I. S. Aranson, K. A. Gorshkov, A. S. Lomov, and M. I. Rabinovich. Nonlinear dynamics of the localized states of multidimensional fields. Well-known examples of the two- and three-dimensional localized states (particle-like structures) of nonlinear fields or media are vortices in the atmosphere, rings in the ocean, different types of defects in crystals and regular wave lattices, localized waves of charge density, plasma streams in controlled fusion installations, localized spirals in liquid crystals, etc. It is possible that elementary particles correspond to localized "singularity-free" solutions of nonlinear multidimensional field equations.

The spatial characteristics of localized states are universal. They are independent of the physical nature of the field (medium) under study, and they also do not depend on whether we are talking about dissipative nonequilibrium media (for example, convective flows), Hamiltonian fields ("particles"), or statistical systems (crystalline lattices). This universality is explained by the fact that in all these situations the localized states result from spontaneous breaking of the symmetry of the system and for this reason they satisfy general topological laws.

The physics of nonequilibrium media is concerned primarily with the dynamics of localized states (charge transfer, interaction of "particles" in field theory, interaction of vortices in turbulence, etc.). The main problem here is to construct basic models of the theory such that stable localized states of multidimensional fields, for example, stable three-dimensional solitons, would exist. It is well known, however, that the traditional models give us examples only of unstable stationary "particles" (the one-dimensional case is a lucky exception). Models in which stable multidimensional "particles" exist were recently proposed by Rabinovich *et al.*[1] This report is concerned with a discussion of these models and the construction, based on them, of a theory of interaction of localized states, encompassing the formation of lattices, "planetary systems," the creation of dynamical space-time chaos, etc.

In the analysis of stationary localized states it is natural to study simultaneously the equations describing dissipative nonequilibrium media and Hamiltonian fields. It is convenient to do this for the example of potential fields, whose stationary states satisfy the equation $\delta F/\delta u = 0$, where u is a physical variable and for a Hamiltonian field F is the Lagrangian (the potential energy, etc.) and for a dissipative nonequilibrium medium F is the free energy. A stable localized structure can be found by setting to zero the first variation, $\delta F = 0$, and requiring the second variation to be positive, $\delta^2 F > 0$ (this corresponds to a minimum of the potential). The dynamic equations have the following form

$$\frac{\partial u}{\partial t} = -\frac{\delta F\{u\}}{\delta u} \qquad (1)$$

for a dissipative medium and

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta F\{u\}}{\delta u} \qquad (2)$$

for a Hamiltonian field. The stationary structures of the fields (1) and (2) are obviously identical. The number of such structures can be arbitrarily large (multistability). On the basis of the gradient model (1) the stationary states are established spontaneously in the limit $t \to \infty$ as a result of the evolution of the system. In the case of the Hamiltonian model (2), however, they must be guessed outright (which is unlikely) or based on the system (1).

To expand the right sides of Eq. (1) or (2) we shall expand the energy density

$$\mathcal{F}\left(\mathrm{F}=\int \mathcal{F}\,\mathrm{d}\mathbf{r}\right)$$

in powers of the field and the gradient of the field near the point where the trivial uniform state is unstable:

$$\mathcal{F}_1=\alpha u^2+\beta u^3+\gamma u^4+\xi(\nabla u)^2+\zeta(\nabla^2 u)^2+\ \dots \qquad (3)$$

As a result we obtain an equation which can be naturally termed the generalized Swift-Hohenberg equation (the coefficients are omitted)

$$\frac{\partial u}{\partial t}=-u(1-\beta u+u^2)-(k_0^2+\nabla^2)^2 u. \qquad (4)$$

Direct computer experiments show that in this model there do indeed exist stable localized states with different topology (they correspond to different local minima of F). These are a "sphere," "torus," and "baseball"—a structure similar to the figure on a tennis ball (Fig. 1). The fundamental difference between these "particles" and the traditional solitons is the character of the decay of the field at the periphery of the structure—in our case the field decays exponentially and oscillates. Because of this, such structures can form extremely diverse stable bound states—chains, lattices, planetary systems with a discrete (infinite) set of orbits, etc. (this is confirmed by computer experiments).

All these stationary states also exist and are stable in the Hamiltonian analog of the model (4)

$$\frac{\partial^2 u}{\partial t^2}+(k_0^2+\nabla^2)^2 u+u(1-\beta u+u^2)=0. \qquad (5)$$

In the gradient model (4) all limiting (in the limit $t\to\infty$) states of the field (medium) are static. In the system (5), however, mutual rotation of the "particles," formation of (periodically, quasiperiodically, and chaotically) oscillating clusters, propagation of waves in lattices, etc. are possible. Assuming that the interaction of the particles is weak an asymptotic theory can be constructed to describe their dynamics (the small parameter here is the ratio of the field onthe "tail" of one particle to the field at the maximum of the other particles).

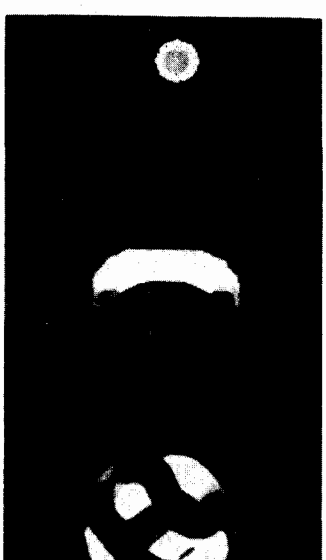

FIG. 1.

For example, for the coordinates of the centers of mass of the interacting spheres we obtain a system of equations similar to the equations of Newtonian mechanics:

$$\frac{\mathrm{d}\mathbf{r}_{0i}}{dt}=\mathbf{v}_i,$$

$$\frac{\mathrm{d}\mathbf{v}_i}{dt}=\nabla_{\mathbf{r}_{0i}}\sum_{j\neq i}\mathrm{Re}\,\frac{\exp(ik\,|\mathbf{r}_{0i}-\mathbf{r}_{0j}|)}{|\mathbf{r}_{0i}-\mathbf{r}_{0j}|}\qquad (k=k'+ik''), \qquad (6)$$

which differs from the classical equations only by the character of the potential. This system can be employed to describe the mutual rotations (periodic and quasiperiodic) of localized states, observed in direct computer experiments with Eq. (5), as well as the formation of diverse bound states.

It is remarkable that on the basis of models of the type (5) it is possible to understand the mechanisms of creation of spatial-temporal disorder in purely dynamical models of a field. One of the main such mechanisms is the appearance of localized states and their random wandering in space (as a result of interaction with other "particles" or regular fields). Thus under certain initial conditions the dynamics of the system

$$\frac{\partial^2 u}{\partial t^2}+(k_0^2+\nabla^2)^2 u+u(1-\beta u+u^2)+uv=0,$$

$$\Delta v+v=0 \qquad (7)$$

can be regarded as the interaction of one "particle" with a periodically nonuniform field v. In particular, for a two-dimensional space Eq. (6) assumes the form

$$\frac{\mathrm{d}^2\mathbf{r}_0}{dt^2}=-\nabla U(X,Y),\quad \mathbf{r}_0=\{X,Y\}, \qquad (8)$$

where X and Y are the coordinates of the center of mass of the localized state. It is well-known[2] that systems such as (8) describe the random walk of a particle in the X, Y space. Such random walks have also been observed in direct experiments with the system (7) (Fig. 2).

An analysis similar to that presented here for real scalar fields can also be performed for complex fields. In this case, analogously to Eq. (3), we have

$$\mathcal{F}_2=\alpha|u|^2+\beta|u|^4+\gamma|u|^5+|(k_0^2+\nabla^2)u|^2+\ \dots \qquad (9)$$

FIG. 2.

(here the fact that the energy is independent of the phase of the field—gauge invariance—is taken into account). Correspondingly, the equation has the form

$$(i\varkappa + \nu)\frac{\partial u}{\partial t} = -u(1 - \beta|u|^2 + |u|^4) - (k_0^2 + \nabla^2)^2 u. \qquad (10)$$

The case $\varkappa = 0$ corresponds to a gradient system (dissipative system) and the case $\nu = 0$ corresponds to a Hamiltonian system.

The model (10) has made it possible to observe localized two- and three-dimensional spirals (in the $3D$ case the spirals are toroidal whorls), to study their bound states, and to describe random walks.

Unsolved problems of particular interest include, first of all, the study of strong interactions of "particles," in which some structures are transformed into other structures, some structures are annihilated, etc. With the help of models similar to those studied above it is apparently possible to transfer systematically from dynamical systems to the systems of statistical mechanics.

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Translated by M. E. Alferieff