

# The Bohr-Sommerfeld quantization rule and quantum mechanics

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In the semiquantum Bohr theory for an arbitrary quasiperiodic motion of a system the quantization condition has the form

$$\oint p \, dq = \int_0^l p \, dq = 2\pi n \hbar \quad (n = 0, 1, 2, \dots). \quad (1)$$

For a hydrogen atom and hydrogenlike atoms the Bohr theory leads to the same emission and absorption spectra as the Heisenberg-Schrödinger quantum mechanics. But in quantum mechanics the generalized coordinate  $q$  and the generalized momentum  $p$  corresponding to it cannot have exact values simultaneously. Therefore equation (1) loses its meaning.

In the case of a circular orbit and when  $q$  is an angle variable condition (1) can be regarded as a requirement that an integral number of de Broglie waves would fit into a stationary orbit of an electron or as a result of quantization of the moment of momentum of the electron. But in the case of an arbitrary quasiperiodic motion and when  $q$  is not an angle variable the interpretation of condition (1) turns out to be unclear.

At the present time there is no generally recognized idea concerning the status of condition (1) even within the framework of the Bohr-Sommerfeld theory. A. Messiah in his book (Ref. 1, p. 35) asserts that "the determination of the "quantization rules" constitutes the central problem of this Old Quantum Theory" and that these rules are postulated on the basis of intuition. In the reference book (Ref. 2, p. 385) the condition (1) is included among Bohr's postulates: "Bohr's second postulate (the rule of quantization of orbits)" reduces to the quantization of the values of the moment of momentum of an electron moving along a circular orbit.

However in the Physics Encyclopedia (Ref. 3, p. 153) equation (1) is said to be a supplementary quantization condition which is "frequently incorrectly included among Bohr's postulates".

The problem arises: can one modify condition (1) in such a way that its meaning would be clear also within a consistent quantum mechanics.

We introduce the "physical quantity"

$$f = \frac{1}{\psi} \hat{f} \psi, \quad (2)$$

where  $f$  is the quantum mechanical operator of a physical

quantity,  $\psi$  is the state function of the object. In the eigen representation (and in the eigen state) of the physical quantity under consideration, i.e., when  $\hat{f}\psi = f\psi$ , the quantity that has been introduced, as follows from (2), coincides with the physical quantity in the usual sense.

Thus, in the  $q$ -representation the "physical quantity"  $q$  is identical with the generalized coordinate of the particle. The same will also apply to an arbitrary function of the coordinates: the corresponding "physical quantity" will be identical with the function  $f(q)$ .

In the quasiclassical approximation  $\psi = \exp(iS/\hbar)$  the generalized momentum in the sense of (2) will have the form  $p = \partial S / \partial q$ , which coincides with the well-known expression for the generalized momentum in terms of the action  $S$  (Ref. 3, p. 399).

The foregoing, apparently, justifies the introduction of "physical quantities" in the sense of (2) and the replacement by them of the usual physical quantities in equation (1). Replacing the momentum  $p$  by the expression  $-i\hbar \partial \ln \psi / \partial q$ , we obtain

$$-i\hbar \int_0^l \frac{\partial}{\partial q} \ln \psi \, dq = n\hbar,$$

from where we can obtain  $\psi(l)/\psi(0) = 1$  or  $\psi(l) = \psi(0)$  (3). This constitutes the meaning of the Bohr-Sommerfeld quantization rule from the point of view of quantum mechanics: for motion that is periodic in the coordinate  $q$  in the classical sense the state function of the object at the initial point  $q = 0$  and at the end of the period  $q = l$  must assume the same value.

In particular cases, for example, the motion of a point along a circle and the linear oscillator, one can verify that "quantization rule" (3) does indeed hold. We also note that "rule" (3) can be taken as the starting point and, arguing in the reverse order, obtain condition (1), in which, however, in the general case one should understand under coordinate and momentum not the usual physical quantities, but the quantities in the sense of (2).

<sup>1</sup>A. Messiah, *Quantum Mechanics*, Vol. 1, Wiley, NY, 1961 [Translated from French; Russ. transl., Nauka, M., 1978].

<sup>2</sup>B. Yavorsky and A. Detlaf. *Handbook of Physics*, Mir, M., 1980 [Russ. original, Nauka, M., 1977, 1985].

<sup>3</sup>*Physics Encyclopedia*, Vol. 1 (In Russian), Sov. entsiklopediya, M., 1988.

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