# Large-scale fractal structure in laboratory turbulence, astrophysics, and the ocean

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It is demonstrated that large-scale fractal structure in laboratory turbulence, the ocean, and the clustering of galaxies may have a common percolation nature. The fractal dimensionality of all these structures is shown to be approximately 4/3 (much smaller than 3, the dimensionality of spatial motion). The author compares laboratory measurements and natural observations of various scientists.

*(Dedicated to the memory of the remarkable scientists Ya. B. Zel'dovich and A. N. Kolmogorov)*

In recent years fractal structures have attracted the interest of specialists in hydrodynamics and astrophysics. Various theoretical models involving fractal structures have been proposed and experimental observations, both in the laboratory and in nature, have been carried out (see, for example, Refs. 1–7). All this research points to a fundamental question: how general are the observed fractal properties? In this note it is shown that similar large-scale fractal structures (of fractal dimension $D \approx 4/3$) are found in laboratory turbulence (on a length scale of $\sim 10^{-1}$ m), in the ocean, and in the clustering of galaxies (on a length scale of up to $10^2$ Mps). These structures have a percolation nature [8–10].

**1**. We can break up the region of turbulent motion of a liquid into cubical cells of edge length $\eta$ (Kolmogorov scale[11]). Vortices of this size are quickly damped out by viscosity. At a given moment in time the motion will be turbulent in some cells and laminar (vortex-free) in others.

We can introduce the probability $p$ that a given cell exhibits turbulence. When $p = 0$ there are no turbulent cells, when $p \ll 1$ the clusters of turbulent cells are small, whereas when $p = 1$ all cells exhibit turbulence. There exists a critical concentration $p_c$, $0 < p_c < 1$, corresponding to the situation when an infinite cluster of turbulent cells first forms. This infinite cluster radically alters the situation. Before its appearance any energy introduced to the system contributes to dissipation and increases the number of turbulent cells. As an infinite cluster forms, additional energy can be channeled to "infinity", i.e. out of the region of turbulence.

In the presence of an infinite cluster, the concentration of turbulent cells can increase due to fluctuations. But the new, fluctuation-induced turbulent cells will be damped out by viscosity, since their energy is not constantly replenished. For the same reason, the only stable part of the infinite cluster is the backbone: the collection of cells belonging to infinite paths through the cluster. The other, dead-end branches of the cluster will decay due to viscosity that is not counteracted by a constant energy input. The critical clusters that are formed in such percolation systems are fractal objects.[8]

**2**. The formation of an infinite cluster is a critical phenomenon. The characteristic extent $l$ of a vortex cluster near $p_c$ diverges[8]

$$l \sim |p_c - p|^{-\nu}. \tag{1}$$

The critical exponent $\nu$ is universal, depending only on the topological dimensionality $d$ of the space.

This exponent has been derived by various authors employing different models. For $d = 3$ one finds, approximately, $\nu \approx 0.9$ (Ref. 8).

**3**. Let us relate the critical exponent to the fractal dimensionality $D_s$ of the cluster backbone. Given an initial large-scale velocity field (only vortices of size $l_0$ are excited), the cascade process of scale division will produce a hierarchy of vortices of size $l_n \sim q^{-n} l_0$ ($q$ is the multiplicity of the scale division). Since the energy transfer over the cascade is chaotic, the anisotropy and large-scale inhomogeneity of the initial velocity field exerts less and less influence on the statistical properties of pulsations on smaller scales. Hence on a sufficiently small scale ($l_0 \gg l_n \gg \eta$) one should observe scale invariance and local isotropy. For isotropic pulsations the energy distribution over length scales ($l \sim k^{-1}$, where $k$ is the wave number) is given by the spectral density $E(k)$. If we consider the characteristic period $T_m$ of pulsations arising at the $m$th division, then from dimensionality arguments or simple physical considerations[11] we have:

$$T_m \sim (E(k)\, k^3)^{-1/2}, \quad k_m \sim l_m^{-1}. \tag{2}$$

The characteristic period $T_m$ can be interpreted as the time required for vortices of size $l_m$ to excite vortices of size $l_{m+1}$. The excitation time of the total vortex cascade is

$$t_\infty \sim \sum_{m=0}^{\infty} T_m. \tag{3}$$

Generally, expression (2) for $T_m$ cannot be substituted into (3) because (2) is valid only for sufficiently large $m$. However, we are really interested not in $t_\infty$, but in the quantity

$$(t_\infty - t_M) \sim \sum_{m=M}^{\infty} T_m, \tag{4}$$

and therefore expression (2) becomes valid for sufficiently large $M$.

If $M$ is large enough, the system exhibits scale invariance and scaling behavior:[11]

$$E(k) \sim k^{-\alpha}. \tag{5}$$

From (5) and (2) we obtain

$$(t_\infty - t_M) \sim \sum_{m=M}^{\infty} l_m^{(3-\alpha)/2}. \tag{6}$$

Taking $l_m \sim q^{-m} l_0$ as before, one then obtains from (6)

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$$(t_\infty - t_M) \sim \sum_{m=M}^{\infty} q^{-m(3-\alpha)/2} \sim q^{M(\alpha-3)/2} \tag{7}$$

(for sufficiently large $M$).

After $M$ divisions a single initial vortex of size $l_0$ will break up into

$$N \sim q^M \tag{8}$$

vortices of size $l_m \sim q^{-M} l_0$. In the intermediate asymptotic region ($l_0 \gg l_M \gg \eta$) we obtain from (8) and (7)

$$N(t_M) \sim (t_\infty - t_M)^{2/(\alpha-3)}. \tag{9}$$

This system of vortices (vortex cluster[3]) will occupy some volume in a region of effective size $l_*$, with

$$N \sim l_*^{D_s}, \tag{10}$$

where $D_s$ is the fractal dimension of the vortex cluster.[3] Then, from (9) and (10) we find

$$l_*(t_M) \sim (t_\infty - t_M)^{2/(\alpha-3)D_s}. \tag{11}$$

If we interpret this result within the framework of percolation theory, $t$ approaches the critical value $t_\infty$ just as $p$ approaches $p_c$. Let us define $(t_\infty - t) = \tau$. Clearly as $\tau \to 0$, $(p_c - p) \to 0$. Under the usual assumption that the dependence of $(p_c - p)$ on $\tau$ is analytic for small $\tau$, we obtain $(p_c - p) \sim \tau$. Comparing expressions (1) and (11) one finds

$$\frac{2}{3-\alpha} = \nu D_s. \tag{12}$$

If we take $D_s$ in (12) to reflect the stable fractal dimensionality of the vorticity (dissipation) field, then it follows from Ref. 3 that $\alpha = D_s$ (for $d = 3$) and hence, from (12),

$$D_s = \frac{3}{2} \pm \left(\frac{9}{4} - \frac{2}{\nu}\right)^{1/2}. \tag{13}$$

Substituting the value $\nu \approx 0.9$ from Ref. 8 obtained from numerous percolation studies into (13) we find

$$D_s \approx \frac{5}{3}, \tag{14}$$

$$D_{ss} = \frac{4}{3}. \tag{15}$$

Since for $d = 3$, $\alpha = D_s$ (Ref. 3), the first value (14) yields the well-known Kolmogorov–Obukhov spectrum.[11] As for the second value $D_{ss} \approx 4/3$ (15), it probably corresponds to the large-scale percolation structures usually described in percolation theory as the "elastic backbone".[12] This substructure of the infinite cluster backbone comprises only the shortest paths connecting sufficiently remote points. Numerical simulations reported by Herrmann *et. al.*[12] yielded $D_{ss} = 1.35 \pm 0.05$, in agreement with (15).

**4.** The spatial scale required for the formation of the "elastic backbone" in percolation turbulence should correspond to the integral scale of the turbulence ($L$) (Ref. 11), i.e. to the characteristic scale of the velocity pulsation field over which there still exists a noticeable correlation between field values at two different points. Consequently, the formation of the "elastic backbone" is blocked by spontaneous scale invariance breakdown over distances of order $L$ (Ref. 13). Nonetheless, it appears that under certain, as-yet-undetermined conditions the "elastic backbone" appears despite this spontaneous scale invariance breakdown. Figure 1, taken from Ref. 14, illustrates the energy spectrum of turbulence observed experimentally beyond a hydrodynamic lattice. The data are plotted logarithmically and the straight lines emphasize the effects of scaling laws with $\alpha \approx 5/3$ and $\alpha \approx 4/3$.

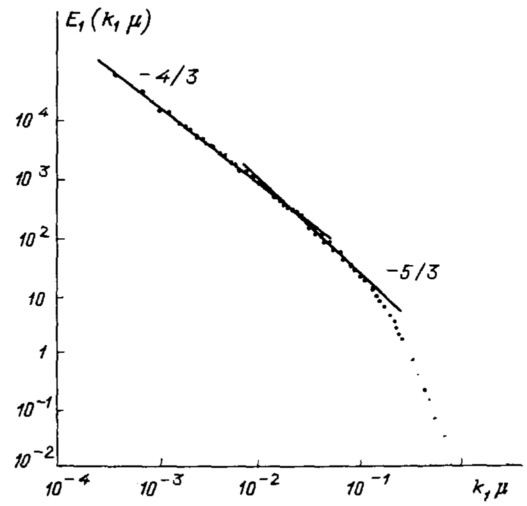


FIG. 1.

In Fig. 2, taken from Ref. 15 (see also Ref. 16, p. 181), we find analogous results in the measurements of oceanic thermoclines on a 100 m horizon. Here also the large-scale end of the spectrum contains regions exhibiting $\alpha \approx 4/3$ scaling (shown by labeled line segments).

In this regard, the direct observation of turbulence-induced fractal structures that apparently occurs in radar measurements of cloud structure[17–19] is of considerable in-

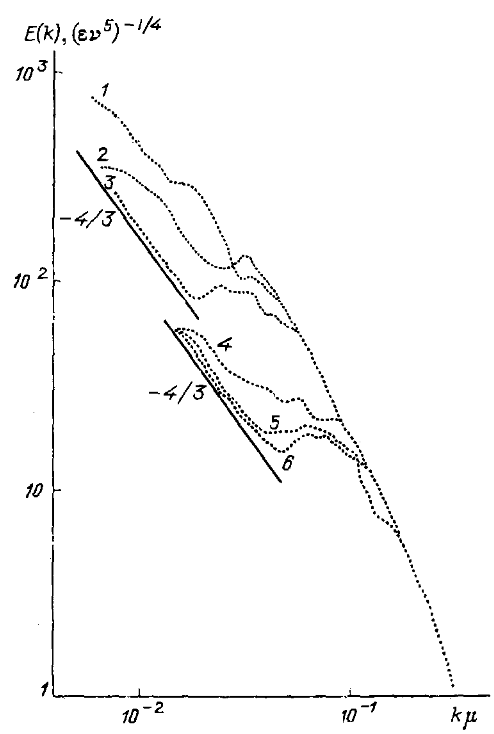


FIG. 2.

terest. Which of the structures: the unstable one, the backbone or the elastic backbone is seen in such a direct observation? Once again, the answer here probably depends on the range of length scales $D$[1)] used in the measurement.

**5**. Another example of a direct observation of large-scale fractal structures has been provided by astrophysicists. If we assume that the matter in the universe is in turbulent motion, then the large-scale aggregations of matter—galaxies and galaxy clusters[3–7]—should occur in regions characterized by active energy dissipation, i.e. the fractal dimensionality of the turbulent dissipation (vorticity) field should coincide with the fractal dimensionality of the matter density field, as given by the galaxies and their clusters. Observations of the fractal character of the large-scale structures in the universe (galaxies and galaxy clusters) have produced[4–7] a fractal dimension $D = 1.3 \pm 0.1$, close to the "4/3" value.

Thus we find that the same type of large-scale fractal structures can be observed in laboratory turbulence, in the ocean and atmosphere, and over length scales comparable to the size of the visible universe.

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