Localization and wave propagation in randomly layered media

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The authors review localization and wave propagation in randomly layered media. They demonstrate that in addition to discrete spectrum waves a disordered open system can support quasihomogeneous waves (metastable states). These quasihomogeneous waves are due to the interference of multiply scattered fields and their existence leads to the appearance of a fluctuation waveguide that can channel energy along the layers for distances exponentially greater than the layer thickness. The authors also examine in detail the statistical characteristics of the field produced in a randomly layered medium by a point source.

1. INTRODUCTION

A consistent treatment of interference by multiply scattered fields is an important current problem in the theory of wave propagation in random media and in quantum theory of disordered solids. The concept of localization arose naturally in the course of research into this problem, as localization phenomena reflect the most general properties of disordered systems, arising from the wave nature of scattered fields and the random character of scattering media. Over thirty years ago, P. Anderson published his seminal paper[1] where he argued that all states of a given disordered three-dimensional system will be localized if the system is sufficiently random. In the subsequent decades the concept of localization became fundamental in the physics of disordered solids, with the appearance of such concepts as the Anderson dielectric, the Anderson transition, scaling theory of localization, and weak localization that now occupy a place of pride in numerous monographs,[2-4] textbooks,[5,6] and physics encyclopedias.[7,8]

In recent years the localization of various types of waves and excitations in media that are not spatially periodic or homogeneous has attracted a great deal of interest. Much research has been devoted to the localization of acoustic waves in continuous media,[9-15] electromagnetic waves in solids and plasmas,[14,16] gravitational waves in shallow water channels with a rough bottom,[17-19] third and fourth sounds in a helium film on a randomly inhomogeneous substrate,[20,21] surface waves in metals,[22-24] and so forth. Studies of the transmission of short, irregular pulses through homogeneous media stimulated much research into the localization aspects of the solutions of the Dirac-type equations[25-27] that describe the properties of single-particle excitations in superconductors and semiconductors,[28,29] as well as wave transmission through a multilayer structure typical of x-ray mirrors.[30]

In the scientific literature devoted to wave propagation the term "localization" is of more recent origin than in the theory of disordered solids and has not yet gained the same acceptance, even though the first studies of localization phenomena in radiophysics appeared as early as the 1950's. Those original studies were motivated by the desire to determine and extend the limits of applicability of the radiation transfer equation. In the late 1950s, Gertsenshteĭn and Vasil'ev[31,32] demonstrated for the first time that the average transmission coefficient for a plane wave propagating through a one-dimensional disordered layer decays exponentially with layer thickness. In the years that followed, research into the transmission of waves through a one-dimensional random medium continued and intensified, as evidenced by several monographs on this topic,[33-35] but the concept of localization did not come into use in this field until very recently. This was probably due to the fact that the main postulates of localization theory had been formulated and proven for closed systems with self-conjugate boundary conditions (see, for example, Ref. 4). In radiophysics and acoustical physics, on the other hand, one usually encounters open systems with non-self-conjugate boundary conditions, such as radiation fields at infinity. This review will address the spectral properties of such systems, as well as localization and associated phenomena that arise when waves propagate in random media whose parameters depend on a single coordinate (see also Refs. 36, 37).

Layered "one-dimensional" structures are frequently employed in optics, radiophysics, and acoustics as models of a propagation medium.[38] In particular, a theoretical understanding of these structures is necessary for the description of wave propagation in naturally occurring media (atmosphere, ionosphere, the ocean) within the framework of the two-region model.[39,40] In these problems the spatial spectrum of the refraction index $n(\mathbf{R})$ or the permittivity $\varepsilon(\mathbf{R})$ can be separated into two statistically independent regions, whose effects can be treated separately and, to some extent, independently. One of the two regions, characterized by turbulence and a small length scale, can be treated within the Markov process approximation as long as the inhomogeneities are not too large.[33] The other region, characterized by a large length scale, is usually strongly anisotropic (because of the boundary interface), making it possible in a number of cases to neglect its variation in the horizontal x–y plane and describe it approximately as a function of a single coordinate z. In this approach solving the one-dimensional problem becomes an essential intermediate step in the calculation of the real spatial dependence $\varepsilon(\mathbf{R})$.

We shall proceed from the scalar Helmholtz equation

$$\Delta u + \varepsilon(\mathbf{R})\frac{\omega^2}{c^2}u = 0, \tag{1.1}$$

in which the permittivity ε is of the form

$$\varepsilon(\mathbf{R}) = \varepsilon_0 + \delta\varepsilon(z),\ \mathbf{R} = (\boldsymbol{\rho}, z); \tag{1.2}$$

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where $\delta\varepsilon(z)$ is a random function of coordinate z and has zero average value. Defining

$$\varepsilon_0 \frac{\omega^2}{c^2} \equiv E_0,$$

Fourier-transforming the field in terms of the coordinate ρ:

$$u(\mathbf{R}) = (2\pi)^{-2} \int e^{i\varkappa\rho} \tilde{u}(\varkappa, z)\, d\varkappa,$$

and taking

$$E_0 - \varkappa^2 = E, \quad -E_0 \frac{\delta\varepsilon(z)}{\varepsilon_0} = v(z), \quad \tilde{u}(\varkappa, z) = \psi(z),$$

we arrive at the one-dimensional Schrödinger equation

$$-\psi'' + v(z)\psi = E\psi. \tag{1.3}$$

When $v(z)$ and consequently $\varepsilon(z)$ are spatially homogeneous on average and exhibit decreasing correlation over large distances, the properties this equation have already been extensively studied, making it possible to employ the full range of results obtained in the theory of disordered solids (see, for example, Ref. 4). The first of these results is the self-averaging of certain extrinsic physical quantities (density of states, free energy density, damping coefficient of the transmissivity of a one-dimensionally disordered random layer, etc.). The fundamental property of self-averaging quantities is that their values in a particular realization of an infinite system will coincide with their mean values system with unity probability. This endows the calculated mean values with the physical meaning of real observables.

The reliable identification of self-averaging quantities with their mean values occurs in the macroscopic limit $V \to \infty$. Every self-averaging quantity f_V will have a characteristic volume V_f in some parameter space, beyond which the distribution of f_V will become Gaussian with a mean value $\langle f_V \rangle$ that is independent of V and a dispersion $\langle f_V^2 \rangle - \langle f_V \rangle^2 \sim V_f/V$. For systems smaller than V_f (but still significantly larger than microscopic), on the other hand, the f_V becomes a random quantity dependent on the particular realization of the system. In this regime the system is considered mesoscopic and the fluctuations of f_V are known as mesoscopic fluctuations.[41-44]

Usually in the theory of wave propagation in random media we are faced with quantities that are not self-averaging. For these quantities the system is mesoscopic regardless of volume and the mean values provide no information about a given realization. In order to endow these mean values with some physical meaning and facilitate comparisons with experiments (that are usually performed in a concrete realization), both the experiment and analysis of data require specialized methods. One of these methods involves the measurement of some observable over a finite time interval and the comparison of the time-averaged value with the mean value calculated over the ensemble of realizations. In real-world radiophysical measurements this procedure comes naturally, as the required variation of the parameters is accomplished by the time evolution of a real medium (for example, the refraction index of the atmosphere [45] or the surface roughness of the sea.[46]

Another method of analyzing experimental data can be employed when the measured result is a function of some additional parameter (frequency, transmitter and receiver design, etc.). In this case one can compare the ensemble mean value of the observable with the value obtained by averaging over one or more parameters. The averaging must be performed over a region that is large compared to the appropriate correlation radius, and yet sufficiently small for the ensemble mean not to vary.[47,48] For instance, in the study of surface roughness in solids by means of light scattering, the averaging parameter is the aperture of the beam, i.e., the area of the illuminated region. If this area is sufficiently large, the mean scattering indicatrix can be obtained from a single sample.[49-52]

Still, there exist physical quantities f_V whose relative fluctuations increase with the volume of the system, rendering the above-described methods inapplicable. The mean values $\langle f_V \rangle$ of these quantities differ strongly from the values measured in typical realizations since they are dominated by improbable representative realizations. As a consequence the mean values $\langle f_V \rangle$ contain little physical information. In this case additional information on the behavior of the random quantity f_V in typical (most probable) realizations can be obtained by investigating the dependence of f_V on other self-averaging quantities. Indeed, if f_V can be expressed in terms of some self-averaging quantity γ_V and the volume $f_V = f(\gamma_V, V)$, then in the limit $V \to \infty$, γ_V tends to some non-random limit

$$\gamma = \lim_{V\to\infty} \gamma_V.$$

Thus, as long as V is large, the function $f(\gamma, V) \neq f(\gamma_V, V)$ describes (at least qualitatively) the behavior of f_V in typical realizations. We shall apply this method below in the analysis of the wave reflection coefficient by a disordered layer and the transmitted wave intensity (Sec. 2), as well as to the distribution of energy flux from a point source in a randomly layered medium (Sec. 4).

Consider the solution $\psi(z)$ of equation (1.3) that satisfies the current-free (self-conjugate) boundary condition, for example at the $z = 0$ interface:

$$\psi(0) + a\psi'(0) = 0, \quad \operatorname{Im} a = 0. \tag{1.4}$$

The reflection coefficient r_- for a wave with energy $E = k^2$ incident on such an interface from the right has the form

$$r_- = \frac{ika - 1}{ika + 1} \tag{1.5}$$

and corresponds to total reflection, $|r_-| = 1$. The solution can be chosen as real and parametrized by

$$\psi(z) = e^{\xi} \sin\varphi, \quad \psi'(z) = ke^{\xi} \cos\varphi. \tag{1.6}$$

In the case when $v(z)$ is spatially homogeneous on average and exhibits decaying correlation with distance, as $z \to \infty$ the ratio $\xi(z)/z$ tends with unity probability to a non-random limit. In other words, the ratio $\xi(z)/z$ becomes a self-averaging quantity:

$$\lim_{|z|\to\infty} \frac{\xi(z)}{z} = \lim_{|z|\to\infty} \left\langle \frac{\xi(z)}{z} \right\rangle > 0. \tag{1.7}$$

Consequently, it is this ratio that is hereafter taken as the quantity $\gamma(z)$ (with the coordinate z acting as the volume):

$$\gamma(z) = \frac{\xi(z)}{z}.$$

In an infinite disordered system, the limiting value (1.7) of $\gamma(z)$ is related to the localization length $l(k^2)$ of the state with energy k^2 by the simple expression

$$\lim_{|z|\to\infty} \gamma(z) = \frac{1}{2l(k^2)}, \tag{1.8}$$

The state itself is exponentially localized in the vicinity of some localization center. In the language of propagation theory this means that, for example, if a waveguide consists of two ideal reflecting planes at $z = 0$ and $z = L$, whenever L is sufficiently large the random stratification of the permittivity will lead to a radical change in the spatial distributions of the field $\psi_n(z)$ of the modes along the transverse sections of the waveguide. In contrast to the one-dimensional case, where this field undergoes regular oscillations [$\psi_n \sim \sin(n\pi z/L)$, n being the mode index], in an irregular waveguide the envelopes of the normal waves decay exponentially on both sides of the random localization centers z_n, $|\psi_n(z)|^2 + |\psi'_n(z)|^2 \sim \exp[-(z - z_n)/l_n]$.

In the limiting case, when the correlation radius r_c of the potential $v(z)$ is small compared to the wavelength

$$r_c \ll k^{-1}, \tag{1.9}$$

the potential can be treated as δ-correlated:

$$B(z) = \langle v(z)v(0)\rangle \approx 2D\delta(z), \quad D \sim B(0)r_c, \tag{1.10}$$

and the localization length becomes

$$l(k^2) \approx \frac{2k^2}{D}, \quad k^2 \gg D^{2/3}. \tag{1.11}$$

If, in addition, the inequality

$$r_c \ll D^{-1/3}, \tag{1.12}$$

is satisfied, then whenever

$$|E| \ll r_c^{-2} \tag{1.13}$$

the potential $v(z)$ becomes Gaussian. The localization length behaves as follows:

$$\begin{aligned} l(E) &\approx \frac{2E}{D}, & D^{2/3} \ll E \ll r_c^{-2}, \\ &\approx D^{-1/3}, & |E| \lesssim D^{2/3}, \\ &\approx \frac{1}{2}|E|^{-1/2}, & D^{2/3} \ll -E \ll r_c^2. \end{aligned} \tag{1.14}$$

By applying the standard method of averaging over rapidly changing variables,[4,34] in the high energy regime

$$E = k^2 \gg D^{2/3} \tag{1.15}$$

one recovers the closed Fokker-Planck equation for the probability density $p(\gamma,z)$ of the quantity $\gamma(z)$, which turns out to have a Gaussian distribution with the mean value (1.8) and a dispersion inversely proportional to the "volume" z. In other words, $\gamma(z)$ behaves like a typical self-averaging quantity in the limit of an infinitely large volume. The mean values of quantities that have the form $\exp(\alpha\xi)$ behave as follows:

$$\langle e^{\alpha\xi}\rangle = \langle e^{\alpha z\gamma}\rangle = \exp\left[\frac{\alpha(\alpha+2)}{4}\frac{z}{l}\right]. \tag{1.16}$$

Clearly, when $\alpha \in [-2,0]$ the mean value $\langle \exp(\alpha\xi)\rangle$ grows exponentially despite the exponential decay of the quantity $\exp(\alpha\xi)$ in all realizations (i.e., with unity probability). This behavior is typical of exponential quantities, indicating that the mean value is dominated by improbable representative realizations rather than by typical ones. The experimental observation of this mean value requires an exponentially extensive ensemble of realizations. Only then will the mean value contain any physical information. Quantities of the form $\exp(\alpha\xi)$ fluctuate strongly: their relative fluctuations are proportional to $[\exp(\alpha^2 z/4l)-1]^{1/2}$ and grow exponentially except for the trivial $\alpha = 0$ case.

Some consequences of the exponential growth (1.7) are well-known in statistical radiophysics. For example, it is responsible for the phenomenon of the stochastic parametric resonance (see Refs. 34, 35). Indeed, the formulae (1.21) in Chap. 6 of the Klyatskin monograph[34] indicate that the mean values of the quadratic combinations of solutions to equation (1.3) grow exponentially with the increment that exactly corresponds to formulae (1.11) and (1.16) with $\alpha = 2$. Moreover the formulae of Ref. 34 were derived precisely in the regime (1.12).

All the above-described results retain their validity when equation (1.3) is derived from the Helmholtz equation for a three-dimensional randomly layered medium. In this case the coefficient D in formulae (1.10)–(1.12) is of the order of magnitude

$$D \sim \lambda_0^{-4} r_c \left(\frac{\sigma_\varepsilon}{\varepsilon_0}\right)^2, \tag{1.17}$$

where λ_0 is the wavelength in the fluctuation-free medium divided by 2π, and σ_ε^2 is the dispersion of the permittivity fluctuations $\delta\varepsilon(z)$. In the small fluctuation regime, where $\sigma_\varepsilon \ll \varepsilon_0$, the potential can be replaced by Gaussian white noise as long as $r_c \ll \lambda_0$. The region of high energies (short wavelengths) is bounded by the inequality

$$\lambda \ll \lambda_0 \left(\frac{r_c}{\lambda_0}\right)^{-1/3} \left(\frac{\sigma_\varepsilon}{\varepsilon_0}\right)^{-2/3}, \quad \lambda \equiv k^{-1}. \tag{1.18}$$

In the one-dimensional Helmholtz equation the spectral parameter appears in the potential $v(z) = -k^2\delta\varepsilon(z)/\varepsilon_0$ and we can identify λ with λ_0 (k with $E_0^{1/2}$) in the formulae (1.17), (1.18). Consequently, in the long wavelength limit $\lambda \to \infty$ ($k \to 0$) the sufficient conditions for replacing the potential by Gaussian white noise and taking the energy k^2 to be high are satisfied automatically. The expression (1.11) for the localization length then becomes

$$l(k^2) = \frac{2}{k^2 r_c}\left(\frac{\sigma_\varepsilon}{\varepsilon_0}\right)^2.$$

Thus we find that in the infinitely long wavelength limit the states are delocalized (the random function $v(z)$ drops out of the dynamical equation). Many researchers who investigated similar continuous[11,53,54] or discrete[55,56] models have obtained this result on the basis of other arguments or by direct computation.

2. WAVE PROPAGATION IN A RANDOM LAYER

First, let us recall the well-known results for the one-dimensional scattering problem for equation (1.3), where a unity amplitude monochromatic wave with wavevector k is incident from the right on a disordered segment $[0,L]$ (see Refs. 4, 34 for details). We shall always consider a sufficient-

ly long segment with $L \gg l(k^2)$. The transmissivity of the disordered segment (that is, the squared modulus of the transmission coefficient) can be written in the form[1)]:[57]

$$|t(L)|^2 = 1 - |r(L)|^2 = 4(2 + e^{2\xi_c(L)} + e^{2\xi_s(L)})^{-1}, \quad (2.1)$$

where the functions $\xi_c(z)$ and $\xi_s(z)$ are determined by equations (1.6) and the boundary conditions

$$\varphi_c(0) = \frac{\pi}{2}, \quad \varphi_s(0) = 0, \quad \xi_{c,s} = 0.$$

Since in every realization $\xi(z)$ undergoes mostly a linear increase (1.7), (1.8), the transmissivity of a typical realization falls exponentially with a damping rate corresponding to the inverse of the localization length:

$$-L^{-1} \ln |t(L)|^2 \sim l^{-1}, \quad L \gg l, \quad (2.2)$$

while the square of the modulus of the reflection coefficient in a typical realization is close to unity.

It appears reasonable that the average transmissivity of the segment should also fall exponentially with its length L. In this particular problem this is indeed the case, even though a decrease in a typical realization need not imply a decrease in the average value, as noted in Sec. 1. The average transmissivity damping rate γ_T does exist[58]

$$\gamma_T = -\lim_{L\to\infty} L^{-1} \ln \langle |t(L)|^2 \rangle$$

and does not exceed the decrement in the transmission in a realization $\bar{\gamma}$:

$$\gamma_T \leqslant \bar{\gamma} = l^{-1}$$

(this is physically obvious, since untypical realizations are improbable and the transmissivity has an upper bound $|t|^2 \leqslant 1$). In the simplest case of equation (1.3) with potential (1.10) in the semiclassical regime (1.5), the average transmissivity is[59]

$$\langle |t(L)|^2 \rangle \approx \frac{\pi^{5/2}}{2} \left(\frac{L}{l}\right)^{-3/2} \exp\left(-\frac{L}{4l}\right), \quad (2.3)$$

and hence[2)]

$$\gamma_T = (4l)^{-1} = \frac{1}{4}\bar{\gamma}. \quad (2.4)$$

It is the exponentially improbable realizations with $\xi_{c,s}(L) \approx 0$ that contribute to the average transmissivity, because in these realizations the transmissivity is nearly total $|t(L)|^2 \approx 1$, i.e., they are the representative realizations for $\langle |t(L)|^2 \rangle$.

Now suppose that an ideal point source radiating at frequency ω is placed at the point z_0 inside the disordered segment $[0,L]$ that has an ideal reflecting boundary ($r_- = 1$) at $z = 0$. The field of this point source coincides with the Green's function $g(z,z_0;E)$ of equation (1.3):

$$\left(E + \frac{d^2}{dz^2} - v(z)\right) g(z, z_0; E) = \delta(z - z_0).$$

This field can be expressed in terms of the functions $\xi(z)$ and $\varphi(z)$ of equation (1.6). The radiative flux density j is given by

$$j = 2\,\mathrm{Im}\,(g^* g') = \frac{2}{E^{1/2}} \sin^2 \varphi(z_0) \cdot \exp[-2(\xi(L) - \xi(z_0))]. \quad (2.5)$$

In the simplest case with $z_0 = 0, \varphi_c(0) = \pi/2, \xi_c(0) = 0$, the expression for flux density simplifies:

$$j = \frac{2}{E^{1/2}} e^{-2\xi_c(L)}. \quad (2.6)$$

This quantity is also of the $\exp(\alpha\xi)$ form with the critical, in the sense of (1.6), value of $\alpha: \alpha = -2$. In other words, in a typical realization with

$$\gamma(L) = \frac{\xi(L)}{L} \approx \frac{1}{2l},$$

the flux density is exponentially small, $j \sim \exp(-L/l)$, whereas the average flux density is dominated by improbable realizations with $\gamma(L) \approx -(2l)^{-1}$ and equals $2/E^{1/2}$.

Even though the average transmissivity falls exponentially with the length of the disordered segment L, other physical quantities that depend on the integral of transmissivity over the entire spectrum can have an entirely different dependence on L. An example of such a quantity is the transmissivity $\mathcal{T}$ of the segment for a wave packet with an envelope $p(k)$

$$\mathcal{T} = \int |t(L, k^2)|^2 p(k)\, dk.$$

If, for example

$$p(k) = (2\eta)^{-1} \operatorname{sech}^2 \frac{k - k_0}{\eta},$$

then as long as the following inequalities are satisfied

$$(DL)^{1/2} \gg \eta \gg L^{-1}, \quad 1 - \frac{\bar{k}}{k_0} \gg \left[\frac{\eta}{(DL)^{1/2}}\right]^{1/3}, \quad \bar{k} = \frac{1}{2}(\eta DL)^{1/3}$$

the transmissivity is given by

$$\langle \mathcal{T} \rangle = \frac{\pi^2 \sqrt{3}}{12} \left(\frac{\eta^2}{DL}\right)^{1/3} \exp\left[\frac{2k_0}{\eta} - \frac{3}{2}\left(\frac{DL}{\eta}\right)^{1/3}\right].$$

Consequently $\mathcal{T}$ exhibits a much slower decrease with L than the transmissivity (2.3) for a monochromatic wave.[3)]

As we mentioned earlier, the average transmissivity (2.3) is dominated by the improbable, nearly totally transparent realizations. In order to understand the physical mechanisms responsible for these realizations, as well as the properties of the corresponding scattering states, we must discuss the phenomenon of resonant transmission.

Let R be the reflection coefficient of the disordered segment. We will define the segment to be resonantly transparent if $1 - |R|$ is of the order unity (rather than $\exp(-L/l)$, as in a typical realization). The case of $R = 0$ corresponds to total transmission. The real and imaginary parts, $-\Delta$ and ϕ, of the logarithm of the reflection coefficient [$R = \exp(-\Delta + i\phi)$] obey the following system of equations

$$\begin{aligned} \Delta' &= \frac{v(z)}{k} \operatorname{sh}\Delta \cdot \sin\phi, \\ \phi' &= 2k - \frac{v(z)}{k}(1 + \operatorname{ch}\Delta \cdot \cos\phi) \end{aligned} \quad (2.7)$$

with the initial condition $\Delta(0) = +\infty$. According to (2.2), in a typical realization we have[4)]

$$\Delta(L) \sim e^{-L/l}.$$

Let us divide the disordered segment into two half-segments, for example $[0,z_0]$ and $[z_0,L]$, and define the reflec-

tion coefficients of the first half-segment as $r_1(r_-)$ for a wave incident from the left (right). The reflections coefficients for the second half-segment will be defined as r_+ (r_2). Then the reflection coefficient from the entire disordered segment for a wave incident from the right (R_+) becomes[36]

$$R_+ = r_1 \frac{1-(r_+/r_-^*)}{1-r_+r_-}. \tag{2.8}$$

(The coefficient R_- for a wave incident from the left is obtained by interchanging the indices $- \leftarrow\rightarrow +$ and $1 \leftarrow\rightarrow 2$).

Now let us introduce the moduli $\rho_\pm = \exp(-\Delta_\pm)$ and phases $\phi_\pm$ of the reflection coefficients $r_\pm = \rho_\pm \exp(i\phi_\pm)$. We find

$$|R_\pm| = \left| \frac{\rho_- - \rho_+ \exp[i(\phi_+ + \phi_-)]}{1-\rho_-\rho_+ \exp[i(\phi_+ + \phi_-)]} \right|. \tag{2.9}$$

In a typical realization of the potential, $\rho_\pm \approx 1$ for both half-segments. If, moreover, $\phi_+ + \phi_- \neq 2\pi n$, then the numerator and denominator of (2.9) coincide with exponential accuracy and $|R_\pm| \approx 1$. If, on the other hand, $\phi_+ + \phi_- = 2\pi n$, then

$$|R_\pm| \approx \left| \frac{\Delta_+ - \Delta_-}{\Delta_+ + \Delta_-} \right|. \tag{2.10}$$

The quantities Δ_+ and Δ_- are both exponentially small with a probability close to unity. With that same probability one of them is exponentially larger than the other and once again $|R_\pm| \approx 1$. Finally, if the realization on one of the two half-segments is improbable, with $1-|\rho| \sim 1$, it still follows from (2.9) and (2.8) that $|R_\pm| \approx 1$.

There are only two possible exceptions to this result. The first corresponds to the typical case of $\rho_\pm \approx 1$, but with an exponential coincidence of the quantities $\Delta_\pm$, such that they are of the same order. The second exception corresponds to the case when both half-segments are resonantly transparent, i.e., both moduli $\rho_\pm$ are simultaneously not exponentially close to unity. Both these exceptions lead to resonant transmission: $1-|R_\pm| \sim 1$. The two mechanisms for the creation of a resonantly transparent realization are distinguishable only for a particular choice of z_0, since in the case of the second mechanism it suffices to shift z_0 slightly and the realization (evidently still resonantly transparent) will be described by the first mechanism.

In the case of total transmission $R_\pm = 0$, the scattering states within the segment are clearly localized (they are still not quadratically integrable over the entire z-axis because of the oscillatory tails). This follows from the matching of the boundary conditions with only the incident wave on one side of the segment and only the transmitted wave on the other. As a result, inside the segment the square of the modulus of this state near each of the boundaries is the sum of the squares of the moduli of two solutions of the same equation (1.3). These solutions satisfy boundary conditions of the type (1.4) and hence grow exponentially as one moves into the segment (see (1.7), (1.8)).

In their pioneering paper,[61] Lifshits and Kirpichenkov examined resonant transmission through a segment in which the potential consisted of a sequence of δ-function wells on a constant repulsive (positive) potential background. They demonstrated that the resonant states in this system were localized and classified the resonant realizations. These included both the first (a segment with a single potential well) and the second (a sequential connection of several such segments) mechanisms discussed above. It was shown subsequently[62] that the transmission probability densities in the case of one or two wells exhibit integrable singularities at the point $|t|^2 = 1$, corresponding to total transmission.

Let us now qualitatively describe the dependence of intensity $I(z) = |\psi(z)|^2$ of a wave propagating through a random segment $[0,L]$ as a function of coordinate z. The intensity is related to the reflection coefficient $r_-(z)$ characterizing the segment $[0,z]$ by the expression[34]

$$I(z) = I(0) \frac{|1+r_-(z)|^2}{1-|r_-(z)|^2}. \tag{2.11}$$

First consider the case when the modulus of the reflection coefficient for $z \gg l$ follows the typical dependence (2.2):

$$1-|r_-(z)| \sim e^{-z/l}.$$

Then the intensity $I(z)$ becomes a quickly oscillating (on the scale of order k^{-1}) function of z whose envelope grows exponentially from $I(0) \sim \exp(-L/l)$ to a value of order unity in the vicinity of the right-hand boundary at $z = L$ (dashed line in Fig. 1). At intermediate points in the $[0,L]$ interval and exactly at the right-hand boundary the intensity can be considerably smaller than the exit intensity because of oscillations. This happens whenever the phase of the reflection coefficient at a given point z is close to $(2n+1)\pi$, meaning that the incident and reflected waves at z add in antiphase. This leveling of intensity at the "entry" point z and the exit point $z = 0$ is not due to intensity enhancement at the exit, but rather to the destructive interference at the "entry" (solid line in Fig. 1).

Fluctuations in the modulus of the reflection coefficient alter the above-described physical picture. It is convenient to classify the modulus fluctuations into two types. The first type of fluctuation renders the segment $[0,z]$ significantly less transparent than in a typical realization

$$1-|r_-(z)|^2 \ll e^{-z/l}.$$

The existence of such a fluctuation in the vicinity of some point z leads to a sharp increase in the intensity envelope by a factor of $\exp(-z/l)/(1-|r_-(z)|)^2$. As a result the intensity at the point z can even exceed the exit intensity (dash-dotted line in Fig. 1). The second type of fluctuation renders the modulus of the reflection coefficient small in the

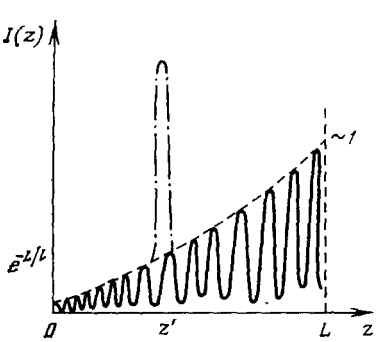

FIG. 1. Schematic plot of intensity as a function of coordinate z for a wave transmitted through a nonresonant realization of a randomly stratified layer.

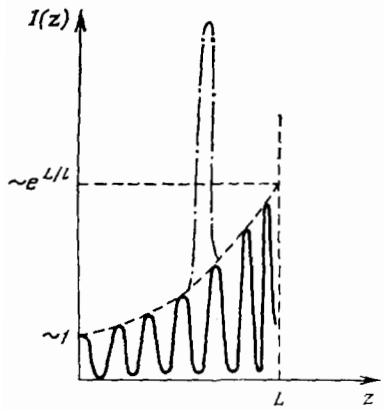

FIG. 2. Schematic plot of intensity as a function of coordinate z for a wave in a resonant realization. The dashed curve is the intensity envelope; the dash-dotted curve is the intensity peak caused by a fluctuation in $|r - (z)|$.

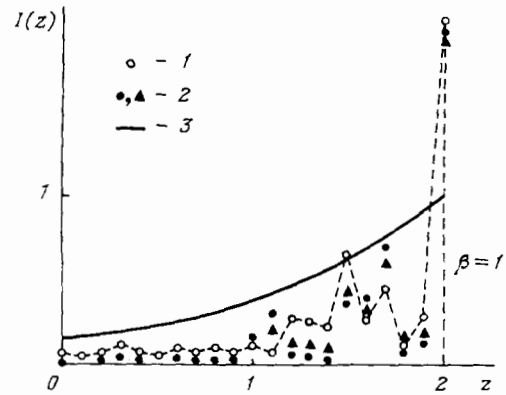

FIG. 4. Numerically computed intensity in the layer. *1*—wave intensity $I(z)$ in a layer of thickness $L = 4(k^2\sigma_c^2 r_c)^{-1}$ and dissipation coefficient $\Gamma = \mathrm{Im}\{\delta\varepsilon\}(\beta = 2\Gamma(k\sigma_c^2 r_c)^{-1}$; *2*—cases when $\mathrm{Re}\{\delta\varepsilon\}$ was replaced by $-\mathrm{Re}\{\delta\varepsilon\}$ (taken to be zero) over an interval of order k^{-1} inside the layer, *3*—the case $\delta\varepsilon = 0$.

vicinity of some point z. More precisely, the difference $(1 - |r \quad |)$ becomes of order unity (rather than exponentially small). Then the intensity envelope at this point will be of the same order of magnitude as the exit intensity $I(0)$. If this point z is located within a localization length from the entry point,[5)] $(L - z) \lesssim l$, one obtains resonant transmission: the intensities at the entry and exit points not only become of the same order of magnitude, but also match the intensity of the incident wave (Fig. 2). As a result $I(0) \sim I(L) \sim 1$, whereas at intermediate points because of (2.11) the intensity is exponentially large even if $r \quad (z)$ behaves in a typical fashion. Near points corresponding to fluctuations of the first type the intensity becomes larger still.

Such a behavior of intensity in a given realization differs markedly from the behavior of the intensity moments described in Ref. 34 (Fig. 3). Nonetheless we can follow qualitatively the formation of these moments. The typical behavior of the reflection coefficient modulus leads to a "monotonic" exponential decay of the intensity envelope from the entry point of the segment, where the intensity is of order unity, to the exit. The improbable resonant realizations lead to the appearance of exponentially large intensity peaks inside the layer (see Ref. 61). As a result the behavior of average intensity deviates from an exponential, while the higher intensity moments behave nonmonotonically and exhibit maxima whose amplitudes increase with moment number. Yet the resonant realizations do not exert any significant influence on the values of the intensity moments at the entry and exit points of the segment (since there $I(0) \sim I(L) \sim 1$). This argument is in agreement with the results[34] shown in Fig. 3.

Numerical studies of plane wave propagation in a randomly stratified layer have also been carried out.[63,64] In particular, the embedding method was employed to investigate the dependence of wave intensity on the coordinate inside the layer in particular realizations constructed using a random number generator. Examples of the calculated dependence of wave intensity on coordinate are shown in Figs. 4, 5.[6)] (For simplicity only data points separated by distances of the order of 10 wavelengths are plotted. The full sets of data points exhibit much greater scatter.) Since the positions and amplitudes of individual peaks are determined by the particulars of a given realization, they cannot be reproduced by the qualitative analysis cited above. Otherwise, the above-discussed behavior of intensity in a typical realization is in good agreement with numerically calculated results.

3. POINT SOURCES IN A RANDOMLY STRATIFIED LAYER

In the preceding section we discussed the transmission of a plane wave through a stratified layer and the field generated in such a layer by an infinite radiating plane. Practical problems, such as the computation of a field generated by a given current distribution in an antenna, require the calcula-

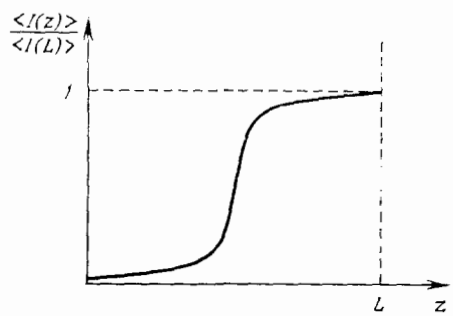

FIG. 3. Average intensity in the layer (the curve crosses the ordinate axis at the origin).

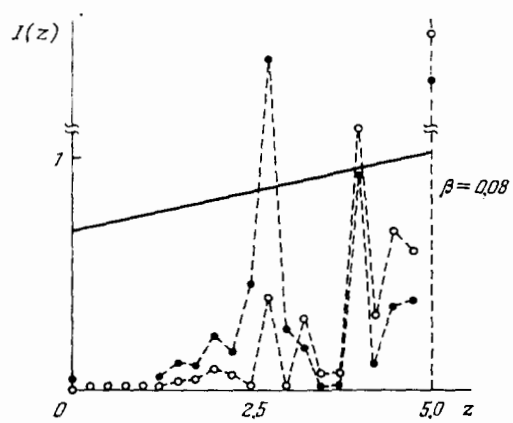

FIG. 5. Same as Fig. 4, but with $L = 10(k^2\sigma_c^2 r_c)^{-1}$.

tion of the field G generated by a point source.

Consider the field $G(\mathbf{R},\mathbf{R}_0)$ generated in the randomly layered medium by a monochromatic source located at $\mathbf{R}_0 = (0,z_0)$ above a perfectly reflecting plane $z = 0$. (This situation adequately describes the propagation of radiowaves in the atmosphere above the earth's surface, for example). The field G satisfies the equation

$$\Delta G(\mathbf{R}, \mathbf{R}_0) + \frac{\omega^2}{c^2}\varepsilon(z)\, G(\mathbf{R}, \mathbf{R}_0) = \delta(\mathbf{R} - \mathbf{R}_0), \quad \mathbf{R} = (\boldsymbol{\rho}, z), \tag{3.1}$$

with the self-conjugate boundary condition at $z = 0$

$$\left(G + a\frac{\partial G}{\partial z}\right)\Big|_{z=0} = 0, \quad \operatorname{Im} a = 0, \tag{3.2}$$

that corresponds to the total reflection of a plane wave regardless of the angle of incidence and to the existence only of outgoing waves at infinity.

The Fourier transform $\widetilde{G}(\varkappa,z)$ of the field $G(\mathbf{R},\mathbf{R}_0)$ in terms of the in-plane coordinate ρ

$$\widetilde{G}(\boldsymbol{\varkappa}, z) = \int G(\boldsymbol{\rho}, z)\, e^{-i\boldsymbol{\varkappa}\boldsymbol{\rho}}\, d\boldsymbol{\rho} \tag{3.3}$$

coincides as a function of z with the Green's function of equation (1.3) with $E = E_0 - \varkappa^2$. Since the Green's function satisfies the same boundary conditions as $G(\mathbf{R},\mathbf{R}_0)$, it is natural to decompose $G(\mathbf{R},\mathbf{R}_0)$ in terms of the eigenfunctions of the one-dimensional problem:

$$\begin{aligned} 4iG(\mathbf{R}, \mathbf{R}_0) = &\sum_j{}' \psi_j^*(z)\, \psi_j(z_0)\, H_0^{(1)}(\rho(E_0 - E_j)^{1/2}) \\ &+ \int_0^E \psi_E^*(z)\, \psi_E(z_0)\, H_0^{(1)}(\rho(E_0 - E)^{1/2})\, dE \\ &+ \int_{E_0}^{\infty} \psi_E^*(z)\, \psi_E(z_0)\, H_0^{(1)}(i\rho(E - E_0)^{1/2}). \end{aligned} \tag{3.4}$$

The dependence of the field on the transverse coordinate z is described by the wavefunctions $\psi_j(z)$ and $\psi_E(z)$ of the discrete ($E_j < 0$) and continuous ($E > 0$) spectrum respectively. These wavefunctions are bounded at infinity and satisfy equation (1.3), boundary condition (1.4) at $z = 0$, and the normalization conditions

$$\begin{aligned} &\int_0^\infty \psi_j^*(z)\, \psi_k(z)\, dz = \delta_{jk}, \\ &\int_0^\infty \psi_E^*(z)\, \psi_{E'}(z)\, dz = \delta(E - E'). \end{aligned} \tag{3.5}$$

If the permittivity fluctuations of the medium are restricted to a layer of finite thickness L, then $\varepsilon(z)$ in (1.2) will have the following form

$$\begin{aligned} \varepsilon(z) &= \varepsilon_0 + \delta\varepsilon(z), \quad 0 < z < L, \\ &= \varepsilon_0, \qquad\qquad L < z. \end{aligned} \tag{3.6}$$

We will assume that $\delta\varepsilon(z)$ satisfies the previously formulated conditions of spatial homogeneity on average and a fall in correlations at infinity over the entire range $-\infty < z < \infty$. By virtue of $\langle\delta\varepsilon(z)\rangle = 0$ and $\varepsilon_0 = \text{const}$ no ordinary refraction takes place in the layer. It is of interest to determine how this randomly stratified, homogeneous on average medium affects the distance over which the wave propagates and, in particular, whether the medium has any channeling, waveguiding properties.

A characteristic feature of channeled propagation is the cylindrical divergence of energy density flux

$$\mathbf{S}(\mathbf{R}) = 2\operatorname{Im}(G^*(\mathbf{R})\nabla G(\mathbf{R}))$$

at large distances away from the source:

$$|\mathbf{S}(\rho, z)|\,\big|_{\substack{z=\text{const},\\ \rho\to\infty}} \sim \rho^{-1}. \tag{3.7}$$

In a homogeneous space with $\delta\varepsilon(z) = 0$ the energy density flux of a point source falls off as ρ^{-2}. The divergence of the energy flux is unambiguously related to the existence of nonzero flux $\Phi_d(z') > 0$ through the side of a cylinder with an infinite radius, bounded by the reflecting plane at $z = 0$ and the plane $z = z'$:

$$\Phi_d(z') = \lim_{\rho\to\infty} \rho \int_0^{z'} dz \int_0^{2\pi} d\varphi S_\rho(\rho, z').$$

In a homogeneous space with a spherical flux divergence this quantity is zero.

The flux $\Phi_c(z')$ through the infinite plane at $z = z' > z_0$

$$\Phi_c(z') = \int S_z(\rho, z')\, d\rho$$

is related to $\Phi_d(z')$ by the simple sum rule

$$\Phi_c(z') + \Phi_d(z') = \Phi_0 = -2\operatorname{Im} G(\mathbf{R}_0, \mathbf{R}_0),$$

where Φ_0 is the total energy flux radiated by the source per unit time. When $z' < z_0$, $\Phi_c(z') + \Phi_d(z') = 0$. In a homogeneous space $\Phi_c(z')$ is independent of z' and equals Φ_0.

By way of standard, albeit cumbersome, calculations the fluxes $\Phi_d(z)$ and $\Phi_c(z')$ can be expressed in terms of the wavefunctions of the one-dimensional problem (which can be chosen as real)

$$\Phi_d(z) = \frac{1}{2}\sum_j{}' \psi_j^2(z_0) \int_0^z \psi_j^2(\zeta)\, d\zeta, \tag{3.8a}$$

$$\begin{aligned} \Phi_c(z) &= -\Phi_d(z) \text{ when } 0 < z < z_0 \\ &= \Phi_c + \frac{1}{2}\sum_j \psi_j^2(z_0) \int_z^\infty \psi_j^2(\zeta)\, d\zeta \text{ when } z_0 < z \end{aligned} \tag{3.8b}$$

$$\Phi_d \equiv \Phi_d(\infty) = \frac{1}{2}\sum_j{}' \psi_j^2(z_0), \tag{3.8c}$$

$$\Phi_c \equiv \Phi_c(\infty) = \frac{1}{2}\int_0^{E_0} \psi_E^2(z_0)\, dE. \tag{3.8d}$$

These formulae are exact (i.e., valid for all realizations) dynamical expressions which provide the framework for studying the flux distribution in a given realization. But if one is to take into account explicitly the random character of fluctuations $\delta\varepsilon(z)$, it is convenient to express the continuous spectrum wavefunctions in terms of the well-known functions $\xi(z)$ and $\varphi(z)$ (1.6) (see Ref. 4). Recall that the quantity $\gamma(z) = \xi(z)/z$ is self-averaging for large z. As a result formula (3.8d) can be recast into the form

$$\Phi_c = \frac{1}{2\pi}\int_0^{E_0} \sin^2\varphi(z_0)\cdot\exp[-2(\xi(L) - \xi(z_0))]\frac{dE}{E^{1/2}}. \tag{3.9}$$

We note that the integrand in expression (3.9) for the total flux out of the layer coincides, as expected, with equation (2.5) for the energy density flux of a plane wave in the one-dimensional problem.

4. THE FLUCTUATION WAVEGUIDE

Now let us analyze the expressions for the energy fluxes derived in the preceding section. It follows from (3.8a) and (3.8c) that waveguide propagation [in the sense of (3.7)] can only take place when the one-dimensional problem (1.3) with the potential created by (3.6) has a discrete spectrum. Indeed, when this is the case $\Phi_d(z) \neq 0$ and

$$\Phi_c'(z) = -\Phi_d'(z) = -\frac{1}{2}\sum_j \psi_j^2(z_0)\,\psi_j^2(z) < 0. \qquad (4.1)$$

This implies that as one moves away from the source (i.e., as z increases) the flux through the infinite plane $z = \text{const}$ decreases precisely because energy is channeled "sideways" by the waveguide modes of the discrete spectrum $E_j < 0$ that represent ordinary waves propagating along the layers. When $\rho(E_0 - E_j)^{1/2} \gg 1$, the dependence of these modes on z is of the form $\sim H_0^{(1)}(\rho(E_0 - E_j)^{1/2}) \sim \exp[i\rho(E_0 - E_j)^{1/2}]$. It is clear from (3.8d) that the energy of these modes in the z direction is confined within the layer and hence the waves of the discrete spectrum do not contribute to the "upward" flux Φ_c. In a homogeneous medium the discrete spectrum does not exist and hence $\Phi_d = 0$, $\Phi_c = \Phi_c^{(0)} = \pi^{-1}E_0^{1/2}$ as discussed above.

In the case of a dielectric waveguide of thickness $L \gg \lambda$ with $\delta\varepsilon(z) = \varepsilon_1 > 0$ (where λ is the wavelength radiated by the source in a homogeneous medium with $\varepsilon = \varepsilon_0 + \varepsilon_1$), the number of levels in the discrete spectrum is proportional to L, while the amplitude of the corresponding wavefunctions $\psi_j(z)$ has the same order of magnitude $\sim L^{-1/2}$ everywhere in the layer. As a result the flux $\Phi_d(z)$ increases linearly with z from zero to $\Phi_d(L) \sim \Phi_d$, whereas the total "sideways" flux Φ_d depends but weakly on layer thickness L. This implies that the layer is, on the average, uniformly "illuminated." The energy flux out of the layer $\Phi_c(L) \sim \Phi_c$ is carried by waves belonging to the continuous spectrum with $E \in [0, E_0]$ and is also only weakly dependent on L.

Moving on to the randomly layered medium, let us note that in the one-dimensional case every potential well must contain at least one discrete level. Those realizations that contain no wells at all correspond to $\delta\varepsilon(z) \leqslant 0 (v(z) \geqslant 0)$ for all $z < L$ and their proportion falls off exponentially with the parameter L/r_c. Every realization with a thick layer $L \gg r_c$ contains normal modes with $E < 0$ with a probability exponentially close to unity. Hence a thick layer supports waveguided propagation. In a sense the disordered system is equivalent to a dielectric layer with a higher index of refraction than the surrounding medium (a finite potential well), except that in our case the channeling of energy is not caused by reflection from a specular boundary or ordinary refraction, but is rather a purely fluctuation-induced effect that disappears when $\delta\varepsilon = 0$.

The spatial dependence of the energy flux radiated by a point source in a randomly layered medium is clearly the same as in a dielectric waveguide: far away from the source the flux falls off as ρ^{-1}. Yet the height distribution of the normal mode fields and energy flux $\Phi_d(z)$ in a randomly stratified layer exhibit a number of specific features. Outside the layer, if $E < 0$ and $z > L, \psi(z) \sim \exp[-(-E)^{1/2}(z - L)]$, which permits us to write the effective boundary condition on ψ at the point L in the "self-conjugate" form

$$\left.\frac{\psi'}{\psi}\right|_{z=L} = -(-E)^{1/2}.$$

For this reason ψ_j have the same properties as the eigenfunctions of closed disordered systems.[4] In particular, the moduli of the wavefunctions $\psi_j(z)$ that describe, in the sense of (3.4), the height distribution of the normal wave fields, deviate strongly from zero only within distances of the order of $l_j = l(E_j)$ from localization centers z_j and decay exponentially thereafter. The characteristic distance between the localization centers for waves belonging to the discrete spectrum is clearly of order $\mathcal{N}^{-1}(0)$, where $\mathcal{N}(E')$ is the number of states with $E < E'$ per unit thickness of the stratified layer.

If conditions (1.9) and (1.15) are fulfilled, the random function $v(z)$ is characterized by a single length parameter $D^{-1/3}$ (1.10) and hence $\mathcal{N}(0) \sim D^{1/3}$ by dimensional analysis. Consequently,

$$\Delta z \sim D^{-1/3} = \lambda_0 \left[\frac{r_c}{\lambda_0}\left(\frac{\sigma_\varepsilon}{\varepsilon_0}\right)^2\right]^{-1/3}. \qquad (4.2)$$

It follows from the same argument that when $|E_j| \ll D^{1/3}$ the localization radius l_j is of that same order. In the opposite limit, when $|E_j| \gg D^{1/3}$, we obtain from (1.14)

$$l_j \sim |E_j|^{-1/2}.$$

This strongly inhomogeneous dependence of wave fields on the transverse coordinate distinguishes the randomly layered medium from a regular dielectric waveguide in which the height coefficients of normal modes oscillate regularly throughout the layer.

The energy flux Φ_d channeled along the layer by discrete spectrum waves is also different in the case of a randomly stratified layer. It is evident from (3.8a) that the jth state of the discrete spectrum contributes to the flux $\Phi_d(z)$ only when $(z - z_j) \geqslant l_j$, because when $z < (z_j - l_j)$ the integral in (3.8a) is exponentially small, whereas when $z > (z_j + l_j)$ it is practically equal to unity. The magnitude of this contribution, which equals $\psi_j^2(z_0)/2$, is noticeably different from zero only when the localization center of this state lies within a localization radius l_j in the z direction, in which case it is of order l_j^{-1} [see the normalization condition (3.5)]. Therefore, in a thick layer $L \gg l$ the total "sideways" flux Φ_d (3.8c) is comprised of a small number of waves for which $|z_0 - z_j| \lesssim l_j$. This flux is of the order

$$\Phi_d \sim (\Delta z)^{-1}; \qquad (4.3)$$

where l is the localization radius at $|E| \sim \sigma_\varepsilon \omega^2/c^2$; Δz is the average separation of neighboring localization centers. The height distribution of the "sideways" flux is strongly nonuniform: only a thin band $|z - z_0| \ll l \ll L$ around the point source is illuminated. In the example (1.10) we find from (4.2), (4.3)

$$\Phi_d \sim D^{1/3}. \qquad (4.4)$$

The "sideways" flux Φ_d is not a self-averaging quantity and hence exhibits a fine structure determined by the partic-

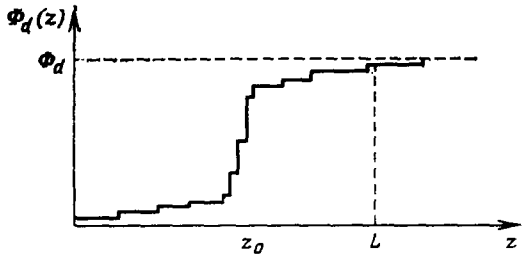

FIG. 6. The flux $\Phi_d(z)$ through the side surface of a cylinder with infinite radius and height z.

ular realization. This fine structure was essentially described in the preceding paragraph; it is schematically illustrated in Fig. 6. A more convenient quantity for characterizing the fine structure is the spatial derivative of Φ_d:

$$\Phi_d'(z) = \frac{1}{2}\sum_j \psi_j^2(z_0)\,\psi_j^2(z),$$

which, in addition to the main peak near z_0 (endowed with its own fine structure), exhibits numerous weaker peaks at all localization centers (Fig. 7). These peaks contribute to the mesoscopic structure of the flux derivative $\Phi_d'(z)$ with a characteristic period of the same order as the separation between localization centers Δz. By varying the position of the source one can, in principle, determine the coordinates z_j of the localization centers for normal modes (i.e., those regions of the inhomogeneous layer that are the most "transparent" in the plane of the layer and hence contribute the most to the channeling of energy) and the amplitudes of the wavefunctions at these centers $\psi_j(z_j)$. The full set of z_j and $\psi_j(z_j)$ uniquely characterizes a realization and becomes a valid form of identification [just as the dependence of conductivity on the magnetic field $\sigma(H)$ in mesoscopic semiconductors has acquired the colorful label of "magnetofingerprints"].

The average total "sideways" flux $\langle\Phi_d\rangle$ agrees in order of magnitude with its value in a particular realization. Indeed, after employing (3.8a) to write $\langle\Phi_d\rangle$ in the alternate form

$$\langle\Phi_d\rangle = \frac{1}{2}\int_{-\infty}^{0}\left\langle\sum_j \delta(E-E_j)\,\psi_j^2(z_0)\right\rangle dE,$$

we find (see Ref. 4) that if z_0, $(L-z_0)\gg r_c$ the integral on the right is the average number of discrete levels per unit thickness. Consequently

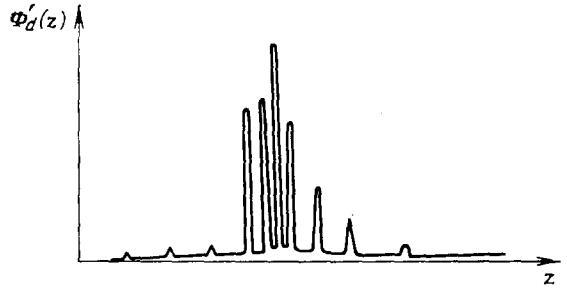

FIG. 7. The fine structure of the flux derivative $\Phi_d'(z)$ (the main peak corresponds to z_0).

$$\langle\Phi_d\rangle \approx \frac{1}{2}\mathcal{N}(0),$$

and in the particular case (1.10) we obtain once again the estimate (4.4).

In conclusion, it follows from expressions (4.1) and (4.2) that in a randomly layered medium all realizations except for some exponentially improbable ones support waveguide propagation. The waveguiding is accomplished by discrete spectrum waves with negative values of the parameter E. The rest of the energy falls in the continuous spectrum whose states are delocalized (i.e., the field of these delocalized states is not confined to the layer but rather radiates outward, demonstrating the "openness" of the system). However, as we shall show in the next section, the specific properties of disordered open systems that result in the formation of a continuous spectrum also induce radical changes in the fields associated with the continuous spectrum. In particular, there appear quasihomogeneous waves (analogous to metastable quantum mechanical states) which strongly enhance the waveguiding effects over those observed in regular structures.

5. QUASIHOMOGENEOUS WAVES

The hypothesis that waves belonging to the continuous spectrum should also be channeled in a randomly stratified medium arises from the following simple argument. The field of a point source can always be decomposed into a superposition of plane waves. It follows from (2.2) that these waves, including the ones propagating perpendicularly to the layer, will be reflected by a sufficiently thick layer that has a reflection coefficient whose modulus is exponentially close to unity. This enhanced reflection should logically lead to the partial confinement of the radiation in the z direction and hence to channeling along the layers.

In order to observe this phenomenon, let us analyze the flux Φ_c radiated out of the layer by continuous spectrum waves. By defining

$$\rho(E) = \sin^2\varphi(z_0)\cdot\exp[-2(\xi(L)-\xi(z_0))] \tag{5.1}$$

we can rewrite formula (3.9) as

$$\Phi_c = \frac{1}{2\pi}\int_0^{E_0}\rho(E)\,\frac{dE}{E^{1/2}}\,. \tag{5.2}$$

The above expression can be understood as the radiated energy flux per unit interval of the spectral parameter E, i.e., the density of the angular distribution ($\vartheta = \arcsin(E/E_0)^{1/2}$, $\int_0^{E_0}\ldots dE = E_0\int_0^{\pi/2}\ldots\sin 2\vartheta\, d\vartheta$) of the "upward" flux.

In nearly all realizations the function $\xi(z)$ is generally linearly increasing. It then follows from (5.1) that when $L-z_0\gg l(E)$ the quantity $\rho(E)$ is exponentially small in the overwhelming majority of the realizations. At first glance this appears to support the qualitative argument cited above. Yet, as we have seen in Sec. 2, in the example (1.10) with $z_0=0$ and $r_c=1$ the mean value is $\langle\rho(E)\rangle = 1$ (2.6) [just as in free space with $\delta\varepsilon(z)=0$], because $\langle\rho(E)\rangle$ is dominated by improbable realizations in which $\rho(E)$ is exponentially large, $\rho\sim\exp(2L/l)$.

There are two alternative schemes for calculating $\langle\rho(E)\rangle$. One makes use of the fact that the integral (5.2) is exponentially small in the overwhelming majority of realiza-

tions, while its average value is dominated by improbable realizations. In this case an arbitrary realization will have a probability exponentially close to unity of having good waveguiding properties: only an exponentially small part of the total flux is radiated out of the layer.

The second scheme is based on the idea that since for a given value of E the estimate $\gamma(L)\sim\xi(L)/L\sim(2l)^{-1}$ is valid for an overwhelming majority of, but not all, realizations, then in every realization there will be such $E\in[0,E_0]$ for which the integrand $\rho(E)$ becomes exponentially large. Accordingly, in every realization the radiated flux carried off by the continuous spectrum waves will be of the same order of magnitude as in a homogeneous medium

$$\Phi_c\sim\lambda_0^{-1}$$

and thus significantly larger than the "sideways" flux (4.4):

$$\Phi_{cl}\sim D^{1/3}\sim\Phi_c\left[\frac{r_c}{\lambda_0}\left(\frac{\sigma_\varepsilon}{\varepsilon_0}\right)^2\right]^{1/3}\ll\Phi_c.$$

In order to decide which of these alternative schemes is valid, let us express the density $\rho(E)$ (5.1) of the flux radiated out of the layer in terms of the reflection coefficient $r_+(E)$ of a segment on which a plane wave is incident from the left. In the special case $z_0=0, r_-=1$, the density $\rho(E)$ has the form

$$\rho(E)=\frac{1-|r_+(E)|^2}{|1-r_+(E)|^2}. \tag{5.3}$$

In the region of sufficiently large $E\gg|v|$, the phase $\phi_+(E)=\mathrm{Arg}\{r_+(E)\}$ of the reflection coefficient obeys the approximate dependence $\phi_+(E)=2LE^{1/2}$, as follows from (2.7). Since in a typical realization $1-|r_+(E)|\sim O(\exp(-L/l))$, for a given E we find, as a rule,

$$\rho(E)\sim e^{-L/l}. \tag{5.4}$$

The exceptions to (5.4) consist of those values E_n of the parameter E for which

$$\varphi_+(E_n)=2\pi n. \tag{5.5}$$

At these points

$$E_n\approx\frac{n^2\pi^2}{L^2} \tag{5.6}$$

the denominator $|1-r_+|^2$ in (5.3) becomes small

$$|1-r_+(E_n)|^2\sim e^{-2L/l}, \tag{5.7}$$

and hence $\rho(E)$ takes on exponentially large values

$$\rho(E_n)\sim e^{L/l(E_n)}. \tag{5.8}$$

In general $\rho(E)$ is a sharply peaked function shown in Fig. 8. The separation $\Delta E_n=E_{n+1}-E_n$ between the peaks of $\rho(E)$, i.e., between the roots of equation (5.5), can be obtained from (5.6) as $\Delta E_n=2n\pi^2/L^2$. The half-width of these peaks, δE_n, produced by the departure of $|r_+|$ from unity, equals

$$\delta E_n\approx\frac{1}{2\pi}\Delta E_n\cdot e^{-L/l(E_n)}. \tag{5.9}$$

In calculating the flux Φ_c (5.2) the function $\rho(E)$ can

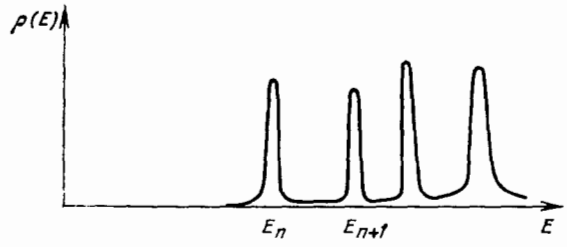

FIG. 8. Effective density of states $\rho(E)$.

be replaced by a smoothed function $\tilde\rho(E)$ obtained by averaging over the interval ΔE where $E^{-1/2}\approx$ const but still contains a large number of peaks

$$\tilde\rho(E)=\frac{1}{\Delta E}\int_{-\Delta E/2}^{\Delta E/2}\rho(E+E')\,dE'.$$

Direct computation using relations (5.8) and (5.9) yields

$$\tilde\rho(E)\approx\frac{1}{2\pi}, \tag{5.10}$$

Consequently, in a typical realization the radiated flux agrees in order of magnitude with its average value $\langle\Phi_c\rangle$ and with the corresponding flux $\Phi_c^{(0)}$ in a homogeneous medium. This implies, in particular, that the system is ergodic in some sense in the parameter E: $\langle\rho(E)\rangle\sim\tilde\rho(E)$. This property accounts for the aforesaid difference between the behavior of $\rho(E)$ in typical realizations and in representative realizations for the given value of E. Indeed, the probability of the density $\rho(E)$ being exponentially small (5.4) in a given realization is the same as the probability of the given value of E not belonging to the interval δE, which is obviously

$$1-\frac{\delta E}{\Delta E}=1-\frac{1}{2\pi}e^{-L/l(E_n)}.$$

In other words, the proportion of typical realizations for a given value of E is exponentially close to unity. On the other hand, in the representative realizations, whose proportion is exponentially small $\sim\exp(-L/l)$, the density $\rho(E)$ is exponentially large (5.8).

In view of (5.2) and (5.10) we find that the second scheme of defining $\langle\rho(E)\rangle$ is the correct one.

It follows from the preceding arguments, that the total flux radiated away from the disordered layer has a strongly inhomogeneous angular distribution: the radiated energy is concentrated near the angles $\vartheta_n=\arcsin(E_n/E_0)^{1/2}$ that correspond to the values E_n at which $\rho(E)$ is sharply peaked. The physical meaning of E_n (5.6) becomes clear if we write the solution of equation (1.3) with the boundary condition (1.4) at a point outside the layer $z>L$

$$\psi(E,z)=(1-r_+(E))\,t^*(E)\exp[-iE^{1/2}(z-L)]+(1-r_+^*(E))\,t(E)\exp[iE^{1/2}(z-L)] \tag{5.11}$$

where $t(E)$ is the transmission coefficient of the disordered segment. Evidently, when E is real the solution contains both incident (from the right) and reflected waves. On the other hand, when $\mathscr{E}_n=E_n-\delta_{1n}-i\delta_{2n}$ is complex and

$$r_+(\mathscr{E}_n)=1 \tag{5.12}$$

the coefficient of the incident wave becomes zero and only

the outgoing wave remains

$$\psi(\mathscr{E}_n, z) = (1 - r_+^*(\mathscr{E}_n))\, t(\mathscr{E}_n) \exp[i\mathscr{E}_n^{1/2}(z - L)]. \quad (5.13)$$

In quantum mechanics the wavefunction (5.13) describes a so-called decay state. Because of the temporal dependence $\sim e^{-i\mathscr{E}_n t} = e^{-iE_n t + i\delta_{1n} t - i\delta_{2n}}$ the square of its modulus decays with the characteristic time $\tau \sim (\delta_{2n})^{-1}$. However, when $\delta_{2n} \ll E_1 - \delta_{1n}$ the decay time becomes large compared to the oscillation period $(E_1 - \delta_{1n})^{-1}$ and the resulting state is known as metastable. Usually metastable states appear because of the specific form of the potential profile $v(z)$ in (1.3) that incorporates a potential well separated from the rest of the space by a sufficiently wide potential barrier that is higher than the energy of the particle [65,66]. In our case the metastable states appear "above the barrier": the energy of the particle is higher than the scattering potential and particle confinement is wave-like in nature, arising from the wave interference due to multiple scattering from potential fluctuations. (The individual scattering events need not be particularly strong.)

For those values of parameter E that correspond to a localization length $l(E)$ smaller than the layer thickness, the modulus of the reflection coefficient is exponentially close to unity. Consequently equation (5.12) can be written as

$$\mathscr{E}_n = E_n - i\delta_{2n}. \quad (5.14)$$

The imaginary part δ_{2n} is smaller by a factor of two than the half-width of the corresponding peak in $\rho(E)$ (5.3) and is exponentially small in the parameter L/l:

$$\delta_{2n} = \frac{1}{2}\delta E_n = \frac{1}{4\pi}\Delta E_n \cdot e^{-L/l(E_n)} \quad (5.15)$$

(on this scale the shift δ_{1n} of the real part E_n is indistinguishable from zero). In this fashion, the values of E_n (5.6) at which $\rho(E)$ is peaked are the real parts of the complex values $\mathscr{E}_n$ (5.14), (5.12) that correspond to metastable states whose lifetime $\tau \sim \exp(L/l)$ is exponentially large. The total flux Φ_c radiated out of the layer is formed precisely from these metastable states.

The wavefunctions $\psi(E_n, z)$ corresponding to E_n are exponentially localized inside the layer (clearly they cannot be normalizable because of the oscillatory tails (5.11) that extend outside). This follows from the proportionality between the eigenfunctions $\psi(E_n, z)$ for z inside the layer with $r_- = 1$ and the cosine solutions $c(E_n, z)$ of equation (1.3) that increase exponentially from $z = 0$ and satisfy the boundary conditions $c(E_n, 0) = 1, c'(E_n, 0) = 0$, and from the identity

$$c^2(E_n, L) + E_n^{-1} c_n'^2(E_n, L) = \frac{|1 - r_+(E_n)|^2}{1 - |r_+(E_n)|^2} \sim e^{-L/l(E_n)}.$$

The characteristic localization length scale for the waves $\psi(E_n, z)$ is the usual localization length $l(E_n) \ll L$ rather than the width of the potential barrier. Therein lies the difference between metastable states in disordered and regular systems.

Our understanding of metastable states not only clarifies their contribution to the flux Φ_c but also makes it possible to analyze the dependence of the source field G on the in-plane coordinate ρ. We can use the formula

$$G(\mathbf{R}, \mathbf{R}_0) = \frac{iE_0^{1/2}}{2} \int_\Gamma \frac{1 + r_-(z_0)}{1 - r_+(z_0)\, r_-(z_0)} H_0^{(1)}(k\rho \sin\vartheta)\, f_{1,2}(\vartheta, z) \sin\vartheta\, d\vartheta, \quad (5.16)$$

$$\rho = |\boldsymbol{\rho} - \boldsymbol{\rho}_0|, \quad z \gtrless z_0;$$

where Γ is a certain contour of integration in the complex θ plane; $E_0^{1/2}$ is the wavenumber for the level of the source; $f_{1,2}(\theta, z)$ are the functions that describe the fields in the lower ($z < z_0$) and upper ($z > z_0$) half-spaces when a plane wave of unit amplitude is incident from the vacuum at an angle θ. An analysis of the expressions (5.16) indicates that they can be reduced to the sum of residues corresponding to the poles of the denominator and to the integrals along the edges of the cuts.[38]

When $\varkappa_n \rho \gg 1$, the sum of the residues that describes the field inside the layer can be written in terms of the variable $\mathscr{E} = E_0 - k^2 \sin^2\theta$ as

$$G = \sum_n F_n(z, z_0; \mathscr{E}_n)\, e^{i\varkappa_n \rho}; \quad (5.17)$$

where $\varkappa_n = (E_0 - \mathscr{E}_n)^{1/2}$; $\mathscr{E}_n$ are the roots of the dispersion equation

$$1 - r_+(z_0)\, r_-(z_0) = 0, \quad (5.18)$$

and $r_\pm(z_0)$ are the reflection coefficients of regions $[z_0, \infty]$ ($[0, z_0]$) for a wave incident from the right (left).

It can be shown that the full set of solutions $\mathscr{E}_n$ of equation (5.18) does not depend on the choice of point z_0. Consequently, when $r_- = 1$ this full set of solutions coincides with the set of solutions of (5.12). We have already shown that the latter set contains solutions that describe metastable states. Since in our case time can be identified with the distance ρ in the x–y plane between the source and the observation point, these states correspond to quasihomogeneous waves that decay over distances greater by $\exp(L/l)$ than the distance $\mathscr{D}$ from the source

$$\mathscr{D}_n \sim (\operatorname{Im} \varkappa_n)^{-1} \sim L e^{L/l}. \quad (5.19)$$

The eventual decay of these waves is not due to dissipation, however, but rather to the flux radiated "upward" (into the $z > L$ region). As long as $\rho < \mathscr{D}_n$ the quasihomogeneous waves are confined inside the layer.

Thus the crucial difference between a randomly stratified layer and an ordinary dielectric waveguide lies in the specific role played by the continuous spectrum. The flux produced by this part of the spectrum is carried by quasihomogeneous waves, whose energy is radiated out of the layer at exponentially greater distances away from the source than the layer thickness. In the total field of the continuous part of the spectrum, described by the integrals in (3.4), we can select a discrete series of metastable states that describe slowly decaying quasihomogeneous waves. These waves channel energy along the layer over enormous distances.

In conclusion let us note that the outgoing flux $\Phi_c(z)$ is a consequence of finite layer thickness. In an infinite, randomly stratified medium, all states are exponentially localized in the z direction. As a result, the average value of the flux $\Phi_c(z)$ decays exponentially as the plane z is taken further from the source[67]

$$\langle \Phi_c(z, z_0)\rangle = 8\pi^{9/2}E_0^{1/2}\left(\frac{l(E_0)}{|z-z_0|}\right)^{5/2}$$

$$\times \quad \exp\left(-\frac{|z-z_0|}{4l(E_0)}\right)\left(1+O\left(\frac{\gamma}{E_0^{1/2}l(E_0)}\right)\right),$$

$$|z-z_0| \gg l(E_0),\ \rho; \quad \gamma \ll (E_0^{1/2}l(E_0))^{-1}.$$

The mean intensity $\langle I(\mathbf{R},\mathbf{R}_0)\rangle$ at the point $\mathbf{R}$ also decays exponentially

$$\langle I(\mathbf{R}, \mathbf{R}_0)\rangle = \frac{1}{2(z-z_0)^2}\exp[-|z-z_0|(l^{-1}(E_0)+\gamma E_0^{1/2})],$$

$$\gamma \gg (E_0^{1/2}l(E_0))^{-1}$$

as does the mean field $\langle G(\mathbf{R},\mathbf{R}_0)\rangle$ (the coherent component of the signal)

$$\langle G(\mathbf{R}, \mathbf{R}_0)\rangle = -|z-z_0|^{-1}\exp\left[\frac{iE_0^{1/2}\rho^2}{2|z-z_0|}+iE_0^{1/2}(z-z_0)\right.$$

$$-\frac{|z-z_0|}{2}\left(1+\frac{\rho^2}{|z-z_0|}\right)$$

$$\left.\times \quad (l^{-1}(E_0)+\tilde{l}^{-1}(E_0)+\gamma E_0^{1/2})\right],$$

$$\gamma \gg (E_0^{1/2}l(E_0))^{-1}$$

where γ is the dimensionless decay coefficient; $\rho = |\rho-\rho_0)|$; and $\tilde{l}(E_0)$ is the extinction length (mean free path) with respect to forward scattering.

Finally, we note that the above-discussed "one-dimensional" interference effects, such as the formation of the fluctuation waveguide, can play an important role in the long-range propagation of UHF radiowaves and the channeling of sound in the ocean. Indeed, it is currently accepted that both the atmosphere and the ocean contain strongly anisotropic, quasi-layered fluctuations of the refraction index (see, for example, Refs. 45, 68–70). If the fluctuation waveguide is to capture the wave effectively, the localization length at the appropriate value of parameter $E = [(\omega/c)\sin\psi]^2$ (where ψ is the capture angle) must not exceed the thickness of the fluctuation layer. In the tropospheric layers adjacent to the water the characteristic parameters are $\sigma_\varepsilon \sim 3\times 10^{-7} \approx 0.3N$ units and $r_c \sim 10^3$ cm. For a wave with $\lambda = 3$ cm and the capture angle $\psi \sim 10^{-3}$ equation (1.11) yields a localization length l of the order of ~ 50 m. If we recall that nonuniformities in the permittivity with typical gradients of the order of tenths of N units/m have been observed in the atmosphere at elevations of 1 km and higher, it appears likely that the fluctuation waveguide will channel energy even more efficiently than ordinary refraction, where the capture angle ψ rarely exceeds $\sim 10^{-4}$.

Although no thorough radiophysical and meteorological measurements aimed at observing the fluctuation waveguide have yet been carried out, there exists a quantity of indirect evidence in favor of this phenomenon. For example, researchers have noted the increasing depth of signal extinction at large distances from the source as the average strength[71] and the propagation path length[78] of signal increase. This behavior of the signal amplitude is uncharacteristic of the ordinary tropospheric waveguides, but it does agree with the behavior of intensity in a randomly stratified layer described in Sec. 2. Another evidence for the existence of the fluctuation waveguide is the frequently observed correlation of over-the-horizon field intensity with the dispersion of the refraction index fluctuations σ_ε^2 in the near-surface layer. The signal intensity increases with σ_ε even if the average gradient $d\varepsilon/dz$ is quite small and ordinary waveguiding is ineffective.

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[1)] In a given realization, the transmission t and reflection r coefficients depend both on the segment length L and on the spectral parameter E. In each concrete case, however, we shall explicitly state only the dependence of t and r on the dominant of these two parameters.

[2)] As was demonstrated in Ref. 60, relation (2.4) is valid for a much more general class of cases than (1.10), (1.15).

[3)] This result was obtained in collaboration with Yu. S. Kivshar.

[4)] Clearly, this estimate and the analogous estimates that follow are valid only with logarithmic accuracy.

[5)] In the regime of large $E = k^2$, where one can employ perturbation theory in terms of the small parameter $v(z)/k^2$, according to (2.7) the phase φ of the coefficient changes rapidly over distances of the order of the wavelength k^{-1}, whereas the modulus ρ (i.e. Δ) changes more slowly over distances of the order of localization length l.

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