Modified criterion for the Landau stabilization of the instability of a tangential velocity discontinuity in a compressible medium

A. M. Fridman

Astronomical Council of the USSR Academy of Sciences
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1. OBJECTIVE AND PLAN OF NOTE[1)]

The objective of this note is to show how significant the Landau stabilization criterion is in real systems with gradient flows.[3] We would also like to indicate the reason why the result of Ref. 3 was removed from Ref. 5 (1954) (although it is contained in the 1953 version) and from Ref. 2 after the critical article of Syrovatskiĭ.[4] Reference 3 is also absent in Ref. 1, and the reason indicated is the critical article of Ref.[4].

This note consists of three short parts. In the first part (section 2) an account is given, on a qualitative level, of the physics of the instability of a tangential discontinuity in the velocity of a subsonic flow and its stabilization in a supersonic flow. In the second part (section 3), the critical comment of Syrovatskiĭ[4] (directed at Ref. 3) is formulated. Syrovatskiĭ's note is correct for an infinitely extended medium. In the third and last part (section 4) it is shown that in *real spatially limited* supersonic flows with a tangential velocity discontinuity there is a stabilizing effect which is quantitatively described by a *modified Landau criterion*. Stabilization of the instability of tangential discontinuity of *quasi-two-dimensional* flows (for example, in shallow water[6,7] and the gas disks of galaxies[8]) occurs in full accordance with the Landau criterion.[3]

2. ON THE PHYSICS OF THE INSTABILITY OF A TANGENTIAL VELOCITY DISCONTINUITY IN A SUBSONIC FLOW AND ITS STABILIZATION IN A SUPERSONIC FLOW

For the adiabatic perturbations examined in Refs. 3 and 4 $S=$ const, and the connection between the thermal function W, the pressure P, and density ρ is determined from the expression $W=\int dP/\rho$. For $P=A\rho^{\gamma}$, where A and γ are constants (γ is an adiabat index, $\gamma=c_P/c_V$; c_P, c_V are the specific heats at constant pressure and volume, respectively) we have

$$W=\frac{\gamma}{\gamma-2}A^{1/\gamma}P^{(\gamma-1)/\gamma}=BP^{\alpha}. \qquad (1)$$

Thus, for any $\gamma>1$ ($\alpha>0$) the pressure P increases as W increases.

Figures 1a and 1b show the perturbations of a tangential discontinuity of velocity v directed along the x axis in two opposite limit cases, where the Mach number $\mathrm{Ma}\equiv v/c\ll 1$ and $\mathrm{Ma}\gg 1$, where c is the speed of sound. In Ref. 3 it is shown that the amplitude of the perturbation on both sides of the z axis from the plane $z=0$ of the tangential discontinuity falls off exponentially, $\sim e^{-z/z_0}$. Thus it is sufficient to restrict ourselves to the region $|z|<z_0$.

Region I (above the "hump" of the perturbation) in Fig. 1a may be seen as the region of the critical cross section of the subsonic jet ($\mathrm{Ma}\ll 1$), where, it is known,[9] the velocity of the flow is maximal. Then, from the Bernoulli equation for isentropic flow,

$$\frac{v^2}{2}+BP^{\alpha}=\text{const} \qquad (2)$$

it follows that the pressure above the hump will be minimal. This leads to a further increase in the amplitude of the perturbation–instability.[2)]

Region I (above the hump) in Fig. 1b may be seen as the region of the narrowing channel of the supersonic diffuser ($\mathrm{Ma}\gg 1$), where the velocity v decreases;[9] consequently, pressure above the hump should increase. This leads to "depression" of the hump back into II.

This is what comprises the *stabilization effect* on the instability of the tangential velocity discontinuity in a supersonic flow, which was first discovered by Landau.[3] However, on what then is Syrovatskiĭ's comment based concerning the absence of this stabilizing effect?

3. SYROVATSKIĬ'S CRITIQUE[4] OF LANDAU'S WORK[3]

Let there be a flow along the x axis with a tangential velocity discontinuity (Fig. 2): $v_0=v_x\Theta(-z)$, where Θ is a unit function. Selecting the perturbation in density ρ and velocity v in the form

$$\rho(x,z,t)\sim v(x,z,t)\sim\exp(ikx-\lambda|z|+\gamma t), \qquad (3)$$

Landau showed[3] the absence of instability for $v_0>v_{\mathrm{cr}}$. If the unperturbed density ρ_0 and speed of sound c_0 are considered to be unvaried on both sides of the discontinuity, $\rho_{01}=\rho_{02}=\rho_0$, $c_{01}=c_{02}=c_0$, then in this simplest case

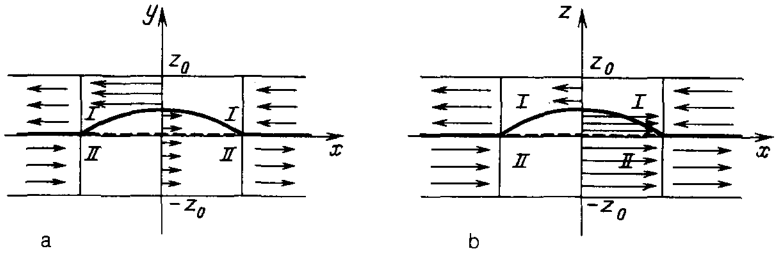

FIG. 1.

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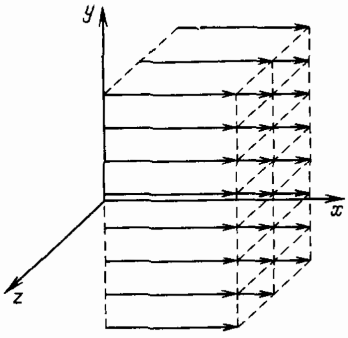

FIG. 2.

$$v_{cr}=2\sqrt{2}c_0. \tag{4}$$

As can be seen from Eq. (3), the wave vector $\mathbf{k}$ was chosen in Ref. 3 to be along the x axis, $k = k_x$. Ten years later, Syrovatskiĭ,[4] solving an analogous problem relative to a general class of perturbations $\mathbf{k} = \{k_x, k_y\} = \{k\cos\theta, k\sin\theta\}$, observed the presence of instability at any v_0.

Assuming that $\rho_{01}=\rho_{02}=\rho_0$, $c_{01}=c_{02}=c_0$, the problem of the instability of a tangential velocity discontinuity in a compressible fluid relative to arbitrary perturbations can be reduced to the following dispersion equation (the temporal dependence is chosen to be $\sim\exp(-i\omega t)$):

$$k^2c_0^2\left[\frac{1}{(\omega-kv_0)^4}-\frac{1}{\omega^4}\right]=\frac{1}{(\omega-kv_0)^2}-\frac{1}{\omega^2}. \tag{5}$$

Canceling out the common factor, which has only the real root $\omega = -kv/2$, we come to the equation

$$f(x)=1, \tag{6}$$

where

$$f(x)\equiv\frac{1}{(x-\mathrm{Ma}\cos\theta)^2}+\frac{1}{x^2},\quad x\equiv\frac{\omega}{kc_0},\quad \mathrm{Ma}\equiv\frac{v_0}{c_0},$$

which differs[4] from Landau's equation[3] by $\cos\theta$. Equation *(6) has four roots. They are all real if the function $f(x)$ is* analogous to the solid line shown in Fig. 3. If $f(x)$ is analogous to the dashed line in Fig. 3, then Eq. (6) has only two real roots. Consequently, the two others are complex conjugates one of which describes the instability. The majorant curve is shown in Fig. 3 by the dot-dash line. This case also has all real roots: $x_1', x_4', x_2' = x_3' = (1/2)\,\mathrm{Ma}\cos\theta$, two of which are multiples. The critical Mach number Ma_{cr} is found from the equation $f((1/2)\,\mathrm{Ma}\cos\theta) = 1$, which defines the point where the majorant curve is tangential to the line $f(x) = 1$. It is found to be equal to

$$\mathrm{Ma}_{cr}=\frac{2\sqrt{2}}{\cos\theta}. \tag{7}$$

Using the expression for $\cos\theta = k_x/|k_\perp|$, where $k_\perp = (k_x^2 + k_y^2)^{1/2}$, we obtain

$$\mathrm{Ma}_{cr}^2=8\left(1+\frac{k_y^2}{k_x^2}\right). \tag{8}$$

In quasi-two-dimensional systems, for example, the gas disks of galaxies and shallow water, only "longitudinal" waves are possible, $k_y/k_x \ll 1$, which were examined by Landau. In this case Ma_{cr} (Eq. (8)) is converted into the Landau[3] Ma_{cr}. The basic comment of Syrovatskiĭ reduced to the fact that arbitrary perturbations allow the examination of the opposite limit case, "transverse" waves, $k_y/k_x \gg 1$. It is obvious that, for example, for

$$\frac{k_y}{k_x}\to\infty \tag{9}$$

stabilization is, in principle, impossible[3)] since, as follows from Eq. (8), $\mathrm{Ma}_{cr}\to\infty$.

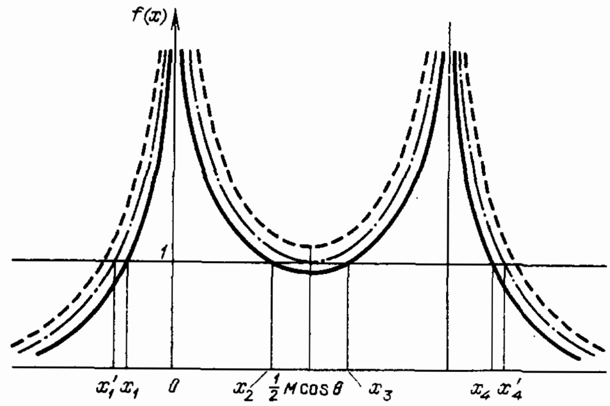

FIG. 3.

4. MODIFIED LANDAU STABILIZATION CRITERION

In an idealized statement of the problem, i.e. a *tangential* velocity discontinuity in a *three-dimensional infinite* space, the condition in Eq. (9) can be satisfied. However, the real situation introduces two significant corrections: 1) the system has finite spatial extent in all three dimensions; 2) the tangential velocity discontinuity is blurred by some amount a.

A consequence of these conditions is the existence of $(k_y/k_x)_{\max}\equiv(k_y)_{\max}/(k_x)_{\min}$. Actually, $(k_x)_{\min}\sim 1/L$, where L is the size of the system in x; $(k_y)_{\max}\sim 1/a$ which follows from the necessary condition of the existence of instability in a flow with an inhomogeneous velocity profile[11] $k_y a < 1$.

Thus, the instability of the "tangential discontinuity" of velocity under real conditions is found to be suppressed when

$$\mathrm{Ma}>\mathrm{Ma}_{cr}\equiv 2\left[2\left(1+\frac{L^2}{a^2}\right)\right]^{1/2}. \tag{10}$$

Usually in practice[4)] $L^2/a^2 \gg 1$, and in this case Ma_{cr} from Eq. (10) exceeds Ma_{crL} (Landau) by a factor of L/a:

$$\mathrm{Ma}_{cr}\approx\frac{L}{a}\mathrm{Ma}_{crL}. \tag{11}$$

Let us now write the condition of "deflection" of the perturbations

$$\frac{1}{\gamma_{\max}}\gg\frac{L}{v}, \tag{12}$$

where $\gamma\equiv\mathrm{Im}\,\omega$ is the increment of instability of the tangential velocity discontinuity. The sense of the criterion in Eq. (12) is that in the time that a gas passes through any region along a system of length L with a velocity v, the perturbations in this region will not have time to increase; when the condition in Eq. (12) is met, instability may be considered to be absent. According to Ref. 10, $\gamma_{max} \approx 0.5 (k_x)_{max} \cdot c \approx 0.5c/a$, which when substituted into Eq. (12) yields

$$\mathrm{Ma} \gg 0{,}18\,\mathrm{Ma}_{cr}. \tag{13}$$

Thus, satisfaction of the condition in Eq. (10) virtually indicates the satisfaction of the condition in Eq. (13).

A flow with a velocity discontinuity characterized by a Mach number $\mathrm{Ma} > \mathrm{Ma}_{cr}$ is stable if the size of the flow satisfies the condition

$$L < a\left(\frac{\mathrm{Ma}^2}{8} - 1\right)^{1/2} \approx \frac{a\,\mathrm{Ma}}{2\sqrt{2}} \quad \text{for } \mathrm{Ma}^2 \gg 8. \tag{14}$$

Thus, the inequality in Eq. (10) defines the *modified* Landau stabilization criterion of a tangential velocity discontinuity *in a real three-dimensional system*. The longitudinal (along the flow velocity) size of the *stable three-dimensional* system is determined from Eq. (14).

[1)] The content of this note was presented by its author at the end of 1983 at a seminar of the USSR Academy of Sciences Astronomical Council dedicated to the 75th birthday of L. D. Landau. E. M. Lifshits, who was present at the seminar, offered to write an article on this theme to introduce some small corrections in future editions of Refs. 1 and 2. The subsequent illness and death of E. M. Lifshits made the timeliness of this note problematic, and only a positive reaction of a recent seminar of V. L. Ginzburg to the comments presented here showed that they might be of interest to the readers of this journal.

[2)] It is clear that there is no need to explain why, in a weakly compressible gas, $\mathrm{Ma} \ll 1$, one should consider $\gamma > 1$ $(\alpha > 0)$.

[3)] We note, however, that for $k_y/k_x \to \infty$ the instability increment[10] $\gamma \to 0$. Below it will be shown that taking into account the increase in perturbations in the deflected flows virtually does not change the stabilization criterion based on Eq. (8).

[4)] For three-dimensional flows, as in the case of two-dimensional flows, as noted above, we get the Landau criterion.

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