# The physics of planetary rings 

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#### Abstract

A review of the collisional, collective, and resonance phenomena in planetary rings is presented. The following questions are examined: the reasons for the existence of planetary rings and the properties of a typical particle, the collisional breaking of loose bodies, and the azimuthal asymmetry effect for the rings of Saturn. A transfer theory is being developed for differentially rotating disks of inelastic particles, and the collective instabilities of planetary rings and a protoplanetary disk are discussed. A model for the resonance origin for the rings of Uranus is described, which enabled one to predict unknown satellites of Uranus that were later discovered by "Voyager-2". The problem of the stability of the rings of Uranus is examined.


## 1. INTRODUCTION

The rings of Saturn discovered in the seventeenth century continually excited the imagination of researchers by their unique form. Such illustrious astronomers, celestial mechanicians, and mathematicians as G. Galilei, C. Huyghens, J. D. Cassini, P. S. de Laplace, J. C. Maxwell, and H. Poincare investigated the rings of Saturn. Kant was the first one who predicted the existence of fine structure of the rings of Saturn. By using his model of a protoplanetary cloud, he imagined a ring in the form of a flat disk of colliding particles revolving differentially around the planet according to Kepler's laws. According to Kant, just such differential revolution is the cause for separating the disk into a series of thin ringlets. Later P. S. de Laplace proved the instability of a solid wide ring. ${ }^{1}$ Many astronomers in the middle of the last century (de Vico in Rome, Bond in the U.S.A., Struve in Russia, Dawes and Lassell in England) discovered a total of ten ringlets around Saturn. At this same time, J. C. Maxwell, who received the Adams Prize for Ref. 2, in which he showed that such narrow rings are also unstable and will fall onto the planet, made an outstanding contribution to the investigation of the stability of the rings of Saturn. One may consider Maxwell's paper as the first investigation of the theory of collective processes carried out at the modern level: the characteristic equation that is now called the dispersion equation was used in the stability analysis in Ref. 2. And although Maxwell's conclusion that a hypothetical solid ice ring would fall onto the planet was incorrect (such a ring must be torn to pieces considerably before this; see Ref. 3 and also Sec. 6.4.4 of this review), the corollary from it, that the rings of Saturn are of meteoritic structure, turned out to be true. Thus, towards the end of the nineteenth century, the hypothesis of the meteoritic structure of the rings of Saturn, which was first expressed by J. D. Cassini, received theoretical and, in 1895, also observational confirmation in the papers of $\mathbf{J}$. Keeler and A. A. Belopol'skii, who measured the velocities of the differential revolution of the rings (see Ref. 4).

A gradual accumulation of new data about planetary rings occurred in the twentieth century: estimates of the sizes and concentration of particles in the rings of Saturn were obtained, ${ }^{4}$ it was determined by spectral analysis that the rings are ice, ${ }^{4.5}$ and the enigmatic phenomenon of the azimuthal variability of the brightness of the rings of Saturn was discovered. ${ }^{6,7}$ The measured pace of scientific activity was changed by the strong rise of general interest in plan-
etary rings at the end of the 1970s, when the narrow and widely separated, coal-black rings of Uranus were independently discovered on March 10, 1977 by several research groups. The discovery was made entirely fortuitously when, in preparing equipment to investigate the parameters of the atmosphere of Uranus by the stellar occultation method and after setting the instruments beforehand, the researchers detected short eclipses during the approach of the star to the planet and as it moved away from the planet. The best tracings were obtained with the telescope of the Kuiper Airborne Observatory. ${ }^{8}$

Two years later, on March 4, 1979, the American interplanetary spacecraft "Voyager-1" also discovered the transparent stony rings around Jupiter. ${ }^{9}$ The rings of Saturn were investigated most intensively at the beginning of the 1980s. A series of American spacecraft operated in their vicinity: "Pioneer-11" (October 1979), "Voyager-1" (November 1980), and "Voyager-2" (August 1981). "Voyager-2" investigated the rings of Uranus in January 1986. In August 1989 this spacecraft encountered Neptune, around which incomplete rings (or "arcs") were detected several years ago by the stellar occultation method ${ }^{10}$ ("Voyager-2" improved on the ground-based observations: the "arcs" turned out to be denser parts of complete rings).

Actually, a new class of solar system objects has been discovered and studied over the last 12 years. Planetary rings turned out to be a necessary element and a regular phenomenon in the satellite systems of the giant (Jovian) planets. Naturally, an abundance of observational material could not fail to cause an intensive development of theoretical models (it is sufficient to cite two voluminous symposia ${ }^{11,12}$ which were published in 1984). This is not merely interest in new astronomical objects. The opinion that planetary rings are the key to understanding the cosmogony of the entire solar system is receiving more and more acceptance. For the pres-ent-day rings are the only representatives accessible for detailed study of differentially revolving disks of inelastic particles. The investigation of such disk systems is of fundamental importance for this cosmogony, since this was the most widespread type of dynamical system (the protoplanetary nebula, protosatellite disks, and protoplanetary rings) in the early stage of the solar system. One must also assign the protoplanetary clouds around other stars (for example, around Beta Pictoris), the accretion disks in binary star systems, and galactic and protogalactic disks, to this
same class of objects. Thus, planetary rings present a unique opportunity to obtain very important information about the collective and other processes which occurred at the stage of the formation of the planets and satellites of the solar system.

One of the tasks of this review is to attract the attention of physicists and specialists of related sciences to planetary rings and to show the importance to physics and astronomy of studying the dynamics of these objects.

Let us list the main problems of the physics of planetary rings:

1. Why do planetary rings exist? Classical models for the formation of rings assumed that rings are in the realm of the tidal disruption of large bodies. But it became clear after the "Voyager" flights that tidal forces are too weak to disrupt particles of the sizes that are observed ( $\varsigma 10 \mathrm{~m}$ ). The question about the reasons for the existence of rings turned out to be directly connected with the mechanical characteristics of a typical particle.
2. What caused the layering of the rings of Saturn? The observed hierarchical structure of the rings of Saturn is built up on the "matrix" principle: broad rings of $\sim 1000 \mathrm{~km}$ width consist of a system of narrower rings of $\sim 100 \mathrm{~km}$ width, etc. The widely accepted opinion that the layering of the rings of Saturn is connected only with negative diffusion instability ${ }^{13,14}$ contradicts the observations; this instability can cause the formation of only the narrowest ringlets (hundreds of meters wide) in fairly dense parts of the disk. ${ }^{15}$
3. How were the rings of Uranus formed and why are they not disrupted? The most popular hypothesis about how the narrow, elliptical rings of Uranus were formed and maintain stability is that this is due to two "shepherd" satellites along the edges of each ring. ${ }^{16}$ However, "Voyager-2" in 1986 did not discover the "shepherd" satellites between the rings of Uranus that are so necessary for this hypothesis. Here the "Voyager-2" data confirmed the alternative hypothesis of the resonance nature of the rings of Uranus. ${ }^{17}$ At present, there exists in the physics of planetary rings a large number of models and hypotheses which often mutually exclude each other. Therefore, it is fairly difficult to present a unified picture for the origin and dynamics of planetary rings. For example, a number of investigators of the stability of planetary rings start with a model of a smooth and very elastic ice particle ${ }^{18}$ without mentioning here the problem of the existence of the rings. In turn, cosmogonists consider an extremely ephemeral formation (ten thousand times less sturdy than a cluster of the flakiest terrestrial snow) as a typical ring particle ${ }^{19}$ without thinking of how such a fragile particle will "work" in other theoretical models.

In this review we tried to give the most consistent and physically complete picture of planetary rings by also critically investigating alternative solutions to a number of problems. In studying the physics of rings, one must turn to the most diverse methods and fields of science: to celestial mechanics and the physics of ice and snow, to impact theory and the kinetic theory of gases, to instability theory and to plasma physics. Analytical calculations and astronomical observations, numerical and full-scale experiments are used here. This is a typical feature of this paper, which is devoted not to an individual method, phenomenon, or experiment, but to a complicated natural object whose "activity" is not subject to the traditional division of the sciences.

The basic, reliably determined observational data are
described in Sec. 2 of the review. The questions, for the understanding of which it is sufficient to know the dynamics of individual particles, are examined in Sec. 3. Sections 4 and 5 are devoted to investigating the collective dynamics of planetary ring particles. The rings of Uranus, whose features are largely determined by the influence of resonant satellites, are examined in Sec. 6 . Section 7 reflects the first attempts to use theoretical models that have been developed in the process of studying planetary rings for solar system cosmogony.

## 2. OBSERVATIONAL DATA ON PLANETARY RINGS

### 2.1. General characteristics

### 2.1.1. Primary and secondary rings

One can divide all ring structures around planets into two classes: the cosmogonically "primary" and "secondary" classes. Dense rings of fairly large particles (up to tens of meters sizes) are the first type of structures. The existence times for such rings is fairly long and is evidently comparable with the cosmogonic time. One can unconditionally assign the classical A, B, and C rings of Saturn and the nine dense rings of Uranus to the primary structures. The lowdensity gas and dust rings, for whose prolonged existence a constant influx of matter is required, are the second type of rings: these are the transparent E ring of Saturn (the source of matter is the satellite Enceladus), the dust rings of Uranus situated between the dense ringlets and connected with the sweeping of fine dust out of the ringlets, and the gas torus of volcanically active $I_{0}$. The ring of Jupiter is evidently secondary, but a low-density layer of large particles may be its source. Data on the rings of Neptune indicate the primary nature of these rings. A characteristic feature is that if the secondary rings can be distributed at any distances from the planet (depending on the arrangement of the "material" source of matter), then the outer radius of the primary rings is clearly bounded and is approximately equal to two radii of "its" planet: 1.8 radii for Jupiter, 2.3 radii for Saturn, 2.0 radii for Uranus, and 2.5 to 2.6 radii for Neptune (from 1989 data). The satellite zone starts beyond the boundary of primary rings, but in a narrow "boundary" zone, the rings and satellites can be arranged "helter-skelter."

### 2.1.2. The size distribution of particles

Secondary rings consist of particles of micron and submicron sizes. The primary rings of Saturn contain particles with sizes from a micron up to 10 to 20 meters, and moreover, meter-size bodies make up the main mass of the rings. The particle size spectrum in the rings of Saturn has a characteristic "cutoff" at radii of about ten meters. The lack of larger size particles is a very important feature of planetary rings and is also observed in the Uranus system. The optical thickness of the rings is determined by both meter-sized and also centimeter-sized particles. ${ }^{20}$

### 2.1.3. Hierarchicallayering of rings

The broad rings of Saturn are divided into narrower ringlets of different scales. ${ }^{20}$ The small-scale division of the $\mathbf{C}$ and B rings is shown in Figs. 1 and 2. The $\mathbf{A}$ ring is more uniform, but possibly has small-scale layering at hundreds of meters. The existence of radial structure in the rings of Saturn is evidently connected with internal evolutionary processes.


FIG. 1. The C Ring of Saturn ("Voyager-2" photo). A regular thousand kilometer-size structure with a small contrast of density is identified in the central part of the ring. Some narrow ringlets in the outer and inner regions of the C Ring are connected with the resonance action of satellites (see Fig. 5).

### 2.1.4. Spiral waves

They are excited by external satellite resonances and have been discovered in large numbers in the rings of Saturn, especially in the A ring. The spiral waves are subdivided into


FIG. 2. The outer part of the B ring (a 6000 km section; a "Voyager-2" photo). The dark corner at the upper left is the Cassini gap. A hierarchy of annular structures of different scales (from thousands to tens of kilometers) is easily visible. We note that these structures are not connected with resonances from satellites.
density waves (Fig. 3) and bending waves, which deflect particles from the equatorial plane. The most powerful (with significant amplitude and extent) spiral waves are

a


FIG. 3. Density waves ("wave trains") in the rings of Saturn (this figure is from the review in Ref. 20). (a) is a linear wave in the Cassini gap connected with the $0: 1$ apsidal resonance from Iapetus (a coincidence of ring particle frequency of precession as a consequence of the non-sphericity of the planet's gravitational field and of the frequency of revolution of a satellite that is, as a rule, very distant, is called as apsidal resonance). (b) is the nonlinear extended spiral wave in the B ring from Janus (the 1:2 Lindblad resonance). (c) is a strongly damping wave from Mimas (3:5) in the A ring.


FIG. 4. Spiral waves at the outer edge of the $A$ ring (this figure is from the review in Ref. 20). Spiral waves are excited even by high order resonances. Mimas causes two types of waves; bending waves (BW), which propagate from the place of resonance towards Saturn, and density waves (DW), which propagate away from the planet. Waves of the first type are caused by resonance between the frequencies of satellite revolution and the vertical oscillations of ring particles, and waves of the second type are caused by resonances between the satellite frequency and the frequency of the radial oscillations of the particles.
caused by low order resonances ( $1: 2,2: 3,3: 4,4: 5,5: 6,3: 5$, and 5:7), but high order resonances (for example, 32:33, etc.) also excite small waves (Fig. 4). A wave removes angular momentum from the particles of a disk, and one associates the occurrence of the Cassini gap with the action of a spiral wave which existed earlier from a 1:2 resonance with Mimas. ${ }^{21}$

## 2. 1.5. Narrow ringlets

The rings of Uranus are a set of narrow (of the order of 10 km width ), dense ringlets which often possess noticeable ellipticities (eccentricities up to 0.01 ) and inclinations to the equatorial plane. The edges of the ringlets are sharp, and the


FIG. 5. A narrow ringlet in the C ring "squeezed" between the resonances from 1980 S 26 (2:1) and Mimas (3:1). The figure is from the review in Ref. 20.
rings precess in the non-spherical field of the planet as a single body. Many ringlets have variable widths (widest at apocenter and narrowest at pericenter ). ${ }^{22}$ Similar rings have also been discovered near Saturn. The location of narrow, dense rings near low order resonances from external satellites is a typical feature. One of Saturn's ringlets is shown in Fig. 5 with its "own" resonances. The relation between the resonance positions and the rings of Uranus is analyzed in Ref. 23. Two narrow rings (the $F$ ring of Saturn and $\varepsilon$ ring of Uranus) are located on the boundaries of primary rings and are circled by "shepherd" satellites.

### 2.2. Individual ring systems

### 2.2.1. The rings of Saturn

The geometric characteristics of the rings are shown in Table I, and their main physical characteristics are in Table II (from Ref. 20). The ring particles consist mainly of water ice, possibly with a small admixture of rocks. Data on the thermal inertia of the particles, ${ }^{29}$ and also spectrometric ${ }^{30}$ and photopolarimetric ${ }^{31}$ data indicate that the particles are covered with a layer of "hoarfrost" or of fine ice dust of micron dimensions. The albedos of the particles of Saturn's rings correspond to the albedo of snow, 0.6 , but some ring structures in the $\mathbf{B}$ ring have albedos different from those of the gaps; the difference can reach $50 \%$, from 0.6 to 0.4 , from the albedo of snow to the reflectivity of rock. ${ }^{20}$ The "spokes", short-lived dust clouds that are elongated radially, are also observed in the $B$ ring; they revolve like rigid bodies, and one associates their origin with the action of the planet's magnetic field on charged dust. ${ }^{21}$ One more interesting effect is observed in the A ring: the azimuthal variability of ring brightness. ${ }^{6,7}$

### 2.2.2. The rings of Uranus

The characteristics of the rings of Uranus are given in Table III. ${ }^{22,28}$ We notice that radial structures of kilometer scales is detected in the widest $\varepsilon$ and $\alpha$ rings. The region between the dense rings is filled with fine dust with optical thicknesses from 0.001 to 0.0001 . This fine dust is distribut-

TABLE I. Geometric characteristics of the rings of Saturn.

| Ring | Orbital Radius |  | Width, $10^{3} \mathrm{~km}$ |
| :---: | :---: | :---: | :---: |
|  | In Saturn Radii | In $10^{3} \mathrm{~km}$ |  |
| D | 1.11-1.235 | 66.97-74.51 | 7.45 |
| ${ }_{\text {c }}$ | 1.235-1.525 | 74.51-92.00 | 17.49 |
| B | 1.525-1.948 | 92.(4)-117.52 | 25.52 |
| The Cassini Gap | 1.948-2. ${ }^{\text {. }} 125$ | 117.52-122.17 | 4.65 |
|  | 2.025-2.267 | 122.17-136. 78 | 14.61 |
| F | 2.324 | 149. 18 | 0.05 |
| G | 2.82 | 170.10 | 1 |
| E | 3-8 | 181-483 | 302 |

ed nonuniformly and forms a series of ring structures which, according to our classification, belong to the secondary rings. ${ }^{28}$ The significant aerodynamic drag from the upper layers of our atmosphere of Uranus is the most important factor which removes dust from the dense rings. ${ }^{33}$

The rings of Uranus are very black, their albedo is about $\mathbf{5 \%}$, and the infrared spectrum of the rings corresponds best to carbonaceous chondrites. ${ }^{22}$

### 2.2.3. The rings of Jupiter

The characteristics of the rings are shown in Table IV. The spectrum of the rings corresponds to rock. ${ }^{34}$ Two small satellites are situated at the outer edge of the main ring. The amount of light reflected indicates the presence of the large particles in the main ring, although all the rings of Jupiter consist mainly of dust. ${ }^{9}$

### 2.2.4. The rings of Nepune

A good many reports of observations of ring arcs near Neptune have accumulated since the mid-1980s. ${ }^{10,24-27}$ The latest images from "Voyager-2" during its recent flyby near Neptune improved the results from ground-based observations. The "Voyager-2" data (for 1989) are shown in Table V . Two dense, complete rings with radii of 52300 km and 62900 km , respectively were recorded (the latter ring has significant azimuthal density variations; it contains three condensations, each with an extent of $6^{\circ}$ to $8^{\circ}$ ).

## 3. PARTICLE COLLISIONS IN RINGS. THE COSMOGONY OF RINGS

The question of the properties of a typical particle is the fundamental question of the physics of planetary rings. Without a detailed investigation of these properties, it is impossible to understand either the origin of rings or the dynamics of collective processes. Therefore, we start from a study of individual ring particles, from the simplest imaginable model, that of a solid sphere.

### 3.1. Collisions of solid particles

A number of authors of theoretical models of the rings of Saturn use a smooth icy sphere as a typical ring particle. ${ }^{18,35-38}$ The restitution coefficient for such a sphere has even been investigated experimentally in Refs. 18 and 37; in a vacuum at low temperatures and typical impact velocities $v \sim 0.1$ to $10 \mathrm{~mm} / \mathrm{sec}$. It has been found that the restitution coefficient $q(v)=v^{\prime} / v$ (where $v^{\prime}$ is the recoil velocity) is close to one for velocities $v \sim 0.1 \mathrm{~mm} / \mathrm{sec}$ and decreases with increasing collision velocity. This dependence agrees with the behavior of $q(v)$ for any kind of smooth bodies in the well investigated range of impact velocities from 1 to $6 \mathrm{~m} / \mathrm{sec} .{ }^{39}$

We investigate a model of hypothetical smooth particles in this section, and we shall show that, because of mutual collisions, they are inevitably covered by a loose, snowlike regolith which qualitatively changes their elastic proper-

TABLE II. Basic physical characteristics of the rings of Saturn.

| Ring | Thickness" | Optical thickness, $\tau^{11}$ | Surface density, g/cm² |
| :---: | :---: | :---: | :---: |
| D | ? | $<0.1$ | ? |
| C | 10 m | 0.08-0.14 | 10 to $15^{41}$ |
| B | 10 m | 1.06 to $1.89{ }^{\text {2) }}$ | 60-70 |
| The Cassini division | 20 m | 0.12 | $10.40^{41}$ |
| A | $40-60 \mathrm{~m}$ | 0.49-0.58 | 20-69 |
| F | $<100 \mathrm{~m}$ | 0.1 to $0.2{ }^{3}$ | ? |
| G | 106 km | $10^{-4}$ to $10^{-5}$ | ? |
| E | $7.5-3510^{3} \mathrm{~km}$ | $10^{-5}$ to $10^{-6}$ | ? |
| ${ }^{1)}$ Thickness and optical thickness are measured transversely to the plane of the rings (parallel to the axis of rotation). The light flux transmitted by the ring is attenuated by a factor of 2.7 at $\tau=1$. |  |  |  |
| ${ }^{2} 10 \%$ of the B ring's surface is completely opaque to observations ( $\tau>2.53$ ). |  |  |  |
| ${ }^{31} 0.8 \leqslant \tau \leqslant 1.2$ for the main part of the $F$ ring with a width of 3 km . |  |  |  |
| Data are for a nominal density of $1 \mathrm{~g} / \mathrm{cm}^{3}$ for the particles. |  |  |  |

TABLE III．The rings of Uranus．

| Ring | Semi－major axis，km | Eccentricity $\times 1000$ | Width，km＇ | Optical thickness ${ }^{\prime \prime}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\begin{gathered} \text { Ring } 1 \\ \text { 1986 U1R } \end{gathered}$ | $\begin{aligned} & 51156 \pm 5 \\ & 50660 \pm 30 \\ & 50030 \pm 30 \end{aligned}$ | 7．924 ${ }_{\text {？}}$ | $\begin{gathered} \text { From20 to } 96 \\ 16 \\ 1-2 \end{gathered}$ | $\left\lvert\, \begin{gathered} \text { From } 1.2 \text { to }>4 \\ 0.1 \\ 0.1 \end{gathered}\right.$ |
| 8 | $48306 \pm 5$ | （0．020 $\pm 0.140)$ | From3 to 9 | From 0.3 to $0.4^{2 \prime}$ |
| $\gamma$ | $47632 \pm 5$ | （0．121＋0．201） | Fromi to 4 | From 1.3 to $2.3^{21}$ |
| 7 | $47184 \pm 5$ | （0．014 0 （0．25） | 0－2 | 0．1－0．4 |
| $\beta$ | 45669 士5 | （0．438 $=0.22$ ） | From7 to 12 | 0.2 |
| $\alpha$ | $44727 \pm 5$ | $0.759 \pm 0.26$ | From 7 to 12 | From 0.3 to 0.4 |
| 4 | 42579 干5 | $1.065 \pm 0.29$ | 2－3 | 0.3 |
| 5 | 42 243士5 | 1.900 －0． 29 | 2－3 | 0．5－0．6 |
| 6 | $41846 \pm 5$ | $1.001+0.24$ | From1 to 3 | From 0.2 to 0.3 |
| Arc 1 | $41760 \pm 30$ | ？ |  | 0.2 |
| Arc 2 | $41470 \pm 30$ | ？ | 4 | 0.2 |
| Arc 3 | $38430 \pm 50$ | ？ | 2 | 0.2 |
| Ring 2 | $38280 \pm 50$ | ？ | 1 | From 0.2 |
| 1986 U2R | $37-39.5 \quad 10^{3} \mathrm{~km}$ | － | 2500 | From 0.001 to 0.000 |

${ }^{11}$ From ．．．to indicates that the ring has（or may have）a variable width and optical thick－ ness（minimum width and maximum optical thickness at pericenter）．
${ }^{2}$ The $\delta$ and $\gamma$ rings are best approximated not by an ordinary ellipse，but by a more complicated combination of an ellipse and circle．${ }^{32}$
${ }^{31} \mathrm{~A}$ dust ring．
ties ${ }^{40,41}$ and makes theoretical models based on the assump－ tion that the particles are smooth unsuitable for rings．

## 3．1．1．Some relatlons from the theory of／mpact of smooth spheres

From the theory of Hertz，we obtain the following expression for the maximum energy of the elastic deforma－ tion of colliding individual smooth spheres（see Ref．42）：

$$
\begin{equation*}
U_{\max }=5,4\left(1-\mu_{\mathrm{p}}^{2}\right) \frac{P_{\max }^{2}}{E} R_{\mathrm{c}}^{3}, \tag{1}
\end{equation*}
$$

where

$$
R_{\mathrm{c}} \approx \frac{\pi}{2}\left(1-\mu_{\mathrm{r}}^{2}\right) \frac{P_{\max }}{E} a
$$

is the maximum radius of the elastic contact zone，$\mu_{\rho}=0.36$ is the Poisson coefficient，$a$ is the radius of the sphere， $P_{\text {max }}=5 \cdot 10^{7}$ to $10^{8} \mathrm{dyn} / \mathrm{cm}$ is the rupture（maximum） stress for ice，and $E=10^{11} \mathrm{dyn} / \mathrm{cm}^{2}$ is Young＇s Modulus．${ }^{43}$ By equating the kinetic energy $m v^{2} / 2$ and $U_{\max }$ ，we find the critical velocity of collision（at which irreversible deforma－ tion of ice in the contact zone begins）

$$
\begin{equation*}
v_{\mathrm{cr}} \approx 3,2 \frac{\left(1-\mu_{\mathrm{p}}^{2}\right)^{2}}{\rho^{1 / 2}} \frac{P_{\max }^{5 / 2}}{E^{2}} \sim 4,5 \cdot 10^{-3} \text { to } 2.5 \cdot 10^{-2} \mathrm{~cm} / \mathrm{sec} \tag{2}
\end{equation*}
$$

where $\rho=0.9 \mathrm{~g} / \mathrm{cm}^{3}$ ．At $v>v_{c r}$ ，the ice undergoes elastic－ brittle fracturing if the rate of deformation $d \varepsilon / d t \geqslant 0.01$ to $0.001 \mathrm{sec}^{-1}$ ，where $\varepsilon$ is the relative deformation．${ }^{43}$ It has
been shown in Ref． 39 that the deformation rate of particles in the rings is $d \varepsilon / d t \gtrsim 0.1$ to $0.01 \mathrm{sec}^{-1}$ ．This indicates that ice particles in the rings of Saturn will be fractured like brit－ tle bodies upon colliding with characteristic velocities $v \sim 1$ $\mathrm{mm} / \mathrm{sec}$ ，with the formation of a fine dust in the contact zone．

## 3．1．2．Est／mates of the fragmentation of ring particles

We estimate the mass of broken ice $\Delta m$ for a single impact of particles with mass $m$ and velocities $v \sim 0.1$ to 0.6 $\mathrm{cm} / \mathrm{sec}$ ，assuming that the entire energy of impact is used up in fracturing：

$$
\begin{equation*}
\frac{\Delta m}{m} \sim \frac{\rho v^{2}}{2 \mathscr{\mathscr { C } _ { v }}} \sim 4.5 \cdot 10^{-11} \text { to } 1,6 \cdot 10^{-8} \tag{3}
\end{equation*}
$$

here $\mathscr{E}_{v}=10^{7}$ to $10^{8} \mathrm{erg} / \mathrm{cm}^{3}$ is the energy needed to frag－ ment a unit volume of ice．${ }^{43}$ Assuming the rate of fracturing to be constant，we obtain the characteristic time for the com－ plete fragmentation of ice particles：
$t_{\mathrm{br}} \sim \frac{m}{\Delta m} t_{\mathrm{ff}} \sim 2.5 \cdot 10^{8}$ to $7 \cdot 10^{6}$ years for $t_{\mathrm{ff}}=10$ hours
where $t_{\mathrm{ff}}$ is the free flight time for particles in rings．From this，one may draw the conclusion that the particles in the rings of Saturn are most likely balls of finely fragmented ice or snow．But one may assume that the ice particles cease

TABLE IV．The rings of Jupiter．

| Ring | Orbit radius， $10^{3} \mathrm{~km}$ | Width， $10^{3} \mathrm{~km}$ | Thickness，km | Optical <br> thickness |
| :--- | :---: | :---: | :---: | :---: |
| The main ring | $122.8-129.2$ | 6.4 | $<30$ | $3 \cdot 10^{-5}$ <br> The faint ring <br> The halo |



TABLE V. New satellites and rings of Neptune.

| Name | Orbit Radius, $10^{3} \mathrm{~km}$ | Orbit Period | Orbital Inclination | Diameter, km |
| :---: | :---: | :---: | :---: | :---: |
| Satellites: |  |  |  |  |
| 1989No 1 | 117.6 | 26.9 hours | $>1^{\circ}$ | 420 |
| 1989№ 2 | 73.6 | 13.3 hours | $>1^{\circ}$ | 200 |
| 1989№ 3 | 62.0 | 9.5 hours | $>1^{\circ}$ | 160 |
| 1989№ 4 | 52.5 | 8.0 hours | $>1{ }^{\circ}$ | 140 |
| 1989No 5 | 50.0 | 7.5 hours | $1^{\circ}$ | 90 |
| 1989№ 6 | 48.2 | 7.1 hours | $1{ }^{\circ}$ | 50 |
| Rings: |  |  |  |  |
| 1989№ 1A | 62.9 | A dense ring of $\leqslant 15 \mathrm{~km}$ with three arc condensations; the length of each one is $\sim 10000 \mathrm{~km}$. |  |  |
| ? | 53.2 | A low optical density ring |  |  |
| 1989 № 2A | 52.3 | A high optical density ring |  |  |
|  | 41.0 | A broad ( $\sim 2500 \mathrm{~km}$ ) low density ring |  |  |

being fractured after the accumulation on their surfaces of a thin, loose layer of finely fragmented ice, which absorbs the energy of impact and protects the single particle from fragmentation. The conclusion that at least a surface layer of loose regolith exists on ring particles is confirmed both by theoretical arguments ${ }^{40,44}$ and also by observations ${ }^{29,30,31}$ (see Sec. 2.2.1). The rate of accretion of micron-sized ice grains is extremely small at temperatures near 70 K , and one can neglect the fusion of grains in the contact zone during impact in connection with the low kinetic energies of the colliding particles; consequently, the surface regolith is a granulated medium that is weakly bound by auto-hesion forces. ${ }^{40}$ The question arises: how similar are particles covered by a thin layer of regolith to smooth spheres in their impact-mechanical properties?

## 3. 1.3. Collision model for particles with regoliths

A three-stage model for the collision of monolithic bodies covered by a thin layer of loose regolith has been suggested in Ref. 40. We denote the energy that is expended on the inelastic deformation of the regolith by $\Delta E$, the energy of elastic deformation of the single particle by $U$, and we denote the critical value of $U$ before fracturing of the single particle by $U_{\text {max }}$. We consider how the collision process changes with increasing particle velocity.
A. The completely inelastic impact stage. The kinetic energy is used up in irreversible compression of the loose surface regolith:

$$
\begin{equation*}
\frac{m v^{2}}{2}<\Delta E, \quad U=0, \quad q(v)=0 . \tag{5}
\end{equation*}
$$

B. Stage of elastic deformation of the monolithic core. The part of the kinetic energy of the particles which remains after compressing the regolith is used up in the reversible deformation of the monolithic core:

$$
\begin{aligned}
& \Delta E<\frac{m v^{2}}{2}<\Delta E+U_{\max }, \quad U=\frac{m v^{2}}{2}-\Delta E \\
& q(v)=\left(1-\frac{2 \Delta E}{m v^{2}}\right)^{1 / 2}
\end{aligned}
$$

C. Stage offracturing. The impact energy is so large that inelastic deformation of the single particle or fragmentation of the regolith grains begins:

$$
\begin{equation*}
\Delta E+U_{\max }<\frac{m v^{2}}{2}, \quad U=U_{\max } \tag{7}
\end{equation*}
$$

$$
q(v)=\left(\frac{2 U_{\max }}{m v^{2}}\right)^{1 / 2}
$$

Recoil, the reconversion of the reversible deformation energy $U_{\text {max }}$ or $U$ back to kinetic energy, is the next natural stage of impact. The values of $\Delta E$ and $U_{\max }$ may depend on $v$, but less strongly than on $v^{2}$. Let us consider the function $U_{\max }(v)$ for a smooth particle.

## 3. 1.4. The restitution coefficlent for a smooth particle

For $\Delta E=0$, at most two stages of impact, B and C, and one unknown quantity, $U_{\max }$, remain. The particles are not fractured ( $q=1$ ) for $v<v_{\mathrm{cr}}$. Local irreversible deformation of the contact zone begins for $v>v_{\mathrm{cr}}$. Let us estimate $U_{\text {max }}$ after the start of fracturing by assuming that Eq. (1) is ap${ }_{r}$ roximately correct if $R_{c}$ is the radius of the contact zone of the particles during fracturing. Let a spherical segment with the base radius $R_{\mathrm{s}}$ and height $l$ in the contact zone be fractured; then $R_{\mathrm{s}}=(2 a l)^{1 / 2}$. Let $R_{\mathrm{c}} \approx R_{\mathrm{s}}$. Then the kinetic energy of a particle will be used up in fragmenting a segment with volume $\pi a l^{2}$, and for the elastic deformation of the particle, we have:

$$
\begin{equation*}
\frac{m v^{2}}{2}=\mathscr{E}_{v} \pi a \mathscr{l}^{2}+5,4\left(1-\mu_{\mathrm{p}}^{2}\right) \frac{p_{\max }^{2}}{E} R_{\mathrm{s}}^{3} . \tag{8}
\end{equation*}
$$

The second term on the right-hand side of Eq. (8) describes the elastic deformation energy $U_{\text {max }}$. According to Eqs. (7),

$$
\begin{equation*}
q=\left[\frac{10.8\left(1-\mu_{p}^{2}\right) p_{\max }^{2}(2 a l)^{3 / 2}}{m v^{2} E}\right]^{1 / 2} \tag{9}
\end{equation*}
$$

By eliminating the quantity $l$ from Eqs. (8) and (9), one can obtain the following equation for $q(v):^{41}$

$$
\begin{equation*}
q^{2}+v^{2 / 3}\left(\frac{q}{L}\right)^{8 / 3}=1 \tag{10}
\end{equation*}
$$

where

$$
L=2,32 \frac{\left(1-\mu_{\mathrm{p}}^{2}\right)^{0.5} P_{\max }}{\rho^{1 / 8} E^{1 / 2 g E_{v}^{3 / 8}}} .
$$

One may call the quantity $L$ the "elasticity parameter": the larger $L$ is, the higher is the restitution coefficient. From Eq. (10) we find

$$
\begin{align*}
q(v) & =1 \quad \text { for } \quad v \rightarrow 0  \tag{11}\\
& =L v^{-0,23} \text { for } v \gg L^{4} .
\end{align*}
$$



FIG. 6. The dependence of the restitution coefficient of a metal sphere on impact velocity. A sphere with a 5 cm diameter falls onto a slab covered with a thin layer of loose regolith. (a) A massive slab of solid (Crimean marble-like) limestone. Fine dry sea sand serves as regolith. Measurement data for $q(v)$ are shown for 1) a clean slab, 2) a 1 mm thick regolith layer, and 3) for a 3 mm thick regolith layer. (b) A concrete slab of dimensions $7 \mathrm{~cm} \times 48 \mathrm{~cm} \times 48 \mathrm{~cm}$ lying on dense soil. The regolith is dry cement dust. The data for 1) a clean slab, and for the three values of $1 \mathrm{~mm}, 2 \mathrm{~mm}$, and 3 mm of regolith thickness (the symbols 2, 3, and 4, respectively).

The relation $q(v) \propto v^{-0.25}$ indicates that, for $v \gg L^{4}$, $U_{\max } \propto v^{3 / 2}$. The solutions of Eq. (10) agree well with the experimental data of Ref. 37 for $L=0.7$ to 1.0 and the data of Ref. 18 for $L=0.55$; such values of $L$ are obtained for $\mu_{\mathrm{p}}$ $=0.36, \rho=0.9 \mathrm{~g} / \mathrm{cm}^{3}, E=10^{11} \mathrm{dyn} / \mathrm{cm}^{2}, \mathscr{E}_{v}=10 \mathrm{erg} /$ $\mathrm{cm}^{3}$ for the strength of ice, and $P_{\text {max }}=10^{8}$ to $1.5 \cdot 10^{8}$ dyn/ $\mathrm{cm}^{2}$ and $0.8 \cdot 10^{8} \mathrm{dyn} / \mathrm{cm}^{2}$, respectively. The solutions of Eq. (10) with $L=1.25$ agree with the approximating expression $q(v)=0.82 v^{-0.047}$ (for ice particles with very smooth surfaces ${ }^{37}$ ) with an accuracy of $\leqslant 4 \%$. ${ }^{41}$

### 3.1.5. The restitution coefficient of particles covered by a regolith layer

It is evident from expressions (5) and (6) that, with a surface regolith present which absorbs part of the impact energy ( $\Delta E \neq 0$ ), the collision pattern changes qualitatively: the restitution coefficient $q(v)=0$ at low velocities (Stage A) and, with increasing velocity, $q(v)$ also begins to increase (Stage B). For a further increase of $v$, the restitution coefficient must again decrease (Stage C). An increase of the restitution coefficient with velocity was not observed before in experiments. In Ref. 45, where a simultaneous increase of $g(v)$ and $v$ was obtained in a formally engineered discrete impact model with a parallel combination of an elastic element and a dry friction element, it is noted that "... the model appears to be doubtful, since... the restitution coefficient increases with increasing velocity, whereas in experiments the opposite tendency is clearly detected".

To check the theoretical laws of the type of Eqs. (5), (6), and (7), an experiment was conducted to measure the restitution coefficient of a steel sphere falling onto a massive stone slab covered with a thin layer of granulated material. ${ }^{41}$ The diameter of the sphere was 5 cm , and its mass 0.5 kg . The velocity of fall was varied from $1 \mathrm{~m} / \mathrm{sec}$ to $6 \mathrm{~m} / \mathrm{sec}$. See Fig. 6 for the data obtained.

In the case of no regolith, the restitution coefficient of a sphere is almost independent of velocity. The presence of a regolith sharply changes the pattern: $q(v) \approx 0$ for $v \leqslant 1 \mathrm{~m} /$ sec, after which $q$ increases sharply. This is found in complete agreement with the first two stages of the model [Eqs. (5) and (6)]. The third stage, when the restitution coefficient again decreases, is also noticed. We show the analytic expressions for two curves of Fig. 6.
A. A Stone Slab. The regolith layer is 1 mm thick:

$$
\begin{align*}
q & =0 \text { for } 0 \leqslant v \leqq 1.5 \mathrm{~m} / \mathrm{sec}  \tag{12}\\
& =\left(1-\frac{1.129}{v^{0,3}}\right)^{1 / 2} \text { for } 1.5 \leqq v \leqq 5 \mathrm{~m} / \mathrm{sec} \tag{13}
\end{align*}
$$

B. A Concrete Slab. The regolith layer is 1 mm thick:

$$
\begin{align*}
q & =0 & \text { for } & 0 \leq v \leq 0.8 \mathrm{~m} / \mathrm{sec}  \tag{14}\\
& =\left(1-\frac{0,946}{v^{0,25}}\right)^{1 / 2} & \text { for } & 0.8 \leq v \leq 2.5 \mathrm{~m} / \mathrm{sec}  \tag{15}\\
& =0,787 v^{-0,5} & \text { for } & 2.5 \leq v \leq 5.5 \mathrm{~m} / \mathrm{sec} \tag{16}
\end{align*}
$$

### 3.1.6. Discussion of the experiment

All the qualitative features of the model [Eqs. (5), (6), and (7)] were confirmed. We notice that one could expect theoretically that $\Delta E$ should be proportional to the volume of the compressed section of regolith, i.e., $\Delta E \propto h^{2}$, where $h$ is the thickness of the regolith layer. The experiment shows that the dependence $\Delta E(h)$ is weaker and is closer to a linear one: $\Delta E \propto h$. This may be caused by the sweeping out observed in the experiment of part of the regolith from under the sphere at the moment of impact, which reduces the volume of the loose material which is deformed. This effect is probably connected with the air wave which arises during compression of the regolith. ${ }^{41}$ The fact of the existence of such an air wave is also confirmed by Hartmann's experiments, ${ }^{46}$ from which it is evident that the scattering of regolith during the impact of a sphere at atmospheric pressure is three orders of magnitude larger than the ejection of regolith from the impact zone in a vacuum.

We estimate the thickness of the regolith layer which is capable of absorbing most of the impact energy. Let the energy that is used up in deforming the layer be $\Delta E=E_{\mathrm{v}} \pi a h^{2}$ ( $E_{\mathrm{v}}$ is the specific energy for deforming the regolith). We assume here that sweeping out does not occur, and by equating $\Delta E$ to the kinetic energy $(2 / 3) \rho \pi a^{3} v^{2}$, we obtain

$$
\begin{equation*}
\frac{h}{a} \sim\left(\frac{2 \rho}{3 E_{\mathrm{v}}}\right)^{1 / 2} v \sim 0.2 \cdot 10^{-3} v \tag{17}
\end{equation*}
$$

$E_{\mathrm{v}}$ in the experiment reached the significant values of $\sim 5 \cdot 10^{7} \mathrm{erg} / \mathrm{cm}^{3}$ for cement dust and $2 \cdot 10^{8} \mathrm{erg} / \mathrm{cm}^{3}$ for sand. These values will evidently be significantly less at low velocities. For planetary ring particles, assuming $\rho \sim 1 \mathrm{~g} /$ $\mathrm{cm}^{3}, E_{\mathrm{v}} \sim 10^{6}$ to $10^{8} \mathrm{erg} / \mathrm{cm}^{3}$, and $v \sim 0.1 \mathrm{~cm} / \mathrm{sec}$, we obtain ( $h / a$ ) $\sim 10^{-4}$ to $10^{-5}$. It follows from this that 10 -metersize planetary ring particles will become completely inelastic even for millimeter regolith thicknesses. For a basalt sphere with $\rho_{3} \approx 3 \mathrm{~g} / \mathrm{cm}^{3}$ and $v \simeq 5 \mathrm{~m} / \mathrm{sec}$ falling onto a regolith with $E_{\mathrm{v}} \sim 5 \cdot 10^{7} \mathrm{erg} / \mathrm{cm}^{3}$, the impact will be practically inelastic at $(h / a) \sim 0.1$. This agrees with the corresponding experiment in Ref. 46.

Theoretical consideration and experimental data show
that smooth particles cannot exist in planetary rings; they are inevitably covered by finely fragmented material which was formed during collisions. Here even a thin regolith layer qualitatively changes the elastic properties of particles: the restitution coefficient is close to zero at low velocities and increases with impact velocity.

The energy balance in models for nongravitating particles that are used in Refs. 18 and 35-38 is stable only for a decrease of the restitution coefficient with increasing velocity (see Sec. 5.2) and for a sufficiently large coefficient, $q>0.63$, which is difficult to expect with a regolith present. Therefore, one may draw the conclusion that the smooth sphere model is unsuitable for planetary ring particles. We notice that one evidently must also take this conclusion into account in models for the accretional growth of satellites and planets, in which the smooth particle approximation is also used. The effect of a regolith can significantly increase the efficiency of adhesion of planetesimals and particles in a protosatellite swarm.

### 3.2. The collisional destruction of loose particles as a reason for the existence of rings

In this section, we shall examine theories of the origin of rings and the limitations which they impose on the mechanical properties of the particles. The question of the reasons for the existence of planetary rings is separated into two questions. 1) Why are the sizes of the particles in rings limited (or why do rings not gather into individual satellites)? 2) By what are the outer boundaries of the rings determined?

Below we shall understand "ring" to mean only primary rings of fairly large particles (see Sec. 2.1.1).

### 3.2.1. Discussion of the traditional point of view for the region of primary rings as being in the Roche zone

E. Roche examined the balance of tidal forces and selfgravitation for a satellite in 1849 and suggested the hypothesis that the rings of Saturn arose as a consequence of the destruction of a large body by tidal tension near the planet. H. Jeffreys showed nearly a century later ${ }^{48}$ that molecular adhesion is more significant than self-gravitation for small satellites, and found the additional condition for destruction:

$$
\begin{equation*}
P_{\mathrm{m}} \leqslant 1,68 \rho a^{2} \Omega^{2} \tag{18}
\end{equation*}
$$

where $P_{\mathrm{m}}$ is the strength of the material of a satellite with density $\rho$ and radius $a$, and $\Omega$ is the angular velocity of its orbital revolution. It follows from condition (18) that an icy body with a strength of $10^{7} \mathrm{dyn} / \mathrm{cm}^{2}$ will not be destroyed if its radius is $a<200 \mathrm{~km}$. Thus, ring particles having a considerably smaller size cannot be obtained by the tidal destruction of a large satellite. The catastrophic Roche hypothesis gave way to the condensation model, according to which the rings are the remnant of a circumplanetary protosatellite cloud. Tidal forces began to appear in the role of a factor which prevented the accretional growth of particles and the formation of satellites. The balance between the accretional growth (adhesion) of particles and their destruction by tidal forces upon reaching a maximum size of $\sim 10 \mathrm{~m}$ has been examined in Ref. 19. It is easy to find from condition (18) that 10 -meter-size particles are efficiently destroyed at a tensile rupture strength of the particle material of $\sim 0.01 \mathrm{dyn} /$
$\mathrm{cm}^{2}$; this is four to five orders of magnitude lower than the strength of loose terrestrial snow ${ }^{49}$ and of granulated lunar regolith. ${ }^{50}$ In order to avoid a conclusion about such anomalous mechanical properties for the particles, the hypothesis is stated in Ref. 19 that particles in rings reach a maximum size of a kilometer, and observers simply did not detect these bodies. Then, from condition (18), we obtain a particle strength of $\sim 100 \mathrm{dyn} / \mathrm{cm}^{2}$ that is completely natural for loose snow. But most researchers think that the upper limit for particle sizes has been reliably determined, and that there are no hypothetical kilometer-sized satellites in the rings. ${ }^{20}$

Thus, tidal forces are too weak to destroy even the loosest snow particles of Saturn's rings.

### 3.2.2. Collisional destruction of particles in tangentlal collisions

Let us examine a fundamentally different mechanism which limits both the sizes of the particles in the rings, and also the radial region of the existence of the rings themselves. The collisional destruction of ring particles has been investigated in Refs. 51 and 52. Particle growth will be limited if there occurs: A) fragmentation of a particle, or B) ejection of fragmented material beyond the particle's gravitational control. Fragmentation of a body is a question of the specific energy of destruction. Below we shall estimate the value of $\mathscr{C}_{v}$ at which mutual particle collisions are efficient for fragmenting material. The question is more complicated with the ejection of material from the particle's sphere of action. Large planetesimals collide in a protoplanetary disk with characteristic velocities $v_{G} \sim(G m / a)^{1 / 2} .{ }^{53}$ These velocities are very substantial for bodies with sizes of kilometers and tens of kilometers and lead to strong fragmentation of the planetesimals right up to catastrophic destruction. Nevertheless, the fragments cannot overcome the gravitation attraction of the body; indeed, for this, they must have velocities larger than the velocity $v_{\mathrm{G}}$, and, notwithstanding the fragmentation, planetesimals grow during mutual collisions. Why then can particle collisions in rings limit the accretional growth of particles? A hypothesis has been suggested in Ref. 51 that two physically different zones exist around a planet:
A. An inner zone of cosmogonically primary rings, where fragments of particles are ejected into space during a collision.
B. An outer satellite zone, where destruction is inefficient; the fragments quickly return to the "mother" particles. The increased state of destruction of particles in the rings is connected with the large value of the relative shear velocities of the particles as a consequence of the differential revolution of the rings. Here quasi-tangential collisions, when the semi-major axes of the orbits of the colliding bodies differ by approximately two particle radii, possess the maximum velocities. By comparing the shear velocity for the particles and the escape velocity from their surfaces, we obtain an expression for the boundary of the rings ${ }^{51}$

$$
\Omega a \approx\left(\frac{G m}{a}\right)^{1 / 2}
$$

consequently,

$$
\begin{equation*}
R_{\mathrm{c}}=\alpha\left(\frac{M}{\rho}\right)^{1 / 3} \tag{19}
\end{equation*}
$$

here $m$ and $M$ are the masses of the particle and of the planet,
respectively, and $\alpha \approx 1$. The Roche zone radius is also described by a similar equation ( $\alpha=1.5$ ), which is not surprising since, in both cases, a balance is considered between the particle's self-gravitation and effects connected with the planet's gravitation; tidal forces or shear velocities. Naturally , the numerical coefficients $\alpha$ must be different.

We now examine the problem of the motions of fragments in a field of colliding particles (and, of course, of the planet itself) more rigorously, following Ref. 52.

### 3.2.3. The motion of fragments which are formed during destruction of particies

We make the following assumptions. 1). Fragments have negligible mass and do not collide with each other. 2). After impact, the large particles move in a single plane along elliptical orbits. 3). The masses of the large particles are small in comparison with the planet mass. 4). The gravitational fields of the large particles are spherically symmetric.

We write the equations for the dynamics of an individual fragment in the $x, y, z$ rotating coordinate system, where the $x$ axis is directed along the radius, the $y$ axis is directed along the vector of the orbital motion of the particles (see Fig. 7), and the origin is the center of the planet. We introduce the following units: the unit of length is $R$, the orbital radius, the unit of time is $T / 2 \pi$, where $T$ is the rotation period of the coordinate system, which coincides with the period of revolution of a body in an orbit of radius $R$, and the unit of mass is the planet mass. The angular velocity of the coordinate system and the gravitational constant are equal to one in this system of units. The system of equations takes the form ${ }^{54}$

$$
\begin{align*}
& \ddot{x}=2 \dot{y}+x+\frac{\partial}{\partial x} \psi(x, y, z)  \tag{20}\\
& \ddot{y}=-2 \dot{x}+y+\frac{\partial}{\partial y} \psi(x, y, z)  \tag{21}\\
& \ddot{z}=\frac{\partial}{\partial z} \psi(x, y, z) \tag{22}
\end{align*}
$$

here


FIG. 7. The flying apart of particles which collided in the $(x, y, z)$ rotating coordinate system at the moment $t=0$. The system rotates counterclockwise, and the planet is situated down below. $a$ is the radius of a particle, and $2 u s \cdot a$ is the initial distance between the centers of the particles along the $y$ axis ( $u s=1$ in the figure).

$$
\psi=\frac{1}{r_{1}}+\frac{m_{2}}{r_{2}}+\frac{m_{3}}{r_{3}}
$$

is the gravitational potential of the planet and of the two large particles with masses $m_{2}$ and $m_{3} ; r_{1}, r_{2}$, and $r_{3}$ are the distances from a fragment to the planet and the large particles, which are $r_{1}=\left(x^{2}+y^{2}+z^{2}\right)^{1 / 2}, r^{2}=\left(x-x_{2}\right)^{2}$ $\left.+\left(y-y_{2}\right)^{2}+z^{2}\right]^{1 / 2}$, and $r_{3}=\left[\left(x-x_{3}\right)^{2}+\left(y-y_{3}\right)^{2}\right.$ $\left.+z^{2}\right]^{1 / 2}$, respectively; $x_{2}(t), y_{2}(t)$ and $x_{3}(t), y_{3}(t)$ are the specified coordinates of the centers of the large particles.

We solve the system of Eqs. (20), (21), and (22) numerically by means of an implicit second order accuracy method (see Ref. 55, for example). The difference equations are available in Ref. 52. An initial uniform network of fragments in the $x, y$ plane is depicted in Fig. 7. The initial velocity of the fragments corresponds to circular Keplerian motion. The calculation showed that the trajectories of individual fragments depend in a very strong way on their position with respect to the bodies at the initial moment of time.

The regions of the initial coordinates of fragments that are captured are shown in Fig. 8.

The different symbols correspond to the different lifetimes for a fragment until it contacts the surface of one of the particles: 1 is up to $1 / 3$ of a revolution, 2 is up to $2 / 3$ of a revolution, 3 is up to 1 revolution, and 4 is up to 2 revolutions. The first and second symbols refer to fragments that are captured by the upper particle (the vertical hatching) and the lower particle (the horizontal hatching), respectively. The following values of the parameter $\alpha=R(\rho / M)^{1 / 3}$ correspond to Figs. 8a through 8f: $0.65,0.81,1.05,1.09$, 1.29 , and 1.62. Four basic classes of fragments are marked by the numbers 1,2,3, and 4 in Fig. 8c: rapidly accreting, slowly accreting, uncaptured leading, and uncaptured lagging, respectively. The initial positions of those fragments which will again be captured by the large bodies in the course of two revolutions are marked by the different symbols. The uncaptured small particles start their motion from a region where there are no symbols. The sequence of figures from 8 a to 8 f corresponds to decreasing influence of the planet. One may achieve this by increasing both $R$, the distance from the planet, and also the mass ratio of the particles to the planet $m / M$. One can reduce the dependence on these quantities of the dynamical pattern of the scattering of fragments to a single parameter: $\alpha=R(\rho / M)^{1 / 3}$; see Eq. (19). Figures 8a through 8 f have been obtained for $\alpha=0.65,0.81,1.05,1.09$, 1.29 , and 1.62 , respectively. According to the data of Sec. 3.3, $\alpha=0.82 \mp 0.05$ at the outer boundaries of planetary rings. The fragments are divided into four classes:

1. Rapidly accreting fragments which returned to the particles practically at once (in the course of $1 / 3$ of a revolution), i.e., those which came in contact with their surfaces. The regions of the initial coordinates of such fragments are marked by the number 1 in Fig. 8c.
2. Slowly accreting fragments, which are captured by the particles after more than $1 / 3$ of a revolution. Regions of such fragments are indicated by the number 2 in Fig. 8c.
3. Uncaptured leading fragments, which entered orbits with semi-major axes larger than $R+a$ or smaller than $R-a$, as a result of which the fragments move away from the site of the collision faster than the large bodies and are not captured by them. Circular motion was assumed for the



FIG. 9. The motion of fragments in the fields of the planet and of the particles being destroyed. (a) Trajectories of an individual fragment at different distances to the planet. The trajectory numbers $1,2,3$, and 4 correspond to the values of the parameter $\alpha=0.81,1.05,1.09$, and 1.29 , respectively. The initial coordinates of the relatively large bodies are the same. (b) The shape of the fragment cloud after $2 / 3$ revolution. Parameter values are $\alpha=1.05$ and $u s=1$. Leading uncaptured particles are marked by " $x$ "'s, and lagging uncaptured ones by plus signs; the outer symbols denote fragments that are captued before two revolutions. The dark or light symbols refer to fragments landing on the dark or light particle, respectively.
large bodies in obtaining Fig. 8. Regions of leading fragments are marked by the number 3 in Fig. 8c.
4. Uncaptured lagging fragments have semi-major axes of their new orbits in the range from $R+a$ to $R-a$, and therefore they lag behind the large particles. The region of the initial coordinates of these fragments is denoted by the number 4 in Fig. 8c.

The trajectories of individual fragments are depicted in Fig. 9a. The initial values of their coordinates have been chosen at the same point in Figs. 8a, 8b, and 8c. Depending on the parameter $\alpha$, this test particle successively enters into all four classes of fragments, from uncaptured lagging to rapidly accreting. The tracks depicted in Fig. 9a are typical for each class of fragments. The shape of the cloud of fragments is shown in Fig. 9b over 2/3 of a revolution after the collision. Later on the cloud will be extended along the orbit.

Let us notice that, besides the four classes of fragments described, there exists a small number of quasi-periodic "wanderers"-small particles which, before settling onto the surfaces of the large bodies, succeed in making several voyages between the latter, being alternately accelerated near the faster particle and decelerated near the more distant and lagging one. The region of the initial coordinates for the "wanderer" fragments is situated in Fig. 8 c on the boundary between the regions of slowly accreting fragments and uncaptured lagging small particles.

### 3.2.4. Eficlency of destroying particles in collislons

The increase of the percentages of captured fragments with increasing orbital radius [or $\alpha=R(\rho / M)^{1 / 3}$ ] and time elapsed after the collision is depicted in Fig. 10. Captures have practically ceased after two revolutions. By assuming a conditional criterion of $50 \%$ for the efficiency of captures, one can obtain an upper estimate for the density of large particles in planetary rings. It is evident from Fig. 10 that, for a density of $0.9 \mathrm{~g} / \mathrm{cm}^{3}$, collisional destruction of particles
in the rings of Saturn is inefficient: more than $80 \%$ to $90 \%$ of the fragments which are formed are captured in the outer $A$ ring. We shall find for the two distances $a$ and $2 a$ (along the $y$ axis) between the large particles at the moment when the scattering of fragments begins that only particles with densities no more than 0.2 to $0.3 \mathrm{~g} / \mathrm{cm}^{3}$ are destroyed efficiently. The results are not changed significantly in a more complicated model, where the fragments move along three-dimensional trajectories; see Figs. 10b and 10d. Allowance for chaotic velocities (with a magnitude of $\approx 0.5 \Omega a$ ) for the small particles does not significantly increase the capture of fragments. The efficiency of destruction is increased significantly by taking into account changes of the trajectories of the large particles in the process of collision. The particles exchange angular momentum during approach and impact as a consequence of the gravitational and contact interactions. The particle further from the planet, having slower orbital revolution, is accelerated and goes into a still more distant orbit (in this lies the meaning of diffusion drifting apart of planetary rings), and this is accompanied by an increase in the particle's eccentricity; in the language of transport theory, this means a transfer of energy of orbital revolution to the energy of chaotic motion. The second particle is decelerated during approach and its orbital radius is reduced. During the exchange of momentum between the particles, the regions of the captured fragments change in the original coordinate space approximately the same way as for a decrease of $\alpha$ or approach to the planet (see Fig. 8).

Thus, a region of intensive collisional destruction of particles exists near a planet, and the efficiency of such destruction decreases sharply as one goes away from the planet; most fragments return to the particles that are destroyed.

### 3.2.5. The balance between accretlonal growth and colllslonal destructlon ${ }^{51}$

Destruction during quasi-tangential collisions reduces the volume of a large particle at the rate


FIG. 10. The dependence of the percentage of captured particles on orbital radius and the time counted. O is the percentage of particles captured in the first third of a revolution, + is the percentage of particles accreted after $2 / 3$ of a revolution, $\Delta$ is the value after one revolution, $X$ is the value after $4 / 3$ revolutions, and is the percentage accreted after two revolutions. (a) For this variant, $u s=0.5$, and the particle densities are $0.9 \mathrm{~g} / \mathrm{cm}^{3}$ and $0.2 \mathrm{~g} / \mathrm{cm}^{3}$. This is a two-dimensional case. (b) is a three-dimensional variant of the previous case, and the initial thickness of the cloud of fragments along the $z$ axis equals the radius $a$ of a body. Only the density of $0.2 \mathrm{~g} / \mathrm{cm}^{3}$ is considered, which corresponds to the lower distance scale. The values of the dimensionless parameter $\alpha$ are shown on the upper scale, (c) $u s=1$ for this variant, and the particle densities are 0.9 and 0.3 g / $\mathrm{cm}^{\prime}$ (the upper and lower curves, respectively). This is a two-dimensional case. (d) is a three-dimensional variant of the previous case for $0.3 \mathrm{~g} /$ $\mathrm{cm}^{3}$ density.

$$
\begin{equation*}
\left(\frac{d V}{d t}\right)^{-} \approx \delta V \omega_{c} \tag{23}
\end{equation*}
$$

where $\delta V$ is the volume of the layer that is swept off during a single collision (we assume that a spherical segment with height $H_{s}$ is swept off; then $\delta V=\pi a H_{s}^{2}$ ), and $\omega_{c}$ is the tangential collision frequency. For an estimate, we assume

$$
\begin{equation*}
\omega_{\mathrm{c}} \approx \frac{2 a H_{\mathrm{s}}}{a^{2}} \omega_{0} \approx \frac{2 H_{\mathrm{s}}}{a} \omega_{0}, \quad \omega_{0}=\frac{3 \Omega \sigma_{\mathrm{a}}}{\rho a}\left(1+\frac{i_{2}^{2}}{i^{2}}\right), \tag{24}
\end{equation*}
$$

where $\omega_{0}$ is the total collision frequency, $u$ is the relative velocity of the particles, $\sigma_{\mathrm{a}}$ is the surface density of the disk of large particles, and $v_{2}^{2}=2 \mathrm{Gm} / a$. Assuming that a volume $\delta V=m(\Omega a)^{2} / \mathscr{C}_{v}$ is destroyed in each tangential collision and allowing for the fact that $(d V / d t)^{-}=4 \pi a^{2}(d a / d t)^{-}$, from Eqs. (23) and (24) we find

$$
\begin{equation*}
\left(\frac{\mathrm{d} a}{\mathrm{~d} t}\right)^{-} \approx 2,3 \frac{\rho^{1 / 2}}{\mathscr{g}_{v}^{3 / 2}} \Omega^{4} a^{3} \sigma_{a}\left(1+\frac{v_{2}^{2}}{u^{2}}\right) \tag{25}
\end{equation*}
$$

The large particles increase their radii as a consequence of adhesions during central collisions and the accretion of small particles. This process is described by the law ${ }^{53}$

$$
\begin{equation*}
\left(\frac{\mathrm{d} a}{\mathrm{~d} t}\right)^{+} \approx \frac{\mathrm{o}_{t} \Omega}{4 \rho}\left(1+\frac{v_{2}^{2}}{u^{2}}\right), \tag{26}
\end{equation*}
$$

where $\sigma_{\mathrm{r}}$ is the surface density of the layer of accreting particles. By comparing expressions (25) and (26), we see that destruction is insignificant for small particles. We write the balance between accretion and fracturing $(d a / d t)^{+}=(d a /$ $d t)^{-}$in the form

$$
\begin{equation*}
\mathscr{E}_{v} \approx 4.4 \rho \Omega^{2} a^{2}\left(\frac{\sigma_{\mathrm{a}}}{\sigma_{\mathrm{r}}}\right)^{2 / 3} \tag{27}
\end{equation*}
$$

It is easy to notice an analogy between expressions (18) and (27), especially if one takes into account that ( $\sigma_{\mathfrak{a}}$ / $\left.\sigma_{r}\right) \sim 1$. It follows from expression (18) that tide destroys a ten-meter size particle if $P_{\mathrm{m}} \sim \rho \Omega^{2} a^{2} \sim 0.01 \mathrm{dyn} / \mathrm{cm}^{2}$; from expression (27), we find for a $\sim 10 \mathrm{~m} \mathscr{E}_{v} \sim \rho \Omega^{2} a^{2} \sim 0.01$ $\mathrm{erg} / \mathrm{cm}^{3}$. The numbers are similar, but the physics is fundamentally different. Condition (27) is considerably "softer" than condition (19), since the destruction energy depends on the sizes of the fragments into which a body is broken up and can be very small, whereas its strength cannot be less than some minimum value (as a consequence of autohesion). The specific destruction energy for a body that is broken up into cubic fragments of size $r_{0}$ is written in the form $\mathscr{C}_{v} \sim 6 \gamma / r_{0}$. The free surface energy is $\gamma \approx 100 \mathrm{erg} / \mathrm{cm}^{2}$ for monolithic ice; $\gamma$ is several orders of magnitude smaller for loose snow. Upon fragmentation, monolithic ice produces many micron-size fragments, which also leads to a large value for $\mathscr{C}_{v}$. If the particles of the rings of Saturn, which are loose clusters of snow, are fractured into centi-meter-sized fragments, then the energy $\mathscr{C}_{v}$ may be very small for them. Let us estimate what fraction of their material particles of different sizes can lose during one collision. From $H_{s} \sim(\delta V / \pi a)^{1 / 2}$ and $\delta V \sim m(\Omega a)^{2} / \mathscr{C}_{v}$, we find

$$
\begin{gather*}
\frac{H_{\mathrm{s}}}{a} \sim 2\left(\frac{\rho}{3 \mathscr{C}_{\mathrm{v}}}\right)^{1 / 2} \mathrm{C} a \sim 0.12-0.16 \text { for } a \sim 3-4 \mathrm{~m} \\
\sim 0.4 \text { for } a \sim 10 \mathrm{~m} \tag{28}
\end{gather*}
$$

here $\mathscr{E}_{v} \sim 0.01 \mathrm{erg} / \mathrm{cm}^{3}$, and $\rho \sim 0.12 \mathrm{~g} / \mathrm{cm}^{3}$ is the density of freshly fallen terrestrial snow. ${ }^{56}$ It is evident from expressions (28) that ten-meter-size particles are destroyed most efficiently (for a reduction of $\mathscr{E}, ~$ by several times, $H_{\mathrm{s}} / a \sim 1$, which indicates simply a catastrophic destruction), whereas only a part of their material is swept off from smaller particles. The material of the rings, which is shaken up many times during continuous fracturings, has, in the process of billions of years of evolution, evidently attained a state of maximum looseness. All the same, the existence of sturdier cores that are protected from fracturing by a thick "snow cover" is entirely possible.

### 3.2.6. The size distribution of particies

An unrealistic strength of material is not the only difficulty for the tidal model of ring formation. One also can not explain the observed spectrum of particle sizes in the framework of this model, in particular, the large number of small particles. Actually, tidal forces can break up a body into only two or three parts, which are stable against further destruction. Just how were the small fragments formed? Also the tidal model says nothing in connection with a specific law for the formation of a spectrum of very large particles.

Let us obtain a spectrum of particle sizes in the collisional destruction model. Until now we have actually looked at the growth and destruction of particles in a two-component model, with an accreting medium of surface density $\sigma_{r}$ and a large particle medium with density $\sigma_{\mathrm{a}}$. Let us now assume that the balance $(d a / d t)^{+}=(d a / d t)^{-}$is fulfilled in each interval of the distribution of the largest particles. This means that large particles of approximately a single size absorb their fraction of accreting particles of all radii, but are destroyed only in colliding with particles of similar size. It is difficult for smaller particles to destroy a larger particle, which effectively controls the parts of space near it (see Fig. 8). It is more likely that the smaller particle will be absorbed by the larger one. Setting $\sigma_{\mathrm{a}} \sim\left(\partial \sigma_{0} / \partial a\right) \Delta a$ in expression (25) and $\sigma_{\mathrm{r}} \sim \sigma=$ const in expression (26), where $\sigma$ is the total surface density of the rings, and setting expressions (25) and (26) equal, we obtain the size distribution for the largest particles [for $\mathscr{E}_{V}(a)=$ const]:

$$
\begin{equation*}
\frac{\partial \sigma_{\mathrm{a}}}{\partial a} \times a^{-3}, \frac{\partial n}{\partial a} \propto a^{-6}, \tag{29}
\end{equation*}
$$

where $n$ is the surface concentration of particles. This estimate agrees with observations, from which it follows that $\partial n / \partial a \propto a^{-5}$ to $a^{-6}$ in the $a>5 \mathrm{~m}$ range. In the small particle range $(a \leqslant 1 \dot{m})$, the ring particle distribution $\partial n / \partial a \propto a^{-3.3}$ to $a^{-3.5}$ agrees with the theoretical models of the distribution of the fragments which are formed as the result of fragmenting large bodies. ${ }^{53}$

Thus, the collisional destruction mechanism limits the sizes of the particles in planetary rings and determines the outer boundary of the rings. The model examined above enables one to estimate the mechanical characteristics of the particles, which are clusters of loose material (of snow in the case of the rings of Saturn), and also to explain the observed spectrum of the particle size distribution.

### 3.3. The azimuthal brightness asymmetry of the rings of Saturn

Thirty years ago, Camichel ${ }^{6}$ discovered the surprising phenomenon of azimuthal brightness asymmetry in the $\mathbf{A}$
ring of Saturn. A number of high-quality observations of ring asymmetry, both from the earth ${ }^{7}$ and also from the "Voyager" interplanetary spacecraft, ${ }^{57}$ exist at present. To explain this phenomenon, a number of hypotheses were suggested (see the review in Ref. 34) that are based on assumptions of synchronous rotation of particles, of particles with asymmetric shapes in the form of elongated ellipsoids pointed at small angles to their orbits, or of particles with asymmetric surface albedos. Synchronous rotation of the particles is unrealistic from the point of view of collisional dynamics, and a tilted orientation of ellipsoidal bodies is unstable. Therefore, the model (see Ref. 57) by which asymmetry of the rings is connected with spiral waves that are caused by the gravitational influence of the large particles is considered the one most preferred. There are no quantitative estimates of the contribution of this effect to the azimuthal asymmetry of the rings.

A mechanism for azimuthal brightness asymmetry that is connected with so fundamental a process for planetary rings as the collisional fracturing of large loose particles is considered in Ref. 58. A cloud of fractured material (see Fig. 9 b ) reflects sunlight well, but large particles diverging after a collision practically completely obstruct a cloud of small fragments both from the Sun and from an observer at definite orbital phase angles (Fig. 11). This is the main factor in the occurrence of the azimuthal asymmetry of the rings of Saturn; more dust and fragment clouds are evident in bright parts of the rings, and there are fewer in their dark parts. Let us take into account the idea that the collisional breaking up of large particles is the only source of the small particles and dust which make the main contribution to ring brightness. Since the time of the existence of a fragment cloud (up to its scattering and absorption by large particles) equals a few hours, as is also the free flight time of a small particle in the rings, one may assume that most small particles, which determine ring brightness, are grouped in the form of clouds which were formed in collisions of large bodies.

The azimuthal asymmetry of the rings of Saturn has been investigated in Ref. 58 for the case when the observer is on the same line as the Sun. This corresponds well to the case for terrestrial observations. ${ }^{6,7}$ The dispersion of a cloud of fragments is calculated for a three-dimensional variant, with


FIG. 11. The arrangement of clouds of dust and small particles with respect to the observer and the sun (down below). It is evident that a dust cloud is obstructed by a large particle (or is projected onto the particle, which also indicates no contribution to ring brightness) in the regions of brightness minimum. The occultation of clouds is small in regions of maximum brightness.
allowance for the thickness of the cloud of fragments along the $z$ axis (see Sec. 3.2).

Since there are clouds at different stages of dispersion and, consequently, with different contributions to asymmetry in each region of the rings, we consider several cloud shapes with times of development from time zero to one revolution. The area projected onto a plane perpendicular to the Saturn-Sun (observer) line is found for each shape of a cloud of fragments. The plane is divided into squares 40 cm on a side, which specifies the effective size of a single fragment. The projected area decreases for eclipses of the small particles by each other and by the large bodies. The projected area of the large particles is not allowed for, since these bodies create only a symmetric brightness background. The projections of the clouds at different stages of development are added up for each value of the angle. The dependences obtained for the projected area on the orbital azimuthal angle and on the other model parameters are compared with observations, since the projected area is proportional to optical thickness and, consequently, it also determines the brightness of the rings.

Theoretical curves for the projected areas of clouds are shown in Fig. 12. The form of a curve depends very strongly on the density of large particles, i.e., on the capability of the latter to disperse a cloud of small fragments. The curves for particles with densities about $0.15 \mathrm{~g} / \mathrm{cm}^{3}$ (for an initial distance of $2 a$ between the particle centers along the $y$ axis in Fig. 12) or about $0.1 \mathrm{~g} / \mathrm{cm}^{3}$ (for a distance of $1.5 a$ between the particle centers) agree best with the observations. ${ }^{7}$ Here the cloud thickness along the $z$ axis was assumed equal to $a$, the number of fragments in a single cloud was $\sim 1500$, and the large particles changed their velocities during the collision; they received the additional (to circular) velocities $\Delta V_{\mathrm{x}}=0.25 \Omega a$ and $\Delta V_{\mathrm{y}}=0.25 \Omega a$ (for the more distant particle; for the nearer one, the additional velocities are negative).

The theory gives the correct positions of minimum and maximum brightness ( $65^{\circ}$ and $160^{\circ}$ ) and the observed difference of the steepness of the wings (a curve increases from minimum more steeply in the region of small angles), and also the presence of a plateau (or minimum) near $80^{\circ}$ to $85^{\circ}$ (see Fig. 12b). It is evident from Fig. 12 that, in summing the contributions from clouds with a "long lifetime" (of the order of one revolution), the brightness minimum is shifted
from $60^{\circ}$ to $85^{\circ}$. The fact that the observed angles of minimum are close to $60^{\circ}$ indicates a lifetime for a cloud of fragments of about half a revolution, after which the cloud is dispersed or it becomes transparent because of adhesion of the fragments to each other. The increase of brightness asymmetry of the rings with decreasing angle of inclination is among the important observational facts. ${ }^{7}$ Calculation also gives this dependence of asymmetry on the angle of inclination; see Fig. 12, where cases $a$ and $b$ correspond to the inclination angles $11.5^{\circ}$ and $16.5^{\circ}$, and moreover, the asymmetry is larger in Fig. 12a. It is also shown in Ref. 58 that the contribution to brightness asymmetry from spiral waves caused by the gravitational influence of large particles is negligibly small.

The model of brightness asymmetry considered above shows how the external integral characteristics of the rings are determined by the elementary processes of the collisions of individual particles. A study of azimuthal asymmetry enables one to determine the ring particle density with high reliability and to calculate the coefficient in Eq. (19) which connects particle density and the outer radius of planetary rings: $\alpha=0.77$ to 0.88 .

The further development of theoretical models and the accumulation of ground-based and satellite observations of the phenomenon of azimuthal brightness asymmetry of the rings of Saturn can serve as a new, powerful method for the remote investigation of planetary rings, the study of many physical characteristics of the particles themselves and of the processes of their interactions.

## 4. THE HYDRODYNAMICS OF LARGE PARTICLES IN PLANETARY RINGS

One can understand the reasons for the formation of rings and azimuthal brightness variability by examining the dynamics of individual particles. But the layering of the rings of Saturn and the occurrence of other spatial structures of planetary rings are caused by collective processes, which one naturally studies in the framework of a hydrodynamical model where the "gas" of colliding large particles is described in the same way as an ordinary molecular gas. The results of Section 3 show that one can take a practically completely inelastic loose sphere of meter dimensions as a typical ring particle. Here one must take into account the gravitational fields of such particles, which play an important role


FIG. 12. Theoretical curves for the projected area of fragment clouds (in units of cells of the plane of projection) ${ }^{58}$ Fig. 12a corresponds to a ring inclination of $11^{\circ} 5$, and Fig. $12 b$ to a ring inclination of $16^{\circ} 5$. The large particle density is $0.15 \mathrm{~g} / \mathrm{cm}^{3}$. The times of evolution of the fragment clouds in fractions of a revolution are indicated near the curves. The small circles and plus signs correspond to Earth-based observations of the rings of Saturn through different filters. ${ }^{7}$
in the processes of collisions and fracturing of large bodies, and also the motions of the small fragments. The fact that the particles undergo continuous fragmentation and adhesion turns out to be insignificant for constructing the hydrodynamics, since the sum of the masses of particles of all sorts turns out to be constant. ${ }^{59}$ One may not talk about the applicability of hydrodynamics to planetary rings without indicating the characteristic scales and times for the processes being described; these must significantly exceed the free flight length and time, respectively, for a particle. These inequalities are fulfilled for the large-scale processes of interest to us; this will be shown in Sec. 5.4.

### 4.1. The transport equations for rotating media

To construct the hydrodynamics for planetary rings, one must obtain a system of moment equations from the kinetic equation for inelastic gravitating particles and close it by means of calculating the transport coefficients (for viscosity and thermal conductivity) by the methods of kinetic theory. ${ }^{59-64}$ Following Ref. 15, we first construct the hydrodynamics of a disk of gravitating elastic particles which rotates like a rigid body, and then we generalize it for the case of a differentially rotating disk of inelastic particles.

### 4.1.1. Obtaining the moment equations

The Lagrangian of a particle rotating in a potential field $\psi_{\mathrm{G}}$ has the form ${ }^{65}$

$$
\begin{equation*}
L=\frac{m}{2}(\mathrm{v}+\mathrm{w})^{2}-m \psi_{\mathrm{G}}, \tag{30}
\end{equation*}
$$

where $w$ is the velocity of the non-inertial reference system $A$, and $v$ is the particle's velocity with respect to $A$. We write the equation of motion:

$$
\begin{equation*}
\frac{\mathrm{d} v}{\mathrm{~d} t}=-\frac{\partial \mathbf{w}}{\partial t}-\nabla\left(\psi_{\mathrm{G}}-\frac{\mathbf{w}^{2}}{2}\right)+[\mathrm{v} \operatorname{rot} \mathrm{w}] . \tag{31}
\end{equation*}
$$

Introducing the notations,

$$
\begin{align*}
& \mathbf{e}=-\frac{\partial \mathbf{w}}{\partial t}-\nabla\left(\psi_{\mathrm{G}}-\frac{\mathbf{w}^{2}}{2}\right),  \tag{32}\\
& \mathbf{h}=\operatorname{rot} \mathbf{w}, \tag{33}
\end{align*}
$$

we obtain

$$
\begin{equation*}
\frac{\partial \mathrm{v}}{\partial t}=\mathrm{e}+[\mathrm{vh}] . \tag{34}
\end{equation*}
$$

The kinetic equation for the system under consideration will take the form

$$
\begin{equation*}
\frac{\partial f}{\partial t}+\mathrm{v} \frac{\partial f}{\partial \mathbf{r}}+(\mathbf{e}+[\mathbf{v h}]) \frac{\partial f}{\partial \mathbf{v}}=\hat{c}, \tag{35}
\end{equation*}
$$

where $\hat{c}$ is the collision integral. The kinetic equation (35) is analogous to the corresponding equation for a charged particle in an electromagnetic field. This analogy ${ }^{66}$ enables one to apply the methods of plasma physics to gravitating media. ${ }^{67.68}$ In the usual manner (see Ref. 63), we obtain the transport equations from Eq. (35) (the chaotic velocity $v_{1}=v-V$ has been introduced, and $T$ is measured in energy units):

$$
\begin{align*}
& \frac{\partial n}{\partial t}+\operatorname{div}(n \mathbf{V})=0 \\
& m n \frac{\mathrm{~d} V_{i}}{\mathrm{~d} t}=-\frac{\partial p}{\partial x_{i}}-\frac{\partial \pi_{i k}}{\partial x_{k}}+m n\left(-\frac{\partial \mathbf{w}}{\partial t}-\nabla\left(\psi \mathrm{G}-\frac{\mathbf{w}^{2}}{2}\right)\right. \\
&+[\mathbf{V} \text { rot } \mathbf{w}])_{i}, \tag{36}
\end{align*}
$$

$$
\frac{3}{2} n \frac{\mathrm{~d} T}{\mathrm{~d} t}+\rho \operatorname{div} \mathbf{V}=-\operatorname{div} \mathbf{q}-\pi_{i k} \frac{\partial V_{i}}{\partial x_{k}},
$$

where $q$ is the thermal flux vector and $\pi_{i k}$ is the viscous stress tensor;

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t}=\frac{\partial}{\partial t}+(\mathrm{V} \nabla), \quad \mathrm{q}=m n\left\langle\frac{\mathrm{v}_{1}^{2}}{2} \mathrm{v}_{1}\right\rangle \\
& \pi_{i k}=m n\left\langle v_{1 i} v_{1 k}-\frac{\mathbf{v}_{1}^{2}}{3} \delta_{i k}\right\rangle,  \tag{37}\\
& p=n T, \quad n=\int f^{(0)} \mathrm{dv} \\
& \mathbf{V}=\frac{1}{n} \int \mathrm{vf} f^{(0)} \mathrm{dv}, \quad T=\frac{1}{n} \int m \frac{\mathbf{v}_{1}^{2}}{3} f^{(0)} \mathrm{dv} . \tag{38}
\end{align*}
$$

All notations are standard; the angular brackets denote averaging over a distribution function.

### 4.1.2. The integro-differential equation for a non-equilibrium correction to a distribution function

We represent the kinetic equation in the form

$$
\begin{equation*}
(\hat{D} f)=\frac{1}{\varepsilon}(\hat{K} f) \tag{39}
\end{equation*}
$$

where $\varepsilon$ is a formal small parameter. We seek a distribution function in the form

$$
\begin{equation*}
f=f^{(0)}+\varepsilon f^{(1)}+\varepsilon^{2} f^{(2)}+\ldots \tag{40}
\end{equation*}
$$

With allowance for Eq. (40), from Eq. (39) we obtain the following series of approximations:

$$
\begin{align*}
& (\hat{K} f)^{(0)}=0,  \tag{41}\\
& (\hat{D} f)^{(0)}=(\hat{K} f)^{(1)} \\
& \ldots \ldots . . . \tag{42}
\end{align*}
$$

If Eq. (41) satisfies a Maxwellian distribution function, then one can construct a transport theory by the ChapmanEnskog method. ${ }^{59,60,61}$ We go over in Eq. (35) to the chaotic velocity: ${ }^{63}$

$$
\begin{align*}
\frac{\mathrm{d} f}{\mathrm{~d} t}+\mathrm{v}_{\mathbf{1}} \nabla f-\left[\frac{\partial \mathrm{w}}{\partial t}+\nabla( \right. & \left.\left.\psi_{\mathrm{G}}-\frac{\mathbf{w}^{2}}{2}\right)+[\operatorname{rot} \mathbf{w}, \mathbf{V}]+\frac{\mathrm{d} \mathbf{V}}{\mathrm{~d} t}\right] \nabla_{v_{1} f} f \\
& -\frac{\partial V_{i}}{\partial x_{k}} v_{1 k} \frac{\partial f}{\partial v_{1 i}}+\left[\mathbf{v}_{1} \operatorname{rot} \mathrm{w}\right] \nabla_{v_{1} f} f=\hat{C} . \tag{43}
\end{align*}
$$

We write Eq. (43) in a form analogous to Eq. (39):

$$
\begin{equation*}
(\hat{D} f)=\hat{C}-\left[\mathbf{v}_{1} \operatorname{rot} w\right] \nabla_{v_{1}} f \tag{44}
\end{equation*}
$$

The right-hand side of Eq. (44) equals zero if $f^{(0)}$ is a Maxwellian distribution:

$$
\begin{equation*}
f^{(0)}=n\left(\frac{m}{2 \pi T}\right)^{3 / 2} e^{-m v_{1}^{2 / 2 T}}, \tag{45}
\end{equation*}
$$

or

$$
\begin{equation*}
\ln f^{(0)}=\ln n-\frac{3}{2} \ln T-\frac{m v_{1}^{2}}{2 T}+\text { const. } \tag{46}
\end{equation*}
$$

In the zeroth order approximation, the system of equations has the form

$$
\begin{align*}
& \frac{\mathrm{d} n}{\mathrm{~d} t}=-n \nabla \mathbf{V}, \\
& \frac{\mathrm{~d} \mathbf{V}}{\mathrm{~d} t}=\mathscr{F}-\frac{1}{n m} \nabla n T, \tag{47}
\end{align*}
$$

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}=-\frac{2}{3} T \nabla \mathrm{~V} .
$$

Substituting Eq. (45) into the left-hand side of Eq. (44), with allowance for Eqs. (46) and (47), and after symmetrization, we obtain

$$
\begin{align*}
(\hat{D} f)^{(0)}= & f^{(0)}\left[\left(\frac{m v_{1}^{2}}{2 T}-\frac{5}{2}\right) \mathbf{v}_{1} \nabla \ln T\right. \\
& \left.+\frac{m}{2 T}\left(v_{1 i} v_{1 k}-\frac{\mathbf{v}_{1}^{2}}{3}\right) \delta_{i k} W_{t k}\right] \tag{48}
\end{align*}
$$

where

$$
\begin{equation*}
W_{l k}=\frac{\partial V_{i}}{\partial x_{k}}+\frac{\partial V_{k}}{\partial x_{i}}-\frac{2}{3} \delta_{i k} \operatorname{div} \mathbf{V} \tag{49}
\end{equation*}
$$

is the shear velocity tensor. We transform the right-hand side of Eq. (44). We represent a nonequilibrium distribution function in the form

$$
\begin{equation*}
f=f^{(0)}(1+\psi), \quad \psi \ll 1 . \tag{50}
\end{equation*}
$$

After linearization, the right-hand side of Eq. (44) will take the form

$$
\begin{equation*}
\hat{C}(\psi)-f^{(0)}\left[\mathbf{v}_{1} \operatorname{rot} w\right] \nabla_{v_{1}} \psi . \tag{51}
\end{equation*}
$$

The linearity of the equation for the correction and considerations of tensor invariance enable one to seek a solution in the form

$$
\begin{equation*}
\psi=\psi_{i}\left(\mathbf{v}_{1}^{2}\right) v_{1 i}+\psi_{i k}\left(v_{1 i} v_{1 k}-\frac{v_{1}^{2}}{3} \delta_{i k}\right) . \tag{52}
\end{equation*}
$$

The first vector term corresponds to the perturbing action of the temperature gradient, and the tensor term is connected with the shear velocity tensor $W_{i k}$. The components $\nabla \ln T$ and $W_{i k}$ are linearly independent, which enables one to calculate $\psi_{i}$ and $\psi_{i k}$ separately. One can find a detailed description of the solution in Refs. 62 and 63, therefore we shall show only a summary of the results. We notice that, for an expansion of the integral equations in Sonin polynomials, an infinite system of algebraic equations is obtained, which we truncate after the first two terms.

### 4.1.3. Summary of the results

In the general case, viscosity is expressed by a tensor of the fourth rank which has only five independent components. ${ }^{61}$ We introduce the following tensors $(x, y, z \rightarrow 1,2,3$; $\operatorname{rot} W \| z):{ }^{63}$

$$
\begin{align*}
& W_{o t h}=\left[\begin{array}{ccc}
\frac{1}{2}\left(W_{x x}+W_{y y}\right) & 0 & 0 \\
0 & \frac{1}{2}\left(W_{x x}+W_{y y}\right) & 0 \\
0 & 0 & W_{z z}
\end{array}\right],  \tag{53}\\
& W_{1 i k}=\left[\begin{array}{ccc}
-\frac{1}{2}\left(W_{x x}-W_{y y}\right) & W_{x y} & 0 \\
W_{x y} & \frac{1}{2}\left(W_{y y}-W_{x x}\right) & 0 \\
0 & 0 & 0-
\end{array}\right], \\
& W_{\mathbf{a} t k}=\left[\begin{array}{ccc}
0 & 0 & W_{x z} \\
0 & 0 & W_{y z} \\
W_{z x} & W_{z} & 0
\end{array}\right] ;
\end{align*}
$$

$$
\begin{aligned}
& W_{3 i k}=\left[\begin{array}{ccc}
-W_{x y} & \frac{1}{2}\left(W_{x x}-W_{y y}\right) & 0 \\
\frac{1}{2}\left(W_{x x}-W_{y y}\right) & W_{x y} & 0 \\
- & 0 & W^{-} \\
0 & 0 & -W_{y z} \\
0 & 0 & W_{x z} \\
-W_{z i y} & W_{z x} & 0
\end{array}\right] .
\end{aligned}
$$

Then one can write the viscosity tensor in the following form:

$$
\begin{equation*}
\pi_{i k}=-\eta_{0} W_{0 i k}-\eta_{1} W_{1 i k}-\eta_{2} W_{2 i k}+\eta_{3} W_{3 i k}+\eta_{4} W_{4 i k} \tag{54}
\end{equation*}
$$

where

$$
\begin{align*}
& \eta_{0}=\frac{b}{s} n T t_{c}, \quad \eta_{1}=\eta_{2}(2 x), \quad \eta_{2}=n T t_{\mathrm{c}} \frac{a x^{2}+b}{\Delta}, \\
& \eta_{3}=\eta_{4}(2 x), \quad \eta_{4}=n T t_{c} x \frac{x^{2}+c}{\Delta},  \tag{55}\\
& \Delta=x^{4}+g x^{2}+s, \quad x=|\operatorname{rot} \mathrm{w}| t_{\mathrm{c}}, \\
& a=\beta_{00}, \quad b=\frac{1}{\xi^{2}} \beta_{11}\left(\beta_{11} \beta_{00}-\beta_{10} \beta_{01}\right), \\
& c=\frac{1}{\xi^{2}}\left(\beta_{11}^{2}+\xi \beta_{01} \beta_{10}\right), \quad \xi=\frac{7}{2}, \\
& g=\frac{1}{\xi^{2}}\left(\beta_{11}^{2}+2 \xi \beta_{01} \beta_{10}+\xi^{2} \beta_{001}^{2}\right),  \tag{56}\\
& s=\frac{1}{\xi^{2}}\left(\beta_{11} \beta_{00}-\beta_{10} \beta_{01}\right)^{2} .
\end{align*}
$$

The thermal flux vector is written in the form
$\mathbf{q}=-\frac{n T t_{c}}{m}\left[\frac{b^{\prime}}{s^{\prime}} \Delta_{\|} T+\frac{a^{\prime} x^{2}+b^{\prime}}{\Delta^{\prime}} \Delta_{\perp} T \frac{x\left(x^{2}+c^{\prime}\right)}{\Delta^{\prime}}\left[\mathbf{h}_{1} \Delta T\right]\right]$,
where $h_{1}$ is the unit vector along rot $w$, and
$a^{\prime}=\frac{5}{2} \alpha_{11}, \quad b^{\prime}=\frac{5}{2} \frac{1}{\theta^{2}}-a_{22}\left(a_{22} \alpha_{11}-\alpha_{12} \alpha_{21}\right)$,
$c^{\prime}=\frac{5}{2} \frac{1}{\theta^{2}}\left(\alpha_{22}^{2}+\theta \alpha_{21} \alpha_{12}\right), \quad g^{\prime}=\frac{1}{\theta^{2}}\left(\alpha_{22}^{2}+2 \theta \alpha_{21} \alpha_{12}+\theta^{2} \alpha_{11}^{\prime}\right)$,
$s^{\prime}=\frac{1}{\theta^{2}}\left(\alpha_{12} \alpha_{21}-\alpha_{22} \alpha_{11}\right)^{2}, \quad \Delta^{\prime}=x^{4}+g^{\prime} x^{2}+s^{\prime}, \quad \theta=\frac{7}{4}$.
The matrix coefficients $\beta_{i k}$ and $\alpha_{i k}$ depend on the form of the collision integral. For the Landau integral, if one adopts the expression

$$
\begin{equation*}
t_{\mathrm{c}}=\frac{v_{1}^{3}}{4 \sqrt{3 \pi}} \frac{1 G^{2} m^{2} n}{}, \tag{59}
\end{equation*}
$$

as the definition of free flight time, where $\Lambda$ is the Coulomb logarithm ( $\Lambda \sim 1$ for planetary rings), then the matrices $\beta_{i k}$ and $\alpha_{i k}$ coincide with the corresponding matrices that have been calculated in plasma transport theory: ${ }^{62,63}$

$$
\begin{align*}
\alpha_{i k} & =\frac{4}{3}\left(\begin{array}{cccc}
0 & 0 & 0 & . \\
0 & 1 & 3 / 4 & . \\
0 & 3 / 4 & 45 / 16 & . \\
\cdots & \cdots & .
\end{array}\right), \\
\beta_{i k} & =\frac{6}{5}\left(\begin{array}{cccc}
1 & 3 / 4 & . \\
3 / 4 & 205 / 48 & . \\
\cdots & \cdots & .
\end{array}\right) \tag{60}
\end{align*}
$$

Then, for the numerical coefficients of Eqs. (56) and (58), we obtain

$$
\begin{aligned}
& a=1,2, \quad b=2,23, \quad c=2,38, \quad g=4,05, \quad s=2,33, b / s=0,96, \\
& \\
& \\
& a^{\prime}=2,0, \quad b^{\prime}=2,645, \quad c^{\prime}=4,65, \quad g^{\prime}=2,70, \\
& \\
& s^{\prime}=0,677, \quad b^{\prime} s^{\prime}=3,906 .
\end{aligned}
$$

Thereby, we expressed $\pi_{i k}$ and $q$ in terms of the macroscopic variables $n, V$, and $T$ and closed the system of transport equations (36). The system of transport equations that has been obtained is analogous to the equations for a plasma, ${ }^{62}$ and the linear oscillations of rotating disks of elastic particles are similar to the normal modes of a plasma in an electromagnetic field. ${ }^{69}$

### 4.1.4. The generalization of transport theory to the case of differentially rotating disks of inelastic particles

We write a kinetic equation for a differentially rotating medium of inelastic particles (in cylindrical coordinates; see Refs. 70 and 71)

$$
\begin{align*}
\frac{\partial f}{\partial t} & +v_{r} \frac{\partial f}{\partial r}+\left(\frac{\partial \psi}{\partial r}+\Omega^{2} r\right) \frac{\partial f}{\partial v_{r}}+\left(\frac{v_{\varphi}^{2}}{r} \frac{\partial}{\partial v_{r}}-\frac{v_{\varphi} v_{r}}{r} \frac{\partial}{\partial v_{\varphi}}\right) f \\
& +\left[2 \Omega v_{\varphi} \frac{\partial}{\partial v_{r}}-\left(2 \Omega+r \frac{\partial \Omega}{\partial r}\right) v_{r} \frac{\partial}{\partial v_{\varphi}}\right] f=\dot{\hat{C}}_{N}(f, f) \tag{61}
\end{align*}
$$

where $\widehat{C}_{N}$ is the collision integral with allowance for inelasticity. Trulsen ${ }^{72}$ obtained $\widehat{C}_{N}$ for the case of nongravitating inelastic spheres, and Shukhman ${ }^{73}$ obtained it for spheres with spin and finite dimensions. We obtain the following series of approximations to calculate the distribution function for particles in the case of a medium with an arbitrary collision frequency; see Eqs. (41) and (42):

$$
\begin{align*}
& {\left[2 \Omega v_{\varphi} \frac{\partial}{\partial v_{r}}-\left(2 \Omega+r \frac{\partial \Omega}{\partial r}\right) v_{r} \frac{\partial}{\partial v_{\varphi}}\right] f^{(0)}-\hat{C}_{N}\left(f^{(0)}, f^{(0)}\right)=0,}  \tag{62}\\
& \begin{aligned}
\hat{D} f^{(0)}+\left[2 \Omega v_{\varphi} \frac{\partial}{\partial v_{r}}-(2 \Omega+\right. & \left.\left.r \frac{\partial \Omega}{\partial r}\right) v_{r}-\frac{\partial}{\partial v_{\varphi}}\right] f^{(1)} \\
& -\hat{C}_{N}\left(f^{(1)}, f^{(0)}\right)-\hat{C}_{N}\left(f^{(0)}, f^{(1)}\right)=0,
\end{aligned}
\end{align*}
$$

The solution of Eq. (62) is unknown. It has been shown in Refs. 35 and 72 that the distribution function corresponding to the solution of Eq. (62) is anisotropic [in the maximally anisotropic case, the ratios of the thermal velocity components are: ${ }^{35}\left(v_{1 \varphi} / v_{1 r}\right)=0.5$, and $\left.\left(v_{1 z} / v_{1 r}\right)=0.65\right]$. This poses serious difficulties for the use of the Chapman-Enskog method.

The Gaussian function is used in Ref. 35 as an anisotropic distribution function. As a result, the scalar energy equation was transformed into a tensor equation. Closure of such a system of transport equations is carried out in Ref. 35 by dropping a tensor of the third rank which, in the isotropic case, corresponds to the thermal flux vector, from the energy equation. As will be shown below, the thermal flux plays an important role in the stability of rings; therefore, such a simplification is undesirable. On the other hand, the system of equations which was obtained remains very cumbersome and of little use for analyzing collective processes in rings. For example, the conversion of the scalar energy equation to a tensor equation increases the number of terms in the dis-
persion equation by 30 times, and it is already fairly complicated (see Paragraph 4.2.2).

Numerous investigations of the dynamics of rings ${ }^{11-15,18,35,37,38}$ and of a protoplanetary disk ${ }^{47,53}$ have not revealed a single collective process connected with the anisotropy of thermal velocities. The desire arises to obtain a system of transport equations that is suitable for use by neglecting the anisotropy of the distribution function. Let us consider the conditions under which one may assume a distribution function to be isotropic. It is obvious that the restitution coefficient for the snow particles of the rings of Saturn is close to zero, and the gravitational interactions of the particles perform the role of elastic collisions. If one does not take into account the influence of the planet on the gravitational cross sections of the particles, then one can show that the frequency of gravitational collisions will be several times higher than the frequency of contact collisions. ${ }^{71,74}$ Consequently, one may write: $\widehat{C}_{N}=\widehat{C}_{G}+\widehat{C}$. and $\widehat{C}_{G} \gg \hat{C}_{*}$, where $\widehat{C}_{\mathrm{G}}$ is the integral of elastic collisions (gravitational collisions for planetary rings), and $\widehat{C}$. is the integral of inelastic interactions (contact ones for the rings).

Two variants are given for an equilibrium Maxwell distribution function:
I. The case of frequent quasielastic collisions. $\Omega t_{\mathrm{c}} \ll 1$ and $\hat{C}_{\mathrm{G}} \gg \widehat{C}$. A typical example is a gaseous disk with partially inelastic collisions of molecules (protoplanetary or protosatellite clouds). In this case, Eq. (62) is reduced to the classical form: $\widehat{C}_{G}=0$.
II. The case of slight differential rotation: $r \Omega^{\prime} \ll 2 \Omega$. Let us examine this variant in more detail, since one can also obtain Case I from it, in the limit $\Omega t_{c} \ll 1$ by transferring the first term of Eq. (62) from the zeroth order approximation to the first order approximation, to Eq. (63), which removes the limitation on the degree of differential rotation.

Setting $r \Omega^{\prime} \ll 2 \Omega$ and $\widehat{C}_{\mathrm{G}} \gg \hat{C}$., we transfer the terms which correspond to differential rotation and inelastic collisions from the zeroth order approximation to the first order. We obtain the following series of approximations:

$$
\begin{gather*}
\left(2 \Omega v_{\varphi} \frac{\partial}{\partial v_{r}}-2 \Omega v_{r} \frac{\partial}{\partial v_{\varphi}}\right) f^{(0)}-\hat{C}_{G}\left(f^{(0)}, f^{(0)}\right)=0,  \tag{64}\\
\hat{D}^{(0)}+\left(2 \Omega v_{\varphi} \frac{\partial}{\partial v_{r}}-2 \Omega v_{r} \frac{\partial}{\partial v_{\varphi}}\right) f^{(1)}-\hat{C}_{G}\left(f^{(1)}, f^{(0)}\right) \\
-\hat{C}_{G}\left(f^{(0)}, f^{(1)}\right)-\hat{C}_{*}\left(f^{(0)}, f^{(0)}\right)=0 . \tag{65}
\end{gather*}
$$

A Maxwellian distribution function will be the solution of Eq. (64) [the first term of Eq. (64) equals zero for any isotropic functions $\left.f\left(v_{1}^{2}, v_{z}\right)\right]$. We use the Chapman-Enskog method to solve Eqs. (64) and (65), allowing for the fact that energy is not conserved in a collision of inelastic particles. Therefore a nonzero third moment from the integral of collisions of inelastic particles appears in the moment equation for energy:

$$
\sigma E^{-}=\int \frac{m \mathbf{v}_{1}^{2}}{2} C . d v
$$

This same term remains also in the zeroth order approximation energy equation [see Eq. (47)].

In deriving from Eq. (65) the equation for the correction to the steady-state distribution function, we shall obtain not only vector and tensor terms as before [see Eq. (48)], but also scalar terms because of the appearance in Eq. (65)
of the term $\hat{C} .\left(f^{(0)}, f^{(0)}\right)$ and also due to the appearance in Eq. (47) for the energy of the system of a term with $\sigma E^{-}$. The general form of the correction to the distribution function will have the following appearance:

$$
\begin{equation*}
\psi=\psi_{0}+\psi_{i} v_{1 i}+\psi_{i k}\left(v_{1 i} v_{1 k}-\frac{v_{1}^{2}}{3} \delta_{i k}\right) . \tag{66}
\end{equation*}
$$

Since the terms in Eq. (66) are linearly independent, then, for each form of the correction (scalar, vector, or tensor), we obtain separate equations that are solvable independently . The vector and tensor corrections are calculated the same way as before. The integral equation for the scalar correction is anaiogous to the equation for the correction determined by the transfer of energy between the components in a nonequilibrium plasma. ${ }^{61}$ Just as in plasma theory, it is not necessary to solve this equation, since the scalar correction caused by the inelasticity of the collisions does not affect the expression for the vector and tensor corrections which determine the coefficients of viscosity and thermal conductivity. Thus, the transport coefficients for a differentially rotating disk of gravitating particles are specific of Eqs. (53) through (60).

### 4.2. The linear oscillations of differentially rotating disks of large inelastic particles

In this section we obtain a general dispersion equation for the linear oscillations of disk systems with inelastic particle collisions.

### 4.2.1. The basic equations

We write in a cylindrical coordinate system a system of transport equations with the calculated $\pi_{i k}$ and $q$.

Of the five viscosity coefficients, only three remain in the two-dimensional case of interest to us: $\eta_{0}, \eta_{1}$, and $\eta_{3}$. We neglect terms with $\eta_{3}$ viscosity coefficients, since, upon perturbation, they give terms that are small in comparison with the $\Omega \widehat{V}_{\varphi}$ and $\Omega \widehat{V}_{r}$ terms. The $\eta_{0}$ coefficient actually gives only a factor $4 / 3$ for the term with viscosity $\eta_{1}$ in the equation for the radial velocity component. In connection with differential rotation, the shear viscosity coefficient in the equations of motion cannot even be removed from behind the sign of a space derivative and must undergo perturbation along with the other terms; this is a consequence of its dependence on the temperature $T$ and density $\sigma$.

Terms connected with the perturbation of $E^{-}$and of the viscosity coefficient in the term $E^{+}=v\left(r \Omega^{\prime}\right)^{2}$, which describes the transfer of energy of orbital revolution to the energy of chaotic motion, are added to the energy equation.

The system of transport equations for planetary rings will take the form

$$
\begin{align*}
& \frac{\partial \sigma}{\partial t}+\frac{\partial}{r \partial r}\left(r \sigma V_{r}\right)+\frac{\partial}{r \partial \varphi}\left(\sigma V_{\varphi}\right)=N^{+}(\sigma, T)-N^{-}(\sigma, T)  \tag{67}\\
& \frac{\partial V_{r}}{\partial t}+V_{r} \frac{\partial}{\partial r} V_{r}+V_{\varphi} \frac{\partial}{r \partial \varphi} V_{r}-\frac{V_{\varphi}^{2}}{r}=-\frac{1}{\sigma} \frac{\partial p}{\partial r}-\frac{\partial \psi_{G}}{\partial r} \\
& +\frac{4}{3} v \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r V_{r}+\frac{4}{3 \sigma} \frac{\partial v \sigma}{\partial r}\left(\frac{\partial V_{r}}{\partial r}-\frac{1}{2} \frac{V_{r}}{r}\right) \\
& \quad-\frac{2}{3 \sigma} \frac{1}{r} \frac{\partial}{\partial r}\left(v \sigma \frac{\partial V_{\varphi}}{\partial \varphi}\right) \\
& \quad-\frac{4}{3} v \frac{1}{r^{2}} \frac{\partial V_{\varphi}}{\partial \varphi}+\frac{1}{r \sigma} \frac{\partial}{\partial \Phi}\left[v \sigma\left(\frac{1}{r} \frac{\partial V_{r}}{\partial \varphi}+r \frac{\partial}{\partial r} \frac{V_{\varphi}}{r}\right)\right] \tag{68}
\end{align*}
$$

$$
\begin{array}{r}
\frac{\partial V_{\varphi}^{\circ}}{\partial t}+V_{r} \frac{\partial V_{\varphi}}{\partial r}+\frac{V_{r} V_{\varphi}}{r}+V_{\varphi} \frac{\partial}{\partial \varphi} V_{\varphi}=-\frac{\partial \rho}{\sigma r \partial \varphi}-\frac{\partial \psi_{G}}{r \partial \varphi} \\
+\frac{1}{\sigma r^{2}} \frac{\partial}{\partial r} r^{2} v \sigma \frac{\partial}{\partial r} \frac{V_{\varphi}}{r}+\frac{1}{\sigma r} \frac{\partial}{\partial r}\left(\nu \sigma \frac{\partial}{\partial \varphi} V_{r}\right)+\nu \frac{\partial}{r^{2} \partial \varphi} V_{r} \\
+\frac{1}{\sigma r} \frac{\partial}{\partial \varphi}\left[\frac{4}{3} v \sigma\left(\frac{1}{r} \frac{\partial V_{\varphi}}{\partial \varphi}+\frac{V_{r}}{r}-\frac{1}{2} \frac{\partial V_{r}}{\partial r}\right)\right]
\end{array}
$$

$$
\begin{align*}
\frac{3}{2}\left(\frac{\partial T}{\partial t}+\right. & \left.V_{r} \frac{\partial}{\partial r} T+V_{\varphi} \frac{\partial}{r \partial \varphi} T\right)+T\left(\frac{1}{r} \frac{\partial}{\partial r} r V_{r}+\frac{1}{r} \frac{\partial V_{\varphi}}{\partial \varphi}\right)  \tag{69}\\
& =\frac{1}{\sigma r} \frac{\partial}{\partial r}\left(\chi r \frac{\partial T}{\partial r}\right)+\frac{1}{\partial r^{2}} \frac{\partial}{\partial \varphi}\left(\chi \frac{\partial T}{\partial \varphi}\right)+v\left(r \frac{\partial}{\partial r} \frac{V_{\varphi}}{r}\right)^{2} \\
& +2 v\left[\left(\frac{\partial V_{r}}{\partial r}\right)^{2}+\left(\frac{V_{r}}{r}\right)^{2}\right]+2 v\left[\left(\frac{1}{r} \frac{\partial V_{\varphi}}{\partial \varphi}\right)^{2}+2 \frac{V_{r}}{r^{2}} \frac{\partial V_{\varphi}}{\partial \varphi}\right. \\
& \left.+\frac{1}{2}\left(\frac{\partial V_{r}}{r \partial \varphi}\right)^{2}+\frac{\partial V_{r}}{\partial \varphi}\left(\frac{\partial}{\partial r} \frac{V_{\varphi}}{r}\right)\right]-\frac{2}{3} v\left[\left(\frac{1}{r} \frac{\partial}{\partial r} r V_{r}\right)^{2}\right. \\
& \left.+2 \frac{1}{r^{2}}\left(\frac{\partial}{\partial r} r V_{r}\right) \frac{\partial V_{\varphi}}{\partial \varphi}+\left(\frac{1}{r} \frac{\partial V_{\varphi}}{\partial \varphi}\right)^{2}\right]-\sigma E^{-}, \tag{70}
\end{align*}
$$

here $v=\eta_{1} / \sigma, \psi$ is the coefficient of thermal conductivity [see Eq. (57)], $\sigma=\int n m d z$ is the surface density of the disk, $p=\sigma T \approx \sigma v_{1}^{2} / 3$, and $\psi_{\mathrm{G}}$ is the gravitational potential. The fact that the density of the disk can change not only during diffusion motions of disk particles but also during external (or non-diffusion) flows of material, for example during accretion of material from a gas and dust cloud at an early stage, is allowed for in the equation of continuity, i.e., the possibility of mass exchange with some reservoir of material is assumed for the disk: the function $N^{+}(\sigma, T)$ describes an increase and $N^{-}(\sigma, T)$ a decrease of the mass of the disk.

### 4.2.2. A dispersion equation for linear oscillations

Let us write for a perturbation of the form $\exp (-i \omega t$ $+i k r+i m \varphi$ ) a system of linearized transport equations in the Wenzel-Kramers-Brillouin approximation (the wavelength is considerably shorter than the characteristic scales of the disk: $\lambda \ll r$ or $k r \gg 1$ ):

$$
\begin{align*}
& (\gamma+i m \Omega) \hat{\sigma}+i k \sigma_{0} \hat{V}_{r}=\left(\frac{\partial N^{+}}{\partial \sigma_{\mathfrak{w}}}-\frac{\partial N^{-}}{\partial \sigma_{0}}\right) \hat{\sigma}+\left(\frac{\partial N^{+}}{\partial T_{0}}-\frac{\partial N^{-}}{\partial T_{n}}\right) \hat{T}, \\
& (\gamma+i m \Omega) \hat{V}_{r}-2 \Omega \hat{V}_{\varphi}=i \frac{2 \pi G \sigma_{0}-k c^{2}}{\sigma_{0}} \hat{\sigma}-i k \hat{T}-\frac{4}{3} v k^{2} \hat{V}_{r}, \\
& (\gamma+i m \Omega) \hat{V}_{\varphi}+\frac{\chi^{2}}{2 \Omega} \hat{V}_{r}=-v k^{2} \hat{V}_{\varphi}-i k \alpha \hat{T}-i k \hat{\sigma} \hat{\sigma} \\
& \frac{3}{2}(\gamma+i m \Omega) \hat{T}+i k c^{2} \hat{V}_{r}=-\chi k^{2} \hat{T}-\Delta E_{\sigma} \hat{\sigma}-\Delta E_{r} \hat{T}-i k \mu \hat{V}_{\varphi} \tag{71}
\end{align*}
$$

where

$$
\begin{align*}
& \alpha=\frac{\partial v}{\partial T_{0}}\left(-r \Omega^{\prime}\right), \quad \beta=\frac{1}{\sigma_{0}} \frac{\partial v \sigma_{0}}{\partial \sigma_{0}}\left(-r \Omega^{\prime}\right), \\
& \mu=2 v\left(-r \Omega^{\prime}\right), \quad \Delta E_{\sigma}=\frac{1}{\sigma_{0}}\left(\frac{\partial E^{-} \sigma_{0}}{\partial \sigma_{0}}-\frac{\partial E^{+} \sigma_{0}}{\partial \sigma_{0}}\right),  \tag{72}\\
& \Delta E_{T}=\frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial E^{+}}{\partial T_{0}}, \quad \gamma=-i \omega, \quad c^{2} \equiv T_{0}
\end{align*}
$$

The amplitudes of perturbed quantities are marked by " $\Lambda$ ". $E^{+}=E^{-}$in a steady state, and also dynamic equilibrium exists along the $z$ (the thickness of the disk $h \approx c / \Omega$ ) and $r$ coordinates. By setting the determinant of the system of Eqs. (71) equal to zero, we shall obtain the dispersion equation for the linear oscillations of a differentially rotating disk of
inelastic particles (without allowance for non-diffusion flows, i.e., $N^{+}=N^{-}=0$ ):

$$
\begin{align*}
& (\gamma+i m \Omega)^{4}+(\gamma+i m \Omega)^{3}\left[\frac{2}{3}\left(\chi k^{2}+\Delta E_{r}\right)+\frac{7}{3} v k^{2}\right] \\
& \quad+(\gamma+i m \Omega)^{2}\left[\frac{4}{3} v^{2} k^{4}+\omega_{0}^{2}+\frac{14}{9} v k^{2}\left(\Delta E_{T}+\chi k^{2}\right)+\frac{2}{3} k^{2} \alpha \mu\right] \\
& \quad+(\gamma+i m \Omega)\left[v k^{2}\left(\frac{5}{3} k^{2} c^{2}-2 \pi G \sigma_{k} k\right)\right. \\
& \quad+\frac{2}{3}\left(\frac{4}{3} v^{2} k^{4}+\omega^{2}\right)\left(\chi k^{2}+\Delta E_{T}\right) \\
& \left.\quad+\frac{8}{9} \alpha \mu v k^{4}+\frac{4!}{3} k^{2} c^{2} \Omega \alpha-\frac{2}{3} k^{2} \sigma_{0} \Delta E_{\sigma}-\frac{\chi^{2}}{3 \Omega} k^{2} \mu+k^{2} \beta \cdot 2 \Omega \sigma_{0}\right] \\
& \quad+\frac{2}{3}\left(\left(\chi \nu k^{4}+v k^{2} \Delta E_{T}+k^{2} \alpha \mu\right)\left(k^{2} c^{2}-2 \pi G \sigma_{0} k\right)\right. \\
& \left.+\sigma_{0} k^{4}\left(\chi \beta \cdot 2 \Omega-\beta \mu-\Delta E_{\sigma} v\right)+k^{2} \sigma_{0} 2 \Omega\left(\Delta E_{T} \beta-\Delta E_{\sigma} \alpha\right)\right]=0, \tag{73}
\end{align*}
$$

where

$$
\begin{aligned}
& \omega_{0}^{2}=\frac{5}{3} k^{2} c^{2}-2 \pi G \sigma_{0} k+x^{2}, \\
& \omega_{0}^{2}=k^{2} c^{2}-2 \pi G \sigma_{0} k+x^{2} .
\end{aligned}
$$

An investigation of Eq. (73) will be carried out in the following section.

## 5. COLLECTIVE INSTABILITIES AND STRUCTURES IN THE RINGS OF PLANETS

### 5.1. The physics of instabilities

### 5.1.1. Grav/tational instablity

If one neglects the dissipative effects in ring dynamics, then the dispersion Eq. (73) is converted to the equation for Jeans (gravitational) oscillations: ${ }^{53}$

$$
\begin{equation*}
\omega_{0}^{2}=\frac{5}{3} k^{2} c^{2}-2 \pi G \sigma_{0} k+x^{2} \tag{74}
\end{equation*}
$$

Jeans instability sets in for $\omega_{0}^{2}<0$. Let us examine the physics of this instability and the condition for stability. Let the original disk be infinitely thin. We choose one of the rings into which we broke up the originally uniform disk of particles. Let a test particle of unit mass be located at a distance $\delta$ from the closest point of the ring with width $d$, and moreover, $\delta \gg d$. Then one can consider the ring as an infinitely thin gravitating filament whose potential is $\psi \sim \ln (1 / \delta)$, and the attractive force for the test particle is: $\partial \psi / \partial \delta \sim 1 / \delta \rightarrow \infty$ as $\delta \rightarrow 0$. Obviously, the last condition can be fulfilled only for an infinitely narrow ring, $d \rightarrow 0$, which is, in principle, allowed by the approximation of an infinitely thin disk. However, if the disk has the original thickness $h$, then $(d \psi / d \delta)_{\max } \sim 1 / h$; the thicker the disk, the larger is the destabilizing force. Consequently, the dimensionless destabilizing factor is $r / h$, where $r$ is the radius of the disk, and the dimensionless stabilizing factor is $M / m_{r}$, where $M$ and $m_{r}$ are the masses of the central body and of the disk, respectively. The meaning of the stabilizing factor $M / m$, lies in the fact that, as it increases, the relative influence of the central body also increases. When the force of attraction of the particles towards the central mass exceeds the force of the particles' mutual attraction, the system is stable for the same reason which ensures that a point revolving in a central field is stable (here we do not allow for other interactions besides gravitational). In other words, a system turns out to be unstable if the destabilizing factor exceeds the stabilizing factor, i.e.,

$$
\begin{equation*}
\frac{r}{h}>\frac{M}{m_{r}} \quad \text { or } \quad Q<1, \quad Q=\frac{M}{m_{r}} \frac{h}{r} \tag{75}
\end{equation*}
$$

The parameter $Q$ is called the Toomre margin coefficient. The condition of disk instability in the form of expressions (75) is valid for very short wavelength perturbations with wavelength $\lambda \sim h$. In this case, the disk is broken up into rings with widths of $d \sim h$. But if $\lambda \sim d>h$, then it follows from our arguments that one must replace $h$ by $d$ in condition (75); the larger the widths of the rings, the more difficult it is to fulfill the instability criterion $Q(d)$ $=\left(M / m_{r}\right)(d / r)<1$. According to the results of processing the "Voyager- 2 " data, $Q \approx 2$ for the B ring of Saturn, i.e., the $B$ ring is near the limit of gravitational instability. This result has been obtained by assuming monolithic ring particles. High porosity of these particles ( $\sim 85 \%$; see Sec. 2) can make $Q$ significantly larger. Nevertheless, the existence in the $B$ ring of hyperfine structure as the result of the development of gravitational instability is entirely possible. Later on we shall examine dissipative instabilities of rings that are stable according to Jeans.

### 5.1.2. Thermal instability

The chaotic motion of the particles of a ring in a rotating reference system is analogous to the motion of molecules in a gas. The chaotic motion of the particles is maintained by mutual gravitational perturbations; the energy of orbital revolution of a viscous, differentially rotating disk is converted into chaotic, "thermal" energy. The inelasticity of the particles does not permit the chaotic velocities to increase without limit. The balance between the inflow and outflow of the energy of chaotic motion can, as for any balance, turn out to be unstable. For example, if during cooling of a certain section of the rings, the energy influx (depending on the temperature of the medium ) increases, then the rings will return to the original temperature; if the same energy influx decreases, then the rings abruptly cool off and will go over to a lower energy state. We shall examine other dissipative instabilities, assuming that there is no thermal instability of the disk.

### 5.1.3. Negative diffusion instability

Let us create a sinusoidal surface density perturbation in the disk: $\sigma \sim \sigma_{0} \cos k x$. Let us examine Region 1 with an increased density $\sigma_{1}$ (for $0<x<x_{0}$ ) and Region 2 with decreased density $\sigma_{2}\left(x_{0}<x<x_{1}\right)$. The density is unchanged on the boundary at the point $x_{0}$. The following amount of material flows across a unit length of the boundary separating Regions 1 and 2: $\sigma_{1} v_{1}-\sigma_{2} v_{2}$, where $v_{1}$ and $v_{2}$ are the diffusion velocities, which are proportional to the mean thermal velocities of the particles in Regions 1 and 2, respectively. Instability sets in when the particle density in Region 1 is increased due to migration of particles from Region 2, i.e., $\sigma_{1} v_{1}-\sigma_{2} v_{2}<0$. Since $\sigma_{1}>\sigma_{2}$, then the condition for instability is fulfilled, for example, when $v \sim \sigma^{\alpha-1}$, where $\alpha<0$. The last condition indicates that the particle velocity must decrease with increasing density of the disk. This is possible in the case of inelastic particles when the frequency of collisions increases with increasing density of the medium and the outflow of kinetic energy increases; the chaotic velocities of the particles decreases. The boundary between Regions 1 and 2 corresponds to the inflection point $x_{0}$ of the function
$\sigma(x)$, i.e., at $x_{0}, \partial^{2} \sigma(x) / \partial x^{2}=0 . \partial^{2} \sigma / \partial x^{2}<0$ in Region 1, and $\partial^{2} \sigma / \partial x^{2}>0$ in Region 2. It follows from the diffusion equation $\partial \sigma / \partial t=D \partial^{2} \sigma / \partial x^{2}$ that if the diffusion coefficient is negative ( $D<0$ ) in both regions, the density will increase in Region $1\left(\partial \sigma_{1} / \partial t>0\right)$ and will decrease in Region $2\left(\partial \sigma_{2} /\right.$ $\partial t<0$ ). Now it is understood why the instability examined above has the name "negative diffusion instability". By representing modulation of the density $\sigma$ in the form of a sinusoidal wave with an amplitude that increases exponentially with time $\sigma \sim \sigma_{0} e^{\gamma t} \cos k x$, we shall obtain $\gamma \approx k^{2}|D|$ from the diffusion equation, i.e., the instability increment turns out to be a maximum for short wavelengths.

### 5.1.4. Accretion instability

The instabilities examined above lead to the growth of short wavelength waves. The large-scale structure of the rings can arise as the result of accretion instability connected with the accretion of "external" material, for example, with the flow of fine dust through the ring system because of aerodynamic braking or interaction with solar radiation (the Poynting-Robertson effect). The mechanism of this instability is related to the mechanism for forming sand-hills in the desert: a flow of particles moving towards the planet (along the plane of the rings) "squeezes" into annular fluctuations with high densities and, consequently, also high absorbing capability. Accretion instability generates large-scale layering of the rings, since small-scale fluctuations do not succeed in gathering into a "sand-hill" as a consequence of rapid diffusion spreading over a time $t \sim \lambda^{2}$. One may notice an analogy between accretion instability and Türing instability in diffusion systems with chemical reactions. ${ }^{75}$

### 5.1.5. Elllpse instability

All the instabilties listed above generate ring structures possessing circular symmetry. Ellipse instability arising in a symmetric disk is an example of the spontaneous disruption of symmetry. In order to understand the physics of this instability, let us examine an individual test particle in a slightly elliptical orbit in a continuous circular disk. Moving away from the planet, the particle enters an environment of particles with large orbital velocities. Interacting with them, the test particle will be accelerated and tend to move even further from the planet at apocenter. On the other hand, in approaching the planet, the test particle will be decelerated by the slower disk particles and will approach the planet even more. As a result of such an alternating accelerating, then decelerating action, the orbit of the particle becomes more and more elliptical as long as inelastic collisions do not limit this process.

One more similar example is the classical Laplace-Maxwell problem ${ }^{1,2}$ of the stability of an absolute rigid ring revolving around a planet. Each element of the ring is balanced by centrifugal and gravitational forces, but the connection between the elements of the ring proves to be fatal: the ring is spontaneously shifted out of its circular orbit which, in a linear approximation, corresponds to a transition to an elliptical orbit. Why does this occur? In the displacement of the ring, all its parts continue to revolve with the same velocity. Therefore, the force of gravity starts to dominate for the parts of the ring closer to the planet, and centrifugal force dominates for the distant parts of the ring. The ring is shifted
more and more, just as is the orbit of the test particle in the disk, only the elements of the ring are accelerated and decelerated not by an external medium but by each other.

A fluctuation in the form of an elliptical ringlet in a disk of inelastic particles also behaves in a similar manner, increasing its eccentricity. The physics of such ellipse instability in planetary rings is clear from what has been described above; it is easier for a particle to increase its eccentricity not by itself, but in a group with other particles by creating an elliptical ring. The elliptical ringlets of Uranus and Saturn evidently serve as examples of ellipse instability.

### 5.2. Diffusion and quasi-secular instabilities

Let us examine radial ( $m=0$ ) oscillations of a disk with no external flows of material ( $N^{+}=N^{-}=0$ ).

### 5.2.1. Establishment of criteria for energy and dissipative instabilities

For a disk that is stable according to Jeans and for low frequency oscillations $\gamma \sim v k^{2} \ll \Omega$, we obtain the following dispersion equation:

$$
\begin{align*}
\gamma^{2} \omega_{0}^{2} & +\gamma\left[\frac{2}{3} \omega_{\cdot}^{2}\left(\chi k^{2}+\Delta E_{T}\right)+\nu k^{2}\left(\frac{5}{3} k^{2} c^{2}-2 \pi G \sigma_{0} k\right)\right. \\
& \left.+\frac{4}{3} k^{2} c^{2} \Omega \alpha-\frac{2}{3} k^{2} \sigma_{0} \Delta E_{\sigma}-\frac{\chi^{2}}{3 \Omega} k^{2} \mu-k^{2} \beta \cdot 2 \Omega \sigma_{0}\right] \\
& +\frac{2}{3}\left[\left(\chi \nu k^{4}+v k^{2} \Delta E_{T}+k^{2} \alpha \mu\right)\left(k^{2} c^{2}-2 \pi G \Gamma_{0} k\right)\right. \\
& +\sigma_{0} k^{4}\left(\chi \beta \cdot 2 \Omega-\beta \mu-\Delta E_{\sigma} v\right) \\
& \left.+k^{2} \sigma_{0} \cdot 2 \Omega\left(\Delta E_{T} \beta-\Delta E_{0} \alpha\right)\right]=0 . \tag{76}
\end{align*}
$$

A negative value of the free term in Eq. (76) will be the general criterion of instability. For long wavelength waves $k h \ll 1$, Eq. (76) has two roots which describe the dynamics of the temperature perturbations, $\gamma \approx-(2 / 3) \Delta E_{T}$, and the dynamics of the diffusion oscillations:

$$
\begin{equation*}
\gamma \approx-D k^{2} \tag{77}
\end{equation*}
$$

where

$$
D=\sigma_{0}\left(\beta-\alpha \frac{\Delta E_{\sigma} j}{\Delta E_{T}}\right) \cdot \frac{2 \Omega}{x^{2}}
$$

here $D$ is the diffusion coefficient. It follows from Refs. 13 and 14 that the disk loses stability when $D$ passes through the point $O$ and becomes negative. We shall show from the general Eq. (76) that this is not so.
5.2.1-1. The case of small positive diffusion. If $D \rightarrow 0$ then, from Eq. (76), we obtain for $k h \ll 1$

$$
\begin{equation*}
\gamma=-\frac{v k^{2}}{\omega_{0}^{2}}\left(F k^{2} c^{2}-f 2 \pi G \sigma_{0} k\right) \tag{78}
\end{equation*}
$$

where

$$
\begin{align*}
f= & 1+\frac{\partial v}{\partial T_{0}} \frac{2\left(r \Omega^{\prime}\right)^{2}}{\Delta E_{T}}, \\
F= & f+\frac{\chi}{v c^{2}} \frac{\partial v \sigma_{0}}{\partial \sigma_{0}} \cdot \frac{2 \Omega\left(-r \Omega^{\prime}\right)}{\Delta E_{T}}-\frac{2}{c^{2}} \frac{\partial v \sigma_{0}\left(r \Omega^{\prime}\right)^{2}}{\partial \sigma_{0}} \frac{\Delta E_{T}}{\Delta E_{T^{2}}} \\
& \left.-\frac{1}{\partial E^{-} \sigma_{0}}-\frac{\partial E^{+} \sigma_{0}}{\partial \sigma_{0}}\right) . \tag{79}
\end{align*}
$$

Equation (78) describes a quasi-secular instability with maximally rapidly increasing wavelengths $\lambda_{0} \sim c^{2} / G \sigma_{0}$ (if $F \sim 1$ and $f \sim 1$ ). It is easy to show from Eq. (76) that this instability sets in for a positive diffusion coefficient when $0<D<v\left(2 \pi G \sigma_{0} / \varkappa c\right)^{2} f^{2} / F$.
5.2.1-2. The case of negative diffusion. If $D<0$, then diffusion instability sets in, for which the criterion has been determined in Refs. 13 and 14 in the form $d(v \sigma) / d \sigma<0$. The characteristic scale for layering of a disk was not determined in Refs. 13 and 14. The limit of diffusion instability in a region of short wavelength waves and the length of maximally unstable waves were found in Refs. 15. We find from Eq. (76) that the term $k^{6} v \psi c^{2}$ will always be dominant for the shortest wavelength waves; this stabilizes the diffusion instability at $k h \sim 1$ or $\lambda \sim 2 \pi h .{ }^{15}$ This is easy to see from the following estimates:

$$
\begin{equation*}
\sigma_{0} \cdot 2 \Omega k^{2}\left(\Delta E_{T} \beta-\Delta E_{\sigma} \alpha\right) \sim \frac{7 v^{2} \Omega^{4} k^{2}}{c^{2}} \tag{80}
\end{equation*}
$$

From Sec. 4.1, we obtain the Eichen relation in the region $\Omega t_{c} \sim 1: \psi \sim 5 v$; from this, the "stabilizing" term is $\psi v c^{2} k^{6} \sim 5 v^{2} c^{2} k^{6}$. This term is comparable with the "unstable" term at $k \sim(1 / h) \sim(\Omega / c)$. Stabilization of diffusion instability at $k h \sim 1$ indicates that the increment of instability is a maximum for the wavelengths $\lambda \gtrsim 2 \pi h$ or $\lambda \sim 10 h .^{15}$

### 5.2.2. The diffusion instability criterlon for non-gravitating smooth particles

We write the energy balance equation for smooth particles: ${ }^{35}$

$$
\begin{equation*}
1-q^{2}=\frac{0,6}{1+\tau^{2}} \tag{81}
\end{equation*}
$$

The criterion of thermal instability takes the form

$$
\begin{equation*}
\frac{d\left(1-q^{4}\right)}{d v^{2}}>0, \quad \text { or } \quad \frac{d q}{d v}<0 \tag{82}
\end{equation*}
$$

Negative diffusion instability sets in if $\tau>\tau_{c r}$, where

$$
\begin{equation*}
\tau_{\mathrm{cr}}=\left[\frac{v^{2}}{1-q^{2}} \frac{\partial\left(1-q^{2}\right)}{\partial v^{2}}\right]^{1 / 2} . \tag{83}
\end{equation*}
$$

If one uses the experimental data of Ref. 18 as a basis, then $q \propto v^{-0.25}$, and we find $\tau_{\mathrm{cr}} \approx 0.5$ from Eq. (83). The velocity dispersion is determined from the intersection point of Eq. (81) and the experimental function $q(v)$, and is close to $0.5 \mathrm{~mm} / \mathrm{sec}$. The experimental data obtained in Ref. 37 correspond to smaller $\tau_{\text {cr }}$ and considerably larger (clearly unrealistic) $v$ values. But, as follows from the results of Sec. 3.1, the restitution coefficient for the snow particles of the rings of Saturn is nearly zero; consequently, Eq. (81) is unfeasible [Eq. (81) is possible if $q>0.63$ ]. Even if Eq. (81) is possible (the regolith layer is for some reason very thin), then $d q / d v>0$ and, according to inequalities (82), the equation is unstable.

### 5.2.3. Thermaland diffuslon instability in a model of gravitating particles

The chaotic velocity of gravitating inelastic particles in a differentially revolving disk increases during mutual gravitational close approaches, and decreases during contact collisions. Let us determine the free flight times with allowance for the dependence of the three-dimensional concentration of particies on disk thickness $n \approx \sigma / m h \approx \sigma \Omega / m v$.

1. For contact collisions ${ }^{53}$

$$
\begin{equation*}
t_{\mathrm{cc}}=\frac{1}{\xi_{\Omega} \tau\left(1+x^{-1}\right)} \tag{84}
\end{equation*}
$$

where $\tau=\sigma \pi a^{2} / m, \xi=2, x \approx v^{2} a / G m$.
2. For gravitational interactions ${ }^{53}$

$$
\begin{equation*}
t_{\mathrm{G}}=\frac{x^{2}}{\psi \Omega \tau} \tag{85}
\end{equation*}
$$

where $\psi \approx 4$. We write the shear viscosity coefficient in a simple form ${ }^{35}$

$$
\begin{equation*}
v=\frac{s r t_{\mathrm{c}}}{\left(b \Omega t_{\mathrm{f}}\right)^{2}+1} \tag{86}
\end{equation*}
$$

where $S=0.9$ and $b=2$. This expression agrees well with the viscosity coefficient obtained in Sec. 4.1.3. ${ }^{15}$ For the quantities $S$ and $b$, the index $G$ will denote viscosity caused by gravitational interaction, and the index $c c$ is for contact collision. We denote the corresponding times as $t_{\mathrm{G}}$ and $t_{\mathrm{cc}}$. Assuming that $1-q^{2}=1$, we write the energy equation in the form

$$
\begin{equation*}
S_{\mathrm{G}} \frac{T t_{\mathrm{G}}}{\left(b_{\mathrm{G}} \Omega t_{\mathrm{G}}\right)^{2}+1}\left(r \Omega^{\prime}\right)^{2}=\left(1-q^{2}\right) \frac{3 T}{t_{\mathrm{cc}}} \tag{87}
\end{equation*}
$$

With the allowance for Eqs. (85) and (86), expression (87) will take the form $\left(r \Omega^{\prime}=1.5 \Omega\right)^{74}$

$$
\begin{equation*}
x^{3}+x^{6}-\alpha x^{3}+\beta \tau^{2}(x+1)=0 \tag{88}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=\frac{3 S_{\mathrm{G}} \psi}{4\left(1-q^{2}\right) b_{\mathrm{G}}^{2} \xi}, \quad \beta=\frac{\psi^{2}}{b_{\mathrm{G}}^{2}} \tag{89}
\end{equation*}
$$

We write the stability condition for the energy Eq. (87) ${ }^{74}$ thus:

$$
\begin{equation*}
x^{8}+\alpha x^{8}+2 \beta \tau^{2} x^{6}-3 \alpha \beta \tau^{2} x^{2}+\beta^{2} \tau^{6}>0 \tag{90}
\end{equation*}
$$

If we take into account that $E^{+}=v_{\mathrm{G}}\left(r \Omega^{\prime}\right)^{2}$, then from Eq. (77) we obtain a simpler instability condition for negative diffusion:

$$
\begin{equation*}
\frac{\partial v \sigma_{0}}{\partial \sigma_{0}} \frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial v}{\partial T_{0}} \frac{\partial E^{-} \sigma_{0}}{\partial \sigma_{0}}<0 \tag{91}
\end{equation*}
$$

With allowance for Eqs. (85), (86), and (87), from inequality (91) we obtain the instability condition in the form ${ }^{74}$

$$
\begin{equation*}
2 x^{3}+x^{4}-3 \beta \tau^{2}(x+1)<0 \tag{92}
\end{equation*}
$$

The balance equations (89) has two real roots in the region $0<\tau<\tau_{\text {max }}$ and none in the $\tau>\tau_{\text {max }}$ region. Only one root, which is a maximum at $\tau=0$ and decreases towards $\tau_{\text {max }}$, is energetically stable. ${ }^{74} d x / d \tau=\infty$ at the point $\tau_{\max }$, and the disk will become energetically unstable and will cool off sharply. But just before this (at a smaller $\tau$ ), the disk undergoes diffusion instability. ${ }^{74}$ One can determine the point of the onset of instability ( $x_{\mathrm{cr}}, \tau_{\mathrm{cr}}$ ) by eliminating $\tau$ from Eqs. (89) and condition (92):

$$
\begin{equation*}
5 x_{\mathrm{cr}}^{2}+4 x_{\mathrm{cr}}-3 \alpha=0 \tag{93}
\end{equation*}
$$

from this,

$$
\begin{equation*}
x_{\mathrm{cr}}=0.4\left[(1+3.75 \alpha)^{1 / 2}-1\right] \tag{94}
\end{equation*}
$$

By substituting the value obtained for $x_{\mathrm{cr}}$ in inequality (92), we find $\tau_{\mathrm{cr}}$. For $\alpha=0.3$ to 0.4 , we obtain the fairly small $\tau_{\mathrm{cr}}$ $=0.01$ to 0.017 . The onset of negative diffusion instability
corresponds to the maximum value of viscosity as a function of $\Omega t_{f f}$ (at $\Omega t_{f f} \approx 0.8$ ). The disk, after reaching the point of maximum velocity, rearranges its structure and breaks up into ringlets, and this leads to an effective reduction of viscosity. Immediately after their formation, the rings undergo thermal instability, while the gaps between the rings, on the other hand, are shifted into a range of stable energy balance.

Let us estimate the dispersion of the large particle velocities. From Eqs. (89), for $\alpha \approx 0.4$ and small $\tau$ values we obtain $v=0.55 v_{\mathrm{G}} \approx 1 \mathrm{~mm} / \mathrm{sec}$ for a particle with $a=5 \mathrm{~m}$ and $\rho=0.15 \mathrm{~g} / \mathrm{cm}^{3}$. This indicates that the centers of the large particles are distributed in a layer of about 10 m thicknesses.

Let us examine the dynamics of small particles, which always possess a stable energy balance, by increasing their chaotic velocity upon scattering in the gravitational fields of the large particles, and by reducing it upon mutual collisions (we assume here that the optical thickness of the small particles is significantly greater than the optical thickness of the large ones). One can write the criterion for diffusion instability for a layer of small particles in the form (the index 2 indicates small particles) ${ }^{74}$

$$
\begin{equation*}
\tau_{2}^{2}-1+2 \frac{T_{2}}{E_{2}^{\dot{r}}} \frac{\partial E_{2}^{+}}{\partial T_{2}}>0 \tag{95}
\end{equation*}
$$

We obtain critical thicknesses $\tau_{\text {cr }}=1$ to $\sqrt{3}$ for various mechanisms for transferring energy from large particles to small ones. ${ }^{67,74}$ The chaotic velocities of the small particles can be several times greater than the velocities of the large particles, as a result of which, the smaller particles form a thicker layer several tens of meters thick.

We notice that the results of Sec. 5.2.1 have a general nature and are independent of specific particle properties or of the type of energy balance. At the same time, the models examined in Secs. 5.2.3 (not to speak of Sec. 5.2.2) are very simplified and do not take into account such important properties of particles as rotation around their own axes and nonlocal effects, which cause the appearance of nonlocal viscosity. Therefore, it is difficult to estimate reliably the critical optical thicknesses at which thermal and diffusion instabilities arise. One may speak more confidently about the characteristic scales of the instabilities: thus, the Jeans and diffusion instabilities cause layering of the disk into ringlets with widths of several thicknesses, and the quasi-secular instability breaks up the disk into ringlets of $\lambda_{0} \approx\left(c^{2}\right)$ $G \sigma_{0}$ ) $\sim 0.1 \mathrm{~km}$ to 1 km widths (for $c \sim 0.1 \mathrm{~cm} / \mathrm{sec}$ and $\sigma_{0} \sim 1$ to $10 \mathrm{~g} / \mathrm{cm}^{2}$ ). $\sigma_{0}$ could be smaller and the layering on a larger scale at an early stage. But these instabilities cannot explain the large-scale (up to $1,000 \mathrm{~km}$ ) layering of the rings.

### 5.3. Accretion instability

The large-scale (from 50 to $1,000 \mathrm{~km}$ ) layering of the rings of Saturn can be caused by a new type of instability connected with non-diffusion flows of material in the rings. ${ }^{76}$

### 5.3.1. The stability of rings with allowance for non-diffusion fows

The system of Eqs. (71) is considerably simplifed $\left(\gamma \sim v k^{2} \ll \Omega\right)$ at the long-wavelength limit $(\lambda \gg h)$ for radial oscillations ( $m=0$ ):
$\hat{\gamma} \hat{\sigma}=-i k \sigma_{0} \hat{v}_{r}-\left(\frac{\partial N^{-}}{\partial \sigma_{0}}-\frac{\partial N^{+}}{\partial \sigma_{0}}\right) \hat{\sigma}-\left(\frac{\partial N^{-}}{\partial T_{0}}-\frac{\partial N^{+}}{\partial T_{0}}\right) \hat{T}$,
$\frac{x^{2}}{2 \Omega} \hat{v}_{r}=-i k\left[\frac{\partial v \sigma_{0}}{\partial \sigma_{0}}\left(-r \Omega^{\prime}\right)\right] \frac{\hat{\sigma}}{\sigma_{0}}-i k\left[\frac{\partial v}{\partial T_{0}}\left(-r \Omega^{\prime}\right)\right] \hat{T}$,
$\left(\frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial E^{+}}{\partial T_{0}}\right) \hat{T}=-\left(\frac{\partial E^{-} \sigma_{0}}{\partial \sigma_{0}}-\frac{\partial E^{+} \sigma_{0}}{\partial \sigma_{0}}\right) \frac{\hat{\sigma}}{\sigma_{0}}-i k \cdot 2 v\left(-r \Omega^{\prime}\right) \hat{v}_{\Phi} ;$ $\hat{v}_{\varphi}=-i \frac{\pi G}{\Omega} \hat{\sigma}$.

We obtain a dispersion equation from Eqs. (96) (retaining terms with $k_{n}, n \leqslant 2$ )

$$
\begin{equation*}
\gamma=-D k^{2}+A k+B, \tag{97}
\end{equation*}
$$

where

$$
\begin{aligned}
D= & {\left[\frac{\partial v \sigma_{0}}{\partial \sigma_{0}}-\frac{\partial v}{\partial T_{0}}\left(\frac{\partial E^{-} \sigma_{0}}{\partial \sigma_{0}}-\frac{\partial E^{+} \sigma_{0}}{\partial \sigma_{0}}\right)\left(\frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial E^{+}}{\partial T_{0}}\right)^{-1}\right] \frac{2 \Omega\left(-r \Omega^{\prime}\right)}{x^{2}}, } \\
A= & \left(\frac{\partial N^{-}}{\partial T_{0}}-\frac{\partial N^{+}}{\partial T_{0}}\right)\left(\frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial E^{+}}{\partial T_{0}}\right)^{-1} v\left(-\frac{r \Omega^{\prime}}{\Omega}\right) \cdot 2 \pi G, \\
B= & \frac{1}{\sigma_{0}}\left(\frac{\partial N^{-}}{\partial T_{0}}-\frac{\partial N^{+}}{\partial T}\right)\left(\frac{\partial E^{-} \sigma_{0}}{\partial \sigma_{0}}-\frac{\partial E^{+} \sigma_{0}}{\partial \sigma_{0}}\right)\left(\frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial E^{+}}{\partial T_{0}}\right)^{-1} \\
& -\left(\frac{\partial N^{-}}{\partial \sigma_{0}}-\frac{\partial N^{+}}{\partial \sigma_{0}}\right) .
\end{aligned}
$$

If $D>0$ (the disk is "diffusion-stable") then, for $A>0$ and $B>-A^{2} / 4 D$, an instability connected with external flows of material develops.

### 5.3.2. Investigation of the criterion of instability

For the fairly rare collisions of large particles, taking Sec. 5.2.3 into account, we obtain

$$
\begin{aligned}
& \frac{\partial E^{-} \sigma_{0}}{\partial \sigma_{0}}-\frac{\partial E^{+} \sigma_{0}}{\partial \sigma_{0}} \approx v, \\
& \frac{\partial E^{-}}{\partial T_{0}}-\frac{\partial E^{+}}{\partial T_{0}} \approx 2 \frac{E^{r}}{T_{0}}=\frac{2 v\left(r \Omega^{\prime}\right)^{2}}{T_{0}}>0 .
\end{aligned}
$$

Here $\partial\left(v \sigma_{0}\right) / \partial \sigma_{0} \approx 2 v$. We find for $D, A$, and $B$

$$
\begin{align*}
& D=6 v,  \tag{98}\\
& A=\left(\frac{\partial N^{-}}{\partial T_{0}}-\frac{\partial N^{+}}{\partial T_{0}}\right) \frac{T_{0}}{3 \Omega^{2}} \cdot 2 \pi G, \\
& B=-\left(\frac{\partial N^{-}}{\partial \sigma_{0}}-\frac{\partial N^{+}}{\partial \sigma_{0}}\right) .
\end{align*}
$$

The criterion for the development of accretion instability in a disk that is far from diffusion instability is written in the form

$$
\begin{align*}
& \frac{\partial N^{-}}{\partial T_{0}}>\frac{\partial N^{+}}{\partial T_{0}},  \tag{99a}\\
& \frac{\partial N^{-}}{\partial \sigma_{0}}<\frac{\partial N^{+}}{\partial \sigma_{0}} . \tag{99b}
\end{align*}
$$

The range $-A^{2} / 4 D<B<0$, in which instability is also possible, is not allowed for in inequality (99b). Analysis shows that conditions (99) are completely realistic. The increment is a maximum at $k_{\max }=A / 2 D$, or

$$
k_{\max } \sim \frac{N^{+}}{3 \Omega^{2}} \cdot \frac{2 \pi G}{12 v}
$$

From this, one may obtain an estimate of the layering scales which are formed:

$$
\begin{equation*}
\lambda_{\max } \sim \frac{\Omega v_{G}^{2}}{G \rho_{\mathrm{du}} v} \sim \frac{\Omega v}{G \rho_{\mathrm{du}}} \tag{100}
\end{equation*}
$$

Let us estimate the density of the layer of "non-diffusion" particles at an early stage (i.e., of the dust particles which are contained in the gas and dust cloud surrounding the ring and from which accretion onto the rings occurs). Typical densities of the protodisks around Jupiter and Saturn were $\sim 10^{6}$ $\mathrm{g} / \mathrm{cm}^{2} ;{ }^{53}$ assuming that the fraction of dust suspended in the disk was 0.01 and 0.001 , we obtain (for $h_{\mathrm{du}} \approx 10^{9} \mathrm{~cm}$ ) $\rho_{\mathrm{du}}$ $\sim 10^{-5}$ to $10^{-6} \mathrm{~g} / \mathrm{cm}^{3}$. From expression ( 100 ), we obtain

$$
\begin{aligned}
& \lambda \sim \frac{100}{\rho_{\mathrm{du}}} \sim 10^{3} \mathrm{~km} \text { for } \rho_{\mathrm{du}} \sim 10^{-8} \mathrm{~g} / \mathrm{cm}^{3} \\
& \sim 50 \mathrm{~km} \text { for } \rho_{\mathrm{du}}-2 \cdot 10^{-5} \mathrm{~g} / \mathrm{cm}^{3}
\end{aligned}
$$

The typical times for the growth of such rings are about $10^{6}$ years and 2,500 years, respectively.

### 5.4. Instability of the ellipse mode

Let us examine the non-axisymmetric modes $m \neq 0$. The $m=1$ mode is the most interesting one (from the point of view of dissipative instability). Setting $\omega \ll \Omega$ and $k h \ll 1$, from Eq. (73) we obtain for $m=1^{77,78}$

$$
\begin{align*}
-\omega:= & \frac{1}{2 \Omega}\left(S k^{2} c^{2}-2 \pi G \sigma_{0} k-2 \Omega \omega_{\mathrm{p}}\right) \\
& -i \frac{\Delta E_{T}}{3 \Omega^{2}}\left(S^{\prime} k^{2} c^{2}-2 \pi G \sigma_{0} k-2 \Omega \omega_{\mathrm{p}}\right) \tag{101}
\end{align*}
$$

where

$$
\begin{aligned}
& S=\frac{5}{3}+\frac{14}{9} \frac{v \Delta E_{T}}{c^{2}}+\frac{2}{3} \frac{\alpha \mu}{c^{2}}-\frac{4}{3} \frac{\sigma_{0}}{\Omega c^{2}}\left(\Delta E_{\gamma} \beta-\Delta E_{\sigma} \alpha\right), \\
& S^{\prime}=1+\left[\Omega\left(2 \alpha c^{2}+3 \beta \sigma_{0}\right)-\sigma_{0} \Delta E_{\gamma}-5 \Omega^{2} v\right]\left(\Delta E_{\tau} c^{2}\right)^{-1}
\end{aligned}
$$

and $\omega_{\mathrm{p}}$ is the rate of precession because of the non-sphericity of the gravitational field of the planet:

$$
\begin{equation*}
\omega_{\mathrm{p}}=\frac{\Omega^{2}-\varkappa^{2}}{2 \Omega} \approx \Omega-x=\frac{3}{2} J_{2}\left(\frac{R_{\mathrm{p}}}{r}\right)^{2} \Omega \tag{102}
\end{equation*}
$$

The condition for instability is written in the form

$$
\begin{equation*}
S^{\prime} k^{2} c^{2}-2 \pi G \sigma_{0} k-2 \Omega \omega_{\mathrm{p}}>0 \tag{103}
\end{equation*}
$$

$\lambda \lesssim c^{2} / G \sigma_{0} \approx 1 \mathrm{~km}$ (for $c \approx 0.1 \mathrm{~cm} / \mathrm{sec}$ and $\sigma_{0} \approx 1 \mathrm{~g} / \mathrm{cm}^{2}$ ) with the increment $\gamma \sim v k^{2}$ are the most unstable wavelengths. The typical time of growth is $\approx \gamma^{-1} \leqslant 0.1$ year. Evidently just this instability of non-axisymmetric perturbations is responsible for the appearance of eccentricity for the rings of Uranus and of certain rings of Saturn. The modes with $m>1$ differ qualitatively from the mode examined from the presence of a real part of the frequency with $\omega \sim \Omega$.

Knowing now the typical scales of the perturbations and the magnitudes of the increments $\gamma$ for the dissipative instabilities described above, one can show the validity of the conditions for the hydrodynamic approximation: $\gamma \ll \omega_{\text {col }}$ and $l \sim\left(v / \omega_{\text {col }}\right) \ll \lambda$, where $\omega_{\text {col }}$ is the frequency of particle collisions, $v$ is a typical particle velocity in a revolving system, and $l$ is the mean free path for the particles. Allowing for the fact that $\gamma \sim v k^{2}$ and the frequency $\omega_{\text {cot }} \sim \Omega$, we obtain from Eq. (87)

$$
v k^{2} \sim \frac{k^{2} c^{2}}{\omega_{\mathrm{c}}} \sim \frac{k^{2} c^{2}}{\Omega} \ll \Omega, \quad \text { or } \quad(k h)^{2} \ll 1
$$

A similar condition ( $k h \ll 1$ ) follows from the inequality
$l \ll \lambda$, since $v \sim c$. Thus the condition for the suitability of the hydrodynamical approximation is valid for wavelengths longer than the disk thickness, i.e., $k h \ll 1$. This condition is fulfilled for all the instabilities listed above (only for the diffusion instability did we use the condition $k h \sim 1$ for the estimates).

## 6. THE RESONANCE ORIGIN OF THE RINGS OF URANUS AND THE PREDICTION OF A SERIES OF UNDISCOVERED SATELLITES

### 6.1. The first hypothesis on the nature of the rings of Uranus

### 6.1.1. The surprising properties of the rings of Uranus

The discovery of the rings of Uranus on March 10, 1977 caused sudden interest among researchers, since the questions of the origin and stability of the narrow elliptical rings turned out to be not so simple. First, as a consequence of exchange of angular momentum colliding particles, a narrow ring must be rapidly dispersed (in a matter of decades), increasing its width and reducing the sharpness of its edges. Second, the non-sphericity of the gravity field of Uranus causes precession of the elliptical orbits; its rate depends on the size of the semi-major axis. Differential precession of particles at the outer and inner edges of a ring must transform a narrow elliptical ring into a circular and wider one after several hundred years. Nevertheless, the rings are not dispersed; they have clearly defined boundaries and precess as a single body.

Let us list the main problems of the origin and dynamics of the rings of Uranus: ${ }^{79}$

1. How were the rings of Uranus formed? What gathered the material near the planet into narrow rings that are widely separated from each other?
2. How did the eccentricity of the rings arise?
3. Why are the rings not destroyed?

These problems have evoked a large number of different hypotheses.

### 6.1.2. Hypotheses about the connection of the rings with the known five satellites of Uranus

In the paper announcing the discovery of the rings of Uranus, the idea was stated that the distances between the rings are explained by resonances with the known five large satellites of Uranus. ${ }^{8}$ The positions of the five rigns discovered in 1977 and designated by $\alpha, \beta, \gamma, \delta$, and $\varepsilon$ were compared in a paper by Dermott and Gold ${ }^{80}$ with a series of three-frequency resonances from Ariel-Titania and ArielOberon (when the frequency $\Omega$ of the revolution of a ring satisfies the equation: $q \Omega-(q+p) \Omega_{2}+p \Omega_{3}=0$, where $\Omega_{2}$ and $\Omega_{3}$ are the frequencies of the revolutions of the two satellites, and $q$ and $p$ are integers; this resonance is observed for three satellites of Jupiter: $I_{0}$, Europa, and Ganymede, where $q=1$ and $p=2$ ). It was assumed that the particles are "stuck" in resonant orbits in motion towards the planet; such a capture model was examined by Gold in Ref. 81. Later Aksnes ${ }^{82}$ and Goldreich and Nicholson ${ }^{83}$ showed that the three-frequency resonances connected with Miranda but not with Ariel are more significant in the zone of the rings; here even the strongest three-frequency resonances can control the motion of particles in only a very narrow zone (in a few tens of meters), considerably narrower than the widths of the narrowest rings. But the critics themselves did not give
up the idea of the resonance nature of the rings of Uranus: Aksnes ${ }^{82}$ states the idea that only definite sorts of resonances capture material. He also mentions Colombo's remark about the approximate resonance relations between the rings themselves. Goldreich and Tremaine ${ }^{21}$ state the hypothesis that the rings of Uranus are strongly nonlinear waves that are excited by resonances in an optically thin disk. Steigman ${ }^{84}$ modifies the Dermott-Gold hypothesis by connecting the arrangement of the rings with three-frequency resonances from Miranda-Ariel and of Miranda with an undiscovered satellite in an orbit of $105,221 \mathrm{~km}$ radius. But four more rings of Uranus ( $\eta, 4,5,6$ ) were discovered in 1978, and it became difficult to compare the positions of all nine rings with three-frequency resonances from outer satellites. In combination with critical remarks ${ }^{82,83}$, this seriously damaged the positions of resonance hypotheses (and the Der-mott-Gold model among them).

### 6.1.3. Hypotheses about unknown satellites in the rings and "shepherd"satellites

Hypotheses which assume the presence of undiscovered satellites within the zone of the rings appeared in 1979. It is assumed in Refs. 85 and 86 that here the rings are either continuously renewable gaseous "traces" of invisible satellites ${ }^{85}$ or clusters of particles in complicated banana-shaped orbits near a satellite ${ }^{86}$; these are situated in each ring near a satellite. Goldreich and Tremaine ${ }^{16}$ assumed that each ring is situated between two "shepherd" satellites, which does not give the ring particles a chance to disperse. The influence of the "shepherd" satellites could also induce the eccentricity of the rings. ${ }^{87}$ The stability of the rings against differential precession was explained well by self-gravitation forces. ${ }^{88}$ In Novembr 1980, "Voyager-1" discovered two "shepherd" satellites (Pandora and Prometheus) beside the narrow, elliptical $F$ ring of Saturn, after which the idea of "shepherd"-satellites gained very wide acceptance.

### 6.2. The hypothesis of the resonance nature of the rings of Uranus and of the existence of a number of undiscovered satellites beyond the boundary of the rings

### 6.2.1. The initlal premises of the hypothesis

The formation of satellites in the rings is ruled out because of the intensive collisional fracturing of the parti-
cles $^{51,52}$ (see Sec. 3.2). A "helter-skelter" coexistence of rings and satellites is possible only in a fairly narrow zone between the regions of the rings and satellites. Models which assume the existence of from 9 to 18 satellites over the entire zone of rings plainly contradict this concept of the formation of rings.

Several narrow rings, sometimes with noticeable eccentricities, that are associated not with "shepherd" satellites, but with resonances from outer satellites (see Figs. 5 and 13), were discovered in the rings of Saturn at the beginning of the 1980s. This raises doubt about the need for a shepherd satellite model even for the rings of Uranus.

Rings and satellites are formed from the condensation of a single protosatellite disk. Here the material of the protodisk is distributed in a continuum. From this point of view, the immense ( $\sim 80000 \mathrm{~km}$ ) empty space between the rings of Uranus and Miranda raised doubt. A number of small satellites have been discovered in recent years near the outer boundaries of the rings of Jupiter and Saturn. It was natural to assume that unknown satellites also exist beyond the outer boundary of the rings of Uranus. Might it be that the resonance effect of these satellites also formed the surprising system of narrow elliptical rings of Uranus?

### 6.2.2. Calculation of the orbital radii of hypothetical satellites

As follows from what has been described above, satellites cannot exist within a zone of rings. A hypthesis was stated by the authors of the present review, according to which the positions of the rings of Uranus correspond to lower order Lindblad resonances ( $1: 2,2: 3$, and $3: 4$ ) with a number of undiscovered satellites beyond the outer boundary of the rings. ${ }^{17}$ The zone where the undiscovered satellites possessing this property are located must be situated between 50000 and 82500 km from the center of Uranus, and more-over, the maximum number of such satellites can be about 30 . Since significantly fewer satellites are required to form nine rings, it would be impossible to indicate any kind of specific orbits except for one remarkable property of the rings. We discovered that several orbits exist in the indicated zone of the hypothetical new satellites of Uranus (from 50000 to 82500 km from Uranus), each of which is simultaneously in resonance with a pair of rings, i.e., the


FIG. 13. Correlations between narrow ringlets and resonances in the Saturn system. $R$ is a narrow ring, RE is a narrow ring with eccentricity, DW is a spiral density wave, and BW is a flexural spiral wave. The optical thickness profile has been taken from Ref. 20.


FIG. 14. A hypothetical system of satellites of Uranus. ${ }^{17}$
resonances from each satellite in all such orbits simultaneously determined the positions of two (or more) rings. There turned out to be five such orbits. The authors determined one of these orbits to be superfluous in preparing Ref. 17; the data on this orbit were published later ${ }^{89}$, since a satellite in such an orbit partly duplicated the effect of another satellite. Besides this, a "shepherd"-satellite was introduced near its outer edge to explain the features of the outermost, widest, and most eccentric ( $\varepsilon$ ) ring; here the satellite also determined the position of Ring 4 by the $3: 4$ resonance. The overall pattern of hypothetical satellites and their resonances are depicted in Fig. 14. It is evident from Fig. 14 that the satellite $z_{0}$ "published" in Ref. 89 duplicates the effect of the satellite $z$ (this occurs because the orbits of $z$ and $z_{0}$ are in resonance with each other in the ratio 9:10). An algorithm for identifying the narrow zones with two resonances by starting from the structure of the ring system of Uranus is depicted in Fig. 15. The diameters of the unknown satellites of Uranus were estimated to be 100 km in Ref. 90 ; the possibility of detecting these satellites by ground-based telescopes by using their predicted orbital radii and periods of revolution of the unknown satellites and the situation of the plane of the satellite system of Uranus almost perpendicular to the Earth-Uranus line allowed one to follow the satellites by the method of superposing photographs taken at the frequencies of revolution of the satellites; this significantly increased the signal/noise ratio. The estimates of the stellar magnitudes of the unknown satellites enabled one to hope for the possibility of discovering them by using modern radiation detectors. ${ }^{90}$

### 6.3. The discovery of new satellites of Uranus. The correlation between rings and resonances from satellites 6.3.1. The "Voyager-2" flyby near Uranus in January 1986

The American spacecraft "Voyager-2" discovered ten new satellites and thereby led to the first summaries of the discussion about the nature of the rings. Only the one outermost and "anomalously" wide $\varepsilon$ ring turned out to be surrounded by shepherd satellites; here "Voyager" completely confirmed the hypothesis of the resonance nature of the rings of Uranus. ${ }^{89.91}$ The general arrangements of the predicted and discovered satellite systems are depicted in Fig. 16. A comparison of the orbital radii of the discovered and predicted satellites is given in Table VI. A comparison of the points of the hypothesis and of the "Voyager" observational data is given in Table VII. We notice that all the satellites giving two resonances in the ring zone were predicted correctly. Here the inner "shepherd"-satellite of the $\varepsilon$ ring fulfills all its predicted functions and simultaneously determines the positions of Ring 4 , only not by the $3: 4$, but by the 4:5 resonance.

The coincidence of the orbits of the predicted and discovered satellites is the main evidence for the resonance origin of the rings of Uranus. Nevertheless, a more detailed analysis of the mutual arrangements of the rings and resonances is necessary, since the positions of the rings are displaced with respect to the resonance orbits. As will be shown below, the last fact has a profound physical basis. Let us make an analysis following Ref. 23.


FIG. 15. An algorithm for identifying zones of location of unknown satellites of Uranus from ring positions. (a) The zone of satellite locations which seem to give one strong (of $1: 2,2: 3$, or $3: 4$ type) resonance in the ring zone. (b) The zone of satellites giving two resonances in the ring zone. (c) The zones of satellites giving resonances for two groups of rings. (d) Zones of the locations of individual satellites (the selected orbital radii and marked by dots). (e) Locations of the satellites discovered by "Voyager-2".

a


FIG. 16. (a) The general arrangement of previously unknown, discovered, and predicted satellites in the Uranus system. (b) Hypothetical (above) satellites and those discovered by "Voyager-2" (below). The vertical solid lines are the boundaries of the zone of satellites with two resonances in the region of the rings. The dashed lines are the zones of individual satellites (see Fig. 15)

### 6.3.2. Distribution of the distances between the rings and resonances

The nine main (most noticeable, discovered in 1977) rings of Uranus are situated in a zone between 40000 and 53000 km from the planet's center (see Table III). "Voyager" discovered another series of less noticeable narrow ring structures. The total number of rings in the 36000 to 53000 km zone has reached 15 . Let us compare the rings of Uranus with low order resonances ( $1: 2,2: 3,3: 4,4: 5,1: 3$, and 3:5); there are 25 of them in the 40000 to 53000 km zone and 31 in the 36000 to 53000 km zone. Let us find the radii $R_{\mathrm{rs}}$ of the resonance orbits, neglecting the small influence of the non-spherical harmonics of the gravitational field of Uranus on the resonance ratio $n \Omega=m \Omega_{s}$, where $\Omega$ is the frequency of revolution in a resonance orbit and $\Omega_{s}$ is the satellite's frequency of revolution. From this, the radius of a resonance orbit $R_{\mathrm{rs}}$ is connectd with the radius of a satellite orbit $R_{\mathrm{s}}: R_{\mathrm{rs}}=(\mathrm{n} / \mathrm{m})^{2 / 3} R_{\mathrm{s}}$.

Let us calculate the distances $\Delta_{r}$ from each resonance orbit to all the very close (no further than 1000 km ) rings. Let us examine the magnitude distribution of $\Delta_{r}$ by dividing 1000 km into several interval. Let $\bar{N}$ be the number of distances $\Delta_{r}$ in each interval divided by the number of rings. If
$\bar{N} \sim 1$ in a certain interval, then this indicates that, on an average, one distance from this interval fits each ring. Histograms of the distribution have been constructed in Figs. 17a and 17 b for intervals of 125 km and 100 km , respectively, for the case of 13 rings in the 40000 to 53000 km zone. Two features of the $\bar{N}$ distribution are clearly evident: a dip in the first interval and a peak in the second one, i.e., there are almost no rings near the resonances, for practically every ring is situated at a distance of 100 to 250 km from a strong resonance. Let us check the statistical significance of these features of the distribution. For this, we "throw" (by means of a random number generator which gives a uniform distribution) a fictitious random system of 13 rings onto the actual set of resonance orbits in the 40000 to 53000 km zone. A total of 5000 such system are generated. Here we calculate the mean value $\bar{N}$ and the value of the error $\sigma$ for each interval. The random distribution of $\bar{N}$ is shown by the dashed line in Figs. 17a and 17b. It is evident in Figs. 17a and 17b that the peaks and dips on the actual distribution significantly exceed the indicated error limits. The amount by which the dip and peak exceed the error $\sigma$, and also the corresponding probability that this feature is non-random are indicated for different cases in Table VIII. We notice that the peak in

TABLE VI. A comparison of the orbits of the predicted and discovered satellites of Uranus.

| Orbital radii for the satellites, km |  | $\begin{gathered} \text { Accuracy of } \\ \text { Agreement, } R_{\mathrm{h}} \\ -R_{\mathrm{s}}, \mathrm{~km} \end{gathered}$ | Number of resonances of type 1:2, 2:3, and 3:1 in the ring zone from 41500 to 52000 km | Satellite <br> Diameters, km |
| :---: | :---: | :---: | :---: | :---: |
| Predicted, $R_{\text {h }}$ | Discovered, $R_{\text {s }}$ |  |  |  |
|  | 86000 |  | 0 | 155 |
|  | 75260 |  | 1 (1:2) | 60 |
|  | 69940 |  | 1 (1:2) | 60 |
| 66450 | 66090 | $+360$ | $2(1: 2), 2: 3)$ | 110 |
|  | 64350 |  | 1 (2:3) | 80 |
| 62470 | 62680 | $-210$ |  | 60 |
| 61860 | 61780 | $\begin{aligned} & +80 \end{aligned}$ | $\frac{2}{2}(2: 3,3: 4)$ | 60 50 |
| 58600 | 59170 | $-570$ | $2(2: 3,3: 4)$ | 50 30 |
| 55380 | 53800 | $(+1580)$ | 1 (3:4) | 30 .25 |
| 51580 | 49770 | $(+1810)$ |  | 25 |

TABLE VII. A comparison of the Gor'kavyĭ-Fridman hypothesis and "Voyager-2" observations.

| Hypothesis | Observations |
| :---: | :---: |
| A series of small satellites exists beyond the outer boundary of the rings of Uranus | Nine of the ten new satellites of Uranus are situated beyond the outer boundary of the rings. |
| Satellites are not formed inside the ring zone | Only one very small satellite is situated in the intermediate zone (near the outer edge of the rings) |
| Ring positions are determined by $1: 2,2: 3$, and 3:4 type resonances from undiscovered satellite (situated in the 50000 to 82500 km zone) | Eight of the ten new satellites are situated in this zone and have resonances of this type in the region of the rings. The correlation coefficient between ring positions and the resonances is very high, $\approx 0.84^{23}$ [see Sect. 6.3 .3 (below)] |
| Each of the five predicted sateilites simultaneously determines the positions of two rings | Four of the ten satelitites simultaneously determine the positions of two (or more) rings; their orbits agree well with the orbits of predicted satellites |
| The features of the outer $\varepsilon$ ring are explained by the presence of "shepherd""satellites <br> The satellites diameters are $\sim 100 \mathrm{~km}$ | The $\varepsilon$ ring is the only one near which "shep-herd"-satellites have been discovered <br> The average diameter of the satellites is $\approx 70$ km |

the 125 to 250 km interval is non-random according to the most rigorous probabalistic criteria. But why do the rings prefer to be situated at definite distances from the resonance? A model for the formation of the rings of Uranus, according to which the rings were formed at the boundaries of spiral waves that are caused by the rings were formed at the boundaries of spiral waves that are caused by the resonance perturbations of satellites in a continuous protoring of Uranus, has been examined in Ref. 89 (see also Sec. 6.4). The distance between the ring which is being formed and the resonance is equal to the propagation wavelength. The distribution of the extent of spiral waves in the rings of Saturn (the waves from resonances of the order of $m+n \leqslant 15$ and $m-n \leqslant 2$ are examined) is shown in Fig. 17c. It is evident that strong resonances cause waves with propagation lengths $\Delta_{w} \sim 100 \mathrm{~km}$ to 200 km . This fact explains well the
features of the distribution of the ring-resonance distances in the Uranus system: the peak of the distribution corresponds to typical lengths for the propagation of spiral waves, and the dip is caused by the absence of short-length spiral waves from strong resonances and by the impossibility of forming rings inside the perturbed zone of a spiral wave. These typical features of the distribution of $\bar{N}$ are hard observational evidence for the resonance origin of the rings of Uranus and, at the same time, they make (because of the displacement of the rings from the resonances) the correspondence of the positions of the rings and resonances less obvious.

### 6.3.3. Correlation between the arrangement of the rings and resonances

Let us analyze the arrangement of the rings and resonances by another, independent method. The general ar-


FIG. 17. The distribution of radial distances between rings and resonances in the Uranus system. $\bar{N}$ is the number of ring-resonance distances from $\Delta_{r}$ to $\Delta_{r}+\delta$, divided by the number of rings in the system. (a) $\delta=125 \mathrm{~km}$. (b) $\delta=100 \mathrm{~km}$. The bar shows the mean square error of the distribution of $\bar{N}$ for the rings situated randomly. (c) The distribution of spiral waves in the rings of Saturn caused by lower order resonances with respect to propagation length $\Delta_{w} \cdot n$ is the number of spiral waves.

TABLE VIII. The statistical significance of features of the ring-resonance distance distribution.

| Variants |  | Dip in the first interval |  | Peak in the second interval |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Number of rings | Region size, km | Size of interval $\delta, \mathrm{km}$ |  |  |  |
|  |  | $\delta=100$ | $\theta=125$ | $\delta=100$ | $8=12 \overline{0}$ |
| 9 | $\begin{aligned} & 40000 \mathrm{to} \\ & 53000 \mathrm{~km} \end{aligned}$ | 1.43\% | 1.26\%, | 1.99\%, | 3,05\%, |
| 13 | 40000 | $84.7 \%$ 1.950 | $79.2 \%$ | 95.3\% | 99.8\%, |
| 13 | 53000 km | 94.95\%, | 94.94\%, | 2.31\%, | $3.45 \%$ $99.95 \%$ |
| 15 | 36000 to | 1.65\%, | 1.70) | $2.53 \sigma$ | 3.52\%, |
|  | 53000 km | 90.1\% | 91.1\% | 98.9\% | 99.95\% |

rangement of the narrow rings of Uranus and of the resonances from the discovered satellites are depicted in Fig. 18b.

If one divides the entire ring zone ( 36000 to 53000 km ) into thousand-kilometer interval, then one may notice that the average number of resonances in an interval containing a ring is more than two times greater than the average number of resonances in an empty interval [in the zone of the main rings ( 40000 to 53000 km ), it is 2.5 times greater]. The general pattern of resonances in the ring zone is depicted in Fig. 18a by arrows of the different heights $H_{r}=3 /(m+n)$, and a histogram (the continuous line) which sums up the heights of the resonance arrows in each interval and characterizes the spatial distribution of the resonance orbits is constructed. The hatched regions form a histogram of the distribution of rings; each component of the ring system makes the same contribution to the histogram. We calculate by a standard procedure ${ }^{92}$ the correlation coefficient between the heights of the two histograms: $Q=0.727 \mp 0.114$ in the 36000 to 53000 km zone, and $Q=0.782 \mp 0.108$ in the 40000 to 53000 km zone. Taking into account higher order resonance orbits of the type 5:6, $6: 7, \ldots, 10: 11$ and $5: 7, \ldots, 9: 11 m+n \leqslant 21, m-n \leqslant 2$ ) in the histogram practically does not change the result: $Q_{36-53}=0.704 \mp 0.122$ and $Q_{40-53}=0.780 \mp 0.109$. We notice that the need to introduce the different weights 3/
( $m+n$ ) for the resonances loses significance in considering only lower order resonances. Here the correlation coefficient between the number of rings and the number of lower order resonances is each interval the impressive value $0.838 \mp 0.083$. This is one more demonstration of the resonance origin of the rings of Uranus. Let us consider which satellites exerted the decisive contribution to the correlation coefficient for each satellite or type of resonance: $\Delta Q_{x}=Q-Q_{x}$, where $Q_{x}$ is the correlation coefficient upon excluding given resonances (of one of the types of resonances or of one of the satellites) from the general pattern. The value of $\Delta Q_{x}$ for each satellite is depicted in Fig. 19a; the order of the positions of the satellite is depicted in Fig. 19a; the order of the positions of the satellites corresponds to the actual one (Uranus is on the left). We compare these data, which reveal the satellites that are significant in forming the rings with the predicted satellite system (see Table VI). The two most distant satellites, 1986U5 and 1985U1 have only one resonance in the ring zone; therefore their orbits could not be calculated from the arrangement of the rings with sufficient reliability. Of the remaining eight satellites, only the five predicted satellites make a positive contribution to the correlation coefficient (for the 40000 to 53000 km zone). One satellite accurately corresponds to a satellite predicted by its function: 1) it is a "shepherd"-satellite for the $\varepsilon$

a


FIG. 18. Correlation of the positions of the rings and resonances in the Uranus system. The open circles, plus sign, " $\times$ "s, and arrows show the positions of low order resonances. (a) The hatched region is the histogram which characterizes the location of the rings, and the solid line is a histogram of the resonances. (b) The solid lines are the main rings (discovered from the Earth, except for 1986U $1 R$ ), and the dashed lines are the rings discovered by "Voyager- 2 ", 1986U 2R is a diffuse dust ring which contains two more dense features of the ring system. The numbers to the right of the ordinate axis show the significance of the given resonance. The Lindblad resonances from each satellite are joined by a dashed line (the year of discovery, 1986, has been omitted from the names of the satellites). Non-Lindblad resonances are marked by plus signs for the $1: 3$ type, and by " $\times$ "'s for the $3: 5$ type.



FIG. 19. Magnitudes of the specific contributions to the correla tion coefficients between the positions of the rings and satellites. (a) The resonances of individual satellites; the solid line is the 40000 to 53000 km zone, and the dashed line is the 36000 to 53000 km zone; the numbers under the satellite names are the number of strong resonances ( $m+n \leqslant 9, m-n \leqslant 2$ ) from the given satellite in the 40000 to 53000 km zone. The satellites are arranged along the abscissa axis in order of their increasing distance from Uranus. The year of satellite discovery, 1986, has been omitted, U1* is the satellite 1985 U1, discovered on December $31,1985$. b) and c) are individual types of resonances from b) 10 satellites, and c) from eight satellites (without 1986 U 2 and $1986 \mathrm{U4}$ ). The dashed line is the nominal contribution of one resonance of each type. The numbers near the break points are the numbers of resonances of a given type in the 40000 to 53000 km zone. The $n /(n+1)$ resonances are situated to the left, and $n /(n+2)$ resonances are to the right.
ring (only not from outer but from the inner side), and 2) it has a strong resonance near Ring 4 (only it is not $3: 4$, but 4:5). If one examines not the $3: 4$ resonance with Ring 4 but the $4: 5$ one, then one can "predict" the orbital radius of the inner shepherd-satellite as 40410 km , which differs from the actual radius by at most 360 km . With allowance for this remark, we find that the mean deviation of the actual orbits from the orbits that are calculated from the ring positions is 316 km . It is obvious that this deviation is mainly connected with the physically determined displacements of the rings from the resonances by approximately 200 kilometers. It is evident from Fig. 19a that the two satellites 1986U2 and 1986U4 clearly did not take part in formation of the rings (evidently because of the later formation of these satellites). The exclusion of the resonances from these two satellites from the general pattern sharply raises the correlation coefficient:

$$
\begin{aligned}
& Q_{36-53}=0.837 \mp 0.073 \\
& Q_{40-53}=0.921 \mp 0.042(\text { for } m+n \leqslant 9, m-n \leqslant 2), \\
& Q_{36-53}=0.820 \mp 0.080, \\
& Q_{40-53}=0.899 \mp 0.053(\text { for } m+n \leqslant 21, m-n \leqslant 2) .
\end{aligned}
$$

The contributions of the different types of resonances to the correlation coefficient (the continuous curve) are shown in Fig. 19b. A clear regularity is evident; resonances of the $1: 2,2: 3,3: 4,4: 5$ and $1: 3,3: 5(m+n \leqslant 9, m-n \leqslant 2)$ types make the main positive contribution to $Q$. The higher order resonances have $\Delta Q_{x}$ values that are near zero or negative (i.e., they are situated randomly). This regularity is main-
tained even if all types of resonances make the same contribution to the histogram $\left[H_{r}=\operatorname{const}(n, m)\right] . \Delta Q_{x}$ is depicted in Fig. 19c for the case when the two satellites 1986U2 and 1986U4 are omitted ( $Q=0.899$ ). The tendency for $\Delta Q_{x}$ to decrease with increasing resonance order is just as clearly evident. This regularity is one more demonstration of the resonance origin of the rings of Uranus.

Thus, the resonance nature of the rings of Uranus is an authentic fact determined from observational data by two independent methods.

### 6.4. The formation and stability of the rings of Uranus

A fairly uniform disk of large particles, and not individual narrow rings, existed around Uranus at an early stage of its development. One can estimate its surface density by assuming that it is close to the mean density in the neighboring small satellite zone, 10 to $20 \mathrm{~g} / \mathrm{cm}^{2}$. Outer satellites cause a number of resonance effects in a continuous ring.

### 6.4.1. The resonance interaction of a satellite with ring particles

We write a simple system of equations.

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}+\mathbf{v} \nabla\right) \mathbf{v}=-\nabla\left(\psi_{\mathrm{p}}+\psi_{\mathrm{s}}\right)  \tag{103'}\\
& \frac{\partial \sigma}{\partial t}+\operatorname{div}(\sigma v)=0
\end{align*}
$$

where $\psi_{p}$ and $\psi_{s}$ are the gravitational potentials of the planet and satellite, respectively:

$$
\begin{aligned}
\Psi_{\mathrm{p}} & =-\frac{G M_{\mathrm{p}}}{r}, \quad \psi_{\mathrm{s}}(r, \varphi, t)=-\frac{G M_{\mathrm{s}}}{\left|r-r_{\mathrm{s}}\right|} \\
& =-G M_{\mathrm{s}}\left(r^{2}+r_{\mathrm{s}}^{2}-2 r r_{\mathrm{s}} \cos \left(\varphi-\Omega_{\mathrm{s}} t\right)\right]^{-1,2} .
\end{aligned}
$$

Let us examine the perturbations $v=v_{0}+v_{1}+\ldots$, $\sigma=\sigma_{0}+\sigma_{1}+\ldots,\left|v_{1}\right| \ll\left|v_{0}\right|,\left|\sigma_{1}\right| \ll\left|\sigma_{0}\right|$. We linearize the system of Eqs. (103'). Omitting the index 1, we obtain

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}+\Omega_{0} \frac{\partial}{\partial \varphi}\right) v_{r}-2 \Omega_{0} v_{\varphi}=-\frac{\partial \psi_{\mathrm{s}}}{\partial r} \\
& \left(\frac{\partial}{\partial t}+\Omega_{0} \frac{\partial}{\partial \varphi}\right) v_{\varphi}+\frac{\Omega_{0}}{2} \tau_{r}=-\frac{1}{r} \frac{\partial \psi_{\mathrm{s}}}{\partial \varphi}  \tag{104}\\
& \left(\frac{\partial}{\partial t}+\Omega_{0} \frac{\partial}{\partial \varphi}\right) \sigma=-\frac{\sigma_{0}}{r}\left[\frac{\partial}{\partial r}\left(r v_{r}\right)+\frac{\partial v_{\varphi}}{\partial \varphi}\right] .
\end{align*}
$$

We expand the potential of the satellite in a Fourier series:

$$
\psi_{\mathrm{s}}(r, \varphi, t)=\sum_{m=-\infty}^{\infty} \psi_{s_{m}}(r) \exp \left[\operatorname{im}\left(\varphi-Q_{s} t\right)\right]
$$

By virtue of the linearity of the system of Eqs. (104), let us choose one of the harmonics: $\sim \psi_{\mathrm{sm}} \exp \left[\operatorname{im}\left(\varphi-\Omega_{\mathrm{s}} t\right)\right]$. One must look for the perturbed functions $v_{r}, v_{\varphi}$, and $\sigma$ in the same form. As a result, we obtain

$$
\begin{align*}
& v_{\mathrm{s} m}(r)=-\frac{i m}{r D}\left[\left(\Omega_{0}-\Omega_{\mathrm{s}}\right) r \frac{\mathrm{~d}}{\mathrm{~d} r}+2 \Omega_{0}\right] \psi_{\mathrm{s} n}(r), \\
& v_{\varphi m}(r)=\frac{1}{2 r D}\left[\Omega_{0} r \frac{\mathrm{~d}}{\mathrm{~d} r}+2 m^{2}\left(\Omega_{0}-\Omega_{s}\right)\right] \psi_{\mathrm{s} \cdot n}(r),  \tag{105}\\
& \sigma_{m}(r)=-\frac{\sigma_{0}}{i m r\left(\Omega_{0}-\Omega_{s}\right)}\left[\frac{\partial}{\partial r}\left(r v_{r m}\right)+i m v_{\varphi m}\right] .
\end{align*}
$$

The going to zero of the denominators of these expressions, $\Omega_{0}(r)-\Omega_{s}=0$ and $D(r)=\Omega_{0}^{2}(r)-m^{2}\left[\Omega_{0}(r)-\Omega_{s}\right]^{2}$, determines three types of resonances of the disk with the satellite:

1) $\Omega_{0}\left(r_{\text {cor }}\right)=\Omega_{s}$ is the co-rotational resonance,
2) $\Omega_{0}\left(r_{\text {in }}\right)=(m / m-1) \Omega_{s}$ is the inner Lindblad resonance, and
3) $\Omega_{0}\left(r_{\text {out }}\right)=(m / m+1) \Omega_{s}$ is the outer Lindblad resonance.
Let $L$ be the angular momentum that is transferred from the satellite to the entire disk:

$$
\begin{aligned}
L & =L^{(0)}+L^{(1)}+\ldots, \\
L^{(0)} & =-\int_{r_{1}}^{r_{2}} \sigma_{0}(r) r \mathrm{~d} r \int_{0}^{2 \pi} \frac{\partial \psi_{\mathrm{s}}}{\partial \varphi^{\prime}} \mathrm{d} \varphi^{\prime} \\
& =-\int_{r_{1}}^{r_{2}} \sigma_{0}(r) r \mathrm{~d} r\left(\psi_{\mathrm{s}}(2 \pi)-\psi_{\mathrm{s}}(0)\right) .
\end{aligned}
$$

We write the perturbed angular momentum that is transferred to the entire disk:

$$
L^{(1)}=-\int_{r_{1}}^{r_{2}} r \mathrm{~d} r \int_{0}^{2 \pi} \sigma_{1}\left(r, \varphi^{\prime}\right) \frac{\partial \psi_{\mathrm{s}}\left(r, \varphi^{\prime}\right)}{\partial \varphi^{\prime}} \mathrm{d} \varphi^{\prime} .
$$

In the vicinity of a resonance ( $r=r_{m}$ ), we find for the Fourier harmonic $L_{m}^{(1)}$

$$
L_{m}^{(1)}=\frac{2 \pi m \sigma_{0} A^{2}}{r_{n} D_{s,}\left(r_{m}\right)} \operatorname{Im} \int_{-\varepsilon}^{p} \frac{\partial x}{x},
$$

where

$$
\begin{aligned}
& A\left(r_{m}\right) \equiv-\frac{G M_{\mathrm{s}}}{2 r_{\mathrm{s}}}\left(2 m b+\beta \frac{\mathrm{d} b}{\mathrm{~d} b}\right)_{\beta}, \frac{r_{m b}}{r_{\mathrm{s}}}, \quad x=\frac{r-r_{m}}{r_{m}}, \\
& b \equiv \frac{2}{r} \int_{0}^{\pi} \frac{\cos m \varphi^{\prime} d \phi^{\prime}}{\left(1+\beta^{2}-2 \beta \cos \varphi^{\prime}\right)^{1 / 2}}, \quad D .\left.\equiv \frac{\partial D}{\partial r}\right|_{r=r_{m k}} .
\end{aligned}
$$

We redefine $x$ in the following manner:

$$
x=\lim _{\alpha \rightarrow 0}(x+i \alpha) .
$$

Then (Fig. 20)

$$
\lim _{\alpha \rightarrow 0} \operatorname{Im} \int_{-\varepsilon}^{\varepsilon} \frac{d x}{x+i \alpha}=-\lim \alpha \int_{-\varepsilon}^{\varepsilon} \frac{d x}{x^{2}+\alpha^{2}}=-\pi \operatorname{sgn} \alpha
$$

Finally,

$$
\begin{equation*}
L_{m}^{(1)}=-\frac{4 \pi^{3} \sigma_{0} A^{2}\left(r_{m}\right)}{3 \Omega_{0}\left(r_{m}\right) \Omega_{\mathrm{s}}} \operatorname{sgn} \alpha . \tag{106}
\end{equation*}
$$

It is shown in Ref. 93 that $\alpha>0$. Here spiral waves which damp out with distance (because of viscosity) diverge on both sides from the inner Lindblad resonance. As a rule, the spiral density wave which moves outwards from the planet is more powerful. ${ }^{93}$ The angular momentum that is transferred from the satellite to the disk plays an important role in the dynamics of the ring particles.

### 6.4.2. Spiral waves and the formation of rings

According to tra ditional ideas of celestial mechanics, the zone of the resonance influence of an outer satellite is very narrow; the natural width of a resonance is ${ }^{21}$
$\Delta_{\mathrm{L}} \sim R_{\mathrm{L}}\left(\frac{M_{\mathrm{s}}}{M_{\mathrm{p}}}\right)^{1 / 2} \sim 30 \mathrm{~km}$ for Saturn's satellite Mimas,
$\sim 4 \mathrm{~km}$ for a satellite of Uranus with $a_{s}=50 \mathrm{~km}$;
here $R_{L}$ is the resonance radius, $a_{s}$ is the radius of the satellite, and $M_{s} / M_{p}$ is the ratio of the masses of the satellite and planet. However, the ideas about the local nature of the reso-


FIG. 20. (a) The contour of integration. (b) The dependence of $1 / \alpha$ on $x$ in the resonance region.
nance effect of a satellite are not true for a medium of large particles with collective properties (pressure, self-gravitation, etc.). A resonance perturbation in such media is propagated over many hundreds of kilometers from the point of resonance. For example, the Cassini gap, which was formed at the site of a resonance spiral wave from Mimas, extends for 4500 km , which is two orders of magnitude larger than the resonance width from Eq. (107). In removing angular momentum from the disk particles, a spiral wave causes drift of particles toward the planet and the formation of a gap for a satellite mass greater than the critical mass: ${ }^{94}$

$$
\begin{equation*}
\frac{M_{\mathrm{s}}}{M_{\mathrm{p}}} \geqslant \frac{1}{n} \frac{c}{\Omega_{\mathrm{s}} R_{\mathrm{s}}}\left(\frac{\tau}{\tau^{2}+1}\right)^{1 / 2} ; \tag{108}
\end{equation*}
$$

here $c$ are the chaotic velocities of the disk particles, $\tau$ is the optical thickness of the disk, and $n$ is the resonance characteristic ( $n /(n-1)$ ). The mass of Mimas is sufficient to form gaps. The new satellites of Uranus with radii of up to 80 km formed spiral waves in the protoring which do not cause divisions; because of particle diffusion, they were closed faster than particles were removed by resonance sweeping. The characteristics of a spiral wave depend not only on the satellite, but also on the parameters of the disk. We illustrate this dependence by means of the simplest system of equations, following Ref. 93 (allowance for viscosity is especially simplified: $\alpha=\beta=0$ ):

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}+\mathrm{v} \nabla\right) \mathbf{v} \\
& \quad=-\nabla\left(\psi_{p}+\psi_{s}+\psi_{\mathrm{d}}\right)-\frac{1}{\sigma} \nabla p+\nu \Delta \mathrm{v}+\frac{v}{3} \nabla(\nabla \mathrm{v}), \\
& \quad \frac{\partial \sigma}{\partial t}+\operatorname{div}(\sigma \mathrm{v})=0, \quad \frac{\mathrm{~d} p}{\mathrm{~d} \sigma}=c^{2}, \\
& \left(\frac{\partial^{2}}{\partial z^{2}}+\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r^{2}} \frac{\partial^{2}}{\partial \varphi^{2}}\right) \Phi_{\mathrm{d}}=4 \pi G \delta(z) \sigma,  \tag{109}\\
& \Phi_{\mathrm{d}}=e^{-k|z|} \psi_{\mathrm{d}} .
\end{align*}
$$

Similarly to Sec. 6.4.1, by linearizing this system of equations, we reduce it to a single equation for the Fourier harmonic $v_{\mathrm{rm}}$ which, near the inner Lindblad resonance, has the form

$$
\begin{equation*}
-\alpha_{\mathrm{W}}^{3} \frac{\mathrm{~d}^{2} v_{\mathrm{r} m}}{\mathrm{~d} x^{2}}+\alpha_{\mathrm{G}}^{2} \frac{\mathrm{~d} v_{\mathrm{r} m}}{\mathrm{~d} x}-i x v_{\mathrm{r} m}=c_{m} \tag{110}
\end{equation*}
$$

where

$$
\begin{align*}
& \alpha_{v}^{3} \equiv i \alpha_{\mathrm{p}}^{3}+\alpha_{v}^{3}, \quad c_{m} \equiv-\frac{\Omega\left(r_{m}\right) A}{r_{m}^{2} D_{*}}, \\
& \alpha_{\mathrm{p}}^{3} \equiv-\frac{c^{2}}{3 m r_{m}^{2} \Omega_{0}\left(r_{m}\right) \Omega_{\mathrm{s}}},  \tag{111}\\
& \alpha_{v}^{3} \equiv \frac{7}{9} \frac{\nu}{m r_{m}^{2} \Omega_{\mathrm{s}}}, \quad \alpha_{G}^{2} \equiv \frac{2 \pi \sigma_{0} G}{3 m r_{m} \Omega_{s} \Omega_{0}\left(r_{m}\right)} .
\end{align*}
$$

We examine some exact solutions of Eq. (110).
a) Let $\left|\alpha_{G}\right|,\left|\alpha_{v}\right| \ll\left|\alpha_{p}\right|$, therefore

$$
\begin{equation*}
v_{\mathrm{r} m}(x)=c_{m} \int_{0}^{\infty} \exp \left[i\left(k x-\frac{\alpha_{\mathrm{p}}^{3} k^{3}}{3}\right)\right] \mathrm{d} k . \tag{112}
\end{equation*}
$$

By virtue of the definition of $\alpha_{\mathrm{p}}^{3}$ in Eqs. (111), we find that $k<0$, i.e., an acoustic wave propagates from the resonance towards the planet. $v_{r m}$ can be represented in the form of


FIG. 21. Graphs of the exact solutions of the equations for spiral waves in the limiting cases when (a) the speed of sound, (b) viscosity, and (c) selfgravitation of the disk are dominant.
combinations of Airy type functions (Fig. 21a):

$$
\begin{equation*}
v_{\mathrm{r} m}^{\mathrm{r}}=\frac{\mathrm{E} \pi c_{m p}}{\left|\alpha_{\mathrm{p}}\right|}\left(\mathrm{Ai}\left(\frac{x}{\left|\alpha_{\mathrm{p}}\right|}\right)+i \mathrm{Gi}\left(\frac{x}{\left|\alpha_{\mathrm{p}}\right|}\right)\right) ; \tag{113}
\end{equation*}
$$

b) Let $\left|\alpha_{G}\right|,\left|\alpha_{p}\right| \ll\left|\alpha_{v}\right| ;$ then (Fig. 21b)
$v_{\mathrm{r} \mu m}(x)=c_{m / 1} \int_{i}^{\infty} \exp \left(i k x-\frac{\alpha_{v}^{3} k^{3}}{3}\right) \mathrm{d} k=\frac{\pi G}{\alpha_{v}} \mathrm{Hi}\left(\frac{i x}{\alpha_{v}}\right)$;
c) If $\left|\alpha_{p}\right|,\left|\alpha_{v}\right| \ll\left|\alpha_{G}\right|$, then we find (Fig. 21c)

$$
\begin{align*}
\boldsymbol{v}_{\mathrm{r} m}(x) & =c_{n} \int_{0}^{\infty} \exp \left[i\left(k x-\frac{\alpha_{\mathrm{G}}^{\mathrm{a}} k^{2}}{2}\right)\right] \mathrm{d} k \\
& =c_{m} \frac{V^{\prime} \bar{\pi}}{\left|\alpha_{\mathrm{G}}\right|}\left[g\left(-\frac{x}{\sqrt{\pi}\left|\alpha_{\mathrm{G}}\right|}\right)+i f\left(-\frac{x}{\sqrt{\pi}\left|\alpha_{\mathrm{G}}\right|}\right)\right], \tag{115}
\end{align*}
$$

where $g$ and $f$ are Fresnel integrals. Thus, spiral waves can propagate on both sides of a resonance. The gravitational field of a wave causes periodic motion of particles (Fig. 22) and can capture and transfer dusty material, "unloading" it beyond the spiral wave zone. But this is not the only mechanism for the growth of rings.


FIG. 22. The phase trajectories of particles moving in the field of a wave.

### 6.4.3. The role of accretional dust flow

Such a material transport factor as a strong flow of dust towards the planet exists near Uranus. Aerodynamic drag alone leads to the sweeping out of micron-sized particles after 100 to 1000 years. ${ }^{33}$ How did this strong planetocentric dust flow interact with the spiral waves in the protoring of Uranus? If a spiral density wave moving outwards from the planet brakes the radial planetocentric dust flow, then a condensation is formed at the encounter site of the wave and dust flow, a ring (see Fig. 23). Such a condensation is accretionally unstable (if the diffusion of particles from it is suppressed; see Sec. 6.4.5) and it grows by holding back the inflowing dust. The bent spiral waves, whose heights above the ring plane reach several hundred meters, and the acoustic waves propagate towards the planet and can, on the other hand, accelerate the dust motion, which will also lead to the formation of a ring at the edge of the wave propagation zone.

We estimate the time for the growth of rings in the planetocentric dust flow. Let the protodisk from which the rings were formed occupy the region ( $r_{1}, r_{2}$ ). The amount of dust which accreted onto the planet over the time of the existence of the protodisk $t_{\mathrm{L}}$ is $2 \pi r_{1} \sigma_{\mathrm{a}} v_{\mathrm{a}} t_{\mathrm{L}}$, where $\sigma_{\mathrm{a}}$ is the surface density of dust and $v_{\mathrm{a}}$ is its radial velocity in the region of the inner radius $r_{1}$ of the disk. Obviously, this amount of dust cannot be less than the protodisk mass $\sigma_{0} \pi\left(r_{2}^{2}-r_{1}^{2}\right) \approx \sigma_{0} \pi r_{2}^{2}$ ( $\sigma_{0}$ is the mean surface density of the protodisk) since, besides the material of the inner protodisk (from which the rings, whose masses we neglect, were formed), part of the material from the outer regions of the protodisk where the satellites are formed, and also interplanetary material, can also accrete onto Uranus. Thus, one can write the inequality: $2 \pi r_{1} \sigma_{\mathrm{a}} v_{\mathrm{a}} t_{\mathrm{L}} \geqslant \sigma_{0} \pi r_{2}^{2}$; from this, $\sigma_{\mathrm{a}} v_{\mathrm{a}} \geqslant \sigma_{0} r_{2}^{2} / 2 r_{1} l_{\mathrm{L}}$. We estimate the maximum time $t_{\text {max }}$ over which the flow of dust towards Uranus $\sigma_{\mathrm{a}} v_{\mathrm{a}}$ will be able to create a ring with density contrast $\Delta \sigma$ and width $\Delta r$. Using the equation $2 \pi r \Delta r \Delta \sigma=2 \pi r v_{a} \sigma_{a} t_{\text {max }}$, we obtain
$t_{\text {max }}=\frac{\Delta r \Delta \sigma}{v_{\mathrm{a}} \sigma_{\mathrm{a}}} \leqslant \frac{\Delta r \Delta \sigma}{\sigma_{0} r_{2}^{2}} \cdot 2 r_{1} t_{\mathrm{L}}=\frac{\Delta r}{r_{2}} \frac{\Delta \sigma}{\sigma_{0}} t_{\mathrm{L}} \approx 2000$ to $10^{\varsigma}$ years
the following values were used here: $\Delta \sigma / \sigma \approx 1, \Delta r \approx 10 \mathrm{~km}$, $r_{2}=50000 \mathrm{~km}, r_{1}=26000 \mathrm{~km}$, and $i_{\mathrm{L}}=10^{7}$ to $10^{9}$ years.

If the formation of a narrow isolated ringlet is the result of the external resonance effect of satellites, then the further evolution of such a ringlet and its acquiring a stable, elliptical form evidently proceeds practically independently.


FIG. 23. Formation of a condensation at the edge of a spiral wave.

### 6.4.4. Eccentricity of the rings and seff-development

Let us examine in more detail the classical LaplaceMaxwell problem of the stability of a continuous ring around a planet. ${ }^{1,2}$ We write an equation for an elastic, absolutely flexible filament in the gravity field of a central body: ${ }^{106}$

$$
\begin{equation*}
\frac{\partial^{2} r}{\partial t^{2}}=-\frac{G M}{r^{3}} \mathbf{r}+\frac{1}{\mu_{0}} \frac{\partial}{\partial a} \mu_{0} c_{E}^{2} \frac{\partial \mathbf{r}}{\partial a}\left(1-\frac{1}{|\partial r / \partial a|}\right) ; \tag{117}
\end{equation*}
$$

here $r$ is the radius vector of a section of ring with the Lagrangian coordinate $a, \mu_{0}$ is the linear density of the ring, and the squares of the velocities of longitudinal elastic oscillations are $c_{E}^{2}=E / \mu_{0}$ ( $E$ is Young's modulus). We obtain the dispersion equation by the usual method of perturbation theory ${ }^{106}$

$$
\begin{gather*}
\left(\omega^{2}-\theta m^{2} \Omega_{\mathrm{E}}^{2}\right)\left[\omega^{2}+3 \Omega_{\mathrm{G}}^{2}+\left(1-m^{2}\right) \Omega_{\mathrm{T}}^{2}-\Omega_{\mathrm{E}}^{2}\right] \\
-\left[2 \Omega_{0} \omega+m\left(\theta \Omega_{\mathrm{E}}^{2}+\Omega_{\mathrm{T}}^{2}\right)\right]^{2}=0, \tag{118}
\end{gather*}
$$

where $\theta$ is the tension ( $\theta \equiv R / A>1$ ), $A$ and $R$ are the unperturbed and perturbed radii of the ring, $T=E(\theta-1)$, $\mu=\mu_{0} / \theta, T$ and $\mu$ are the stress and linear density, respectively, of the ring $\Omega_{\mathrm{T}}^{2} \equiv c_{\mathrm{T}}^{2} / R^{2}, c_{\mathrm{T}}^{2}=T / \mu$, and $\Omega_{\mathrm{E}}^{2}=c_{\mathrm{E}}^{2} / R^{2}$. Maxwell's dispersion equation is obtained from Eq. (118) in the absolutely rigid body limit, ${ }^{2}$ from which it follows that such a ring is unstable with respect to $m=1$ perturbations; the ring is displaced and falls onto the planet. But if one allows for the finite strength of a real ring, ${ }^{107}$ then Maxwell's conclusion turns out to be incorrect; the ring doesn't fall onto the planet, but is torn to pieces during a very small displacement. ${ }^{3}$

As was evident from the results of Secs. 5.1 and 5.4, an analog of such an instability is also observed in a disk of inelastic particles: the ellipse instability of a non-axisymmetric ( $m=1$ ) perturbation of a circular ringlet. Evidently just this dissipative instability caused the appearance of eccentricity for the narrow rings of Uranus and Saturn. We notice that resonance perturbations from satellites are too small and cannot be responsible for the observed eccentricities of the rings. It is evident from Sec. 5.4 that ring perturbations of the most different widths can be unstable for a suitable density. But one must consider that the time for differential precession to destroy eccentricity for wide rings is shorter than the time for the growth of eccentricity with ellipse instability present. Therefore, only the densest (and narrowest, no more than a few tens of kilometers in width ) ringlets, in which self-gravitation stabilizes the differential precession, can maintain their eccentricities. ${ }^{88}$ The following unsolved problem remains:

### 6.4.5. Why do rings not disperse because of diffusion? Why are ring edges sharp?

The main reason for diffusion stability of rings is in their non-circular form. A circular ring disperses by the drift of particles into neighboring quasi-circular orbits which do not intersect the main ring. In the dispersion of an elliptical ring, the closest orbits also will be elliptical with similar eccentricities. In a non-spherical field, a ring precesses with a rate different from the precession rate of the orbit of a particle which broke away from it (because of the difference in the semi-major axes). As a consequence of this, the orbit of the particle inevitably intersects the ring, and the particle, after
using up the energy of its relative motion in inelastic collisions, is again squeezed into the main mass of particles. We estimate how far a ring "allows" particles to get before recapturing them. This distance will also be an estimate of the sharpness of the edge of a ring. We examine two embedded non-intersecting ellipses (with coinciding foci and almost coinciding apocenter directions) with the semi-major axes $a$ and $a+\delta a$ and eccentricities $e$ and $e+\delta e$ ( $\delta a \ll a$ and $\delta e \ll e)$. We find the angle $\Delta \varphi_{\text {min }}$, the minimum angle between the lines of apsides at which the ellipses are in contact:
$\Delta \varphi_{\min } \approx \arccos \left[-\left(\frac{\delta a}{2 a e}+\frac{a \delta e}{\delta a}\right)\right]-\arccos \left(\frac{\delta a}{2 a e}+\frac{a \delta e}{\delta a}\right)$.

Taking into account that $\delta e$ is associated with collisions and is comparable with thermal eccentricities ( $\delta e \sim h / a$ ), we obtain the condition for "catching up": $h \leqslant \delta a \leqslant 2 a e$. For the strong inequality $h \ll \delta a \ll 2 a e, \Delta \varphi_{\text {min }} \approx(\delta a / 2 a e)+(a \delta e /$ $\delta a)$. The edge of the ring catches up to a particle because of the differential precession $\Delta \omega_{\mathrm{N}}$ after the characteristic time

$$
\begin{equation*}
t_{\mathrm{p}} \approx \frac{\Delta \varphi_{\min }}{\Delta \omega_{\mathrm{p}}} \approx \frac{(\delta a / 2 a e)+(a \delta e / \delta a)}{(21 / 4) J_{2} \Omega\left(R_{\mathrm{p}} / a\right)^{2} \delta a / a} . \tag{120}
\end{equation*}
$$

Comparing Eq. (120) and the diffusion dispersion time for the edge $t_{\mathrm{a}}-(\delta a)^{2} / v$, we find the equilibrium $\delta a$. For example, the sharpness of the edge of the $\varepsilon$ ring $\delta a$ is 400 to 750 meters for a particle free flight time $t_{\mathrm{ff}}-1 / \Omega$ to $10 / \Omega$. This edge sharpness agrees well with the observational data. We notice that even a very small eccentricity ( $e \sim 10^{-5}$ to $10^{-6}$ ) must sharply increase the diffusion stability of the edge of the ring. A variable ring width can also play an important role in the diffusion stability of elliptical rings, i.e., an increased eccentricity of the outer edge in relation to the inner edge. Such an eccentricity gradient leads to a situation where the outer ring layers move faster than the inner ones near the pericenter and transfer their angular momentum to the inner layers.

### 6.4.6. The rings of Uranus and aerodynamic drag

The stability problems examined above are common to the elliptical rings of both Uranus and Saturn. But one more strong disruptive factor, discovered by "Voyager-2", acts on the rings in the Uranus system: aerodynamic drag on the particles by the upper layers of the very distended atmosphere of Uranus. Micron-size particles fall onto Uranus after a few hundred years, ${ }^{33}$ and the lifetimes of the rings themselves are a few million years. ${ }^{33}$ As is shown in Ref. 95, only the massive $\varepsilon$ ring, situated furthest of all from Uranus and surrounded by the two shepherd-satellites Cordelia and Ophelia, is stable. The stability of the other rings was incomprehensible. Furthermore, even if one assumes the existence of shepherd-satellites with radii about ten kilometers near the inner rings (larger satellites would have been discovered by "Voyager- 2 "), then, all the same, one cannot "save" the rings; the angular momentum flux from the satellites is two orders of magnitude less than that needed to neutralize the aerodynamic drag. ${ }^{95}$

Following Ref. 96, we examine a fundamentally different stabilization mechanism. A trail of fine dust which stretched towards Uranus from the inner edge of the $\varepsilon$ ring was visible on "Voyager-2" images. ${ }^{97}$ The optical thickness
of the dust layer is $\tau \approx 0.001$. The origin of this trail is connected with such factors as aerodynamic drag, braking on magnetic lines (for charged dust particles), and pulverization of $\varepsilon$ ring material by micro-meteorites and fluxes of magnetosphere particles. In moving towards Uranus, the dust flow from the $\varepsilon$ ring is captured and is again emitted by the other rings of Uranus. It is shown in Ref. 96 that the stability of the rings of Uranus is connected with their ability to extract from the dusty planetocentric flux the angular momentum necessary for their own stability. Actually, the velocities of dust particles in a circular orbit exceed the orbital velocities $V_{0}$ of the particles of an elliptical ring with eccentricity $e$ near apocenter by the amount $\Delta V=V_{0} e$, or $\Delta V=\Omega R e$. If the dust particles themselves move along elliptical orbits, then the relative velocity between the dust at pericenter and the ring particles at apocenter reaches the value $\Delta V=\Omega R\left(e+e_{\mathrm{du}}\right)$, where $e_{\mathrm{d} \text { a }}$ is the eccentricity of the dust particle orbits. The momentum which is imparted to the ring by the dust flow can neutralize the effect of aerodynamic drag. We introduce a self-stabilization coefficient for the rings.

$$
\begin{equation*}
S=\frac{2 \Delta V \partial m / \partial t}{V_{\mathrm{rn}} M_{\mathrm{rn}} \Omega} \tag{121}
\end{equation*}
$$

here $\partial m / \partial t$ is the rate of absorption of dust by a ring with mass $M_{\mathrm{rn}}$ and angular velocity $\Omega$, and $V_{\mathrm{r} n}$ is the radial drift velocity of the ring itself, which is mainly determined by the aerodynamic drag. If the coefficient $S \approx 1$, then the ring is stable. If $S \approx 0$, then the self-stabilization effect is small. We examine the following factors of ring stability.

### 6.4.7. The absorption of dust moving towards a planet under the influence of aerodynamic drag

Then $\partial m / \partial t=2 \pi R V_{\mathrm{du}} \sigma_{\mathrm{du}}$, where $V_{\mathrm{d} u}$ is the drift velocity of a dust layer with surface density $\sigma_{\mathrm{du}}$. The ratio of the drift velocities of the dust and ring equals ( $V_{d u} / V_{\mathrm{rl}}$ ) $=3 \sigma / 4 \rho \tau_{\mathrm{rn}} a_{\mathrm{du}}$; here $a_{\mathrm{du}}$ is the size of a dust particle, $\sigma$ is the surface density of a ring with optical thickness $\tau_{\mathrm{rn}}$ and width $\Delta r$. As a result, we obtain a simple expression for $S\left(\tau_{\mathrm{du}}\right.$ $=3 \sigma_{\mathrm{du}} / 4 \rho a_{\mathrm{du}}$ )

$$
\begin{equation*}
S=\frac{2 R\left(e+e_{\mathrm{du}}\right)}{\Delta r} \frac{\tau_{\mathrm{du}}}{\tau_{\mathrm{m}}} . \tag{122}
\end{equation*}
$$

We notice that the stability of a ring ceases to depend on the amount of aerodynamic drag, since the drag acts on both the ring and the dust flow. We examine $e+e_{\mathrm{du}}$, the sum of the eccentricities of the ring and dust. The eccentricity of neutral dust that is swept out of an elliptical ring by aerodynamic drag depends on two factors. First, the dust retains the eccentricity of the ring itself. This is connected with the fact that uniform drag reduces the eccentricity by $\Delta e$ and the semi-major axis by $\Delta R$, maintaining the proportion $\Delta R /$ $R \sim \Delta e / e$. Consequently, if one talks about dust particle displacements by $\Delta R \sim 1000 \mathrm{~km}$ (to the next ring), then $\Delta R$ / $r \sim \Delta e / e \sim 0.02$, which one can neglect. Second, the eccentricity of collision-free dust can increase during resonance interaction with the non-axisymmetric gravitational potential of an elliptical ring. Without allowing for the last factor, let us estimate the self-stabilization coefficient for the nearest circular ring to the $\varepsilon$ ring with $R=50660 \mathrm{~km}, \Delta r=16$ $\mathrm{km}, \tau_{\mathrm{rn}}=0.1$, and $e_{\mathrm{du}} \approx 0.008$, i.e., equal to the eccentricity of the $\varepsilon$ ring. We find $S \sim 0.5$. Consequently, this ring is close
to a stable state. $S$ decrease rapidly for the rings further in due to the fact that the dust loses its original significant eccentricity acquired by it upon leaving the $\varepsilon$ ring. $S$ again increases for the inner Rings 4,5, and 6, reaching values from 0.25 to 0.75 for Ring $6(R=42000 \mathrm{~km}, \Delta r=1$ to 3 km , $\tau_{\mathrm{rn}}=0.2$ to $0.3, e \approx 0.001$, and $e_{\mathrm{du}} \approx 0.002$ ). We notice that we did not allow for charged dust, which drifts towards the planet faster than neutral dust and can acquire significant orbital eccentricity by the action of the magnetic field (especially at places of resonance between the frequencies of magnetic field rotation and orbital revolution of the dust ). ${ }^{9}$ This difficult to estimate factor can significantly increase the stability of rings. Micrometeorites are one more source of a stabilizing dust flow.

### 6.4.8. Micrometeorite erosion of the outere ring

We recall that the width of the $\varepsilon$ ring exceeds the total width of the other dense rings by approximately a factor of two. Micrometeorites and fast magnetosphere particles create, in pulverizing the material of the $\varepsilon$ ring, flows of broken material with significant eccentricities. The apocenters of these flows will be distributed outside the $\varepsilon$ ring, and their pericenters will be inside the $\varepsilon$ ring orbit, in the zone of the remaining rings. The absorption by the inner ring of the flows of broken material from the $\varepsilon$ rings is a significant source of angular momentum. We estimate the dust absorption rate for a ring with width $\Delta r_{\text {rn }}$ and optical thickness $\tau_{\text {rn }}$ in the following manner: $\partial m / \partial t=2 \pi R \Delta r \sigma_{\mathrm{du}} \Omega \tau_{\mathrm{rn}}$, where $\sigma_{\mathrm{du}}$ is the surface density of the dust cloud created by micrometeorite pulverization of the $\varepsilon$ ring. Let us calculate $\sigma_{d u}$ by equating the rate of mass loss by the $\varepsilon$ ring to the rate of capture of this mass by the inner rings. We obtain

$$
\begin{equation*}
\sigma_{\mathrm{it}}=\frac{\sigma_{\mathbf{e}} \Delta r_{\varphi}}{t_{\mathrm{SS}} \Omega \sum_{i}^{\prime} \tau_{\mathrm{irn}} \Delta r_{\mathrm{irn}}} \tag{123}
\end{equation*}
$$

where $\sigma_{\varepsilon}$ is the decrease of the surface density of the $\varepsilon$ ring with its width $\Delta r_{\varepsilon}$ over its entire lifetime $t_{\mathrm{ss}}$, and $\Sigma \tau_{\mathrm{irn}} \Delta r_{\text {irn }}$ is the sum of the products of the widths of the inner rings times their optical thicknesses. With allowance for Eq. (123), we obtain the self-stabilization coefficients of the rings

$$
\begin{equation*}
S=\frac{2 R e_{\mathrm{du}}}{V_{\mathrm{ru}} t_{\mathrm{ss}}} \frac{M_{\mathrm{e}}}{\sum_{i} M_{\mathrm{irn}}}, \tag{124}
\end{equation*}
$$

where $M_{\varepsilon}$ is the mass of material lost by the $\varepsilon$ ring over its entire lifetime, and $\Sigma M_{\mathrm{rn}}$ is the total mass of the inner rings. Not only the mass of the material ejected from the $\varepsilon$ ring, but also that from the outer satellites is also included in $\boldsymbol{M}_{c}$. For $R=4.5 \cdot 10^{9} \quad \mathrm{~cm}, \quad e_{\mathrm{du}} \sim 0.1$ to $0.2, t_{\mathrm{ss}} \sim 10^{9}$ years, and $V_{\mathrm{rn}} \sim 10^{-6} \mathrm{~cm} / \mathrm{sec},{ }^{33}$ we find $S \sim 0.3$ to 0.6 for $M_{\varepsilon} / \Sigma M_{\mathrm{rn}} \sim 10$. If one assumes that the mass of the $\varepsilon$ ring equals $6.1 \cdot 10^{18} \mathrm{~g}$, and the total mass of the two widest of the inner rings, the $\alpha$ and $\beta$ rings (see Table III), is $8 \cdot 10^{16} \mathrm{~g}$ (from Ref. 95), then such a ratio appears to be completely realistic. The quoted estimates show that the action of all the above-listed factors for forming dust flows provides stability for the inner rings of Uranus, which turn out to be "trainbearers" "holding on" to the dust train of the most massive and stable $\varepsilon$ ring. The stabilizing action of the dust flow sig-
nificantly exceeds the action of the hypothetical shepherdsatellites whose existence had to be assumed in Ref. 95. We note that the stability of the rings is also the result of natural evolutionary selection; rings with unstable forms or insufficient influx of angular momentum have already been destroyed.

## 7. DISSIPATIVE STRUCTURES IN A PROTOPLANETARY DISK

### 7.1. Dissipative instabilities and the planetary distance relation

### 7.1.1. Introduction

The planetary distance relation has a history of many centuries. The possibility of explaining the planetary distance relation on the basis of the idea of gravitational instability in the protoplanetary cloud is investigated in Ref. 98. Gravitational instability of the solar protoplanetary disk is improbable according to modern cosmogonic ideas. ${ }^{99}$ Following Ref. 100, we examine dissipative instabilities of the protodisk as the cause of the origin of the regular arrangement of the planets.

If the planets were formed from annular perturbations which grew as a result of the action of an instability, then the distance between planets is the wavelength of the most unstable perturbations at a given point of the disk. ${ }^{98}$ Since the lengths of the most unstable waves are connected with the characteristics of the disk then, from the change of wavelength with orbital radius, it is easy to calculate the dependence of the disk characteristics on radius. ${ }^{98}$ Comparison of the dependences obtained with those hypothesized from modern cosmogonic models ${ }^{99}$ will allow one to estimate the reality of a scenario that is discussed for the formation of the regular structure of the Solar System.

The stability of a differentially rotating viscous disk is investigated in Refs. 13, 14, 15, 67, 68, 76, 77, 78, 79, 101, 102, and 108 (see Secs. 4 and 5). We apply the results of this analysis to the protoplanetary disk. We note that transport equations of the type of Eqs. (67) through (70) can describe both a laminar and also a turbulent disk. ${ }^{101.102}$ In the case of turbulent gas, the velocities $V_{\mathrm{r}}$ and $V_{4}$ will refer to the largescale motions of the gas, and $T$ will characterize not the thermal, but the turbulent velocities. The meaning of energy sources and sinks, and the expressions for transport coefficients are changed correspondingly. But the general form of the transport equations is maintained, and an analysis of them will give the same instabilities that are characteristic of a differentially rotating viscous medium. The dissipative instabilities that have been examined in Sec. 5 lead to two characteristic scales of layering of the disk that are comparable with the observed distances between the planets: $\lambda_{0} \approx 2 \pi h$ (diffusion instability) and $\lambda_{0}=c^{2} / G \sigma_{0}$ (quasi-secular instability). ${ }^{100}$ The characteristic time for the development of the instabilities is $\gamma \sim v k^{2}$. It is easy to show that, in the case of molecular viscosity, the time for the growth of even the shortest [ 0.3 astronomical units (a.u.)] waves is longer than the cosmogonic time. Thus, annular perturbations with $\lambda_{0} \sim 1$ a.u. can be formed sufficiently rapidly only for turbulent viscosity of the disk. From estimates of the turbulent dispersion of the protodisk, ${ }^{99}$ one may adopt $v_{T} \sim 10^{12} \mathrm{~cm}^{2} /$ sec for turbulent viscosity, which gives the following characteristic times of growth $t_{\mathrm{ch}} \sim 2 \cdot 10^{4}$ years for $\lambda_{0} \sim 0.3 \mathrm{a}$.u., and $t_{\mathrm{ch}} \sim 2 \cdot 10^{7}$ years for $\lambda_{0} \sim 10 \mathrm{a} . \mathrm{u}$. The basic time in the
existence of the protodisk, when it could be in a stable state and "awaiting" the onset of conditions for instability, could occur only in a definite, fairly short period of this evolution.

### 7.1.2. The hypotheticai characteristics of the protodisk

The Bode-Titius relation essentially describes a simple fact: the distance between planets increases proportionally to the distance from the Sun, $R_{n}=0.4+0.3 \cdot 2^{n}$ a.u. ( $n=0,1, \ldots, 6$ ). If One writes the interplanetary distance relation in the form

$$
\begin{equation*}
\lambda=\frac{2}{3}(R-0.4) \tag{125}
\end{equation*}
$$

where $\lambda=R_{n+1}-R_{n}$, and $R=\frac{1}{2}\left(R_{n+1}+R_{n}\right)$, then Eq. (125) corresponds to the Bode-Titius relation for all the interplanetary distances except the interval between Mercury and Venus. Let us examine the instability with the wavelength $\lambda_{0}=c^{2} / G \sigma_{0}$, where $c$ is the velocity of turbulent gas motions. One can obtain the observed interplanetary distances only by assuming that the turbulent motions are an order of magnitude slower than the speed of sound. By using Eq. (125), one can obtain a hypothetical dependence of $c$ on $R$, but this dependence is difficult to verify, both because of the simultaneous presence of the two parameters $c$ and $\sigma_{0}$, and because of the lack of detailed models for turbulence in the protodisk. The situation is considerably simpler with diffusion instability, whose scale depends only on the thickness of the disk and, consequently, on the molecular temperature of the gas.

According to the calculations of Safronov, ${ }^{53}$ the temperature in the central plane of the gas and dust disk can be described by the equation $T=100 / R^{1 / 2} K(R$ is in a.u.). If the arrangement of the planets corresponds to the subdivision of the protoplanetary disk into rings as the result of the action of diffusion instability (with the characteristic layering scale $2 \pi h$ ) then, setting $\lambda_{0}$ equal to the distance between the planets, one can calculate from $\lambda_{0}$ the thickness of the disk and, consequently, also its temperature

$$
\begin{equation*}
T=\frac{m h^{2} \Omega^{2}}{8 k_{\mathrm{B}}}=\frac{m G M}{8 k_{\mathrm{B}} \alpha_{\cdot}^{2}} \frac{\lambda_{0}^{2}}{R^{3}} ; \tag{126}
\end{equation*}
$$

$k_{\mathrm{B}}$ is the Boltzmann constant, here $T$ is the temperature in degrees Kelvin, $m$ is the mass of a hydrogen molecule (assuming a predominantly hydrogen composition for the protodisk), $M$ is the mass of the Sun, and the coefficient $\alpha .=\lambda_{0} / h$ determines the exact scale of the layering. We assume that the thickness of the disk is determined by the gas temperature, since the turbulent motions evidently have subsonic velocities. ${ }^{99}$ The results of solving this inverse problem are shown in Fig. 24, when the temperature of the protodisk is calculated over the actual distances between the planets at the moment of ring formation in a given region of the disk. It is evident from Fig. 24 that the values obtained for the temperature agree well with the temperature profile calculated in Ref. 53 (also see Ref. 99). The regions from Mercury to Venus and from Mars to the asteroids are exceptions. The heating of the regions of the disk closest to the Sun to almost the highest possible temperature values is natural, whereas the more distant regions of the gas disk are shielded and are heated only by the tangential solar radiation. We note that the temperature values calculated from Eq. (126) do not provide a picture at one given moment of time, since ring formation did not occur simultaneously. For example, due only to the increase of $\lambda_{0}$, the characteristic time for the formation of rings in the zone from Mars to the asteroids is seven times longer than that in the zone from the Earth to Mars, and is 21 times longer than that in the zone from Venus to the Earth. One must add to this that, according to the estimates of Refs. 53 and 103, the time for the accretional growth of a planet increases sharply with orbital radius from Venus to Mars. Thus, the mutual accretion of dust near the moment of the formation of the Martian ring and other factors could have increased the transparency of the zone of the Earth, and this is what led to an increased heating of the zone of Mars. In this way a temperature increase from the zone from Venus to the Earth to the zone from Mars to the asteroids can be explained. The data of Fig. 24 can serve as an important source of information about temperature and other conditions in the protocloud. One can draw the conclusion from the results of this section, that the regularity of the arrangement of the planets in the Solar System is connected


FIG. 24. Dependence of the protoplanetary disk temperature on distance from the sun. The upper two curves describe the decrease of temperature with distance for a black plate perpendicular of the sun's rays and for a black sphere (the open and dark small squares, respectively). The two lower curves are the temperature of a gas and dust disk according to Safronov. ${ }^{53}$ The dark circles correspond to the central plane of the disk, and the open circles correspond to the height of a uniform atmosphere. The hatched rectangles indicate the temperatures calculated from the observed locations of the planets by means of Eq. (126) for $\alpha=2 \pi$.
with a smooth change of the thickness of the protodisk as a function of radius and with the action of diffusion instability.

### 7.2. Self-organization of the solar system and prospects for investigations in planetary physics

The process examined in the previous section of selforganization of dissipative annular structures in the protoplanetary disk, which formed a regular series of planets, shows the degree of the possible influence of instabilities on cosmogony. Allowance for all the self-organizing factors (the development of the diffusion, quasi-secular, accretion, and ellipse instabilities, and the generation of spiral waves and of other structures) in modern cosmogonic theories will enable one to approximate closely a detailed model for the formation and evolution of our planetary system. Such a model can give valuable information about terrestrial processes on planet-wide scales and will further the formulation of a process for exploring the solar system on a scientific basis, with allowance for the prediction of the future evolution of the planetary system under conditions of a) natural development, b) technogenic loading, and c) guided evolution.

The wealth of self-organization processes in the solar
system is determined by the following features: ${ }^{104}$

1. The presence of long-lasting energy sources; solar radiation (thermonuclear reactions); radioactive heating of large bodies (nuclear decay reactions);
gravitational compression of planets (especially of the giant planets);
the energy of orbital revolution and rotation;
impact heating (by meteorites and planetesimals); and the gravitational effect of neighboring bodies (tides and resonances).
2. A variety of distributed media in which collective self-organization processes can proceed:
the solid media of terrestrial type planets, satellites, comets, etc.;
the liquid material of the terrestrial oceans, and the envelopes of satellites and planets;
gaseous media of different densities, the atmospheres of the planets, satellites, and comets;
the interplanetary and solar plasmas;
the gas and dust media of protosatellite and protoplanetary clouds; and the large particle media of planetary rings, protosatellite swarms, and disks of planetesimals.

TABLE IX. 1) Investigations of planets and satellites by means of automatic interplanetary stations.

| Planet | Number of automatic interplanetary stations which investigated the planet |  | Automatic interplanetary station mames and the years they investigated a planet (USSR; USA) |
| :---: | :---: | :---: | :---: |
|  | USSR | USA |  |
| Mercury | - | 1 | "Mariner-10", 1974 and 1975 |
| Venus | 15 | 5* | "Venera-4 through Venera-16" and "Vega-1 and -2"; ( 1967 through 1985): "Mariner- $2,-5$. and -10 , "Pioneer-Venus-I and -2 " ( 1962 through 1968) |
| Mars | 6 | 6 | "Mars-2, -3, -4, -5, and -6" ( 1971 through 1974), and "Fobos-2" (1989); "Mariner-4, -6, -7, and -9 ", and "Viking-1, and -2 " (1965 through 1976) |
| Jupiter | - | 4* | "Pioncer- 10 and -11 " and "Voyager- 1 and -2 " ( 1973 through 1979) |
| Saturn | - | 3 | "Pioneer-11" and "Voyager-1 and -2" ( 1979 through 1981) |
| Uranus | - | 1 | "Voyager-2" (1986) |
| Neptune | - | 1 | "Voyager-2". (1989) |
| Pluto | - | - |  |

2) Summary.

| Efforts and results | USSR | USA | Comments |
| :---: | :---: | :---: | :---: |
| Number of automatic interplanetary stations launched | 30 | 18 | No automatic interplanetary stations have been launched in the USA since 1978* |
| Number of automatic interplanetary stations which transmitted information | 21 | 15 | 12 failed automatic interplanetary stations (loss of radio communication, unsuccessful launches) were launched to investigate Mars and Venus |
| Number of planet encounters with results | 21 | 21 | A "Planet encounter" is an investigation of one planet by one spacecraft |
| Number of planets studied | 2 | 7 | Neptune is the seventh planet (not counting the Earth) investigated by American Automatic Interplanetary Stations |
| Number of satellites studied |  | 48 | American Automatic Interplanetary Stations discovered! 27 satellites and obtained photographs of the surfaces of 23 large satellites of planets. |
| *Launches made recently: towards Venus, the "Magellan" Automatic Interplanetary Station (1989) and "Galileo" Automatic Interplanetary Station (1989); Towards Jupiter, "Galileo" (1989). |  |  |  |

## 3. Two types of structure formation:

the origin of structures that are ordered in space and time in the protoplanetary and protosatellite disks; and the appearance of individual subsystems (planets, satellites, comets, etc).

The subsystems are a consequence of structures which selforganize and themselves generate the structures of a new order.

Thus, modern cosmogonic theory must consider the solar system as a complicated hierarchy of subsystems which self-organize, are saturated with energy sources, and possess a surprising wealth of collective processes.

The role of the cosmogonic approach in investigating the solar system is now growing strongly. This is connected with the completion of the epoch of "great geographic discoveries' out to the known limits of the solar system (eight of the nine planets of the solar system had been investigated "close-up", by means of spacecraft by the end of 1989), and by the accumulation of a gigantic volume of observational and theoretical information. A transition to systematic investigations of the solar system, which requires a preliminary estimate of the effectiveness of space projects, and the development of observations and experiments which give key information to solve fundamental problems of planetary physics, is inevitable. The consideration of modern theoretical models of the evolution of the solar system, which will indicate the most promising objects for investigation and experiments, is important here. In our opinion, among the latter is Project "Kronos" that is proposed in Ref. 105 and which proposes the building of a long-lasting satellite of Saturn and direct probing of its rings (a flyby investigation of the Jupiter system and of one of the asteroids can also be planned in this connection). An investigation of planetary rings can give unique information on collective processes in disks of many particles and can play the role of a catalyst for the development of cosmogonic models. It seems that Project "Kronos" could compensate to a small degree for the lag of the USSR in the field of investigating the outer planets (see Table IX from Ref. 105). We note that the building of orbital probes for both Jupiter (Project "Galileo") and for Saturn (Project "Cassini") is being planned in the USA.

## 8. CONCLUSION

The investigation of the physics of planetary rings during the last 10 to 12 years has led to very important results and has revealed a number of previously unknown processes and effects. The reason for the very existence of rings has been determined, and we have succeeded for the first time in determining the density of particles of the rings of Saturn and in studying the process of collisional destruction of the loose bodies of which planetary rings consist by means of the theory of the azimuthal brightness asymmetry effect. Papers on collective processes in rings turned out to be very fruitful: the diffusion, quasi-secular, thermal and accretion instabilities have been discovered, and also the ellipse-mode instability, which is responsible for the eccentricity of the narrow rings of Uranus. Apparently this is still not a complete list of the possible structure-forming instabilities in differentially rotating disks of inelastic particles. The dynamics of such uniquely stable objects as the rings of Uranus, which arose due to the resonance action of its outer satellites, became
better understood. Already the first application of the models developed to the entire solar system enabled one to evaluate in a new way the reasons for the regularity of the planetary orbits, which are subject to some planetary distance relation.

Planetary rings have turned out to be unique sanctuaries for natural processes of self-organization, which play a fundamental role in the cosmogony of the solar system.
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