial\Psi}{\partial t} = \omega^2_{\mathbf{k}_0}\Psi + \left[V_{\mathbf{k}_0} - \left(\frac{\partial\omega^2}{\partial\rho}\right)^2 \frac{\rho}{c_1^2} - \frac{k_0^2}{\rho}\right]|\Psi|^2\Psi; \qquad (2.39)$$

Here $V_{\mathbf{k}_0}$ is the vertex of the process



equal to

$$\left(\frac{c_2^2}{8}\,\frac{\partial^3 T/\partial S^3}{\partial T/\partial S} + \frac{\partial^2 S^2}{\partial S^2 2\rho_n}\right) k_0^2.$$

It is easy to see that Eq. (2.39) has the following solution:

$$\Psi = \Psi_0 e^{i\tilde{\omega}^2_{\mathbf{k}_0} t}; \qquad (2.40)$$

Here $\tilde{\omega}^2_{\mathbf{k}_0}$ is the renormalized frequency, equal to

$$\tilde{\omega}^2_{\mathbf{k}_0} = \omega^2_{\mathbf{k}_0} + \left[V_{\mathbf{k}_0} - \left(\frac{\partial\omega^2}{\partial\rho}\right)^2 \frac{\rho}{c_1^2} - \frac{k_0^2}{\rho}\right]|\Psi_0|^2. \qquad (2.41)$$

The relation (2.44) describes a monochromatic wave with the wave vector k_0 and the frequency $\tilde{\omega}^2_{\mathbf{k}_0}$. The perturbation of the density ρ' and velocity v are equal to

$$\rho' = -\frac{\partial\omega^2}{\partial\rho}\frac{\rho}{c_1^2}|\Psi_0|^2, \quad \mathbf{v} = -\frac{\mathbf{k}_0}{\rho}|\Psi_0|^2, \qquad (2.42)$$

i.e., the velocity v is oriented in the direction opposite to the direction of propagation of the wave. The velocity of propagation of the wave $c_2 = \tilde{\omega}^2_{\mathbf{k}_0}/k_0$ depends on the amplitude. Numerical analysis shows that the nonlinear correction to the velocity of sound Δc_2 at $T_\alpha \approx 1.885$ K is equal to

$$\Delta c_2 \approx -1{,}09\frac{k_0}{\rho}|\Psi_0|^2. \qquad (2.43)$$

Thus the nonlinear wave moves more slowly than the linear wave. The Doppler interaction of the wave under study with the first sound engendered by it makes the main contribution to the change in the velocity of the wave. Thus effective interaction between quanta of second sound is realized through ordinary sound. This situation is reminiscent of the theory of superconductivity, where the interaction between the electrons is realized through lattice vibrations.

3. MULTI-DIMENSIONAL WAVE PACKETS

3.1. Self-focusing of a monochromatic wave

In Sec. 2 we studied some nonlinear phenomena involved in the propagation of one-dimensional waves. The one-dimensionality of a wave means that the surfaces of constant phase are planes in the y, z coordinates. In reality, of course, waves have a finite transverse size, as a result of which nonuniformity arises in the y and z directions. For waves with an infinitesimal amplitude taking the transverse nonuniformity into account results in diffraction phenomena. In the case of finite amplitudes a number of fundamentally new effects, connected with the combined action of nonlinear and diffraction terms in the equations of motion, can arise. In this section we shall study the propagation of a nonlinear wave of second sound, weakly modulated in the transverse direction, i.e., under the assumption that the nonlinear and diffraction terms are of the same order of smallness.

We shall study the case of a cubically nonlinear medium (see Sec. 2.9), for which it has been shown that a monochromatic nonlinear wave of second sound can exist in a neighborhood of the temperature $T_\alpha \approx 1.88$ K. We shall study the effect of a transverse nonuniformity on the evolution of such a wave.

As in Sec. 2.9, we shall give the description in terms of the function $\Psi(\mathbf{r},t)$, which, unlike the case studied previously, depends on the spatial coordinate $\mathbf{r}$. The weak dependence of the function $\Psi(\mathbf{r},t)$ on $\mathbf{r}$ can be represented as the appearance of harmonics with wave vectors $\mathbf{k}$ close to $\mathbf{k}_0$ in the main wave. This changes the quadratic part of the Hamiltonian H_2, a consequence of which is that the equations of motion contain terms describing linear diffraction. The complete Hamiltonian H can be derived by simply going through all the cases, as done in the preceding section. Referring the reader to Ref. 30 for the details of the calculations, we write out the equation for the complex envelope $\Psi(\mathbf{r},t)$ (in the steady-state case):

$$2ik_0\frac{\partial\Psi}{\partial x}-\Delta_\perp\Psi=\frac{2k_0}{c_2}\left[\left(\frac{k_0^2}{\rho}+\frac{\Omega'^2\rho}{c_1^2}-V_{\mathbf{k}_0}\right)|\Psi|^2\right.$$
$$\left.+\left(\frac{3}{2}\frac{\Omega'\Omega''\rho^2}{c_1^2}+W_{\mathbf{k}_0}-\frac{1}{2}\frac{\partial V_{\mathbf{k}_0}}{\partial\rho}\frac{\Omega'\rho}{c_1^2}-\frac{k_0^2\Omega'}{\bar{\rho}c_1^2}\right)|\Psi|^4\right]\Psi.\quad(3.1)$$

Here $W_{\mathbf{k}_0}$ is the matrix element of a six-wave process and Ω' and Ω'' are the first and second derivatives of the frequency $\omega_{\mathbf{k}_0}^2$ with the respect to the density ρ.

A relation of the type (3.1) was first derived for the passage of laser radiation through matter (see Ref. 20 and the references cited there) and is called the nonlinear parabolic equation. It has been studied in a number of papers (see, for example, Ref. 32). Following the results of Ref. 32, we shall describe the behavior of the wave under study.

The coefficient in front of the term $|\Psi|^2\Psi$ (the expression in the first brackets) is a positive quantity and is equal to $\approx 2.18k_0^3/c_2\rho>0$. The fact that this quantity is positive means that the velocity of propagation of the wave decreases with the amplitude (see Sec. 2.9). As a result at the periphery of the wave packet, where the amplitude is smaller, the velocity of the wave is higher than on the axis of the beam. The wavefront bends, as shown in Fig. 10, and focusing of the packet starts. As a result of the focusing action the amplitude on the axis increases, which results in an even greater difference between the velocity of propagation of the peripheral and central sections. This intensifies the focusing effect even more. Thus the nonlinear self-focusing of the wave packet is a consequence of the fact that the coefficient in front of $|\Psi|^2\Psi$ is positive.[5)]

The diffraction term $\Delta_\perp\Psi$ in Eq. (3.1) results (in the absence of nonlinear effects) in spreading of the packet in the transverse direction. For this reason, to observe self-focusing the energy flux must exceed some threshold value I_{cr}. Calculation gives the result $I_{cr}\approx 0.6\cdot 10^7\nu^{-2}$w, where ν is the frequency of the sound in hertz (see Ref. 30). We note that I_{cr} can be evaluated from Eq. (3.1) by equating in order of magnitude the diffraction and nonlinear terms. The intensity I_{cr} can be achieved experimentally, and self-focusing can realistically be observed.

Thus the fourth-order nonlinear terms in the Hamiltonian describe the compression of a wave beam. As a result the amplitude on the axis increases rapidly, and the further behavior depends on the higher-order terms, in our case, sixth-order terms. As the calculations show (see Ref. 30), the coefficient in front of the term $|\Psi|^4\Psi$ (the expression in the second set of square brackets in Eq. (3.1)) is negative. This is equivalent to the six-wave Hamiltonian H_6 being positive, which corresponds to three-particle repulsion.

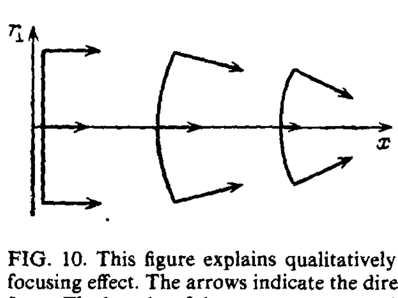


FIG. 10. This figure explains qualitatively the appearance of the self-focusing effect. The arrows indicate the direction of motion of the wavefront. The lengths of the arrows correspond to the velocities of propagation of the sections of the wave.

Competing with two-particle attraction, the latter process stabilizes the wave packet and prevents it from collapsing to zero dimensions. This qualitative argument is confirmed by the results of the numerical solution of an equation of the type (3.1) (see Ref. 32). Ultimately the beam is compressed to a size for which the fourth- and sixth-order diffraction and nonlinear terms "balance" one another and the packet, as it is said, enters the regime of self-channelization. The size of the channel δ can be estimated from the following relation:

$$k_0\delta\sim\left(-\frac{\sigma_1 I}{I-I_{cr}}\right)^{1/2};\quad(3.2)$$

here σ_1 is the dimensionless coefficient in front of the term $|\Psi|^4\Psi$ in Eq. (3.1), $\sigma_1\approx -2\cdot 10^{-2}$. The condition (3.2) means that for values of I greater than I_{cr} the beam can be compressed to a very small size. In this case, high energy is concentrated on the axis of the beam, and this can give rise to significant overheating and make it easier to observe the effect.

3.2. Wave beams in a quadratically nonlinear medium

In this section we shall describe the behavior of the wide beams of second sound in quadratically nonlinear media, where the self-focusing phenomena are accompanied by steepening of the wave and the formation of a shock front.

We shall assume that in the evolution of the wave the dissipative and diffraction effects are of the same order of smallness as the nonlinear effects. Actually, the smallness parameter, related with the transverse nonuniformity, must satisfy the relation $k_\perp^2/k_x^2=O(\Delta v_n/c_2)$, where $k_\perp^{-1}$ and k_x^{-1} are the transverse and longitudinal dimensions of the wave pulse and Δv_n is its amplitude. The equation for the evolution of sound pulses in ordinary media was derived in Refs. 33 and 34 under these assumptions. Analogous calculations give an evolutionary equation for nonuniform disturbances of second sound (in the reference frame moving with velocity c_2):

$$\frac{\partial}{\partial x}\left(\frac{\partial v_n}{\partial t}+\alpha_2(T)v_n\frac{\partial v_n}{\partial x}-\mu_2\frac{\partial^2 v_n}{\partial x^2}\right)=\frac{c_2}{2}\Delta_\perp v_n.\quad(3.3)$$

Compared with the nonlinear parabolic equation (3.1), Eq. (3.3) has not been studied much. One reason for this is that there are no steady-state waves with constant profile. Indeed, the nonlinear term $\alpha_2(T)v_n\partial v_n/\partial x$ describes the formation of shock fronts, on which strong dissipation occurs and the amplitude of the wave must decrease. In addition, in the case of Eq. (3.1) the effect of the nonlinear term can be reduced (at least qualitatively) to the bending of the front (see Fig. 10), whereas here the situation is more confusing. Indeed, the quantity $\alpha_2(T)v_n$, which plays the role of a nonlinear correction to the velocity of sound, is different at different points in the wave, as a result of which nonuniform deformation of the wave profile, also complicated by diffraction phenomena, will occur.

There are, however, quite a large number of investigations of the numerical solution of Eq. (3.3). These studies are reviewed in detail in the book of Ref. 35 (see also Ref. 55). We shall describe, following Ref. 35, the main characteristics of the evolution of the wave, whose characteristic transverse size is k^{-1}. The calculations show that one of the main parameters affecting the evolution of waves is the quantity $N = L_{\rm diff}/L_{\rm dis}$; here $L_{\rm dis} = c_2/\alpha_2(T)\Delta v_{\rm n} k_x$ is the distance along the x axis at which a discontinuity appears in a one-dimensional wave (see Sec. 2.5) and $L_{\rm diff} = k_x/k_\perp^2$ is the characteristic distance of diffraction spreading of the packet. For $N \leqslant 1$ there is enough time for the wave to be transformed into a diverging wave before nonlinear effects become significant. The further evolution of the packet proceeds as in a nonlinear spherical wave (see Ref. 55). As N increases the nonlinear effects come into play earlier. As in the one-dimensional case, they lead to deformation of the wave profile and formation of a shock front. This deformation, however, does not occur identically for different distances from the axis. The discontinuity forms first on the axis of the beam ($\mathbf{r}_\perp = 0$). Next the peripheral sections are drawn in. The distance at which a shock front forms on the axis of the packet is of the order of $L_{\rm dis}$, though it fluctuates somewhat as a function of N and the starting distribution. After the discontinuity forms intense dissipation of wave energy starts and, just as for small values of N, the packet transforms into a diverging wave. It should be noted that prior to the formation of the shock front some similarity to self-focusing is observed, namely, the amplitude on the axis increases and for beams with sharp edges the amplitude increases by a factor of two and more, while the transverse distribution becomes narrower, i.e, the beam is compressed. This process, however, does not last very long and steady-state self-focusing, apparently, does not occur in a quadratically nonlinear medium.

Analogous phenomena occur with isolated (positive or negative) pulses (see Ref. 55). For them, just as for periodic waves, nonuniform distortion of the profile occurs and a shock front forms. Prior to the formation of the shock front the width of the packet decreases (if $\alpha_2(T)\Delta v_{\rm n} < 0$) and the amplitude on the axis increases. Just as for periodic waves, however, this does not result in transverse collapse. Indeed, the characteristic length $L_{\rm col} = (-c_2/\alpha_2(T)\Delta v_{\rm n} k_\perp^2)^{1/2}$, at which nonlinear compression of the beam occurs (see Ref. 55), satisfies the condition $L_{\rm col}^2 = L_{\rm dis} L_{\rm diff}$, i.e., either $L_{\rm col} > L_{\rm diff}$ or $L_{\rm col} > L_{\rm dis}$. In the first case diffraction effects, resulting in spreading of the packet, predominate. In the second case the shock front forms before the beam is focused. After the discontinuity forms strong dissipation arises, the amplitude decreases, and the wave packet transforms into a diverging wave in accordance with the laws of linear acoustics.

4. STABILITY OF NONLINEAR WAVES

4.1. Nonlinear transformation of first sound into second sound

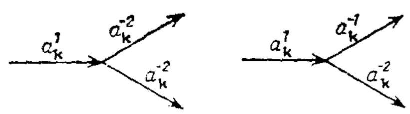
The study of the stability of solutions is an important part of the theory of nonlinear waves. First of all, this permits determining the region of values of the parameters for which the solution found is realized. Second, the possible instability of the wave is related with the nonlinear character of the equations, and for this reason stability questions are ideologically close to the nonlinear theory. The stability is studied within the framework of the standard scheme (see Ref. 36). The variables describing the system are represented as a sum of two parts, one of which is the starting solution and the other is a small correction, connected, for example, with fluctuations. Next the general equations are linearized with respect to the small corrections, and in this manner the problem is reduced to studying a system of linear equations (with coefficients which depend on both the starting solution and the properties of the medium). At this stage the question of the time dependence of the starting solution is very important. For time-dependent solutions the problem turns out to be very complicated and can be solved in only very few cases. Problems in which the starting solutions do not depend on the time are relatively simple (traveling waves with constant profile, in which the variables are functions of the combination $x - Ut$, are also of this type). In this case the coefficients of the linearized system do not depend on the time and the equations have solutions in the form of a superposition of exponentials $\Sigma_i e^{\lambda_i t}$, where λ_i are functionals of the starting solution and the properties of the system. If in the set of λ_i there are functionals such that $\mathrm{Re}\lambda_i > 0$, then small corrections grow exponentially, which means that the solution under study is unstable.

There is one other interesting aspect to the question of the stability of the waves. Suppose that a wave of first sound is excited in helium. Suppose further that small disturbances, associated with the second-sound mode, are unstable in the presence of this first sound. Then they grow and can be detected experimentally. This effect can be interpreted as the generation of second sound by first sound.[6)] It is from this viewpoint that the stability of first sound was studied in Refs. 6 and 56. Assume that we have initially a monochromatic wave of first sound. It can be written as follows in the Hamiltonian variables:

$$a_{\mathbf{k}}^1 = a\delta(\mathbf{k} - \mathbf{k}_0)\, e^{-i\omega^1_{\mathbf{k}_0} t}, \quad a_{\mathbf{k}}^2 = 0. \tag{4.1}$$

Generally speaking, a solution of the type (4.1) does not satisfy the nonlinear equations (1.17) with the Hamiltonian (1.18) (see the beginning of this section). The results obtained in Refs. 6 and 56 nonetheless are very important for understanding the nonlinear interaction between the different types of sound, and the problem at hand is thus of a model character.

When three-wave processes are taken into account there are two possible mechanisms for development of instability which are associated with the appearance of second sound. These are the decomposition and Cherenkov processes, which have the following schematic form:



(4.2)

It is well known from the general results (see Ref. 2 and 3) that the instability develops under conditions of resonance, i.e., aside from the satisfaction of the condition $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ (this condition is a consequence of the spatial uniformity and is a consequence of the δ functions in the Hamiltonian H (1.18)), the analogous condition for the frequencies must also be satisfied:

$$\omega_{\mathbf{k}}^1 = \omega_{\mathbf{k}_1}^2 + \omega_{\mathbf{k}_2}^2, \quad \omega_{\mathbf{k}}^1 = \omega_{\mathbf{k}_1}^1 + \omega_{\mathbf{k}_2}^2. \tag{4.3}$$

The simultaneous satisfaction of the conditions on the frequencies $\omega_{\mathbf{k}}^\nu$ and the wave vectors $\mathbf{k}$ delimits the region in $\mathbf{k}$ space in which Cherenkov and decomposition processes are allowed, and this gives a basis for studying them separately.

Consider, for example, the decomposition process. We

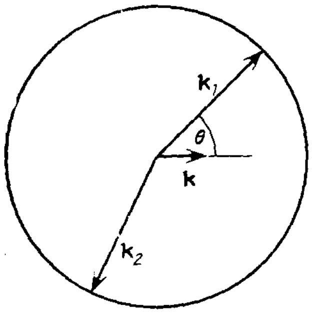


FIG. 11. Section of the surface on which lie the tips of the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, satisfying the resonance conditions (4.4) in the decomposition process.

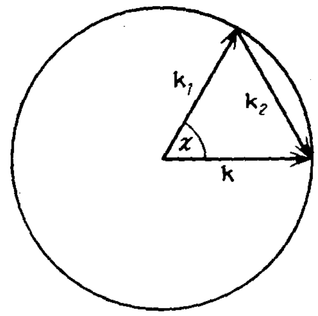


FIG. 12. Section of the surface on which lies the tip of the vector $\mathbf{k}_1$ of the quantum of first sound in the process of Cherenkov emission. The surface is determined by the relation (4.8).

shall write the conditions of resonance in the following form:

$$c_1|\mathbf{k}_0| = c_2|\mathbf{k}_1| + c_2|\mathbf{k}_2|, \quad \mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2. \tag{4.4}$$

From here one can see that the momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ of the emitted quanta lie on the surface of an ellipsoid of revolution (Fig. 11). As a result of the condition $c_2 \ll c_1$ the corresponding ellipse has a small eccentricity, and the momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are almost oppositely oriented. The equations for the small disturbances $\beta^{(2)}_{\mathbf{k}_1}$ and $\beta^{(2)}_{\mathbf{k}_2}$ comprise a system of two ordinary linear differential equations, which have solutions of the form $\beta^{(2)}_{\mathbf{k}}(t) \propto \exp(\nu^{\mathrm{dec}} t - i\omega^2_{\mathbf{k}} t)$, and in addition

$$\nu^{\mathrm{dec}} = -\frac{\gamma^{(2)}_{\mathbf{k}_1} + \gamma^{(2)}_{\mathbf{k}_2}}{2} + \left(|V^{1-2-2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a|^2 + \frac{\gamma^{(2)}_{\mathbf{k}_1} - \gamma^{(2)}_{\mathbf{k}_2}}{2}\right)^{1/2}; \tag{4.5}$$

here $\gamma^{(2)}_{\mathbf{k}}$ is the damping of second sound. One can see from the relation (4.5) that above some critical amplitude $a > a^{\mathrm{dec}}$ the increment ν^{dec} acquires a positive real part and the perturbations grow exponentially. The threshold value a^{dec} is equal to

$$a^{\mathrm{dec}} = \left(\frac{\gamma^{(2)}_{\mathbf{k}_1}\gamma^{(2)}_{\mathbf{k}_2}}{V^{\mathrm{dec}}(\omega, \theta)}\right)^{1/2}; \tag{4.6}$$

here $V^{\mathrm{dec}}(\omega,\theta)$ is the vertex of the decomposition process V^{1-2-2}, expressed in terms of the starting frequency and the angle of emergence θ of a quantum of second sound,

$$V^{\mathrm{dec}}(\omega, \theta) = \left(\frac{2\omega^3}{\rho c_1^2}\right)^{1/2} \left(-\frac{\rho}{2\rho_s} - \cos^2\theta - \frac{\rho_n\rho}{2\rho_s}\frac{\partial\rho_n^{-1}}{\partial\rho} + \frac{1}{2}\frac{\partial^2 p/\partial\sigma^2}{\partial T/\partial\sigma}\right). \tag{4.7}$$

The threshold amplitude a^{dec} depends on the direction of emergence and its minimum value obtains at $\theta = 0$, i.e., emergence in the forward direction occurs. The second emerging quantum moves in almost the opposite direction.

Analogous calculations and relations also are obtained for Cherenkov emission. For this process the resonance conditions have the following form:

$$c_1|\mathbf{k}_0| = c_1|\mathbf{k}_1| + c_2|\mathbf{k}_2|, \quad \mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2. \tag{4.8}$$

Since the velocity of second sound is much less than the velocity of first sound, $c_2 \ll c_1$, the conditions (4.8) hold when $|\mathbf{k}_0| \approx |\mathbf{k}_1|$, i.e., the tips of the vectors $\mathbf{k}$ and $\mathbf{k}_1$ lie on an almost spherical surface (Fig. 12). We denote by x the angle between $\mathbf{k}_0$ and $\mathbf{k}_1$. As a function of ω and χ the vertex of the Cherenkov process has the form[6]

$$V(\omega, \chi) = \frac{\omega^{3/2}}{2\rho^{3/2}}\frac{\partial\rho/\partial T}{(\partial\sigma/\partial T)^{1/2}}\left(\frac{c_2}{c_1}\sin\frac{\chi}{2}\right)^{1/2} \times\left[\rho\left(\frac{\partial}{\partial\rho}\ln\frac{\rho c_1^2}{\partial p/\partial\sigma}\right) + \cos\chi\right]. \tag{4.9}$$

The vertex of the Cherenkov process (4.9) depends on the angle χ. Its maximum value occurs for χ close to π, i.e., the quantum of first sound is emitted almost backwards. The frequency of the quantum of second sound is, in this case, close to $2\omega c_2/c_1$. The threshold value of the amplitude for which the increment ν^{Cher} acquires a positive real part satisfies the following condition:

$$|V^{\mathrm{Cher}}(\omega,\chi)a^{\mathrm{Cher}}_{\mathrm{cr}}| = \left[\gamma_1(\omega)\,\gamma_2\left(\frac{2\omega c_2}{c_1}\right)\right]^{1/2}. \tag{4.10}$$

Figure 13 shows the behavior of the ratio of the threshold amplitudes in the Cherenkov and decomposition processes $a^{\mathrm{Cher}}_{\mathrm{cr}}/a^{\mathrm{dec}}_{\mathrm{cr}}$. One can see from the figure that with the exception of a narrow "window" near the temperature $T \approx 1.15$ K the Cherenkov process precedes the decomposition process.

An analogous study was performed for ^{3}He–^{4}He mixtures in a recently published paper[84] (see also Ref. 85). In contradistinction to pure ^{4}He the Cherenkov process is much more efficient in the ^{3}He–^{4}He mixture. The author suggests that the corresponding nonlinear process be used to observe a specific phenomenon: wavefront reversal.

4.2. Stability of a pressure shock wave

As we have already mentioned, a monochromatic wave of first sound is not a solution of nonlinear equations of motion. On the contrary, an initially sinusoidal wave becomes steeper and transforms into a sawtooth wave (from the for-

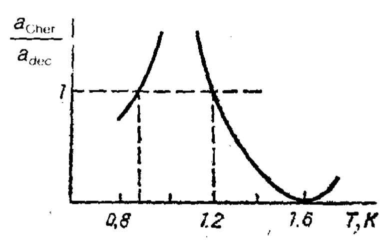


FIG. 13. The ratio of the threshold amplitudes of Cherenkov and decomposition processes $a_{\mathrm{Cher}}/a_{\mathrm{dec}}$ as a function of the temperature.

mal viewpont steepening is a consequence of nonlinear processes occurring "inside" the first sound which were neglected in Refs. 6 and 56). It is of great interest to study the conditions under which this sawtooth wave can emit second sound. However the corresponding stability problem is very complicated, first, because the starting solution depends on the time and, second, because it contains singularities (discontinuities).

It is nonetheless possible to alter the emphasis in this problem and to clarify the role of discontinuities in the stability of the wave, i.e., to raise the question of the stability of an idealized shock-wave step, moving from left to right and having definite values of the variables on the shock front. This question is also of interest for the following reason. The point is that the equation of entropy transport is important in the study of the stability of shock waves in ordinary media (see, for example, Refs. 3, 37, and 38). Some differences can arise in HeII, where entropy transport is realized, unlike in all other media, by a wave mechanism.

The question of the stability of a pressure shock wave was studied in Ref. 39. The problem was solved based on the Hamiltonian equations of motion in the class of generalized functions for the starting solution and small corrections. This made it possible, in contrast to the traditional methods (see Refs. 37 and 38), to study the stability of the shock wave not only relative to weak distortions of the wavefront ("ripples"), but also with respect to disturbances incident on the discontinuity. In an idealized shock wave—a step moving from left to right—the hydrodynamic variables are proportional to a unit step function $\theta(-x+Ut)$, where U is the velocity of the discontinuity. In the Hamiltonian variables such a step is described by the following relation:[39]

$$a_{\mathbf{k}}^{\pm 1}=\frac{2}{3}\left(\frac{\pi\rho}{c_1|k_x|}\right)^{1/2}\frac{\Delta v\delta(k_\perp)\theta(k_x)\exp(-ik_xUt)}{k_x\pm i0},\quad a_{\mathbf{k}}^{\mp 2}=0; \tag{4.11}$$

here Δv is the amplitude of the discontinuity and the term $\pm i0$ in the denominator gives the rule for circumscribing the pole: it corresponds to a wave moving from left to right. Because the Cherenkov vertex is small the system of equations for the small disturbances of the first $\beta_{\mathbf{k}}^1$ and second $\beta_{\mathbf{k}}^2$ sound modes separates into two structurally identical relations of the following form:

$$\frac{\partial\beta_{\mathbf{k}}^{\nu}}{\partial t}+i(\omega_{\mathbf{k}}^{\nu}-Uk_x)\beta_{\mathbf{k}}^{\nu}+\frac{1}{\pi i}\int\frac{A_{\mathbf{k}\mathbf{k}_2}^{+\nu}\beta_{\mathbf{k}_2}^{\nu}\,dk_{2x}}{k_{2x}-k_x+i0}+\frac{1}{\pi i}\int\frac{A_{\mathbf{k}\mathbf{k}_2}^{-\nu}\beta_{\mathbf{k}_2}^{-\nu}\,dk_{2x}}{k_{2x}+k+i0}=0\quad(\nu=1,2); \tag{4.12}$$

here $A_{\mathbf{k}\mathbf{k}_2}^{\pm}$ are some functions of $\mathbf{k}$ and $\mathbf{k}_2$, related with the vertices of the nonlinear processes. The denominators $k_{2x}\pm k_x+i0$ arose formally from the Fourier representations of the step function (see Ref.39). By means of special transformations the relation (4.12) can be reduced to a system of singular integral equations with rational coefficients. The solution of such equations (see Ref. 57) is related with the reconstruction of a piecewise-analytical function (of the complex variable k_x) from its discontinuity on the real axis in the k_x plane. The calculations performed in Ref. 39 lead to the following result. The disturbances $\beta_{\mathbf{k}}^{\nu}(0)$ arising, for example, owing to the fluctuation "force" $G(\mathbf{k},t)$, later develop according to the law

$$\beta_{\mathbf{k}}^{\nu}(t)\propto\int e^{\lambda t}\hat{F}(\lambda,\mathbf{k})G(\mathbf{k},\lambda)\,d\lambda;$$

here $G(\mathbf{k},\lambda)$ is the Laplace transform of the function $G(\mathbf{k},t)$, and $\hat{F}(\lambda,k)$ is a linear operator whose form can be determined from the complete solution of the problem. Correspondingly, $\beta_{\mathbf{k}}^{\nu}\propto e^{\lambda_i t}$, where λ_i is the pole of this operator and thus the stability problem reduces to finding the poles of the operator $F(\lambda,\mathbf{k})$. Investigations show that there are three branches for λ_i, two of which are as follows:

$$\lambda=iUk_x\pm ic_\nu(k_x^2+k_\perp^2)^{1/2},$$
$$\lambda=i(U-\Delta v)k_x\pm i\left[\left(c_\nu^2+\frac{\partial c_\nu^2}{\partial v}\Delta v\right)(k_x^2+k_\perp^2)\right]^{1/2}. \tag{4.13}$$

The physical significance of these branches is that they describe oscillations behind and in front of the shock wave (in a coordinate system tied to the discontinuity). The quantity $\mathrm{Re}\lambda=0$, i.e., the branches found do not lead to instability. The third branch of the poles of the operator $\hat{F}(\lambda,\mathbf{k})$ is discrete (it does not depend on k_x). Its physical significance is that it describes the evolution of disturbances on the shock front. For disturbances of the second sound type the operator $\hat{F}(\lambda,\mathbf{k})$ does not have poles of the indicated type. As regards disturbances of the first-sound type, they, as is well known (see Ref. 9), lead to acceleration and distortion of the wavefront, creating "ripples" on it, whose evolution is described by the branch under study. In other words, the indicated branch of the poles of the operator describes the stability of the shock wave relative to weak distortions of the surface of discontinuity. This formulation of the problem is identical with investigations of the stability of shock waves in ordinary liquids (see Refs. 37 and 38). The calculations performed in Ref. 39 lead to the following simple result. The starting distortions $\xi(r_\perp,t)|_{t=0}$ of the shock front will subsequently decay exponentially with decrement $U_1k_\perp$, ($U_1=U-\Delta v$ is the velocity of the liquid behind the discontinuity), which means that the shock wave is absolutely stable. This result is identical to the results of Refs. 37 and 38, where the stability of shock waves in ordinary liquids was studied.

Summarizing the results of this section we can say that pressure shock waves in HeII are stable relative to distortions of the surface of discontinuity, and they are a very stable type of flow, as in the case of ordinary media.

We call attention to the fact that if the specific form of the initial disturbance $G(\mathbf{k},t)$ is given (for example, if it is assumed that $G(\mathbf{k},t)\propto\delta(\mathbf{k}-\mathbf{k}_0)\exp(i\omega_{\mathbf{k}_0}^{\nu}t)$), then the method described in this section can be used to describe the interaction of this disturbance (sound) with a shock wave.

5. STOCHASTIC NONLINEAR WAVE PROCESSES

5.1. Solution of the kinetic equations

In the preceding sections we studied some questions regarding the propagation of nonlinear waves based on the dynamical equations of motion. A somewhat different approach arises in the problem when a wave packet containing a large number of harmonics which are not correlated with one another is excited in HeII. This can actually happen as a result of the instability of waves or, for example, random pumping of pulses (heat or pressure) into the volume of the liquid.

Suppose that we have some source of wave energy that generates harmonics with a characteristic wave number of the order of k_+, which is usually assumed to be (in order of magnitude) the inverse size of the system, $k_+ \sim L^{-1}$. As a result of nonlinear processes harmonics with higher values of k, which, in their turn, generate still higher harmonics, appear in the system. For very large values of k, of the order of k_-, viscous terms come into play in the equations of motion, and waves with momenta $k \gtrsim k_-$ decay rapidly. Ultimately some distribution of waves, which is characterized by the transfer of energy from large-scale motions to small-scale motions, is established in **k** space. This picture is typical for turbulent phenomena, and since we are talking about sound waves it is called acoustical turbulence (AT).

In Sec. 5 we shall study acoustical turbulence in HeII, a characteristic feature of which is that the cross interaction of first and second sounds is added to the above-described wave interaction. Stochastic wave processes can be described systematically on the basis of the nonequilibrium diagrammatic technique (DT).[40,41] The diagrammatic technique for the case of HeII was developed in Ref. 42. Under the assumption that the nonlinearity is small, the relations of the proposed DT (Dyson's equation) reduce to a system of kinetic equations (KEs) for $n_{\mathbf{k}}^{\nu}$. defined as $n_{\mathbf{k}}^{\nu}\delta(\mathbf{k}-\mathbf{k}') = \langle a_{\mathbf{k}}^{\nu} a_{\mathbf{k}'}^{-\nu}\rangle I$—pair correlation functions of the complex amplitudes or spectral densities (spectra). In the stationary and spatially homogeneous case the system of KEs has the following form: (Ref. 42) [7)]

$$I_{st}^{\nu}\{n\} = \sum_{\nu_1,\nu_2=1,2} \int d\mathbf{k}_1\, d\mathbf{k}_2\, [D_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\nu\nu_1\nu_2}(n_{\mathbf{k}_1}^{\nu_1} n_{\mathbf{k}_2}^{\nu_2} - n_{\mathbf{k}}^{\nu} n_{\mathbf{k}_1}^{\nu_1} - n_{\mathbf{k}}^{\nu} n_{\mathbf{k}_2}^{\nu_2})$$
$$- D_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}}^{\nu_1\nu_2\nu}(n_{\mathbf{k}_2}^{\nu_2} n_{\mathbf{k}}^{\nu} - n_{\mathbf{k}_1}^{\nu_1} n_{\mathbf{k}_2}^{\nu_2} - n_{\mathbf{k}_1}^{\nu_1} n_{\mathbf{k}}^{\nu})$$
$$- D_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}^{\nu_2\nu\nu_1}(n_{\mathbf{k}}^{\nu} n_{\mathbf{k}_1}^{\nu_1} - n_{\mathbf{k}_2}^{\nu_2} n_{\mathbf{k}}^{\nu} - n_{\mathbf{k}_2}^{\nu_2} n_{\mathbf{k}_1}^{\nu_1}) = 0] \quad (\nu = 1, 2); \tag{5.1}$$

here

$$D_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\nu\nu_1\nu_2} = \frac{\pi}{2} |V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\nu-\nu_1-\nu_2}|^2 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\omega_{\mathbf{k}}^{\nu} - \omega_{\mathbf{k}_1}^{\nu_1} - \omega_{\mathbf{k}_2}^{\nu_2}).$$

The equations (5.1) are identical to the kinetic equations employed to describe phonon systems (see Refs. 36 and 43) with the difference that in Eqs. (5.1) the spontaneous processes are omitted (owing to the fact that $\hbar \ll n_{\mathbf{k}}^{\nu}$). It can be verified in the standard manner (see Ref. 41) that the KEs (5.1) have a solution of the form:

$$n_{\mathbf{k}}^1 = \frac{\widetilde{T}}{c_1|\mathbf{k}|}, \quad n_{\mathbf{k}}^2 = \frac{\widetilde{T}}{c_2|\mathbf{k}|}; \tag{5.2}$$

here $\widetilde{T}$ is a constant, which plays the role of temperature. The solution (5.2) is the equilibrium Rayleigh-Jeans distribution; it is characterized by the absence of any flows (in **k** space) and, correspondingly, it is not appropriate for the problem of acoustic turbulence.

In the nonequilibrium situation the formulation of the problem of finding the spectra of acoustic turbulence $n_{\mathbf{k}}^{\nu}$ presupposes, aside from Eq. (5.1), the existence of a source and sink of waves. Generally speaking, the solution $n_{\mathbf{k}}^{\nu}$ depends on the specific form of the source (and sink). As often happens, however, the regions where the source and sink are influential are strongly separated in **k** space, i.e., $k_+ \ll k_-$. In addition, in the range of wave numbers **k** far from both k_+ and k_- (i.e., $k_+ \ll k \ll k_-$), in the so-called inertial interval (II) some distribution $n_{\mathbf{k}}^{\nu}$ which does not depend on the form of the source (and sink) and is determined solely by nonlinear interaction of the waves can be established.

Thus the problem of finding the spectra of acoustic turbulence reduces to finding solutions of the KES (5.1), which make the energy flux $P_{\mathbf{k}}$ in **k** space constant. This problem is very difficult, since it is related with finding the solution of a system of nonlinear integral equations. Even in the simpler case of a single wave mode the exact solution can be found only in the isotropic situation and with very stringent restrictions on the form of the functions $\omega_{\mathbf{k}}$ and $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$, namely, these quantities must be homogeneous functions of their arguments. This requirement as well as the condition $k_+ \ll k_-$, by virtue of which we can set $k_+ = 0$, $k_- = \infty$, leads to the assumption that the problem is scale invariant, i.e., there are no characteristic dimensions for **k**. This suggests that the solution $n_{\mathbf{k}}$ has a power-law form: $n_{\mathbf{k}} = Ak^s$. The exponent s can be calculated with the help of the so-called Zakharov transformations (see Ref. 44 for a detailed discussion). With the help of these transformations it is possible to factorize the integrand in the kinetic equations (5.1), and in the process one of the cofactors leads to the solution $n_{\mathbf{k}} \sim k^s$, characterized by a constant flow of energy in **k** space. In particular, Zakharov and Sagdeev[45] found by this method the spectrum of acoustical turbulence in an ordinary liquid. They found that $s = -9/2$, and calculated the relation between the amplitude spectrum A and the power P of the source of wave energy.

In HeII, owing to the existence of several types of nonlinear processes, the collision integral cannot be factorized directly. Nonetheless, as we shall now show, the system of kinetic equations (5.1) has an isotropic scale-invariant solution of the following form:

$$n_{\mathbf{k}}^1 = Ak^s, \quad n_{\mathbf{k}}^2 = Bk^s \tag{5.3}$$

with the same exponent s (see Ref. 42).

This assertion can be proved as follows. Substitution of the spectra (5.3) into the kinetic equations (5.1) reveals the following important fact. Because all vertices $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3}$ (1.20) have the same degree of homogeneity and because both dispersion laws $\omega_{\mathbf{k}}^{\nu}$ are linear the external argument **k** appears in all terms in the form of the factor k^{5+2s}. As a result, after canceling k^{5+2s}, the kinetic equations (5.1) reduce to a system of bilinear algebraic equations for the quantities A and B:

$$X_{AA}A^2 + X_{AB}AB + X_{BB}B^2 = 0,$$
$$Y_{AA}A^2 + Y_{AB}AB + Y_{BB}B^2 = 0. \tag{5.4}$$

The quantities X and Y can be calculated from the integrals appearing in the kinetic equations; they are functions of the parameter s. Since the system (5.4) is homogeneous (with respect to A and B) a solution exists only for certain values of s, which play the role of eigenvalues. It will be shown below that $s = -9/2$ is an eigenvalue of the system (5.4).

We shall write out the terms in the kinetic equations (5.1) that correspond to decomposition processes (it will be seen from the presentation given below that for Cherenkov processes the situation is completely analogous). Denoting them by J^{122} and J^{212}, where the first index indicates the number of the equation and the other two indices denote the

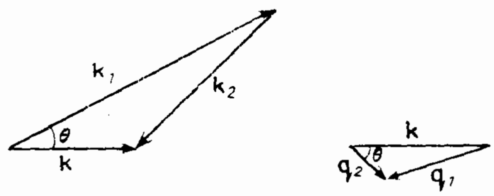


FIG. 14.

type of process, we obtain

$$J^{122} = \int D^{122}_{kk_1k_2}(n^2_{k_1}n^2_{k_2} - n^1_k n^2_{k_2} - n^1_k n^2_{k_1})\, dk_1\, dk_2, \qquad (5.5)$$

$$J^{212} = -\int D^{122}_{k_1k_2k}(n^2_{k_2}n^2_k - n^1_{k_1}n^2_k - n^1_{k_1}n^2_{k_2})\, dk_1\, dk_2 - \int D^{122}_{q_2kq_1}(n^2_k n^2_{q_1} - n^1_{q_2}n^2_k - n^1_{q_2}n^2_{q_1})\, dq_1\, dq_2. \qquad (5.6)$$

Consider the second term in Eq. (5.6). The conservation laws appearing in the factor $D^{122}_{q_2kq_1}$ require that the following conditions be satisfied:

$$c_1|\mathbf{q}_2| = c_2|\mathbf{k}| + c_2|\mathbf{q}_1|, \quad \mathbf{q}_2 = \mathbf{k} + \mathbf{q}_1. \qquad (5.7)$$

Analogously, the conservation laws in the integral J^{222} have the form

$$c_1|\mathbf{k}| = c_2|\mathbf{k}_1| + c_2|\mathbf{k}_2|, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2. \qquad (5.8)$$

We illustrate the triplet of vectors coupled by the conditions (5.7)–(5.8) in Fig. 14; the triangles are chosen to be similar. We rotate the triangle $\mathbf{q}_2\mathbf{k}\mathbf{q}_1$ so as to orient $\mathbf{k}$ along $\mathbf{k}_1$ in the first triangle $\mathbf{k}\mathbf{k}_1\mathbf{k}_2$ and stretch it by a factor of k/k_1, after which both triangles coincide, as shown in Fig. 15. These operations are equivalent to the following substitution of variables:

$$q_1 = \frac{k}{k_1}k_2, \quad q_2 = \left(\frac{k}{k_1}\right)^2 k_1. \qquad (5.9)$$

Using the substitution (5.9) and the homogeneity of the quantities $V^{\nu_1\nu_2\nu_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$, $\omega^\nu_\mathbf{k}$, and $n^\nu_\mathbf{k}$, the term under study can be put into the form

$$\int \left(\frac{k}{k_1}\right)^{8+2s} D^{122}_{kk_1k_2}(n^2_{k_1}n^2_{k_2} - n^1_k n^2_{k_1} - n^1_k n^2_{k_2})\, dk_1\, dk_2, \qquad (5.10)$$

which differs from J^{122} by the factor $(k/k_1)^{8+2s}$ in the integrand. Analogous calculations show that the first term in Eq. (5.6) assumes the form of J^{122} with the factor $(k/k_2)^{8+2s}$ in the integrand. Next, multiplying J^{122} by c_1 and J^{212} by c_2 and adding the two expressions we obtain

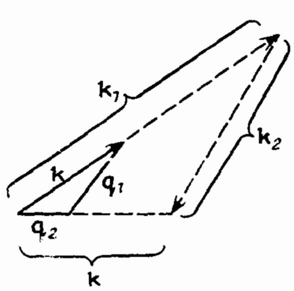


FIG. 15.

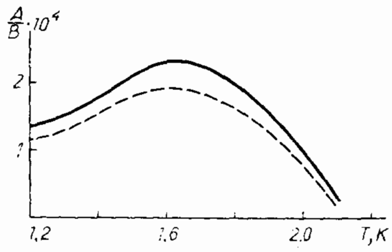


FIG. 16.

$$c_1J^{122} + c_2J^{212} = \int D^{122}_{kk_1k_2}\left[c_1 - c_2\left(\frac{k}{k_1}\right)^{8+2s} - c_2\left(\frac{k}{k_1}\right)^{8+2s}\right] \times (n^2_{k_2}n^2_{k_1} - n^1_k n^2_{k_1} - n^1_k n^2_{k_2})\, dk_1\, dk_2. \qquad (5.11)$$

It is not difficult to see that for $8 + 2s = -1$ (i.e., for $s = -9/2$) the expression in the brackets in Eq. (5.11) is identical to the argument of the frequency δ function appearing in D^{122}, so that the integral vanishes. Thus, irrespective of the dependence on the amplitudes A and B of the spectra for $s = -9/2$ the quantities J^{122} and J^{212} are related with one another by the expression $c_1J^{122} + c_2J^{212} = 0$. In particular, if a relation is chosen between A and B such that J^{122} vanishes, then J^{212} will also vanish. As a result the contributions of decomposition processes to $I^\nu_{st}\{n\}$ vanish in both equations of the system (5.1). Contributions from the nonlinear processes "inside" each wave mode vanish (for $s = -9/2$) automatically, since the situation is entirely analogous to acoustical turbulence in ordinary media, as described in Ref. 45. Thus by selecting the relation between A and B (and setting $s = -9/2$) both collision integrals in the kinetic equations (5.1) can be made to vanish; in other words, a solution of the form (5.3) can be obtained. The relation between A and B can be established with the help of any of the equations (5.4). The results of the calculations are presented in Fig. 16, where the ratio A/B is shown as a function of the temperature. We call attention to the fact that in order of magnitude A/B is close to $(1/2)(2c_2/c_1)^{9/2}$ (the latter is shown in Fig. 16 by the dashed line). This fact can also be given a physical explanation. As calculations show, the contribution of decomposition processes to the quantities X and Y [see Eq. (5.4)] can be much greater than the contribution of the Cherenkov processes. But in decomposition processes, as shown in Sec. 4.1, one quantum of first sound with momentum $\mathbf{k}$ decomposes into two quanta of second sound with momenta $\mathbf{k}_2$ and $\mathbf{k}_1$, and in addition $k_1, k_2 \approx c_1k/2c_2$. Thus the relation $2n^1_\mathbf{k} \sim n^2_{c_1k/2c_2}$ is satisfied in order of magnitude, whence, taking into account the fact that $n^u_\mathbf{k} \propto k^{-9/2}$, we obtain

$$\frac{A}{B} \sim \frac{1}{2}\left(\frac{2c_2}{c_1}\right)^{9/2}.$$

We shall illustrate schematically the spectra of the first and second sounds as functions of the wave number $\mathbf{k}$ (Fig. 17). Region I is the region of influence of the source and in region III viscous damping of the waves is significant. Region II is the inertial interval, in which the spectra $n^1_\mathbf{k}$ and $n^2_\mathbf{k}$ found above are realized.

We shall evaluate the energies $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ stored in the wave fields. These quantities can be obtained by averaging the quadratic part of the Hamiltonian H (1.18). For $\varepsilon^{(1)}$ we

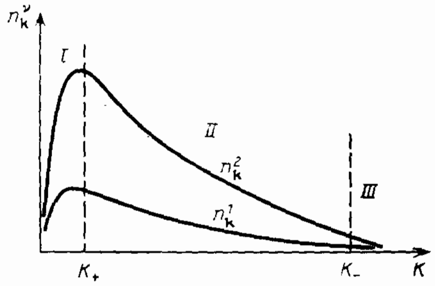


FIG. 17.

have

$$\varepsilon^{(1)} = \int \omega_k^1 n_k^1 \, dk = 4\pi c \; A \int k^{-3/2} \, dk \sim \frac{8\pi c_1 A}{k_+^{1/2}}. \qquad (5.12)$$

The integral is cut off at $k_+ \sim L^{-1}$, where L, we repeat, is the size of the system. Analogously, for second sound we have $\varepsilon^{(2)} \sim 8\pi c_1 B / k_+^{1/2}$. The ratio of the energies $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ is equal to $\varepsilon^{(1)}/\varepsilon^{(2)} \sim (2c_2/c_1)^{7/2}$, i.e., the energy stored in the second, softer mode greatly exceeds the energy of the harder first sound. This is a characteristic feature of a nonequilibrium distribution, since for the Rayleigh solution (5.2) the energies $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are equal, which agrees with the law of equipartition of energy (see Ref. 46).

As regards energy transfer, calculations show that irrespective of the method of excitation of the wave energy the flows $P_k^{(1)} + P_k^{(2)}$ are distributed in the same manner along the spectrum of the first and second modes and they are related with the amplitudes of the spectra A and B by the following expressions:

$$P_k^{(1)} \approx \frac{6\alpha_1^2(T)\, c_1}{\rho} A^2, \quad P_k^{(2)} \approx \frac{6\alpha_2^2(T)\, c_2 \rho_s}{\rho\rho_n} B^2, \qquad (5.13)$$

where $\alpha_1(T)$ and $\alpha_2(T)$ are the coefficients of nonlinearity of the first and second sounds (see Sec. 2.5). The total energy flux $P_k^{(1)} + P_k^{(2)}$, of course, is equal to the pumping energy (or dissipation) per unit volume. It is interesting to note that in spite of the general nonequilibrium nature of the system the "gases" of the quanta of first and second sounds are in equilibrium with one another and the total energy flux from one mode to the other is equal to zero.

5.2. Acoustical properties of turbulent helium

The acoustical properties of helium II in which acoustical turbulence has been excited are different from those of the undisturbed liquid. Indeed, any sound wave propagating in turbulent helium will interact with the wave fields. The result of this interaction is some additional damping Γ_k and dispersion Δ_k.

It can be shown with the help of the diagrammatic technique (see Refs. 41, 42, and 47) that the damping Γ_k^ν and dispersion Δ_k^ν of the wave of the νth mode are the imaginary and real part of the following expression:[8)]

$$\sum_{\nu_1,\nu_2=\pm 1,2} \int \frac{|V_{kk_1k_2}^{\nu\nu_1\nu_2}|^2 n_{k_1}^\nu \, dk_1 \, dk_2}{\omega_k^\nu + \omega_{k_1}^{\nu_1} \operatorname{sign} \nu_1 + \omega_{k_2}^{\nu_2} \operatorname{sign} \nu_2 + i0}. \qquad (5.14)$$

The integration in Eq. (5.14) must be performed in accordance with the formula

$$\frac{1}{x+i0} = P\left(\frac{1}{x}\right) - \pi i \delta(x).$$

We shall first perform calculations for first sound, i.e., we set $\nu = 1$. We shall study the contribution to Γ_k^1 and Δ_k^1 of the interaction with the wave field of the first mode, i.e., we assume $\nu_1, \nu_2 = \pm 1$. The calculations of the damping Γ_k^{111} and dispersion Δ_k^{111} (the first index is the number of the external mode and the other two indices denote the form of the process contributing to the quantity under study) are practically identical to the calculations of these quantities for sound propagating in a system of phonons with a linear dispersion law.[9)] Referring the reader to Refs. 42 and 43 for the details of the calculations, we present the final result:

$$\Gamma_k^{111} = \frac{\alpha_1^2(T)\, A}{2\pi\rho} \frac{k}{k_+^{1/2}}, \quad \Delta_k^{111} = \frac{\alpha_1^2(T)\, A}{\pi^2\rho} \frac{k}{k_+^{1/2}} \ln \frac{c_1 k}{\Gamma_k^1}. \qquad (5.15)$$

In the calculation of the relations (5.15) the fact that the spectra n_k^ν have the form (5.3) was used and the integrals were cut off at the lower limit k_+ $(\sim L^{-1})$.

We shall now study the contribution of the decomposition interaction of the external wave to Γ_k^1, i.e., we set $\nu = -2$ and $\nu_2 = -2$ in the formula (5.14). The damping Γ_k^{122} has the following form:

$$\Gamma_k^{122} = \pi \int |V_{kk_1k-k_1}^{1-2-2}|^2 n_{k_1}^2 \delta(\omega_k^1 - \omega_{k_1}^2 - \omega_{k-k_1}^2)\, dk_1. \qquad (5.16)$$

The calculation of the integral in (5.16) reduces to integration over the surface (4.4) (see also Fig. 4.1). Carrying out the integration and neglecting terms of order $(c_2/c_1)^2$ with respect to the retained terms, we obtain

$$\Gamma_k^{122} = \frac{\langle\alpha_p\rangle}{16\pi\rho} \left(\frac{2c_2}{c_1}\right)^{3/2} B k^{1/2}; \qquad (5.17)$$

here $\langle\alpha_p\rangle$ is the angle-averaged expression in the brackets in the formula (4.7). It is of the order of unity.

We shall now calculate the contribution of decomposition processes to the dispersion of first sound Δ_k^{122}, equal to

$$\Delta_k^{122} = \fint |V_{kk_1k-k_1}^{122}|^2 \frac{n_{k_1}^2 dk_1}{\omega_k^1 - \omega_{k_1}^2 - \omega_{k-k_1}^2}. \qquad (5.18)$$

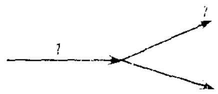
The integral in Eq. (5.18) is a principal value integral. Near the resonance surface the integrand has a singularity of the type $(k - k_{res})^{-1}$ of different sign on both sides of the surface. The integral nonetheless is different from zero in view of the fact that the integrand contains a rapidly decaying function $n_{k_1}^2 \propto k_1^{-9/2}$. We shall estimate the dispersion Δ_k^{122}, neglecting the directional dependence of the matrix element V^{1-2-3}, i.e., we assume that $V_{k_1k_2k_3}^{1-2-2} = \mathrm{const} \cdot (k_1 k_2 k_3)^{1/2}$. In this case it is possible to perform the integration over the angle between the vectors $\mathbf{k}$ and $\mathbf{k}_1$ analytically. As a result the integration of singularities of the form $(k - k_{res})^{-1}$ splits into two singularities of the logarithmic type. The infrared region $k_1 \to 0$ as well as the region near the logarithmic singularities make the main contribution to the integral. The first (infrared) region leads to the following (in order of magnitude) expression for the dispersion:

$${}^{1}\Delta_k^{122} \sim \frac{B}{\pi^2\rho} \frac{k}{k_+^{1/2}}. \qquad (5.19)$$

Estimates made for the regions with logarithmic singularities give the following expression for ${}^2\Delta_{\mathbf{k}}^{122}$:

$$ {}^2\Delta_{\mathbf{k}}^{122} = \frac{\langle\alpha_p\rangle}{8\pi\rho c_1} Bk^{1/2}. \tag{5.20} $$

The contributions of the Cherenkov processes to the damping $\Gamma_{\mathbf{k}}^{122}$ and the dispersion $\Delta_{\mathbf{k}}^{112}$ can be calculated analogously. The results are qualitatively similar, i.e., the quantities $\Gamma_{\mathbf{k}}^{122}$ and $\Delta_{\mathbf{k}}^{122}$ have a square-root dependence on the wave number **k**, similarly to the case of decomposition processes. Quantitatively, however, these quantities are much less than in the case of decomposition processes owing to the fact that the Cherenkov vertices are small and the phase volume of integration in the processes



is also small.

We shall now consider the problem of calculating the correlation characteristics of second sound. By analogy to first sound here we have the contributions $\Gamma_{\mathbf{k}}^{222}$ and $\Delta_{\mathbf{k}}^{222}$ connected with nonlinear processes inside the second mode and equal to

$$ \Gamma_{\mathbf{k}}^{222} = \frac{\alpha_2^2(T)\,\rho_s B}{2\pi\rho\rho_s} \frac{k}{k_+^{1/2}}, \quad \Delta_k^{222} = \frac{\alpha_2^2(T)\,\rho_s B}{\pi^2\rho\rho_n} \frac{k}{k_+^{1/2}} \ln \frac{c_2 k}{|\Gamma_{\mathbf{k}}^2|}. \tag{5.21} $$

Analogously to the case of first sound cross nonlinear processes contribute to the damping and the dispersion. As calculations show (see Ref. 42), however, these contributions are small compared to $\Gamma_{\mathbf{k}}^{222}$ and $\Delta_{\mathbf{k}}^{222}$ and they can be neglected.

We shall briefly summarize the results obtained. A wave of first sound propagating in turbulent HeII is subject to additional damping and dispersion. The decomposition processes ($\Gamma_{\mathbf{k}}^{122} \gg \Gamma_{\mathbf{k}}^{111}, \Delta_{\mathbf{k}}^{122} \gg \Delta_{\mathbf{k}}^{111}$) make the largest contribution here. The damping $\Gamma_{\mathbf{k}}^{122}$ has a square-root dependence on the wave number **k** and can be easily observed experimentally. The dispersion consists of two parts [see Eqs. (5.19) and (5.20)]. The first part is linear in the wave number **k** and thus reduces to renormalization of the velocity of sound. The second part has a square-root dependence on **k** and results in the "true" dispersion of the velocity of sound, and in addition $\Delta c_2^{122} \propto k^{-1/2}$. The damping and dispersion of second sound are determined primarily by nonlinear processes occurring "inside" the second wave mode. The quantities $\Gamma_{\mathbf{k}}^{222}$ and $\Delta_{\mathbf{k}}^{222}$ are proportional to the wave number **k**. The dispersion $\Delta_{\mathbf{k}}^{222}$ results in renormalization of the velocity of sound, $\Delta c^{222} = \Delta_{\mathbf{k}}^{222} = \Delta_{\mathbf{k}}^{222}/|\mathbf{k}|$. As regards the damping, first of all, it exceeds the viscous damping $\gamma_{\mathbf{k}}^{(2)} \propto k^2$ and, second, it differs from the viscous damping in that it depends on **k** and can also be observed experimentally.

6. EXPERIMENTAL INVESTIGATIONS

6.1. Brief review of experimental work

As mentioned in the introduction, the experimental study of nonlinear waves is not as well developed as the theoretical work. Most experimental investigations concern the dynamics of intense pulses of second sound. It is important to note that in the course of these studies the investigators gradually shifted their attention toward observing phenomena connected with quantum vortices, such as critical velocities, the dynamics of a vortex cluster, overheating, boiling, etc. These phenomena are undoubtedly important and interesting, but on the one hand their description falls outside the framework of nonlinear acoustics (and even outside the framework of classical two-velocity hydrodynamics) and, on the other hand, being mixed with purely nonlinear effects, they complicate the overall picture. For this reason, in describing the experimental work we shall concentrate only on the results which are associated with purely nonlinear processes.

Furthermore, although the dynamics of nonlinear heat pulses is undoubtedly interesting from the viewpoint of nonlinear waves, as one can see from the foregoing discussion it is only a small part of the nonlinear theory. Most other questions and effects predicted by the theory have remained outside the field of view of experimenters. It is thus best to divide this section, devoted to experiments, into two parts. The first part will be devoted to existing experimental investigations and a discussion of the results obtained there from the viewpoint of the theory. The second part (see Sec. 6.2) consists of suggestions for experiments which are of interest in themselves or are concerned with the study of the properties of HeII (phase transition, dissipation, Kapitsa jump, etc.) by the method of nonlinear acoustics.

Osborne's experiment must apparently be regarded as the classical experiment providing the impetus for studying nonlinear waves in HeII.[14] He was the first to observe the steepening of the trailing edge of a wave of second sound—a phenomenon which at that time had no analog in classical gas dynamics.[10)] An explanation of this effect was given, in a somewhat altered form, in the papers of Khalatnikov.[5,11] These results are described in Secs. 2.5–2.6. The experimental study of nonlinear waves was continued by Dessler and Fairbank.[15] The main goal of their work was to measure the coefficient of nonlinearity $\alpha_2(T)$ (see Sec. 2.5), which was calculated theoretically by Khalatnikov. Dessler and Fairbank employed a novel method for measuring $\alpha_2(T)$. A short "rider" pulse was applied from above to the main pulse of second sound, which had a square shape. Depending on the temperature, the "rider" moved along the main pulse forward or backward with a velocity proportional to the amplitude of the main pulse. The coefficient of proportionality, evidently also equal to $\alpha_2(T)$, is shown in Fig. 2. It should be noted that the accuracy of the experiment was quite low.

The cited works refer to the mid-1950s. After a hiatus of almost ten years interest in this subject has reappeared. One of the first investigations was the series of papers by Gulyaev. His results are described in detail in Ref. 65. Using the method of Schlieren photography Gulyaev obtained shadow patterns of acoustic disturbances in HeII with pulsed heating on flat and cylindrical heaters. One of the main results is the conclusion that there are two types of waves propagating with velocities c_1 and c_2 (compare with Sec. 2.4). True, the shadow method, based on recording changes in the density, does not permit identifying in detail the structure of the disturbances. The author himself believes that the waves traveling with velocity c_1 are pressure waves which arise as a result of boiling of the liquid and formation of a film. On this basis he questions the results of Refs. 11, 14, and 15, where he believes the distortion of the shape of the waves of second sound are related with boiling and not with nonlinear effects. In 1979 there appeared a paper by Pomerants,[66] who also recorded two waves with pulsed heating at the source. He recorded both waves with the help of a thermometer. The leading wave had the form of a cooling pulse, from which the author concludes that the coefficient of expansion is negative.

Since the second half of the 1970s the number of experimental and theoretical works started to grow rapidly. This was due both to the increased interest in nonlinear effects in HeII and to the wider experimental possibilities. Thus Cummings, Schmidt, and Wagner[16] employed a quite unusual, for that time, sensor to record short temperature waves. The sensor consisted of a film of superconducting material deposited on a quartz substrate. The superconducting transition in such a system occurs in an extended temperature range, and such a device can serve as a sensitive and, which is very important, short-time-constant thermometer. In Ref. 16 oscillograms of the nonlinear pulses are presented; the pulses have the distinct form of the characteristic Burgers triangles with steep leading or trailing edges. To compare their experiments with theoretical predictions the authors measured the transit time of a nonlinear wave between two sensors. The low time constant makes it possible to do this with high accuracy. The average transit time, which the authors called the nonlinear velocity of the signal, depends on the injected power, but by no means linearly, which, one would think, contradicts the theory (see the relation (2.21)). The contradiction obtained was resolved experimentally by Tsoĭ *et al.*[13] In this work it was pointed out that although the relation (2.21) is correct on the discontinuity of the wave, the dynamics of a separate pulse, in particular, its velocity and transit time, satisfies more complicated dependences (compare with Sec. 2.5 as well as Ref. 73). The computed transit time agrees well with the measured value. In this article it is pointed out that for more intense pulses ($W \gtrsim$ W/cm^2) systematic deviations from the theoretical predictions appear. Having a method for calculating the dependence of the transit time, Tsoĭ solved experimentally the problem of measuring the nonlinearity coefficient $\alpha_2(T)$.[67] The results obtained determine the quantity sought with an accuracy somewhat higher than that achieved by Dessler and Fairbanks. In particular, the temperature T_α at which $\alpha_2(T)$ vanishes was determined with high accuracy. As already mentioned, it was pointed out in Ref. 13 that for very high intensities there are deviations from the classical nonlinear theory. It was suggested that these deviations are connected with the generation of quantum vortices. It must be said that previously vortices were observed only in steady-state flows (or transitional flows). The characteristic heat fluxes were of the order of fractions of W/cm^2, while the development time of a vortex structure is of the order of tens or hundreds of seconds. It seems that if the heat fluxes are significantly increased, these times can decrease to milliseconds. This proposition was confirmed experimentally in Ref. 49. It is interesting to note that the additional damping of the probe wave, owing to the presence of vortices, was recorded by the methods of nonlinear acoustics (the dependence of the transit time on the amplitude of the wave was employed).

A large series of experimental studies of nonlinear waves was performed by Iznankin and Mezhov-Deglin.[17,18] In these investigations the increase in the pulse width predicted by the theory (see Sec. 2.5) was observed qualitatively. The authors observed the interesting phenomenon of the appearance of a negative temperature pulse behind the main signal. This effect, connected with the nonplanar nature of the wave, analogous to a rarefaction wave in ordinary acoustics (see, for example, Ref. 9), was first observed for second sound. The problem of a spherical nonlinear pulse of second sound was studied theoretically by Atkins and Fox.[72] In the papers cited the dynamics of waves accompanying pulsed heating in HeII was studied thoroughly in a wide range of intensities and durations. It was shown that for moderate intensities the dynamics of the waves is in good agreement with the theoretical predictions. In particular, steepening of pulses on the trailing or leading edges (as a function of the temperature) is observed. It was shown that the form of the pulse remains unchanged at T_α. Waves propagating with the velocity of first sound were also recorded. As in Ref. 13, deviation from the theoretical predictions was observed at high powers. A paper by Kitabatake and Sawada[68] appeared at approximately the same time. In this article the dynamics of a pulse of second sound was studied experimentally and it was shown that it agrees with Burgers equation (see Sec. 2.6). Nonlinear waves of both first and second sounds were studied in a series of investigations by Tsvetkov *et al.*,[69–71] which were performed during the same period. Unlike the traditional methods, the sounds were excited with the help of optical pumping of a germanium crystal immersed in HeII. The parameters of the sound waves had record-high values, for example, the duration was of the order of 10 ns. It is unclear whether or not hydrodynamic equations can be used at all for second sound with very high intensity and short duration. As regards first sound, it was shown that the velocity of propagation of shock waves of first sound is proportional to the square root of the injected power, i.e., it is linear with respect to the jump in the pressure (compare with Sec. 2.5). The coefficient of proportionality is close to the coefficient given by the relation (2.20).

It should be noted that during this period investigators gradually shifted their attention from purely nonlinear effects to phenomena occurring in transcritical (relative to vortex formation) regimes. A series of investigations carried out at the California Institute of Technology was performed in this spirit (see, for example, the works by Turner).[74,75] In these investigations Turner observed the typical nonlinear phenomena, such as steepening of the trailing and leading edges or generation of a pressure wave with thermal excitation (compare with Secs. 2.2–2.4). For example, for pumping intensities of the order of 10 W/cm^2 he recorded a pressure pulse $\Delta p \sim -0.1$ bar, which agrees with the formulas (2.8) and (2.16). By further increasing the pumping amplitude Turner observed a deviation from the theory, based on which he concluded that there exists a limiting intensity of the thermal wave, which he attributes to an "internal critical velocity." The papers of Torczynski[51,76] are devoted to the same subject. The experiment with a converging shock wave of second sound is interesting. This formulation of the problem made it possible to obtain very high intensities in the wave, which refutes Turner's conclusion that there exist limiting amplitudes. It seems that Turner observed the additional damping of a wave of second sound caused by damping on vortices generated by the second-sound wave itself.

An analogous series of investigations was performed recently at the Max Planck Institute (Göttingen).[77,78] The authors of these papers studied the dynamics of very powerful heat pulses in planar and cylindrical geometries. They showed that the experimental results are described well by the nonlinear theory, in which additional terms owing to the appearance of vortices in the wave are introduced (the equations of hydrodynamics of superfluid turbulence (see also Refs. 48 and 60)). It is interesting to note that to calibrate the temperature sensor they employed a nonlinear wave of second sound with known parameters.

A recent paper by Borisenko, Efimov, and Mezhov-Deglin[79] is devoted to purely nonlinear effects. In this investigation the nonlinear waves of second sound in a resonator were studied experimentally, but the theoretical questions were not studied. However, if one ignores the nonlinear transformation of waves of first and second sounds, then it should be expected that the dynamics of the waves will be close to that of the nonlinear waves of ordinary sound in a resonator. Indeed, the authors observe saw-tooth waves however, not standing waves (as in the linear case), but rather waves traveling along the resonator. This picture corresponds to the theoretical predictions of Chester.[80] The orientation of the peaks depended on the temperature. It is interesting that at T_α the waves had a sinusoidal form. This means that the nonlinearity coefficient $\alpha_2(T)$ vanishes not only for traveling but also for standing waves.

Among other experiments we must call attention to the recent work of Danil'chenko *et al.*, in which powerful pulses of second sound in HeII are studied with the help of phonon signals reflected into the heater.[86] The authors of this paper arrive at the unexpected conclusion that the mechanism of heat transfer at the helium-solid boundary is identical for HeI and HeII. The work of Kotsubo and Swift,[87] in which intense second sound is obtained, unlike in traditional methods, by forcing liquid through a porous barrier, is also interesting from the experimental viewpoint.

6.2. Experimental applications

The brief review of experimental investigations shows that most experiments are devoted to the dynamics of intense pulses of second sound. Meanwhile, as follows from the content of this review, the physics of nonlinear phenomena in HeII is much "richer" and more diverse. In this section we shall discuss the type of experiments that must be performed in order to observe the effects predicted by the theory.

Two of the most important theoretical results are the conclusion that the initial disturbances of the temperature or pressure decay nonlinearly and the determination of the quantitative characteristics of this process (see Secs. 2.4–2.5). Generally speaking, the formation of two types of waves accompanying pulsed liberation of heat has been observed by many authors (see the preceding section), though different interpretations were given for this phenomenon. However the experimental methods for recording waves did not permit determining the detailed structure of the disturbances. In all probability, calibrated pressure and temperature sensors, which permit determining quantitatively all quantities transported in the waves formed, must be used simultaneously in an experiment. Another interesting, from our viewpoint, result is the conclusion that a nonlinear monochromatic wave of second sound is self-focused at a temperature $T_\alpha \approx 1.885$ K (see Sec. 3). To observe this effect it is necessary to have a flat heater with nonuniform heating over its surface and a mobile temperature sensor. The theory predicts that the width of the wave beam will decrease away from the heater and at the same time the amplitude on the axis of the packet will increase. It would also be interesting to observe single heat pulses of finite width at other temperatures. Here transverse deformation of the packet and partial self-focusing are also possible (see Sec. 3.2).

The questions of decomposition and merging of monochromatic waves owing to instability under resonance conditions are classical questions in the nonlinear theory (see Sec. 4.1). It would undoubtedly be interesting to observe the generation of second sound accompanying the passage of a monochromatic pressure wave in HeII. Such an experiment could be performed in the inverse arrangement also. For example, two almost oppositely directed beams of second sound should merge into one wave of first sound with an approximately five times lower frequency. The problem of the interaction of weak acoustic disturbances with a pressure shock wave was described in Sec. 4.2. It would be interesting to perform an experiment on determining the laws of refraction and reflection of waves of second sound at a surface of discontinuity.

The existence of power laws for the correlation functions of hydrodynamic quantities in turbulent phenomena has always been quite intriguing. Especially interesting is the manifestation of these laws in acoustically "coupled" systems, of which HeII is one. The wave turbulence described in Sec. 5 can in principle be created by stochastic pumping of low-frequency first and second sounds. The correlation functions can be measured directly using temperature and pressure sensors by methods which are well known in the theory of turbulence. These quantities can be determined indirectly by measuring the additional damping of the sounds (see Sec. 5).

The experiments proposed above pertain directly to the nonlinear effects predicted by the theory. Another aspect of the experimental applications is connected with the use of the methods of nonlinear acoustics for studying the properties of superfluid helium.

We shall present a number of examples of which of the results obtained can be used for this purpose.

One of the main features of the nonlinear theory is that the amplitude of the wave is related with other acoustic characteristics, such as, the velocity of propagation of the waves, the size of the wave channel, the damping, etc. This is very important, because there are serious and often insurmountable difficulties in performing direct measurement of the temperature (especially in non-steady-state cases). The relations of nonlinear acoustics make it possible to proceed from measuring perturbations of the temperature δT to measuring other characteristics, for example, the transit time of an intense pulse over a fixed distance, which is a simple problem. For example, the following, fundamentally new, method for measuring the Kapitsa resistance R_K was proposed in Ref. 48. Analysis of the boundary conditions shows that when a surface bounding HeII is heated a relaxation process with the characteristic time $\tau = R_k C$ where C is the heat capacity per unit surface area, occurs at the wall. For this reason, if, for example, heat with intensity W is released at the wall in a time $t<\tau$, then not all of this heat enters the helium; part of the heat is retained in the substrate. As a result the amplitude of the heat pulse Δv_n will be less than expected $\Delta v_n^0 = W/ST$, and in addition the decrease in the amplitude is functionally related with the Kapitsa resistance R_K. Furthermore, since the transit time of a nonlinear pulse depends on the amplitude it will also be functionally related with R_K. Thus the Kapitsa resistance R_K can be determined by performing an experiment on the propagation of a heat pulse in HeII. Another example of this approach is to use nonlinear second sound to probe HeII containing vortex filaments. Here a relationship can also be established between the transit time of a nonlinear signal and the characteristics of a vortex cluster (see Refs. 49 and 50). The transit-time method can be used in a similar manner to measure the parameters of some processes, for example, to determine the heat released on some heated surfaces. This ideology could find application in metrology. In particular, in Ref. 59 it is proposed that the temperature $T_\alpha = 1.885$ (see Sec. 2.9) be used as a reference temperature point. The procedure for establishing this reference point is connected with the property that the transit time of a nonlinear pulse of second sound (at $T = T_\alpha$) is, with high accuracy, independent of its amplitude.

Furthermore, it was shown in Sec. 2.4 that in the case of pulsed heating of a wall a "precursor"—a pressure wave—propagates in the helium in addition to entropy waves. The intensity of the heat pulse released at the wall can be determined from the amplitude δp of this "precursor." This could be useful in applied problems, for example, for developing a control system in cryogenic installations.

Nonlinearity acoustics gives new possibilities for studying the damping of second sound. For example, such damping can be determined from the "spreading" of the shock front of a nonlinear pulse (see Secs. 2.6 and 2.7). The threshold amplitudes of the decomposition and Cherenkov processes are related with the coefficients of viscosity, and the latter coefficients can be determined from measurements of the threshold amplitudes. The damping of spin waves in ferromagnets is measured in a similar manner (see Ref. 3).

Many quantitative relations presented in this review could serve as a basis for measuring thermodynamic quantities. We note, by the way, that the good agreement between the experimentally determined value of $\alpha_2(T)$ (see Refs. 13 and 15) and the value calculated using the formula (2.27) is confirmed by the dependence found for the thermodynamic quantities as a function of the relative velocity $\mathbf{v}_n - \mathbf{v}_s|$. In this connection we once again call attention to the results of Secs. 2.7–2.8. The coefficients $\alpha_2(T)$, D_2, D_3, and μ_2, appearing in Eq. (2.29), carry very rich information about fundamental processes in HeII, such as the interaction of quasiparticles, the dynamics of fluctuations, etc. Comparing the analytical solution found with the experimentally observed dynamics of heat pulses makes it possible to elucidate in this manner the basic features of these processes. In this way it is possible to measure the kinetic indices of the thermodynamic quantities near T_λ, to determine their absolute values, etc. The examples given above illustrate to a certain extent the assertion that the methods of nonlinear acoustics provide extensive possibilities for studying the properties of superfluid helium.

As mentioned in the Introduction, HeII is a unique acoustic system. Properties such as the existence of two modes of different physical nature, which makes it possible to record separately both types of sound, the strong temperature dependence of the acoustical parameters (such as the coefficient of nonlinearity, the coefficient of viscosity, the dispersion, the velocity of second sound, etc.), the steepening of the trailing and leading edges, the existence of a monochromatic wave in a nondispersive medium—all this makes HeII an extremely interesting object from the viewpoint of the nonlinear theory of waves. This diversity of the properties of HeII makes it possible to model in an experiment a large number of different exotic situations for propagation and interaction of nonlinear waves.

CONCLUSIONS

This review encompasses practically all traditionally studied problems in the theory of nonlinear waves. Thus the subject is in some sense complete. In experimental studies of the dynamics of intense waves, however, there arises a new approach. The nonlinear acoustics of superfluid helium, presented in this review, is based on the classical equations of two-fluid hydrodynamics and, as a consequence, it is valid only on the basis of the assumptions under which these equations were derived. In real situations, for example, in the study of wide powerful pulses or, conversely, waves with wavelengths comparable to the free path of quasiparticles, these assumptions do not hold, and there are a number of ways in which one can move beyond the model employed.

One such possibility that we want to discuss here is connected with vortex formation in intense pulses of second sound. In such pulses the relative velocity $|\mathbf{v}_n - \mathbf{v}_s|$ has a large value, which for a short time is capable of developing a vortex structure (see Refs. 18, 49, 51, and 52).

The generated vortices affect the wave that engendered them, as a result of which this wave will evolve differently than in the manner described in this review (see Sec. 6.1). A number of experimental articles on the study of powerful heat pulses in HeII indeed indicate that the observed effects disagree with the predictions of the nonlinear theory. These disagreements include the facts that the pulse does not have the shape of a Burgers triangle (see Secs. 2.5 and 2.6) or that the measured correction to the velocity of second sound is quantitatively different from the computed value. The disagreements also include the observed formation of a film of vapor with intense heating of a wall for a short period of time. Indeed, according to the formulas of acoustics (see Sec. 2), even in very strong pulses (up to 100 W/cm^2), the amplitude of the temperature in them does not exceed a value of the order of 0.05 K. This is often not enough simply to reach the HeII-vapor equilibrium curve (in p–T coordinates), while it is known that much larger overheatings (up to 2 K) are necessary in order to form a vapor film. The experiments of Refs. 53 and 54, concerning not very strong (up to 1 W/cm^2) but very long heat pulses[11] also cannot be explained on the basis of the classical two-velocity hydrodynamics. It seems that the phenomena observed in the cited papers can be explained on the basis of the equations of hydrodynamics of superfluid turbulence, i.e., the equations of motion of HeII containing randomly oriented vortex filaments. Some results of this theory are presented in Refs. 50 and 60.

Vortex formation in sound waves is an example of the impossibility of describing the evolution of thermal disturbances on the basis of the classical hydrodynamics of a superfluid liquid. Another example are problems associated with very short durations or length scales, comparable to the free path of quasiparticles. In this case the dynamics of thermal disturbances can be described correctly only on the basis of the kinetic approximation. What we have said above pertains especially to nonlinear waves, since in the process of evolution of such waves the scales over which the parameters change are significantly smaller. An example is the description in Sec. 2 of the steepening of the wave profile and the formation of a shock front. Based on the kinetic theory one should apparently solve problems concerning the propagation of thermal disturbances near solid surfaces, in particular, the question of a boundary heat wave (see, for example, Ref. 61).

In the process of evolution of the temperature field accompanying the propagation of thermal disturbances in HeII the helium can be heated up to temperatures close to T_λ. In the neighborhood of T_λ care must be taken in using the classical two-velocity hydrodynamics. Aside from the fact that because of the singularities of the thermodynamic quantities the acoustics problems become very specific (we recall, for example, the divergence of $\alpha_2(T)$ in Sec. 2.7 and Fig. 3), the hydrodynamics itself may become inapplicable owing to the increase in the correlation radius. In this case the methods employed for studying disturbances involve very complicated and subtle questions regarding the dynamics of fluctuations or the use of the Ψ theory of superfluidity (see Ref. 62).

A number of fundamentally new problems arising in the study of thermal disturbances concern the study of waves in which the conditions for the phase transitions HeI–HeII, HeII-vapor, or HeII–HeI-vapor are satisfied. These problems are the analogs of Stefan's problem for HeII and, for all practical purposes they have not been studied (see, however, Refs. 63 and 64).

Thus it can be suggested that the further study of the dynamics of thermal disturbances in HeII should be conducted outside the "framework" of the standard two-velocity hydrodynamics, while the description of such disturbances by the methods of classical nonlinear acoustics is a stage that has been completed.

In conclusion I thank I. M. Khalatnikov and V. V. Lebedev for numerous discussions and a number of helpful remarks. I also thank A. N. Tsoĭ for selecting and discussing the experimental investigations.

[1] The variables γ and f describe nonpotential flows of the normal component and can be set equal to zero for acoustics problems.

[2] Everywhere in what follows in the variables a_k^ν and ω_k^ν as well as n_k^ν, Γ_k^ν and Δ_k^ν (see Sec. 5) the quantities $\nu = \pm 1$ and ± 2 are indices (not powers)!

[3] In Ref. 7 the problem, described below, of calculating the Riemann invariants is solved neglecting the terms containing the coefficient of expansion $\beta_T = -\rho^{-1} \times \partial\rho/\partial T$, which is valid in the range 0.8 K $< T <$ 1.6 K. Taking into account the terms related with β_T, this problem can be solved in the particular case when either only the disturbance of the density ρ' or only the disturbance of the entropy σ' is given as the boundary conditions. However the corresponding calculations are extremely unwieldy, and in this review we confine ourselves to presenting the results of Ref. 7.

[4] The solution of the system of n quasilinear equations, in which $n-2$ variables are functionally related with 2 (of the n) variables, is called a simple wave of rank 2.[10] Correspondingly, the waves described by Riemann are called simple waves of rank 1.

[5] This fact can be given a different interpretation. The positiveness of the quantity under study means that the fourth-order Hamiltonian H_4 is negative, i.e., the two-particle interaction between quanta of second sound is attractive, which is what leads to the self-focusing of the wave.

[6] This process of generation of second sound by first sound, connected with the instability of the waves, must be distinguished from the nonlinear decomposition of waves of entropy and density (see Sec. 2.4), which was a consequence of the fact that the variables ρ' and σ' do not describe "pure" wave modes.

[7] This equation can also be derived by calculating directly the changes in the number of sound quanta in the nonlinear processes.[58]

[8] The relation given below for the damping Γ_k^ν can also be derived from the system of kinetic equations (see Refs. 36 and 43).

[9] With the difference that the form of the spectrum $n_k^{1,2}$ is different from the Planck distribution.

[10] An analogous phenomenon was recently observed in classical media near the critical point.[89]

[11] A detailed review of the cited and analogous works is given in Ref. 88.

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Translated by M. E. Alferieff