Optical superradiance: new ideas and new experiments

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An orderly account is given of the processes of coherent propagation, amplification, and generation of pulses in polyatomic resonant media. The characteristic features of the propagation of solitons in resonant media of finite length are considered. Comparisons are made of the exponential and "lethargic" amplification of pulses in resonant media. It is shown that an allowance for the reflection of the field from the boundaries of an active medium gives rise to a qualitatively different dependence of the parameters of the pulses on the reflection coefficients under superradiance and lasing conditions, which makes it possible to introduce a criterion to distinguish lasing from superradiance. An analysis is made of the characteristics of collective superradiance, which is a new case of coherent generation of pulses. Experimental and theoretical investigations of rf superradiance emitted by systems of nuclear spins are reviewed.

1. INTRODUCTION

Optical superradiance is one of the most striking examples of the cooperative behavior of polyatomic systems. It is different from other collective processes because the system itself goes over from an uncorrelated to a correlated state as a result of internal interactions and then the correlation radius of the moments of the transitions of the particles assumes a macroscopic value. Superradiance was predicted by Dicke[1] back in 1954 before the discovery of lasers. However, superradiance was detected experimentally only after lasers became available[2] and the requirements which pumping of the active medium had to satisfy were fulfilled specifically by laser radiation. It is well known that superradiance and lasing (stimulated emission) represent two limiting cases of the same process. Any process generating oscillations is based on the interaction of two subsystems, one of which is the medium in which energy is stored and the other is the field to which this medium transfers the stored energy. Coherent radiation is generated if at least one of the interacting subsystems is ordered, i.e., if it has the fewest possible number of the degrees of freedom between which its energy may be distributed. In the ideal case both the radiators in the active medium and the field have the same resonance frequency ω_0 with infinitesimally small widths $\Delta\omega_a$ and $\Delta\omega_f$ and, moreover, the angular distribution of the radiation field can be extremely narrow. In lasers such ordering is attained by discrimination of the radiation field modes. Narrowness of the field line $\Delta\omega_f$ is achieved by the use of high-Q resonators. Therefore, the field should remain sufficiently long in the resonator, compared with the transit time across the resonator, so that it would make at least several round trips. We shall be interested in the free oscillation case, when an initially excited medium generates a radiation pulse as a result of spontaneous decay. The field line width is then governed by the length of the medium l. If no reflection of the field takes place at the boundary of the medium, then $\Delta\omega_f = 1/\tau$, where $\tau = L/c$ is the transit time of a photon crossing the medium. However, if reflection of the field does occur at the boundaries of the medium, then

$$\Delta\omega_f = \frac{1}{\tau}\ln\frac{1}{r_1 r_2},$$

where r_1 and r_2 are the amplitude reflection coefficients. If the length of the medium is much less than the resonator length, then the resonator length L occurs in the above expressions. However, in this case the active medium has to be excited synchronously with the arrival of a pulse so that we are then dealing with the amplification of an external pulse which should be considered separately. The line width of an atomic transition is $\Delta\omega_a = 1/T_2$. If $\Delta\omega_f \ll \Delta\omega_a$, the ordering is produced by the field. In the opposite limiting case $\Delta\omega_f \gg \Delta\omega_a$, it is produced by the medium. The former case is close to lasing and in the case of mirror-free systems is called superluminescence or amplification of spontaneous radiation. The second case, when the ordering is due to the narrowness of the atomic transition line, is called superradiance. Discrimination of the field modes (along the direction of emission of the photons) is achieved in both cases by selection of an acicular shape of the active medium. A wave traveling along the axis is then amplified much more strongly than waves traveling along other directions.

A system of atoms interacting with a radiation field can be described by the following parameters: N/V is the density of atoms; $\Delta\omega_a = 1/T_2$ is the width of an atomic transition line; L is the length of the medium governing the width of the field line $\Delta\omega_f = c/L$; T_1 is the spontaneous decay time of an individual atom; r_1 and r_2 are the reflection coefficients at the boundaries of the medium. If the medium is characterized by inhomogeneous broadening, we have to add an inhomogeneous broadening parameter $\Delta\omega^* = 1/T_2^*$. In the present review we shall try to answer the following questions. How do the parameters of free oscillation pulses depend on the parameters just listed? How and to within what limits can the parameters of the output pulses be controlled? Which regime—superradiance or superluminescence—is more convenient from the point of view of the highest rate of energy transfer from an excited medium to the radiation field? In comparisons of the efficiencies of the different re-

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gimes it is, in our opinion, convenient to select such a normalization of the field amplitude $a(x, t)$ that the field energy density is given by $W(x,t) = \hbar\omega_0 (N/V) n(x, t)$. In this case $n(x, t) = |a(x, t)|^2$ represents the density of the number of photons in units of the density of the number of particles. Consequently, the time-integrated value of $\overline{n(x, t)}$ determines the number of the emitted photons, calculated per one atom. The pulse duration unit can be conveniently the transit time $\tau = L/c$. Clearly, the minimum duration of a radiation pulse or a train of pulses is a quantity of the order of τ. Such normalization is introduced in Sec. 2. The answers to the questions posed above will provide information on the feasibility of controlling the parameters of radiation pulses and suggest ways of optimization of the oscillation process.

There are several methods which can be used to generate pulses with specified parameters (see, for example, Refs. 3–10). In all these cases a pulse is formed in a multistage system. However, in our opinion, all the possibilities of generation of pulses with given parameters already in the first stage have not yet been exhausted. This problem is becoming even more urgent because the appearance of femtosecond pulses has opened up extensive opportunities for ultrafast excitation of media, which should make it possible to utilize the regime of coherent collective oscillation. Theoretical and experimental investigations of superradiance[9–26] have been usually carried out independently of the theory of lasing. In our opinion, the time has come to unify and compare the various theories.

The optimization of the oscillation process in terms of parameters of the active media does not cover all the possibilities of control of radiation parameters. In fact, the general problem of the interaction of the atomic and field subsystems depends also on the initial and boundary conditions. The boundary conditions are taken into account above by including the two reflection coefficients r_1 and r_2 as parameters of the system. However, the initial conditions may influence greatly the nature of radiative decay of our system. For example, supersymmetric solutions of the general system of equations describing the interaction of an ensemble of two-level atoms of the resonant field were obtained in Ref. 27 and it was shown there that the selection of the initial conditions corresponding to these solutions results in generation of pulses of maximum intensity. The process is essentially as follows. The high rate of decay of superradiance systems is due to the fact that the atomic subsystem reaches a state which is fully symmetric under the transpositions of atoms. It is known that this situation can occur only in two cases: in a system with dimensions much less than the radiation wavelength when an extended polyatomic system interacts with one of the field modes. In real extended systems an inhomogeneity of the field amplitude along a sample disturbs the symmetry of the atomic subsystem. This increases the duration of the output radiation pulses. It was shown in Ref. 27 that there are states which allow for the symmetry of the complete atomic–field system and not only of the atomic subsystem. The process of radiative decay in an extended polyatomic system then occurs as if in a lumped system: it is characterized by the maximum rate and the minimum duration for a given inversion density.

This range of topics is closely related to the problems of propagation of pulses in resonant media under coherent interaction conditions.[28–37] One of the most striking effects occurring in the course of such propagation is the self-induced transparency, which had been the subject of thorough experimental[10,11,28,29] and theoretical[30–35] investigations. However, the recent developments in the theory of propagation of 2π pulses in resonantly absorbing media had been accompanied by the growth of a theory of pulse propagation in resonantly amplifying media (see Ref. 36 and the bibliography given there). In the latter case, a theory of semiinfinite media yields, for certain values of the parameters of the problem, seemingly nonphysical predictions such as the propagation of pulses at superluminal velocities, etc. It was pointed out already in Ref. 36 that a correct interpretation can be provided if we consider bounded media. We shall derive soliton solutions for spatially inhomogeneous media (including those which are spatially confined), which provide a clear interpretation of all the characteristics of soliton propagation in amplifying media. We shall demonstrate the feasibility of control of the state of a medium and extensive opportunities for the generation of pulses with a given profile particularly in multilevel systems.

A theory of superradiance has grown out of an analysis of the problem of radiative decay of a polyatomic system of dimensions much less than the radiation wavelength. For a long time it has been assumed that this problem is of purely model nature and its attractiveness is simply due to the feasibility of providing an exact description of its dynamics. However, recent experiments[38–40] have shown that the model has a physical realization, which is undoubtedly of interest in the theory of superradiance because it makes it possible to test the fundamentals of this theory.

A discussion of the topics mentioned above represents the bulk of the present paper. In Sec. 2, we shall derive equations of a semiclassical theory of resonant interaction of radiation with matter, reduce the equations obtained to the dimensionless form, and discuss both the initial and boundary conditions. We shall also determine the integrals of motion. In Sec. 3 we shall analyze a linear stage of the oscillation pulses. We shall show that this analysis allows us to find the delay time of a pulse, the amplitude of the field at its maximum, and qualitative influence of inhomogeneous broadening on the parameters of the radiation pulses. We shall consider the dynamics of the correlation properties of the field in the process of oscillation and find the dependence of the correlation radius of the field on the characteristic parameters of the problem. In Sec. 4 we shall discuss the problems of coherent amplification of pulses in resonant media and consider the conditions for realization of the "lethargic" and exponential amplification. We shall consider the characteristics of propagation of soliton pulses in resonantly amplifying media of finite length. We shall demonstrate that it is possible to control the motion of level populations in an active medium and the parameters of amplified pulses at the exit from the medium. In Sec. 5 we shall present graphically the dependences of the main parameters of the output pulses (peak intensity and pulse duration) on the parameters of the medium. We shall determine the limits of regions of saturation of the intensity within which the peak value of the intensity does not increase when the inversion density is increased. We shall determine the dependences of these parameters of the pulses on the amplitude reflection coefficients at the boundaries of the medium, demonstrating the feasibility of optimization of the superradiance pulse parameters and allowing for introduction of mathematical criteria of the oscillation regime. An analysis of the feasibility of optimization of the parameters of the output pulses by variation of the initial conditions will be shown to lead to a prediction of a new regime which is the collective superradiance. A discussion of the characteristics of this regime is given in subsection 5.3. In Sec. 6 we shall discuss the characteristics of the superradiance emitted by a system of nuclear spins generating coherent radiation with a wavelength much greater than the size of the active medium. We shall propose in the same section a variant of a theory of spin superradiance, which relates the main parameters of the output pulses to the microscopic characteristics of an ensemble of nuclear spins.

2. SEMICLASSICAL SYSTEM OF EQUATIONS FOR THE INTERACTION OF A TWO-LEVEL SYSTEM OF ATOMS WITH A RESONANT ELECTROMAGNETIC FIELD

2.1. Wave equation

A system of semiclassical equations describing the interaction of radiation with matter is obtained by averaging a quantum system of equations (A1.10) (see Appendix 1) and by semiclassical decoupling of operators:

$$\langle\sigma_\pm A\rangle=\langle\sigma_\pm\rangle\langle A\rangle,\quad \langle\sigma_3 A\rangle=\langle\sigma_3\rangle\langle A\rangle. \tag{2.1}$$

If the interaction between radiators is via a transverse electromagnetic field, the system of equations becomes

$$\begin{aligned}
&\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A(\mathbf{r},t)=-\frac{4\pi}{c}(j^+ + j^-),\\
&\frac{\partial j^+}{\partial t}-i\omega_0 j^+ + \frac{j^+}{2T_2}=\frac{i|m|^2}{\hbar c}A\rho,\\
&\frac{\partial j^-}{\partial t}+i\omega_0 j^- + \frac{j^-}{2T_2}=-\frac{i|m|^2}{\hbar c}A\rho,\\
&\frac{\partial\rho}{\partial t}+\frac{\rho-\rho_e}{T_1}=\frac{2i}{\hbar c}(j^+ - j^-)A,
\end{aligned} \tag{2.2}$$

where

$$j^\pm = m\sum_{i=1}^{N}\langle\sigma_{i\pm}\rangle\delta(\mathbf{r}-\mathbf{r}_i),\quad \rho=\sum_{i=1}^{N}\langle\sigma_{i3}\rangle\delta(\mathbf{r}-\mathbf{r}_i),\quad m=\mathbf{e}m,$$

$\mathbf{e}$ is a unit vector of the wave polarization, T_2 is the transverse relaxation time, T_1 is the longitudinal relaxation time, and ρ_e is the equilibrium value of the difference between the populations. In the derivation of the system (2.2) we ignored the effects of inhomogeneous broadening and assumed that $m_i = m_0$, which is satisfied in the absence of level degeneracy (i.e., in the case of a true two-level system). In the absence of relaxation, i.e., when $T_2 = T_1 = \infty$, the system of equations (2.2) has a well-known integral of motion

$$j^+(\mathbf{r},t)\,j^-(\mathbf{r},t)+\frac{|m|^2}{4}\rho^2(\mathbf{r},t)=\text{const}. \tag{2.3}$$

Using the analogy between a system of classical oscillators and a two-level system of atoms, the equations (2.2) are frequently written in the following equivalent form:

$$\begin{aligned}
&\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A(\mathbf{r},t)=-\frac{4\pi}{c}j(\mathbf{r},t),\\
&\frac{\partial^2 j}{\partial t^2}+\frac{1}{T_2}\frac{\partial j}{\partial t}+\left(\frac{1}{4T_2^2}+\omega_0^2\right)j=-\frac{2\omega_0|m|^2}{\hbar c}A\rho,\\
&\frac{\partial\rho}{\partial t}+\frac{\rho-\rho_e}{T_1}=\frac{2}{\hbar\omega_0 c}A\frac{\partial j}{\partial t},
\end{aligned} \tag{2.4}$$

where $j=j^+ + j^-$. In this case if $T_2 = T_1 = \infty$, the integral of motion (2.3) becomes

$$\left(\frac{\partial j}{\partial t}\right)^2+\omega_0^2 j^2+|m|^2\omega_0^2\rho^2=\text{const}. \tag{2.5}$$

Therefore, the system of equations (2.2) or (2.4) is characterized in the $T_2 = T_1 = \infty$ case by the law of conservation (2.3) or (2.5), which is the basis of the description of the processes of coherent resonant interaction of radiation with matter. It should be pointed out particularly that Eqs. (2.3) and (2.5) contain quantities that oscillate at the optical frequency. Traditionally these conservation laws are deduced from the equations for slowly varying amplitudes. Moreover, it should be pointed out that if $j^\pm$ are functions of the complex variables in a system (2.2), then j in the system (2.4) is a real quantity.

We shall adopt the plane wave approximation. We shall introduce slowly varying amplitudes $A_{1,2}(x,t)$ for the right- and left-hand waves, described by the expression

$$A(\mathbf{r},t)=A_1(x,t)\sin(\omega t-kx)+A_2(x,t)\sin(\omega t+kx+\varphi), \tag{2.6}$$

where the wave vector k is determined by the condition $k=2\pi n/L$, L is the length of the medium, and $\omega = kc$; we shall assume that $\omega=\omega_0$. In this case the system of equations for slowly varying amplitudes assumes the form (the corresponding system for the case when $\omega\neq\omega_0$ can be found in the Appendix 2):

$$\begin{aligned}
&\frac{\partial A_1}{\partial t}+c\frac{\partial A_1}{\partial x}=\frac{2\pi}{\varkappa}j_1,\quad \frac{\partial A_2}{\partial t}-c\frac{\partial A_2}{\partial x}=\frac{2\pi}{\varkappa}j_2,\\
&\frac{\partial j_1}{\partial t}+\frac{j_1}{2T_2}=\frac{|m|^2}{\hbar c}(A_1\rho_0+A_2\rho_1),\\
&\frac{\partial j_2}{\partial t}+\frac{j_2}{2T_2}=\frac{|m|^2}{\hbar c}(A_2\rho_0+A_1\rho_1),\\
&\frac{\partial\rho_0}{\partial t}+\frac{\rho_0-\rho_l}{T_1}=-\frac{1}{\hbar c}(A_1 j_1+A_2 j_2),\\
&\frac{\partial\rho_1}{\partial t}+\frac{\rho_1}{T_1}=-\frac{1}{2\hbar c}(A_1 j_2+A_2 j_1),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
&j(\mathbf{r},t)=j_1(x,t)\cos(\omega t-kx)+j_2(x,t)\cos(\omega t+kx+\varphi),\\
&\rho(\mathbf{r},t)=\rho_0(x,t)+2\rho_1(x,t)\cos(2kx+\varphi).
\end{aligned} \tag{2.8}$$

It is clear from the last expression that the system (2.7) allows for the possibility of inducing a population grating at the spatial frequency $K=2k$, which—in principle—can occur when oscillation takes place in a resonator under the cooperative decay conditions. In fact, one of the conditions for the observation of cooperative effects is

$$T_2,\ T_2^*\gg\tau_p, \tag{2.9a}$$

where T_2^* is the inhomogeneous transverse relaxation time and τ_p is the characteristic duration of a radiation pulse. If we assume that T_2^* is due to the Doppler broadening, the condition (2.8) becomes

$$T_2^*=\frac{\lambda}{v_T}\gg\tau_p. \tag{2.9b}$$

Since λ/v_T is the transit time of a particle crossing the distance between the two consecutive nodes of an electromagnetic field in a resonator, it follows from the condition (2.9a) that the thermal motion of atoms cannot destroy the population inversion grating induced in the active medium during one output pulse. However, these effects can have a significant influence on the dynamics of decay of the system only for specific parameters of the problem (see, for example, Secs. 2.5 and 4.1). An analysis of some of the aspects of the dynamics of such decay is given in Refs. 41 and 42.

2.2. Normalized equations and main parameters of the problem

We shall introduce a dimensionless coordinate $x'=x/L$ and a dimensionless time $t'=t/\tau$, where $\tau=L/c$ is the time taken by a photon to cross the sample, and we shall also normalize the unknown functions as follows:

$$A_{1,2}(x,\ t)=a_{1,2}(x,\ t)\left(\frac{8\pi\hbar c^2}{\omega}\frac{N}{V}\right)^{1/2},$$
$$p_{1,2}(x,\ t)=\frac{2\pi\tau}{\varkappa}\,j_{1,2}(x,\ t)\left(\frac{8\pi\hbar c^2}{\omega}\frac{N}{V}\right)^{-1/2}, \qquad (2.10)$$
$$R_{0,1}(x,\ t)=\rho_{0,1}(x,\ t)\left(\frac{N}{V}\right)^{-1}.$$

In terms of the new dimensionless variables, the system (2.7) becomes

$$\frac{\partial a_1}{\partial t}+\frac{\partial a_1}{\partial x}=p_1,\quad \frac{\partial a_2}{\partial t}-\frac{\partial a_2}{\partial x}=p_2,$$
$$\frac{\partial p_1}{\partial t}+\alpha p_1=\beta\,(a_1R_0+a_2R_1),$$
$$\frac{\partial p_2}{\partial t}+\alpha p_2=\beta\,(a_2R_0+a_1R_1), \qquad (2.11)$$
$$\frac{\partial R_0}{\partial t}+\alpha_1\,(1+R_0)=-4\,(a_1p_1+a_2p_2),$$
$$\frac{\partial R_1}{\partial t}+\alpha_1R_1=-2\,(a_1p_2+a_2p_1),$$

where

$$\beta=\frac{2\pi\,|m|^2}{\hbar\omega}\frac{N}{V}\tau,\quad \alpha=\frac{\tau}{2T_2},\quad \alpha_1=\frac{\tau}{T_1}. \qquad (2.12)$$

The system (2.11) is simplified by dropping the primes of the dimensionless variables x' and t'. The ratio of the coefficients β and α, deduced from the system (2.11), represents the steady-state gain. In fact, if

$$\frac{\partial a_n}{\partial t}=\frac{\partial p_n}{\partial t}=0,$$

it follows from the system (2.11) that

$$\frac{\partial a_1}{\partial x}=\frac{\beta}{\alpha}R_0(0)\,a_1=\mu_0La_1.$$

Consequently, in the case of complete inversion $[R_0(0)=1]$, we have

$$\mu_0L=\frac{\beta}{\alpha}=\frac{4\pi\,|m|^2}{\hbar\omega}\tau\frac{N}{V}T_2=\frac{3\lambda^2}{4\pi}\frac{N}{V}\frac{T_2}{T_1}L. \qquad (2.13)$$

Under the normalization conditions given by the system (2.10), the quantity $n(x,\ t)=|a(x,\ t)|^2$ represents the density of the number of photons normalized to the density of resonant atoms. In particular, the energy density of the electromagnetic field can be expressed in terms of n_1 and n_2 as follows:

$$W_{\mathrm{F}}(x,\ t)=\hbar\omega_0\frac{N}{V}\,(n_1(x,\ t)+n_2(x,\ t)), \qquad (2.14)$$

whereas the energy flux density (Poynting vector) is

$$\mathrm{S}(x,\ t)=\mathrm{e}_x\hbar\omega_0\frac{N}{V}\,(n_1(x,\ t)-n_2(x,\ t)). \qquad (2.15)$$

2.3. Initial and boundary conditions

In investigations of spontaneous decay of polyatomic systems the initial conditions for the field and polarization should be homogeneous:

$$a_1(x,\ 0)=a_2(x,\ 0)=p_1(x,\ 0)=p_2(x,\ 0)=0. \qquad (2.16)$$

However, the solution of the system (2.11) subject to the initial conditions (2.16) is identically zero. This is due to the fact that the semiclassical decoupling of the operators described by Eq. (2.1) suppresses the spontaneous radiation sources. Therefore, we either have to supplement the system of equations (2.11) by effective spontaneous polarization sources or we have to select nonzero initial conditions. For example, we can assume that

$$a_n(x,\ 0)=a_{0n}(x),\quad p_n(x,\ 0)=p_{0n}(x), \qquad (2.17)$$

where either a_{0n} or p_{0n} may vanish. The quantity p_{0n} can be found if during the initial stage we do not use the decoupling of Eq. (2.1), but allow for two-particle correlations of collective atomic operators $R^{\pm}$ (for details, see, for example, Ref. 1):

$$p_{01}^2+p_{02}^2=\frac{\beta}{2N^2}\langle R^+R^-+R^-R^+\rangle\big|_{t=0}=\frac{\beta}{2N};$$

we therefore have

$$p_{01}=p_{02}=\left(\frac{\beta}{2N}\right)^{1/2}. \qquad (2.18)$$

In the case of more rigorous theories the initial conditions are specified in the form of a correlation matrix (see, for example, Ref. 43):

$$K_{nm}(x,x')=\langle a_n^*(x,0)\,a_m(x',0)\rangle, \qquad (2.19)$$

where the angular brackets denote quantum averaging within the framework of quantum theories or the averaging over realizations in phenomenological theories.

The boundary conditions for slowly varying amplitudes $a_n(x,\ t)$ are of the form

$$a_1\left(-\frac{1}{2},\ t\right)=r_1a_2\left(-\frac{1}{2},\ t\right), \qquad (2.20a)$$
$$a_2\left(\frac{1}{2},\ t\right)=r_2a_1\left(\frac{1}{2},\ t\right), \qquad (2.20b)$$

where $r_{1,2}$ are the amplitude reflection coefficients at the left-hand $(x=-1/2)$ and right-hand $(x=1/2)$ ends of the active medium. It should be pointed out that introduction of a phase φ in Eqs. (2.6) and (2.8) makes it possible to allow for the difference between the phases of the reflection coefficients: $r_{1,2}=|r_{1,2}|\exp(i\varphi_{1,2})$, which leads to $\varphi=\varphi_2-\varphi_1$.

2.4. Integrals of motion

Equation (2.11) readily yields the following expression which has the meaning of a local law describing the change in energy:

$$\frac{\partial}{\partial t}\,(W_{\mathrm{F}}+W_{\mathrm{A}})+\operatorname{div}\mathbf{S}=0, \qquad (2.21)$$

where

$$W_A = \frac{\hbar\omega_0}{2} R_0. \tag{2.22}$$

Integrating Eq. (2.21) with respect to time and over the length of a sample, and also allowing for the boundary conditions described by Eqs. (2.20a) and (2.20b), we obtain

$$\bar{n}_1 + \bar{n}_2 + (1 - |r_2|^2) \int_0^t n_1\left(\frac{1}{2}, t'\right) dt'$$
$$+ (1 - |r_1|^2) \int_0^t n_2\left(-\frac{1}{2}, t'\right) dt'$$
$$= \frac{1}{2}(1 - \bar{R}_0), \text{ where } n_{1,2}(t) = \int_{-1/2}^{1/2} n(x, t)\, dx. \tag{2.23}$$

The above expression gives the law of conservation of energy in its integral form.

The second integral expression can be obtained from the last four equations in the system (2.11). This expression is

$$\sum_{n=1}^{2} |p_n(x, t)|^2 + \frac{\beta}{4}(R_0^2(x, t) + 2R_1^2(x, t))$$
$$+ 2\int_0^t \left\{\alpha \sum_{n=1}^{2} |p_n(x, t')|^2 + \alpha_1 \frac{\beta}{4}[(1 + R_0) R_0 + 2R_1^2]\right\} dt' = \text{const}. \tag{2.24}$$

If $\alpha = \alpha_1 = 0$, it reduces to the law of conservation of the length of a Bloch vector:

$$p_1^2(x, t) + p_2^2(x, t) + \frac{\beta}{4}(R_0^2(x, t) + 2R_1^2(x, t)) = \text{const}. \tag{2.25}$$

It follows directly from the system (2.11) and also as a result of substitution of Eqs. (2.8) and (2.10) in Eq. (2.5) and averaging of the latter over the period $T = 2\pi/\omega_0$ and over the wavelength $\lambda = 2\pi/k$.

2.5. Equations for slowly varying amplitudes

As pointed out in Sec. 2.1, one of the conditions for the appearance of self-diffraction processes due to four-photon scattering is the condition (2.9). However, this condition is insufficient to ensure that a light-induced population grating is readily established. It follows from the last equation of the system (2.11) that R_1 is close to R_0 if $|a_1 p_1| \sim |a_1 p_2|$. We shall show in the next section that in the case of spontaneous decay of open systems ($r_1, r_2 < 0.9$ we as a rule have $|a_1 p_1| \gg |a_1 p_2|$. In this case the system of equations (2.11) assumes the simpler form

$$\frac{\partial a_1}{\partial t} + \frac{\partial a_1}{\partial x} = p_1, \quad \frac{\partial a_2}{\partial t} - \frac{\partial a_2}{\partial x} = p_2,$$
$$\frac{\partial p_1}{\partial t} + \alpha p_1 = \beta a_1 R, \quad \frac{\partial p_2}{\partial t} + \alpha p_2 = \beta a_2 R, \tag{2.26}$$
$$\frac{\partial R}{\partial t} = -4(a_1 p_1 + a_2 p_2).$$

The laws of conservation (2.21) and (2.23) retain their previous form, whereas Eq. (2.25) becomes

$$p_1^2 + p_2^2 + \frac{\beta}{4} R^2 = \text{const}. \tag{2.27}$$

3. DYNAMICS OF A SYSTEM DURING THE LINEAR STAGE OF OSCILLATION

We shall consider the dynamics of our system during the linear stage when the condition $1 - R(t) \ll R(0) = 1$ is obeyed.

3.1. HOMOGENEOUS BOUNDARY CONDITIONS

If we can ignore the reflection of the field at the boundaries of the medium, the boundary conditions given by the system (2.20) become

$$a_1\left(-\frac{1}{2}, t\right) = a_2\left(\frac{1}{2}, t\right) = 0. \tag{3.1}$$

In this case during the linear stage the right- and left-hand waves grow independently of one another. If we assume that $R_0(x) = 1$, then the right-hand wave can be deduced from the system (2.26) and it is then described by the following system of two linear equations

$$\frac{\partial a}{\partial t} + \frac{\partial a}{\partial x} = p, \quad \frac{\partial p}{\partial t} + \alpha p = \beta a. \tag{3.2}$$

Applying the Laplace–Carson transformation

$$a(x, u) = \int_0^\infty a(x, t) e^{-ut}\, dt \tag{3.3}$$

we readily find that the transform $a(x, u)$ is described by

$$\frac{\partial a(x, u)}{\partial x} + \left(u - \frac{\beta}{u + \alpha}\right) a(x, u) = a(x, 0) + \frac{p(x, 0)}{u + \alpha}. \tag{3.4}$$

Let us assume that $p(x, 0) = 0$; then, applying the inverse Laplace–Carson transformation, we find that $a(x, t)$ is described by

$$a(x, t) = \int_{-1/2}^{x} a(x', 0)\, G_a(x - x', t)\, dx', \tag{3.5}$$

where the Green's function $G_a(x - x', t)$ is given by the expression

$$G_a(x - x', t) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \exp\left\{u[t - (x - x')] + \frac{\beta(x - x')}{u + \alpha}\right\} du$$
$$= \left[\frac{\beta(x - x')}{t - (x - x')}\right]^{1/2} I_1(2\{\beta(x - x')[t - (x - x')]\}^{1/2})$$
$$\times \exp\{-\alpha[t - (x - x')]\}\, \theta(t - (x - x')); \tag{3.6}$$

here, $\theta(z)$ is the Heaviside step function:

$$\theta(z) = 1, \quad z \geqslant 0,$$
$$= 0, \quad z < 0. \tag{3.7}$$

If we initially have $a(x, 0) = 0$ and the spontaneous polarization $p(x, 0)$ differs from zero, the solution of the system (3.2) is

$$a(x, t) = \int_{-1/2}^{x} p(x', 0)\, G_p(x - x', t)\, dx', \tag{3.8}$$

where

$$G_p(x - x', t) = I_0(2\{\beta(x - x')[t - (x - x')]\}^{1/2})$$
$$\times \exp\{-\alpha[t - (x - x')]\}\, \theta(t - (x - x')). \tag{3.9}$$

3.2. Delay time

We shall now estimate the dependences of the characteristic times of the change in the field amplitude on the parameters of the problem. We shall assume, for the sake of simplicity, that

$$a(x, 0) = 0, \quad p(x, 0) = \left(\frac{\beta}{2N}\right)^{1/2} \delta(x). \tag{3.10}$$

We then have

$$a\left(\frac{1}{2}, t\right) = \left(\frac{\beta}{2N}\right)^{1/2} I_0\left(\left[2\beta\left(t-\frac{1}{2}\right)\right]^{1/2}\right) \times \exp\left[-\alpha\left(t-\frac{1}{2}\right)\right] \theta\left(t-\frac{1}{2}\right). \tag{3.11}$$

If $\beta t \gg 1$, we can use the asymptotic form of a modified Bessel function. We then have

$$a\left(\frac{1}{2}, t\right) = \left(\frac{\beta}{2N}\right)^{1/2} \frac{\exp[\{2\beta[t-(1/2)]\}^{1/2} - \alpha[t-(1/2)]]}{[2\pi\{2\beta[t-(1/2)]\}^{1/2}]^{1/2}}. \tag{3.12}$$

It then follows from the law of conservation (2.23) that

$$\int_{1/2}^{\infty} n\left(\frac{1}{2}, t\right) dt \leqslant \frac{1}{2},$$

so that half the total number of photons is emitted in each side if the initial conditions are symmetric. Using Eq. (3.12), we readily obtain

$$\int_{1/2}^{t_0+1/2} n\left(\frac{1}{2}, t\right) dt = \frac{1}{8\sqrt{\pi}N} G^{1/2} e^{G} [\Phi(G^{1/2}) - \Phi(G^{1/2} - (2t_0^{*}\alpha)^{1/2})]$$

$$\approx \frac{1}{8\pi N}\left[\frac{\exp\left\{2\alpha\left[\left(\frac{2G}{\alpha}t_0^{*}\right)^{1/2} - t_0^{*}\right]\right\}}{1-(2\alpha t_0^{*}/G)^{1/2}} - 1\right] = \frac{1}{4}, \tag{3.13}$$

where

$$t_0^{*} = t_0 + \frac{1}{2}.$$

The delay time is determined by the moment when the intensity reaches its peak value. The symmetric profile of the radiation pulse means that up to that moment half of the total number of photons is emitted. The following notation is used in Eq. (3.13):

$$G(\Gamma) = \frac{\beta}{\alpha} = \frac{3\lambda^2}{8\pi}\frac{N}{V}\frac{1}{\Gamma T_1}L, \tag{3.14}$$

where $\Gamma = 1/2T_2$ is the homogeneous line width. The delay time t_0 is described by the following expression deduced from Eq. (3.13):

$$t_0 = \frac{1}{2} + \left\{\left(\frac{G}{2\alpha}\right)^{1/2} - \left[\frac{G}{2\alpha} - \frac{1}{2\alpha}\ln(2\pi N)\right]^{1/2}\right\}^2. \tag{3.15}$$

On the other hand, the right-hand side of Eq. (3.12) reaches its maximum at

$$t_m = \frac{1}{2} + \frac{2G}{\alpha}. \tag{3.16}$$

Therefore, as long as $G > \ln(2\pi N)$ the delay time is given by Eq. (3.15). It should be noted that $G = \mu_0 L$, i.e., in the case of strongly amplifying media the expression for the delay time is given by Eq. (3.15). In particular, if

$$G(\Gamma) \gg \ln(2\pi N)$$

we obtain from Eq. (3.15)

$$t_0 = \frac{1}{2} + \frac{1}{8\beta}\ln^2(4\pi N). \tag{3.17}$$

If $G < \ln(2\pi N)$, the left-hand side of Eq. (3.13) does not reach 1/4, i.e., the total number of the emitted photons becomes less than the number of atoms and we are dealing with the case of superradiance in weakly amplifying media.[44] The delay time is then described by Eq. (3.16). In the region of $G \sim \ln(2\pi N)$ the delay time must be determined directly from Eq. (3.13). It should be pointed out that if $G \sim \ln(2\pi N)$, we can write Eq. (3.16) in the form

$$t_0 = \frac{1}{2} + \frac{2}{\alpha}\ln(2\pi N_{\text{eff}}), \tag{3.18}$$

where N_{eff} is the effective number of the light-emitting atoms (see Ref. 44).

3.3. Superradiance in a resonator

In contrast to the boundary conditions given by Eq. (3.1) the boundary conditions specified by Eqs. (2.20a) and (2.20b) relate the amplitudes of the counterpropagating waves also during the linear stage of the evolution of the system. If we assume that $p_n(x, 0) = 0$ and repeat the procedure described in Sec. 3.1, we can readily show that the amplitude of the right-hand wave is then given by the expression

$$a_1(x, t) = \int_{-1/2}^{x} a_1(x', 0) G_a(x-x', t) dx' + \int_{-1/2}^{1/2} a_1(x', 0) \sum_{k=1}^{\infty} (r_1 r_2)^k G_a(x-x'+2k, t) dx' + \int_{-1/2}^{1/2} a_2(x', 0) r_1 \sum_{k=0}^{\infty} (r_1 r_2)^k G_a(x+x'+2k+1, t) dx'. \tag{3.19}$$

The meaning of the additional, compared with Eq. (3.5), terms in Eq. (3.19) is obvious. The second term describes a wave formed as a result of k reflections of the right-hand wave from both ends of the medium. The third term describes a wave formed on reflection of the left-hand wave from the left-hand end of the medium and by subsequent k reflections from both ends. The upper limit of the sums on the right-hand side of Eq. (3.19) is determined, in accordance with Eq. (3.6), by the number of reflections which are completed by the time t. In the special case when $r_1 = r_2 = r$ and $a_2(x, 0) = a_1(-x, 0)$, Eq. (3.19) simplifies to

$$a_1(x, t) = \int_{-1/2}^{x} a(x', 0) G_a(x-x', t) dx' + \int_{-1/2}^{1/2} a(x', 0) \sum_{k=1}^{\infty} r^k G_a(x-x'+k, t) dx'. \tag{3.20}$$

In the case when $a_1(x, 0) = a_2(x, 0) = 0$ and nonzero values of $p_1(x, 0)$ and $p_2(x, 0)$, Eqs. (3.19) and (3.20) retain their form when the substitutions $a_n(x, 0) \to p_n(x, 0)$ and $G_a(x-x', t) \to G_p(x-x', t)$, are made, where $G_p(x-x', t)$ is given by Eq. (3.9).

3.4. Correlation properties of a superradiance pulse

We shall assume that the correlation properties of the unrenormalized (bare) field $a_n(x, 0)$ are given by a correlation matrix of the following type

$$K_{nm}(x,x',0)=\langle a_n(x,0)a_m(x',0)\rangle=A(x-x')\delta_{nm}. \quad (3.21)$$

For simplicity, we shall assume that all the amplitudes are real. Generalization to the case of complex fields presents no difficulty.

The dynamics of the correlation function is given by the expression

$$K_{nm}(x,x',t)=\int_{-1/2}^{x} d\xi \int_{-1/2}^{x'} d\eta A(\xi-\eta)G(x-\xi,t)G(x'-\eta,t).$$

We can see that on increase in time t the initial correlation properties of the field are "forgotten" and the correlation radius begins to be governed by the values of the parameters α and β.

The main difference between the case of superradiance in free space ($r_1=r_2=0$) and its decay in a resonator ($r_1\neq 0$, $r_2\neq 0$) is that in the former case the correlation function $K_{12}^{(0)}(1/2,-1/2,t)$ is always zero, i.e., the superradiance pulses emerging from both ends of the active medium are uncorrelated if they are due to uncorrelated sources of the renormalized field or due to spontaneous polarization. However, if $r_1\neq 0$ and $r_2\neq 0$, it follows from Eq. (3.19) that

$$K_{12}^{(2)}\left(\frac{1}{2},-\frac{1}{2},t\right)=\int_{-1/2}^{1/2} dx \int_{-1/2}^{1/2} dx' A(x-x')\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(r_1r_2)^{k+l}$$
$$\times\left[r_2G_a\left(\frac{1}{2}-x+2k,t\right)G_a'\left(-\frac{1}{2}+x'+2l+1,t\right)\right.$$
$$\left.+r_1G_a\left(\frac{1}{2}+x+2k+1,t\right)G_a\left(-\frac{1}{2}-x'+2l,t\right)\right]. \quad (3.22)$$

We can see that if $t>1$, a correlation is established between the waves emitted from the opposite ends of the medium. The degree of mutual correlation rises on increase in the parameter β and on increase in the reflection coefficients $r_{1,2}$, and decreases on increase in the parameter α.

3.5. Inhomogeneous broadening

We have ignored so far the effects of inhomogeneous broadening of atomic lines. If we allow for such broadening, the system of equations (2.11) becomes

$$\frac{\partial a_1}{\partial t}+\frac{\partial a_1}{\partial x}=\int p_1(\Delta)f(\Delta)\,d\Delta,\ \frac{\partial a_2}{\partial t}-\frac{\partial a_2}{\partial x}=\int p_2(\Delta)f(\Delta)\,d\Delta,$$
$$\frac{\partial p_1(\Delta)}{\partial t}+(\alpha+i\alpha^*\Delta)p_1(\Delta)=\beta(a_1R_0(\Delta)+a_2R_1(\Delta)),$$
$$\frac{\partial p_2(\Delta)}{\partial t}+(\alpha+i\alpha^*\Delta)p_2(\Delta)=\beta(a_2R_0(\Delta)+a_1R_1^*(\Delta)), \quad (3.23)$$
$$\frac{\partial R_0(\Delta)}{\partial t}+\alpha_1(1+R_0(\Delta))=-2(a_1p_1^*(\Delta)+a_2p_2^*(\Delta)+\text{c.c.}),$$
$$\frac{\partial R_1(\Delta)}{\partial t}+\alpha_1R_1(\Delta)=-2(a_1p_2^*(\Delta)+a_2^*p_1(\Delta)),$$

where $f(\Delta)$ is the profile of an inhomogeneously broadened line satisfying the condition

$$\int_{-\infty}^{\infty} f(\Delta)\,d\Delta=1, \quad (3.24)$$

$\alpha^*=\tau\Delta_0$, and Δ_0 is the characteristic width of the function $f(\Delta)$.

We shall now consider how the main parameters of the radiation pulses emitted in the linear stage change under the influence of the inhomogeneous broadening. Applying the Laplace–Carson transformation, by analogy with the procedure adopted in Sec. 2.1, we find that the transform of the amplitude of the field of the right-hand wave is readily described by the following equation

$$\frac{da_1(x,u)}{dx}+\left(u-\beta\int\frac{f(\Delta)R_0(\Delta)\,d\Delta}{u+\alpha+i\alpha^*\Delta}\right)a_1(x,u)$$
$$=a_1(x,0)+p_1(x,0)\int\frac{f(\Delta)\,d\Delta}{u+\alpha+i\alpha^*\Delta}, \quad (3.25)$$

where $R_0(\Delta)=R(\Delta,x,0)$. Let us assume that $p_1(x,0)=0$; the Green's function $G_a(x-x',t)$ used above is then characterized by a singularity $u_0=-\alpha$ in the case of homogeneously broadened systems [see Eq. (3.6)]. If a medium is inhomogeneously broadened, the function $G_a(x,x',t)$ is characterized by singularities of the following function

$$F(u)=\int\frac{f(\Delta)R_0(\Delta)\,d\Delta}{u+\alpha+i\alpha^*\Delta}. \quad (3.26a)$$

We shall now assume that $R_0(\Delta)=1$, i.e., that all the atoms within the inhomogeneous line width of the transition are excited equally.

For convenience in interpretation of the above expressions, let us return for a moment to the dimensional quantities, when α is replaced with $\Gamma=1/(2T_2)$, and the function $F(u)$ is

$$F(u)=\int\frac{f(\Delta)\,d\Delta}{u+\Gamma+i\Delta}; \quad (3.26b)$$

we shall assume that $f(\Delta)$ is of the form

$$f(\Delta)=\frac{\Delta_0}{\pi}\frac{1}{\Delta^2+\Delta_0^2}, \quad (3.27)$$

where Δ_0 is the half-width of the function $f(\Delta)$ and the function $f(\Delta)$ still satisfies the condition (3.24). Substituting Eq. (3.27) into Eq. (3.26a), we obtain

$$F(u)=\frac{1}{u+\Gamma+\Delta_0}. \quad (3.28)$$

Consequently, the expression for the field amplitude $a_1(x,t)$ again retains the form of Eqs. (3.5) and (3.6), but instead of the homogeneous line width Γ it now contains the following expression:

$$\Gamma+\Delta_0=\frac{1}{2T_2}+\Delta_0=\frac{1}{2T_2^*}.$$

Therefore, in the case when the whole inhomogeneous spectrum is excited uniformly, the nature of growth of the field is governed not by the ratio β/α, but by β/α^*, which is of the form

$$G(\Gamma+\Delta_0)=\frac{\beta}{\alpha}\frac{\Gamma}{\Gamma+\Delta_0}=\frac{3\lambda^2}{4\pi}\frac{N}{V}\frac{T_2^*}{T_1}L. \quad (3.29)$$

If initially only a part, of half-width Δ_1, of the inhomogeneous spectrum is excited, so that

$$R_0(\Delta)=\frac{\Delta_1^2}{\Delta^2+\Delta_1^2}, \quad (3.30)$$

the function $F(u)$ becomes

$$F(u)=\frac{\Delta_1}{\Delta_0^2-\Delta_1^2}\left(\frac{\Delta_0}{u+\Gamma+\Delta_1}-\frac{\Delta_1}{u+\Gamma+\Delta_0}\right). \quad (3.31)$$

In particular, if $\Delta_0 \gg \Delta_1$, we obtain

$$F(u) = \frac{\Delta_1}{\Delta_0} \frac{1}{u + \Gamma + \Delta_1}. \qquad (3.32)$$

Consequently, the nature of growth of the field is now determined by the parameter

$$G(\Gamma + \Delta_1, \Delta_0) = \frac{\beta}{\alpha} \frac{\Delta_1}{\Delta_0} \frac{\Gamma}{\Gamma + \Delta_1} = \frac{3\lambda^2}{4\pi} \frac{N}{V} \frac{\Delta_1}{\Delta_0} \frac{T_{21}^*}{T_1} L, \qquad (3.33)$$

where $2T_{21}^* = (\Gamma + \Delta_1)^{-1}$. The process of superradiance now involves only the atoms with frequencies lying within the band $\Gamma + \Delta_1$ near the resonance line of the transition under consideration. The number of these atoms is given by $N(\Delta_1/\Delta_0)$.

If the part of an inhomogeneous spectrum of a line of width Δ_1 is excited and this happens not at the center of a frequency, but as a result of detuning Δ_2 from the line center, i.e.,

$$R_0(\Delta) = \frac{\Delta_1^2}{(\Delta - \Delta_2)^2 + \Delta_1^2}, \qquad (3.34)$$

then if $\Delta_0 \gg \Delta_1$, we obtain

$$F(u) = \frac{\Delta_0 \Delta_1}{\Delta_0^2 + \Delta_2^2} \frac{1}{u + \Gamma + \Delta_1 - i\Delta_2}. \qquad (3.35)$$

Consequently, the wave amplitude is an oscillatory function with the frequency Δ_2 in the time domain and with the wave vector Δ_2/c in space. The dynamics of oscillation during the initial stage is given by the parameter

$$G(\Gamma + \Delta_1, \Delta_0, \Delta_2) = \frac{3\lambda^2}{4\pi} \frac{N}{V} \frac{\Delta_0 \Delta_1}{\Delta_0^2 + \Delta_2^2} \frac{T_{21}^*}{T_1} L. \qquad (3.36)$$

Therefore, it follows from a comparison of Eqs. (3.11)–(3.17) and Eqs. (3.29), (3.33), and (3.36) that the dynamics of oscillation during the initial stage is given by Eqs. (3.11)–(3.17) when the substitutions $\beta \to \beta_{\text{eff}}$ and $\alpha \to \alpha_{\text{eff}}$ are made and the new parameters α_{eff} and β_{eff} are described by the following expressions:

a) if $R_0(\Delta) = 1$,

$$\beta_{\text{eff}} = \beta, \quad \alpha_{\text{eff}} = \alpha \frac{\Gamma + \Delta_0}{\Gamma};$$

b) if $R_0(\Delta) = \Delta_1^2/(\Delta^2 + \Delta_1^2)$,

$$\beta_{\text{eff.}} = \beta \frac{\Delta_1}{\Delta_0}, \quad \alpha_{\text{eff}} = \alpha \frac{\Gamma + \Delta_1}{\Gamma};$$

c) if $R_0(\Delta) = \Delta_1^2/[(\Delta - \Delta_2)^2 + \Delta_1^2]$,

$$\beta_{\text{eff}} = \beta \frac{\Delta_0 \Delta_1}{\Delta_0^2 + |\Delta_2^2}, \quad \alpha_{\text{eff}} = \alpha \frac{\Gamma + \Delta_1}{\Gamma}.$$

A change in β corresponds to a reduction in the density of the number of the excited atoms and a change in α corresponds to an increase in the line width.

4. ONE-WAVE PROBLEMS

It follows from the results of the preceding section that, in the absence of reflection on the boundaries of an active medium, two counterpropagating waves evolve independently of one another during the linear stage. In this case the characteristic parameters of the radiation pulses can be determined using the one-wave model. The class of one-wave tasks includes also the tasks of amplification and propagation of coherent pulses.

4.1. Interaction of counterpropagating waves

In the one-wave case the system of equations (2.26) becomes

$$\frac{\partial a}{\partial t} + \frac{\partial a}{\partial x} = p, \quad \frac{\partial p}{\partial t} + \alpha p = \beta a R,$$
$$\frac{\partial R}{\partial t} + \alpha_1 (1 + R) = -4ap, \qquad (4.1)$$

subject to the initial conditions

$$a(x, 0) = a_0(x), \quad p(x, 0) = 0 \qquad (4.2a)$$

or

$$a(x, 0) = 0, \quad p(x, 0) = p_0(x) \qquad (4.2b)$$

and subject to the boundary condition

$$\alpha\left(\frac{1}{2}, t\right) = A(t). \qquad (4.3)$$

In the case of spontaneous decay, we have $A(t) = 0$. The solution of the system of equations (4.1) is identical with the solution of the system (2.26) if in the latter the initial and boundary conditions for the left-hand wave are selected in the form

$$a_2(x, 0) = p_2(x, 0) = 0, \ a_2\left(\frac{1}{2}, t\right) = 0. \qquad (4.4)$$

By way of illustration, Fig. 1 shows—for the sake of comparison—the radiation profiles in two cases:

$$p_1(x, 0) = p_0, \quad p_2(x, 0) = 0, \qquad (4.5a)$$
$$p_1(x, 0) = p_2(x, 0) = p_0, \qquad (4.5b)$$

when in both cases the initial and boundary conditions for the field amplitudes are the same. It follows from Fig. 1 that in the case described by Eq. (4.5a), we have $a_2(x, t) = 0$, whereas in the case described by Eq. (4.5b), we find that $a_2(x, t) = a_1(-x, t)$. It is also clear from this figure that the interaction between counterpropagating waves is quite unimportant up to the moment of formation of the first maximum of the radiation intensity and begins to manifest itself only in the region of formation of the subsequent maxima. This is due to the fact, as can be demonstrated by the system (2.26), that the interaction of the counterpropagating waves is entirely due to the change in the population difference. It follows from Fig. 2a that the bulk of the energy of the right-hand wave up to its maximum is concentrated in the interval $0 \leqslant x \leqslant 1/2$ and only depletion of the inversion near the right-hand end of the medium causes the maximum of the ampli-

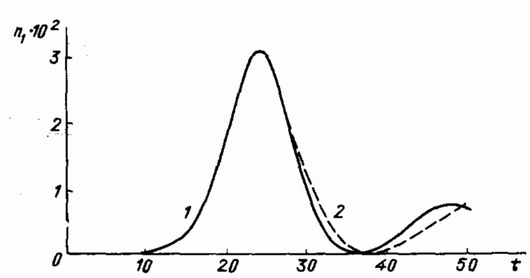

FIG. 1. Influence of the initial conditions on the radiation intensity profile. Curve *1* corresponds to the initial conditions of Eq. (4.5a), whereas curve *2* corresponds to Eq. (4.5b).

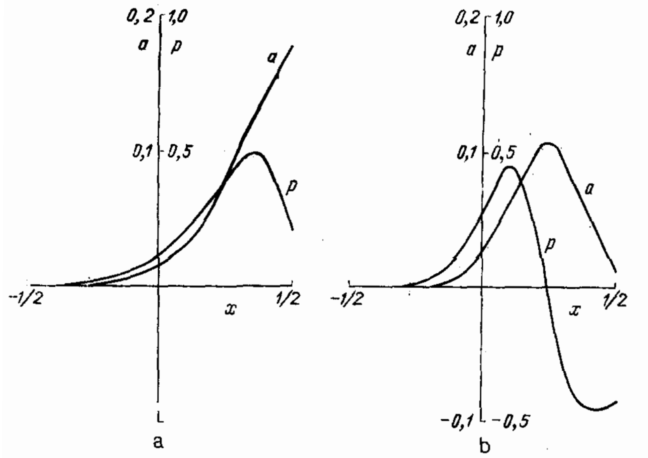

FIG. 2. Spatial distributions of the amplitude of the field of the right-hand wave (a) and of the corresponding polarization (b) at different moments in time: a) $t = 24$; b) $t = 35$.

tude of the right-hand wave to shift to the center of a sample (Fig. 2b), so that the effects of the interaction of the waves via mutual depletion of the inversion begin to manifest themselves.

Figure 2 reflects one other characteristic of spontaneous decay of the open systems. Since in the case of symmetric initial conditions of Eq. (4.5b) we have $a_2(x, 0) = a_1(-x, 0)$ and $p_2(x, 0) = p_1(-x, 0)$, it follows from Fig. 2 that within the limits of the first radiation pulse we obtain $|a_1(x, t)p_1(x, t)| \gg |a_1(x, t)p_2(x, t)|$. In particular, for the case shown in Fig. 2, we have

$$\frac{|a_1(x, t_0)\, p_1(x, t_0)|_{\max}}{|a_1(x, t_0)\, p_2(x, t_0)|_{\max}} \approx 30.$$

Consequently, the effects of a light-induced grating of the population inversion do not influence the formation of the first radiation pulse and can manifest themselves only in the afterglow.

4.2. Soliton propagation of pulses in unbounded media

One of the most interesting phenomena described by the system (4.1) is the soliton propagation of pulses. The soliton solutions appear in the coherent regime if $\alpha = \alpha_1 = 0$. In this case the Bloch integral of motion is

$$p^2(x, t) + \frac{\beta}{4} R^2(x, t) = p^2(x, 0) + \frac{\beta}{4} R^2(x, 0). \qquad (4.6)$$

Hence, assuming that

$$R = R_0 \cos\theta, \qquad (4.7)$$

we obtain

$$p = \frac{\beta^{1/2}}{2} R_0 \sin\theta, \quad a = \frac{1}{2\beta^{1/2}} \frac{\partial\theta}{\partial t} \qquad (4.8)$$

and the next equation for the Bloch angle $\theta(x, t)$ is

$$\frac{\partial^2\theta}{\partial t^2} + \frac{\partial^2\theta}{\partial t\, \partial x} = \beta R_0 \sin\theta. \qquad (4.9)$$

The one-soliton solution of Eq. (4.9) for an unexcited resonant medium, when $R_0 = -1$, has the familiar form[28-33]

$$\theta(t, x) = 4 \operatorname{arctg} \exp \frac{t - (x/v)}{\tau_0}, \qquad (4.10)$$

where the soliton velocity and its duration are related by (Fig. 3a)

$$\tau_0 = \left(\frac{1}{\beta} \frac{1 - v}{v}\right)^{1/2}, \text{ or } \quad v = \frac{1}{1 + \beta\tau_0^2}. \qquad (4.11)$$

In an excited resonant medium ($R_0 = 1$) we can again expect soliton propagation of the pulses (for a review see Ref. 36), but in this case their propagation is characterized by a whole range of special features. For example, the relationship between the soliton velocity and its duration now becomes

$$\tau_0 = \left[\frac{1}{\beta}\left(1 - \frac{1}{v}\right)\right]^{1/2}, \text{ or } \quad v = \frac{1}{1 - \beta\tau_0^2}. \qquad (4.12)$$

Therefore, it follows from Fig. 3b that whereas in the case $R_0 = -1$ the velocity is $v \leqslant 1$ for all values of τ_0, in the case when $R_0 = 1$ the velocity is $v > 1$ if $\beta\tau_0^2 < 1$, but $v < 0$ if $\beta\tau_0^2 > 1$. Consequently, the velocity of the maximum of a pulse is either higher than the velocity of light or is negative. In the latter case we have $|v| > 1$ if $1 < \beta\tau_0^2 < 2$ and $|v| \leqslant 1$ if $\beta\tau_0^2 \geqslant 2$, i.e., the negative velocity can also exceed the velocity of light.

The field amplitude is described by the same expression in both cases ($R_0 = \pm 1$):

$$a(x, t) = \frac{1}{\beta^{1/2}\tau_0} \operatorname{ch}^{-1}\left[\frac{1}{\tau_0}\left(t - \frac{x}{v}\right)\right]. \qquad (4.13)$$

4.3. Exponential and lethargic amplification

The solution of Eq. (3.4) for $a(x, u)$ is generally

$$a(x, u) = A_0(u)\, e^{-\lambda x} + \int_0^x \left(a(x', 0) + \frac{p(x', 0)}{u + \alpha}\right) \exp[-\lambda(x - x')]\, dx', \qquad (4.14)$$

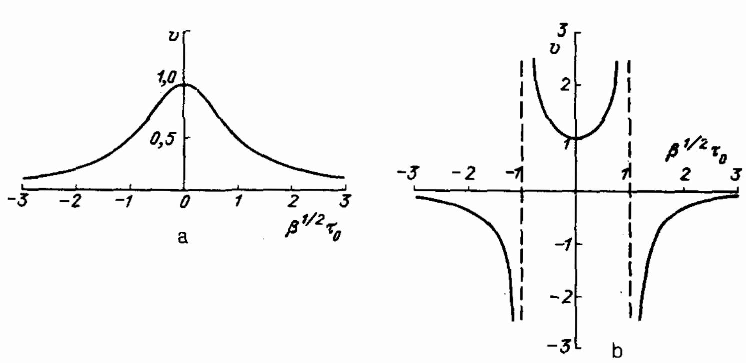

FIG. 3. Dependence of the soliton velocity v on its duration τ_0: a) absorbing media, $R_0 = -1$; b) amplifying media, $R_0 = 1$.

where

$$\lambda = u - \frac{\beta}{u+\alpha}.$$

Let us now assume that $a(x, 0) = 0$ and $p(x, 0) = 0$, so that

$$a(x, u) = e^{-\lambda x}\int_0^\infty e^{-ut} a_0(0, t)\,dt. \quad (4.15)$$

Consequently, we have

$$a(x, t) = \int_0^\infty a(0, t')\,G_b(t-t', x)\,dt' + a(0, t), \quad (4.16)$$

where

$$G_b(t-t', x) = \left(\frac{\beta x}{t-t'-x}\right)^{1/2} I_1(2[\beta x(t-t'-x)]^{1/2}) \times \exp[-\alpha(t-t'-x)]\,\theta(t-t'-x). \quad (4.17)$$

In this section we shall consider, for the sake of convenience, the problem in question in the interval $[0, \infty]$ and not in the interval $[-1/2, 1/2]$, which we have used so far. Therefore, $a(0, t)$ determines the shape of a pulse incident on the boundary of the active medium.

If the boundary condition is of the form

$$a(0, t) = a_0 e^{\delta t}, \quad (4.18)$$

then using Eqs. (3.16) and (3.17), we obtain

$$a(x, t) = \Big\{a_0 \exp\left[\delta(t-x) + \frac{\beta x}{\alpha+\delta}\right] + a_0 \exp[-\alpha(t-x)]\sum_{n=0}^{\infty}\left[(\alpha+\delta)\left(\frac{t-x}{\beta x}\right)^{1/2}\right]^n \times I_n(2[\beta x(t-x)]^{1/2})\Big\}\,\theta(t-x). \quad (4.19)$$

If a pulse of constant amplitude ($\delta = 0$) is incident on the medium, it then follows from Eq. (4.19) that

$$a(x, t) = \Big\{a_0 \exp\frac{\beta x}{\alpha} + a_0 \exp[-\alpha(t-x)]\sum_{n=0}^{\infty}\left[\alpha\left(\frac{t-x}{\beta x}\right)^{1/2}\right]^n \times I_n(2[\beta x(t-x)]^{1/2})\,\theta(t-x)\Big\}. \quad (4.20)$$

In the case when $\alpha = 0$, Eq. (4.19) transforms to

$$a(x, t) = \Big\{a_0 \exp\left[\frac{\beta x}{\delta} + \delta(t-x)\right] + a_0 \sum_{n=0}^{\infty}\left[\delta\left(\frac{t-x}{\beta x}\right)^{1/2}\right]^n \times I_n(2[\beta x(t-x)]^{1/2})\Big\}\,\theta(t-x). \quad (4.21)$$

If $\alpha = \delta = 0$, then

$$a(x, t) = a_0 I_0(2[\beta x(t-x)]^{1/2})\,\theta(t-x). \quad (4.22)$$

It follows from the above expressions that if at least one of the parameters α or δ differs from zero, then in the $\beta x \ll 1$ case we have the usual exponential amplification of a pulse characterized by the gain

$$\mu_0 L = \frac{\beta}{\alpha}.$$

As the product $\beta x(t-x)$ rises this type of amplification changes to one characterized by a lower rise along x:

$$a(x, t) \propto \exp\{2[\beta x(t-x)]^{1/2}\},$$

which is known as the lethargic amplification regime. If $\alpha = \delta = 0$, it follows from Eq. (4.22) that the lethargic amplification regime applies right from the beginning.

4.4. Soliton solutions in spatially inhomogeneous media

The results obtained in Secs. 4.2 and 4.3 allow us to formulate the following problems which require further thought. Can the soliton velocity exceed the velocity of light, as shown in Fig. 3? What is the meaning of the divergence of the velocity v when $\beta\tau_0^2 = 1$ (Fig. 3b)? The soliton is an eigenvalue solution of Eq. (4.9) for an infinite medium. What happens if a radiation pulse of the type described by Eq. (4.13) is transmitted by an amplifying medium bounded along the coordinate x? Is a soliton obtained again at the exit or is its profile deformed by the boundaries of the medium?

We shall show that the answer to the last question provides also answers to the preceding questions. For example, we shall consider the propagation of radiation pulses in spatially inhomogeneous media under the coherent amplification conditions ($\alpha = 0$). The Bloch integrals of motion described by Eqs. (2.24), (2.25), and (4.6) have the property that they are satisfied at each point x. Therefore, assuming that the medium has a certain profile of the particle density,

$$R(x, 0) = R_0(x), \quad (4.23)$$

we can rewrite Eqs. (4.7) and (4.8) in the form

$$R(x, t) = R_0(x)\cos\theta(x, t),$$
$$p(x, t) = \frac{\beta^{1/2}}{2} R_0(x)\sin\theta(x, t), \quad (4.24)$$
$$a(x, t) = \frac{1}{2\beta^{1/2}}\frac{\partial\theta(x, t)}{\partial t}.$$

As a result of substitution of these expressions into the system (4.1) in the case when $\alpha = \alpha_1 = 0$, we obtain the following equation for the Bloch vector:

$$\frac{\partial^2\theta(x, t)}{\partial t^2} + \frac{\partial^2\theta(x, t)}{\partial x\,\partial t} = \beta R_0(x)\sin\theta(x, t). \quad (4.25)$$

We shall seek the solution of Eq. (4.25) in the form

$$\theta(x, t) = 4\,\mathrm{arctg}\, e^{\Phi(x,t)}. \quad (4.26)$$

Then, in the case of the variable $\Phi(x, t)$, we obtain

$$\frac{\partial\Phi}{\partial t}\left(\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x}\right) = \beta R_0(x),$$
$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x}\right) = 0. \quad (4.27)$$

It follows from the last equation that

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x} = C(x), \quad (4.28)$$

whereas the first equation yields

$$\frac{\partial\Phi}{\partial t} = \frac{\beta R_0(x)}{C(x)}$$

or

$$\Phi(x, t) = \frac{\beta R_0(x)}{C(x)}t + \Phi_0(x). \quad (4.29)$$

Substituting Eq. (4.29) into Eq. (4.28), and integrating the latter, we obtain

$$\Phi(x, t) = \int_{-\infty}^{x}\left(C(x') - \frac{\beta R_0(x')}{C(x')}\right)dx' + \Phi_1(t). \quad (4.30)$$

Comparing Eqs. (4.29) and (4.30), we can readily demonstrate that

$$C(x)=\beta R_0(x)\tau_0,$$

where τ_0 is a constant. Finally, we find that $\Phi(x, t)$ is described by

$$\Phi(x, t)=\frac{t-x}{\tau_0}+\beta\tau_0\int_{-\infty}^{x} R_0(x')\,dx'. \tag{4.31}$$

It therefore follows that the constant τ_0 determines the pulse duration outside the medium where $R_0(x)=0$.

We shall determine the form of $\Phi(x, t)$ for a number of $R_0(x)$ profiles:

a) $R_0(x)=R_0/\mathrm{ch}^2\,\alpha_0 x$:

$$\Phi(x, t)=\frac{t-x}{\tau_0}+\frac{\beta\tau_0 R_0}{\alpha_0}\,\mathrm{th}\,\alpha_0 x; \tag{4.32a}$$

b) $R_0(x)=R_0\,[\mathrm{th}\,\alpha_0 x+\mathrm{th}\,\alpha_0(L-x)]/2$:

$$\Phi(x, t)=\frac{t-x}{\tau_0}+\frac{\beta\tau_0 R_0}{2\alpha_0}\ln\frac{\mathrm{ch}\,\alpha_0 x}{\mathrm{ch}\,\alpha_0(L-x)}; \tag{4.32b}$$

c) $R_0(x)=R_0\,[\mathrm{sgn}\,x+\mathrm{sgn}\,(L-x)\,]/2$:

$$\Phi(x, t)=\frac{t-x}{\tau_0}+\frac{\beta\tau_0 R_0}{2}\,x\,[\mathrm{sgn}\,x+\mathrm{sgn}\,(L-x)] + \frac{\beta\tau_0 R_0 L}{2}\,[1+\mathrm{sgn}\,(x-L)]; \tag{4.32c}$$

d) $R_0(x)=R_0+R_1\cos\alpha_0 x$:

$$\Phi(x, t)=\frac{t-x}{\tau_0}+\beta\tau_0\left(R_0 x+\frac{R_1}{\alpha_0}\sin\alpha_0 x\right). \tag{4.32d}$$

Since Eq. (4.25) is a consequence of the equations for slowly varying amplitudes, it follows that (within the limits of the validity of this approximation) we can ignore the reflections of the field at the boundaries of the medium. The length of the medium is designated L in Eqs. (4.32b) and (4.32c) and, in the normalization adopted by us above, we have $L=1$.

The expressions relating $a(x, t)$ and $R(x, t)$ to $\Phi(x, t)$ are obtained by substituting Eq. (4.26) into the expressions in the system (4.24), and are as follows:

$$a(x, t)=\frac{1}{\beta^{1/2}\tau_0}\frac{1}{\mathrm{ch}\,\Phi(x, t)}, \tag{4.33a}$$

$$R(x, t)=R_0\left(1-\frac{2}{\mathrm{ch}^2\,\Phi(x, t)}\right). \tag{4.33b}$$

Hence, it follows that if $\Phi(x, t)=0$, the field amplitude $a(x, t)$ reaches its maximum and we have $R(x, t)=-R_0$. We shall represent $\Phi(x, t)$ in the form

$$\Phi(x, t)=\frac{1}{\tau_0}(t-\varphi(x)). \tag{4.34}$$

A graph of the dependence $\varphi(x)$ in the case of media excited right from the beginning ($R_0>0$), corresponding to the profiles described by Eqs. (4.32b) and (4.32d), is given in Fig. 4. It is clear from Fig. 4a that at a moment t_1 in the region defined by $-\infty<x<\infty$ we have a solitary pulse with its maximum at a point x_1, found from the condition $\varphi(x_1)=t_1$, and this pulse is moving in the positive direction of the x axis. At a moment $t=t_2$ we have three pulses, two of which are outside the amplifying medium and move in the positive direction of the x axis, whereas the third, which is inside the medium, is traveling opposite to the incident pulse. At a moment t_3 we again have one pulse traveling in the direction of the incident pulse. The occurrence of three maxima at the moment t_2 is due to the fact that as a result of amplification of the leading edge of a pulse of the type (4.33a) incident on a sample at a moment t_0 the difference between the populations at the end of the sample becomes $R(L, t_0)=-R_0$. If $\beta\tau_0^2R_0=2$ and $\alpha_0=10$, which is the case illustrated in Fig. 4a, we have $t\simeq 0$. Figure 5 shows the transformation of a pulse near the end of the sample. It is clear that if $t>0$, a pulse splits into two parts, one of which leaves the sample and the other travels opposite to the incident pulse. If $\beta\tau_0^2R_0=2$, these two halves are symmetric relative to a point $x=1$. This is due to the fact that the rising and falling parts of the $\varphi(x)$ curve in the case when $\beta\tau_0^2R_0=2$ have the same slope relative to the ordinate. Therefore, the projections onto the x axis of those parts of the $\varphi(x)$ curve which are cut off by the straight lines t_2 and $t_2+\tau_0$, have the same values. If $\beta\tau_0^2R_0>2$, the linear dimensions of a pulse become shorter in the active region, whereas if $\beta\tau_0^2R_0<2$, the pulse becomes elongated in the sample.

It should be pointed out that at $t=t_0$ the amplitude of the incident or input pulse at the entry face to the active medium is

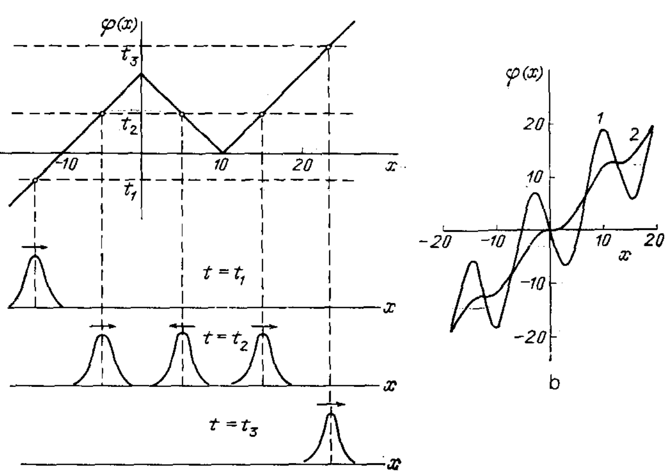

FIG. 4. Interpretation of Eqs. (4.32)–(4.34): a) function $\varphi(x)$ in the case described by Eq. (4.32b) when $\alpha_0=10$, $L=10$, and $\beta\tau_0^2R_0=2$, and the spatial distribution of the field at three different moments in time; b) function $\varphi(x)$ in the case described by Eq. (4.32d), calculated assuming that $\alpha_0=0.5$, $R_0=0$, and $\beta\tau_0^2R_1=5$ in the case of curve *1* and that $\alpha_0=0.5$, $R_0=0$, and $\beta\tau_0^2R_1=1$ for curve *2*.

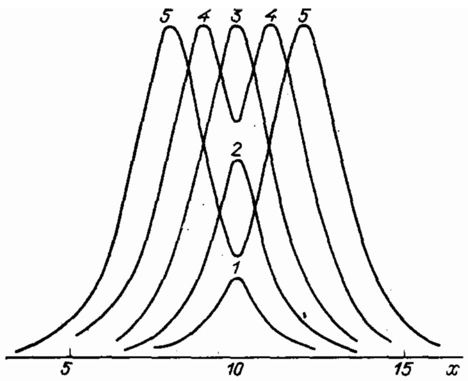

FIG. 5. Dynamics of formation of a pulse at the exit end of the active medium: *1*) $t=-2$; *2*) $t=-1$; *3*) $t=0$; *4*) $t=1$; *5*) $t=2$.

$$a_{\rm in}(0,t_0)=\frac{1}{\tau_0\beta^{1/2}}\frac{1}{\operatorname{ch}[(1/\tau_0)(\beta\tau_0^2R_0-1)]}$$
$$\approx\frac{2}{\tau_0\beta^{1/2}}\exp\left(-\frac{\beta\tau_0^2R_0-1}{\tau_0}\right),$$

whereas at the exit face of the active medium the amplitude of the same pulse is

$$a_{\rm in}(L,t_0)=\frac{1}{\tau_0\beta^{1/2}}\frac{1}{\operatorname{ch}(\beta\tau_0R_0)}\approx\frac{2}{\tau_0\beta^{1/2}}e^{-\beta\tau_0R_0}.$$

Since the amplitude of the output pulse at $t=0$ is

$$a_{\rm out}(L,t_0)=\frac{1}{\tau_0\beta^{1/2}},$$

we can define the gain as follows:

$$a_{\rm out}(L,t_0)=e^{GL}a_{\rm in}(L,t_0),$$

which yields

$$G=\beta\tau_0R_0-\ln 2\approx\beta\tau_0R_0.$$

Comparing this expression with Eq. (4.21), we can readily see that the value of the gain corresponds to the first term in Eq. (4.21). In fact, it follows from Eq. (4.18) that $\delta=1/\tau_0$; assuming that $R_0=1$, we obtain $\beta/\delta=\beta\tau_0$. Therefore, amplification of an exponential input signal of the form $a(0,t)=a_0\exp(t/\tau_0)$ results in the formation of a soliton pulse of the form given by Eq. (4.33a) at the exit from the medium; the duration of this soliton pulse τ_0 is governed by the growth increment of the input signal.

If $\beta\tau_0^2R_0=1$, the curve $\varphi(x)$ assumes the form shown in Fig. 6. In this case we can expect uniform motion of the population inversion over the whole length of the sample. This pulse propagation regime is undoubtedly of interest for three-level systems of pulse generation and amplification. For certain durations of a pulse due to a resonant transition from the ground state we can ensure a homogeneous, over the whole length of the sample, inversion for the next transition.

Further opportunities for controlling the pulse profile in a medium are provided by active media with a periodic profile of the excitation or of the concentration of active atoms. It is clear from Fig. 4b that variation of the parameters R_0, R_1, τ_0, and α_0 [see Eq. (4.32d)] can create a periodic excitation profile for the adjacent transition in a three-level medium. This profile can be realized simultaneously over the whole length as well as in several local regions of the active medium.

It therefore follows from the above analysis that a soliton pulse can propagate across resonantly amplifying or absorbing media of finite length without a change in its profile. The shape of the profile in the medium can then be modified significantly, but it is still described by the same expression [Eq. (4.33a)]. A pulse of the type described by Eq. (4.33a) may have several maxima (Fig. 5) or it can reach its maximum value within the limits of the macroscopic region along x, as in Fig. 6. The soliton velocity becomes infinite, as demonstrated by an analysis of the problem in an unbounded medium (Fig. 3) when maxima appear at the leading edge of a pulse because of the amplification in the medium. In fact, it follows from Eq. (4.34) that the velocity of the pulse maximum is

$$v_{\max}=\frac{dx}{dt}=\left(\frac{d\varphi}{dx}\right)^{-1}.$$

At the point x_0, defined by the condition

$$\frac{d\varphi}{dx}(x_0)=0,$$

at a moment $t_0=\varphi(x_0)$ a pulse of the type described by Eq. (4.33a) reaches its maximum value which is $a(x_0,t_0)=1/\tau_0\beta^{1/2}$. This corresponds to an infinitely fast jump to the pulse maximum. However, as we can see from Fig. 5, the rise of the pulse at the exit of the sample takes a finite time in reality. The amplitude of the output pulse $1/\tau_0\beta^{1/2}$ is limited by the fact that the maximum polarization for a resonant transition is restricted by the law of conservation of the Bloch vector [Eq. (4.6)].

5. TWO-WAVE PROBLEMS

In the present section we shall study the dependence of the nature of generation and amplification of pulses on the parameters of a medium and we shall employ models that allow for the interaction of one-dimensional counterpropagating waves. The system of equations describing the dy-

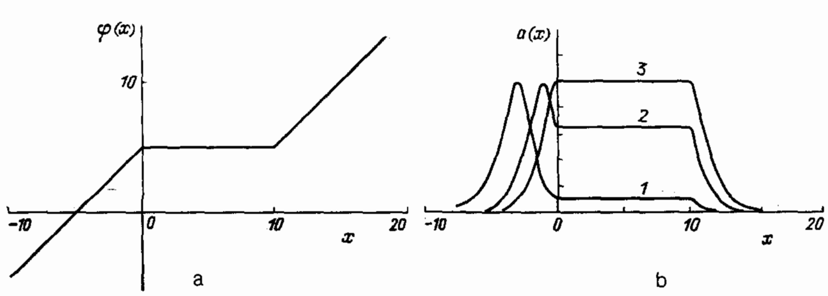

FIG. 6. a) Form of the function $\varphi(x)$ in the case described by Eq. (4.23b), calculated on the assumption that $\alpha_0=10$, $L=10$, and $\beta\tau_0^2R_0=1$; b) dynamics of the pulse profile in the case when $\beta\tau_0^2R_0=1$, shown for $t=1$, 4, and 5 (curves *1*, *2*, and *3*, respectively).

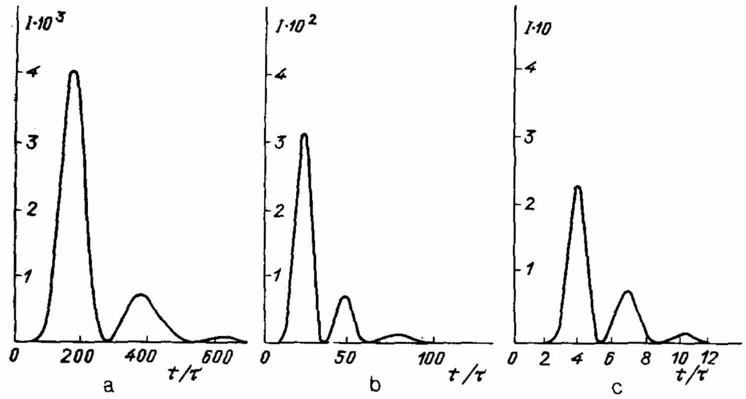

FIG. 7. Radiation intensity profile plotted for three values of the parameter β at a fixed value of the gain $\mu_0 L = \beta/\alpha$: a) $\alpha = 10^{-3}$, $\beta = 10^{-1}$; b) $\alpha = 10^{-2}, \beta = 1$; c) $\alpha = 10^{-1}, \beta = 10$.

namics of the field amplitudes, polarizations, and difference between the populations of inhomogeneously broadened media is

$$\frac{\partial a_1}{\partial t} + \frac{\partial a_1}{\partial x} = \int p_1(\Delta) f(\Delta)\,d\Delta,$$
$$\frac{\partial a_2}{\partial t} - \frac{\partial a_2}{\partial x} = \int p_2(\Delta) f(\Delta)\,d\Delta, \tag{5.1}$$
$$\frac{\partial p_{1,2}(\Delta)}{\partial t} + (\alpha + i\alpha^*\Delta)\, p_{1,2}(\Delta) = \beta a_{1,2} R(\Delta),$$
$$\frac{\partial R(\Delta)}{\partial t} = -2\sum_{n=1}^{2} (a_n^* p_n(\Delta) + a_n p_n^*(\Delta)).$$

The initial conditions corresponding to spontaneous decay of a system of excited atoms can be selected in the form

$$a_1(x, 0) = a_2(x, 0) = 0, \quad R(x, 0, \Delta) = R_0(x, \Delta),$$
$$p_1(x, 0, \Delta) = p_{01}(x, \Delta), \quad p_2(x, 0, \Delta) = p_{02}(x, \Delta), \tag{5.2}$$

where $p_{01}(x, \Delta)$ and $p_{02}(x, \Delta)$ determine the initial spontaneous polarization. The boundary conditions are generally given by

$$a_1\left(-\frac{1}{2}, t\right) = r_1 a_2\left(-\frac{1}{2}, t\right), \quad a_2\left(\frac{1}{2}, t\right) = r_2 a_1\left(\frac{1}{2}, t\right). \tag{5.3}$$

It follows from Eq. (5.1) that the main parameters influencing the nature of dynamics of decay of the system are the ratios of the transit time ($\tau = L/c$) to the homogeneous $\alpha = \tau/2T_2$ and inhomogeneous $\alpha^* = \tau/2T_2^*$ relaxation times, and the ratio of τ to the coherence time $\beta = \tau/\tau_c$. We shall investigate the influence of these parameters on the intensities and shapes of radiation pulses.

5.1. Superradiance and superluminescence

We shall consider first the decay occurring in free (with no mirrors) systems. We then have

$$r_1 = r_2 = 0, \tag{5.4}$$

and the boundary conditions of Eq. (5.3) become homogeneous. Figure 7 shows the intensity profile of the radiation generated in media excited completely and uniformly at the initial moment

$$R_0(x, 0) = 1 \tag{5.5}$$

in the absence of inhomogeneous broadening

$$f(\Delta) = \delta(\Delta). \tag{5.6}$$

In all three cases the ratio β/α has the same value $\beta/\alpha = 100$, whereas the parameter β varies as follows: $\beta = 0.1$ in Fig. 7a, $\beta = 1$ in Fig. 7b, and $\beta = 10$ in Fig. 7c. It is clear from this figure that the intensity profile is practically unaffected by variation of β, but the maximum intensity of the pulse and the time when it occurs (delay time t_0) depend strongly on β. The intensity rises on increase in β and the delay time falls (approximately as β^{-1}). Figure 8 illustrates the nature of the dependence of the peak intensity of the first pulse on the value of the parameter β for a fixed value of α (Figs. 8a and 8b) and on the parameter α for a fixed β (Fig. 8c). We can see that an increase in the parameter β in the range $\beta > \alpha$ first increases the pulse intensity linearly, which then reaches saturation as I_0 reaches $I_{0|\max} = 1$. We recall that the field amplitude in the system (5.1) is normalized in such a way that

$$n(x, t) = |a(x, t)|^2$$

is the photon number density normalized to the density of atoms. In turn, we have

$$I(t) = n\left(\frac{1}{2}, t\right).$$

Consequently, the fact that $I(t)$ becomes equal to unity means that the photon density at the end of a sample reaches the density of resonant atoms. It is clear from Fig. 8c that in

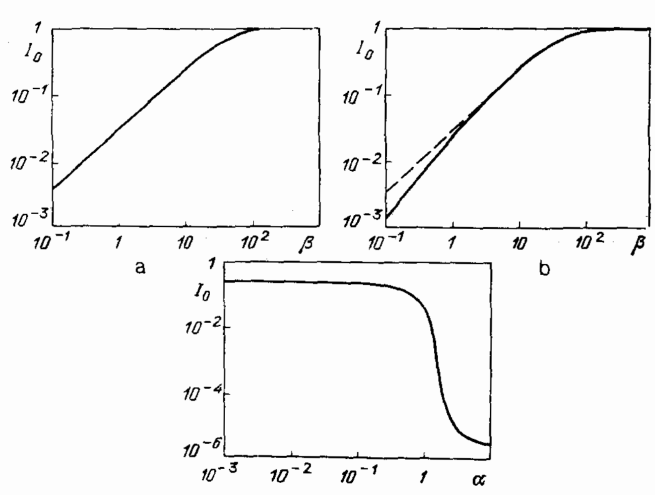

FIG. 8. Dependence of the peak value of the intensity on the parameter β for fixed values of $\alpha = 10^{-3}$ (a) and $\alpha = 10^{-2}$ (b), and on the value of α for fixed $\beta = 10$ (c).

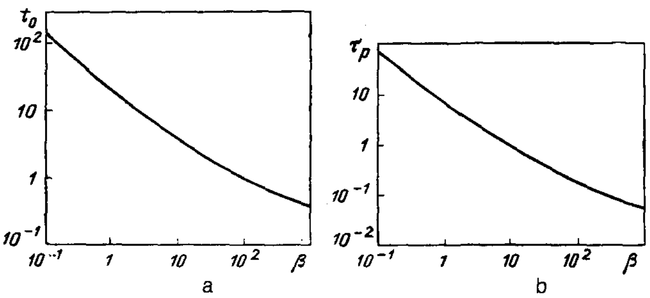

FIG. 9. Dependences of the delay time (a) and of the pulse duration (b) on the parameter β for a fixed $\alpha = 10^{-3}$.

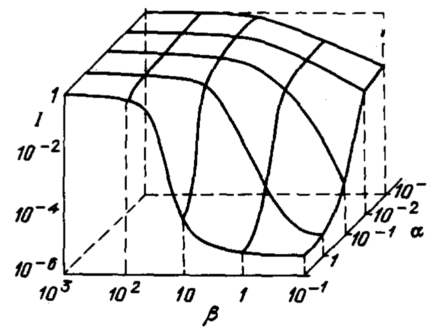

FIG. 10. Peak intensity of a pulse plotted as a function of the parameters α and β.

the range $\alpha \ll \beta$ the maximum intensity of a pulse is independent of the parameter α, but the "saturation" intensity depends on β. As α tends to β, the maximum radiation intensity begins to fall strongly and vanishes at $\alpha \approx \beta$. The dashed curve in Fig. 8b represents the curve in Fig. 8a. We can therefore see that in the range $\beta \gg \alpha$ the value of I_0 is independent of α and is influenced only by the parameter β. As β approaches α, the gain $\mu_0 L = \beta/\alpha$ falls and the radiation intensity decreases. Figure 9 shows the dependence of the delay time t_0 (Fig. 9a) and of the pulse duration τ_p (Fig. 9b) on the parameter β for a fixed value of α. As pointed out earlier, $t_0 \propto \beta^{-1}$ and $\tau_p \propto \beta^{-1}$ in the range $\beta \gg \alpha$. Figure 10 illustrates the behavior of the peak value of the intensity in the plane of the parameters (α, β). It is clear from this figure that the coherent radiation intensity exceeds the spontaneous radiation background when $\beta > \alpha$. The pulse intensity rises further in the range $\beta > \alpha$ along straight lines parallel to the linear dependence $\log \alpha = -\log \beta$. The intensity is approximately constant along lines parallel to the line $\log \alpha = \log \beta$.

Figures 11 and 12 illustrate the influence of inhomogeneous broadening on the profile and intensity of the radiation pulses. It is clear from these figures that, in full agreement with our analysis in Sec. 3.5, coherent radiation is generated when $\beta/\alpha^* > 1$. If $\alpha^* \propto \beta$, the peak value of the pulse intensity is close to the spontaneous background level.

The results of our analysis allow us to draw the following conclusions. Firstly, in the case of media characterized by homogeneous broadening the threshold of generation of coherent radiation is given by the condition $\beta/\alpha > 1$. Secondly, in the case of media characterized by inhomogeneous broadening, when excitation is distributed uniformly across the whole inhomogenous line profile (Figs. 11 and 12 illustrate this case) the condition of generation of coherent radiation becomes $\beta/(\alpha + \alpha^*) > 1$. Thirdly, the nature of the dependence of the peak value of the intensity on the parameter β allows us to distinguish two ranges of β which differ qualitatively. We can see from Fig. 8 that if $\beta < 10$, the peak intensity is a quadratic function of the initial inversion density. In fact, if $\beta < 10$, then $I_0 \propto \beta$. Using the normalization introduced by us earlier [see Eqs. (2.10) and (2.14)] and Eq. (2.12), we obtain

$$I_{\max} \propto cW = c\hbar\omega \frac{N}{V} I_0 \propto \frac{N}{V} \beta \propto \left(\frac{N}{V}\right)^2. \qquad (5.7)$$

The regime with the quadratic dependence of the peak intensity on the inversion density is usually called superradiance. We can also see from Fig. 8 that in the range $\beta > 10$ the peak intensity of the first pulse rises more slowly on increase in β, tending in the limit to the linear dependence $I_{\max} \propto N/V$. This is usually called superluminescence or amplification of spontaneous radiation.

5.2. Superradiance, superluminescence, and self-excitation in an optical resonator

We shall now consider the dependence of the intensity and profile of a radiation pulse on the parameters of a medium when generation of radiation occurs in a resonator. We must point out straightaway that we are not dealing here

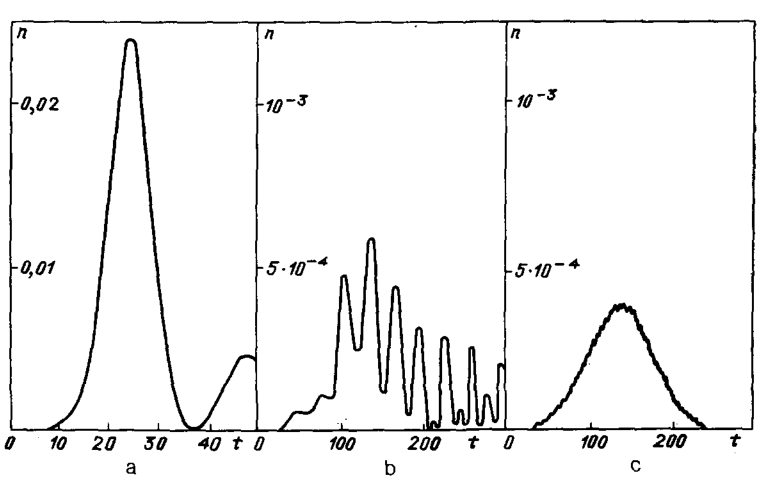

FIG. 11. Influence of inhomogeneous broadening on the profile of the radiation intensity plotted for $\alpha = 10^{-2}$, $\beta = 1$, and $\alpha^* = 0.1$ (a), $\alpha^* = 1$ (b), and $\alpha^* = 5$ (c).

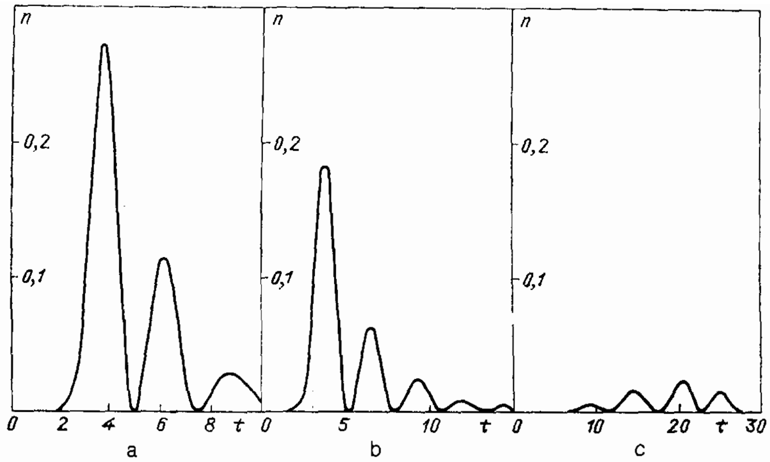

FIG. 12. Influence of inhomogeneous broadening on the profile of the radiation intensity plotted for $\alpha = 10^{-2}$, $\beta = 10$, and $\alpha^* = 0.1$ (a), $\alpha^* = 1$ (b), and $\alpha^* = 5$ (c).

with the case of an active medium inside a high-Q optical resonator, but with the more general problem of the influence on the nature of the dynamics of reflection of the field at the boundaries of the active medium on the nature of the dynamics of generation of radiation. We shall assume that the reflection coefficients $r_{1,2}$ occurring in the boundary conditions of Eq. (5.3) vary from 0 to 1. Therefore, generation of radiation in a high-Q optical resonator is a special case of the general problem considered by us. It is necessary to allow for the reflection of the field because in all cases of practical importance the coefficient of reflection at the boundary of an active medium does not vanish, because the permittivity of such a medium always differs from the permittivity of the environment. This difference is important even when an active medium is a low-pressure gas and it can alter considerably the nature of dynamics of generation of radiation in the case of solid-state active media. Therefore, decay in open systems ($r_1 = r_2 = 0$) and decay in a high-Q resonator ($r_1 \approx r_2 \approx 1$) are two limiting cases of our general discussion.

Figure 13 illustrates changes in the radiation intensity profile when the reflection coefficients are varied. Figure 14 gives the resultant dependences of the peak intensity on the reflection coefficient for various values of the parameters α and β. Figures 14a–14d show the dependence of the field inside the resonator $I_0 = |a_1(1/2, t)|^2$ and outside the resonator $I = I_0(1 - r_2^2)$ in two limiting cases: when both mirrors have the same reflection coefficient ($r_1 = r_2 = r$) and when the left-hand mirror is totally reflecting ($r_1 = 1$) and the reflection coefficient of the right-hand mirror varies from 0 to 1. It is clear from these figures that in superradiant media ($\alpha \ll \beta < 1$) the peak intensity rises quadratically on increase in r. On the other hand, in superluminescent media ($1 < \alpha \ll \beta$) an increase in r reduces the peak value of the intensity. The behavior of the superradiant and superluminescent media differs also qualitatively as r approaches 1. It is clear from Fig. 15a that in the case of superradiant media when $r = 1$ the energy is transferred regularly from the field to the medium and back again. The peak value $I_0 = 1/2$, i.e., the maximum total number of photons in two counterpropagating waves, is equal to the number of atoms. On the other hand, when $r = 1$ in superluminescent media, several consecutive oscillations do not alter the field ampli-

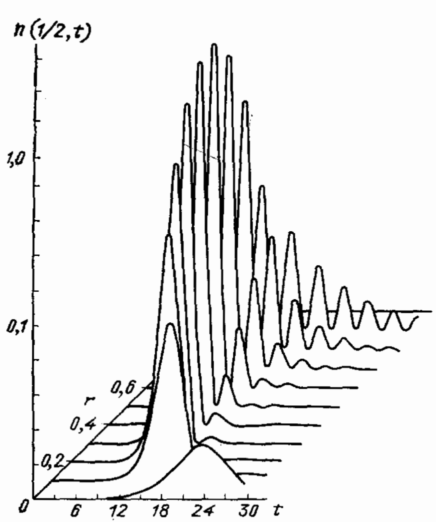

FIG. 13. Dependence of the radiation pulse profile on the reflection coefficient $r_1 = r_2 = r$ in the case when $\alpha = 10^{-2}$ and $\beta = 1$.

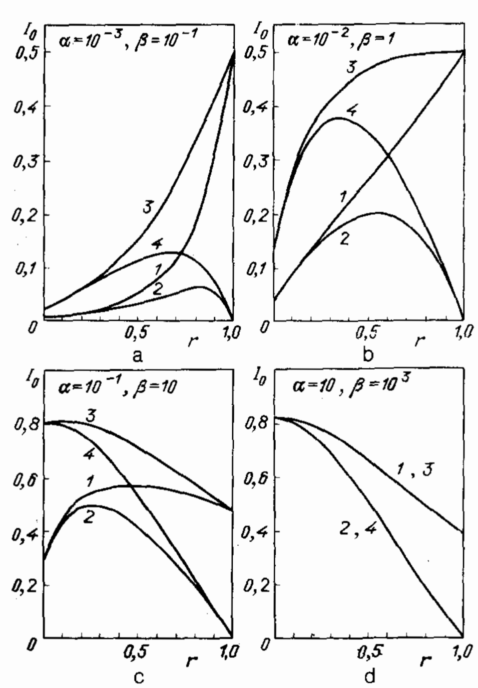

FIG. 14. Peak value of the radiation intensity inside (curves *1* and *3*) and outside (curves *2* and *4*) the active medium, plotted as a function of the reflection coefficient. Curves *1* and *2* correspond to $r_1 = r_2 = r$, whereas curves *3* and *4* correspond to $r_1 = 1$, $r_2 = r$.

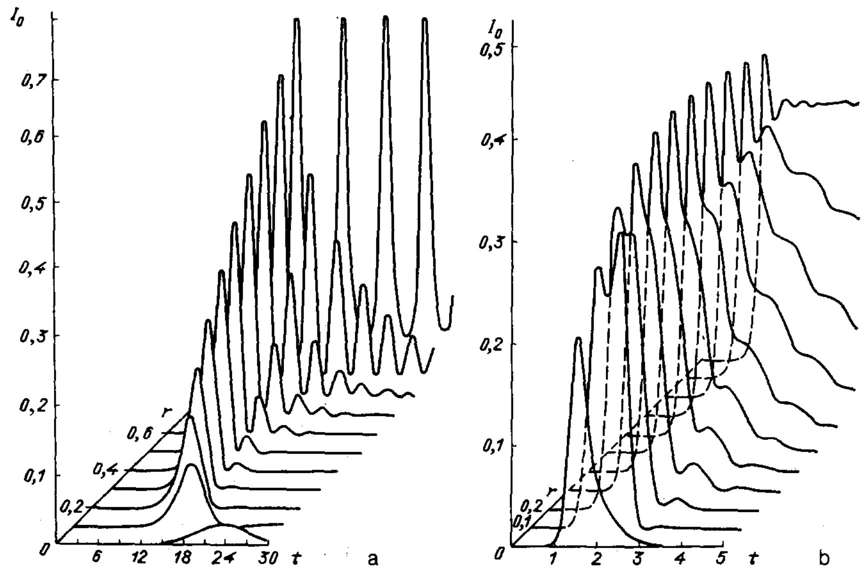

FIG. 15. Influence of the reflection coefficient r on the intensity profile inside the active region $I_0(t) = |a(1/2,\ t)|^2$: a) $\alpha = 10^{-2}, \beta = 1$; b) $\alpha = 10$, $\beta = 100$.

tude and I_0 assumes the value 1/4. This means that the resonant atoms emit $N/2$ photons and then the population difference vanishes so that the medium no longer interacts with the field inside the resonator.

Figure 16 shows the dependence of the delay time on the reflection coefficient r for the same parameters α and β as in Fig. 14. We can see that whereas in superradiant media the delay time is inversely proportional to r, in superluminescent media it is practically independent of r.

It is traditional to divide media into superradiant and superluminescent in accordance with the value of the parameter $\alpha = \tau/T_2$. When the transit time of a photon across a sample $\tau = L/c$ is less than the homogeneous relaxation time T_2, then superradiance appears in such media. If $\alpha > 1$, i.e., when $T_2 < \tau$, it is assumed that superluminescence is emitted. It is clear from Fig. 14 that the division of media

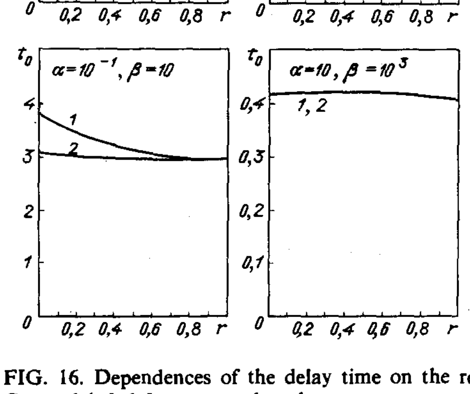

FIG. 16. Dependences of the delay time on the reflection coefficient. Curves labeled *1* correspond to the case $r_1 = r_2 = r$, whereas curves labeled *2* correspond to the case $r_1 = 1$, $r_2 = r$.

into these two classes in accordance with the parameter α alone is quite rough. The parameter β is very important. Figure 17 gives the dependences of I_0 and I on r for media with $\alpha = 10^{-2}$ and different values of the parameter β. We can see that if $\beta < 1$, the value of I_0 increases with r and for media with $\beta > 1$, we have a reduction. Therefore, introduction of a self-consistent definition of the regime of generation of coherent radiation should be based on the use of qualitative differences between the dependences of the characteristic parameters of the output pulses on the parameters of the medium. In our opinion a suitable criterion is that based on the sign of the derivatives $\partial I_0/\partial r|_{r=0}$ and $\partial I_0/\partial r|_{r=1}$:

1) $\left.\frac{\partial I_0}{\partial r}\right|_{r=0} > 0, \quad \left.\frac{\partial I_0}{\partial r}\right|_{r=1} \geqslant 0$ — superradiance,

2) $\left.\frac{\partial I_0}{\partial r}\right|_{r=0} \leqslant 0, \quad \left.\frac{\partial I_0}{\partial r}\right|_{r=1} < 0$ — superluminescence,

3) $\left.\frac{\partial I_0}{\partial r}\right|_{r=0} > 0, \quad \left.\frac{\partial I_0}{\partial r}\right|_{r=1} < 0$ — transition regime .

The rise of I_0 with r in superradiant media is due to the fact that in this case the width of a field line $\Delta\omega_f \approx 1/\tau$ is greater than the width of an atomic transition line $\Delta\omega_a = 1/T_2$. Placing of a medium inside the resonator then reduces the field line width

$$\Delta\omega_f = \frac{1}{\tau} \ln \frac{1}{r}$$

on increase in r, which results in a stronger interaction of the field and atomic subsystems. In the case of superluminescent media we have $\Delta\omega_f < \Delta\omega_a$ already for $r = 0$, so that an increase in r—which reduces $\Delta\omega_f$—simply reduces the strength of the interaction between the field and atomic subsystems.

We shall now consider the influence of inhomogeneous broadening on the nature of the dependence $I_0(r)$. Figure 18 shows the dependences of I on r for various values of the inhomogeneous broadening parameter α^*. We can see that if $\alpha^* \gg \alpha$, the radiation intensity decreases on increase in α^*. However, the qualitative nature of the dependence $I_0(r)$ is practically unaffected. In particular, the sign of the derivative $\partial I_0/\partial r|_{r=1}$ remains unaltered. The reduction in the radiation intensity on increase in α^* is related, as shown in Sec.

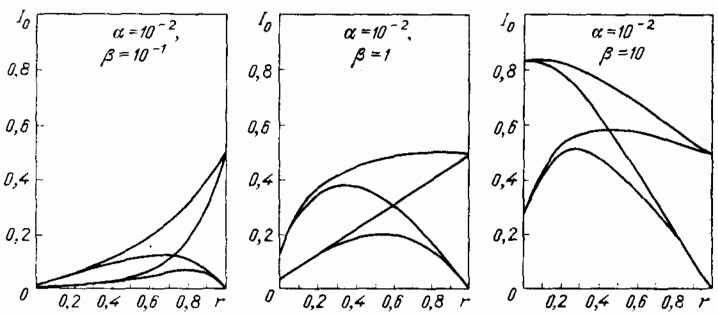

FIG. 17. Influence of the parameter β on the nature of the dependences $I_0(r)$ and $I(r)$.

2, to a reduction in the density of resonantly interacting atoms.

5.3. Collective superradiance

We considered above the dependences of the intensity and profile of the output pulses on the main parameters of a light-emitting medium: α, β, α^*, r_1, r_2. The first three are the internal energy parameters, for example, an increase in β requires an increase in the pump power, whereas the parameters α and α^* can be reduced by supplying energy so as to establish a more ordered state in the atomic subsystem. The parameters r_1 and r_2 are external. Variation of these parameters in a certain range available in experiments makes it possible to control the parameters of the output pulses, i.e., to some extent it is possible to optimize the process of extraction of the energy stored initially in the atomic subsystem.

Optimization in respect of these parameters does not exhaust all the possibilities for optimization of the process of generation of radiation by extended polyatomic inverted media. One other opportunity is provided by control of the initial conditions described by Eq. (5.2). This is more difficult than the method of control described above, in which case the dependence of the solution on the parameters of the problem can be obtained simply by varying these parameters in a certain range, as was done above. Therefore, I_0 is a function of the parameters α, β, α^*, r_1, r_2, and it is also a functional in the space of the initial conditions. This problem cannot be solved simply by suitable selection of the initial conditions. However, we shall be interested only in the extremal regions of the functional. We can therefore adopt a more direct approach involving an analysis of the symmetry of the investigated mathematical or physical problem. For example, in the case of point-like systems, i.e., systems with dimensions much less than the radiation wavelength, the symmetry of the atomic subsystem in the superradiant state ensures the maximum decay rate proportional to N^2. We demonstrated above that in the case of extended systems this symmetry breaks down. This is due to the fact that in the case of point-like systems the symmetry of the atomic subsystem determines the symmetry of the total system, since the field intensity is the same at all the points in the active medium and field photons leave the light-emitting region almost instantaneously. However, in extended systems an inhomogeneity of the field along the active medium results in breaking of the symmetry of the atomic subsystem, because the rate of emission of stimulated radiation depends on the field amplitude. Therefore, the states symmetric only relative to the variables of the atomic subsystem are not characterized by the highest radiative decay rates, which is true only of the states that allow for the symmetry of the complete atomic–field system.

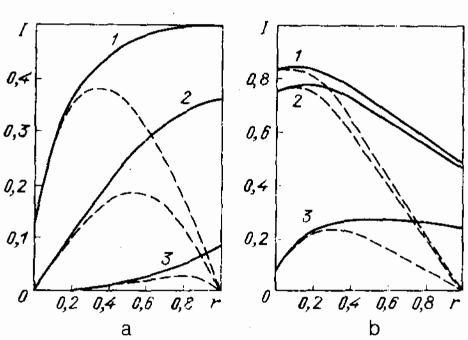

FIG. 18. Influence of inhomogeneous broadening on the nature of the dependences $I_0(r)$ and $I(r)$: a) $\alpha=10^{-2}$, $\beta=1$, $r_1=1$, $\alpha^*=0.1$, 1, and 5 for curves *1, 2,* and *3,* respectively; b) $\alpha=10^{-2}$, $\beta=10$, $r_1=1$, and $\alpha^*=0.1$, 1, and 5, for curves *1, 2,* and *3,* respectively.

Such high-symmetry solutions of the system of equations (5.1), subject to the boundary conditions (5.3), were found in Ref. 27. These solutions are

$$
\begin{aligned}
a_1(x,t) &= \frac{1}{2\beta^{1/2}}\frac{\partial\theta(x,t)}{\partial t}\cos\varphi(x),\\
a_2(x,t) &= \frac{1}{2\beta^{1/2}}\frac{\partial\theta(x,t)}{\partial t}\sin\varphi(x),\\
p_1(x,t) &= \frac{\beta^{1/2}}{2}R_0\sin\theta(x,t)\cdot\cos\varphi(x),\\
p_2(x,t) &= \frac{\beta^{1/2}}{2}R_0\sin\theta(x,t)\cdot\sin\varphi(x),\\
R(x,t) &= R_0\cos\theta(x,t),
\end{aligned}
\tag{5.8}
$$

where the Bloch angle $\theta(x,t)$ is given by the expression

$$\theta(x,t)=A(\operatorname{ch}\Phi(x))^{1/2}u(t);\tag{5.9}$$

here,

$$A^2=\frac{4\ln(1/r_1r_2)^{1/2}}{[(1-r_1^2)/r_1]+[(1-r_2^2)/r_2]},\tag{5.10}$$

$$\Phi(x)=x\ln\frac{1}{r_1r_2}-\ln\left(\frac{r_2}{r_1}\right)^{1/2},\tag{5.11}$$

$$\sin 2\varphi(x)=\frac{1}{\operatorname{ch}\Phi(x)}.\tag{5.12}$$

It is clear from the expressions (5.8)–(5.12) that the spatial dependence of the difference between the populations, field amplitudes, and polarization waves is determined uniquely by the boundary conditions. The dynamics of all five unknown functions $a_1(x,t)$, $a_2(x,t)$, $p_1(x,t)$, $p_2(x,t)$, $R(x,t)$ is described by one equation:

$$\frac{d^2u}{dt^2}+\gamma\frac{du}{dt}=\beta R_0\sin u,\tag{5.13}$$

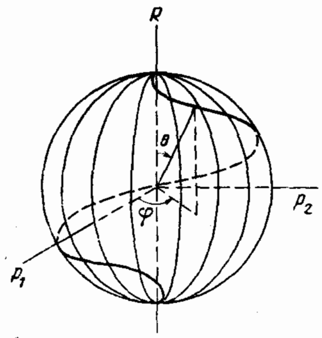

FIG. 19. Angles $\theta(x, t)$ and $\varphi(x, t)$ specify uniquely the position of the Bloch vector in the pseudospin space (p_1, p_2, R). In general, the end of the Bloch vector winds itself on the Bloch sphere, which increases the radiative decay time. The highest decay rate is exhibited by the states corresponding to the motion of the end of the Bloch vector along the azimuth.

where

$$\gamma = \ln \frac{1}{(r_1 r_2)^{1/2}}.$$

It is clear from the system (5.8) that $\varphi(x)$ is the azimuthal angle in the energy pseudospin space (p_1, p_2, R), where the Bloch angle $\theta(x, t)$ is the polar angle. These two angles specify completely the position of the polarization vector on a sphere of radius R_0. The difference between the supersymmetric solution of Eq. (5.8) and the arbitrary solution of the system of equations (5.1) characterized by $\alpha = \alpha^* = 0$ is that in general the angle φ depends on time $\varphi(x, t)$. We can therefore say that in general the end of the Bloch vector winds itself on a Bloch sphere in the course of radiative decay of our system (Fig. 19) which accounts for an increase in the radiative decay time and a consequent reduction in the intensity. On the other hand, in the case of the solution (5.8) the end of the Bloch vector moves from the uppermost point on the Bloch sphere to the lowest one following the shortest path along the azimuth, which ensures the maximum radiative decay rate.

Using Eqs. (5.8)–(5.12), we can show readily that the intensity of the radiation emitted from the right-end of the medium, where the reflection coefficient has the value r_2 is given by the expression

$$I_1(t) = \left| a_1\left(\frac{1}{2}, t\right)\right|^2 (1 - r_2^2) = 2\gamma a^2(t) \frac{(1 - r_2^2) r_1}{(1 - r_1^2) r_2 + (1 - r_2^2) r_1}, \tag{5.14}$$

where

$$a(t) = \frac{1}{2\beta^{1/2}} \frac{du}{dt}.$$

The total intensity of the radiation emitted from both ends of the medium is

$$I(t) = I_1(t) + I_2(t) = 2\gamma a^2(t). \tag{5.15}$$

The intensity of the field within the active medium is

$$n(x, t) = n_1(x, t) + n_2(x, t) = A^2 a^2(t) \operatorname{ch} \Phi(x). \tag{5.16}$$

The integrated intensity of the field in the medium is given by

$$I_0(t) = \int_{-1/2}^{1/2} n(x, t)\, dx = a^2(t). \tag{5.17}$$

Let us assume that initially a state of complete inversion is established in the medium, so that $R_0 = 1$. We shall now introduce a new dimensionless time

$$\tau = \beta^{1/2} t,$$

so that Eq. (5.13) can be rewritten in the form

$$\frac{d^2 u}{d\tau^2} + \delta \frac{du}{d\tau} = \sin u, \tag{5.18}$$

where

$$\delta = \frac{\gamma}{\beta^{1/2}}. \tag{5.19}$$

Consequently, the profile of a radiation pulse depends on

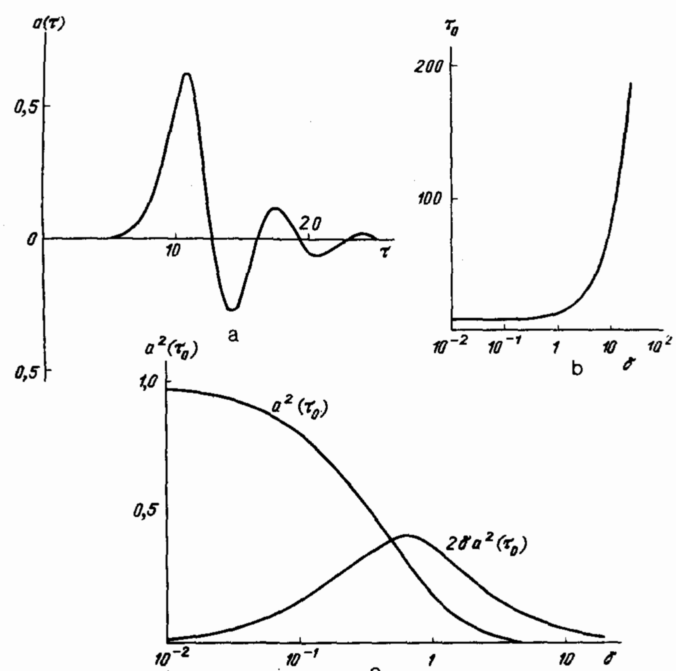

FIG. 20. a) Profile of a collective superradiance pulse, $\delta = 0.5$; b) dependence of the delay time τ_0 on the value of the parameter σ; c) dependence of the peak value $I_0(\tau_0) = a^2(\tau_0)$ and $I(\tau_0) = 2\delta a^2(\tau_0)$ on the parameter δ.

just one parameter δ. Figure 20 shows the pulse profile for one of the values of the parameters δ, as well as the dependence of the delay time τ_0 and of the square of the maximum amplitude $a^2(\tau_0)$ on the parameter δ. The quantity $2\delta a^2(\tau_0)$, determines, as can be deduced from Eq. (5.15), the intensity of the radiation emitted from both ends of the medium. We can see that it has an extremum where the maximum of the peak value of the radiation intensity is reached.

It should be pointed out that if $r_1 = r_2 = 1$, the solution of Eq. (5.13) assumes the familiar form

$$a(t) = \frac{1}{\operatorname{ch}[\beta^{1/2}(t-t_0)]},$$

where

$$t_0 = \frac{1}{\beta^{1/2}} \ln \frac{1}{\theta_0},$$

from which it follows that the expressions for the wave amplitudes $a_{1,2}(x, t)$ have the following form in the interval $(0, 2t_0)$:

$$a_1(x, t) = a_2(x, t) = \frac{1}{\operatorname{ch}[\beta^{1/2}(t-t_0)]} \cdot \frac{1}{\sqrt{2}}$$

and they are then repeated periodically at intervals $[2nt_0, 2(n+1)t_0]$.

Systems exhibiting inhomogeneous broadening were also investigated in Ref. 27. In this case we have

$$\theta(x, t, \Delta) = A(\operatorname{ch}\Phi(x))^{1/2} u(t, \Delta),$$

where the functions $u(t, \Delta)$ satisfy the equations

$$\frac{d^2u(t, \Delta)}{dt^2} + (\gamma - i\alpha^*\Delta)\frac{du(t, \Delta)}{dt} = \beta \int \exp[-i(\Delta - \Delta')\alpha^* t] \sin u(t, \Delta') \cdot f(\Delta')\, d\Delta'. \qquad (5.20)$$

It is clear from Eq. (5.8) that $a_n(x, t)$ and $p_n(x, t)$ can be represented in the form

$$a_1(x, t) = a(x, t)\cos\varphi(x, t), \quad a_2(x, t) = a(x, t)\sin\varphi(x, t),$$
$$p_1(x, t) = p(x, t)\cos\varphi(x, t), \quad p_2(x, t) = p(x, t)\sin\varphi(x, t).$$

The high symmetry of the solutions given by Eq. (5.8) and of the corresponding states of the atomic–field system is reflected also in the fact that for these solutions, in addition to the Bloch conservation integral

$$p^2(x, t) + \frac{\beta}{4} R^2(x, t) = p^2(x, 0) + \frac{\beta}{4} R^2(x, 0),$$

there are two further integrals of motion:

$$\varphi(x, t) = C_1(x),$$
$$a(x, t)[\sin(2\varphi(x, t))]^{1/2} = C_2(t).$$

After finding our supersymmetric solutions, we can readily identify the initial conditions that ensure that these solutions are obtained. It follows from Eq. (5.13) that we can select two main types of the initial conditions:

$$\text{(I)}\ u(0) = u_0, \quad \frac{du}{dt}(0) = 0,$$
$$\text{(II)}\ u(0) = 0, \quad \frac{du}{dt}(0) = \frac{du_0}{dt}.$$

In the former case the initial conditions for $a_n(x, t)$, $p_n(x, t)$ and $R(x, t)$ are

$$a_1(x, 0) = a_2(x, 0) = 0,$$
$$p_1(x, 0) = \frac{\beta^{1/2}}{2} R_0 \frac{\sin[A(\operatorname{ch}\Phi(x))^{1/2} u_0]}{(2e^{-\Phi}\operatorname{ch}\Phi)^{1/2}} \approx \frac{\beta^{1/2}R_0}{2}\frac{Au_0}{\sqrt{2}} e^{\Phi/2},$$
$$p_2(x, 0) = \frac{\beta^{1/2}}{2} R_0 \frac{e^{-\Phi}\sin[A(\operatorname{ch}\Phi(x))^{1/2} u_0]}{(2e^{-\Phi}\operatorname{ch}\Phi)^{1/2}} \approx \frac{\beta^{1/2}R_0}{2}\frac{Au_0}{\sqrt{2}} e^{-\Phi/2}, \qquad (5.21)$$
$$R(x, 0) = R_0 \cos[A(\operatorname{ch}\Phi(x))^{1/2} u_0].$$

In the latter case, the initial conditions are given by the following set of equations:

$$a_1(x, 0) = \frac{A}{2\beta^{1/2}} \frac{du_0}{dt} \frac{e^{\Phi/2}}{\sqrt{2}},$$
$$a_2(x, 0) = \frac{A}{2\beta^{1/2}} \frac{du_0}{dt} \frac{e^{-\Phi/2}}{\sqrt{2}},$$
$$p_1(x, 0) = p_2(x, 0) = 0,$$
$$R(x, 0) = R_0. \qquad (5.22)$$

In the former case we have to create a certain spatial distribution of the initial population inversion and of the polarization current density. In the latter case we have to find the initial field distribution given by Eq. (5.22).

A clear illustration of the feasibility of optimization of the process of energy extraction from an excited medium under conditions of collective spontaneous radiation is provided by Figs. 21–23. Figure 21 shows how the profile of a radiation pulse depends on the parameter β (for example, the inversion density in the medium) for a fixed value of the parameter $\alpha = \tau/T_2$ and a uniform excitation of the medium

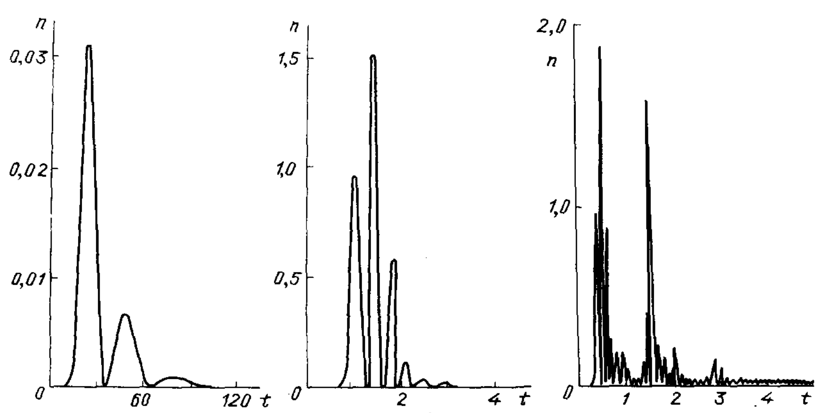

FIG. 21. Dependence of the radiation intensity profile on the parameter $\beta = 1$ (a), 100 (b), and 10^3 (c) for fixed $\alpha = 10^{-3}$ and the initial conditions $a_1(x, 0) = a_2(x, 0) = a_0$, $P_n(x, 0) = 0$, and $R(x, 0) = R_0$.

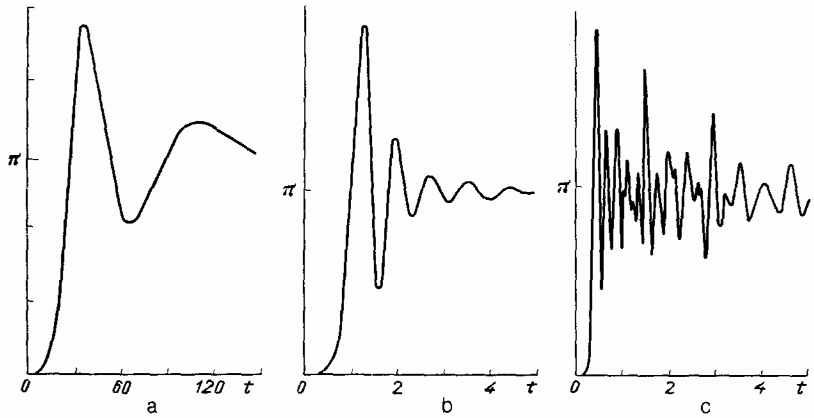

FIG. 22. Dynamics of the area under the pulses shown in Fig. 21.

$R(x, 0) = 1$. We can see that as the inversion density rises, a radiation pulse is transformed into a sequence of pulses with an irregular variation of the peak intensity of the separate pulses. In all the cases considered above, we are dealing with generation of coherent radiation, which is confirmed by the time dependence of the area under a pulse shown in Fig. 22. We can see that

$$\theta\left(\frac{1}{2}, t\right)\Big|_{t\to\infty} \to \pi.$$

Figure 23 shows, for the sake of comparison, the profiles of the radiation pulses corresponding to the initial conditions of Eqs. (5.21) and (5.22). We can see that, instead of a train of pulses, we can again have a pulse of the shape shown in Figs. 20a and 21a. The intensity of the main peak exceeds considerably the intensity of any pulse in a train and the energy in this pulse is equal to the total energy of a train of pulses.

6. SUPERRADIANCE OF A SYSTEM OF NUCLEAR SPINS

One of the remarkable recent results of experimental investigations of superradiance is that emitted by a system of proton spins.[38-40] These experiments demonstrated the validity of the model, regarded for a long time as artificial, of superradiance of a point-like system which allowed Dicke to predict superradiance and which has continued to attract the interest of theoreticians for several decades. The importance of these experiments is that a detailed comparison of their results with theoretical predictions makes it possible to test the validity of the models proposed earlier. In this section we shall derive the equations of the dynamics and analyze the characteristic features of the superradiance emitted by a system of proton spins.

6.1. Experimental investigations

Investigations of rf superradiance emitted by a system of nuclear spins were carried out using a high-Q oscillatory circuit with an induction coil containing a solid sample characterized by an initial negative polarization of the nuclear spins. The active medium was an insulator with a high proton concentration in the form of propanediol ($C_3H_8O_2$) containing paramagnetic Cr^{+V} impurity ions. Cooled spheres 1 mm in diameter were poured into a chamber where ^3He was dissolved in ^4He and they were magnetized by a static field. The spin concentration was $N/V = 3.8\times 10^{22}$ cm^{-3} in the experiments reported in Ref. 38 and $N/V = 4.5\times 10^{22}$ cm^{-3} in those reported in Ref. 39. In the latter case the population inversion of the Zeeman proton levels was induced by the method of dynamic polarization of nuclei in a field of 2.45 T, which corresponded to a proton magnetic resonance frequency $\nu_p = 104.3$ MHz. A sample was located in a multimode cylindrical microwave resonator of 14 cm^3 volume. An inductance coil was placed inside the resonator and it consisted of three turns of an isolated wire. The diameter of the turns was 12 mm and the length of the coil was 8 mm. The polarization measurements were made

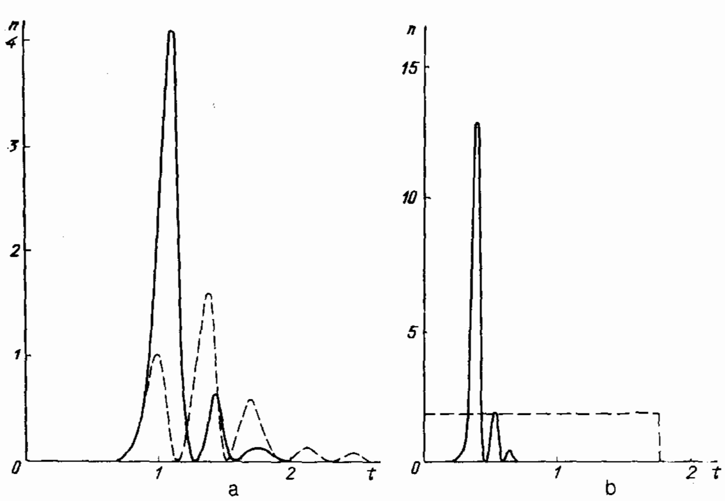

FIG. 23. Comparison of the profiles of the collective superradiance pulses (continuous curve) and conventional superradiance (see Figs. 21b and 21c): a) $\alpha = 10^{-3}, \beta = 10^2$; b) $\alpha = 10^{-3}, \beta = 10^3$. In case b the height of the dashed rectangle is equal to the maximum peak value of the intensity, whereas the width is equal to the characteristic duration of a train of pulses shown in Fig. 21c.

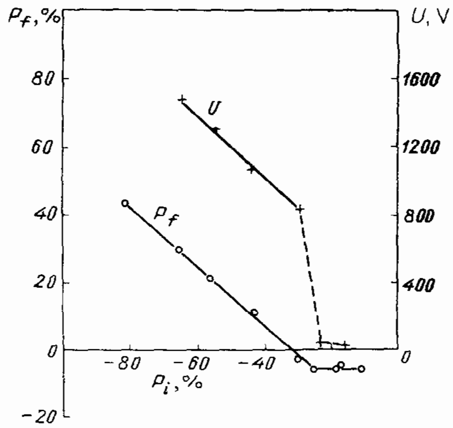

FIG. 24. Dependences of the amplitude of the rf pulse U and of the final value of the polarization P_f on the initial polarization P_i observed in the experiments reported in Ref. 39.

by connecting the coil to a Q meter. The initial polarization was first measured and then the Q meter was disconnected and the applied magnetic field was reduced at the rate of 0.005 T/s. The inductance coil in a section of the cable inside a cryostat formed an oscillatory circuit resonating at a frequency $\nu_r = 54$ MHz. When the proton resonance frequency ν_p approached the resonance frequency ν_r, an rf pulse appeared in the oscillatory circuit and this was recorded with a storage oscilloscope. Figure 24 shows the dependences, on the initial polarization, of the amplitude of the rf pulse and of the finite value of the polarization observed in Ref. 39. It is clear from Fig. 24 that there was a critical value of the initial polarization above which the final polarization began to rise linearly. Reversal of the polarization was observed for $|R_0| > |R_{0c}| = 0.32$.

6.2. Theory of spin superradiance

As pointed out above, the spin superradiance obeys the Dicke model for a point-like system of radiators. However, the actual model has a number of features which distinguish it from an open system of two-level radiators. These features are discussed partly in Refs. 38–40. One of the main differences is the presence of a high-Q resonator ensuring frequency discrimination of the radiation modes. The other is that the characteristic duration of superradiance pulses is comparable with the homogeneous transverse relaxation time T_2. In this case the duration of the radiation pulses depends strongly on T_2. A theory of superradiance allowing for the finite nature of the time T_2 is developed specifically for optical systems in Ref. 44.

We shall derive equations describing the dynamics of a system of spins interacting with one another and with currents in a resonant circuit. If the Hamiltonian of this interaction, described by Eqs. (A1.1)–(A1.4), is modified by substituting the expression for the current density $\mathbf{j}$ in the form

$$\mathbf{j} = c \operatorname{curl} \mathbf{M}, \tag{6.1}$$

where $\mathbf{M}$ is the magnetization vector

$$\mathbf{M} = \sum_{i=1}^{N} \boldsymbol{\mu}_i \delta(\mathbf{r} - \mathbf{r}_i), \tag{6.2}$$

the Hamiltonian of Eq. (A1.4) becomes

$$H_S = -\frac{1}{c}\int \mathbf{j}\mathbf{A}\,dV = -\int \mathbf{M}\mathbf{B}\,dV; \tag{6.3}$$

here, $\mathbf{B}$ is the magnetic induction vector defined by $\mathbf{B} = \operatorname{curl} \mathbf{A}$. Assuming that the magnetic field $\mathbf{B}$ includes a constant component B_0 directed along the z axis and two transverse components oscillating at a radio frequency, we can reduce the Hamiltonian of Eq. (6.3) to

$$H_S = -\hbar\gamma \sum_{i=1}^{N} \mathbf{I}_i \mathbf{B}(\mathbf{r}_i, t) = -\frac{\hbar\gamma}{2}\sum_{i=1}^{N} B_0(\mathbf{r}_i)\sigma_{i3} - \frac{\hbar\gamma}{2}\sum_{i=1}^{N}(\sigma_{i1}B_1(\mathbf{r}_i, t) + \sigma_{i2}B_2(\mathbf{r}_i, t)), \tag{6.4}$$

where $\mathbf{I}$ is the spin operator related to the magnetic moment operator $\boldsymbol{\mu}$ and the spin matrices $\boldsymbol{\sigma}$ by

$$\mathbf{I} = \frac{1}{2}\boldsymbol{\sigma} = \frac{\boldsymbol{\mu}}{\hbar\gamma}. \tag{6.5}$$

Changing the components B_1 and B_2 to circularly polarized components

$$\mathbf{B}_\perp = \mathbf{e}_+ B_+ + \mathbf{e}_- B_-, \tag{6.6}$$

where

$$B_\pm = \frac{1}{2}(B_1 \pm iB_2), \quad \mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2,$$

we can rewrite Eq. (6.4) in the form

$$H_S = -\frac{\hbar\gamma}{2}\sum_{i=1}^{N} B_0(\mathbf{r}_i)\sigma_{i3} - \hbar\gamma\sum_{i=1}^{N}(\sigma_{i+}B_-(\mathbf{r}_i) + \sigma_{i-}B_+(\mathbf{r}_i)), \tag{6.7}$$

where

$$\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2). \tag{6.8}$$

Applying the commutation relationships for the Pauli matrices, we can readily obtain the following equations of motion for $\sigma_\pm$ and σ_3:

$$\begin{aligned} \frac{\partial\sigma_{i+}}{\partial t} &= -i\gamma B_{0i}\sigma_{i+} + i\gamma B_{i+}\sigma_{i3}, \\ \frac{\partial\sigma_{i3}}{\partial t} &= 2i\gamma(\sigma_{i+}B_{-i} - B_{+i}\sigma_{i-}). \end{aligned} \tag{6.9}$$

The equation for the vector potential of the rf field

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}(\mathbf{j}_c + \mathbf{j}_s) \tag{6.10}$$

has two terms on the right-hand side: $\mathbf{j}_c$ is the conduction current density and $\mathbf{j}_s$ is the magnetization current density. The last is described by Eqs. (6.1) and (6.2). The current $\mathbf{j}_c$ is due to the motion of free charges along wires, whereas the current $\mathbf{j}_s$ is due to the rotation of spins. In view of the linearity of Eq. (6.10) the vector potential can also be represented by two terms. The former is related to the conduction current given by

$$\mathbf{A}_c(\mathbf{r}, t) = \frac{1}{c}\int dV' \frac{\mathbf{j}_c(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{J_0(t)\,e^{-i\omega t}}{c}\oint_{\mathscr{L}} d\mathbf{l}' \frac{\exp(i\varkappa|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}, \tag{6.11}$$

where $t' = t - (|\mathbf{r}-\mathbf{r}'|/c)$, $J_0(t)$ is the amplitude of the current in a resonant circuit, and $\mathscr{L}$ is the integration contour which is identical with the lines of flow of the current in the resonant circuit. Equation (6.11) is derived on the assumption that $\mathbf{j}_c(\mathbf{r}', t')$ can be represented in the form

$$\mathbf{j}_c(\mathbf{r}', t') = \mathbf{j}_0(r', t)\exp(-i\omega t + i\varkappa|\mathbf{r}-\mathbf{r}'|). \qquad (6.12)$$

Thus, part of the magnetic field due to the conduction currents in the resonant circuit can be described by

$$\mathbf{B}_c = \operatorname{curl}\mathbf{A}_c = \frac{J(t)}{c}\oint_{\mathscr{L}}\frac{[\mathrm{d}\mathbf{l}', \mathbf{r}-\mathbf{r}']}{|\mathbf{r}-\mathbf{r}'|^3}(1 - i\varkappa|\mathbf{r}-\mathbf{r}'|)\exp(i\varkappa|\mathbf{r}-\mathbf{r}'|). \qquad (6.13)$$

If the magnetization vector $\mathbf{M}$ is represented in a form similar to Eq. (6.12),

$$\mathbf{M}(\mathbf{r}', t') = \mathbf{M}_0(\mathbf{r}', t)\exp(-i\omega t + i\varkappa|\mathbf{r}-\mathbf{r}'|), \qquad (6.14)$$

the following expression is readily obtained for the field $\mathbf{B}_s$:

$$B_{s+} = \frac{e^{-i\omega t}}{2}\int \mathrm{d}V'\mathbf{M}_0(\mathbf{r}', t)\left[[\mathbf{e}_+^*\nabla]\frac{(\mathbf{r}-\mathbf{r}')\exp(i\varkappa|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|^3}\right]. \qquad (6.15)$$

Therefore, if $\varkappa r \ll 1$, the positive-frequency part of the magnetic field can be represented by

$$B_+(r, t) = b_1 J_+ + b_2 R^+, \qquad (6.16)$$

where

$$b_1 = \frac{1}{2c}\oint_{\mathscr{L}}\frac{e_+^*[\mathrm{d}\mathbf{l}', \mathbf{r}-\mathbf{r}']}{|\mathbf{r}-\mathbf{r}'|^3},$$

$$b_2 = \frac{\hbar\gamma}{4N}\sum_{i=1}^{N}\left([\mathbf{e}_+[\mathbf{e}_+^*\nabla]]\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}\right),$$

$$R^+ = \sum_{i=1}^{N}\langle\sigma_{i+}\rangle e^{-i\omega_0 t}. \qquad (6.17)$$

Using Eqs. (6.11)–(6.17), we obtain the following expression for the current in the oscillatory circuit:

$$L\frac{\mathrm{d}^2J}{\mathrm{d}t^2} + R\frac{\mathrm{d}J}{\mathrm{d}t} + \frac{1}{C}J = -\frac{1}{c}\frac{\mathrm{d}^2\Phi_S}{\mathrm{d}t^2}, \qquad (6.18)$$

where L is the inductance of the circuit

$$L = \frac{1}{c^2}\oint_{\mathscr{L}}\oint_{\mathscr{L}}\frac{\mathrm{d}\mathbf{l}\,\mathrm{d}\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|},$$

R is the active resistance, and C is the capacitance of the circuit. The magnetic field flux $\mathbf{B}_s$ through a plane Σ supported by the contour $\mathscr{L}$ is

$$\Phi_s = \int_{\Sigma}\mathbf{B}_s\,\mathrm{d}\mathbf{S} = \oint_{\mathscr{L}}\mathbf{A}_s\,\mathrm{d}\mathbf{l} = g_2 R^+, \qquad (6.19)$$

where

$$g_2 = \int_{\Sigma} b_2(\mathbf{r})\,\mathbf{l}_+\,\mathrm{d}S = \langle b_2\rangle\Sigma_+.$$

Finally, the equation for the current in the resonant circuit becomes

$$\frac{\mathrm{d}^2J}{\mathrm{d}t^2} + 2\delta\frac{\mathrm{d}J}{\mathrm{d}t} + \Omega^2 J = -\nu\frac{\mathrm{d}^2R^+}{\mathrm{d}t^2}, \qquad (6.20)$$

where

$$2\delta = \frac{R}{L}, \quad \Omega^2 = \frac{1}{LC}, \quad \nu = \frac{g_2}{Lc}.$$

Assuming that L can be represented in the form

$$L = \frac{1}{c}\int_{\Sigma} b_1(r)\,e_+\,\mathrm{d}S = \frac{\langle b_1\rangle}{c}\Sigma_+,$$

the expression for ν can be rewritten as follows:

$$\nu = \frac{\langle b_2\rangle\Sigma_+}{Lc} = \frac{\langle b_2\rangle\Sigma_+}{\langle b_1\rangle\Sigma_+} = \frac{\langle b_2\rangle}{\langle b_1\rangle}. \qquad (6.21)$$

It should be noted that the quantities ν and b_1 can be complex, whereas b_2 is real. Therefore, according to Eq. (6.16) there may be a phase shift in the fields B_c and B_s.

Combining the equations for the spins [Eq. (6.9)] with the equations for the conduction current, we obtain the final result in the form of a system of equations

$$\frac{\mathrm{d}R^+(t,\Delta)}{\mathrm{d}t} + \left(i\Delta + \frac{1}{2T_2}\right)R^+(t,\Delta) = i\gamma B_+ R_3(t,\Delta),$$

$$\frac{\mathrm{d}R_3(t,\Delta)}{\mathrm{d}t} = 2i\gamma(R^+(t,\Delta)B_- - B_+R^-(t,\Delta)),$$

$$\frac{\mathrm{d}J^+}{\mathrm{d}t} + [\delta + i(\Omega - \omega_0)]J_+ = i\frac{\omega_0\nu}{2}\int R^+(t,\Delta)f(\Delta)\,\mathrm{d}\Delta,$$

$$B_+(t) = b_1 J_+(t) + b_2\int R^+(t,\Delta)f(\Delta)\,\mathrm{d}\Delta. \qquad (6.22)$$

The system (6.22) allows for an inhomogenous broadening of the proton magnetic resonance lines: $f(\Delta)$ is the profile of an inhomogeneously broadened line, ω_0 is the central frequency of this profile, and $\Delta = \omega - \omega_0$. Neglect of the delay effects, justified for systems with dimensions much less than the radiation wavelength, reduces the wave equation (6.10) to a simple functional dependence of $B_+(t)$ on $J_+(t)$ and $R^+(t)$. This approximation is used in Eqs. (6.12) and (6.14).

In an analysis of the qualitative characteristics of the spin superradiance we shall confine ourselves to the case when $f(\Delta) = \delta(\Delta)$. The system of equations (6.22) contains two characteristic relaxation times: T_2 which is the relaxation time of the spin system and δ^{-1} which is the lifetime of the field in a resonator. The relationship between these times will determine, in the first approximation, the nature of generation of radiation. If $\delta T_2 \gg 1$, we obtain superradiance, whereas superluminescence is limited if $\delta T_2 \ll 1$. A more detailed classification of the radiation generation regimes was given earlier in Sec. 5 and it can be applied directly to the present case if the decay constant δ and the reflection coefficients $r_{1,2}$ are expressed in terms of the Q factor of the resonator. During the initial (linear) stage when $R_3 \approx -|R_{30}|$, the equation for the amplitude of the current in the oscillatory circuit is

$$\frac{\mathrm{d}J}{\mathrm{d}t} + (\delta + i\Delta)J_+ = \omega b_2\gamma T_2|R_{30}|J_+.$$

Consequently, J_+ is given by

$$J_+(t) = J_+(0)\exp[(g-1)\delta t - i(\Omega - \omega)t], \qquad (6.23)$$

where

$$g = \frac{T_2\omega_0\gamma b_2|R_{30}|}{\delta}. \qquad (6.24)$$

The condition for the growth of the current amplitude in the circuit is

$$g>1. \tag{6.25}$$

We shall consider the case when $\delta T_2 \gg 1$. We can then assume that

$$J_+(t) = \frac{i\omega v}{2(\delta + i(\Omega-\omega))} R^+(t),$$

so that the expression for R_3 becomes

$$R_3(t) = |R_{30}| \left[-\frac{1}{g(\omega)} + \left(1 - \frac{1}{g(\omega)}\right) \operatorname{th} \frac{t-t_0}{2\tau_p} \right], \tag{6.26}$$

where

$$g(\omega) = g \frac{\delta^2}{\delta^2 + (\Omega-\omega)^2}, \tag{6.27}$$

$$t_0 = \tau_p \ln \left[\frac{|R_{30}|}{|R_{+0}|} \left(1 - \frac{1}{g(\omega)}\right) \right], \tag{6.28}$$

$$\frac{1}{\tau_p} = \frac{|R_{30}| \omega \gamma b_3}{\delta} \frac{\delta^2}{\delta^2 + (\Omega-\omega)^2} \left(1 - \frac{1}{g(\omega)}\right). \tag{6.29}$$

The pulse of a current in the circuit is in the form of a classical superradiance pulse:

$$|J_+(t)|^2 = \frac{|R_{30}|^2}{4} \left(1 - \frac{1}{g(\omega)}\right)^2 \frac{\omega^2 |v|^2}{4} \frac{1}{\delta^2 + (\Omega-\omega)^2} \frac{1}{\operatorname{ch}^2[(t-t_0)/2\tau_p]}. \tag{6.30}$$

Therefore, it follows from the above expressions that we are dealing here with a regime of superradiance in weakly amplifying media, predicted in Ref. 44. For optical systems the quantity $g(\omega)$ is the gain. The threshold of emission of radiation is given by the condition $g(\omega) > 1$. Only a certain effective number of atoms

$$N_{\text{eff}} = |R_{30}| \left(1 - \frac{1}{g(\omega)}\right),$$

participates in the superradiance process, so that in the course of generation of a superradiance pulse only N spins are inverted. Therefore, if the initial value of the polarization of the target is given by

$$P_i = -\frac{|R_{30}|}{N},$$

the final value of the polarization of the target is described, in accordance with Eq. (6.26), by the expression

$$= \omega(\frac{R_{31}}{N} \left(1 -)P \frac{2}{} \right). \quad \text{“Left part of Eq. missing”} \tag{6.31}$$

Consequently, if $g(\omega) > 2$, the polarization is reversed. Such dependences are in agreement with the experimental results reported in Ref. 39.

In the opposite limiting case ($\delta T_2 \ll 1$) the rotation of spins occurs in a time much shorter than the time taken to establish oscillations in a circuit, which amounts to δ^{-1}. Consequently, the duration of an output radiation pulse is of the order of δ^{-1} and its leading edge exhibits a peak or a sequence of pulses due to the motion of spins.

APPENDICES

1. Derivation of the equations for the interaction of a system of two-level atoms with radiation

The Hamiltonian of a system of two-level atoms, interacting with an electromagnetic field, is of the form

$$H = H_a + H_f + H_{\text{int}}, \tag{A1.1}$$

where H_a is the Hamiltonian of the atomic system:

$$H_a = \sum_{i=1}^{N} \frac{\hbar\omega_{0i}}{2} \sigma_{i3}, \tag{A1.2}$$

H_f is the Hamiltonian of the free field

$$H_f = \int \left[2\pi c^2 \mathbf{B}^2(\mathbf{r}, t) + \frac{1}{8\pi} (\operatorname{curl} \mathbf{A}(\mathbf{r}, t))^2 \right] dV, \tag{A1.3}$$

and H_{int} is the interaction Hamiltonian

$$H_{\text{int}} = -\frac{1}{c} \int \mathbf{jA}\, dV. \tag{A1.4}$$

In Eq. (A1.3) the quantity $\mathbf{A}(\mathbf{r}, t)$ is the operator of the vector potential of the electromagnetic field and $\mathbf{B}(\mathbf{r}, t)$ is a canonically conjugate generalized momentum

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{4\pi c^2} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}. \tag{A1.5}$$

The density of the current representing a transition in Eq. (A1.4) is related to the Pauli matrices σ_+ and σ_- used in describing the dynamics of two-level atoms:

$$\mathbf{j}(\mathbf{r}, t) = \sum_{i=1}^{N} (\langle + | \mathbf{j}_0^+(\mathbf{r} - \mathbf{r}_i, t) | - \rangle \sigma_{i+}(t) + \langle - | \mathbf{j}_0^-(\mathbf{r} - \mathbf{r}_i, t) | + \rangle \sigma_{i-}(t)), \tag{A1.6}$$

where $\mathbf{j}_0^+(\mathbf{r})$ is the positive-frequency part of the current density operator describing a transition in a single atom; $|+\rangle$ and $|-\rangle$ are the wave functions of the excited and ground states of an atom. In a model of point-like atoms, the expression (A1.6) becomes

$$\mathbf{j}(\mathbf{r}, t) = \sum_{i=1}^{N} (\mathbf{m}_i \sigma_{i+}(t) + \mathbf{m}_i^* \sigma_{i-}(t))\, \delta(\mathbf{r} - \mathbf{r}_i), \tag{A1.7}$$

where $\mathbf{m}_i$ is the matrix element of the current representing a transition in the ith atom.

Using the commutation relationships for the operators $\mathbf{A}$ and $\mathbf{B}$

$$\begin{aligned} [A_\alpha(\mathbf{r}, t), A_\beta(\mathbf{r}', t)] &= [B_\alpha(\mathbf{r}, t), B_\beta(\mathbf{r}', t)] = 0, \\ [A_\alpha(\mathbf{r}, t), B_\beta(\mathbf{r}', t)] &= i\hbar \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}'), \quad \alpha, \beta = x, y, z, \end{aligned} \tag{A1.8}$$

and the Pauli spin matrices

$$[\sigma_{i+}, \sigma_{j-}] = \sigma_{i3}\delta_{ij}, \quad [\sigma_{i\pm}, \sigma_{j3}] = \mp 2\sigma_{i\pm}\delta_{ij}, \tag{A1.9}$$

we can readily obtain the following system of the operator equations:

$$\begin{aligned} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} &= 4\pi c^2 \mathbf{B}(\mathbf{r}, t), \\ \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} &= -\frac{1}{4\pi} \operatorname{curl}\operatorname{curl} \mathbf{A}(\mathbf{r}, t) + \frac{1}{c} \mathbf{j}(\mathbf{r}, t), \\ \frac{\partial \sigma_{i+}}{\partial t} &= i\omega_{0i}\sigma_{i+} + \frac{i}{\hbar c} \mathbf{m}_i^* \mathbf{A}(\mathbf{r}_i, t)\, \sigma_{i3}, \\ \frac{\partial \sigma_{i3}}{\partial t} &= \frac{2i}{\hbar c} (\mathbf{m}_i \sigma_{i+} - \mathbf{m}_i^* \sigma_{i-})\, \mathbf{A}(\mathbf{r}_i, t). \end{aligned} \tag{A1.10}$$

2. System of slowly varying equations in the nonresonance case

We shall represent the vector potential $A(\mathbf{r}, t)$ and the density of the polarization current $j(\mathbf{r}, t)$, occurring in Eq. (2.4), in the following form

$$A(\mathbf{r}, t) = A_1(x, t)\sin(\omega t - kx) + A_2(x, t)\sin(\omega t + kx + \varphi) + A_3(x, t)\cos(\omega t - kx) + A_4(x, t)\cos(\omega t + kx + \varphi), \quad (A2.1)$$

$$j(\mathbf{r}, t) = j_1(x, t)\cos(\omega t - kx) + j_2(x, t)\cos(\omega t + kx + \varphi) + j_3(x, t)\sin(\omega t - kx) + j_4(x, t)\sin(\omega t + kx + \varphi),$$

where $A_n(x, t)$ and $j_n(x, t)$ are slowly varying amplitudes. Substituting the expressions from the system (A2.1) into the system of equations (2.4), we obtain the following system for slowly varying amplitudes

$$\begin{aligned}
&\frac{\partial A_n(x, t)}{\partial t} - (-1)^n c\frac{\partial A_n(x, t)}{\partial x} = \frac{2\pi}{\varkappa} j_n(x, t),\\
&\frac{\partial j_1(x, t)}{\partial t} + \frac{1}{2T_2} j_1(x, t) + (\omega - \omega_0) j_3(x, t)\\
&\quad = \frac{|m|^2}{\hbar c}(A_1\rho_0 + A_2\rho_1 - A_4\rho_2),\\
&\frac{\partial j_2(x, t)}{\partial t} + \frac{1}{2T_2} j_2(x, t) + (\omega - \omega_0) j_4(x, t)\\
&\quad = \frac{|m|^2}{\hbar c}(A_2\rho_0 + A_1\rho_1 + A_3\rho_2),\\
&\frac{\partial j_3(x, t)}{\partial t} + \frac{1}{2T_2} j_3(x, t) - (\omega - \omega_0) j_1(x, t)\\
&\quad = -\frac{|m|^2}{\hbar c}(A_3\rho_0 + A_4\rho_1 + A_2\rho_2),\\
&\frac{\partial j_4(x, t)}{\partial t} + \frac{1}{2T_2} j_4(x, t) - (\omega - \omega_0) j_2(x, t)\\
&\quad = -\frac{|m|^2}{\hbar c}(A_4\rho_0 + A_3\rho_1 - A_1\rho_2),\\
&\frac{\partial \rho_0}{\partial t} + \frac{\rho_0 - \rho_e}{T_1} = -\frac{1}{\hbar c}(A_1 j_1 + A_2 j_2 - A_3 j_3 - A_4 j_4),\\
&\frac{\partial \rho_1}{\partial t} + \frac{\rho_1}{T_1} = -\frac{1}{2\hbar c}(A_1 j_2 + A_2 j_1 - A_3 j_4 - A_4 j_3),\\
&\frac{\partial \rho_2}{\partial t} + \frac{\rho_2}{T_1} = \frac{1}{2\hbar c}(A_1 j_4 + A_3 j_2 - A_2 j_3 - A_4 j_1),
\end{aligned} \quad (A2.2)$$

where

$$\rho(x, t) = \rho_0(x, t) + \rho_+(x, t)e^{i2kx} + \rho_-(x, t)e^{-i2kx} = \rho_0(x, t) + 2\rho_1(x, t)\cos(2kx + \varphi) - 2\rho_2(x. t)\sin(2kx + \varphi). \quad (A2.3)$$

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