Predictions of the general theory of relativity are free from ambiguity

S.I. Chermyanin

Scientific-Research Radiophysics Institute, Gorki Usp. Fiz. Nauk 160, 127–131 (May 1990)

The set of the complete Schwarzschild solutions for a static spherically symmetric homogeneous ideal liquid of finite volume is examined. Using the example of the phenomenon of the red shift of frequency in a gravitational field it is shown that the use of different types of solutions leads, contrary to the assertion of authors of a relativistic theory of gravitation, to unambiguous predictions of gravitational effects by the general theory of relativity.

1. The general theory of relativity has been criticized over the past few years in several papers by A. A. Logunov and his colleagues.¹⁻⁶ In particular, they have raised the question of an alleged ambiguity in predictions for gravitational effects in the structure of the general theory.³⁻⁶ Although recent papers by Zel'dovich and Grishchuk^{7,8} and Ginzburg^{9,10} have already outlined why this criticism is groundless, it is still worthwhile to take another, more-detailed look at this question.

One goal of the present paper is to show that the predictions of the general theory are unambiguous in the specific example of the gravitational frequency shift, through the use of two types of complete solutions of the Schwarzschild problem. A second goal is to demonstrate that the assertions of the authors of the relativistic theory of gravitation are groundless.

2. We assume that the source of the gravitational field is a static ideal liquid which fills the space inside a sphere of physical radius l_0 . For simplicity we assume that the liquid is homogeneous, i.e., that the self-mass density is a constant ($\mu = \text{const}$). The energy-momentum tensor for this medium, without any macroscopic motion ($\mu^i = 0$), is written in the form¹)

$$T^{\alpha}_{\beta} = (\mu + P) u_{\beta} u^{\alpha} - P \delta^{\alpha}_{\beta} = (\mu + P) \delta_{\alpha 4} \delta^{\beta}_{4} - P \delta^{\alpha}_{\beta}, \qquad (1)$$

where *P* is the pressure of the medium, and $\delta^{\alpha}_{\beta} = \delta_{\alpha\beta}$ is the Kronecker 4-delta.²

We place the center of the source at the origin of coordinates, and we seek a set of complete solutions in the general orthogonal form

$$\mathrm{d}s^2 = e^{\nu}\mathrm{d}t^2 - e^{\lambda}\mathrm{d}r^2 - \eta^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\varphi^2\right), \tag{2}$$

where ν , λ , and η are functions of the radial coordinate r alone which have the limits $\nu \rightarrow 0$, $\lambda \rightarrow 0$, and $\eta \rightarrow r$ as $r \rightarrow \infty$.

After the metric tensor from the quadratic form in (2) and the energy-momentum tensor in (1) are substituted into Einstein's equations

$$R^{\alpha}_{\beta} - \frac{1}{2} \delta^{\alpha}_{\beta} R = \varkappa T^{\alpha}_{\beta} , \qquad (3)$$

the latter are satisfied either identically or by virtue of the three independent equations

$$e^{-\lambda}\left(\frac{\lambda'\eta'}{\eta}-\frac{{\eta'}^{*}}{\eta^{*}}-2\frac{\eta''}{\eta}\right)+\frac{1}{\eta^{*}}=\varkappa\mu, \qquad (4a)$$

$$e^{-\lambda}\left(\frac{\nu'\eta'}{\eta}+\frac{\eta'}{\eta^2}\right)-\frac{1}{\eta^2}=\kappa P,$$
(4b)

$$e^{-\lambda}\left(\nu'' + \frac{\nu''}{2} - \frac{\nu'\lambda'}{2} + 2\frac{\nu'\eta'}{\eta}\right) = \varkappa (\mu + P), \qquad (4c)$$

where the prime means differentiation with respect to r.

If the boundary of the source is assumed to be free, the pressure is zero at this surface, and the mass density of the liquid is generally discontinuous. Using the joining conditions at the boundary of the source for the components of the metric $g_{\alpha\beta}$, we can write a set of complete solutions for all space:

$$\eta'^{*} = e^{\lambda} \left(1 - \frac{\alpha}{\eta} \right), \qquad (5a)$$

$$e^{\nu} = 1 - \frac{\alpha}{\eta}, \quad \eta \ge \eta_0,$$
 (5b)

and

$$\eta'^* = e^{\lambda} \left(1 - \frac{\eta^2}{A^3} \right), \tag{6a}$$

$$e^{\gamma} = \frac{1}{4} \left[3 \left(1 - \frac{\eta_0^2}{A^2} \right)^{1/2} - \left(1 - \frac{\eta^2}{A^3} \right)^{1/2} \right], \ \eta \leqslant \eta_0, \qquad (6b)$$

where

$$\frac{1}{4^2} = \frac{\varkappa \mu}{3} = \frac{8\pi}{3} \, \sigma \mu, \quad \alpha = \varkappa \int_{0}^{\eta_0} \mu \eta^2 \mathrm{d}\eta = 2\sigma M,$$

M is the gravitational mass of the source, and η_0 is the value of the function η at the boundary of the source. The expression for the pressure becomes

$$P = \mu \frac{\left[1 - (\eta_0^2/A^3)\right]^{1/2} - \left[1 - (\eta^3/A^3)\right]^{1/2}}{\left[1 - (\eta^3/A^3)\right]^{1/2} - 3\left[1 - (\eta_0^2/A^3)\right]^{1/2}}.$$
(7)

The pressure inside the liquid is negative here, but this point is of no importance for our model problem.

We adopt the differential condition $dr = d\eta$, which fixes the type of solution. The metric of this solution is written in the form

$$\mathrm{d}s^2 = \left(1 - \frac{\alpha}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - (\alpha/r)} - r^2\mathrm{d}\Omega^2, \quad r \ge r_0, \quad (8a)$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2,$$

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and

$$ds^{2} = \frac{1}{4} \left[3 \left(1 - \frac{r_{0}^{2}}{A^{2}} \right)^{1/2} - \left(1 - \frac{r^{2}}{A^{2}} \right)^{1/2} \right]^{2} dt^{2} - \frac{dr^{2}}{1 - (r^{2}/A^{2})} - r^{2} d\Omega^{2}, \ r \leqslant r_{0},$$
(8b)

where r_0 is the value of the coordinate at the source boundary. From the set of solutions in (5)-(6) we can select a solution of a different type if we set³⁾ $dr^*/r^* = e^{\lambda/2} d\eta/\eta$. The metric then becomes

$$ds^{2} = \left[\frac{1 - (\alpha/4r^{*})}{1 + (\alpha/4r^{*})}\right]^{2} dt^{2} - \left(1 + \frac{\alpha}{4r^{*}}\right)^{4} (dr^{**} + r^{**}d\Omega^{2}),$$

$$r^{*} > r_{0}^{*}, \qquad (9a)$$

and

$$ds^{2} = \frac{1}{4} \left(3 \frac{c^{2} - r_{0}^{*}}{c^{2} + r_{0}^{*}} - \frac{c^{2} - r^{*}}{c^{2} + r^{*}} \right)^{2} dt^{2} - \frac{4c^{2}A^{2}}{(c^{2} + r^{*})^{2}} (dr^{*} + r^{*}d\Omega^{2}), \ r^{*} \leqslant r_{0}^{*},$$
(9b)

where c is an arbitrary constant, and r_0^* is the value of the coordinate at the source boundary. At the surface of the source we have the relation

$$\left(1 + \frac{\alpha}{4r_0^*}\right)^2 = \frac{2cA}{c^2 + r_0^{**}}$$
 (10)

On the other hand, we have

$$\alpha = \frac{r_0^3}{A^3} = \frac{r_0^{**}}{A^2} \frac{8c^3A^3}{(c^2 + r_0^{**})^3} \,. \tag{11}$$

From the relation

$$4r_{0}^{\bullet}\left[\left(\frac{2cA}{c^{2}+r_{0}^{*}}\right)^{1/2}-1\right]=\frac{r_{0}^{\bullet}}{A^{2}}\frac{8c^{3}A^{3}}{(c^{2}+r_{0}^{*})^{3}}$$
(12)

we can then find the constant c. The coordinates of solution (8) are related to those of solution (9) inside the source¹¹ by

$$r^{\bullet} = r \frac{c}{A\{1 + [1 - (r^2/A^2)]^{1/2}\}}, \quad r = r^{\bullet} \frac{2cA}{c^2 + r^{*}}, \quad (13a)$$

and outside the source by

$$r^{*} = \frac{1}{2} \left[\left(r^{2} - \alpha r \right)^{1/2} + r - \frac{\alpha}{2} \right], \ r = r^{*} \left(1 + \frac{\alpha}{4r^{*}} \right)^{3}.$$
(13b)

3. Expressions (8)-(9) are different solutions of Einstein equations (3) for a spherically symmetric source (an ideal homogeneous liquid). The difference between these solutions lies in the specification of different differential conditions which determine $\eta(r)$. In the general theory of relativity, however, these solutions are completely equivalent! In the usual terminology, these solutions are one and the same, but written in two different coordinate systems. It is for this reason that the solutions lead to the same value in a prediction of a gravitational effect in the general theory of relativity. We can demonstrate this point in a very simple example: the gravitational frequency shift of our source in an external field.

Let us assume that an observer and a transmitter of electromagnetic radiation of frequency ω are on a common straight line which passes through the spherical center of our liquid. We assume that the transmitter is on the sphere of this source at the coordinate r_0 (r_0^*) , while the observer is at a distance l from the source, at the coordinate $r(r^*)$. The relative gravitational frequency shift for solutions (8), (9) is then written in the following respective forms:

$$\frac{\Delta\omega}{\omega} = \left(\frac{g_{00}(r_0)}{g_{00}(r)}\right)^{1/2} - 1, \qquad (14a)$$

$$\frac{\Delta^*\omega}{\omega} = \left(\frac{g_{00}^*(r_0^*)}{g_{00}^*(r^*)}\right)^{1/2} - 1,$$
(14b)

where

$$g_{00}(r) = 1 - \frac{\alpha}{r}, \quad g_{00}^{\bullet}(r^{\bullet}) = \left[\frac{1 - (\alpha/4r^{*})}{1 + (\alpha/4r^{*})}\right]^{2}.$$

If we wish to use these expressions in the theoretical prediction of a gravitational shift, we need to know the coordinates $(r_0 r_0^* \text{ and } r, r^*)$ at the positions of the measurement apparatus. From the physically measurable quantities l_0 and l we can find the values of these coordinates if we work from the integral relations

$$l_0 = \int_0^{r_0} \frac{\mathrm{d}r}{\left[1 - (r^2/A^2)\right]^{1/2}} = \int_0^{r_0} \frac{2cA}{c^2 + r^{\star^2}} \,\mathrm{d}r^\star \tag{15}$$

and

$$l = \int_{r_0}^{r} \frac{\mathrm{d}r}{[1 - (\alpha/r)]^{1/2}} = \int_{r_0}^{r^*} (1 + \frac{\alpha}{4r^*})^2 \,\mathrm{d}r^*.$$
(16)

The coordinates r_0 and r_0^* at the source boundary have different numerical values, in contrast with the suggestion in Refs. 3-6. We write the relationship between these coordinates as follows:

$$r_{0} = r_{0}^{*} \left(1 + \frac{\alpha}{4r_{0}^{*}} \right)^{2}.$$
 (17)

This relation expresses the obvious requirement that the surface area of the boundary sphere must be the same in the two forms of the solution. In other words, the physical distance from the center to the boundary sphere must be identical in these solutions. The coordinates r and r^* , which fix the position of the observer, are related in a corresponding way. Taking these circumstances into account, we then find the equations

$$g_{00}(r_0) = g_{00}^*(r_0^*), \quad g_{00}(r) = g_{00}^*(r^*),$$
 (18)

from which it follows that the relative gravitational shifts at given spatial measurement points found from the different solutions will nevertheless have a common definite value:

$$\Delta \omega = \Delta^* \omega. \tag{19}$$

4. We thus reach the conclusion that a mathematically rigorous analysis of the Schwarzschild problem leads, in contradiction of the assertions of the authors of the relativistic theory of gravitation, to the equivalence of solutions (8) and (9) in terms of their predictions of the gravitational frequency shift. This assertion can also be made for other gravitational effects. The common coordinate principle in the general theory of relativity expresses not only the trivial point that it is possible to write the mathematical relations in covariant form, as can be done even in physics without gravity, but also the point that the solutions of Einstein's equations are themselves equivalent in each specific physical situation.

The choice of the arithmetization of the space here is common for these solutions; nevertheless, the coordinates of different solutions at any spatial point (except the center of the source) will give different values. These values will differ to exactly the extent required to eliminate any ambiguity in the predictions of the general theory! Only at the center of the source will the values of these coordinates be the same: $r = r^* = 0$.

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¹⁾We are assuming everywhere that the velocity of light is $c_0 = 1$.

³⁾In Ref. 3, the choice of coefficients N(r) = 0, $c(r) = r^{*2} [1 + (\alpha/4r^*)]^4$

corresponds to a solution of this type if we consider only the external solution.

¹A. A. Vlasov, A. A. Logunov, and M. A. Mestvirishvili, Teor. Mat. Fiz.

- 61, 323 (1984) [Theor. Math. Phys. (USSR) 61, 1167 (1984)].
- ²A. A. Logunov and M. A. Mestvirishvili, Teor. Mat Fiz. **61**, 327 (1984) [Theor. Math. Phys. (USSR) **61**, 1170 (1984)].
- ³A. A. Logunov, Yu. M. Loskutov, and Yu. V. Chugreev, Teor. Mat. Fiz.
- 69, 328 (1986) [Theor. Math. Phys. (USSR) 69, 1179 (1986)].
- ⁴A. A. Logunov, Yu. M. Loskutov, and M. A. Mestvirishvili, Teor. Mat.
- Fiz. 73, 163 (1987) [Theor. Math. Phys. (USSR) 73, 1131 (1987)].
- ⁵A. A. Logunov, Yu. M. Loskutov, and M. A. Mestvirishvili, Usp. Fiz.
- Nauk 155, 369 (1988) [Sov. Phys. Usp. 31, 581 (1988)].
- ⁶A. A. Logunov and Yu. M. Loskutov, Teor. Mat. Fiz. **76**, 163 (1988) [Theor. Math. Phys. (USSR) **76**, 779 (1988)].
- ⁷Ya. B. Zel'dovich and L. P. Grishchuk, Usp. Fiz. Nauk **149**, 695 (1986) [Sov. Phys. Usp. **29**, 780 (1986)].
- ⁸Ya. B. Zel'dovich and L. P. Grishchuk, Usp. Fiz. Nauk **155**, 517 (1988) [Sov. Phys. Usp. **31**, 666 (1988)].
- °V. L. Ginzburg, Nauka i zhizn' No. 4, 91 (1987).
- ¹⁰V. L. Ginzburg, Nauka i zhizn' No. 6, 114 (1988).
- ¹¹C. Møller, *Theory of Relativity*, Oxford Univ. Press, London, 1972 [Russ. transl., Atomizdat, M., 1975].

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²¹To satisfy the condition that our source be static, we choose the parameters μ , P, and l_0 here from the range of values for ordinary celestial bodies of the solar system, e.g., the earth.