A new representation of quantum mechanics

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In order to endow the Bohr–Sommerfeld quantization rule

$$\oint \rho \mathrm{d}q = n\hbar,\tag{1}$$

with physical meaning, the consistent quantum theory presented in Ref. 1 introduced "physical quantities" defined by

$$[\boldsymbol{\psi}(q)]^{-1}\boldsymbol{f}\boldsymbol{\psi}(q), \tag{2}$$

where q represents one or several variables that determine the state of the system. If the momentum p of formula (1) is understood in the sense of (2), the quantization rule becomes equivalent to the condition that the state function $\psi(q)$ assume the same value at the endpoints of some interval of the generalized coordinate q.

Let us demonstrate that definition (2) makes it possible to construct a new representation of quantum mechanics in which every dynamical variable is identified with a generally complex function of coordinates f(q).

Recall that "all elements needed to describe a quantum system can be assembled by defining the fundamental dynamical variables, the commutation relations obeyed by the corresponding observables, and the explicit representation of the Hamiltonian \hat{H} , which determines the time evolution of the system, in terms of these fundamental variables" (Ref. 2, p. 313). Note that the various quantum mechanical representations can be obtained from a given representation via unitary transformations of the following type:

$$\hat{f}' = \hat{S}\hat{f}\hat{S}^{-1}, \quad \psi' = S\psi$$

where $\hat{S}^+ = \hat{S}^{-1}$. In the general case, when the function $\psi(q)$ is complex, definition (2) is not a unitary transformation of the operator \hat{f} . If, on the other hand, the action of operator \hat{S} consists of multiplication by a phase factor $e^{i\varphi}(q)$, the value of the function f(q) will not change after a unitary transformation of the operator f and the state function ψ (gauge invariance).

Further, let us consider the fundamental properties of the "physical quantities" f(q) that follow from the standard quantum mechanics. We will demonstrate that if these properties of the "physical quantity" f(q) and the relations between the various "quantities" f_i are taken as fundamental, one can retrace the derivation and arrive at the original representation of the theory.

The expression for the average (expectation) value of a physical quantity is:²

$$\bar{f} = \int \psi^* \hat{f} \psi dq,$$

where $dq = dq_1 dq_2 \dots dq_5$. It follows from (2) that the aver-

age value can also be written as follows:

$$f = \int f(q) |\psi|^2 \, \mathrm{d}q. \tag{3}$$

Since the operator f is Hermitian we have

$$f = \int f^*(q) |\psi|^2 \mathrm{d}q$$

In other words

f⁺ =**1**

and

$$\operatorname{Im}\overline{f(q)} = 0. \tag{4}$$

We find that

$$\int \psi^* \hat{f} \, \hat{g} \psi \mathrm{d}q = \int (\hat{f} \psi)^* \hat{g} \psi \mathrm{d}q = \int f^* (q) \, g(q) \, |\psi|^* \, \mathrm{d}q,$$

i.e., the product of physical quantities f and g corresponds to a product of "quantities" $f^*(q)g(q)$ in some specific order. If the operators \hat{f} and \hat{g} commute, the following condition holds

$$\operatorname{Im}\overline{f(q)\,g(q)}=0.$$

Let us introduce the commutator

$$[f,g] = f^*g - g^*f. \tag{5}$$

We will take the "physical quantities" f and g to commute if [f,g] = 0 and to commute on average if $[\overline{f,g}] = 0$.

Since observables are Hermitian it follows from (2) that

$$[\overline{f,g}] = \int \psi^*[\widehat{f},\ \widehat{g}] \ \psi dq. \tag{6}$$

From the commutation relations for observables

$$[\hat{f}_l, \ \hat{f}_k] = C_{lk}^l \hat{f}_l \tag{7}$$

and definition (2) we obtain

$$[\overline{f_i, f_k}] = C_{ik}^{l} \overline{f_l}.$$
(8)

From the equations of motion in the Schrödinger form

$$\mathscr{B}\psi = H\psi,\tag{9}$$

where $\hat{\mathscr{C}} = ih \partial / \partial t$ and \hat{H} is a given function of the fundamental observables, we can write

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial t} + \frac{t}{\hbar} [\hat{H}, \hat{f}]. \tag{10}$$

in the Heisenberg representation. Using definition (2) we obtain

$$\overline{\mathscr{E}(q)} = \overline{H(f(q))},\tag{11}$$

$$\frac{d\bar{f}}{dt} = \frac{\partial\bar{f}}{\partial t} + \frac{i}{\hbar} [\overline{H}, \overline{f}].$$
(12)

It follows from the Hermiticity and linearity of operator \hat{f} together with the condition that the possible values of the dynamical variable f must belong to the set of eigenvalues of the observable \hat{f} , that the physical quantity f(q) can only take on a particular set of constant, coordinate-independent, necessarily real values λ_i . Moreover, in the general case

$$\bar{f} = |c_l|^2 \lambda_l. \tag{13}$$

Now we can postulate that each dynamical variable can be associated with a coordinate function f(q) that depends on the state of the system and satisfies the condition

$$\overline{f} = \int f(q) w(q) \, \mathrm{d}q,$$

with $\overline{\mathrm{Im}f} = 0$. The function $f[\psi(q)]$ takes on a particular set of constant (real) values λ_i which coincides with the set of possible values of the dynamical variable. In an arbitrary state of the system

$$\bar{f} = \lambda_i \omega_i. \tag{14}$$

The product of simultaneously measurable dynamical variables f and g corresponds to either $f^*(q)g(q)$ or $g^*(q)f(q)$.

Let us postulate the commutation relations

$$[f_l, f_k] = C_{lk}^l f_l \tag{15}$$

and the equations of motion

$$\mathscr{E}(q) = H(q) \tag{16}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} + \frac{i}{\hbar} [H, f].$$
(17)

We can also postulate the existence of a probability density w of localizing the system in configuration space that satisfies the normalization condition $\int w(q) dq = 1$ and the conservation law

$$-\frac{\partial w}{\partial t} = \operatorname{div} \mathbf{j},\tag{18}$$

where

$$\mathbf{j} = \frac{\mathbf{w}}{2m} \left(\mathbf{p} + \mathbf{p}^* \right). \tag{19}$$

Now we can show that in a particular case the postulates and relations (14)–(19) are sufficient to arrive at the usual representation of quantum theory. Suppose that there exists a function $\psi(q)$ that satisfies $w = |\psi|^2$, together with operators defined by (2) that satisfy the postulates and conditions (14)–(19). Then the condition $\overline{\text{Im}f} = 0$ for any function $\psi(q)$ ensures the Hermiticity of operator \hat{f} . Assuming that this operator is linear and that its eigenfunctions φ_i form a complete, linearly independent set, we obtain

$$\overline{f} = |c_i|^2 \lambda_i, \tag{20}$$

where c_i are the Fourier coefficients of the decomposition $\psi = c_i \varphi_i$.

By substituting (14) into equation (20) we obtain $w_i = |c_i|^2$.

Furthermore, by performing aveaging on equations

(15)-(17) we obtain equations (8), (11), and (12). Since $\psi(q,t)$ is an arbitrary function of coordinates, we can derive at a given instant in time equations (7), (9), and (10), which together with the aforesaid operator properties and equation $w(q) = |\psi(q)|^2$ describe the standard representation of quantum mechanics.

The question arises whether the new representation of quantum theory holds any advantages over the usual representations beyond the aforesaid possibility of interpreting the quantization rule (1) within the framework of quantum theory. Some physicists, such as R. Feynman³, consider the search of new representations of quantum mechanics and quantum electrodynamics justified because they can lead to new theoretical interpretations and generalizations. We believe the above representation to be advantageous at least from the methodological viewpoint.

Let us write the commutation relation

$$[\overline{x, p}] = i\hbar$$

for the coordinate x and the corresponding momentum p. This can also be written as

$$\int (xp - p^*x) w(x) dx.$$
(21)

The uncertainty relation $\overline{\Delta x^2} \,\overline{\Delta p^2} \ge \hbar^2/4$ can be deduced from (21) by the usual means.

If we assume that (21) is true locally, as well as on average, we can solve for the imaginary part of the momentum

$$\operatorname{Im} p = \frac{i\hbar}{2x} \, .$$

Let us postulate the commutation relation

$$[\Delta x, \, \Delta p] = i\hbar \tag{22}$$

for the uncertainties

$$\Delta x = x - \overline{x}, \ \Delta p = p - \overline{p}.$$

Then, from equation (21), we obtain

$$\operatorname{Im} p = \frac{i\hbar}{2\Delta x} . \tag{23}$$

It follows from (23) that if Δx increases, the imaginary part (and consequently the modulus) of momentum uncertainty decreases. Conversely, as $\Delta x \rightarrow 0$, the imaginary part and the modulus of the momentum uncertainty grow without bound.

Note also, that equation (23) together with the condition that the average value of the imaginary part of any physical variable go to zero, imposes an additional constraint on the state function

$$\int_{-\infty}^{+\infty} |\psi|^2 (x - \overline{x})^{-1} dx = 0.$$

A characteristic feature of quantum mechanics is the existence of two classes of physical quantities and states of the system: measurable and "computable" physical quantities, real and virtual states of the system. An experimentally measured value of a physical quantity is unique-a single measurement cannot yield any "superposition" of several values. The measured physical quantity is always given by a mathematical quantity that is a constant or uniquely defined function of time. This mathematical quantity belongs to the set of real numbers. Any state of an object can be defined by fixing the values of a maximally large (complete) set of physical variables. The real state of an object can only be defined, whether in a real or a *gedanken* experiment, by unique and precise values of the complete set of "measurable" physical quantities. In other words, the real state is necessarily an eigenstate of some complete set of observables that belong to the corresponding complete set of eigenvalues.

Sometimes it is erroneously asserted that the state of an object may be defined experimentally by establishing through repeated measurement all possible values and their relative frequencies of all the dynamical variables that make up an arbitrary complete set of physical quantities. In fact, in order to perform this operation one must find a means of repeatedly preparing the "same" state of an object. The question arises, how can we ascertain that a given state is actually the state we want to measure? Statistical methods are of no use here, because any ensemble statistics might include different states of the object. Evidently, before applying statistical methods we require some experimentally valid criterion for identifying a given state and establishing whether two such states are actually identical.

Bohr's complementarity principle of quantum mechanics implies that the state of an object also includes either a description of the experimental situation, i.e., measurement apparatus, or the enumeration of the particular complete set of dynamical variables the experiment intends to measure. If we measure one complete set of variables and then switch to another measurement instrument or another complete set of variables, we will find that in the meantime the object changed from a real into an intermediate or virtual (i.e., computable) state characterized by computable variables. These variables may be multiple-valued, the corresponding mathematical objects may be operators, matrices, etc. Virtual states may follow the system's motion along a classical trajectory, but the amplitude summation rules will be different. By changing simultaneously the type of intermediate states and the alternative summation rules, one obtains different variants of virtual states and different rules for computing the possible final states selected by a measurement.

Intermediate states and the corresponding physical quantities invariably have some "peculiar" or exotic properties, no matter how close they are to classical counterparts. For example, the appearance of virtual particles in the intermediate states of quantum field theory violates either energy conservation (within the limits of the energy-time uncertainty product) or the relativistic relation between energy and momentum. Yet the measurement of a complete set of physical variables and description of the new state in terms of the quantities that had just been measured will transform the intermediate, virtual state into a real state and will remove all "peculiarities" from both the state description and the values of physical quantities.

Any assertion about computable quantities or virtual states can always turn out to be conditional, in other words dependent on the particular representation of quantum theory. For example, a number of authors have discussed the introduction of the concept of phase space into quantum mechanics, i.e., the possibility of simultaneously describing the state of an object in momentum and coordinate spaces. In our representation this is indeed possible: equation (1) already presupposes that momentum can be considered a function of coordinates. The "peculiarity" of definition (2) consists of the fact that this function is generally complex.

At this time, the author cannot point to any particular advantages of the above-described representation in solving concrete problems that cannot be approached by other means. Perhaps such advantages will surface later. Recall that Feynman's method, which eventually proved successful precisely in solving concrete problems, "provided no practical advantages in the early years".⁴

¹A. S. Tarnovskii, Usp. Fiz. Nauk **160**, 155 (1990) [Sov. Phys. Usp. **33**, 86 (1990)].

²A. Messiah, *Quantum Mechanics*, Vol. 1, Wiley, N. Y., 1961 [Translated from French; Russ. transl., Nauka, M., 1978].

³R. P. Feynman, Phys. Rev. 84, 108 (1951).

⁴M. B. Menskii, Path Group: Measurements, Fields, Particles [in Russian], Nauka, M., 1983.

Translated by A. Zaslavsky